Pacific Journal of Mathematics

SIGMA THEORY AND TWISTED CONJUGACY, II: HOUGHTON GROUPS AND PURE SYMMETRIC AUTOMORPHISM GROUPS

DACIBERG L. GONÇALVES AND PARAMESWARAN SANKARAN

Volume 280 No. 2 February 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 280, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.280.349

SIGMA THEORY AND TWISTED CONJUGACY, II: HOUGHTON GROUPS AND PURE SYMMETRIC AUTOMORPHISM GROUPS

DACIBERG L. GONÇALVES AND PARAMESWARAN SANKARAN

Let $\phi : \Gamma \to \Gamma$ be an automorphism of a group Γ. We say that $x, y \in \Gamma$ are in the same ϕ-twisted conjugacy class and write $x \sim_\phi y$ if there exists an element $\gamma \in \Gamma$ such that $y = \gamma x \phi(\gamma^{-1})$. This is an equivalence relation on Γ called the ϕ-twisted conjugacy. Let $R(\phi)$ denote the number of ϕ-twisted conjugacy classes in Γ. If $R(\phi)$ is infinite for all $\phi \in \mathrm{Aut}(\Gamma)$, we say that Γ has the R_∞-property.

The purpose of this note is to show that the symmetric group S_∞, the Houghton groups and the pure symmetric automorphism groups have the R_∞-property. We show, also, that the Richard Thompson group T has the R_∞-property. We obtain a general result establishing the R_∞-property of the finite direct product of finitely generated groups.

This is a sequel to an earlier work by Gonçalves and Kochloukova, in which it was shown using the sigma theory of Bieri, Neumann and Strebel that, for most of the groups Γ considered here, $R(\phi) = \infty$ where ϕ varies in a finite index subgroup of the automorphisms of Γ.

1. Introduction

Let Γ be a group and let $\phi : \Gamma \to \Gamma$ be an endomorphism. Then ϕ determines an action Φ of Γ on itself where, for $\gamma \in \Gamma$ and $x \in \Gamma$, we have $\Phi_{\gamma(x)} = \gamma x \phi(\gamma^{-1})$. The orbits of this action are called the ϕ-twisted conjugacy classes. We write $x \sim_\phi y$ if x and y are in the same ϕ-twisted conjugacy class. Note that when ϕ is the identity automorphism, the orbits are the usual conjugacy classes of Γ. We denote by $\mathcal{R}(\phi)$ the set of all ϕ-twisted conjugacy classes and by $R(\phi)$ the cardinality $\#\mathcal{R}(\phi)$ of $\mathcal{R}(\phi)$. We say that Γ has the R_∞-property if $R(\phi) = \infty$, that is, if $\mathcal{R}(\phi)$ is infinite, for every automorphism ϕ of Γ.

The problem of determining which groups have the R_∞-property — more briefly the R_∞-problem — has attracted the attention of many researchers since it was discovered that all nonelementary Gromov-hyperbolic groups have the R_∞-property.

MSC2010: primary 20E45; secondary 20E36.

Keywords: twisted conjugacy, Reidemeister number, sigma theory, Houghton groups, infinite symmetric group, pure symmetric automorphism groups.

See [Levitt and Lustig 2000; Felshtyn 2001]. It is particularly interesting when the group in question is finitely generated or countable. The notion of twisted conjugacy arises naturally in fixed point theory, representation theory, algebraic geometry and number theory. In recent years the R_∞-problem has emerged as an active research area.

Recall that Houghton introduced a family of groups H_n, $n \geq 2$, defined as follows: let $M_n := \{1, 2, \dots, n\} \times \mathbb{N}$. Then the group H_n consists of all bijections $f : M_n \to M_n$ for which there exist integers $t_1, \dots, t_n$ such that $f(j, s) = (j, s+t_j)$ for all $s \in \mathbb{N}$ sufficiently large and all $j \leq n$. Note that necessarily $\sum_{1\leq j\leq n} t_j = 0$. Let $Z = \big\{(t_1, \dots, t_n) \mid \sum_{1\leq j\leq n} t_j = 0\big\} \subset \mathbb{Z}^n \cong \mathbb{Z}^{n-1}$. One has a surjective homomorphism $\tau : H_n \to Z \cong \mathbb{Z}^{n-1}$ sending f to its *translation part* $(t_1, \dots, t_n)$ (with notation as above). It is easily verified that τ is surjective with kernel the group of all *finitary* permutations of M_n. K. S. Brown [1987a] showed that H_n is finitely presented for $n \geq 3$ and that it is FP_{n-1} but not FP_n. Note that the above definition of H_n makes sense even for $n = 1$ and that we have $H_1 \cong S_\infty$. However, we treat the group S_∞ separately and we shall always assume that $n \geq 2$ while considering the family H_n.

Next we recall the group G_n, the group of *pure symmetric automorphisms* of the free group F_n of rank $n \geq 2$. Fix a basis x_k, $1 \leq k \leq n$, of F_n. Denote by $\alpha_{ij} \in \mathrm{Aut}(F_n)$, $1 \leq i \neq j \leq n$, the automorphism defined as $x_i \mapsto x_j x_i x_j^{-1}$, $x_k \to x_k$, $1 \leq k \leq n$, $k \neq i$. The group G_n is the subgroup of $\mathrm{Aut}(F_n)$ generated by α_{ij}, $1 \leq i \neq j \leq n$. McCool [1986] showed that G_n is finitely presented where the generating relations are:

(i) $[\alpha_{ij}, \alpha_{kl}] = 1$ whenever i, j, k, l are all different;

(ii) $[\alpha_{ik}, \alpha_{jk}] = 1$ and $[\alpha_{ij}\alpha_{kj}, \alpha_{ik}] = 1$ whenever i, j, k are all different.

It was shown by Gonçalves and Kochloukova [2010] that $R(\phi) = \infty$ for all ϕ in a finite index subgroup of the group of all automorphisms of Γ where $\Gamma = H_n, G_n$. Our main result is the following theorem. We give two proofs for the case of Houghton groups, neither of which use Σ-theory. However, we still need to use the results of [Gonçalves and Kochloukova 2010] in the case of G_n.

Theorem 1.1. *The following groups have the R_∞-property*:

(i) *the group S_∞ of finitary permutations of* $\mathbb{N}$,

(ii) *the Houghton groups H_n, $n \geq 2$, and*

(iii) *the group G_n, $n \geq 2$, of pure symmetric automorphisms of a free group of rank n.*

Recall that Richard Thompson constructed three finitely presented infinite groups $F \subset T \subset V$ around 1965 and showed that T and V are simple. The groups F, T, and V arise as certain homeomorphism groups of the reals, the circle, and the

Cantor set respectively. Since then these constructions have been generalized by G. Higman [1974]. See also Brown [1987a], R. Bieri and R. Strebel [2014], and M. Stein [1992]. For an introduction to the Thompson groups F, T, V see [Cannon et al. 1996].

Theorem 1.2. *The Richard Thompson group T has the R_∞-property.*

As the group T is simple, Σ-theory yields no information about the R_∞-property. The above theorem was first proved by Burillo, Matucci, and Ventura [Burillo et al. 2013]. Shortly thereafter, Gonçalves and Sankaran [2013] also independently obtained the same result.

In Section 2 we make some preliminary observations concerning the R_∞-property which will be needed for our purposes. Theorem 1.1 will be established in Section 3. The R_∞-property of the group T will be proved in Section 4. In Section 5 we consider the R_∞-property of finite direct products of groups and obtain a strengthening of a result of Gonçalves and Kochloukova [2010].

This is a sequel to the paper [Gonçalves and Kochloukova 2010]. We reassure the reader that this paper can be read independently of it. Although results from [Gonçalves and Kochloukova 2010] are used, we develop our own proof techniques to go forward.

Note. Just after this paper was submitted, J. H. Jo, J. B. Lee, and S. R. Lee [Jo et al. 2015] have announced almost simultaneously a proof of the R_∞-property for the Houghton groups.

If $f : X \to Y$ is a map of sets, we shall always write the argument to the right of f; thus $f(x)$ denotes the image of $x \in X$ under f.

2. Preliminaries

We begin by recalling some general results concerning twisted conjugacy classes of an automorphism of a group and that of its restriction to a normal subgroup. We obtain a criterion for a periodic automorphism to have infinitely many twisted conjugacy classes. We shall also briefly recall the notion of the Bieri–Neumann–Strebel invariant and give its known description in the case of Houghton groups and the pure symmetric automorphism groups.

2A. ***Addition formula.*** The following addition formula is found in [Gonçalves and Wong 2003, Lemma 2.1]. This is a special case of a more general formula proved in [Gonçalves and Wong 2005, §2]. For any element $g \in G$, we shall denote by ι_g the inner automorphism $x \mapsto gxg^{-1}$ of G. When N is a normal subgroup of G, we shall abuse notation and denote by the same symbol ι_g the automorphism of N obtained by restriction of ι_g to N.

Lemma 2.1. *Suppose that we have a commutative diagram of homomorphisms of groups where the vertical arrows are isomorphisms and horizontal rows are short exact sequences*:

$$\begin{array}{ccccccccc} 1 & \to & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \to & 1 \\ & & \downarrow \theta' & & \downarrow \theta & & \downarrow \bar{\theta} & & \\ 1 & \to & N & \xrightarrow{i} & G & \xrightarrow{p} & G/N & \to & 1 \end{array}$$

Then:

(i) *One has an exact sequence of* (*pointed*) *sets* $\mathcal{R}(\theta') \xrightarrow{i_*} \mathcal{R}(\theta) \xrightarrow{p_*} \mathcal{R}(\bar{\theta}) \to \{0\}$. *That is*, p_* *is surjective and* $\mathrm{Im}(i_*)$ *equals* $p_*^{-1}(\{N\})$.

(ii) (*Addition formula*) *Suppose* $R(\bar{\theta}) < \infty$ *and* $\mathrm{Fix}(\iota_{\alpha N} \circ \bar{\theta}) = \{N\}$ *for all* $\alpha \in G$. *Then* $R(\theta) < \infty$ *if and only if* $R(\iota_\alpha \theta') < \infty$ *for all* $\alpha \in G$. *Moreover*, *the following addition formula holds if* $R(\theta) < \infty$: $R(\theta) = \sum_{[\alpha N] \in \mathcal{R}(\bar{\theta})} R(\iota_\alpha \theta')$. □

We omit the proof. Part (i) is trivial. As mentioned above, the addition formula is also a known result. In any case, it can be proved in a straightforward manner. It can also be proved easily using the fixed point version of the following six-term exact sequence of sets due to P. R. Heath [2015, equation (2), p. 4] (cf. [Heath 1985, Theorem 1.8]), where $\bar{\alpha}$ denotes $\alpha N \in G/N$:

$$1 \to \mathrm{Fix}(\iota_\alpha \theta') \to \mathrm{Fix}(\iota_\alpha \theta') \to \mathrm{Fix}(\iota_{\bar{\alpha}} \bar{\theta}) \to \mathcal{R}(\iota_\alpha \theta') \to \mathcal{R}(\iota_\alpha \theta) \to \mathcal{R}(\iota_{\bar{\alpha}} \bar{\theta}) \to 1.$$

Remark 2.2. Note that if $G/N \cong \mathbb{Z}^n$, $n < \infty$, and if 1 is not an eigenvalue of the matrix of $\bar{\theta}$ with respect to a basis of G/N, then, for any $\alpha \in G$, we know that $\mathrm{Fix}(\iota_{\alpha N} \circ \bar{\theta}) = \mathrm{Fix}(\bar{\theta})$ consists only of the trivial element. So the lemma implies that if $R(\theta') = \infty$, then $R(\theta) = \infty$.

