THE SECOND CR YAMABE INVARIANT

Pak Tung Ho
THE SECOND CR YAMABE INVARIANT

PAK TUNG HO

Let \((M, \theta)\) be a compact strictly pseudoconvex CR manifold of real dimension \(2n + 1\) with a contact form \(\theta\). Motivated by the work of Ammann and Humbert, we define the second CR Yamabe invariant, which is a natural generalization of the CR Yamabe invariant, and study its properties in this paper.

1. Introduction

Let \((M, g)\) be an \(n\)-dimensional compact Riemannian manifold where \(n \geq 3\). The Yamabe problem is to find a Riemannian metric \(\tilde{g}\) conformal to \(g\) such that the scalar curvature of \(\tilde{g}\) is constant. Yamabe [1960] claimed to solve it. However, Trudinger [1968] realized that Yamabe’s proof was incomplete, and he was able to solve the Yamabe problem when the scalar curvature of \(g\) is nonpositive. When the scalar curvature of \(g\) is positive, Aubin [1976] solved the case when \(n \geq 6\) and \(M\) is not locally conformally flat, and Schoen [1984] solved the remaining cases by using the positive mass theorem.

The method to solve the Yamabe problem was the following. If \(\tilde{g} = u^{\frac{4}{n-2}} g\), where \(u \in C^\infty(M)\) and \(u > 0\), then

\[
L_g(u) = R_{\tilde{g}} u^{\frac{n+2}{n-2}},
\]

where

\[
L_g = -\frac{4(n-1)}{n-2} \Delta_g + R_g.
\]

Here \(\Delta_g\) is the Laplacian of \(g\), and \(R_g\) and \(R_{\tilde{g}}\) are the scalar curvatures of \(g\) and \(\tilde{g}\). The Yamabe problem is to solve (1-1) with \(R_{\tilde{g}}\) being constant. The Yamabe invariant \(Y(M, g)\) of \((M, g)\) is defined as

\[
Y(M, g) = \inf_{u \neq 0, u \in C^\infty(M)} E(u),
\]

where

\[
E(u) = \frac{\int_M u L_g(u) \, dV_g}{\left(\int_M |u|^{\frac{2n}{n-2}} \, dV_g\right)^{\frac{n-2}{n}}}.
\]

MSC2010: primary 32V20, 32V05; secondary 53C17.

Keywords: CR manifold, CR Yamabe problem, CR Yamabe invariant.
The key point of the resolution of the Yamabe problem is the following theorem due to Aubin [1976].

**Theorem 1.1.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). If \(Y(M, g) < Y(S^n)\), then there exists a positive smooth function \(u\) satisfying (1-1). Here \(Y(S^n)\) is the Yamabe invariant of the sphere \(S^n\) with respect to the standard metric.

The strict inequality was used to show that a minimizing sequence does not concentrate at any point. Aubin [1976] and Schoen [1984] proved the following:

**Theorem 1.2.** Let \((M, g)\) be a compact Riemannian manifold of dimension \(n \geq 3\). Then \(Y(M, g) \leq Y(S^n)\). Moreover, the equality holds if and only if \((M, g)\) is conformally diffeomorphic to the sphere.

These theorems solve the Yamabe problem. See also [Brendle 2005; 2007a; 2007b; Chow 1992; Schwetlick and Struwe 2003; Ye 1994] for using the flow approach to solve the Yamabe problem.

Ammann and Humbert [2006] defined the \(k\)-th Yamabe invariant as a generalization of the Yamabe invariant. More precisely, let

\[
\lambda_1(g) < \lambda_2(g) \leq \lambda_3(g) \leq \ldots \leq \lambda_k(g) \cdots \to \infty
\]

be the eigenvalues of \(L_g\) appearing with multiplicities. Let \([g]\) be the conformal class of \(g\). For any positive integer \(k\), the \(k\)-th Yamabe invariant \(Y_k(M, g)\) is defined by

\[
Y_k(M, g) = \inf_{[\tilde{g}] \in [g]} \frac{\lambda_k(\tilde{g}) \operatorname{Vol}(M, \tilde{g})^{\frac{2}{n}}}{[g]}.
\]

In particular, \(Y_1(M, g) = Y(M, g)\) when the Yamabe invariant \(Y(M, g)\) is nonnegative.

One can consider the following CR analogue of the Yamabe problem, the CR Yamabe problem. Suppose that \((M, \theta)\) is a compact strictly pseudoconvex CR manifold of real dimension \(2n + 1\) with a contact form \(\theta\). The CR Yamabe problem is to find a contact form \(\tilde{\theta}\) conformal to \(\theta\) such that the Webster scalar curvature of \(\tilde{\theta}\) is constant. Jerison and Lee [1987; 1988; 1989] solved the CR Yamabe problem when \(n \geq 2\) and \(M\) is not locally CR equivalent to the sphere. The remaining cases, namely when \(n = 1\) or \(M\) is locally CR equivalent to the sphere, were studied respectively by Gamara and Yacoub [2001] and by Gamara [2001]. See also the recent work of Cheng, Chiu and Yang [Cheng et al. 2014] and Cheng, Malchiodi and Yang [Cheng et al. 2013]. See also [Chang and Cheng 2002; Chang et al. 2010; Ho 2012; Zhang 2009] for using the flow approach to solve the Yamabe problem.

Motivated by the result of Ammann and Humbert [2006], we study the \(k\)-th CR Yamabe invariant in this paper. In Section 2, we define the \(k\)-th CR Yamabe invariant and the generalized contact form. In Section 3, we give the variational
characterization of $Y_k(M, \theta)$. In Section 4, we derive the Euler–Lagrange equation for $Y_2(M, \theta)$. Sections 5 and 6 will be devoted to proving a lower bound and an upper bound for $Y_2(M, \theta)$ respectively. In Section 7, we study whether $Y_2(M, \theta)$ is attained by some contact form or generalized contact form. Finally, in Section 8, we study the properties of the $k$-th CR Yamabe invariant $Y_k(M, \theta)$.

2. Definitions

Suppose that $(M, \theta)$ is a compact strongly pseudoconvex CR manifold of real dimension $2n + 1$ with a given contact form $\theta$. Let $u \in C^\infty(M)$, $u > 0$. Then $\tilde{\theta} = u^{\frac{2}{n}} \theta$ is a contact form conformal to $\theta$, and the Webster scalar curvature $R_{\tilde{\theta}}$ of $\tilde{\theta}$ is given by

\begin{equation}
L_\theta(u) = R_{\tilde{\theta}} u^{1 + \frac{2}{n}}.
\end{equation}

Here

\begin{equation}
L_\theta = -\left(2 + \frac{2}{n}\right) \Delta_\theta + R_\theta,
\end{equation}

where $\Delta_\theta$ is the sub-Laplacian of $\theta$ and $R_\theta$ is the Webster scalar curvature of $\theta$. The CR Yamabe invariant is defined as

\begin{equation}
Y(M, \theta) = \inf_{u \neq 0, u \in C^\infty(M)} E(u),
\end{equation}

where

\begin{equation}
E(u) = \frac{\int_M (2 + \frac{2}{n}) |\nabla_\theta u|_{\tilde{\theta}}^2 + R_\theta u^2 \, dV_\theta}{(\int_M |u|^{2 + \frac{2}{n}} \, dV_\theta)^{\frac{n}{n+1}}}.\end{equation}

It is well known that $L_\theta$ has discrete spectrum

\begin{equation}
\text{Spec}(L_\theta) = \{\lambda_1(\theta), \lambda_2(\theta), \ldots\},
\end{equation}

where the eigenvalues

\begin{equation}
\lambda_1(\theta) < \lambda_2(\theta) \leq \lambda_3(\theta) \leq \cdots \leq \lambda_k(\theta) \cdots \to \infty
\end{equation}

appear with multiplicities. The variational characterization of $\lambda_1(\theta)$ is given by

\begin{equation}
\lambda_1(\theta) = \inf_{u \neq 0, u \in C^\infty(M)} \frac{\int_M (2 + \frac{2}{n}) |\nabla_\theta u|_{\tilde{\theta}}^2 + R_\theta u^2 \, dV_\theta}{\int_M u^2 \, dV_\theta}.
\end{equation}

Let $[\theta]$ be the conformal class of $\theta$, i.e.,

\begin{equation}
[\theta] = \{\tilde{\theta} = u^{\frac{2}{n}} \theta \mid u \in C^\infty(M), u > 0\}.
\end{equation}

If $Y(M, \theta) \geq 0$, then it is easy to check that

\begin{equation}
Y(M, \theta) = \inf_{\tilde{\theta} \in [\theta]} \lambda_1(\tilde{\theta}) \text{Vol}(M, \tilde{\theta})^{\frac{1}{n+1}}.
\end{equation}
Following the definition of the $k$-th Yamabe invariant in [Ammann and Humbert 2006], we have the following:

**Definition.** For any positive integer $k$, the $k$-th CR Yamabe invariant is defined by

\[ Y_k(M, \theta) = \inf_{\tilde{\theta} \in [\theta]} \lambda_k(\tilde{\theta}) \text{Vol}(M, \tilde{\theta})^{\frac{4}{n+1}}. \]  

Then it follows from (2-3) and Theorem 8.2 that

\[ Y_1(M, \theta) = \begin{cases} Y(M, \theta) & \text{if } Y(M, \theta) \geq 0, \\ -\infty & \text{if } Y(M, \theta) < 0. \end{cases} \]

We write $L^{2+\frac{2}{n}}_+(M) = \{ u \in L^{2+\frac{2}{n}}(M) | u \geq 0, u \neq 0 \}$. For $u \in L^{2+\frac{2}{n}}_+(M)$, we define $\text{Gr}^u_k(C^\infty(M))$ to be the set of all $k$-dimensional subspaces of $C^\infty(M)$ such that the restriction operator to $M \setminus u^{-1}(0)$ is injective. More precisely, we have

\[ \text{span}(v_1, \ldots, v_k) \in \text{Gr}^u_k(C^\infty(M)) \iff v_1|_{M \setminus u^{-1}(0)}, \ldots, v_k|_{M \setminus u^{-1}(0)} \text{ are linearly independent} \]

\[ \iff u^{\frac{1}{n}} v_1, \ldots, u^{\frac{1}{n}} v_k \text{ are linearly independent}. \]

Similarly, replacing $C^\infty(M)$ by $S^2_1(M)$, we obtain the definition of $\text{Gr}^u_k(S^2_1(M))$. Hereafter, $S^2_1(M)$ denotes the Folland–Stein space, which is the completion of $C^1(M)$ with respect to the norm

\[ \|u\|_{S^2_1(M)} = \left( \int_M (|\nabla_{\tilde{\theta}} u|^2_{\tilde{\theta}} + u^2) \, dV_{\tilde{\theta}} \right)^{\frac{1}{2}}. \]

(For more properties about the Folland–Stein space, see [Folland and Stein 1974].)

