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COMPLEX HYPERBOLIC (3, 3, n) TRIANGLE GROUPS

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Let p, q, r be positive integers. Complex hyperbolic (p, q, r) triangle groups are representations of the hyperbolic (p, q, r) reflection triangle group to the holomorphic isometry group of complex hyperbolic space $H^2_{\mathbb{C}}$, where the generators fix complex lines. In this paper, we obtain all the discrete and faithful complex hyperbolic (3, 3, n) triangle groups for $n \geq 4$. Our result solves a conjecture of Schwartz in the case when $p = q = 3$.

1. Introduction

An abstract (p, q, r) reflection triangle group for positive integers p, q, r is the group

$$\Delta_{p,q,r} = \langle \sigma_1, \sigma_2, \sigma_3 \mid \sigma_1^2 = \sigma_2^2 = \sigma_3^2 = (\sigma_2\sigma_3)^p = (\sigma_3\sigma_1)^q = (\sigma_1\sigma_2)^r = \mathrm{id} \rangle.$$

We sometimes take (at least) one of p, q, r to be ∞, in which case the corresponding relation does not appear.

It is interesting to seek geometrical representations of $\Delta_{p,q,r}$. An extremely well-known fact is that $\Delta_{p,q,r}$ may be realised geometrically as the reflections in the side of a geodesic triangle with internal angles π/p, π/q, π/r. Furthermore, if $1/p + 1/q + 1/r > 1$, $= 1$ or < 1 then this triangle is spherical, Euclidean or hyperbolic respectively. Moreover, up to isometries (or similarities in the Euclidean case) there is a unique such triangle and the representation is rigid. In the case where (at least) one of p, q, r is ∞, we omit the relevant term from $1/p+1/q+1/r$ and we insist that the sides of the triangle are asymptotic. Thus the (∞, ∞, ∞) triangle is a triangle in the hyperbolic plane with all three vertices on the boundary.

In contrast, if we choose a geometrical representation of $\Delta_{p,q,r}$ in a space of nonconstant curvature then more interesting things can happen; see, for example, [Brehm 1990]. In this paper, we consider representations of $\Delta_{p,q,r}$ to SU(2, 1), which is (a triple cover of) the group of holomorphic isometries of complex hyperbolic space $H^2_{\mathbb{C}}$. A convenient model of $H^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$ with the

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Bergman metric, having constant holomorphic sectional curvature and $1/4$-pinched real sectional curvatures.

A *complex hyperbolic triangle group* will be a representation of $\Delta_{p,q,r}$ to SU(2, 1) where the generators fix complex lines. Note we could have made other choices. For example, we could choose the generators to be antiholomorphic isometries, or we could choose reflections in three complex lines but with higher order. These choices lead to interesting results, but we will not consider them here. A crucial observation is that when $\min\{p, q, r\} \geq 3$, there is a one (real) dimensional representation space of complex hyperbolic triangle groups with $1/p+1/q+1/r < 1$ (either make a simple dimension count or see [Brehm 1990] for example). This means that the representation is determined up to conjugacy by p, q, r and one extra variable. This variable is determined by certain traces; see, for example, [Pratoussevitch 2005].

In order to state our main results, we need a little terminology. Elements of SU(2, 1) act on complex hyperbolic space $\boldsymbol{H}^2_{\mathbb{C}}$ and its boundary (see below). An element $A \in \mathrm{SU}(2, 1)$ is called *loxodromic* if it fixes two points, both of which lie on $\partial\boldsymbol{H}^2_{\mathbb{C}}$; *parabolic* if it fixes exactly one point, and this point lies on $\partial\boldsymbol{H}^2_{\mathbb{C}}$; *elliptic* if it fixes at least one point of $\boldsymbol{H}^2_{\mathbb{C}}$. Discrete groups cannot contain elliptic elements of infinite order. Therefore in a representation of an abstract group to SU(2, 1), if an element of infinite order in the abstract group is represented by an elliptic map then the representation is not discrete or not faithful (or both); compare with [Goldman and Parker 1992].

Complex hyperbolic triangle groups have a rich history; see Schwartz's ICM survey [2002] for an overview. In particular, he presented the following conjectural picture:

Conjecture 1.1 [Schwartz 2002]. *Let $\Delta_{p,q,r}$ be a triangle group with $p \leq q \leq r$. Then any complex hyperbolic representation Γ of $\Delta_{p,q,r}$ is discrete and faithful if and only if $W_A = I_1I_3I_2I_3$ and $W_B = I_1I_2I_3$ are not elliptic. Furthermore*:

(i) *If $p < 10$ then Γ is discrete and faithful if and only if $W_A = I_1I_3I_2I_3$ is nonelliptic.*

(ii) *If $p > 13$ then Γ is discrete and faithful if and only if $W_B = I_1I_2I_3$ is nonelliptic.*

The initial step towards solving this conjecture is the following result of Grossi.