2B. ***Periodic outer automorphisms.*** Let Γ be a group with infinitely many conjugacy classes. Then, for any automorphism $\phi : \Gamma \to \Gamma$ and any $g \in G$, we have $R(\phi) = R(\iota_g \circ \phi)$ where ι_g denotes the inner automorphism $x \mapsto gxg^{-1}$. Indeed, it is readily seen that the ϕ-twisted conjugacy classes are the same as the left translation by g of the $\iota_g \circ \phi$-twisted conjugacy classes. Thus Γ has the R_∞-property if and only if $R(\phi) = \infty$ for a set of coset representatives of $\mathrm{Out}(\Gamma) = \mathrm{Aut}(\Gamma)/\mathrm{Inn}(\Gamma)$. We have the following lemma. Compare with [Gonçalves and Sankaran 2013, Lemma 3.4].

Lemma 2.3. *Let* $\theta \in \mathrm{Aut}(\Gamma)$ *and let* $n \geq 1$. *Suppose that* $\{x^n \mid x \in \mathrm{Fix}(\theta)\}$ *is not contained in the union of finitely many* θ^n*-twisted conjugacy classes of* Γ. *Then* $R(\theta) = \infty$.

Proof. Let $x \sim_\theta y$ in Γ where $x, y \in \mathrm{Fix}(\theta)$. Thus there exists a $z \in \Gamma$ such that $y = z^{-1} x \theta(z)$. Applying θ^i to both sides, we obtain $y = \theta^i(z^{-1}) x \theta^{i+1}(z)$, since $x, y \in \mathrm{Fix}(\theta)$. Write $\phi := \theta^n$. Multiplying these equations successively for $0 \leq i < n$, we obtain

$$y^n = \prod_{0 \le i < n} \theta^i(z^{-1}) x \theta^{i+1}(z) = z^{-1} x^n \theta^n(z) = z^{-1} x^n \phi(z).$$

That is, $y^n \sim_\phi x^n$. Our hypothesis says that there are infinitely many elements $x_k \in \mathrm{Fix}(\theta)$, $k \ge 1$, such that the x_k^n are in pairwise distinct ϕ-twisted conjugacy classes of Γ. Hence we conclude that $R(\theta) = \infty$. □

Remark 2.4. When $\theta^n = \iota_\gamma$ is an inner automorphism, we see from the above lemma that $R(\theta) = \infty$ if $\{x^n \gamma \mid x \in \mathrm{Fix}(\theta)\}$ is not contained in a finite union of conjugacy classes of Γ. When $\theta^n = \mathrm{id}$, we see that $R(\theta) = \infty$ if $\mathrm{Fix}(\theta)$ contains elements of order k for arbitrarily large values of $k \in \mathbb{N}$.

2C. *Σ-theory of H_n and G_n.* Bieri, Neumann, and Strebel [Bieri et al. 1987] introduced, for any finitely generated group Γ, an invariant $\Sigma(\Gamma)$ which is a certain open subset — possibly empty — of the character sphere $S(\Gamma) := \mathrm{Hom}(\Gamma, \mathbb{R}) \setminus \{0\}/\mathbb{R}_{>0}$ where the action of the multiplicative group of positive reals is via scalar multiplication. The automorphism group $\mathrm{Aut}(\Gamma)$ acts on $S(\Gamma)$ where $\phi^* : S(\Gamma) \to S(\Gamma)$ is defined as $[\chi] \mapsto [\chi \circ \phi]$, $[\chi] \in S(\Gamma)$, for $\phi \in \mathrm{Aut}(\Gamma)$. This action preserves the subspace $\Sigma(\Gamma)$ and hence also its complement $\Sigma^c(\Gamma)$. If the image of the antihomomorphism $\eta : \mathrm{Aut}(\Gamma) \to \mathrm{Homeo}(\Sigma^c(\Gamma))$ defined as $\phi \mapsto \phi^*$ is a finite group, then $K = \ker(\eta)$ is a finite index subgroup of $\mathrm{Aut}(\Gamma)$ which fixes every character class in $\Sigma^c(\Gamma)$. This happens, for example, if $\Sigma^c(\Gamma)$ is a nonempty finite set. If $\Sigma^c(\Gamma)$ contains a *discrete* character class $[\chi]$, that is, a class represented by a character χ whose image $\chi(\Gamma) \subset \mathbb{R}$ is infinite cyclic, then it was observed by Gonçalves and Kochloukova [2010] that the character χ itself is fixed by the action of K on $\mathrm{Hom}(\Gamma, \mathbb{R})$. That is, $\chi \circ \phi = \chi$ for all $\phi \in K \subset \mathrm{Aut}(\Gamma)$. This easily implies that $R(\phi) = \infty$ by Lemma 2.1(i), taking $G = \Gamma$, $N = \ker \chi$, $\theta = \phi$ in the notation of that lemma, so that $\bar{\theta} = \mathrm{id}$.

When Γ is G_n, $n \ge 3$, the group of pure symmetric automorphisms of F_n, L. Orlandi-Korner [2000] has determined $\Sigma^c(\Gamma)$. When Γ is H_n, the Houghton group, Brown [1987b] computed the set $\Sigma^c(\Gamma)$. Using these results, Gonçalves and Kochloukova [2010], showed that if Γ is any one of the groups H_n, $n \ge 2$, G_m, $m \ge 3$, then the image of $\eta : \mathrm{Aut}(\Gamma) \to \mathrm{Homeo}(\Sigma^c(\Gamma))$ is finite.

In the case of the Houghton group H_n, $n \ge 2$, it turns out that $\Sigma^c(H_n)$ is a finite set of discrete character classes $[\chi_j]$, $1 \le j \le n$. Explicitly, $\chi_j : H_n \to \mathbb{Z}$ may be taken to be $-\pi_i \circ \tau$ where $\tau : H_n \to Z$ is the translation part (see Section 1) and $\pi_i : Z \to \mathbb{Z}$ is the restriction to $Z \subset \mathbb{Z}^n$ of the i-th projection (see [Brown 1987b]). (Recall from Section 1 that $Z = \big\{(t_1, \ldots, t_n) \in \mathbb{Z}^n \mid \sum_{1 \le j \le n} t_j = 0\big\}$.) Thus $\mathrm{Homeo}(\Sigma^c(H_n)) \cong S_n$ is finite and so is the image of $\eta : \mathrm{Aut}(H_n) \to \mathrm{Homeo}(\Sigma^c(H_n))$. As already remarked,

$R(\phi) = \infty$ for all $\phi \in \ker(\eta)$. The lemma below will not be used in this paper but is included here for illustrative purposes.

Lemma 2.5. *Suppose $\eta(\phi) : \Sigma^c(H_n) \to \Sigma^c(H_n)$ is not an n-cycle. Then $R(\phi) = \infty$.*

Proof. Since $\eta(\phi)$ is not an n-cycle, the orbit of $[\chi_1]$ under $\eta(\phi)$ consists of at most $n-1$ elements. Since χ_1 is discrete, the orbit of $\chi_1 \in \mathrm{Hom}(H_n, \mathbb{R})$ consists of discrete elements. In fact, the orbit of χ_1 is a subset of $\{\chi_j \mid 1 \le j \le n\}$. Now the orbit sum $\lambda := \sum_{1 \le j \le k} \chi_1 \phi^j$ is a *nonzero* character since any $n-1$ elements of χ_j, $1 \le j \le n$, form a *basis* of $\mathrm{Hom}(H_n, \mathbb{R})$. It follows, since $\phi^*(\lambda) = \lambda$, that $R(\phi) = \infty$. □

If $\phi^* : \Sigma^c(H_n) \to \Sigma^c(H_n)$ is an n-cycle, the orbit sum is zero and the above argument fails. In fact, it is easily seen that every possible permutation of $\Sigma^c(H_n)$ may be realized as $\eta(\phi)$ for some $\phi \in \mathrm{Aut}(H_n)$; that is, $\eta : \mathrm{Aut}(H_n) \to \mathrm{Homeo}(\Sigma^c(H_n)) \cong S_n$ is surjective.

3. Proof of Theorem 1.1

Let X be an infinite set. We will only be concerned with the case when X is countably infinite. We shall denote by $S_\infty(X)$ the group of all finitary permutations of X, that is, those permutations which fix all but finitely many elements of X. The group of *all* permutations of X will be denoted by $S(X)$. We shall denote $S(X)$ (resp. $S_\infty(X)$) simply by S_ω (resp. S_∞) when X is clear from the context. If $x = (x_k)_{k \in \mathbb{Z}}$ is a doubly infinite sequence in X of pairwise distinct elements, we regard it as an element of $S(X)$ where $x(x_k) = x_{k+1}$ and $x(a) = a$ if $a \neq x_k$ for all $k \in \mathbb{Z}$. Two such sequences $x = (x_k)$ and $y = (y_k)$ define the same permutation if and only if y is a shift of x, that is, there exists an n such that $x_k = y_{k+n}$ for all $k \in \mathbb{Z}$. Thus, the sequence $x = (x_k)_{k \in \mathbb{Z}}$ is just the infinite cycle $x \in S(X)$. Any $f \in S(X)$ is uniquely expressible as a product of disjoint cycles. Such an expression of f is its *cycle decomposition*. The *cycle type* of an $f \in S(X)$ is the function $c(f) : \mathbb{N} \cup \{\infty\} \to \mathbb{Z}_{\ge 0} \cup \{\infty\}$ where $c(f)(\alpha)$ is the number of α-cycles in the cycle decomposition of f if that number is finite; otherwise it is ∞ for $\alpha \in \mathbb{N} \cup \{\infty\}$. As in the case $S_\infty(X)$, if f and g have the same cycle type, then they are conjugate in $S(X)$. We need a criterion for f and g to be conjugate by an element of $S_\infty(X)$.

