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We study two questions. When does a function belong to the union of Lebesgue spaces, and when does a function have an A_1 majorant? We provide a systematic study of these questions and show that they are fundamentally related. We show that the union of $L^p_w(\mathbb{R}^n)$ spaces with $w \in A_p$ is equal to the union of all Banach function spaces for which the Hardy–Littlewood maximal function is bounded on the space itself and its associate space.

1. Introduction and statement of the main results

While the L^p spaces are considered fundamental spaces of interest in analysis, the weighted L^p spaces and the related study of A_p weights are perhaps part of a more specialized area of analysis. It is the goal of this article to show that the L^p spaces considered in aggregate are intimately linked to these latter topics and to the notion of an A_1 majorant. By recent developments our results indicate that weighted Lebesgue spaces with A_p weights may be good candidates for ambient spaces for operators in harmonic analysis.

We begin with the following question.

Question 1.1. When does a function belong to the union of L^p spaces?

Question 1.1 is vaguely stated on purpose. By union, we mean either the union of L^p as p varies or the union of L^p_w as w varies with p fixed. The union of L^p spaces often arises when considering a general domain to define operators in harmonic analysis. Several such operators are bounded on L^p for all $1 < p < \infty$, and hence take functions from $\bigcup_{p>1} L^p$ into itself.

It turns out Question 1.1 is closely related to the theory of weighted Lebesgue spaces and the action of the Hardy–Littlewood maximal operator on these spaces.

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For our purposes, a weight is a positive locally integrable function. An A_1 weight is one that satisfies

$$Mw \leq Cw \quad \text{a.e.}$$

Here M denotes the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f| \, dx.$$

We exclude the weight $w \equiv 0$ from belonging to A_1, and in this case we see that if $w \in A_1$ then $w > 0$ a.e. The A_1 class of weights characterizes when M maps L^1_w into $L^{1,\infty}_w$. When $1 < p < \infty$, M is bounded on L^p_w exactly when $w \in A_p$:

$$\left(\frac{1}{|Q|} \int_Q w \, dx\right)\left(\frac{1}{|Q|} \int_Q w^{-1/(p-1)} \, dx\right)^{p-1} \leq C$$

for all cubes Q. At the other endpoint the A_∞ class is defined to be the union of all A_p for $p \geq 1$. We now come to our second question.

Question 1.2. Given a measurable function f, when does there exist an A_1 weight w such that

$$|f| \leq w? \tag{1}$$

We call a weight satisfying (1) an A_1 majorant of f and write $\mathcal{M}_{A_1}$ for the set of measurable functions possessing an A_1 majorant. As stated, Question 1.2 does not seem to have been considered before. As far as we can tell, the first notion of an A_1 majorant appeared in an article by Rutsky [2011]. In Rutsky's paper, however, a different definition of an A_1 majorant is given — one which requires the function and the weight to a priori belong to a more restrictive class of functions.

If we examine weights locally, say on the interval [0, 1], then our problem has a remarkably simple answer which reveals a close connection between traditional L^p spaces, weighted L^p spaces, and A_1 majorants:

$$\mathcal{M}_{A_1}([0, 1]) = \bigcup_{p>1} L^p([0, 1]) = \bigcup_{w \in A_2} L^2_w([0, 1]). \tag{2}$$

The proof of (2) is a synthesis of known important results for Muckenhoupt weights. This equivalence reinforces the saying attributed to Antonio Córdoba, "There are no L^p spaces, only weighted L^2 spaces."

The local theory has several extensions including an application to Hardy spaces on the unit disk. In [McCarthy 1990], while studying the range of Toeplitz operators, the second author showed that the Smirnov class, N^+, can be realized as a union of weighted Hardy spaces:

$$N^+ = \bigcup_{w \in \mathcal{W}} H^2_w$$

where $\mathscr{W}$ is the Szegő class of weights (see Section 2 for relevant definitions). The class $A_\infty(\mathbb{T})$ is a proper subset of $\mathscr{W}$ (as $\bigcup_{p>0} H^p$ is a proper subspace of N^+). Using our techniques we are able to give a characterization of $\bigcup_{p>0} H^p$ in terms of weighted H^2 spaces:

$$\bigcup_{p>0} H^p = \bigcup_{w\in A_\infty} H^2_w. \tag{3}$$

We refer the reader to Section 4 for more on the local case.

For functions on $\mathbb{R}^n$, the theory is not as nice. In the local case the $L^p([0,1])$ spaces are nested in p, whereas the $L^p(\mathbb{R}^n)$ spaces are not. We are not able to obtain equality of $\bigcup_{p>1} L^p(\mathbb{R}^n)$ and $\mathcal{M}_{A_1}(\mathbb{R}^n)$. Remarkably, even the much larger union over weak-$L^p(\mathbb{R}^n)$ spaces is not equal to $\mathcal{M}_{A_1}(\mathbb{R}^n)$. As a consequence of our results, if p_0 is any exponent satisfying $1 < p_0 < \infty$ then

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subsetneqq \bigcup_{w\in A_{p_0}} L^{p_0}_w(\mathbb{R}^n) \subsetneqq \mathcal{M}_{A_1}(\mathbb{R}^n). \tag{4}$$

The class $\mathcal{M}_{A_1}(\mathbb{R}^n)$ can be thought of as a generalization of $L^\infty(\mathbb{R}^n)$ — i.e., functions that are majorized by constants, which are A_1 weights — while $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ is a generalization of $L^1(\mathbb{R}^n)$. With this in mind we obtain the following theorem.

Theorem 1.3. *Suppose* $1 < p < \infty$. *Then*

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \left(\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n) \right).$$

Considering the basic fact

$$L^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n) \subset \bigcap_{1<p<\infty} L^p(\mathbb{R}^n),$$

Theorem 1.3 shows that if we enlarge both $L^\infty(\mathbb{R}^n)$ to $\mathcal{M}_{A_1}(\mathbb{R}^n)$ and $L^1(\mathbb{R}^n)$ to $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ and intersect the two, then we pick up an even bigger class of functions, one that by (4) properly contains the union of all $L^p(\mathbb{R}^n)$ for $p > 1$. As a consequence to Theorem 1.3, we see that for all $1 < p, q < \infty$,

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) = \bigcup_{u\in A_q} L^q_u(\mathbb{R}^n).$$

The proof of Theorem 1.3 uses the extrapolation theory of Rubio de Francia [1984; 1987] (see also the book [Cruz-Uribe et al. 2011]).

The union $\bigcup_{p>1} L^p$ is a good candidate for a natural collection of functions on which to iterate the Hardy–Littlewood maximal function. Rutsky [2014, Theorem 1] showed that Banach function spaces $\mathscr{X}$ on $\mathbb{R}^n$ (see Section 2) for which the Hardy–Littlewood maximal function is bounded on both the space $\mathscr{X}$ and the associate space $\mathscr{X}'$ act as a natural domain for the set of all Calderón–Zygmund operators. We end the introduction with our main result which says a function belongs to a function space $\mathscr{X}$ for which the Hardy–Littlewood maximal function is bounded on $\mathscr{X}$ and $\mathscr{X}'$ if and only if $f \in L^p_w(\mathbb{R}^n)$ for some $p > 1$ and $w \in A_p(\mathbb{R}^n)$.