**Proposition 2.1.** Suppose $\tilde{\theta}$ is a contact form conformal to $\theta$. Then we have

\[ \lambda_k(\tilde{\theta}) = \inf_{V \in \text{Gr}^u_k(S^2_1(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_{\tilde{\theta}} v \, dV_{\tilde{\theta}}}{\int_M u^{\frac{2}{n}} v^2 \, dV_{\tilde{\theta}}}. \]

**Proof.** Let $u \in C^\infty(M)$, $u > 0$. For all $f \in C^\infty(M)$, $f \neq 0$, we set $\tilde{\theta} = u^{\frac{2}{n}} \theta$ and

\[ F'(u, f) = \frac{\int_M f L_{\tilde{\theta}} f \, dV_{\tilde{\theta}}}{\int_M f^2 \, dV_{\tilde{\theta}}}. \]

The operator $L_{\theta}$ is conformally invariant in the following sense:

\[ u^{1+\frac{2}{n}} L_{\tilde{\theta}}(u^{-1} f) = L_\theta(f), \]
because
\[ u^{1 + \frac{2}{n}} L_{\tilde{\theta}}(u^{-1} f) = -\left(2 + \frac{2}{n}\right)u^{1 + \frac{2}{n}} \Delta_{\tilde{\theta}}(u^{-1} f) + R_{\tilde{\theta}}(u^{1 + \frac{2}{n}}(u^{-1} f)) \]
\[ = -\left(2 + \frac{2}{n}\right)(u \Delta_{\theta}(u^{-1} f) + 2(\nabla_{\theta} u, \nabla_{\theta}(u^{-1} f)) + \left(-\left(2 + \frac{2}{n}\right)\Delta_{\theta} u + R_{\theta} u\right)(u^{-1} f) \]
\[ = -\left(2 + \frac{2}{n}\right)\Delta_{\theta} f + R_{\theta} f = L_{\theta}(f), \]
where we have used (2-1) and (2-2). Combining (2-6) with the fact that
\[ (2-7) \quad dV_{\tilde{\theta}} = u^{2 + \frac{2}{n}} dV_{\theta}, \]
we get
\[ (2-8) \quad F'(u, f) = \frac{\int_M f L_{\tilde{\theta}} f dV_{\tilde{\theta}}}{\int_M f^2 dV_{\tilde{\theta}}} = \frac{\int_M f u^{-(1 + \frac{2}{n})} L_{\theta}(uf) u^{2 + \frac{2}{n}} dV_{\theta}}{\int_M f^2 u^{2 + \frac{2}{n}} dV_{\theta}} = \frac{\int_M (uf) L_{\theta}(uf) dV_{\theta}}{\int_M u^{\frac{2}{n}}(uf)^2 dV_{\theta}}. \]

Using the min-max principle, we have
\[ (2-9) \quad \lambda_k(\tilde{\theta}) = \inf_{V \in \text{Gr}_k(S^2_1(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_{\tilde{\theta}} v dV_{\tilde{\theta}}}{\int_M v^2 dV_{\tilde{\theta}}}. \]

Since \( u > 0 \), we have \( \text{Gr}_k(S^2_1(M)) = \text{Gr}^u_k(S^2_1(M)) \). Therefore, it follows from (2-8) and (2-9) that
\[ \lambda_k(\tilde{\theta}) = \inf_{V \in \text{Gr}_k(S^2_1(M))} \sup_{f \in V \setminus \{0\}} F'(u, f). \]

Now replacing \( uf \) by \( v \), we obtain (2-5) by (2-8). \( \square \)

Now we can define the generalized contact form:

**Definition.** The generalized contact form \( \tilde{\theta} \) is defined as \( \tilde{\theta} = u^{\frac{2}{n}} \theta \), where \( u \) is no longer necessarily positive or smooth, but \( u \in L^{2 + \frac{2}{n}}_{+}(M) \).

We enlarge the conformal class \( [\theta] \) of \( \theta \) by including all the generalized contact forms conformal to \( \theta \), as follows:
\[ [\theta] = \{ \tilde{\theta} = u^{\frac{2}{n}} \theta \mid u \in L^{2 + \frac{2}{n}}_{+}(M) \}. \]

In view of Proposition 2.1, for a generalized contact form \( \tilde{\theta} = u^{\frac{2}{n}} \theta, u \in L^{2 + \frac{2}{n}}_{+}(M) \), conformal to \( \theta \), we define
\[ (2-10) \quad \lambda_k(\tilde{\theta}) = \inf_{V \in \text{Gr}^u_k(S^2_1(M))} \sup_{v \in V \setminus \{0\}} \frac{\int_M v L_{\theta} v dV_{\theta}}{\int_M u^{\frac{2}{n}} v^2 dV_{\theta}}. \]
Using (2-10), we can generalize the definition of $k$-th CR Yamabe invariant to the generalized contact form by using (2-4).

3. Variational characterization of $Y_k(M, \theta)$

For all $u \in L^{2+\frac{2}{n}}_+(M)$, $v \in S^2_1(M)$ such that $u\frac{1}{n} v \neq 0$, we set

$$F(u, v) = \frac{\int_M (2 + \frac{2}{n})|\nabla_\theta v|^2 + R_\theta v^2 \, dV_\theta}{\int_M u^2 \frac{2}{n} v^2 \, dV_\theta} \left( \int_M u^{2+\frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}}.$$ 

**Proposition 3.1.** If $[\theta]$ contains all the contact forms conformal to $\theta$, then

$$Y_k(M, \theta) = \inf_{u \in C^\infty(M)} \sup_{v \in V \setminus \{0\}} F(u, v). \quad (3-1)$$

Similarly, if $[\theta]$ contains all the generalized contact forms conformal to $\theta$, then

$$Y_k(M, \theta) = \inf_{u \in L^{2+\frac{2}{n}}_+(M)} \sup_{v \in V \setminus \{0\}} F(u, v). \quad (3-2)$$

**Proof.** Using the definition of $Y_k(M, \theta)$ and the fact that $\text{Vol}(M, \tilde{\theta}) = \int_M u^{2+\frac{2}{n}} \, dV_\theta$, we obtain from (2-5) that

$$Y_k(M, \theta) = \inf_{\tilde{\theta} \in [\theta]} \lambda_k(\tilde{\theta}) \text{Vol}(M, \tilde{\theta})^{\frac{1}{n+1}}$$

$$= \inf_{u \in C^\infty(M), u > 0} \lambda_k(\tilde{\theta}) \left( \int_M u^{2+\frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}}$$

$$= \inf_{u \in C^\infty(M), u > 0} \sup_{V \in \text{Gr}_k^H(S^2_1(M))} \int_M u^\frac{2}{n} v \, dV_\theta$$

which proves (3-1). Similarly, we can prove (3-2) by using the same arguments as above, except we need to replace $C^\infty(M)$ by $L^{2+\frac{2}{n}}_+(M)$. \qed

4. Generalized contact form and the Euler–Lagrange equation

We will need the following:

**Lemma 4.1.** Let $u \in L^{2+\frac{2}{n}}_+(M)$ and $v \in S^2_1(M)$. We assume that

$$L_{\theta} \, v = u^\frac{2}{n} \, v \quad (4-1)$$

holds in the sense of distributions. Then $v \in L^{2+\frac{2}{n}+\varepsilon}(M)$ for some $\varepsilon > 0$. 

Proof. Without loss of generality, suppose \( v \neq 0 \). We define \( v_+ = \sup(v, 0) \). We let \( q \in (1, (n + 1)/n] \) be a fixed number and \( l > 0 \) be a large real number which will tend to \(+\infty\). We let \( \beta = 2q - 1 \). We then define for \( x \in \mathbb{R} \),

\[
G_l(x) = \begin{cases} 
0 & \text{if } x < 0, \\
x^\beta & \text{if } 0 \leq x < l, \\
l^{q-1}(ql^{q-1}x - (q-1)l^q) & \text{if } x \geq l, 
\end{cases}
\]

\[
F_l(x) = \begin{cases} 
0 & \text{if } x < 0, \\
x^q & \text{if } 0 \leq x < l, \\
l^{q-1}x - (q-1)l^q & \text{if } x \geq l. 
\end{cases}
\]

It is easy to check that for all \( x \in \mathbb{R} \),

\begin{align*}
(F_l'(x))^2 &\leq qG_l'(x), \\
(F_l(x))^2 &\geq xG_l(x), \\
xG_l'(x) &\leq \beta G_l(x).
\end{align*}

Since \( F_l \) and \( G_l \) are uniformly Lipschitz continuous functions, \( F_l(v_+) \) and \( G_l(v_+) \) belong to \( S^2(M) \). Let \( x_0 \in M \). Denote by \( \eta \) a \( C^2 \) nonnegative function supported in \( B(x_0, 2\delta) \), where \( \delta > 0 \) is a small fixed number such that \( 0 \leq \eta \leq 1 \) and \( \eta(B(x_0, \delta)) = \{1\} \). Multiply (4-1) by \( \eta^2G_l(v_+) \) and integrate over \( M \). Since the supports of \( v_+ \) and \( G_l(v_+) \) coincide, we get

\begin{equation}
(2 + \frac{2}{n}) \int_M \langle \nabla_\theta v_+, \nabla_\theta \eta^2 G_l(v_+) \rangle_\theta dV_\theta + \int_M R_\theta v_+ \eta^2 G_l(v_+) dV_\theta = \int_M u^{\frac{2}{n}} v_+ \eta^2 G_l(v_+) dV_\theta.
\tag{4-5}
\end{equation}

We are going to estimate the terms in (4-5). In the following, \( C \) will denote a positive constant depending possibly on \( \eta, q, \beta, \delta \), but not on \( l \). Note that

\begin{align*}
\int_M \langle \nabla_\theta v_+, \nabla_\theta \eta^2 G_l(v_+) \rangle_\theta dV_\theta \\
&= \int_M G_l(v_+) \langle \nabla_\theta v_+, \nabla_\theta \eta^2 \rangle_\theta dV_\theta + \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 dV_\theta \\
&= -\int_M G_l(v_+)v_+ \Delta_\theta (\eta^2) dV_\theta - 2\int_M v_+ G_l'(v_+)(\eta \langle \nabla_\theta v_+, \nabla_\theta \eta \rangle_\theta) dV_\theta \\
&\quad + \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 dV_\theta \\
&\geq -C \int_M v_+ G_l(v_+) dV_\theta - 2\int_M v_+^2 G_l'(v_+)|\nabla_\theta \eta|_\theta^2 dV_\theta \\
&\quad + \frac{1}{2} \int_M G_l'(v_+) \eta^2 |\nabla_\theta v_+|_\theta^2 dV_\theta,
\end{align*}

where
where the last inequality follows from $|\langle \nabla \theta v_+ , \nabla \theta \eta \rangle \rangle_\theta \leq |\nabla \theta \eta |^2_\theta + \frac{1}{4} |\nabla \theta v_+|^2_\theta$.