Lemma 3.1. *Let $x = (x_k)_{k \in \mathbb{Z}}$, $y = (y_k)_{k \in \mathbb{Z}} \in S_\omega(X)$ be two disjoint infinite cycles and let $(a, b) \in S_\infty$.*

(i) *If $a = x_0$, $b = x_k$, $k > 0$, then $(a, b)x = uv$, where $u = (u_j)_{j \in \mathbb{Z}} \in S_\omega$, $v \in S_\infty$ are disjoint cycles defined by*

$$u_j = \begin{cases} x_j & j < 0, \\ x_{j+k} & j \ge 0, \end{cases}$$

and $v = (x_0, \ldots, x_{k-1}) \in S_\infty$.

(ii) *If* $a = x_0,\ b = y_0$, *then* $(a,b)xy = uv$, *where* $u = (u_j)_{j\in\mathbb{Z}},\ v = (v_j)_{j\in\mathbb{Z}}$ *are disjoint infinite cycles defined by*

$$u_j = \begin{cases} x_j & j < 0, \\ y_j & j \geq 0, \end{cases} \quad \text{and} \quad v_j = \begin{cases} y_j & j < 0, \\ x_j & j \geq 0. \end{cases} \qquad \square$$

If $k \in \mathbb{N}$, we denote by $\mathbb{N}_{>k}$ the set of all integers greater than k. Note that $S_\infty = \bigcup_{k\geq 2} S_k$ where S_k is the subgroup consisting of permutations of $\mathbb{N}$ which fix all $n > k$. In particular, the group S_∞ is generated by transpositions $(i, i+1),\ i \geq 1$. The alternating group A_∞ equals the commutator subgroup $[S_\infty, S_\infty]$, has index 2 in S_∞ and is simple. The conjugacy class of any element of S_∞ is determined by its cycle type, as in the case of finite symmetric groups. The group S_∞ is a normal subgroup of $S_\omega = S(\mathbb{N})$. In particular, any bijection $f : \mathbb{N} \to \mathbb{N}$ defines an automorphism $\iota_f \in \mathrm{Aut}(S_\infty)$ by restricting the inner automorphism determined by $f \in S_\omega$. Moreover ι_f is the identity automorphism only if f equals the identity map. The following result is well-known. See [Scott 1987, §11.4].

Theorem 3.2. *The homomorphism* $\iota : S_\omega \to \mathrm{Aut}(S_\infty)$ *is an isomorphism of groups.*

The following corollary is a special case of a more general result established in [Dixon and Mortimer 1996, Theorem 8.2A]. We include a proof, which is simpler in our special case.

Corollary 3.3. *Suppose that* S_∞ *is a characteristic subgroup of a group* H *contained in* S_ω. *Then the automorphism group of* H *is isomorphic to the normalizer* $N(H)$ *of* H *in* S_ω. *In particular, every automorphism of* H *is the restriction to* H *of a unique inner automorphism of* S_ω.

Proof. We shall use the same symbol ι_f to denote the conjugation by $f \in S_\omega$ or its restriction to any subgroup normalized by f.

It is evident that $\iota : N(H) \to \mathrm{Aut}(H)$ defined as $f \mapsto \iota_f$ defines an homomorphism. (Here $\iota_f(h) = fhf^{-1}$ for all $h \in H$.) This is a monomorphism since ι_f is nontrivial on $S_\infty \subset H$ if f is not the identity.

Let $\phi : H \to H$ be any automorphism and let $f \in S_\omega$ be the element such that $\phi|_{S_\infty} = \iota_f$. We claim that $\phi = \iota_f$. Suppose that $u := \phi(h),\ \iota_f(h) = fhf^{-1} =: v$ for some $h \in H$. We must show that $u(i) = v(i)$ for all $i \in \mathbb{N}$. It suffices to show that $\{u(i), u(j)\} = \{v(i), v(j)\}$ for all $i, j \in \mathbb{N},\ i \neq j$. Let $i, j \in \mathbb{N},\ i \neq j$. Now consider the transposition $(a,b) \in S_\infty$ such that $\iota_f(a,b) = \phi(a,b) = (i, j)$. We have

$$\phi(h(a,b)h^{-1}) = \phi(h)\phi(a,b)\phi(h^{-1}) = u(i, j)u^{-1} = (u(i), u(j)),$$

while

$$\iota_f(h(a,b)h^{-1}) = \iota_f(h)\iota_f(a,b)\iota_f(h^{-1}) = v(i, j)v^{-1} = (v(i), v(j)).$$

Therefore $(u(i), u(j)) = (v(i), v(j)) \in S_\infty$ since ι_f and ϕ agree on S_∞. This implies that $\{u(i), u(j)\} = \{v(i), v(j)\}$, completing the proof. $\square$

3A. ***S_∞ has the R_∞-property.*** Let $\theta \in \mathrm{Aut}(S_\infty)$. In view of Theorem 3.2, $\theta = \iota_f$ for some $f \in S_\omega$. Let $x, y \in S_\infty$ and suppose that $y = zx\theta(z^{-1}) = zxfz^{-1}f^{-1}$ for some $z \in S_\infty$. Then we have $yf = z(xf)z^{-1}$ in S_ω for some $z \in S_\infty$. For any cycle (finite or infinite) $u = (u_j)$, we have that zuz^{-1} is the cycle $(z(u_j))$. Any $z \in S_\infty$ moves only finitely many elements of $\mathbb{N}$. Hence when u is an infinite cycle we have $z(u_j) = u_j$ for all but finitely many $j \in \mathbb{Z}$. For an arbitrary element u expressed as a product of pairwise disjoint cycles, $u(\alpha) = (u(\alpha)_j)$, the element zuz^{-1} being a product of $zu(\alpha)z^{-1}$, we see that $zu(\alpha)z^{-1} = u(\alpha)$ for all but a finitely many α, and, moreover, if $u(\alpha) = (u(\alpha)_j)_{j\in\mathbb{Z}})$ is an infinite cycle, then $z(u(\alpha)_j) = u(\alpha)_j$ for all but finitely many $j \in \mathbb{Z}$.[1]

Lemma 3.4. *If $f \in S_\omega$ has an infinite cycle u, then there exist infinitely many transpositions $\tau_k \in S_\infty$ such that $\tau_j f \neq z\tau_k f z^{-1}$ for any $z \in S_\infty$.*

Proof. Fix an infinite cycle $u = (u_\alpha)_{\alpha\in\mathbb{Z}}$ that occurs in the cycle decomposition of f. Let $\tau_\alpha = (u_0, u_\alpha)$, $\alpha \geq 1$. Then we claim that $\tau_\alpha f$ and $\tau_\beta f$ are not conjugates if $\alpha \neq \beta$. To see this, we apply Lemma 3.1 to compute $\tau_\alpha u$, $\alpha \geq 1$. Note that the cycles that occur in $\tau_\alpha u$ also occur in the cycle decomposition of $\tau_\alpha f$. This is true in particular of the infinite cycle, denoted $v(\alpha)$, that occurs in $\tau_\alpha u$.

Now $v(\alpha)_p = v(\beta)_p = u_p$ for all $p < 0$ and $\alpha, \beta \geq 1$, and, when $\alpha \neq \beta$, we have $u_{p+\alpha} = v(\alpha)_p \neq v(\beta)_p = u_{p+\beta}$, $p \geq 0$. This implies that the $zv(\beta)z^{-1}$ cannot occur in $\tau_\alpha f$ for any $z \in S_\infty$ if $\alpha \neq \beta$ in its cycle decomposition, by the assertion made in the paragraph above the statement of the lemma. Hence $\tau_\alpha f \neq z\tau_\beta f z^{-1}$ for any $z \in S_\infty$. $\square$

We are now ready to prove part (i) of Theorem 1.1, restated below:

Theorem 3.5. *The group S_∞ has the R_∞-property.*

Proof. Let $\theta = \iota_f \in \mathrm{Aut}(S_\infty)$ where $f \in S_\omega$. We need to show that there exist pairwise distinct elements $\tau_j \in S_\infty$, $j \in \mathbb{N}$, such that $\tau_j f \neq z\tau_k f z^{-1}$ for any $z \in S_\infty$ if $j \neq k$. Since S_∞ has infinitely many conjugacy classes, the assertion holds for $f \in S_\infty$; thus we need only consider the case $f \notin S_\infty$. In the cycle decomposition of f, either (i) there exists an infinite cycle, or (ii) all the cycles are finite and there are infinitely many of them.

Case (i). In this case the assertion has already been established in Lemma 3.4.

Case (ii). Suppose that $f = \prod_{\alpha\in\mathbb{N}} u(\alpha)$ where the $u(\alpha)$ are all finite cycles having length $\ell(\alpha)$ at least 2 for every $\alpha \in \mathbb{N}$. Let $J := \{\alpha \in \mathbb{N} \mid \ell(\alpha) \geq 3\}$. We break up the proof into two subcases depending on whether J is infinite or not.

[1] There is a mild abuse of notation here; $u(\alpha)$ is not to be confused with the value of u at α. We will use Greek letters as labels in such situations.

Subcase (a). J is infinite. Let $J_k \subset J$ be the set consisting of the first k elements of J (with respect to the usual ordering on $J \subset \mathbb{N}$). Write $u(\alpha) = (u(\alpha)_1, \ldots, u(\alpha)_{\ell(\alpha)})$ and set $U(\alpha) := \{u(\alpha)_i \mid 1 \le i \le \ell(\alpha)\}$, $\alpha \in \mathbb{N}$. Consider the collection of pairwise disjoint transpositions $\lambda_\alpha = (u(\alpha)_1, u(\alpha)_2)$, $\alpha \in J$, and let $\tau_k = \prod_{\alpha \in J_k} \lambda_\alpha$. Note that

$$\lambda_\alpha u(\alpha) = (u(\alpha)_1) \cdot (u(\alpha)_2, \ldots, u(\alpha)_{\ell(\alpha)}) = (u(\alpha)_2, \ldots, u(\alpha)_{\ell(\alpha)})$$

fixes only $u(\alpha)_1$ in the set $U(\alpha)$, as $\ell(\alpha) \ge 3$. Then $\tau_k \cdot \prod_{\alpha \in J_k} u(\alpha)$ fixes only $u(\alpha)_1 \in \mathbb{N}$, $\alpha \in J_k$, in the set $\bigcup_{\alpha \in J_k} U(\alpha)$. Let $F_0 = \mathrm{Fix}(f)$. Then $\mathrm{Fix}(\tau_k f) = F_0 \cup \{u(\alpha)_1 \mid \alpha \in J_k\} =: F_k$.

Suppose that $\tau_j f = z \tau_k f z^{-1}$ with $z \in S_\infty$ and $j \neq k$. Then z defines a bijection $\zeta : F_j \to F_k$ between the fixed sets of $\tau_j f$ and $\tau_k f$. Clearly this is a contradiction if $\mathrm{Fix}(f) = F_0$ is finite. Assume that $F_0 \subset \mathbb{N}$ is infinite. Since z is in S_∞, it fixes all but finitely many elements of F_0. Let $L := \{m \in F_0 \mid z(m) \neq m\}$. Note that ζ restricts to the identity on $F_0 \setminus L$. Therefore ζ restricts to a bijection between $L \cup \{u(\beta)_1 \mid \beta \in J_j\}$ and $L \cup \{u(\beta)_1 \mid \beta \in J_k\}$. Since $j \neq k$, we have that L is finite and $L \subset F_0$ is disjoint from $\{u(\beta)_1 \mid \beta \in J_n\}$, $n = j, k$, which is a contradiction.