Theorem 1.4. *Suppose* $1 < p < \infty$. *Then*

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \bigcup \{\mathscr{X} : M \in \mathscr{B}(\mathscr{X}) \cap \mathscr{B}(\mathscr{X}')\},$$

where the second union is over all Banach function spaces such that the Hardy–Littlewood maximal operator is bounded on $\mathscr{X}$ *and* $\mathscr{X}'$.

Banach function spaces for which $M \in \mathscr{B}(\mathscr{X}) \cap \mathscr{B}(\mathscr{X}')$ are also related to the Fefferman–Stein inequality. Define the sharp maximal function $M^\#$ by

$$M^\# f(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f - f_Q| \, dx,$$

where $f_Q = \frac{1}{|Q|} \int_Q f \, dx$. Lerner [2010] proved that if $M \in \mathscr{B}(\mathscr{X})$, then the Fefferman–Stein inequality

$$\|f\|_{\mathscr{X}} \le c \|M^\# f\|_{\mathscr{X}} \tag{5}$$

holds for all nice functions in $\mathscr{X}$ if and only if $M \in \mathscr{B}(\mathscr{X}')$. In particular, Theorem 1.4 shows that if f belongs to a Banach function space for which $M \in \mathscr{B}(\mathscr{X})$ and the Fefferman–Stein inequality (5) holds on $\mathscr{X}$, then for any $1 < p < \infty$, there exists $w \in A_p$ for which $f \in L^p_w(\mathbb{R}^n)$.

The outline of this paper is as follows. In Section 2 we state preliminary results that are necessary for the rest of the paper. In Section 3 we study the classes of functions with A_1 and A_p majorants. In Section 4 we give a treatise of local theory with applications to Hardy spaces on the unit disk. Section 5 is devoted to the theory on $\mathbb{R}^n$, in particular the proofs of Theorems 1.3 and 1.4. We finish the article with some open questions in Section 6.

2. Preliminaries

In this section, Ω denotes either $\mathbb{R}^n$ or a cube Q with sides parallel to the coordinate planes in $\mathbb{R}^n$. For $0 < p < \infty$, $L^p(\Omega)$ is the set of measurable functions such that

$$\|f\|^p_{L^p} = \int_\Omega |f|^p \, dx < \infty.$$

Given p with $1 \le p \le \infty$, we use p' to denote the dual exponent defined by the equation $1/p + 1/p' = 1$. A weight defined on a cube Q is a positive function in $L^1(Q)$. A weight on $\mathbb{R}^n$ is a positive function in $L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Given a weight, w, define $L^p_w(\Omega)$ to be the collection of functions satisfying

$$\|f\|^p_{L^p_w} = \int_\Omega |f|^p w\,dx < \infty.$$

We define $L^\infty_w(\Omega)$ to be the space of functions for which $f/w \in L^\infty(\Omega)$. This space is normed by

$$\|f\|_{L^\infty_w} = \|f/w\|_\infty = \operatorname*{ess\,sup}_{x\in\Omega} \frac{|f(x)|}{w(x)}.$$

If $\mathbb{T}$ is the unit circle in the complex plane, then $L^p(\mathbb{T})$ and $L^p_w(\mathbb{T})$ are identified as the space of 2π periodic functions that belong to $L^p([0,2\pi])$ and $L^p_w([0,2\pi])$, respectively.

We also examine the "complex analyst's Hardy space", as opposed to the real analyst's Hardy space defined in terms of maximal functions. Let $\mathbb{D}$ denote the unit disk in the plane with boundary $\mathbb{T}$. Given p with $0 < p < \infty$, let $H^p = H^p(\mathbb{D})$ be the space of analytic functions "normed" by

$$\|f\|_{H^p} = \sup_{0<r<1} \left(\int_0^{2\pi} |f(re^{i\theta})|^p \,\frac{d\theta}{2\pi} \right)^{1/p}.$$

"Norm" is in quotes since this is not a norm for $0 < p < 1$, but we use norm notation $\|\cdot\|$ nonetheless. The Nevanlinna class, denoted N, is the collection of analytic functions on $\mathbb{D}$ such that

$$\|f\|_N = \sup_{0<r<1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \,\frac{d\theta}{2\pi} < \infty.$$

Functions in N have nontangential limits almost everywhere on the boundary, so we may treat them as functions on the disk or the circle. The Smirnov class N^+ consists of functions $f \in N$ such that

$$\lim_{r\to 1} \int_0^{2\pi} \log^+ |f(re^{i\theta})| \,\frac{d\theta}{2\pi} = \int_0^{2\pi} \log^+ |f(e^{i\theta})| \,\frac{d\theta}{2\pi}.$$

It is well known that

$$\bigcup_{p>0} H^p \subsetneqq N^+ \subsetneqq N$$

(see, e.g., the books by Duren [1970] or Rudin [1964]). The Smirnov class is often considered a natural limit of H^p as $p \to 0$.

The weighted Hardy space $H^p_w = H^p_w(\mathbb{D})$ is the closure of analytic polynomials in $L^p_w(\mathbb{T})$. While there are real variable definitions of weighted Hardy spaces, this classical definition has an intuitive appeal.

Let M_Ω be the Hardy–Littlewood maximal operator restricted to Ω, i.e.,

$$M_\Omega f(x) = \sup_{\substack{Q\subset\Omega\\ x\in Q}} \frac{1}{|Q|}\int_Q |f|\,dy.$$

When $\Omega = \mathbb{R}^n$ we write $M_{\mathbb{R}^n} f = Mf$.

We define $A_1(\Omega)$ to be the class of all weights on Ω such that $M_\Omega w(x) \le Cw(x)$ a.e. $x \in \Omega$. For $p > 1$, $A_p(\Omega)$ is the class of all weights on Ω such that

$$\sup_{Q\subset\Omega}\left(\frac{1}{|Q|}\int_Q w\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}\,dx\right)^{p-1} < \infty.$$

Given an A_p weight w we refer to the weight $\sigma = w^{1-p'}$ as the dual weight. For the endpoint, $p = \infty$, we use the definition

$$A_\infty(\Omega) = \bigcup_{p\ge 1} A_p(\Omega).$$

There are several other definitions of A_∞, e.g., weights satisfying a reverse Jensen inequality, a reverse Hölder inequality, or a fairness condition with respect to Lebesgue measure [Duoandikoetxea 2001; Grafakos 2008].

A weight on the torus is a positive function in $L^1(\mathbb{T})$. The classes $A_1(\mathbb{T})$, $A_p(\mathbb{T})$, and $A_\infty(\mathbb{T})$ are defined analogously on $\mathbb{T}$. The Szegő class of weights, denoted $\mathcal{W}$, are weights on $\mathbb{T}$ satisfying

$$\int_{\mathbb{T}} \log w\,d\theta > -\infty.$$

We notice that if $w \in A_\infty(\mathbb{T})$, then we have

$$\left(\int_{\mathbb{T}} w\,\frac{d\theta}{2\pi}\right)\exp\left(-\int_{\mathbb{T}} \log w\,\frac{d\theta}{2\pi}\right) < \infty.$$

In particular, $A_\infty(\mathbb{T}) \subset \mathcal{W}$.

Example 2.1. Let $x_0 \in \Omega$, $1 \le p \le \infty$, and $w_{x_0}(x) = |x - x_0|^\alpha$. Then $w_{x_0} \in A_p(\Omega)$ if and only if $-n < \alpha < n(p-1)$.

We will need some elementary properties of A_p weights, most of which follow from the definition (see [Duoandikoetxea 2001, Proposition 7.2]).

Theorem 2.2. *The following hold*:

(i) $A_1 \subset A_p \subset A_q \subset A_\infty$ *if* $1 < p < q < \infty$.