Hence, we have

\begin{align}
(4-7) \quad & \int_M \langle \nabla \theta v_+ , \nabla \theta \eta^2 G_l(v_+) \rangle_\theta dV_

\geq -C \int_M v_+ G_l(v_+) dV_

\geq -C \int_M (F_l(v_+))^2 dV_

= -C \int_M (F_l(v_+))^2 dV_

\geq -C \int_M (F_l(v_+))^2 dV_

\end{align}

where the first inequality follows from (4-6), the second inequality follows from (4-4), the third inequality follows from (4-2) and (4-3), and the fourth inequality follows from

\[ |\nabla \theta (\eta F_l(v_+))|_{\theta}^2 = |F_l(v_+) \nabla \theta \eta + \eta \nabla \theta F_l(v_+)|_{\theta}^2 \leq 2\eta^2 |\nabla \theta F_l(v_+)|_{\theta}^2 + 2|\nabla \theta \eta|^2_{\theta} (F_l(v_+))^2. \]

By the Folland–Stein embedding from $S^2_1(M)$ into $L^{2+\frac{2n}{n}}(M)$, there exists a constant $A > 0$ depending only on $(M, \theta)$ such that

\[ \int_M |\nabla \theta (\eta F_l(v_+))|_{\theta}^2 dV_\theta \geq A \left( \int_M (\eta F_l(v_+))^{2+\frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}} - \int_M (\eta F_l(v_+))^2 dV_\theta. \]

From this, together with (4-7), we obtain

\begin{align}
(4-8) \quad & \int_M \langle \nabla \theta v_+ , \nabla \theta \eta^2 G_l(v_+) \rangle_\theta dV_

\geq -C \int_M (F_l(v_+))^2 dV_\theta + \frac{A}{4q} \left( \int_M (\eta F_l(v_+))^{2+\frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}}. \end{align}
Independently, we choose \( \delta > 0 \) small enough such that

\[
\int_{B(x_0, 2\delta)} u^{2 + \frac{2}{n}} dV_\theta \leq \left( \left( 2 + \frac{2}{n} \right) \frac{A}{8q} \right)^{n+1}.
\]

Then it follows from (4-3), (4-9) and Hölder’s inequality that

\[
\int_M u^{\frac{2}{n}} v_+ \eta^2 G_l(v_+) dV_\theta \\
\leq \int_M u^{\frac{2}{n}} \eta^2 (F_l(v_+))^2 dV_\theta \\
\leq \left( \int_{B(x_0, 2\delta)} u^{2 + \frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M (\eta F_l(v_+))^{2 + \frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}} \\
\leq \left( 2 + \frac{2}{n} \right) \frac{A}{8q} \left( \int_M (\eta F_l(v_+))^{2 + \frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}}.
\]

On the other hand, it follows from (4-3) that

\[
\int_M R_\theta v_+ \eta^2 G_l(v_+) dV_\theta \geq -\left( \max_M |R_\theta| \right) \int_M v_+ \eta^2 G_l(v_+) dV_\theta \\
\geq -\left( \max_M |R_\theta| \right) \int_M \eta^2 (F_l(v_+))^2 dV_\theta \\
\geq -C \int_M (F_l(v_+))^2 dV_\theta.
\]

Substituting (4-8), (4-10), (4-11) into (4-5), we obtain

\[
\left( 2 + \frac{2}{n} \right) \frac{A}{8q} \left( \int_M (\eta F_l(v_+))^{2 + \frac{2}{n}} dV_\theta \right)^{\frac{n}{n+1}} \leq C \int_M (F_l(v_+))^2 dV_\theta.
\]

Now, by the Folland–Stein embedding, \( v_+ \in L^{2 + \frac{2}{n}}(M) \). Since \( 2q \leq 2 + \frac{2}{n} \) and \( C \) does not depend on \( l \), the right-hand side of the inequality is bounded when \( l \to \infty \), and we obtain

\[
\limsup_{l \to \infty} \int_M (\eta F_l(v_+))^{2 + \frac{2}{n}} dV_\theta < \infty.
\]

This proves that \( v_+ \in L^{q(2 + \frac{2}{n})}(B(x_0, \delta)) \). Since \( x_0 \) is arbitrary, we get that \( v_+ \in L^{q(2 + \frac{2}{n})}(M) \). Doing the same with \( v_- = \sup(-v, 0) \) instead of \( v_+ \), we get that \( v \in L^{q(2 + \frac{2}{n})}(M) \). This proves Lemma 4.1.

**Proposition 4.2.** For any generalized contact form \( \bar{\theta} = u^{\frac{2}{n}} \theta \), \( u \in L^{2 + \frac{2}{n}}_+(M) \), conformal to \( \theta \), there exist two functions \( v, w \in S^1_+(M) \) with \( v \geq 0 \) such that in the
sense of distributions

\[ L_\theta v = \lambda_1(\tilde{\theta})u^{2/n}_\theta v, \]
\[ L_\theta w = \lambda_2(\tilde{\theta})u^{2/n}_\theta w. \]

Moreover, we can normalize \( v \) and \( w \) such that

\[ \int_M u^{2/n}_\theta v^2 dV_\theta = \int_M u^{2/n}_\theta w^2 dV_\theta = 1 \quad \text{and} \quad \int_M u^{2/n}_\theta vw dV_\theta = 0. \]

Proof. Let \( (v_m)_m \) be a minimizing sequence for \( \lambda_1(\tilde{\theta}) \), i.e., a sequence \( v_m \in S^1_1(M) \) such that

\[ \lim_{m \to \infty} \int_M (2 + \frac{2}{n}) |\nabla v_m|^2 + R v_m^2 dV_\theta = \lambda_1(\tilde{\theta}). \]

It is well known that \( (|v_m|)_m \) is also a minimizing sequence. Hence we can assume that \( v_m \geq 0 \). If we normalize \( v_m \) by \( \int_M u^{2/n}_\theta v_m^2 dV_\theta = 1 \), then \( (v_m)_m \) is bounded in \( S^1_1(M) \) and after passing to a subsequence, we may assume that there exists \( v \in S^1_1(M) \), \( v \geq 0 \) such that \( v_m \to v \) weakly in \( S^1_1(M) \) and strongly in \( L^2(M) \) almost everywhere. If \( u \) is smooth, then

\[ \int_M u^{2/n}_\theta v^2 dV_\theta = \lim_{m \to \infty} \int_M u^{2/n}_\theta v_m^2 dV_\theta = 1, \]

and by standard arguments, \( v \) is nonnegative minimizer of the functional associated to \( \lambda_1(\tilde{\theta}) \).

We must show that (4-15) still holds if \( u \in L^{2+2/n}_2(M) \). Let \( A > 0 \) be a large real number and set \( u_A = \inf(u, A) \). Then

\[ \left| \int_M u^{2/n}_\theta (v_m^2 - v^2) dV_\theta \right| \leq \int_M u^{2/n}_A |v_m^2 - v^2| dV_\theta + \int_M (u^{2/n}_\theta - u^{2/n}_A) (|v_m| + |v|)^2 dV_\theta \leq A^{2/n} \int_M |v_m^2 - v^2| dV_\theta + \left( \int_M (u^{2/n}_\theta - u^{2/n}_A)^{n+1} dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M (|v_m| + |v|)^{2+2/n} dV_\theta \right)^{\frac{n}{n+1}}, \]

where we have used Hölder’s inequality in the last inequality. Since

\[ |u^{2/n}_\theta - u^{2/n}_A|^{n+1} \leq u^{2+2/n}_2 \in L^1(M), \]

by Lebesgue’s dominated convergence theorem we have

\[ \lim_{A \to \infty} \int_M (u^{2/n}_\theta - u^{2/n}_A)^{n+1} dV_\theta = \int_M \lim_{A \to \infty} (u^{2/n}_\theta - u^{2/n}_A)^{n+1} dV_\theta = 0. \]
Since \((v_m)_m\) is bounded in \(S^2_1(M)\), it is bounded in \(L^{2+\frac{2}{n}}(M)\), and hence there exists \(C > 0\) such that
\[
\int_M \left(|v_m| + |v|\right)^{2+\frac{2}{n}} dV_\theta \leq C.
\]
By strong convergence in \(L^2(M)\),
\[
\lim_{m \to \infty} \int_M |v_m^2 - v^2| dV_\theta = 0.
\]
Combining (4-16)–(4-19), we obtain (4-15). Therefore \(v\) is a nonnegative minimizer of the functional associated to \(\lambda_1(\tilde{\theta})\). Writing the Euler–Lagrange equation of \(v\), we find that \(v\) satisfies (4-12).

Now we define
\[
\lambda'_1(\tilde{\theta}) = \inf \frac{\int_M \left(2 + \frac{2}{n}\right)|\nabla_{\theta} w|_{\tilde{\theta}}^2 + R_{\theta} w^2 \, dV_\theta}{\int_M \frac{2}{n} |w|^2 \, dV_\theta},
\]
where the infimum is taken over smooth functions \(w\) such that \(u^{\frac{1}{n}} w \neq 0\) and such that
\[
\int_M u^{\frac{2}{n}} v w \, dV_\theta = 0.
\]
With the same method, we find a minimizer \(w\) of this problem that satisfies (4-13) with \(\lambda'_2(\tilde{\theta})\) instead of \(\lambda_2(\tilde{\theta})\). However, it is not difficult to see that \(\lambda'_2(\tilde{\theta}) = \lambda_2(\tilde{\theta})\) and Proposition 4.2 easily follows.

**Lemma 4.3.** Let \(u \in L^{2+\frac{2}{n}}(M)\) with \(\int_M u^{2+\frac{2}{n}} \, dV_\theta = 1\). Suppose that \(w_1, w_2 \in S^2_1(M) \setminus \{0\}, w_1, w_2 \geq 0\) satisfy
\[
\int_M \left(\left(2 + \frac{2}{n}\right)|\nabla_{\theta} w_1|_{\tilde{\theta}}^2 + R_{\theta} w_1^2 \right) \, dV_\theta \leq Y_2(M, \theta) \int_M \frac{2}{n} w_1^2 \, dV_\theta,
\]
\[
\int_M \left(\left(2 + \frac{2}{n}\right)|\nabla_{\theta} w_2|_{\tilde{\theta}}^2 + R_{\theta} w_2^2 \right) \, dV_\theta \leq Y_2(M, \theta) \int_M \frac{2}{n} w_2^2 \, dV_\theta,
\]
and suppose that \((M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))\) has measure zero. Then \(u\) is a linear combination of \(w_1\) and \(w_2\), and we have equality in (4-20) and (4-21).