Subcase (b). The set J is finite; we set $K = \mathbb{N} \setminus J$ and define K_j, $j \in \mathbb{N}$, to be the set of first α elements of K. Again we set $\lambda_\alpha = (u(\alpha)_1), u(\alpha)_2) = u(\alpha)$, $\alpha \in K$. Now, if $\alpha \in K$, we have $\lambda_\alpha u(\alpha) = \mathrm{id}$; that is, $\lambda_\alpha u(j)$ fixes both points of $U(\alpha)$. We set $\tau_j := \prod_{\alpha \in K_j} \lambda_\alpha$ and $F_j := \mathrm{Fix}(\tau_j f) = F_0 \bigcup_{\alpha \in K_j} U(\alpha)$. Arguing exactly as above, for any $z \in S_\infty$, we see that $\tau_j f = z \tau_k f z^{-1}$ implies $j = k$, completing the proof. □

3B. ***Houghton groups.*** As in the introduction, H_n, $n \ge 2$, denotes the Houghton group. We first describe the group of outer automorphisms of H_n. Recall from Section 1 that one has an exact sequence

$$1 \to S_\infty(M_n) \hookrightarrow H_n \xrightarrow{\tau} Z \to 1$$

where $\tau : H_n \to Z$ sends $f \in H_n$ to the translation part $(t_1, \ldots, t_n) \in Z$ of f. The group $S_\infty(M_n)$ is the commutator subgroup of H_n if $n \ge 3$. When $n = 2$, the commutator subgroup is the alternating group $A_\infty(M_2)$ which has index 2 in $S_\infty(M_2)$. In any case, $S_\infty = S_\infty(M)$ is characteristic in H_n as H_n/S_∞ is the maximal *torsion-free* abelian quotient of H_n.

Lemma 3.6. *Let $\phi : H_n \to H_n$, $n \ge 2$, be an automorphism. Then ϕ is inner if and only if $\bar{\phi} : Z \to Z$ is the identity automorphism.*

Proof. It is trivial to see that any inner automorphism of H_n induces the identity automorphism of Z. For the converse, suppose that $\phi : H_n \to H_n$ induces the identity automorphism of Z.

Let $f \in S(M_n)$ be such that $\iota_f(H_n) = H_n$.

Consider the element $h_p : M_n \to M_n,\ 1 \le p < n$, in H_n defined as follows:[2]

$$h_p(i,k) = \begin{cases} (p, k+1) & \text{if } i = p,\ k \ge 1, \\ (n, k-1) & \text{if } i = n,\ k > 1, \\ (p, 1) & \text{if } i = n,\ k = 1, \\ (i, k) & \text{if } i \ne p, n. \end{cases}$$

Thus h_p permutes $\{p, n\} \times \mathbb{N}$ in a single cycle,

$$h_p = (\ldots, (n,2), (n,1), (p,1), (p,2), \ldots, (p,k), \ldots),$$

and so $f h_p f^{-1}$ is the cycle

$$f h_p f^{-1} = (\ldots, f(n,2), f(n,1), f(p,1), f(p,2), \ldots, f(p,k), \ldots) \in H_n.$$

The only infinite cycles in H_n are those whose terms, except for a finite part of the cycle, are *consecutive* numbers along two rays, say $\{i_n\} \times \mathbb{N}$ and $\{i_p\} \times \mathbb{N}$, in the negative and positive directions respectively of the cycle $f h_p f^{-1}$. Therefore we have $\tau(f h_p f^{-1}) = e_{i_p} - e_{i_n}$. Moreover, there exist integers $t_n,\ t_p$ such that $f(n,k) = (i_n, k + t_n)$ and $f(p,k) = (i_p, k + t_p)$ for sufficiently large k. Clearly i_n and t_n are independent of p. Since f is a bijection, the association $p \mapsto i_p$ is a permutation $\pi_f \in S_n$, and consequently $\sum_{1 \le q \le n} t_q = 0$. Note that $\pi_f = \mathrm{id}$ if and only if $f \in H_n$.

Since S_∞ is characteristic in H_n, by Corollary 3.3, $\phi = \iota_g$ for a unique $g \in S(M_n)$. *We claim that* $g \in H_n$. Since $\tau(g h g^{-1}) = \tau(\phi(h)) = \tau(h)$ for all $h \in H_n$, we have $\pi_g(q) = q$ for all $q \le n$ and so we have $g \in H_n$. □

The group S_n acts on the set $M_n = \{1, \ldots, n\} \times \mathbb{N}$ in the obvious manner, by acting via the identity on $\mathbb{N}$. This defines an action ψ of S_n on the group $S(M_n)$ defined as $f \mapsto \sigma \circ f \circ \sigma^{-1}$ which preserves the subgroup H_n. Thus we obtain a homomorphism $\psi : S_n \to \mathrm{Aut}(H_n)$. It is readily seen that $\tau(\psi_\sigma(h)) = \sigma(\tau(h))$ for all $h \in H_n$, where σ acts on $Z \subset \mathbb{Z}^n$ by permuting the standard basis elements $e_1, \ldots, e_n$. In particular ψ is a monomorphism. Let $\bar{\psi} : S_n \to \mathrm{Out}(H_n)$ be the composition of ψ with the projection $\mathrm{Aut}(H_n) \to \mathrm{Out}(H_n)$.

Proposition 3.7. *The homomorphism* $\bar{\psi} : S_n \to \mathrm{Out}(H_n)$ *is an isomorphism and so* $\mathrm{Aut}(H_n) = \mathrm{Inn}(H_n) \rtimes S_n \cong H_n \rtimes S_n$.

Proof. Lemma 3.6 shows that $\bar{\psi}$ is a monomorphism. We shall show that it is surjective.

Let $\phi \in \mathrm{Aut}(H_n)$. Write $\phi = \iota_f$ for a (unique) $f \in S(M_n)$. With notation as in the proof of Lemma 3.6, let $\pi := \pi_f \in S_n$.

[2]The element $(p, k) \in M_n$ should not be confused with the transposition in $S(\mathbb{N})$.

Consider the automorphism $\psi_\pi^{-1}\phi =: \theta$. We have

$$\tau(\theta(h_p)) = \pi^{-1}(\tau(\phi(h_p))) = \pi^{-1}(\tau(f h_p f^{-1})) = \pi^{-1}(e_{\pi(p)} - e_{\pi(n)}) = e_p - e_n = \tau(h_p)$$

for $1 \le p < n$. Since the group Z is generated by $\tau(h_p)$, $1 \le p < n$, it follows by Lemma 3.6 that θ is inner. Hence $\bar{\psi}(\pi) = \phi \pmod{\mathrm{Inn}(H_n)}$.

Finally, note that $\mathrm{Inn}(H_n) \cong H_n$ since the center of H_n is trivial. □

The above description of $\mathrm{Aut}(H_n)$ has been obtained by Burillo, Cleary, Martino, and Röver [Burillo et al. 2014, Theorem 2.2] and also by Cox [2014, §2.2]. All the proofs make essential use of Theorem 3.2 and Corollary 3.3. The proof given by Burillo et al. and our proof seem to be based on the same idea, although conceived of independently.

Theorem 3.8. *The Houghton group H_n has the R_∞-property for any $n \ge 2$.*

We shall give two proofs for part (ii) of Theorem 1.1, restated above. The first one uses the structure of the automorphism group of H_n and is more direct. The second one uses the result of Theorem 3.5 and the addition formula (Lemma 2.1).

First proof. Observe that there are infinitely many conjugacy classes in H_n since two elements in $S_\infty = S_\infty(M_n) \subset H_n$ are conjugates in H_n only if they have the same cycle type. It follows that $R(\phi) = \infty$ for any inner automorphism ϕ of H_n. Therefore, to show that $R(\phi) = \infty$ for an *arbitrary* $\phi \in \mathrm{Aut}(H_n)$, it suffices to show that $R(\phi) = \infty$ for all ϕ in a set of coset representatives of elements of $\mathrm{Out}(H_n)$. Thus we need only show that $R(\psi_\sigma) = \infty$ for any $\sigma \in S_n$, where $\psi : S_n \to \mathrm{Aut}(H_n)$ is as defined in the paragraph above Proposition 3.7. We shall use Lemma 2.3 and Remark 2.4 to achieve this.

For $k \ge 1$, consider the element ξ_k which is defined as the product of k-cycles $((i, 1), \ldots, (i, k)) \in H_n,\ 1 \le i \le n$. Explicitly,

$$\xi_k(i, j) = \begin{cases} (i, j+1) & \text{if } 1 \le j < k, \\ (i, 1) & \text{if } j = k, \\ (i, j) & \text{if } j > k, \end{cases}$$

for all $i \le n$. Then ξ_k is fixed by ψ_σ for every $\sigma \in S_n$. Thus, $\{\xi_k^n \mid k \ge 1\}$ contains elements of arbitrarily large orders and so by Remark 2.4 it follows that $R(\psi_\sigma) = \infty$ for all $\sigma \in S_n$, completing the proof. □

Second proof. Consider the exact sequence $1 \to S_\infty(M_n) \to H_n \to Z \to 0$. As remarked already, $S_\infty(M_n)$ is characteristic in H_n and we have $Z \cong \mathbb{Z}^{n-1}$. Thus any automorphism θ of H_n restricts to an automorphism θ' of $S_\infty(M_n)$ and induces an automorphism $\bar{\theta}$ of Z. If $R(\bar{\theta}) = \infty$ then, by Lemma 2.1(i), we have $R(\theta) = R(\bar{\theta}) = \infty$. Now suppose that $R(\bar{\theta}) < \infty$. Then $\mathrm{Fix}(\bar{\theta}) = 0$. Since Z is abelian and since $R(\theta') = \infty$ by Theorem 3.5, the addition formula (Lemma 2.1(ii)) yields $R(\theta) = R(\theta') = \infty$, completing the proof. □

3C. ***The group of pure symmetric automorphisms.*** Recall that $G_n \subset \mathrm{Aut}(F_n)$, $n \geq 2$, denotes the group of pure symmetric automorphisms of the free group F_n of rank n. A presentation for G_n, obtained by McCool [1986], was recalled in Section 1. It is immediate from this presentation that the abelianization $G_n^{\mathrm{ab}} = G_n/[G_n, G_n]$ is isomorphic to $\mathbb{Z}^{n^2-n}$ with basis the images $\bar{\alpha}_{ij}$, $1 \leq i \neq j \leq n$. We denote by $\{\chi_{ij} \mid 1 \leq i \neq j \leq n\}$ the basis of $\mathrm{Hom}(G_n^{\mathrm{ab}}, \mathbb{Z})$ dual to the basis $\{\bar{\alpha}_{ij} \mid 1 \leq i \neq j \leq n\}$. We shall denote by the same symbol χ_{ij} the composition $G_n \to G_n^{\mathrm{ab}} \xrightarrow{\chi_{ij}} \mathbb{Z} \hookrightarrow \mathbb{R}$. We will assume that $n \geq 3$, leaving out G_2 which is isomorphic to a free group of rank 2 and which is known to have the R_∞-property.