Proposition 1.2 [Grossi 2007]. *Let $\Delta_{p,q,r}$ be a triangle group with $p \leq q \leq r$. Define $W_A = I_1I_3I_2I_3$ and $W_B = I_1I_2I_3$. Then for complex hyperbolic representations of $\Delta_{p,q,r}$*:

(i) *If $p < 10$ and $W_A = I_1I_3I_2I_3$ is nonelliptic then W_B is nonelliptic.*

(ii) *If $p > 13$ and $W_B = I_1I_2I_3$ is nonelliptic then W_A is nonelliptic.*

A motivating example, initially considered by Goldman and Parker [1992] and completed by Schwartz [2001b; 2005], concerns complex hyperbolic ideal triangle groups, that is, representations of $\Delta_{\infty,\infty,\infty}$. This result may be summarised as follows:

Theorem 1.3 [Goldman and Parker 1992; Schwartz 2001b; 2005]. *Let* $\Gamma = \langle I_1, I_2, I_3\rangle$ *be a complex hyperbolic* (∞, ∞, ∞) *triangle group. Then* Γ *is a discrete and faithful representation of* $\Delta_{\infty,\infty,\infty}$ *if and only if* $I_1 I_2 I_3$ *is nonelliptic.*

Note that this gives a complete solution to Schwartz's conjecture in the case $p = q = r = \infty$. Furthermore, Schwartz [2001a] gives an elegant description of the group where $I_1 I_2 I_3$ is parabolic.

Theorem 1.4 [Schwartz 2001a]. *Let* $\Gamma = \langle I_1, I_2, I_3\rangle$ *be the* (∞, ∞, ∞) *complex hyperbolic triangle group for which* $I_1 I_2 I_3$ *is parabolic. Let* $\Gamma_2 = \langle I_1 I_2, I_1 I_3\rangle$ *be the index*-2 *subgroup of* Γ *with no complex reflections. Then* $\boldsymbol{H}^2_{\mathbb{C}}/\Gamma_2$ *is a complex hyperbolic orbifold whose boundary is a triple cover of the Whitehead link complement.*

Schwartz [2007] proves his conjecture for $\min\{p, q, r\}$ sufficiently large (but unfortunately with no effective bound on this minimum).

Theorem 1.5 [Schwartz 2007]. *Let* $\Gamma = \langle I_1, I_2, I_3\rangle$ *be a complex hyperbolic* (p, q, r) *triangle group with* $p \le q \le r$. *If* p *is sufficiently large*, *then* Γ *is a discrete and faithful representation of* $\Delta_{p,q,r}$ *if and only if* $I_1 I_2 I_3$ *is nonelliptic.*

Our main result solves Schwartz's conjecture in the case when $p = q = 3$.

Theorem 1.6. *Let* n *be an integer at least* 4. *Let* $\Gamma = \langle I_1, I_2, I_3\rangle$ *be a complex hyperbolic* $(3, 3, n)$ *triangle group. Then* Γ *is a discrete and faithful representation of* $\Delta_{3,3,n}$ *if and only if* $I_1 I_3 I_2 I_3$ *is nonelliptic.*

Note that the "only if" part is a consequence of our observation about elliptic elements above. The "if" part will follow from Corollary 4.4 below.

For the representation where $I_1 I_3 I_2 I_3$ is parabolic, when $n = 4$ and 5 we have the following description of the quotient orbifold from the census of Falbel, Koseleff and Rouillier [Falbel et al. 2015]. The case $n = 4$ combines work of Deraux, Falbel and Wang [Deraux and Falbel 2015; Falbel and Wang 2014]. The cleanest statement may be found in [Deraux 2015, Theorem 4.2], which also treats the case $n = 5$.

Theorem 1.7 [Deraux 2015, Theorem 4.2]. (i) *Let* $\Gamma = \langle I_1, I_2, I_3\rangle$ *be the complex hyperbolic* $(3, 3, 4)$ *triangle group for which* $I_1 I_3 I_2 I_3$ *is parabolic. Let* $\Gamma_2 = \langle I_1 I_2, I_1 I_3\rangle$ *be the index*-2 *subgroup of* Γ *with no complex reflections. Then* Γ_2 *is conjugate to both* $\rho_{1-1}(\pi_1(M_4))$ *and* $\rho_{4-1}(\pi_1(M_4))$ *from [Falbel et al. 2015]. In particular*, $\boldsymbol{H}^2_{\mathbb{C}}/\Gamma_2$ *is a complex hyperbolic orbifold whose boundary is the figure eight knot complement.*

(ii) *Let* $\Gamma = \langle I_1, I_2, I_3 \rangle$ *be the complex hyperbolic* $(3, 3, 5)$ *triangle group for which* $I_1 I_3 I_2 I_3$ *is parabolic. Let* $\Gamma_2 = \langle I_1 I_2, I_1 I_3 \rangle$ *be the index-*2 *subgroup of* Γ *with no complex reflections. Then* Γ_2 *is conjugate to both* $\rho_{4-3}(\pi_1(M_9))$ *and* $\rho_{3-3}(\pi_1(M_{15}))$ *from [Falbel et al. 2015].*

It should be possible to give a similar description of the other complex hyperbolic $(3, 3, n)$ triangle groups for which $I_1 I_3 I_2 I_3$ is parabolic.