**Proof.** We let \(\tilde{u} = a w_1 + b w_2\), where \(a, b > 0\) are chosen such that
\[
\frac{b^2}{a^2} \int_M \frac{2}{n} w_1^2 w_2^2 \, dV_\theta = \frac{\int_M w_1^{2+\frac{2}{n}} \, dV_\theta}{\int_M w_2^{2+\frac{2}{n}} \, dV_\theta},
\]
\[
\int_M \tilde{u}^{2+\frac{2}{n}} \, dV_\theta = a^{2+\frac{2}{n}} \int_M \frac{2}{n} w_1^{2+\frac{2}{n}} \, dV_\theta + b^{2+\frac{2}{n}} \int_M \frac{2}{n} w_2^{2+\frac{2}{n}} \, dV_\theta = 1.
\]
Because of the variational characterization of $Y_2(M, \theta)$ in Proposition 3.1, we have

\begin{equation}
Y_2(M, \theta) \leq \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(\tilde{u}, \lambda w_1 + \mu w_2).
\end{equation}

By (4-20), (4-21), (4-23), and since $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero, we obtain

\begin{equation}
F(\tilde{u}, \lambda w_1 + \mu w_2)
= \int_M \left( \frac{2}{n} \right) \| \nabla (\lambda w_1 + \mu w_2) \|^2 + R_\theta (\lambda w_1 + \mu w_2)^2 \, dV_	heta
\end{equation}

\begin{equation}
= \left[ \frac{\lambda^2}{2} \int_M \left( 2 + \frac{2}{n} \right) |\nabla w_1|^2 + R_\theta w_1^2 \, dV_	heta + \mu^2 \int_M \left( 2 + \frac{2}{n} \right) |\nabla w_2|^2 + R_\theta w_2^2 \, dV_	heta \right]
\end{equation}

\begin{equation}
\leq Y_2(M, \theta) \frac{\lambda^2 \int_M w_1^2 \, dV_\theta + \mu^2 \int_M w_2^2 \, dV_\theta}{\lambda^2 a^2 \int_M w_1^2 + \mu^2 b^2 \int_M w_2^2}.
\end{equation}

By (4-22), the right-hand side of (4-25) does not depend on $\lambda$ and $\mu$. Hence we can choose $\lambda = a$ and $\mu = b$ on the right-hand side of (4-25) to get

\begin{equation}
\sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(\tilde{u}, \lambda w_1 + \mu w_2)
\leq Y_2(M, \theta) \frac{a^2 \int_M w_1^2 \, dV_\theta + b^2 \int_M w_2^2 \, dV_\theta}{a^{2+\frac{2}{n}} \int_M w_1^{2+\frac{2}{n}} \, dV_\theta + b^{2+\frac{2}{n}} \int_M w_2^{2+\frac{2}{n}} \, dV_\theta}
\end{equation}

\begin{equation}
= Y_2(M, \theta) \int_M w_1^2 \, dV_\theta + b^2 \int_M w_2^2 \, dV_\theta
\end{equation}

\begin{equation}
= Y_2(M, \theta) \int_M w_1^2 \, dV_\theta + b^2 \int_M w_2^2 \, dV_\theta
\end{equation}

\begin{equation}
\leq Y_2(M, \theta) \left( \int_M w_1^{2+\frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M w_2^{2+\frac{2}{n}} \, dV_\theta \right)^{\frac{n}{n+1}}
\end{equation}

\begin{equation}
= Y_2(M, \theta),
\end{equation}

where we have used (4-23) in the first equality, the assumption that $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero in the second equality, Hölder’s inequality in the second inequality, and the assumption $\int_M u^{2+\frac{2}{n}} \, dV_\theta = 1$ and (4-23) in the last equality.

Combining (4-24) and (4-26), we have

\begin{equation}
\sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(\tilde{u}, \lambda w_1 + \mu w_2) = Y_2(M, \theta).
\end{equation}
This implies the equality in Holder’s inequality in (4-26), which implies that there exists a constant $c > 0$ such that $u = c \tilde{u}$ almost everywhere. Since $\int_M u^{2 + \frac{2}{n}} \, dV_\theta = \int_M \tilde{u}^{2 + \frac{2}{n}} \, dV_\theta = 1$ by (4-23), we have $c = 1$, i.e., $u = \tilde{u} = aw_1 + bw_2$. Also, equality in (4-25) implies equality in (4-20) and (4-21). This proves the assertion. □

**Theorem 4.4** (Euler–Lagrange equation). Assume $Y_2(M, \theta) \neq 0$ and that $Y_2(M, \theta)$ is attained by a generalized contact form $\tilde{\theta} = u^{\frac{2}{n}} \theta$ with $u \in L^{2 + \frac{2}{n}}(M)$. Let $v$ and $w$ be as in Proposition 4.2. Then $u = |w|$. In particular,

$$L_\theta w = Y_2(M, \theta)|w|^{\frac{2}{n}} w.$$  

Moreover, $w$ has alternating sign and $w \in C^{2,\alpha}(M)$ for all $\alpha \in [0, \frac{2}{n}]$.

**Proof.** Without loss of generality, we can assume that $\int_M u^{2 + \frac{2}{n}} \, dV_\theta = 1$. By assumption and by Proposition 3.1, we have $\lambda_2(\tilde{\theta}) = Y_2(M, \theta)$. Let $v, w \in S_1^2(M)$ be the functions satisfying (4-12), (4-13), and (4-14).

**Step 1.** We have $\lambda_1(\tilde{\theta}) < \lambda_2(\tilde{\theta})$.

We prove this by contradiction. Suppose that $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$. After possibly replacing $w$ by a linear combination of $v$ and $w$, we can assume that the function $u^{\frac{1}{n}} w$ changes sign. If we define $w_1 = \sup(w, 0)$ and $w_2 = \sup(-w, 0)$, then they satisfy the assumption of Lemma 4.3 since $w$ satisfies (4-13) and $\lambda_2(\tilde{\theta}) = Y_2(M, \theta)$.

Applying Lemma 4.3, we find $a, b > 0$ such that $u = aw_1 + bw_2$. Now, by Lemma 4.1, $w \in L^{2 + \frac{2}{n} + \epsilon}(M)$. By a standard bootstrap argument, (4-13) shows that $w \in C^{2,\alpha}(M)$ for all $\alpha \in (0, 1)$. Since $u = aw_1 + bw_2 = a \sup(w, 0) + b \sup(-w, 0)$, we have $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0, 1)$.

Since $\lambda_1(\tilde{\theta}) = \lambda_2(\tilde{\theta})$ and by the definition of $\lambda_1(\tilde{\theta})$, $w$ is a minimizer of the functional $\bar{w} \mapsto F(u, \bar{w})$ among the functions in $S_1^2(M)$ with $u^{\frac{1}{n}} \bar{w} \not\equiv 0$ by Proposition 3.1. Since $F(u, w) = F(u, |w|)$, we have that $|w|$ is a minimizer for the functional associated to $\lambda_1(\tilde{\theta})$, and $|w|$ satisfies same equation as $w$. As a consequence, $|w|$ is $C^2$. By the maximum principle, we have $|w| > 0$ everywhere, which is false since $u^{\frac{1}{n}} w$ changes sign.

**Step 2.** The function $w$ changes sign.

Assume $w$ does not change sign. Then after possibly replacing $w$ by $-w$, we can assume that $w \geq 0$. Setting $w_1 = v$ and $w_2 = w$, we have (4-20) and (4-21). Using (4-14), we can conclude that $(M \setminus w_1^{-1}(0)) \cap (M \setminus w_2^{-1}(0))$ has measure zero. Applying Lemma 4.3, we have equality in (4-20). On the other hand, Step 1 implies that inequality (4-20) is strict since $\lambda_1(\tilde{\theta}) < \lambda_2(\tilde{\theta}) = Y_2(M, \theta)$. This contradiction shows that $w$ changes sign.

**Step 3.** There exist $a, b > 0$ such that $u = a \sup(w, 0) + b \sup(-w, 0)$. Moreover, $w \in C^{2,\alpha}(M)$ and $u \in C^{0,\alpha}(M)$ for all $\alpha \in (0, 1)$.
As in the proof of Step 1, we apply Lemma 4.3 with \( w_1 = \sup(w, 0) \) and \( w_2 = \sup(-w, 0) \). We get \( a, b > 0 \) such that \( u = aw_1 + bw_2 \). As in Step 1, we get \( w \in C^{2,\alpha}(M) \) and \( u \in C^{0,\alpha}(M) \) for all \( \alpha \in (0, 1) \).

**Step 4.** Conclusion.

Let \( h \in C^\infty(M) \) such that \( \text{supp}(h) \subseteq M \setminus u^{-1}(0) \). For \( t \) close to 0, set \( u_t = |u + th| \). Since \( u > 0 \) on the support of \( h \), and since \( u \) is continuous, we have for \( t \) close to 0, \( u_t = u + th \). As \( \text{span}(v, w) \in \text{Gr}^u_2(S^2_1(M)) \), by Proposition 3.1 we have

\[
Y_2(M, \theta) \leq \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(u_t, \lambda v + \mu w).
\]

Note also that the functions \( u \in C^{2,\alpha}(M) \) and \( u \in C^{0,\alpha}(M) \) for all \( \alpha \in (0, 1) \).

(4-28)

\[
F(u_t, \lambda v + \mu w) = \int_M (2 + \frac{2}{n})|\nabla(\lambda v + \mu w)|^2 + R_{\theta}(\lambda v + \mu w)^2 \, dV_{\theta} \left( \int_M u_t^{2 + \frac{2}{n}} \, dV_{\theta} \right)^{\frac{1}{n+1}}
\]

\[
= \frac{\lambda^2 \lambda_1(\tilde{\theta}) \int_M u_t^{2 + \frac{2}{n}} \, dV_{\theta}}{\lambda^2 a_t + \lambda \mu b_t + \mu^2 c_t}
\]

\[
= \frac{\lambda^2 \lambda_1(\tilde{\theta}) + \mu^2 \lambda_2(\tilde{\theta}) \int_M u_t^{2 + \frac{2}{n}} \, dV_{\theta}}{\lambda^2 a_t + \lambda \mu b_t + \mu^2 c_t}
\]

where we have used (4-12), (4-13), and (4-14). Here

\[
a_t = \int_M u_t^{\frac{2}{n}} v^2 \, dV_{\theta}, \quad b_t = 2 \int_M u_t^{\frac{2}{n}} vw \, dV_{\theta} \quad \text{and} \quad c_t = \int_M u_t^{\frac{2}{n}} w^2 \, dV_{\theta}.
\]

Note also that the functions \( a_t, b_t, \) and \( c_t \) are smooth for \( t \) close to 0. Furthermore, \( a_0 = c_0 = 1 \) and \( b_0 = 0 \) by (4-14). Define \( f(t, \alpha) = F(u_t, \sin(\alpha)v + \cos(\alpha)w) \), which is smooth for small \( t \). By (4-28), we have

(4-29)

\[
f(t, \alpha) = F(u_t, \sin(\alpha)v + \cos(\alpha)w)
\]

\[
= \frac{\sin^2(\alpha)\lambda_1(\tilde{\theta}) + \cos^2(\alpha)\lambda_2(\tilde{\theta})}{\sin^2(\alpha)a_t + \sin(\alpha)\cos(\alpha)b_t + \cos^2(\alpha)c_t} \left( \int_M u_t^{2 + \frac{2}{n}} \, dV_{\theta} \right)^{\frac{1}{n+1}}.
\]

Hence, using \( \lambda_1(\tilde{\theta}) < \lambda_2(\tilde{\theta}) \), we can see that \( f(0, (n + \frac{1}{2})\pi) \) is minimum and \( f(0, n\pi) \) is maximum for any integer \( n \). This implies that

\[
\frac{\partial}{\partial \alpha} f(0, \alpha) = 0 \quad \text{if and only if} \quad \alpha \in \frac{\pi}{2} \mathbb{Z},
\]

\[
\frac{\partial^2}{\partial \alpha^2} f(0, \alpha) < 0 \quad \text{if} \quad \alpha \in \pi \mathbb{Z} \quad \text{and} \quad \frac{\partial^2}{\partial \alpha^2} f(0, \alpha) > 0 \quad \text{if} \quad \alpha \in \pi \mathbb{Z} + \frac{\pi}{2}.
\]