We begin by recalling the explicit description of $\Sigma^c(G_n)$ due to Orlandi-Korner [2000].

Let $A_{ij} := \mathbb{R}\chi_{ij} + \mathbb{R}\chi_{ji}$ and $B_{ijk} := \mathbb{R}(\chi_{ij} - \chi_{kj}) + \mathbb{R}(\chi_{jk} - \chi_{ik}) + \mathbb{R}(\chi_{ki} - \chi_{ji})$, with i, j, k pairwise distinct. Note that $A_{ij} = A_{ji}$ and $B_{ijk} = B_{pqr}$ if $\{i, j, k\} = \{p, q, r\}$. Let S be the union of vector subspaces $S = \bigcup A_{pq} \cup \bigcup B_{ijk} \subset \mathrm{Hom}(G_n, \mathbb{R})$ where the unions are over all pairs of distinct numbers $p, q \leq n$ and all pairwise distinct numbers $i, j, k \leq n$. It was shown by Orlandi-Korner [2000] that $\Sigma^c(G_n)$ is the image of $S \setminus \{0\} \subset \mathrm{Hom}(G_n, \mathbb{R}) \setminus \{0\}$.

Let $\mathcal{S}_n$ denote the semidirect product $C_2^n \rtimes S_n$ where S_n acts on C_2^n by permuting the coordinates. Here $C_2 = \{1, -1\}$. The group $\mathcal{S}_n$ acts effectively on F_n, the free group with basis $\{x_1, \ldots, x_n\}$ where $\pi \in S_n$ permutes the generators: we have the equality $\pi(x_j) = x_{\pi(j)}$, $1 \leq j \leq n$, and the action of the k-th factor of C_2^n is given by the automorphism $t_k(x_k) = x_k^{-1}$, $t_k(x_j) = x_j$, $j \neq k$. Thus $\mathcal{S}_n$ is a subgroup of $\mathrm{Aut}(F_n)$. It is readily verified that $\mathcal{S}_n$ normalizes G_n: $t_k \alpha_{i,j} t_k^{-1} = \alpha_{i,j}^{-1}$ if $k = j$ and equals $\alpha_{i,j}$ otherwise; if $\pi \in S_n$, then $\pi \alpha_{i,j} \pi^{-1} = \alpha_{\pi(i),\pi(j)}$ for all i, j. In particular, $\pi^*(A_{ij}) = A_{\pi(i)\pi(j)}$ and $\pi^*(B_{ijk}) = B_{\pi(i)\pi(j)\pi(k)}$ for all $\pi \in S_n$. Thus we have the following lemma:

Lemma 3.9. *Let $n \geq 3$. The action of the group $\mathcal{S}_n \subset \mathrm{Aut}(F_n)$ on $\mathrm{Hom}(G_n, \mathbb{R})$ and on $\Sigma^c(G_n)$ is defined by $\pi^*(\chi_{i,j}) = \chi_{\pi(i),\pi(j)}$, $t^*(\chi_{i,j}) = t_i t_j \chi_{i,j}$, for all $\pi \in S_n$, $t = (t_1, \ldots, t_n) \in C_2^n$.* □

The following proposition is a refinement of a statement in the proof of [Gonçalves and Kochloukova 2010, Theorem 4.11].

Proposition 3.10. *There exists a surjective homomorphism $\eta : \mathrm{Aut}(G_n) \to \mathcal{S}_n$ such that $\phi^*(\chi_{i,j}) = \epsilon_{i,j} \chi_{\sigma(i),\sigma(j)}$, $1 \leq i \neq j \leq n$, where $\epsilon_{i,j} \in \{1, -1\}$ and $\sigma = \eta(\phi) \in S_n$. In particular*, $\mathrm{Aut}(G_n) \cong K \rtimes \mathcal{S}_n$ *where* $K = \ker(\eta)$.

Proof. We see that ϕ^* preserves the collections of subspaces $\mathcal{A} := \{A_{ij} \mid 1 \leq i < j \leq n\}$ and $\mathcal{B} := \{B_{ijk} \mid 1 < i < j < k \leq n\}$, since ϕ^* is a linear isomorphism of $\mathrm{Hom}(G_n, \mathbb{R})$ and since $\phi^* : \Sigma^c(G_n) \to \Sigma^c(G_n)$ is a homeomorphism. Note that $\mathcal{B}$ is nonempty since $n \geq 3$. *In our notation A_{pq}, B_{pqr}, it is not assumed that $p < q < r$.*

It is readily seen that $(A_{pq}+A_{rs})\cap B_{ijk}=0$ unless $\{p,q,r,s\}=\{i,j,k\}$. On the other hand $(A_{ij}+A_{ik})\cap B_{ijk}=\mathbb{R}(\chi_{k,i}-\chi_{j,i})$. It follows that ϕ^* preserves the collection of 1-dimensional spaces $\mathcal{C}:=\{\mathbb{R}(\chi_{k,i}-\chi_{j,i}) \mid i,j,k \text{ pairwise distinct}\}$.

Let $\phi^*(A_{ij})=A_{pq}$, $\phi^*(A_{ik})=A_{rs}$, where i,j,k are pairwise distinct. Then $\{p,q\}\cap\{r,s\}$ is a singleton, say $s=p$ — so that $\phi^*(A_{ik})=A_{pr}$ — and we have $\phi^*(B_{ijk})=B_{pqr}$. For, otherwise, $(A_{ij}+A_{ik})\cap B_{ijk}$ is one-dimensional, whereas $\phi^*((A_{ij}+A_{ik})\cap B_{ijk})=(A_{pq}+A_{pr})\cap\phi^*(B_{ijk})=0$.

In view of the fact that ϕ^* stabilizes $\mathcal{C}$, we have

$$\phi^*(\chi_{k,i}-\chi_{j,i})=a(\chi_{r,p}-\chi_{q,p}). \tag{$*$}$$

On the other hand, we have $\chi_{k,i}\in A_{ik}$ and so $\phi^*(\chi_{k,i})\in\phi^*(A_{ik})=A_{pr}$ and so $\phi^*(\chi_{k,i})=b\chi_{p,r}+c\chi_{r,p}$ for some $b,c\in\mathbb{R}$; similarly, $\phi^*(\chi_{j,i})=b'\chi_{q,p}+c'\chi_{p,q}$ for some $b',c'\in\mathbb{R}$. Therefore,

$$\phi^*(\chi_{k,i}-\chi_{j,i})=b\chi_{p,r}+c\chi_{r,p}-b'\chi_{q,p}-c'\chi_{p,q}. \tag{$**$}$$

Comparing $(*)$ and $(**)$ we see that $b=0=c'$, that is, $\phi^*(\chi_{k,i})=c\chi_{r,p}$ and $\phi^*(\chi_{j,i})=b'\chi_{q,p}$. Since $\phi^*:\mathrm{Hom}(G_n;\mathbb{R})\to\mathrm{Hom}(G_n,\mathbb{R})$ preserves the lattice $\mathrm{Hom}(G_n,\mathbb{Z})$ and since $\chi_{k,i},\chi_{j,i}$ are part of a $\mathbb{Z}$-basis of $\mathrm{Hom}(G_n,\mathbb{Z})$, we see that $c,b'=\pm1$.

To complete the proof, we define the permutation $\sigma\in S_n$ associated to $\phi\in\mathrm{Aut}(G_n)$ as $\sigma(i)=p$ (with notation as above). Note that σ is indeed a bijection since ϕ^* is an isomorphism. We define $\eta:\mathrm{Aut}(G_n)\to S_n$ by $\eta(\phi)=\sigma$. Then η is a homomorphism of groups. It is surjective since its restriction to $S_n\subset\mathcal{S}_n$ is the identity by Lemma 3.9. This also shows that η splits, completing the proof. □

Remark 3.11. It seems plausible that there exists a surjective homomorphism $\tau:\mathrm{Aut}(G_n)\to\mathcal{S}_n$ that satisfies $\phi^*(\chi_{i,j})=t_it_j\chi_{\sigma(i),\sigma(j)}$, $1\le i\ne j\le n$, where $\tau(\phi)=(t_1,\dots,t_n)\in C_2^n$, $\sigma=\eta(\phi)\in S_n$. This would imply that $\mathrm{Aut}(G_n)\cong N\rtimes\mathcal{S}_n$ for a suitable subgroup $N\subset\mathrm{Aut}(G_n)$.

The above proposition says that the matrix of ϕ^*, with respect to the basis $\{\chi_{i,j}\mid 1\le i\ne j\}$ (ordered by, say, the lexicographic ordering of the indices i,j), is of the form $\phi^*=DP$ where D is a diagonal matrix with eigenvalues ±1 and P is a permutation matrix.