(ii) *For* $1 < p < \infty$, $w \in A_p$ *if and only if* $\sigma = w^{1-p'} \in A_{p'}$.

(iii) *If* $0 < s \le 1$ *and* $w \in A_p$, *then* $w^s \in A_p$.

(iv) *If* $u, v \in A_1$, *then* $uv^{1-p} \in A_p$.

It is interesting to note that the converse of (iv) also holds, but the proof is much more intricate. This was shown by Jones [1980] and later by Rubio de Francia [1982]. We emphasize that we do not need this converse statement, only the statement (iv).

We also need the following deeper property of A_∞ weights known as the reverse Hölder inequality. See [Hytönen et al. 2012] for a simple proof with nice constants.

Theorem 2.3. *If $w \in A_\infty(\Omega)$, then there exists $s > 1$ such that for every cube $Q \subset \Omega$,*

$$\frac{1}{|Q|}\int_Q w^s\,dx \le \left(\frac{2}{|Q|}\int_Q w\,dx\right)^s.$$

As a corollary to Theorem 2.3 we have the following openness properties of A_p classes.

Theorem 2.4. *Let $1 \le p \le \infty$. The following hold*:

(i) *If $p > 1$ then $A_p(\Omega) = \bigcup_{1\le q<p} A_q(\Omega)$.*

(ii) *If $w \in A_p(\Omega)$ then $w^s \in A_p(\Omega)$ for some $s > 1$.*

For the results on $\mathbb{R}^n$ we need the notion of a Banach function space. We refer the reader to the book by Bennett and Sharpley [1988, Chapter 1] for an excellent reference on the subject. A mapping ρ, defined on the set of nonnegative $\mathbb{R}^n$-measurable functions and taking values in $[0, \infty]$, is said to be a Banach function norm if it satisfies the following properties:

(i) $\rho(f) = 0 \Leftrightarrow f = 0$ a.e., $\rho(af) = a\rho(f)$ for $a > 0$, $\rho(f+g) \le \rho(f)+\rho(g)$;

(ii) if $0 \le f \le g$ a.e., then $\rho(g) \le \rho(f)$;

(iii) if $f_n \uparrow f$ a.e., then $\rho(f_n) \uparrow \rho(f)$;

(iv) if $B \subset \mathbb{R}^n$ is bounded, then $\rho(\chi_B) < \infty$;

(v) if $B \subset \mathbb{R}^n$ is bounded, then

$$\int_B f\,dx \le C_B\rho(f)$$

for some constant C_B with $0 < C_B < \infty$.

We note that our definition of a Banach function space is slightly different from that found in [Bennett and Sharpley 1988]. In particular, in the axioms (iv) and (v) we assume that the set B is a bounded set, whereas it is sometimes assumed that B merely satisfy $|B| < \infty$. We do this so that the spaces $L^p_w(\mathbb{R}^n)$ with $w \in A_p$ satisfy items (iv) and (v). (See also the discussion at the beginning of Chapter 1 on page 2 of [Bennett and Sharpley 1988].)

Given Banach function norm ρ, $\mathscr{X} = \mathscr{X}(\mathbb{R}^n, \rho)$ is the collection of measurable functions such that $\rho(|f|) < \infty$. In this case we may equip $\mathscr{X}$ with the norm

$$\|f\|_{\mathscr{X}} = \rho(|f|).$$

The associate space $\mathscr{X}'$ is the set of all measurable functions g such that $fg \in L^1(\mathbb{R}^n)$ for all $f \in \mathscr{X}$. This space is normed by

$$\|g\|_{\mathscr{X}'} = \sup\left\{\int_{\mathbb{R}^n} |fg|\,dx : \|f\|_{\mathscr{X}} \le 1\right\}. \tag{6}$$

Equipped with this norm $\mathscr{X}'$ is also a Banach function space and

$$\int_{\mathbb{R}^n} |fg|\,dx \le \|f\|_{\mathscr{X}} \|g\|_{\mathscr{X}'}.$$

Typical examples of Banach function spaces are $L^p(\mathbb{R}^n)$ for $1 \le p \le \infty$, whose associate spaces are $L^{p'}(\mathbb{R}^n)$. Other Banach spaces include weak type spaces $L^{p,\infty}(\mathbb{R}^n)$, the Lorentz space $L^{p,q}(\mathbb{R}^n)$, and Orlicz spaces $L^{\Phi}(\mathbb{R}^n)$ defined for a Young function Φ (see [Bennett and Sharpley 1988; Cruz-Uribe et al. 2011]). When $w \in A_p(\mathbb{R}^n)$ and $1 \le p \le \infty$, the spaces $L^p_w(\mathbb{R}^n)$ are also Banach function spaces with respect to Lebesgue measure. To see this, it suffices to check property (v). Suppose $f \ge 0$, $1 < p < \infty$, and B is bounded. Then $B \subset Q$ for some cube Q so $\sigma(B) < \infty$, and Hölder's inequality implies

$$\int_B f\,dx = \int_B f w^{1/p} w^{-1/p}\,dx \le \sigma(B)^{1/p'} \left(\int_B f^p w\,dx\right)^{1/p} \le \sigma(B)^{1/p'} \|f\|_{L^p_w}.$$

To see that $L^1_w(\mathbb{R}^n)$ is a Banach function space when $w \in A_1(\mathbb{R}^n)$, note that

$$\int_B f\,dx = \int_B f w w^{-1}\,dx \le (\inf_B w)^{-1} \|f\|_{L^1_w}. \tag{7}$$

Finally, if $f \in L^\infty_w$, then

$$\int_B f\,dx = \int_B (f/w) w\,dx \le w(B) \|f\|_{L^\infty_w},$$

showing L^∞_w is a Banach function space.

When $1 < p < \infty$ and $w \in A_p$, the associate space of $L^p_w(\mathbb{R}^n)$ defined by the pairing in (6) is given not by $L^{p'}_w(\mathbb{R}^n)$ but by $L^{p'}_\sigma(\mathbb{R}^n)$ for $\sigma = w^{1-p'}$. When $p = 1$ and $w \in A_1$, the associate space of L^1_w is given by $L^\infty_w(\mathbb{R}^n)$. We are particularly interested in Banach function spaces $\mathscr{X}$ for which

$$\|Mf\|_{\mathscr{X}} \le C\|f\|_{\mathscr{X}},$$

in which case we write $M \in \mathscr{B}(\mathscr{X})$.

We end this section with the classical result of Coifman and Rochberg [1980] (see also [García-Cuerva and Rubio de Francia 1985, Theorem 3.4, p. 158]). This result requires a definition.

Definition 2.5. We say that a function $f(x)$ belongs to $\mathcal{M}_F(\Omega)$ if

$$M_\Omega f(x) < \infty \quad \text{for a.e. } x \in \Omega.$$

If f belongs to a Banach function space for which $M \in \mathcal{B}(\mathcal{X})$, then $f \in \mathcal{M}_F$.

Theorem 2.6. *If $f \in \mathcal{M}_F(\Omega)$ and $0 < \delta < 1$, then $(M_\Omega f)^\delta \in A_1(\Omega)$.*

We leave the reader with a table of the notation used throughout the article.