Applying the implicit function theorem to \( \partial f/\partial \alpha \) at the point \( (0,0) \), we see that there exists a smooth function \( t \mapsto \alpha(t) \), defined on a neighborhood of 0 with
\( \alpha(0) = 0 \) such that
\[
f(t, \alpha(t)) = \sup_{\alpha \in \mathbb{R}} f(t, \alpha) = \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}} F(u_t, \lambda v + \mu w),
\]
where the last equality follows from the fact that
\[
F(u_t, c\lambda v + c\mu w) = F(u_t, \lambda v + \mu w)
\]
for any nonzero constant \( c \) by (4-28). Since \( \alpha(0) = 0 \), we have
\[
\left. \frac{d}{dt} \sin^2 \alpha(t) \right|_{t=0} = \left. \frac{d}{dt} \cos^2 \alpha(t) \right|_{t=0} = \left. \frac{d}{dt} (a_t \sin^2 \alpha(t)) \right|_{t=0} = \left. \frac{d}{dt} (b_t \sin \alpha(t) \cos \alpha(t)) \right|_{t=0} = 0.
\]
Hence, by (4-29), we have
\[
(4-30) \quad \frac{d}{dt} f(t, \alpha(t)) \bigg|_{t=0} = \frac{d}{dt} \left( \frac{\sin^2(\alpha(t)) + \cos^2(\alpha(t))}{\sin^2(\alpha(t))a_t + \sin(\alpha(t))\cos(\alpha(t))b_t + \cos^2(\alpha(t))c_t} \right)
\]
\[
\times \left( \int_M u_t^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} \bigg|_{t=0}
\]
\[
= \lambda_2(\tilde{\theta}) \left( \left. -\frac{d}{dt} c_t \bigg|_{t=0} \right) \left( \int_M u_t^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} + \left. \frac{d}{dt} \left( \int_M u_t^{2+\frac{2}{n}} dV_\theta \right) \right|_{t=0} \right)
\]
\[
= \lambda_2(\tilde{\theta}) \frac{2}{n} \left( -\int_M u^{-1+\frac{2}{n}} h w^2 dV_\theta + \int_M u^{1+\frac{2}{n}} h dV_\theta \right).
\]
By the definition of \( Y_2(M, \theta) \) and \( \lambda_2(\tilde{\theta}) = Y_2(M, \theta) \), \( f \) admits a minimum at \( t = 0 \) because
\[
f(0, \alpha(0)) = f(0, 0) = F(u, w)
\]
and \( w \) satisfies (4-13). Since \( \lambda_2(\tilde{\theta}) = Y_2(M, \theta) \neq 0 \), it follows from (4-30) that
\[
\int_M u^{-1+\frac{2}{n}} h w^2 dV_\theta = \int_M u^{1+\frac{2}{n}} h dV_\theta.
\]
Since \( h \) is arbitrary (we just have to ensure that its support is contained in \( M \setminus u^{-1}(0) \)), we get
\[
u^{-1+\frac{2}{n}} w^2 = u^{1+\frac{2}{n}}
\]
and hence \( u = |w| \) on \( M \setminus u^{-1}(0) \). Together with Step 3, we have \( u = |w| \) everywhere. \( \square \)
5. Lower bound for $Y_2(M, \theta)$

For any compact CR manifold $(M, \theta)$ of the real dimension $2n + 1$, by the definition of the CR Yamabe invariant $Y_1(M, \theta)$, we have

\[
Y_1(M, \theta) = \inf_{u \in S_1^2(M) \setminus \{0\}} \frac{\int_M (2 + \frac{2}{n})|\nabla u|^2 g + R\theta u^2 dV}{(\int_M |u|^{2 + \frac{2}{n}} dV)\frac{n}{n+1}}.
\]

**Theorem 5.1.** We have

\[
Y_2(M, \theta) \geq 2^{\frac{1}{n+1}} Y_1(M, \theta).
\]

Moreover, if $M$ is connected and if $Y_2(M, \theta)$ is attained by a generalized contact form, then this inequality is strict.

**Proof.** The functional

\[
F(u, v) = \frac{\int_M (2 + \frac{2}{n})|\nabla v|^2 g + R\theta v^2 dV}{\int_M u^{\frac{2}{n+1}} v^2 dV}
\]

is continuous on $L^{\frac{2}{n+1}}(M) \times (S_1^2(M) \setminus \{0\})$. As a consequence, $I(u, V) := \sup_{v \in V \setminus \{0\}} F(u, v)$ depends continuously on $u \in L^{\frac{2}{n+1}}(M)$ and $V \in \text{Gr}_2^u(S_1^2(M))$. To prove Theorem 5.1, it suffices to show that $I(u, V) \geq 2^{\frac{1}{n+1}} Y_1(M, \theta)$ for all smooth $u > 0$ and $V \in \text{Gr}_2^u(S_1^2(M))$ thanks to Proposition 3.1. Without loss of generality, we can assume that

\[
\int_M u^{2 + \frac{2}{n}} dV = 1.
\]

The operator

\[v \mapsto P(v) := -(2 + \frac{2}{n})u^{-\frac{1}{n}} \Delta v(u^{-\frac{1}{n}} v) + R\theta u^{-\frac{2}{n}} v\]

is self-adjoint with respect to the $L^2$-scalar product and elliptic. Hence, $P$ has discrete spectrum $\lambda_1 \leq \lambda_2 \leq \cdots$ and the corresponding eigenfunctions $\varphi_1, \varphi_2, \ldots$ are smooth. Setting $v_i = u^{-\frac{1}{n}} \varphi_i$, we obtain

\[
( -(2 + \frac{2}{n}) \Delta + R\theta) (v_i) = -(2 + \frac{2}{n}) \Delta (u^{-\frac{1}{n}} \varphi_i) + R\theta u^{-\frac{1}{n}} \varphi_i = u^{-\frac{1}{n}} P(\varphi_i) = \lambda_i u^{-\frac{1}{n}} \varphi_i = \lambda_i u^{-\frac{2}{n}} v_i
\]

and

\[
\int_M u^{\frac{2}{n}} v_i v_j dV = \int_M \varphi_i \varphi_j dV = 0 \text{ if } i \neq j.
\]

The maximum principle implies that an eigenfunction to the smallest eigenvalue $\lambda_1$ has no zeros. Hence, $\lambda_1 < \lambda_2$ and we can assume that $v_1 > 0$. 
We define $w_+ = a_+ \sup(v_2, 0)$ and $w_- = a_- \sup(-v_2, 0)$, where $a_+, a_- > 0$ are chosen such that
\begin{equation}
\int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta = \int_M u^{\frac{2}{n}} w_-^2 \, dV_\theta = 1.
\end{equation}
We let $\Omega_- = \{v_2 < 0\}$ and $\Omega_+ = \{v_2 \geq 0\}$. By Hölder’s inequality, we have
\begin{equation}
2 = \int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta + \int_M u^{\frac{2}{n}} w_-^2 \, dV_\theta \\
\leq \left( \int_{\Omega_+} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M w_+^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{n}{n+1}} \\
+ \left( \int_{\Omega_-} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M w_-^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{n}{n+1}}.
\end{equation}
Using the inequality (5-1), we get
\begin{equation}
\int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta \geq Y_1(M, \theta) \left( \int_M w_+^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{n}{n+1}},
\end{equation}
which implies that
\begin{equation}
\left( \int_{\Omega_+} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta \right) \\
\geq Y_1(M, \theta) \left( \int_M w_+^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{n}{n+1}} \left( \int_{\Omega_+} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \\
\geq Y_1(M, \theta) \int_M u^{\frac{2}{n}} w_+^2 \, dV_\theta = Y_1(M, \theta),
\end{equation}
where we have used Hölder’s inequality in the last inequality, and (5-5) in the last equality. Similarly, we have
\begin{equation}
\left( \int_{\Omega_-} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M u^{\frac{1}{n}} w_- P(u^{\frac{1}{n}} w_-) \, dV_\theta \right) \geq Y_1(M, \theta).
\end{equation}
Adding (5-7) and (5-8) together, we obtain
\begin{equation}
2Y_1(M, \theta) \leq \left( \int_{\Omega_+} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M u^{\frac{1}{n}} w_+ P(u^{\frac{1}{n}} w_+) \, dV_\theta \right) \\
+ \left( \int_{\Omega_-} u^{2 + \frac{2}{n}} \, dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M u^{\frac{1}{n}} w_- P(u^{\frac{1}{n}} w_-) \, dV_\theta \right).
\end{equation}
Since $w_-$, respectively $w_+$, are multiples of $v_2$ on $\Omega_-$, respectively $\Omega_+$, they satisfy the same equation as $v_2$. Hence, we obtain from (5-4) and (5-9) that

\begin{equation}
2Y_1(M, \theta) \leq \left( \int_{\Omega_+} u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M \lambda_2 u^{\frac{2}{n}} w_+^2 dV_\theta \right) \left( \int_{\Omega_-} \lambda_2^{-\frac{2}{n}} w_-^2 dV_\theta \right)^{\frac{1}{n+1}} \left( \int_M \lambda_2^{-\frac{2}{n}} w^-_2 dV_\theta \right)^{\frac{1}{n+1}}
\end{equation}

\begin{equation}
= \lambda_2 \left( \int_{\Omega_+} u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} + \left( \int_{\Omega_-} u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}},
\end{equation}

where the last equality follows from (5-5). Now, for any nonnegative numbers $a, b \geq 0$, Hölder’s inequality yields

\begin{equation}
a + b \leq 2^{\frac{n}{n+1}} (a^{n+1} + b^{n+1})^{\frac{1}{n+1}}.
\end{equation}

Applying this inequality with

\begin{equation}
a = \left( \int_{\Omega_+} u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} \quad \text{and} \quad b = \left( \int_{\Omega_-} u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}},
\end{equation}

we derive from (5-10) that

\begin{equation}
2Y_1(M, \theta) \leq \lambda_2^{\frac{n}{n+1}} \left( \int_{\Omega_+} u^{2+\frac{2}{n}} dV_\theta \right) + \left( \int_{\Omega_-} u^{2+\frac{2}{n}} dV_\theta \right)
\end{equation}

\begin{equation}
= \lambda_2^{\frac{n}{n+1}} \left( \int_M u^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}} = \lambda_2 2^{\frac{n}{n+1}},
\end{equation}

where the last equality follows from (5-3). This implies that $\lambda_2 \geq 2^{\frac{1}{n+1}} Y_1(M, \theta)$. Since $\lambda_2 = I(u, \text{span}(v_1, v_2))$, this finishes the proof of the first part of Theorem 5.1.

Moreover, if $M$ were connected and if $Y_2(M, \theta)$ were attained by a generalized contact form, then inequality (5-9) would be an equality and we would have that $w_+$ or $w_-$ is a function for which equality in (5-1) is attained. By the maximum principle, we would get that $w_+$ or $w_-$ is positive on $M$, which is impossible. □

6. Upper bound for $Y_2(M, \theta)$

Hereafter, we denote $Y_k(\mathbb{S}^{2n+1})$ the $k$-th Yamabe invariant of $(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}})$, where $\theta_{\mathbb{S}^{2n+1}}$ is the standard contact form on $\mathbb{S}^{2n+1}$ given by

\begin{equation}
\theta_{\mathbb{S}^{2n+1}} = \sqrt{-1} \sum_{j=1}^{n+1} (z_j \ d\bar{z}_j - \bar{z}_j \ dz_j),
\end{equation}

where $(z_1, \ldots, z_{n+1}) \in \mathbb{S}^{2n+1} \subset \mathbb{C}^{n+1}$. 