Lemma 3.12. *Let $T=DP$ where $D,P\in M_m(\mathbb{R})$ are such that D is a diagonal matrix and P is a permutation matrix. If $P=P_1\cdots P_k$ is a cycle decomposition then there exist eigenvectors $v_1,\dots,v_k$ which are linearly independent.*

Proof. The cycle decomposition allows us to express $\mathbb{R}^n$ as a direct sum $V_1\oplus\cdots\oplus V_k$ where V_j is spanned by $\{e_i\mid P_j(i)\ne i\}$. Specifically, if $P_j=(i_1,\dots,i_k)$. Then $v_j:=e_{i_1}+d_{i_1}e_{i_2}+\cdots+d_{i_1}\cdots d_{i_{k-1}}e_{i_k}$, which is the sum of the vectors in the DP-orbit of e_{i_1}, is an eigenvector of T with eigenvalue $d_{i_1}\cdots d_{i_k}$. Evidently $v_1, \ldots, v_k$ are linearly independent. $\square$

We will use the above lemma to construct two linearly independent eigenvectors of ϕ^* (with further properties that are relevant for our purposes). Let $\sigma = \eta(\phi) \neq \mathrm{id}$ and $\phi^* = DP$ with D diagonal and P a permutation transformation (with respect to the basis $\{\chi_{i,j}\}$). Suppose that σ has a k-cycle in its cycle decomposition, where $k > 2$. Choose any i that occurs in the k-cycle and let $j := \sigma(i)$. Then $\chi_{i,j}$ and $\chi_{j,i}$ do not occur in the same orbit of DP and therefore $v_{i,j} := \sum_{0\leq r<k}(DP)^r(\chi_{i,j})$ and $v_{j,i} := \sum_{0\leq j<k}(DP)^r\chi_{j,i}$ are eigenvectors of the same eigenvalue $\epsilon \in \{1, -1\}$. Without loss of generality, we assume that $i = 1,\ j = 2$ and define $v_{1,2} =: u$, $v_{2,1} =: v$. Suppose there is no such k-cycle in σ. Then σ is a product of disjoint transpositions. Without loss of generality, suppose that the transposition $(1, 3)$ occurs in the decomposition. Since $n > 2$, either σ has a fixed point, say 2, or $n > 3$ and, say, the transposition $(2, 4)$ occurs in the decomposition. In the first case, $u := \chi_{1,2} + d_{1,2}\chi_{3,2}$ and $v := \chi_{2,1} + d_{2,1}\chi_{2,3}$ are eigenvectors of P and in the latter case, $u := \chi_{1,2}+d_{1,2}\chi_{3,4}$ and $v := \chi_{2,1}+d_{2,1}\chi_{4,3}$ are eigenvectors of P. Thus in *all* cases, $\chi_{1,2}$ occurs in u and $\chi_{2,1}$ occurs in v where u, v are eigenvectors of ϕ^*. If 1 is an eigenvalue of ϕ^*, then $\bar{\phi}$ has a nonzero fixed element and so $R(\phi) = \infty$. Assume that $\phi^*(u) = -u,\ \phi^*(v) = -v$. Then there exist elements $\beta, \gamma \in G_n$ such that $\bar{\phi}(\bar{\beta}) = -\bar{\beta},\ \bar{\phi}(\bar{\gamma}) = -\bar{\gamma}$, where $\bar{\alpha}_{1,2}$, $\bar{\alpha}_{2,1}$ occur in $\bar{\beta}$, $\bar{\gamma}$ respectively, with coefficient 1.

Denote by $\Gamma_2 := \Gamma_2(G_n)$ the commutator subgroup of G_n and by $\Gamma_3 := \Gamma_3(G_n)$ the subgroup $[G_n, \Gamma_2] \subset \Gamma_2$. Thus G_n/Γ_3 is a two-step nilpotent group and we have the following exact sequences:

$$1 \to \Gamma_3 \to G_n \to G_n/\Gamma_3 \to 1,$$
$$1 \to \Gamma_2/\Gamma_3 \to G_n/\Gamma_3 \to G_n/\Gamma_2 \to 1.$$

Since Γ_2 and Γ_3 are characteristic in G_n, any automorphism of G_n restricts to automorphisms of Γ_2 and Γ_3 and hence induces automorphisms of the quotients $G/\Gamma_3,\ \Gamma_2/\Gamma_3$ and $G_n/\Gamma_2 = G_n^{\mathrm{ab}}$.

Let $\theta \in \mathrm{Aut}(G_n/\Gamma_3)$ be the automorphism defined by ϕ and θ', the restriction of θ to Γ_2/Γ_3. With notation as above, $[\beta, \gamma]\Gamma_3 \in \Gamma_2/\Gamma_3$ satisfies $\theta'([\beta, \gamma]\Gamma_3) = [\beta, \gamma]\Gamma_3$. By using the addition formula (Lemma 2.1), we conclude that $R(\theta) = \infty$, *provided* $[\beta, \gamma]/\Gamma_3$ is of infinite order. Granting this for the moment, by the first part of the same lemma we conclude that $R(\phi) = \infty$ using the first exact sequence above. Since $\phi \in \mathrm{Aut}(G_n)$ was arbitrary, we conclude that G_n has the R_∞-property. So all that remains is to show that $[\beta, \gamma]\Gamma_3$ is not a torsion element.

We use the fact that, under the surjection $\psi : G_n \to G_2$ that maps $\alpha_{i,j}$ to $\alpha_{i,j}$ when $\{i, j\} = \{1, 2\}$ and the remaining $\alpha_{i,j}$ to 1, we have that Γ_k maps onto $\Gamma_k(G_2)$,

$k = 2, 3$. Let $\beta_2, \gamma_2 \in G_2$ be the images of β, γ respectively under ψ. Then $\bar{\beta}_2 = \bar{\alpha}_{1,2}$, $\bar{\gamma_2} = \bar{\alpha}_{2,1} \in G_2^{\mathrm{ab}}$. Therefore, $[\beta_2, \gamma_2]\Gamma_3(G_2) = [\alpha_{1,2}, \alpha_{2,1}]\Gamma_3(G_2)$. Since G_2 is a free group with basis $\{\alpha_{1,2}, \alpha_{2,1}\}$ we see that $[\alpha_{1,2}, \alpha_{2,1}]\Gamma_3(G_2)$ generates an infinite cyclic group. Hence the same is true of $[\beta, \gamma]\Gamma_3$. This completes the proof of part (iii) of Theorem 1.1, which is restated below:

Theorem 3.13. *The group G_n, $n \geq 2$, has the R_∞-property.* □

4. The Thompson group T

Recall from Section 1 the description of the Richard Thompson group T as the group of all orientation-preserving piecewise linear homeomorphisms of $\mathbb{S} = I/\{0, 1\}$ with slopes in the multiplicative group generated by $2 \in \mathbb{R}_{>0}$ and break points in $\mathbb{Z}[1/2]$. We regard the Thompson group F as the subgroup of T consisting of elements which fix the element $1 \in \mathbb{S}^1$. In this section we prove the following result.

Theorem 4.1 [Burillo et al. 2013; Gonçalves and Sankaran 2013]. *The Richard Thompson group T has the R_∞-property.*

The fact that T has the R_∞-property was proved first by Burillo, Matucci, and Ventura [Burillo et al. 2013] (see also [Gonçalves and Sankaran 2013]). The crucial point in the proofs of the result above is the same in both of these papers and both the proofs rely on the description of the outer automorphism of T (recalled in Theorem 4.2 below). However, since the approaches before getting to the main point are slightly different, we provide our proof here which may contain some features that are useful for other situations (such as in Remark 4.7 below).

It is readily seen that the reflection map r defined as $r(x) = 1 - x$, $x \in [0, 1]$, induces an automorphism $\rho : T \to T$ defined as $\rho(f) = r \circ f \circ r^{-1} = r \circ f \circ r$. We now state the following result of Brin.

Theorem 4.2 [Brin 1996]. *The group of inner automorphisms of T is of index* 2 *in* $\mathrm{Aut}(T)$ *and the quotient group* $\mathrm{Out}(T)$ *is generated by ρ.*

As observed in Section 2B, for any group Γ and any automorphism $\phi \in \mathrm{Aut}(\Gamma)$, and any $g \in \Gamma$, it is true that $R(\phi) = \infty$ if and only if $R(\phi \circ \iota_g) = \infty$. Therefore, to establish the R_∞-property for Γ, it is enough to show that $R(\phi) = \infty$ for a set of coset representatives of $\mathrm{Out}(\Gamma)$. In the case $\Gamma = T$, in view of Theorem 4.2 due to Brin, we need only show that $R(\rho) = \infty$ and $R(\mathrm{id}) = \infty$. The latter equality is established in Proposition 4.5 as an easy consequence of Lemma 4.4 below. Since $\rho^2 = \mathrm{id}$, we may apply Remark 2.4 to show that $R(\rho) = \infty$. The main idea is to make use of homeomorphisms in $\mathrm{Fix}(\rho)$, whose supports have an arbitrarily large number of disjoint intervals in $\mathbb{S}^1$. (This was also the idea used in the proof by Burillo, Matucci, and Ventura [Burillo et al. 2013].)

Definition 4.3. Let X be a Hausdorff topological space.

(i) The *support of* $f \in \mathrm{Homeo}(X)$ is the open set $\mathrm{supp}(f) := \{x \in X \mid f(x) \neq x\}$.

(ii) Let $\sigma : \mathrm{Homeo}(X) \to \mathbb{N} \cup \{\infty\}$ be defined as follows: $\sigma(\mathrm{id}) = 0$, if $f \neq \mathrm{id}$; $\sigma(f)$ is the number of connected components of $\mathrm{supp}(f)$, if that number is finite; otherwise $\sigma(f) = \infty$.

Lemma 4.4. *Let* $\Gamma \subset \mathrm{Homeo}(X)$ *and let* σ *be as defined above. Suppose that* $\theta \in \mathrm{Homeo}(X)$ *normalizes* Γ. *Then* $\sigma(f) = \sigma(\theta f \theta^{-1})$.

Proof. It is clear that the number of connected components of an open set $U \subset X$ remains unchanged under a homeomorphism of X. The lemma follows immediately from the observation that $\mathrm{supp}(\theta f \theta^{-1}) = \theta(\mathrm{supp}(f))$. □

Proposition 4.5. *The groups* F *and* T *have infinitely many conjugacy classes.*

Proof. This follows from Lemma 4.4 on observing that F has elements f for which $\sigma(f)$ is any prescribed positive integer. Since $F \subset T$, the same is true of T. □

Lemma 4.6. *Suppose that* $h : \mathbb{R} \to \mathbb{R}$ *is an orientation-preserving homeomorphism. Then* $\mathrm{supp}(h) = \mathrm{supp}(h^k)$ *for any nonzero integer* k.

Proof. Since $\mathrm{supp}(h) = \mathrm{supp}(h^{-1})$ we may assume that $k > 0$. Since h is orientation-preserving, it is order-preserving. Suppose $x \in \mathrm{supp}(h)$ so that $h(x) \neq x$, and suppose $x < h(x)$. Then applying h to the inequality we obtain $h(x) < h^2(x)$ so that $x < h(x) < h^2(x)$. Repeating this argument yields $x < h(x) < \cdots < h^k(x)$ and so $x \in \mathrm{supp}(h^k)$. The case when $x > h(x)$ is analogous. Thus $\mathrm{supp}(h) \subset \mathrm{supp}(h^k)$. On the other hand, if $x \notin \mathrm{supp}(h)$, then $h(x) = x$ and so $h^k(x) = x$ for all k. Therefore, equality should hold, completing the proof. □

Proof of Theorem 4.1. By Theorem 4.2, $\mathrm{Out}(T) \cong \mathbb{Z}/2\mathbb{Z}$ is generated by ρ. By Proposition 4.5, $R(\mathrm{id}) = \infty$. It only remains to verify that $R(\rho) = \infty$. We apply Remark 2.4 with $\theta = \rho$, $n = 2$, $\gamma = 1$. It remains to show that $\mathrm{Fix}(\rho)$ has infinitely many elements h such that the h^2 are pairwise nonconjugate.