Ω	Domain of interest, either $\mathbb{R}^n$ or a cube $Q \subset \mathbb{R}$;
M_Ω	Hardy–Littlewood maximal operator restricted to Ω;
$A_p(\Omega)$	class of A_p weights on Ω;
$\mathcal{M}^r_{A_p}(\Omega)$	functions on Ω with $\lvert f\rvert^r$ majorized by an A_p weight;
$\mathcal{M}_F(\Omega)$	functions on Ω such that $M_\Omega f < \infty$ a.e.;
$A_p^F(\Omega)$	$A_p(\Omega) \cap \mathcal{M}_F(\Omega)$;
$\mathcal{M}_{A_p^F}(\Omega)$	functions majorized by $A_p^F(\Omega)$ weights.

3. The classes $\mathcal{M}^r_{A_p}$

Let us now define a general class of functions majorized by A_p weights and establish some properties of such classes. We remind the reader that a domain Ω will denote throughout either all of $\mathbb{R}^n$ or a cube Q in $\mathbb{R}^n$.

Definition 3.1. Let r and p satisfy $0 < r < \infty$ and $1 \le p \le \infty$. Define $\mathcal{M}^r_{A_p}(\Omega)$ to be the collection of all measurable functions f on Ω such that

$$|f(x)|^r \le w(x) \quad \text{for a.e. } x \in \Omega$$

for some $w \in A_p(\Omega)$. When $r = 1$ we simply write $\mathcal{M}_{A_p}(\Omega)$.

Theorem 2.4 implies the following general facts about the $\mathcal{M}^r_{A_p}$ classes.

Theorem 3.2. *Suppose r and p satisfy $0 < r < \infty$ and $1 \le p \le \infty$. Then*

$$\mathcal{M}^r_{A_p}(\Omega) = \bigcup_{s>r} \mathcal{M}^s_{A_p}(\Omega) \tag{8}$$

and if $p > 1$,

$$\mathcal{M}^r_{A_p}(\Omega) = \bigcup_{1\le q<p} \mathcal{M}^r_{A_q}(\Omega). \tag{9}$$

Proof. We first prove (8). It is clear from (iii) of Theorem 2.2 that the union $\bigcup_{r<s} \mathcal{M}^s_{A_p}(\Omega) \subset \mathcal{M}^r_{A_p}(\Omega)$. On the other hand, if $f \in \mathcal{M}^r_{A_p}(\Omega)$ then $|f|^r \le w \in A_p$. By (ii) of Theorem 2.4, there exists $t > 1$ such that $w^t \in A_p(\Omega)$. But then, taking $s = rt > r$ and $u = w^t$, we have $|f|^s \le u \in A_p$, so $f \in \bigcup_{r<s} \mathcal{M}^s_{A_p}(\Omega)$. The proof of equality (9) follows directly from (i) of Theorem 2.4. □

Our next theorem shows that for a function to have an A_1 majorant it is equivalent for its maximal function to have an A_1 majorant.

Theorem 3.3. *We have $f \in \mathcal{M}_{A_1}(\Omega)$ if and only if $M_\Omega f \in \mathcal{M}_{A_1}(\Omega)$.*

Proof. If $f \in \mathcal{M}_{A_1}(\Omega)$, then $M_\Omega f \le M_\Omega w \le Cw$ since $w \in A_1(\Omega)$, which is to say $M_\Omega f \in \mathcal{M}_{A_1}(\Omega)$. The converse statement follows from the fact that $|f| \le M_\Omega f$. □

Using the exact same reasoning it is easy to prove that $f \in \mathcal{M}^r_{A_1}(\Omega)$ if and only if $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$. However, there is a better result when $r \ge 1$.

Theorem 3.4. *If $r \ge 1$ then the following are equivalent*:

(i) $f \in \mathcal{M}^r_{A_1}(\Omega)$.

(ii) $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$.

(iii) $M_\Omega f \in \mathcal{M}^r_{A_1}(\Omega)$.

Proof. The equivalence (i) ⇔ (ii) follows from Theorem 3.3. We will prove (ii) ⇒ (iii) and (iii) ⇒ (i).

Suppose that $w \in A_1(\Omega)$ and $M_\Omega(|f|^r) \le w$. Since $r \ge 1$, we know that $(M_\Omega f)^r \le M_\Omega(|f|^r) \le w$, which is to say that $M_\Omega f \in \mathcal{M}^r_{A_1}$.

On the other hand if $(M_\Omega f)^r \le w \in A_1(\Omega)$, then $M_\Omega f < \infty$ a.e., and hence f is locally integrable on Ω. By the Lebesgue differentiation theorem we have

$$|f|^r \le (M_\Omega f)^r \le w.$$ □

In the case $0 < r < 1$, we still have $f \in \mathcal{M}^r_{A_1}(\Omega)$ if and only if $M_\Omega(|f|^r) \in \mathcal{M}_{A_1}(\Omega)$. However, it is not true that this is equivalent to $(M_\Omega f)^r \in \mathcal{M}_{A_1}(\Omega)$. Consider the following simple example.

Example 3.5. Let $f(x) = |x|^{-n}$ on $Q = [-1, 1]^n$. If $0 < r < 1$, then $f \in M^r_{A_1}(Q)$ but $M_Q f \equiv \infty$.

Of course, if $0 < r < 1$ and $M_\Omega f < \infty$ a.e., then $(M_\Omega f)^r \in A_1(\Omega)$ (and hence $M_\Omega f \in \mathcal{M}^r_{A_1}(\Omega)$) automatically by Theorem 2.6.

We now study the class $\mathcal{M}_{A_p}$. Since the A_p classes are nested, we have

$$\mathcal{M}_{A_1} \subset \mathcal{M}_{A_p} \subset \mathcal{M}_{A_q} \subset \mathcal{M}_{A_\infty}$$

for $1 \le p \le q \le \infty$. In the local case we have the following characterization.

Theorem 3.6. *If Q is a cube in $\mathbb{R}^n$ then*

$$\mathcal{M}_{A_1}(Q) = \mathcal{M}_{A_\infty}(Q).$$

Proof. It suffices to show $\mathcal{M}_{A_\infty}(Q) \subset \mathcal{M}_{A_1}(Q)$. Suppose that $f \in \mathcal{M}_{A_\infty}(Q)$, so that there exists $w \in A_\infty(Q)$ with

$$|f| \leq w.$$

Since $w \in A_\infty(Q)$, the reverse Hölder inequality implies that there exists $s > 1$ such that

$$(M_Q w^s)^{1/s} \leq 2M_Q w \leq 2(M_Q w^s)^{1/s}.$$

Moreover, since $w \in L^1(Q)$, we have $M_Q w < \infty$ a.e. By Theorem 2.6, $M_Q w$ is bounded above and below by an $A_1(Q)$ weight, and hence is in $A_1(Q)$ itself. $\square$

In the global case we have $\mathcal{M}_{A_1}(\mathbb{R}^n) \subsetneq \mathcal{M}_{A_p}(\mathbb{R}^n)$ for any $p > 1$, as the following example indicates.

Example 3.7. Let $p > 1$ and $0 < \alpha < n(p-1)$. Now consider the function $f(x) = |x|^\alpha$. Then $f \in A_p(\mathbb{R}^n) \subset \mathcal{M}_{A_p}(\mathbb{R}^n)$, but $f \notin \mathcal{M}_F(\mathbb{R}^n)$ so in particular, $f \notin \mathcal{M}_{A_1}(\mathbb{R}^n)$. To see this, notice that for every $x \in \mathbb{R}^n$ and $r > |x|$,

$$Mf(x) \geq \frac{c}{r^n} \int_{|x| \leq r} |x|^\alpha \, dx \simeq r^\alpha$$

so $Mf \equiv \infty$.