Theorem 6.1. Suppose \((M, \theta)\) is a compact CR manifold of real dimension \(2n + 1\) with \(Y_1(M, \theta) \geq 0\). Then

\[
Y_2(M, \theta) \leq \left( Y_1(M, \theta)^{n+1} + Y_1(\mathbb{S}^{2n+1})^{n+1} \right)^{\frac{1}{n+1}}
\]

when \(Y_1(M, \theta) > 0\) and \(n \geq 3\), or \(Y_1(M, \theta) = 0\) and \(n \geq 4\). On the other hand, the inequality in (6-1) is strict when

(i) \(Y_1(M, \theta) > 0\), \(n \geq 7\) and \(M\) is not locally CR equivalent to \(\mathbb{S}^{2n+1}\), or
(ii) \(Y_1(M, \theta) = 0\), \(n \geq 4\) and \(M\) is not locally CR equivalent to \(\mathbb{S}^{2n+1}\).

To prove Theorem 5.4, we have the following:

Lemma 6.2. For any \(\alpha > 2\), there exists a constant \(C > 0\) such that

\[
|a + b|^\alpha \leq a^\alpha + b^\alpha + C(a^{\alpha-1}b + ab^{\alpha-1})
\]

for all \(a, b > 0\).

Proof. Dividing both sides by \(a\), without loss of generality, we can assume that \(a = 1\). Then we set for \(x > 0\),

\[
f(x) = \frac{|1 + x|^\alpha - (1 + x^{\alpha})}{x^{\alpha-1} + x}.
\]

By L’Hôpital’s rule, we have

\[
\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} \frac{\alpha(1 + x)^{\alpha-1} - \alpha x^{\alpha-1}}{(\alpha - 1)x^{\alpha-2} + 1} = \alpha,
\]

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\alpha(1 + x)^{\alpha-1} - \alpha x^{\alpha-1}}{(\alpha - 1)x^{\alpha-2} + 1} = \alpha.
\]

Since \(f\) is continuous, \(f\) is bounded by a constant \(C\) on \((0, \infty)\). Clearly, this constant is the desired \(C\) is the inequality of Lemma 6.2. \(\square\)

Proof of Theorem 6.1. For \(u \in S_1^2(M) \setminus \{0\}\), let

\[
E(u) = \frac{\int_M \left( 2 + \frac{2}{n} \right) |\nabla_\theta u|^2_\theta^2 + R_{\theta} u^2 \ dV_\theta}{\left( \int_M |u|^{2 + \frac{2}{n}} \ dV_\theta \right)^{\frac{n}{n+1}}}.\]

The solution of the CR Yamabe problem provides the existence of a smooth positive minimizer \(v\) of \(E\), and we can assume

\[
\int_M v^{2 + \frac{2}{n}} \ dV_\theta = 1.
\]

Then \(v\) satisfies the CR Yamabe equation

\[
L_\theta(v) = Y_1(M, \theta)v^{1 + \frac{2}{n}}.
\]
Let $x_0 \in M$ be fixed and choose pseudohermitian normal coordinates $(z, t)$ near $x_0$. Let $\delta > 0$ be a fixed number. If $\theta$ is well chosen in the conformal class and if $x_0$ is well chosen in $M$, it was proved by Jerison and Lee [1989, Theorem 4.1] that when $n \geq 3$, there exists a function $v_\varepsilon \geq 0$ with supp($v_\varepsilon$) $\subseteq B(x_0, 2\delta)$ such that

\begin{equation}
E(v_\varepsilon) = Y_1(\mathbb{S}^{2n+1}) - c(M)\varepsilon^4 + O(\varepsilon^5),
\end{equation}

where $c(M) \geq 0$ is a positive constant. In fact, $c(M)$ is the square of the norm of the Chern tensor at $x_0$ up to a dimensional constant. Therefore, we can assume that the constant $c(M)$ in (6-4) satisfies

\begin{equation}
c(M) > 0
\end{equation}

if $(M, \theta)$ is not locally CR equivalent to $\mathbb{S}^{2n+1}$. It follows from (6-4) that

\begin{equation}
\lim_{\varepsilon \to 0} E(v_\varepsilon) = Y_1(\mathbb{S}^{2n+1}).
\end{equation}

More precisely, $v_\varepsilon$ is given by (see [Jerison and Lee 1989, p. 326])

\begin{equation}
v_\varepsilon = C_\varepsilon \eta \left( \frac{\varepsilon^2}{t^2 + (|z|^2 + \varepsilon^2)^2} \right)^{\frac{n}{2}},
\end{equation}

where $\eta$ is a smooth cut-off function such that

\begin{equation}
0 \leq \eta \leq 1, \quad \eta(x) = \begin{cases} 
1 & \text{if } x \in B(x_0, \delta), \\
0 & \text{if } x \not\in B(x_0, 2\delta), 
\end{cases}
\end{equation}

and $C_\varepsilon > 0$ is a constant chosen such that

\begin{equation}
\int_M v_\varepsilon^{2 + \frac{2}{n}} dV_\theta = 1.
\end{equation}

It follows from [Jerison and Lee 1989, Proposition 4.2] that

\begin{equation}
C_\varepsilon = c(n) + O(\varepsilon^4)
\end{equation}

for some positive constant $c(n)$ depending only on $n$. In the following, $C$ will denote a positive constant depending possibly on $\delta, n$, but not on $\varepsilon$. Let

\begin{equation}
\delta_\varepsilon(z, t) = (\varepsilon z, \varepsilon^2 t).
\end{equation}

Note that

\begin{equation}
\delta_\varepsilon^*(\frac{1}{t^2 + (\varepsilon^2 + |z|^2)^2}) = \varepsilon^{-4} \left( \frac{1}{t^2 + (1 + |z|^2)^2} \right)
\end{equation}
\( \delta^* dz \, dt = \varepsilon^{2n+2} \, dz \, dt \). Hence,

\[
(6-9) \quad \int_M |v_\varepsilon|^p \, dV_\theta \leq C_\varepsilon^p \int \left\{ \frac{e^{np} \, dz \, dt}{(t^2 + (\varepsilon^2 + |z|^2)^{np})} \right\} \, dt
\]

\[
= C_\varepsilon^p \int \left\{ \frac{e^{2n+2-np} \, dz \, dt}{(t^2 + (1 + |z|^2)^{np})} \right\} \, dt
\]

\[
\leq C_\varepsilon^p \varepsilon^{2n+2-np} \int_{|z| \leq 2\delta/\varepsilon} \left( \int_{-\infty}^{\infty} \frac{dt}{1 + t^2} \right) \, dz
\]

\[
= C_\varepsilon^p \pi \varepsilon^{2n+2-np} \int_{|z| \leq 2\delta/\varepsilon} \frac{dz}{(1 + |z|^2)^{np-2}}
\]

where we have used (6-8). Note that for \( \varepsilon \ll 1 \),

\[
\int_0^{2\delta/\varepsilon} \frac{r^{2n-1} \, dr}{(1 + r^2)^{np-2}} \leq \int_0^{2\delta/\varepsilon} r^{2n+2-2np} \, dr \leq \frac{C}{\varepsilon^{2n+4-2np}}
\]

if \( p \leq 1 + \frac{3}{2n} \), and

\[
\int_0^{2\delta/\varepsilon} \frac{r^{2n-1} \, dr}{(1 + r^2)^{np-2}} \leq \int_0^1 r^{2n-1} \, dr + \int_1^{2\delta/\varepsilon} \frac{dr}{r^{np-2n-3}}
\]

\[
= \int_0^1 r^{2n-1} \, dr + \int_1^{2\delta/\varepsilon} \frac{dr}{r} = \frac{1}{2n} + \log \varepsilon
\]

if \( p = 1 + \frac{2}{n} \). Combining these with (6-9), we obtain

\[
(6-10) \quad \int_M |v_\varepsilon|^p \, dV_\theta \leq \begin{cases} 
C_\varepsilon^{np-2} & \text{if } p \leq 1 + \frac{3}{2n}, \\
C_\varepsilon^n \log \varepsilon & \text{if } p = 1 + \frac{2}{n}.
\end{cases}
\]

Similarly, for \( \varepsilon \ll 1 \), we have

\[
(6-11) \quad \int_M |v_\varepsilon|^p \, dV_\theta \geq C_\varepsilon^p \int \left\{ \frac{e^{np} \, dz \, dt}{(t^2 + (\varepsilon^2 + |z|^2)^{np})} \right\} \, dt
\]

\[
= C_\varepsilon^p \int \left\{ \frac{e^{2n+2-np} \, dz \, dt}{(t^2 + (1 + |z|^2)^{np})} \right\} \, dt
\]

\[
\geq C_\varepsilon^p \varepsilon^{2n+2-np} \int_{|z| \leq \delta/2\varepsilon} \left( \int_{-\delta/2\varepsilon}^{\delta/2\varepsilon} \frac{dt}{1 + t^2} \right) \, dz
\]

\[
\geq 2C_\varepsilon^p \tan^{-1}(\delta/2\varepsilon) \varepsilon^{2n+2-np} \int_{|z| \leq \delta/2\varepsilon} \frac{dz}{(1 + |z|^2)^{np}}
\]

\[
= C_\varepsilon^p \varepsilon^{2n+2-np} \int_0^{\delta/2\varepsilon} \frac{r^{2n-1} \, dr}{(1 + r^2)^{np}}.
\]
where we have used
\[ t^2 + (1 + |z|^2)^2 \leq (1 + t^2)(1 + |z|^2)^2 \]
and
\[ \{|z| \leq \delta/2\varepsilon\} \cap \{|t| \leq \delta/2\varepsilon\} \subset \left\{ \frac{4}{\sqrt{t^2 + |z|^2}} \leq \delta/\varepsilon \right\} \]
in the second inequality, and (6-8) in the last equality. Note that for \( \varepsilon \ll 1 \),
\[ \int_0^{\delta/2\varepsilon} \frac{r^{2n-1} dr}{(1 + r^2)^{n+1}} \geq \int_0^1 \frac{r^{2n-1} dr}{2^n} + \int_1^{\delta/2\varepsilon} \frac{r^{2n-1} dr}{(2r)^{2n}} = C + \frac{C}{\varepsilon^{2n-2n}} \]
if \( \leq 1 - \frac{1}{2n} \), and
\[ \int_0^{\delta/2\varepsilon} \frac{r^{2n-1} dr}{(1 + r^2)^{n+1}} \geq \int_0^1 \frac{r^{2n-1} dr}{2^n} + \int_1^{\delta/2\varepsilon} \frac{r^{2n-1} dr}{(2r)^{2n}} \]
\[ \geq \frac{1}{2^n} \left( \int_0^1 \frac{r^{2n-1} dr}{2^n} + \int_1^{\delta/2\varepsilon} \frac{dr}{r^{2n-2n-1}} \right) = C + C\varepsilon^{2n-2n} \]
if \( p > 1 \). Combining these with (6-11), we obtain
(6-12)  
\[ \int_M |v_\varepsilon|^p dV_\theta \geq \begin{cases} C\varepsilon^{np+2} & \text{if } p \leq 1 - \frac{1}{2n}, \\ C\varepsilon^{2n+2-2n} & \text{if } p > 1. \end{cases} \]

First we assume that \( Y_1(M, \theta) > 0 \). We set
\[ u_\varepsilon = E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon + Y_1(M, \theta)^{\frac{n}{2}} v. \]