Note that Theorem 1.6 holds in the case $n = \infty$. This follows from recent work of Parker and Will [2015b] (see also [Parker and Will 2015a]). Furthermore, if as above $\Gamma_2 = \langle I_1 I_2, I_1 I_3 \rangle$ is the index-2 subgroup of representation of the $(3, 3, \infty)$ triangle group for which $I_1 I_3 I_2 I_3$ is parabolic, then $\boldsymbol{H}^2_{\mathbb{C}} / \Gamma_2$ is a complex hyperbolic orbifold whose boundary is the Whitehead link complement. This is one of the representations in [Falbel et al. 2015].

Finally, we note some further interesting groups in this family.

Theorem 1.8 [Thompson 2010]. *The complex hyperbolic* $(3, 3, 4)$ *triangle group with* $I_1 I_3 I_2 I_3$ *of order* 7 *and the complex hyperbolic* $(3, 3, 5)$ *triangle group with* $I_1 I_3 I_2 I_3$ *of order* 5 *are both lattices.*

Our method of proof will be to construct a Dirichlet domain based at the fixed point of the order-n elliptic map $I_1 I_2$. Since this point has nontrivial stabiliser, this domain is not a fundamental domain for Γ, but it is a fundamental domain for the coset space of the stabiliser of this point in Γ. Of course, in order to prove directly that this is a Dirichlet domain, we would have to check infinitely many inequalities. Instead, we construct a candidate Dirichlet domain and then use the Poincaré polyhedron theorem for coset decompositions (see [Mostow 1980, Theorem 6.3.2] or [Deraux et al. 2015, Theorem 3.2], for example).

In the case of a Fuchsian $(3, 3, n)$ triangle group acting on the hyperbolic plane, a fundamental domain is a hyperbolic triangle with internal angles $\pi/3$, $\pi/3$ and π/n. The Dirichlet domain with centre the fixed point of an order-n elliptic map is a regular hyperbolic $2n$-gon with internal angles $2\pi/3$. This $2n$-gon is made up of $2n$ copies of the triangular fundamental domain for the $(3, 3, n)$ group; see Figure 1. The stabiliser of the order-n fixed point, which is a dihedral group of order $2n$, fixes the $2n$-gon and permutes the triangles.

For the complex hyperbolic $(3, 3, n)$ triangle groups, we will see that the combinatorial structure of the Dirichlet domain D is the same as that in the Fuchsian case. Namely, D has $2n$ sides, each of which is contained in a bisector. Each side meets exactly two other sides (in the case where $I_1 I_3 I_2 I_3$ is parabolic, there are some additional tangencies between sides on the ideal boundary). The sides are permuted by the dihedral group $\langle I_1, I_2 \rangle$.

In Section 2 we give the necessary background on complex hyperbolic geometry and the Poincaré polyhedron theorem. In Section 3 we normalise the generators of Γ and discus the parameters this involves. Finally, in Section 4 we consider the bisectors and their intersection properties. This is the heart of the paper.

2. Background

Complex hyperbolic space. Let $\mathbb{C}^{2,1}$ be the three-dimensional complex vector space equipped with a Hermitian form H of signature $(2, 1)$. In this paper we consider the diagonal Hermitian form $H = \mathrm{diag}(1, 1, -1)$. Thus if $\boldsymbol{u} = (u_1, u_2, u_3)^t$ and $\boldsymbol{v} = (v_1, v_2, v_3)^t$ then the Hermitian form is given by

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle = \boldsymbol{v}^* H \boldsymbol{u} = u_1\bar{v}_1 + u_2\bar{v}_2 - u_3\bar{v}_3.$$

Define

$$V_- = \{\boldsymbol{v} \in \mathbb{C}^{2,1} : \langle \boldsymbol{v}, \boldsymbol{v}\rangle < 0\}, \quad V_0 = \{\boldsymbol{v} \in \mathbb{C}^{2,1} - \{0\} : \langle \boldsymbol{v}, \boldsymbol{v}\rangle = 0\}.$$