To obtain positive results on $\mathbb{R}^n$ for the classes $\mathcal{M}_{A_p}(\mathbb{R}^n)$ and $\mathcal{M}_{A_\infty}(\mathbb{R}^n)$ similar to Theorem 3.6, we must restrict to A_p majorants whose maximal function is finite. Given $w \in A_\infty$, a simple way to create a weight in A_∞^F is to take a truncation: let $w_\lambda = \min(w, \lambda)$ for $\lambda > 0$. Then $w_\lambda \in A_\infty \cap L^\infty \subset A_\infty^F$. We end our study of the class $\mathcal{M}_{A_1}$ with the following characterizations.

Theorem 3.8. $\mathcal{M}_{A_1}(\mathbb{R}^n) = \mathcal{M}_{A_\infty^F}(\mathbb{R}^n)$.

Proof. Since $A_1(\mathbb{R}^n) \subset A_\infty(\mathbb{R}^n)$ and $A_1(\mathbb{R}^n) \subset \mathcal{M}_F(\mathbb{R}^n)$, we have the inclusion $\mathcal{M}_{A_1}(\mathbb{R}^n) \subset \mathcal{M}_{A_\infty^F}(\mathbb{R}^n)$. On the other hand, if f is dominated by a weight w in $A_\infty^F(\mathbb{R}^n) = A_\infty(\mathbb{R}^n) \cap \mathcal{M}_F(\mathbb{R}^n)$, then by Theorem 2.3 we have

$$M(w^s)^{1/s} \leq 2Mw < \infty \quad \text{a.e.}$$

for some $s > 1$. So in particular, $|f| \leq M(|f|^s)^{1/s} \leq M(w^s)^{1/s} \in A_1(\mathbb{R}^n)$. $\square$

Theorem 3.9. *A function f belongs to $\mathcal{M}_{A_1}(\mathbb{R}^n)$ if and only if there is an $s > 1$ such that $|f|^s \in \mathcal{M}_F(\mathbb{R}^n)$.*

Remark 3.10. Given $r > 0$, if one defines $\mathcal{M}^r_F(\mathbb{R}^n)$ to be the class of functions such that $M(|f|^r) < \infty$ a.e. (equivalently $|f|^r \in \mathcal{M}_F(\mathbb{R}^n)$), then Theorem 3.9 can be stated as

$$\mathcal{M}_{A_1}(\mathbb{R}^n) = \bigcup_{r>1} \mathcal{M}^r_F(\mathbb{R}^n).$$

Proof of Theorem 3.9. Let w be an $A_1(\mathbb{R}^n)$ majorant of f. Since $w \in A_1(\mathbb{R}^n)$, $w^s \in A_1(\mathbb{R}^n)$ for some $s > 1$, which implies $|f|^s \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. By Theorem 3.4 we have $M(|f|^s) \in \mathcal{M}_{A_1}(\mathbb{R}^n) \subset L^1_{\text{loc}}(\mathbb{R}^n)$. On the other hand, if there exists $s > 1$ such that $M(|f|^s) < \infty$ a.e., then $M(|f|^s)^{1/s} \in A_1(\mathbb{R}^n)$ by Theorem 2.6, and $|f| \le M(|f|^s)^{1/s}$. □

4. The local case

For this section Q will be a fixed cube in $\mathbb{R}^n$. We begin with the following extension of the equivalences in (2).

Theorem 4.1. *Let Q be a cube in $\mathbb{R}^n$ and r, p_0 satisfy $0 < r < p_0 < \infty$. Then*

$$\mathcal{M}^r_{A_1}(Q) = \bigcup_{p>r} L^p(Q) = \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q).$$

Proof. We will prove the chain of containments

$$\bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{p>r} L^p(Q) \subset \mathcal{M}^r_{A_1}(Q) \subset \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q).$$

- $\big(\bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{p>r} L^p(Q)\big)$: Suppose we have $f \in L^{p_0}_w(Q)$ for some $w \in A_{p_0/r}(Q)$. Set $q_0 = p_0/r$. By (ii) of Theorem 2.2, $\sigma = w^{1-q_0'} \in A_{q_0'}(Q)$. By Theorem 2.3, σ satisfies a reverse Hölder inequality:

$$\left(\frac{1}{|Q'|}\int_{Q'} \sigma^s\,dx\right)^{1/s} \le \frac{2}{|Q'|}\int_{Q'} \sigma\,dx$$

  for some $s > 1$ and all $Q' \subseteq Q$. This implies that $\sigma \in L^s(Q)$. Define $\frac{1}{q} = \frac{1}{q_0} + \frac{1}{sq_0'}$ so that $q > 1$, and let $p = rq > r$. Then

$$\begin{aligned}\left(\int_Q |f|^p\,dx\right)^{1/p} &= \left(\int_Q |f|^{rq} w^{q/q_0} w^{-q/q_0}\,dx\right)^{1/p}\\ &\le \left(\int_Q |f|^{p_0} w\,dx\right)^{1/p_0}\left(\int_Q \sigma^s\,dx\right)^{1/(sq_0')}.\end{aligned}$$

- $\big(\bigcup_{p>r} L^p(Q) \subset \mathcal{M}^r_{A_1}(Q)\big)$: If $f \in L^p(Q)$ for some $p > r$, then Theorem 2.6 implies $|f|^r \le M_Q(|f|^p)^{r/p} \in A_1(\Omega)$.

- $\big(\mathcal{M}^r_{A_1}(Q) \subset \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q)\big)$: Set $q_0 = p_0/r > 1$ and suppose we have $g = |f|^r \le w \in A_1(Q)$. Then $w^{1-q_0} \in A_{q_0}(Q)$ by (iv) of Theorem 2.2 and
$$\int_Q |f|^{p_0} w^{1-q_0}\,dx = \int_Q g^{q_0} w^{1-q_0}\,dx \le \int_Q w\,dx < \infty. \qquad \square$$

Next, we extend Theorem 4.1 to A_∞ weights.

Theorem 4.2. *Let Q be a cube in $\mathbb{R}^n$ and p_0 be an exponent with $0 < p_0 < \infty$. Then*
$$\bigcup_{r>0} \mathcal{M}^r_{A_1}(Q) = \bigcup_{p>0} L^p(Q) = \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$

Proof. We first prove
$$\bigcup_{r>0} \mathcal{M}^r_{A_1}(Q) = \bigcup_{p>0} L^p(Q).$$

- ($\subset$): If $f \in \mathcal{M}^r_{A_1}(Q)$ for some $r > 0$, and $w \in A_1(Q)$ is such that $|f|^r \le w$, then $f \in L^r(Q) \subset \bigcup_{p>0} L^p(Q)$.
- ($\supset$): If $f \in L^p(Q)$ for some $p > 0$, let r be such that $0 < r < p$. Then $|f|^r \le M_Q(|f|^p)^{r/p} \in A_1(Q)$.