Let $k \geq 1$. Let $f_k \in F \subset T$ be such that $\mathrm{supp}(f_k)$ is a subset of $(0, 1/2)$ which has exactly k components. Thus, $\sigma(f_k) = k$. (It is easy to construct such an element.) Then $\mathrm{supp}(\rho(f_k)) = \mathrm{supp}(r f_k r^{-1}) = r(\mathrm{supp}(f_k)) \subset (1/2, 1)$ is disjoint from $\mathrm{supp}(f_k) \subset (0, 1/2)$. In particular, we have $f_k \cdot \rho(f_k) = \rho(f_k) \cdot f_k =: h_k$ and $\mathrm{supp}(h_k) = \mathrm{supp}(f_k) \cup r(\mathrm{supp}(f_k))$ and so $\sigma(h_k) = 2k$. Moreover, since $\rho^2 = 1$, we see that $h_k \in \mathrm{Fix}(\rho)$. By Lemma 4.6, we have $\sigma(h_k^2) = \sigma(h_k) = 2k$. It follows that h_k^2 are pairwise nonconjugate in T, completing the proof. □

Remark 4.7. In the case of the generalized Thompson groups $T_{n,r}$, suppose that $\theta \in \mathrm{Aut}(T_{n,r})$ is a torsion element, say of order m. Then our method of proof of Theorem 4.1 can be applied to show that $R(\theta) = \infty$. In fact, applying a theorem of McCleary and Rubin [2005] to the group $T_{n,r}$, we obtain that the automorphism group of $T_{n,r}$ equals its normalizer in the group of all homeomorphisms of the circle $\mathbb{S}^1 = [0, r]/\{0, r\}$. Let $\theta \in \mathrm{Aut}(T_{n,r})$ and $f \in \mathbb{S}^1$ such that $\theta(x) = fxf^{-1}$ with $f \in \mathrm{Homeo}(\mathbb{R}/r\mathbb{Z})$. Suppose $f^m = \gamma \in T_{n,r}$ so that θ represents a torsion element of $\mathrm{Out}(T_{n,r})$. If $\gamma = 1$, our method of proof of Theorem 4.1 can be applied to show that $R(\theta) = \infty$. See [Gonçalves and Sankaran 2013] for details. However, when $\gamma \neq 1$, it is not clear to us how to find elements of $\mathrm{Fix}(\theta)$ satisfying the hypotheses of Lemma 2.3. Our approach yields no information about automorphisms which represent nontorsion elements in the outer automorphism group. The study of the R_∞-property for the groups $T_{n,r}$ is a work in progress.

5. Direct product of groups

It was shown in [Gonçalves and Kochloukova 2010, Theorem 4.8] that if we have $G = G_1 \times \cdots \times G_n$, where each G_i is a finitely generated group with the property that $\Sigma^c(G_i)$ is a finite set of discrete character classes, not all of them empty, then there exists a finite index subgroup H of $\mathrm{Aut}(G)$ such that $R(\phi) = \infty$ for all $\phi \in H$. Further, when each G_i is a generalized Richard Thompson group $F_{n_i,\infty}$, $n_i \geq 2$, then G itself has the R_∞-property.

We shall strengthen the above result here. We make use (as did Gonçalves and Kochloukova [2010]) of a result of Meinert, recalled below, that describes the Σ-invariant of a direct product. (Meinert's theorem describes the Σ-invariant in the more general setting of a graph product of groups.)

Let $G = G_1 \times \cdots \times G_n$ and $r_j = rk(G_j^{\mathrm{ab}})$ so that $S(G_j) \cong \mathbb{S}^{r_j - 1}$. We assume that $r_1 \geq 1$. Then $S(G) = \prod_{1 \leq j \leq n} \mathrm{Hom}(G_j, \mathbb{R}) \setminus \{0\}/\sim \cong \mathbb{S}^{r-1}$ and so $S(G) \cong \mathbb{S}^{r-1}$, where $r := \sum_{1 \leq j \leq n} r_j$. It is understood that $S(G_j) = \varnothing$ if $r_j = 0$. The sphere $S(G_i)$ is identified with the subspace of $S(G)$ comprising the set of points with j-th coordinate equal to zero for all $j \neq i$. Observe that $S(G_i) \cap S(G_j) = \varnothing$ if $i \neq j$. In order to emphasize this, we shall write $S(G_i) \sqcup S(G_j)$ to denote their union, where $S(G_i)$ and $S(G_j)$ are thought of as subspaces of $S(G)$.

Recall that $\Sigma^c(G)$ denotes the complement of $\Sigma^1(G) \subset S(G)$.

Theorem 5.1 [Meinert 1995]. *Let $G = G_1 \times \cdots \times G_n$ be finitely generated and let $r_1 = rk(G_1^{\mathrm{ab}})$ be positive. With the above notation*, $\Sigma^c(G) = \bigsqcup_{1 \leq j \leq n} \Sigma^c(G_j)$. □

We will exploit the fact that any $\phi \in \mathrm{Aut}(G)$ induces a homeomorphism of the character sphere $S(G)$ which preserves its rational structure. Recall that an element $[\chi] \in S(G)$ is called *discrete* (or *rational*) if $\mathrm{Im}(\chi) \subset \mathbb{R}$ is infinite cyclic; equivalently, χ may be chosen to take values in $\mathbb{Q} \subset \mathbb{R}$. The set of rational points in $S(G)$ is denoted by $S_{\mathbb{Q}}(G)$. We denote by $D_{\mathbb{Q}}(G)$ the set of isolated rational points in $\Sigma^c(G)$. The set of all limit points of $D_{\mathbb{Q}}(G)$ which are contained in $S_{\mathbb{Q}}(G)$ is denoted by $L_{\mathbb{Q}}(G)$. Also, we denote by $L(G)$ the set of all limit points of $\Sigma^c(G)$. Since $\Sigma^c(G)$ is closed, $L_{\mathbb{Q}}(G)$ and $L(G)$ are subsets of $\Sigma^c(G)$.

Any homeomorphism of $\Sigma^c(G)$ induced by an automorphism of G maps $D_{\mathbb{Q}}(G)$, $L_{\mathbb{Q}}(G)$, $L(G)$ respectively onto itself.

We are now ready to prove the following theorem. The proof is essentially the same in spirit as that of [Gonçalves and Kochloukova 2010, Theorem 3.3]. See also [Gonçalves and Kochloukova 2010, §4c].

Theorem 5.2. *Suppose that $G = G_1 \times \cdots \times G_n$, $n \geq 1$, is finitely generated and that any one of the following holds*:

(i) *the set $D_{\mathbb{Q}}(G_1)$ is nonempty, finite, and contained in an open hemisphere and $D_{\mathbb{Q}}(G_j)$ is finite (possibly empty) for $2 \leq j \leq n$*;

(ii) *the set $L_{\mathbb{Q}}(G_1)$ is nonempty, finite, and contained in an open hemisphere and $L_{\mathbb{Q}}(G_j)$ is finite (possibly empty) for $2 \leq j \leq n$*;

(iii) *the set $L(G_1) \cap S_{\mathbb{Q}}(G_1)$ is nonempty, finite, and contained in an open hemisphere and $L(G_j) \cap S_{\mathbb{Q}}(G_j)$ is finite (possibly empty) for $2 \leq j \leq n$.*

Then G has the R_∞-property.

Proof. Suppose $\phi \in \operatorname{Aut}(G)$. We shall show that there exists a discrete character $\lambda \in \operatorname{Hom}(G, \mathbb{R})$ such that $\lambda \circ \phi = \lambda$. By the discussion in Section 2C, it follows that $R(\phi) = \infty$ and it follows that G has the R_∞-property.

First we suppose that $n = 1$. The theorem, then, is essentially due to Gonçalves and Kochloukova [2010]. Let $\phi^* : \Sigma^c(G) \to \Sigma^c(G)$ be the induced map, defined as $\phi^*([\chi]) = [\chi \circ \phi]$. Since ϕ^* is a homeomorphism, it maps isolated points to isolated points. Moreover, ϕ^* preserves the set of all rational points in $\Sigma^c(G)$. It follows that $\phi^*(W) = W$, where W is one of the sets $D_{\mathbb{Q}}(G)$, $L_{\mathbb{Q}}(G)$ or $L(G) \cap S_{\mathbb{Q}}(G)$.

In each of the cases (i)–(iii), we see that there is a nonempty finite set of rational character classes $W(G) \subset S_{\mathbb{Q}}(G)$ that is contained in an open hemisphere and that is mapped to itself by ϕ^*. Suppose that $[\chi] \in W(G)$ and that the orbit of $[\chi]$ under ϕ^*, namely the set $\{(\phi^*)^j([\chi] = [\chi \circ \phi^j] \mid j \in \mathbb{N}\}$, has k elements. Then the orbit sum $\lambda := \sum_{0 \leq j < k} \chi \circ \phi^j \in \operatorname{Hom}(G, \mathbb{R})$ is a nonzero discrete character invariant under ϕ^*, as was to be shown.

Now let $n = 2$. By Meinert's theorem (Theorem 5.1) $D_{\mathbb{Q}}(G) = D_{\mathbb{Q}}(G_1) \sqcup D_{\mathbb{Q}}(G_2)$, $L_{\mathbb{Q}}(G) = L_{\mathbb{Q}}(G_1) \sqcup L_{\mathbb{Q}}(G_2)$ and $L(G) = L(G_1) \sqcup L(G_2)$.

Case (i). Suppose $[\chi] \in D_{\mathbb{Q}}(G_1)$, and consider the ϕ^*-orbit of $[\chi]$, namely, the set $\{(\phi^k)^*([\chi]) = [\chi \circ \phi^k] \mid k \in \mathbb{Z}\}$. This set is finite since it is contained in $D_{\mathbb{Q}}(G) = D_{\mathbb{Q}}(G_1) \sqcup D_{\mathbb{Q}}(G_2)$, which is finite. Suppose that $[\chi \circ \phi^j]$, $0 \leq j < q$, are the distinct rational points in the orbit. Then we claim that the orbit sum $\lambda := \sum_{0 \leq j < q} \chi \circ \phi^j$ is a *nonzero* character such that $\lambda \circ \phi = \lambda$. To see that $\lambda \in \operatorname{Hom}(G, \mathbb{R})$ is nonzero, we note that its restriction to G_1 is the character $\lambda_J = \sum_{j \in J} \chi \circ \phi^j$ where $J := \{j < q \mid [\chi \circ \phi^j] \in D_{\mathbb{Q}}(G_1)\}$. Since $D_{\mathbb{Q}}(G_1)$ is contained in an open hemisphere, the characters $\chi \circ \phi^j$, $j \in J$, are in an open half-space of $\mathrm{Hom}(G_1, \mathbb{R})$. Therefore the same is true of their sum, λ_1, and we conclude that $\lambda \neq 0$. It is clear that $\lambda \circ \phi = \lambda$ since $[\lambda \circ \phi] = [\lambda]$ and since λ is rational. As observed in the first paragraph of Section 2C, this implies that $R(\phi) = \infty$.