Let us find estimates for \( F(u_\varepsilon, \lambda v_\varepsilon + \mu v) \). Let \( (\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\} \). Then
(6-13)  
\[ F(u_\varepsilon, \lambda v_\varepsilon + \mu v) \]
\[ = \frac{\lambda^2}{\lambda} \int_M v_\varepsilon \lambda_\theta v_\varepsilon dV_\theta + \frac{\mu^2}{\mu} \int_M vL_\theta v dV_\theta + 2\lambda\mu \int_M v_\varepsilon \lambda_\theta v dV_\theta + \frac{\lambda^2}{\lambda} \int_M |u_\varepsilon|^\frac{n}{2} (\lambda v_\varepsilon + \mu v)^2 dV_\theta \]
\[ = \frac{\lambda^2}{\lambda} E(v_\varepsilon) + \frac{\mu^2}{\mu} Y_1(M, \theta) + 2\lambda\mu Y_1(M, \theta) \int_M v + \frac{\lambda^2}{\lambda} \int_M |u_\varepsilon|^\frac{n}{2} v^2 dV_\theta + 2\lambda\mu \int_M \int_M |u_\varepsilon|^\frac{n}{2} v dV_\theta \]
where \( U = \left( \int_M u_\varepsilon^{2n+2/n} dV_\theta \right)^{1/(n+1)} \) and where we have used (6-2), (6-3) and (6-7). Using the definition of \( u_\varepsilon \), we have
(6-14)  
\[ u_\varepsilon \geq E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon \quad \text{and} \quad u_\varepsilon \geq Y_1(M, \theta)^{\frac{n}{2}} v, \]
which implies that

\begin{equation}
\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta \\
\geq \lambda^2 E(v_\varepsilon) \int_M v_\varepsilon^{2+\frac{2}{n}} dV_\theta + \mu^2 Y_1(M, \theta) \int_M v_\varepsilon^{2+\frac{2}{n}} dV_\theta \\
+ 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta \\
= \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta,
\end{equation}

where the last equality follows from (6-2) and (6-7).

If \( \lambda \mu \geq 0 \), then we have

\begin{equation}
2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta \geq \lambda \mu Y_1(M, \theta) \int_M v_\varepsilon^{1+\frac{2}{n}} v_\varepsilon dV_\theta
\end{equation}

by (6-14). Therefore, (6-15) and (6-16) imply that

\[
\frac{\lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) + 2\lambda \mu Y_1(M, \theta) \int_M v_\varepsilon^{1+\frac{2}{n}} v_\varepsilon dV_\theta}{\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta} \leq 1.
\]

If \( \lambda \mu < 0 \), then

\[
|u_\varepsilon|^2 \leq (E(v_\varepsilon)^\frac{2}{n} v_\varepsilon + Y_1(M, \theta)^\frac{2}{n} v_\varepsilon^2) \leq E(v_\varepsilon)^\frac{2}{n} + Y_1(M, \theta)v_\varepsilon^2
\]

when \( n \geq 2 \). Combining this with (6-14) and (6-15), we get

\[
\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta \\
\geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) - C \left( \int_M v_\varepsilon^{1+\frac{2}{n}} v_\varepsilon dV_\theta + \int_M v_\varepsilon^{1+\frac{2}{n}} v_\varepsilon dV_\theta \right) \\
\geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) - C \left( \int_M v_\varepsilon^{\frac{1}{n}+\frac{2}{n}} dV_\theta + \int_M v_\varepsilon^{\frac{1}{n}+\frac{2}{n}} dV_\theta \right),
\]

where \( C > 0 \) is a positive real number independent of \( \varepsilon \). This, together with (6-10), gives

\[
\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta \\
\geq \lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) - O(\varepsilon^n \log \varepsilon) - O(\varepsilon^{n-2}).
\]

This, together with the assumption that \( \lambda \mu < 0 \), implies that

\[
\frac{\lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) + 2\lambda \mu Y_1(M, \theta) \int_M v_\varepsilon^{1+\frac{2}{n}} v_\varepsilon dV_\theta}{\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v_\varepsilon v dV_\theta} \leq 1 + O(\varepsilon^{n-2}).
\]
In any case, we have

\[(6-17)\]
\[
\sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} \frac{\lambda^2 E(v_\varepsilon) + \mu^2 Y_1(M, \theta) + 2\lambda \mu Y_1(M, \theta) \int_M v^{1+\frac{2}{n}}v_\varepsilon dV_\theta}{\lambda^2 \int_M |u_\varepsilon|^2 v_\varepsilon^2 dV_\theta + \mu^2 \int_M |u_\varepsilon|^2 v^2 dV_\theta + 2\lambda \mu \int_M |u_\varepsilon|^2 v dV_\theta} \leq 1 + O(\varepsilon^{n-2}).
\]

On the other hand,
\[
\int_M u_\varepsilon^{2+\frac{2}{n}} dV_\theta = \int_M (E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon + Y_1(M, \theta) \frac{n}{2} v)^{2+\frac{2}{n}} dV_\theta
\]
\[
\leq E(v_\varepsilon)^{n+1} \int_M v^{2+\frac{2}{n}} dV_\theta + Y_1(M, \theta)^{n+1} \int_M v^{2+\frac{2}{n}} dV_\theta
\]
\[
+ C \left( \int_M v_{\varepsilon}^{1+\frac{2}{n}} v dV_\theta + \int_M v^{1+\frac{2}{n}} v_\varepsilon dV_\theta \right)
\]
\[
= E(v_\varepsilon)^{n+1} + Y_1(M, \theta)^{n+1} + C \left( \int_M v_{\varepsilon}^{1+\frac{2}{n}} v dV_\theta + \int_M v^{1+\frac{2}{n}} v_\varepsilon dV_\theta \right),
\]
where the first inequality follows from Lemma 6.2 with
\[a = E(v_\varepsilon)^{\frac{n}{2}} v_\varepsilon \quad \text{and} \quad b = Y_1(M, \theta)^{\frac{n}{2}} v,\]
and the last equality follows from (6-2) and (6-7). This, together with (6-4) and (6-10), implies that

\[(6-18)\]
\[
\left( \int_M u_\varepsilon^{2+\frac{2}{n}} dV_\theta \right)^{\frac{1}{n+1}}
\]
\[
\leq \left( Y_1(\mathbb{S}^{2n+1})^{n+1} + Y_1(M, \theta)^{n+1} \right)^{\frac{1}{n+1}} - c(M)\varepsilon^4 + o(\varepsilon^4) + O(\varepsilon^{n-2}).
\]

If \(\varepsilon > 0\) is small enough, it follows from (6-13), (6-17), and (6-18) that

\[(6-19)\]
\[
Y_2(M, \theta)
\]
\[
\leq \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0,0)\}} F(u_\varepsilon, \lambda v_\varepsilon + \mu v)
\]
\[
\leq \left( Y_1(\mathbb{S}^{2n+1})^{n+1} + Y_1(M, \theta)^{n+1} \right)^{\frac{1}{n+1}} - c(M)\varepsilon^4 + o(\varepsilon^4) + O(\varepsilon^{n-2}).
\]

Since \(n \geq 3\), (6-1) follows from (6-19) by letting \(\varepsilon\) go to zero. On the other hand, if \((M, \theta)\) is not locally CR equivalent to \(\mathbb{S}^{2n+1}\), then (6-5) holds. Hence, if \(n \geq 7\), the strict inequality in (6-1) follows from (6-19) by letting \(\varepsilon\) go to zero.
Now we assume that $Y_1(M, \theta) = 0$. We set $u_\varepsilon = v_\varepsilon$. Then we obtain for $(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}$,

$$F(u_\varepsilon, \lambda v_\varepsilon + \mu v) = \frac{\lambda^2 E(v_\varepsilon)(\int_M v_\varepsilon^{2+\frac{2}{n}} dV_\theta)^{1/(n+1)}}{\lambda^2 \int_M |v_\varepsilon|^{\frac{2}{n}} v^2 dV_\theta + \mu^2 \int_M |v_\varepsilon|^{\frac{2}{n}} v^2 dV_\theta + 2\lambda \mu \int_M |v_\varepsilon|^{\frac{2}{n}} v^2 dV_\theta},$$

by (6-7) and (6-13). Let $\lambda_\varepsilon, \mu_\varepsilon$ such that $\lambda_\varepsilon^2 + \mu_\varepsilon^2 = 1$ and

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = \sup_{(\lambda, \mu) \in \mathbb{R}^2 \setminus \{(0, 0)\}} F(u_\varepsilon, \lambda v_\varepsilon + \mu v).$$

If $\lambda_\varepsilon = 0$, we obtain that $F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = 0$ and the theorem would be proved. Then we assume that $\lambda_\varepsilon \neq 0$ and we can write

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) = \frac{E(v_\varepsilon)}{1 + 2x_\varepsilon b_\varepsilon + x_\varepsilon^2 a_\varepsilon},$$

where $x_\varepsilon = \mu_\varepsilon / \lambda_\varepsilon$ and

$$C \varepsilon^n \leq b_\varepsilon = \int_M v_\varepsilon^{1+\frac{2}{n}} dV_\theta \leq C \varepsilon^{n-1} \log \varepsilon \quad \text{as } \varepsilon \to 0,$$

$$a_\varepsilon = \int_M v_\varepsilon^{\frac{2}{n}} v^2 dV_\theta \geq C \varepsilon^4 \quad \text{as } \varepsilon \to 0$$

by (6-10) and (6-12). Maximizing this expression in $x_\varepsilon$ and using (6-4), we obtain (6-21)

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) \leq \frac{Y_1(\mathbb{S}^{2n+1}) - c(M) \varepsilon^4 + o(\varepsilon^4)}{1 - b_\varepsilon^2/2 a_\varepsilon} = \frac{Y_1(\mathbb{S}^{2n+1}) - c(M) \varepsilon^4 + o(\varepsilon^4)}{1 - C \varepsilon^{2n-6} \log^2 \varepsilon},$$

since $\varepsilon \log \varepsilon \to 0$ as $\varepsilon \to 0$. For $n \geq 4$, it follows from (6-21) that

$$F(u_\varepsilon, \lambda_\varepsilon v_\varepsilon + \mu_\varepsilon v) \leq Y_1(\mathbb{S}^{2n+1}),$$

which proves (6-1) for the case $Y_1(M, \theta) = 0$. On the other hand, if $(M, \theta)$ is not locally CR equivalent to $\mathbb{S}^{2n+1}$, then (6-5) holds. Hence, the strictly inequality in (6-1) follows from (6-21) by letting $\varepsilon$ go to zero. This proves Theorem 6.1. \qed

7. Some properties of $Y_2(M, \theta)$

We have the following questions:

1. Is $Y_2(M, \theta)$ attained by a contact form?
2. Is $Y_2(M, \theta)$ attained by a generalized contact form?
For question 1, we have the following:

**Proposition 7.1.** Let $S^{2n+1} \cup S^{2n+1}$ be the disjoint union of two copies of the sphere equipped with the standard contact form induced from $\theta_{S^{2n+1}}$. Then $Y_2(S^{2n+1} \cup S^{2n+1}) = 2^{\frac{1}{n+1}} Y_1(S^{2n+1})$ and it is attained by the standard contact form.