Next we show
$$\bigcup_{p>0} L^p(Q) = \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$

- ($\subset$): Suppose $f \in L^p(Q)$ for some $0 < p < \infty$. Then if $r < \min(p, p_0)$ we have
$$f \in L^p(Q) \subset \bigcup_{r<p} L^p(Q) = \bigcup_{w\in A_{p_0/r}} L^{p_0}_w(Q) \subset \bigcup_{w\in A_\infty} L^{p_0}_w(Q).$$
- ($\supset$): Suppose $f \in L^{p_0}_w(Q)$ for some $w \in A_\infty$. Then $w \in A_q$ for some $q > 1$. Set $p = p_0/q$ and notice that $p < p_0$. Then
$$\int_Q |f|^p\,dx = \int_Q |f|^p w^{1/q} w^{-1/q}\,dx \le \left(\int_Q |f|^{p_0} w\,dx\right)^{1/q} \left(\int_Q w^{1-q'}\,dx\right)^{1/q'}. \qquad \square$$

Example 4.3. The function

$$f(x) = x^{-1}(\log x)^{-2}\chi_{(0,1/2)}(x) \tag{10}$$

does not belong to $\mathcal{M}_{A_1}([0,1])$. This follows from Theorem 4.1 since it can be readily checked that
$$f \in L^1([0,1]) \Big\backslash \Big(\bigcup_{p>1} L^p([0,1])\Big).$$

However, $f \in \mathcal{M}_F([0,1])$ since $f \in L^1([0,1])$.

Remark 4.4. Suppose $0 < p < \infty$. Then

$$L^p(Q) = \bigcup_{w \in A_1} L^p_w(Q).$$

The proof of the equality in Remark 4.4 follows from the fact that $1 \in A_1$ and from inequality (7) with $B = Q$.

We define $H_{A_1}(\mathbb{T})$ as the set of functions in N^+ whose boundary function is majorized by an $A_1(\mathbb{T})$ weight. Since we may identify the torus $\mathbb{T}$ with $Q = [0, 2\pi]$, it is obvious that Theorems 4.1 and 4.2 hold for $L^p(\mathbb{T})$ and $L^p_w(\mathbb{T})$ spaces. We have the following analogs for Hardy spaces.

Theorem 4.5. *If p_0 is an exponent satisfying $1 < p_0 < \infty$, then*

$$H_{A_1}(\mathbb{T}) = \bigcup_{p>1} H^p = \bigcup_{w \in A_{p_0}} H^{p_0}_w.$$

Theorem 4.6. *If p_0 is an exponent satisfying $0 < p_0 < \infty$, then*

$$\bigcup_{p>0} H^p = \bigcup_{w \in A_\infty} H^{p_0}_w.$$

Proof of Theorems 4.5 and 4.6. Since $N^+ \cap L^p(\mathbb{T}) = H^p$ for $p > 0$ [Duren 1970, Theorem 2.11], we see that

$$H_{A_1}(\mathbb{T}) = N^+ \cap \mathcal{M}_{\mathcal{A}_1}(\mathbb{T}) = N^+ \cap \bigcup_{p>1} L^p(\mathbb{T}) = \bigcup_{p>1} H^p.$$

This is the first part of Theorem 4.5.

To go from equality of the analogous L^p spaces to the Hardy spaces is a matter of using two facts for $0 < p_0 < \infty$:

(a) $\int_{\mathbb{T}} \log w \, d\theta > -\infty$ and $w \in L^1(\mathbb{T})$ implies that $w = |h|^{p_0}$ for some outer function $h \in H^{p_0}$.

(b) If $h \in H^{p_0}$ is outer, then the set $h\mathbb{C}[z] = \vee\{z^j h : j \geq 0\}$ is dense in H^{p_0}.

Item (a) comes from the standard construction of an outer function [Duren 1970, Section 2.5]. As for item (b), when $1 \leq p_0 < \infty$ this is a standard generalization of Beurling's theorem [Duren 1970, Theorem 7.4]. When $0 < p_0 < 1$, this is a less well known result that can be found in Gamelin [1966, Theorem 4].

For Theorem 4.5 we must show for $1 < p_0 < \infty$ that

$$\bigcup_{p>1} H^p = \bigcup_{w \in A_{p_0}} H^{p_0}_w.$$

Now, for $f \in H^p \subset L^p$, we know there exists $w \in A_{p_0}(\mathbb{T})$ such that $f \in L^{p_0}_w(\mathbb{T})$ by (2). Factor $w = |h|^{p_0}$ with outer $h \in H^{p_0}$. Then, $fh \in N^+ \cap L^{p_0}(\mathbb{T}) = H^{p_0}$

while $h\mathbb{C}[z]$ is dense in H^{p_0} so that there exist polynomials Q_n satisfying

$$\int |fh - Q_n h|^{p_0}\, d\theta = \int |f - Q_n|^{p_0} w\, d\theta \to 0$$

as $n \to \infty$. This shows $f \in H^{p_0}_w$ (since it is initially defined as the closure of the analytic polynomials in $L^{p_0}_w(\mathbb{T})$).

Conversely, we have seen that if $f \in H^{p_0}_w$, then $f \in L^p(\mathbb{T})$ for some $p > 1$. Factor $w = |h|^{p_0}$ as before. Then, $fh \in H^{p_0}$ and $1/h$ is outer, so that $f = fh(1/h) \in N^+$. Since $f \in L^p(\mathbb{T})$, we can then conclude that $f \in H^p$.

The proof of Theorem 4.6, which claims for $0 < p_0 < \infty$ that

$$\bigcup_{p>0} H^p = \bigcup_{w \in A_\infty} H^{p_0}_w,$$

is similar once we know the corresponding fact for $L^p(\mathbb{T})$ spaces. Indeed, take $f \in H^p$ for some $p > 0$. There exists $w \in A_\infty$ such that $f \in L^{p_0}_w(\mathbb{T})$ by Theorem 4.2. Factor $w = |h|^{p_0}$ with outer $h \in H^{p_0}$. Then, $f \in H^{p_0}_w$ as above using Gamelin's result. The converse is similar to the previous proof. □

5. The global case

In this section we address the case when our functions are defined on all of $\mathbb{R}^n$. Let us first prove Theorem 1.3, which states that for any $1 < p < \infty$,

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n).$$

Proof of Theorem 1.3. First we show

$$\mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) \subset \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

Suppose w is an A_1 majorant of f and $f \in L^1_u(\mathbb{R}^n)$ for some $u \in A_1(\mathbb{R}^n)$. By Theorem 2.2, $uw^{1-p} \in A_p(\mathbb{R}^n)$ and

$$\int_{\mathbb{R}^n} |f|^p w^{1-p} u\, dx \le \int_{\mathbb{R}^n} |f| u\, dx.$$

To see the reverse containment suppose that $f \not\equiv 0$ belongs to $L^p_w(\mathbb{R}^n)$ for some $w \in A_p(\mathbb{R}^n)$. We will use the fact that $w \in A_p(\mathbb{R}^n)$ implies $M \in \mathcal{B}(L^p_w)$ to apply the Rubio de Francia algorithm:

$$Rf = \sum_{k=0}^{\infty} \frac{M^k f}{2^k \|M\|^k_{\mathcal{B}(L^p_w)}}.$$

Then Rf is an A_1 majorant of f so $f \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. Also let g be any function in $L^{p'}_\sigma(\mathbb{R}^n)$ where $\sigma = w^{1-p'}$ satisfying $\|g\|_{L^{p'}_\sigma(\mathbb{R}^n)} = 1$. Again, since $\sigma \in A_{p'}(\mathbb{R}^n)$, we apply the Rubio de Francia algorithm

$$Rg = \sum_{k=0}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathcal{B}(L^{p'}_\sigma)}},$$

so that Rg is in $A_1(\mathbb{R}^n)$ and $\|Rg\|_{L^{p'}_\sigma(\mathbb{R}^n)} \le 2$. Hence

$$\int_{\mathbb{R}^n} |f| Rg\,dx = \int_{\mathbb{R}^n} |f| w^{1/p} Rg w^{-1/p}\,dx \le \|f\|_{L^p_w(\mathbb{R}^n)} \|Rg\|_{L^{p'}_\sigma(\mathbb{R}^n)} \le 2\|f\|_{L^p_w(\mathbb{R}^n)},$$

showing that $f \in \bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ as well. □