Case (ii). The proof in this case is almost identical, starting with $[\chi] \in L_{\mathbb{Q}}(G_1)$. We need only note that $\phi^*(L_{\mathbb{Q}}(G))$ equals $L_{\mathbb{Q}}(G)$ and that $L_{\mathbb{Q}}(G) = L_{\mathbb{Q}}(G_1) \sqcup L_{\mathbb{Q}}(G_2)$ is finite, as in case (i). The orbit sum $\lambda := \sum_{0 \leq j < q} \chi \circ \phi^j$ is again a nonzero character which is discrete and satisfies $\lambda \circ \phi = \lambda$. Again we conclude that $R(\phi) = \infty$.

Case (iii). Again we start with $\chi \in L(G_1) \cap S_{\mathbb{Q}}(G_1)$ and proceed as in case (ii). We leave the details to the reader.

Finally, let $n \geq 3$ be arbitrary, and let $H = G_2 \times \cdots \times G_n$. Again by Meinert's theorem, we have $D_{\mathbb{Q}}(H) = \bigsqcup_{2 \leq j \leq n} D_{\mathbb{Q}}(G_j)$; similar expressions hold for $L_{\mathbb{Q}}(H)$ and $L(H) \cap S_{\mathbb{Q}}(H)$. Our hypotheses on G_j imply that one of the sets $D_{\mathbb{Q}}(H)$, $L_{\mathbb{Q}}(H)$, or $L(G) \cap S_{\mathbb{Q}}(G)$ is finite depending on case (i), (ii), and (iii), respectively. Since $G = G_1 \times H$, we are now reduced to the situation where $n = 2$, which has just been established. This completes the proof. $\square$

We conclude the paper with the following examples.

Examples 5.3. (i) Examples of groups with $D_{\mathbb{Q}}(G)$ nonempty, finite, and contained in an open hemisphere are known. These include nonpolycyclic nilpotent-by-finite groups of type FP_∞, the generalized Richard Thompson groups $F_{n,\infty}$, the double of a knot group K with nonfinitely generated commutator subgroup (thus $G \cong K \star_{\mathbb{Z}^2} K$). For details see [Gonçalves and Kochloukova 2010, §4].

(ii) Examples of groups with $D_{\mathbb{Q}}(G)$ and $L_{\mathbb{Q}}(G)$ being finite sets are finite groups, the Houghton groups [Brown 1987a], the pure symmetric automorphism groups [Orlandi-Korner 2000], finitely generated infinite groups with finite abelianization (which include the generalized Richard Thompson groups $T_{n,r}$; see [Brown 1987a, p. 64]), $\mathbb{Z}^n$, $n \geq 1$, and the free groups of rank $n \geq 2$. Another class of such groups is provided by [Bieri et al. 1987, Theorem 8.1]. Consider a finitely generated group G which is a subgroup of the group of all orientation-preserving PL-homeomorphisms of the interval [0, 1]. The group G is said to be *irreducible* if there is no G fixed point in (0, 1). The logarithms of the slopes near the end points 0, 1, define characters $\chi_0, \chi_1 : G \to \mathbb{R}$ respectively. We recall that two characters λ, χ are independent if $\lambda(\ker(\chi)) = \lambda(G)$ and $\chi(\ker(\lambda)) = \chi(G)$. It was shown in [Bieri et al. 1987, Theorem 8.1] that $\Sigma^c(G) = \{[\chi_0], [\chi_1]\}$ if G is irreducible and χ_0, χ_1 are independent. (These points may not be in $S_{\mathbb{Q}}(G)$; see [Bieri et al. 1987, p. 470].)

(iii) Let $G = G_1 \times G_2$ where G_1 is a finite product of groups (with G_1 nontrivial) as in example (i), and where G_2 is a finite product of groups as in example (ii) above. Then G has the R_∞-property. Since there are continuously many pairwise nonisomorphic 2-generated infinite simple groups, taking G_2 to be any one of them, we obtain a continuous family of groups with R_∞-property.

Acknowledgments

We thank Dessislava Kochloukova for pointing out to us the papers [Cox 2014] and [Burillo et al. 2014] concerning the Houghton groups. We thank the referee for a careful reading of our paper and for pointing out to us the paper [Heath 2015] and references to Theorem 3.2, Corollary 3.3, namely [Scott 1987] and [Dixon and Mortimer 1996]. Gonçalves was partially supported by Fundação de Amparo a Pesquisa do Estado de São Paulo (FAPESP), Projeto Temático Topologia Algébrica, Geométrica e Diferencial 2012/24454-8. Sankaran was partially supported by Department of Atomic Energy, Government of India, under a XII Plan Project.

References

[Bieri and Strebel 2014] R. Bieri and R. Strebel, "On groups of PL-homeomorphisms of the real line", preprint, 2014. arXiv 1411.2868

[Bieri et al. 1987] R. Bieri, W. D. Neumann, and R. Strebel, "A geometric invariant of discrete groups", *Invent. Math.* **90**:3 (1987), 451–477. MR 89b:20108 Zbl 0642.57002

[Brin 1996] M. G. Brin, "The chameleon groups of Richard J. Thompson: automorphisms and dynamics", *Inst. Hautes Études Sci. Publ. Math.* **84** (1996), 5–33. MR 99e:57003 Zbl 0891.57037

[Brown 1987a] K. S. Brown, "Finiteness properties of groups", *J. Pure Appl. Algebra* **44**:1-3 (1987), 45–75. MR 88m:20110 Zbl 0613.20033

[Brown 1987b] K. S. Brown, "Trees, valuations, and the Bieri–Neumann–Strebel invariant", *Invent. Math.* **90**:3 (1987), 479–504. MR 89e:20060 Zbl 0663.20033

[Burillo et al. 2013] J. Burillo, F. Matucci, and E. Ventura, "The conjugacy problem in extensions of Thompson's group F", preprint, 2013. arXiv 1307.6750v2

[Burillo et al. 2014] J. Burillo, S. Cleary, A. Martino, and C. E. Röver, "Commensurations and metric properties of Houghton's groups", preprint, 2014. arXiv 1403.0026

[Cannon et al. 1996] J. W. Cannon, W. J. Floyd, and W. R. Parry, "Introductory notes on Richard Thompson's groups", *Enseign. Math.* (2) **42**:3-4 (1996), 215–256. MR 98g:20058 Zbl 0880.20027

[Cox 2014] C. Cox, "Twisted conjugacy in Houghton's groups", preprint, 2014. arXiv 1410.7051v1

[Dixon and Mortimer 1996] J. D. Dixon and B. Mortimer, *Permutation groups*, Graduate Texts in Mathematics **163**, Springer, New York, 1996. MR 98m:20003 Zbl 0951.20001

[Felshtyn 2001] A. L. Felshtyn, "Число Райдемайстера любого автоморфизма громовской гиперболической группы бесконечно", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **279**:6 (2001), 229–240. Translated as "The Reidemeister number of any automorphism of a Gromov hyperbolic group is infinite" in *Journal of Mathematical Sciences* (*New York*), 119:1 (2004) 117–123. MR 2002e:20081 Zbl 1070.20050

[Gonçalves and Kochloukova 2010] D. L. Gonçalves and D. H. Kochloukova, "Sigma theory and twisted conjugacy classes", *Pacific J. Math.* **247**:2 (2010), 335–352. MR 2011m:20099 Zbl 1207.20037

[Gonçalves and Sankaran 2013] D. L. Gonçalves and P. Sankaran, "Twisted conjugacy in Richard Thompson's group T", preprint, 2013. arXiv 1309.2875v2

[Gonçalves and Wong 2003] D. L. Gonçalves and P. N.-S. Wong, "Twisted conjugacy classes in exponential growth groups", *Bull. London Math. Soc.* **35**:2 (2003), 261–268. MR 2003j:20054 Zbl 1030.20020

[Gonçalves and Wong 2005] D. L. Gonçalves and P. N.-S. Wong, "Homogeneous spaces in coincidence theory II", *Forum Math.* **17**:2 (2005), 297–313. MR 2006e:55004 Zbl 1103.55002

[Heath 1985] P. R. Heath, "Product formulae for Nielsen numbers of fibre maps", *Pacific J. Math.* **117**:2 (1985), 267–289. MR 86h:55002 Zbl 0571.55002

[Heath 2015] P. R. Heath, "Groupoids and relations among Reidemeister and among Nielsen numbers", *Topology Appl.* **181** (2015), 3–33. MR 3303931 Zbl 1308.55001

[Higman 1974] G. Higman, *Finitely presented infinite simple groups*, Notes on Pure Mathematics **8**, Department of Pure Mathematics, Australian National University, Canberra, 1974. MR 51 #13049

[Jo et al. 2015] J. H. Jo, J. B. Lee, and S. R. Lee, "The R_∞ property for Houghton's groups", preprint, 2015. arXiv 1412.8767

[Levitt and Lustig 2000] G. Levitt and M. Lustig, "Most automorphisms of a hyperbolic group have very simple dynamics", *Ann. Sci. École Norm. Sup.* (4) **33**:4 (2000), 507–517. MR 2002d:20066 Zbl 0997.20043

[McCleary and Rubin 2005] S. McCleary and M. Rubin, "Locally moving groups and the reconstruction problem for chains and circles", preprint, 2005. arXiv math/0510122

[McCool 1986] J. McCool, "On basis-conjugating automorphisms of free groups", *Canad. J. Math.* **38** (1986), 1525–1529. Zbl 0613.20024

[Meinert 1995] H. Meinert, "The Bieri–Neumann–Strebel invariant for graph products of groups", *J. Pure Appl. Algebra* **103**:2 (1995), 205–210. MR 96i:20047 Zbl 0838.20027

[Orlandi-Korner 2000] L. A. Orlandi-Korner, "The Bieri–Neumann–Strebel invariant for basis-conjugating automorphisms of free groups", *Proc. Amer. Math. Soc.* **128**:5 (2000), 1257–1262. MR 2000k:20046 Zbl 0963.20016

[Scott 1987] W. R. Scott, *Group theory*, 2nd ed., Dover, New York, 1987. MR 88d:20001 Zbl 0641.20001

[Stein 1992] M. Stein, "Groups of piecewise linear homeomorphisms", *Trans. Amer. Math. Soc.* **332**:2 (1992), 477–514. MR 92k:20075 Zbl 0798.20025

Received December 27, 2014. Revised June 15, 2015.

DACIBERG L. GONÇALVES
DEPARTMENT OF MATHEMATICS
INSTITUTE OF MATHEMATICS AND STATISTICS
UNIVERSITY OF SÃO PAULO
CAIXA POSTAL 66281
05314-970 SÃO PAULO-SP
BRAZIL

dlgoncal@ime.usp.br

PARAMESWARAN SANKARAN
THE INSTITUTE OF MATHEMATICAL SCIENCES
CIT CAMPUS
TARAMANI, CHENNAI 600113
INDIA

sankaran@imsc.res.in

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 280 No. 2 February 2016