**Proof.** Let $\tilde{\theta}$ be an arbitrary smooth contact form on $S^{2n+1} \cup S^{2n+1}$. We write $S^{2n+1}_1$ for the first $S^{2n+1}$ and $S^{2n+1}_2$ for the second $S^{2n+1}$. Then we have

(7-1) $\lambda_2(S^{2n+1}_1 \cup S^{2n+1}, \tilde{\theta})$

$$= \min\{\lambda_2(S^{2n+1}_1, \tilde{\theta}), \lambda_2(S^{2n+1}_2, \tilde{\theta}), \max\{\lambda_1(S^{2n+1}_1, \tilde{\theta}), \lambda_1(S^{2n+1}_2, \tilde{\theta})\}\}.$$

Therefore,

(7-2) $Y_2(S^{2n+1} \cup S^{2n+1}) \leq \lambda_2(S^{2n+1} \cup S^{2n+1}) \operatorname{Vol}(S^{2n+1} \cup S^{2n+1})^{\frac{1}{n+1}}$

$$= \lambda_2(S^{2n+1} \cup S^{2n+1})(2 \operatorname{Vol}(S^{2n+1}))^{\frac{1}{n+1}}$$

$$= 2^{\frac{1}{n+1}} \lambda_1(S^{2n+1}) \operatorname{Vol}(S^{2n+1})^{\frac{1}{n+1}}$$

$$= 2^{\frac{1}{n+1}} Y_1(S^{2n+1}),$$

where we have used (7-1) in the second equality.

On the other hand, we have

(7-3) $\lambda_2(S^{2n+1}_1, \tilde{\theta}) \operatorname{Vol}(S^{2n+1}_1 \cup S^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}} \geq \lambda_2(S^{2n+1}_1, \tilde{\theta}) \operatorname{Vol}(S^{2n+1}_1, \tilde{\theta})^{\frac{1}{n+1}}$

$$\geq Y_2(S^{2n+1}_1)$$

$$= 2^{\frac{1}{n+1}} Y_1(S^{2n+1}_1),$$

where the last equality follows from Corollary 7.3. Similarly, we have

(7-4) $\lambda_2(S^{2n+1}_2, \tilde{\theta}) \operatorname{Vol}(S^{2n+1}_2 \cup S^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}} \geq 2^{\frac{1}{n+1}} Y_1(S^{2n+1}_2).

By the definition of $Y_1(S^{2n+1}_i)$, we have

$$\lambda_1(S^{2n+1}_i, \tilde{\theta}) \operatorname{Vol}(S^{2n+1}_i, \tilde{\theta})^{\frac{1}{n+1}} \geq Y_1(S^{2n+1}_i)$$

for $i = 1, 2$,

which implies

$$2Y_1(S^{2n+1})^{n+1}$$

$$\leq \sum_{i=1}^{2} \lambda_1(S^{2n+1}_i, \tilde{\theta})^{n+1} \operatorname{Vol}(S^{2n+1}_i, \tilde{\theta})$$

$$\leq \max\{\lambda_1(S^{2n+1}_1, \tilde{\theta})^{n+1}, \lambda_1(S^{2n+1}_2, \tilde{\theta})^{n+1}\} \sum_{i=1}^{2} \operatorname{Vol}(S^{2n+1}_i, \tilde{\theta})$$

$$= \max\{\lambda_1(S^{2n+1}_1, \tilde{\theta})^{n+1}, \lambda_1(S^{2n+1}_2, \tilde{\theta})^{n+1}\} \operatorname{Vol}(S^{2n+1} \cup S^{2n+1}, \tilde{\theta}),$$
which gives

\[(7-5) \quad 2 \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1}) \leq \max\{\lambda_1(\mathbb{S}^{2n+1}, \tilde{\theta}), \lambda_1(\mathbb{S}^{2n+1}, \tilde{\theta})\} \text{Vol}(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}}.\]

Combining (7-3), (7-4), and (7-5), we can derive from (7-1) that

\[2 \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1}) \leq \lambda_2(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta}) \text{Vol}(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}, \tilde{\theta})^{\frac{1}{n+1}}.\]

Since \(\tilde{\theta}\) is an arbitrary smooth contact form on \(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}\), we have

\[(7-6) \quad 2 \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1}) \leq Y_2(\mathbb{S}^{2n+1} \cup \mathbb{S}^{2n+1}).\]

Now Proposition 7.1 follows from combining (7-2) and (7-6).

On the other hand, we have the following:

**Proposition 7.2.** If \(M\) is connected, then \(Y_2(M, \theta)\) cannot be attained by a contact form.

**Proof.** Otherwise, if \(Y_2(M, \theta)\) were attained by a contact form \(\tilde{\theta} = u \tilde{\theta}\), then by Theorem 4.4, we would have \(u = |w|\), and hence \(u\) cannot be positive since \(w\) has alternating sign.

For question 2, we have the following:

**Corollary 7.3.** We have

\[Y_2(\mathbb{S}^{2n+1}) = 2 \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1}).\]

**Proof.** This follows from (6-1) and Theorem 5.1.

**Corollary 7.4.** \(Y_2(\mathbb{S}^{2n+1})\) is not attained by a generalized contact form.

**Proof.** This follows from Theorem 5.1 and Corollary 7.3.

**8. The \(k\)-th CR Yamabe invariant \(Y_k(M, \theta)\)**

In view of Corollary 7.3, it is natural to conjecture that

\[Y_k(\mathbb{S}^{2n+1}) = k \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1})\]

for all \(k\). However, the following result shows that it is false.

**Proposition 8.1.** For \(n \geq 3\), we have

\[Y_{2n+3}(\mathbb{S}^{2n+1}) < (2n + 3) \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1}).\]

**Proof.** Consider \(\mathbb{S}^{2n+1} \subseteq \mathbb{C}^{n+1}\). Let \(z_i\), where \(i = 1, 2, \ldots, n+1\), be the coordinates of \(\mathbb{C}^{n+1}\). Since \(-\Delta_{\theta_{s2n+1}} z_i = \frac{n}{2} z_i\) and \(-\Delta_{\theta_{s2n+1}} \bar{z}_i = \frac{n}{2} \bar{z}_i\),

\[L_{\theta_{s2n+1}}(z_i) = \frac{(n+2)(n+1)}{2} z_i \quad \text{and} \quad L_{\theta_{s2n+1}}(\bar{z}_i) = \frac{(n+2)(n+1)}{2} \bar{z}_i\]
for \( i = 1, 2, \ldots, n + 1 \), and hence
\[
\lambda_{2n+3}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \leq \frac{(n + 2)(n + 1)}{2}.
\]
This shows by the definition of \( Y_{2n+3} \) that
\[
(8-1) \quad Y_{2n+3}(\mathbb{S}^{2n+1}) \leq \lambda_{2n+3}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \frac{n+1}{n+1}
\leq \frac{(n + 2)(n + 1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \frac{n+1}{n+1}.
\]
Since
\[
\frac{(n + 2)(n + 1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \frac{n+1}{n+1}
\leq (2n + 3) \frac{n+1}{n+1} \frac{n(n + 1)}{2} \operatorname{Vol}(\mathbb{S}^{2n+1}, \theta_{\mathbb{S}^{2n+1}}) \frac{n+1}{n+1}
= (2n + 3) \frac{n+1}{n+1} Y_1(\mathbb{S}^{2n+1})
\]
when \( n \geq 3 \), Proposition 8.1 follows from (8-1). \( \square \)

For the case when the \( k \)-th CR Yamabe invariant is negative, we have this:

**Theorem 8.2.** Let \( k \) be an positive integer. Assume that \( Y_k(M, \theta) < 0 \). Then \( Y_k(M, \theta) = -\infty \).

**Proof.** After a possible change of contact form in the conformal class, we can assume that \( \lambda_k(\theta) < 0 \). This implies that we can find smooth functions \( v_1, \ldots, v_k \) satisfying
\[
L_\theta(v_i) = \lambda_i(\theta)v_i \quad \text{for all } i = 1, 2, \ldots, k
\]
and such that
\[
\int_M v_i v_j \, dV_\theta = 0 \quad \text{for all } i, j = 1, 2, \ldots, k \text{ and } i \neq j.
\]
Let \( v_k \) be defined as in the proof of Theorem 6.1. We define \( u_\varepsilon = v_\varepsilon + \varepsilon \). We set \( V = \text{span}\{v_1, \ldots, v_k\} \). For \( v \in V \), we have
\[
\int_M u_\varepsilon^n v^2 \, dV_\theta \leq \varepsilon^{\frac{n}{2}} \int_M v^2 \, dV_\theta + \int_M v_\varepsilon^n v^2 \, dV_\theta
\leq C \varepsilon^{\frac{n}{2}} + C \int_M v_\varepsilon^{\frac{n}{2}} v^2 \, dV_\theta
\leq \begin{cases} 
C \varepsilon^{\frac{n}{2}} + C \left( \int_M v_\varepsilon^{\frac{3}{2}} \, dV_\theta \right)^{\frac{2}{3}} \operatorname{Vol}(M, \theta)^{\frac{1}{3}} = C \varepsilon^{\frac{n}{2}} + C \varepsilon^{\frac{2}{3}} \quad & \text{if } n \geq 2, \\
C \varepsilon^{\frac{3}{2}} + C \left( \int_M v_\varepsilon^{\frac{5}{2}} \, dV_\theta \right)^{\frac{1}{3}} \operatorname{Vol}(M, \theta)^{\frac{4}{3}} = C \varepsilon^{2} + C \varepsilon^{\frac{10}{10}} \quad & \text{if } n = 1
\end{cases}
\]
by (6-10) and Hölder’s inequality. From this, we have
\[
\lim_{\varepsilon \to 0} \int_M \frac{\tilde{u}_\varepsilon}{n} \tilde{u}_\varepsilon^2 \, dV_{\sqrt{g}} = 0
\]
uniformly in \( v \in V \). Since \( \lambda_k(\theta) < 0 \), it is then easy to see that
\[
\sup_{v \in V} F(u_\varepsilon, v) = -\infty.
\]
Together with the variational characterization of \( Y_k(M, \theta) \) in Proposition 3.1, we get that \( Y_k(M, \theta) = -\infty \).

References


Received June 3, 2014. Revised May 18, 2015.
Topological Molino’s theory

JESÚS A. ÁLVAREZ LÓPEZ and MANUEL F. MOREIRA GALICIA

Equivariant principal bundles and logarithmic connections on toric varieties

INDRANIL BISWAS, ARIJIT DEY and MAINAK PODDAR

On a spectral theorem in paraorthogonality theory

KENIER CASTILLO, RUYMÁN CRUZ-BARROSO and FRANCISCO PERDOMO-PÍO

Sigma theory and twisted conjugacy, II: Houghton groups and pure symmetric automorphism groups

DACIBERG L. GONÇALVES and PARAMESWARAN SANKARAN

The second CR Yamabe invariant

PAK TUNG HO

No hyperbolic pants for the 4-body problem with strong potential

CONNOR JACKMAN and RICHARD MONTGOMERY

Unions of Lebesgue spaces and $A_1$ majorants

GREG KNESE, JOHN E. MCCARTHY and KABE MOEN

Complex hyperbolic $(3, 3, n)$ triangle groups

JOHN R. PARKER, JIEYAN WANG and BAOHUA XIE

Topological aspects of holomorphic mappings of hyperquadrics from $\mathbb{C}^2$ to $\mathbb{C}^3$

MICHAEL REITER

2-Blocks with minimal nonabelian defect groups III

BENJAMIN SAMBALE

Number of singularities of stable maps on surfaces

TAKAHIRO YAMAMOTO