Before moving on, we remark that the intersection of $\mathcal{M}_{A_1}(\mathbb{R}^n)$ and $\bigcup_{w\in A_1} L^1_w(\mathbb{R}^n)$ is necessary for the result on $\mathbb{R}^n$. We did not encounter this phenomenon in the local case since for a fixed cube, $\mathcal{M}_{A_1}(Q) \subset L^1(Q)$. To see that the intersection is necessary, notice that the function in Example 4.3 viewed as a function on $\mathbb{R}$ belongs to $L^1(\mathbb{R}) \subset \bigcup_{w\in A_1} L^1_w(\mathbb{R})$, but does not belong to $L^p_w(\mathbb{R})$ for any $p > 1$ and $w \in A_p(\mathbb{R})$ since it is not in $L^p_{\mathrm{loc}}(\mathbb{R})$ for any $p > 1$. Theorem 1.3 shows that for $1 < p < \infty$,

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) \subset \mathcal{M}_{A_1}(\mathbb{R}^n).$$

Below we will show this containment is proper (see Example 5.2).

We now prove Theorem 1.4.

Proof of Theorem 1.4. By Theorem 1.3 it suffices to show

$$\bigcup_{w\in A_p} L^p_w(\mathbb{R}^n) \subset \bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')\} \tag{11}$$

and

$$\bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')\} \subset \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w\in A_1} L^1_w(\mathbb{R}^n). \tag{12}$$

However, the containment (11) is immediate, since

$$M \in \mathcal{B}(L^p_w(\mathbb{R}^n)) \Leftrightarrow w \in A_p(\mathbb{R}^n) \Leftrightarrow \sigma \in A_{p'}(\mathbb{R}^n) \Leftrightarrow M \in \mathcal{B}(L^{p'}_\sigma(\mathbb{R}^n)).$$

On the other hand, for containment (12), if $f \not\equiv 0$, then $f \in \mathcal{X}$ for some Banach function space $\mathcal{X}$ such that $M \in \mathcal{B}(\mathcal{X}) \cap \mathcal{B}(\mathcal{X}')$. Then we may use the Rubio de Francia algorithm to construct an $A_1(\mathbb{R}^n)$ majorant:

$$Rf = \sum_{k=0}^{\infty} \frac{M^k f}{2^k \|M\|^k_{\mathcal{B}(\mathcal{X})}}.$$

Then $Rf \in A_1$ and $|f| \le Rf$, so $f \in \mathcal{M}_{A_1}(\mathbb{R}^n)$. Given $g \in \mathscr{X}'$ let

$$Rg = \sum_{k=0}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathscr{B}(\mathscr{X}')}},$$

so that $Rg \in A_1(\mathbb{R}^n) \cap \mathscr{X}'$ and $\|Rg\|_{\mathscr{X}'} \le 2\|g\|_{\mathscr{X}'}$. Then

$$\int_{\mathbb{R}^n} |f| Rg\, dx \le \|f\|_{\mathscr{X}} \|Rg\|_{\mathscr{X}'} \le 2\|f\|_{\mathscr{X}} \|g\|_{\mathscr{X}'},$$

which yields $f \in \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$. □

When $p > 1$, $L^{p,\infty}(\mathbb{R}^n)$ is a Banach function space on which M is bounded (see [Grafakos 2008]), and likewise, its associate $(L^{p,\infty}(\mathbb{R}^n))' = L^{p',1}(\mathbb{R}^n)$, the Lorentz space with exponents p' and 1, is also a Banach function space on which M is bounded (see [Ariño and Muckenhoupt 1990]).

Corollary 5.1. *Suppose* $1 < p_0 < \infty$. *Then*

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subset \bigcup_{w \in A_{p_0}} L^{p_0}_w(\mathbb{R}^n).$$

From Corollary 5.1 we see that the analogous version of the equivalences in (2) are not true on $\mathbb{R}^n$. This follows since

$$\bigcup_{p>1} L^p(\mathbb{R}^n) \subsetneqq \bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n).$$

For example, $f(x) = |x|^{-n/2} \in L^{2,\infty}(\mathbb{R}^n)$ but $f \notin \bigcup_{p>0} L^p(\mathbb{R}^n)$.

We also remark that the techniques required for $\mathbb{R}^n$ are completely different than the local case. For example, to prove the containment

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subset \mathcal{M}_{A_1}(\mathbb{R}^n)$$

it is not enough to simply dominate $|f|$ by $M(|f|^p)^{1/p}$. However, for $f \in L^{p,\infty}(\mathbb{R}^n)$, $M(|f|^p)$ may not be finite (take $f(x) = |x|^{-n/p}$, in which case $M(|f|^p) \equiv \infty$). Instead we must refine our construction of an A_1 majorant using the techniques of Rubio de Francia [1984].

We now provide examples to show that the inclusions in (4) are proper. We first show that the second inclusion is proper, i.e.,

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) \subsetneqq \mathcal{M}_{A_1}(\mathbb{R}^n).$$

Since

$$\bigcup_{w \in A_p} L^p_w(\mathbb{R}^n) = \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n),$$

it suffices to find a function in $\mathcal{M}_{A_1}(\mathbb{R}^n) \setminus \big(\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)\big)$.

Example 5.2. The function $f(x) = 1$ belongs to $\mathcal{M}_{A_1}(\mathbb{R}^n) \setminus \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$. To prove this we need the fact that if $w \in A_\infty$ then $w \notin L^1(\mathbb{R}^n)$. One way to see this (pointed out by the referee) is to notice that A_∞ weights are doubling, and doubling measures have infinite total mass. We can also give an ad hoc argument using the reverse Hölder inequality. If w satisfies

$$\left(\frac{1}{|Q|}\int_Q w^s\,dx\right)^{1/s} \le \frac{2}{|Q|}\int_Q w\,dx$$

for some $s > 1$ and all cubes Q, then by taking $Q_N = [-N, N]^n$, we have

$$\left(\frac{1}{|Q_N|}\int_{Q_1} w^s\,dx\right)^{1/s} \le \left(\frac{1}{|Q_N|}\int_{Q_N} w^s\,dx\right)^{1/s} \le \frac{2}{|Q_N|}\int_{Q_N} w\,dx \le \frac{2}{|Q_N|}\|w\|_{L^1(\mathbb{R}^n)}.$$

Letting $N \to \infty$ we arrive at a contradiction. Finally, to see $1 \notin \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$, notice that $1 \in L^1_w(\mathbb{R}^n)$ if and only if $w \in L^1(\mathbb{R}^n)$.

Next we show that

$$\bigcup_{p>1} L^{p,\infty}(\mathbb{R}^n) \subsetneq \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

For this example we need the following lemma.

Lemma 5.3. *Suppose $u, v \in A_1(\mathbb{R}^n)$. Then*

$$\max(u, v) \in A_1(\mathbb{R}^n) \quad \textit{and} \quad \min(u, v) \in A_1(\mathbb{R}^n).$$

Proof. To see that $\max(u, v)$ is in $A_1(\mathbb{R}^n)$ note that $\max(u, v) \le u + v \le 2\max(u, v)$, and hence

$$M(\max(u, v)) \le Mu + Mv \le C(u + v) \le 2C\max(u, v).$$

To prove $\min(u, v) \in A_1(\mathbb{R}^n)$ we use the equivalent definition of $A_1(\mathbb{R}^n)$:

$$w \in A_1(\mathbb{R}^n) \Leftrightarrow \frac{1}{|Q|}\int_Q w\,dx \le C\inf_Q w \quad \forall Q \subset \mathbb{R}^n$$

where the infimum is the essential infimum of w over the cube Q. Set $w = \min(u, v)$ and let Q be a cube. Notice that $\inf_Q u > \inf_Q v$ implies $\inf_Q w = \inf_Q v$ and hence

$$\frac{1}{|Q|}\int_Q w\,dx \le \frac{1}{|Q|}\int_Q v\,dx \le C\inf_Q v = C\inf_Q w.$$

On the other hand, if $\inf_Q u \le \inf_Q v$ then $\inf_Q w = \inf_Q u$ and so

$$\frac{1}{|Q|}\int_Q w\,dx \le \frac{1}{|Q|}\int_Q u\,dx \le C\inf_Q u = C\inf_Q w.$$

So $w \in A_1(\mathbb{R}^n)$. $\square$

Example 5.4. Consider $f(x) = \max(|x|^{-\alpha n}, |x|^{-\beta n})$. If $0 < \alpha < \beta < 1$ then $f \notin \bigcup_{p>0} L^{p,\infty}(\mathbb{R}^n)$. However,

$$|f(x)| \leq w(x)$$

where $w(x) = \max(|x|^{-\beta n}, 1)$, and $f \in L^1_u(\mathbb{R}^n)$ where $u(x) = \min(|x|^{-\gamma n}, 1)$ when $1 - \alpha < \gamma < 1$. By Lemma 5.3, both u and w belong to $A_1(\mathbb{R}^n)$. Thus

$$f \in \mathcal{M}_{A_1}(\mathbb{R}^n) \cap \bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) = \bigcup_{w \in A_p} L^p_w(\mathbb{R}^n).$$

Finally, we end with brief descriptions of $\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n)$ and $\mathcal{M}_{A_1}(\mathbb{R}^n)$ in terms of Banach function spaces.

Theorem 5.5. $\displaystyle\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) = \bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}')\} = \bigcup \{\mathcal{X} : \mathcal{X}' \cap A_1(\mathbb{R}^n) \neq \varnothing\}.$

Proof. It is clear that

$$\bigcup_{w \in A_1} L^1_w(\mathbb{R}^n) \subset \bigcup \{\mathcal{X} : M \in \mathcal{B}(\mathcal{X}')\},$$

since the associate space of $L^1_w(\mathbb{R}^n)$ is

$$L^\infty_w(\mathbb{R}^n) = \{f : f/w \in L^\infty\}$$

with norm $\|f\|_{L^\infty_w} = \|f/w\|_{L^\infty}$. For any cube Q,

$$\frac{1}{|Q|} \int_Q |f| \, dx \leq \|f/w\|_{L^\infty} \frac{1}{|Q|} \int_Q w \, dx.$$

Hence if $w \in A_1$, then

$$Mf \leq \|f\|_{L^\infty_w} Mw \leq C \|f\|_{L^\infty_w} w,$$

and dividing through by w we obtain $M \in \mathcal{B}(L^\infty_w)$.

The associate space is always a closed subspace of the dual space [Bennett and Sharpley 1988; Rubio de Francia 1987]. Suppose $\mathcal{X}$ is such that $M \in \mathcal{B}(\mathcal{X}')$. Given $g \in \mathcal{X}'$ with $g \not\equiv 0$ (notice Banach function spaces always contain nonzero functions by property (iv) of Banach function norms), let

$$w = \sum_{k=1}^{\infty} \frac{M^k g}{2^k \|M\|^k_{\mathcal{B}(\mathcal{X}')}}$$

so that $w \in A_1(\mathbb{R}^n)$ and $\|w\|_{\mathcal{X}'} \leq \|g\|_{\mathcal{X}'}$. Thus $w \in \mathcal{X}' \cap A_1(\mathbb{R}^n)$, showing that

$$M \in \mathcal{B}(\mathcal{X}') \Rightarrow \mathcal{X} \cap A_1(\mathbb{R}^n) \neq \varnothing.$$

Finally, suppose $f \in \mathscr{X}$ for some $\mathscr{X}$ such that $\mathscr{X}'$ contains an A_1 weight. Let $w \in \mathscr{X}' \cap A_1(\mathbb{R}^n)$. Then

$$\int_{\mathbb{R}^n} |f| w \, dx \le \|f\|_{\mathscr{X}} \|w\|_{\mathscr{X}'},$$

so that $f \in L^1_w(\mathbb{R}^n)$. □

Finally we refer to a result of Chu [2013] which gives the final characterization of $\mathcal{M}_{A_1}(\mathbb{R}^n)$.

Theorem 5.6 [Chu 2013]. $\mathcal{M}_{A_1}(\mathbb{R}^n) = \bigcup \{\mathscr{X} : M \in \mathscr{B}(\mathscr{X})\}$.

6. Questions

We leave the reader with some open questions.

1. Let $A_p^* = \bigcap_{q>p} A_q$. Is there a characterization of the union

$$\bigcup_{w \in A_p^*} L^p_w?$$

In general $A_p \subsetneqq A_p^*$. For example $w(x) = \max((\log|x|^{-1})^{-1}, 1)$ belongs to A_1^* but not A_1. Moreover,

$$\{w : w, 1/w \in A_1^*\} = \mathrm{clos}_{BMO} L^\infty$$

(see [García-Cuerva and Rubio de Francia 1985; Johnson and Neugebauer 1987]). In the local case we have

$$\bigcup_{w \in A_p^*} L^p_w(Q) \subset \bigcap_{s<p} \bigcup_{r>s} L^r(Q) = \limsup_{r \to p^-} L^r(Q).$$

Are these two sets equal?

2. It is well known that

$$L^1 \cap L^\infty \subset \bigcap_{1<p<\infty} L^p \subset \bigcup_{1<p<\infty} L^p \subset L^1 + L^\infty.$$

When can we write a function as the sum of a function in $\mathcal{M}_{A_1}$ and $\bigcup_{w \in A_1} L^1_w$? That is, what conditions on a function guarantee it belongs to $\mathcal{M}_{A_1} + \bigcup_{w \in A_1} L^1_w$?

3. What can one say about

$$\bigcup_{w \in A_p} L^{p,\infty}_w?$$

If $w \in A_1$ and $p > 1$ then $M \in \mathscr{B}(L^{p,\infty}_w)$, so for $p > 1$,

$$\bigcup_{w \in A_1} L^{p,\infty}_w \subset \mathcal{M}_{A_1}.$$

4. Do these results transfer to more general domains? It is possible to consider a general open set Ω as our domain of interest. We may define the $A_p(\Omega)$ classes, $\mathcal{M}_{A_1}(\Omega)$, and the Hardy–Littlewood maximal operator M_Ω exactly as before. However, the openness results, Theorems 2.3 and 2.4, may not hold for Ω, even if it is bounded [Cruz-Uribe et al. 2011]. In the local case we assume that weights belong to $L^1(\Omega)$. What happens if we only assume $L^1_{\mathrm{loc}}(\Omega)$?

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