There is a natural projection map $\mathbb{P}$ from $\mathbb{C}^{2,1} - \{0\}$ to $\mathbb{CP}^2$ that identifies all nonzero (complex) scalar multiples of a vector in $\mathbb{C}^{2,1}$. *Complex hyperbolic space* is defined to be $\boldsymbol{H}^2_{\mathbb{C}} = \mathbb{P}V_-$ and its boundary is $\partial \boldsymbol{H}^2_{\mathbb{C}} = \mathbb{P}V_0$. Clearly, if $\boldsymbol{v}$ lies in V_- or V_0 then $v_3 \neq 0$ and so $\boldsymbol{H}^2_{\mathbb{C}} \cup \partial \boldsymbol{H}^2_{\mathbb{C}}$ is contained in the affine chart of $\mathbb{CP}^2$ with $v_3 \neq 0$. We canonically identify this chart with $\mathbb{C}^2$ by setting $z = v_1/v_3$ and $w = v_2/v_3$. Thus a vector $(z, w) \in \mathbb{C}^2$ corresponds to $[z : w : 1]^t$ in $\mathbb{CP}^2$. Evaluating the Hermitian form at this point gives $|z|^2 + |w|^2 - 1 = (|v_1|^2 + |v_2|^2 - |v_3|^2)/|v_3|^2$. Therefore

$$\boldsymbol{H}^2_{\mathbb{C}} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 < 1\}, \quad \partial \boldsymbol{H}^2_{\mathbb{C}} = \{(z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1\}.$$

In other words, $\boldsymbol{H}^2_{\mathbb{C}}$ is the unit ball in $\mathbb{C}^2$ and its boundary is the unit sphere S^3.

The Bergman metric on $\boldsymbol{H}^2_{\mathbb{C}}$ is given in terms of the Hermitian form. Let u and v be points in $\boldsymbol{H}^2_{\mathbb{C}}$ and let $\boldsymbol{u}$ and $\boldsymbol{v}$ be vectors in V_- so that $\mathbb{P}\boldsymbol{u} = u$ and $\mathbb{P}\boldsymbol{v} = v$. The Bergman metric is given as a Riemannian metric ds^2 or a distance function $\rho(u, v)$ by the formulae

$$ds^2 = \frac{-4}{\langle \boldsymbol{u}, \boldsymbol{u}\rangle^2} \det \begin{pmatrix} \langle \boldsymbol{u}, \boldsymbol{u}\rangle & \langle d\boldsymbol{u}, \boldsymbol{u}\rangle \\ \langle \boldsymbol{u}, d\boldsymbol{u}\rangle & \langle d\boldsymbol{u}, d\boldsymbol{u}\rangle \end{pmatrix}, \quad \cosh^2\left(\frac{\rho(u, v)}{2}\right) = \frac{\langle \boldsymbol{u}, \boldsymbol{v}\rangle\langle \boldsymbol{v}, \boldsymbol{u}\rangle}{\langle \boldsymbol{u}, \boldsymbol{u}\rangle\langle \boldsymbol{v}, \boldsymbol{v}\rangle}.$$

The formulae for the Bergman metric are homogeneous and so the ambiguity in the choice of $\boldsymbol{u}$ and $\boldsymbol{v}$ does not matter.

Let $\mathrm{SU}(2, 1)$ be the group of unimodular matrices preserving the Hermitian form H. An element A of $\mathrm{SU}(2, 1)$ acts on $\boldsymbol{H}^2_{\mathbb{C}}$ as $A(u) = \mathbb{P}(A\boldsymbol{u})$, where $\boldsymbol{u}$ is any vector in V_- with $\mathbb{P}\boldsymbol{u} = u$. It is clear that scalar multiples of the identity act trivially. Since the determinant of A is 1, such a scalar multiple must be a cube root of unity. Therefore, we define $\mathrm{PU}(2, 1) = \mathrm{SU}(2, 1)/\{\omega I : \omega^3 = 1\}$. Since the Bergman metric is given in terms of the Hermitian form, it is clear that elements of $\mathrm{SU}(2, 1)$ or $\mathrm{PU}(2, 1)$, act as isometries of $\boldsymbol{H}^2_{\mathbb{C}}$. Indeed, $\mathrm{PU}(2, 1)$ is the full group of holomorphic isometries of $\boldsymbol{H}^2_{\mathbb{C}}$. In what follows, we choose to work with matrices in $\mathrm{SU}(2, 1)$.

There are two kinds of totally geodesic two-dimensional submanifolds in $\boldsymbol{H}^2_{\mathbb{C}}$: complex lines and totally real totally geodesic subspaces. Let $\boldsymbol{c} \in \mathbb{C}^{2,1}$ be a vector with $\langle \boldsymbol{c}, \boldsymbol{c}\rangle > 0$. Then a *complex line* is the projection of the set $\{\boldsymbol{z} \in \mathbb{C}^{2,1} : \langle \boldsymbol{z}, \boldsymbol{c}\rangle = 0\}$. The vector $\boldsymbol{c}$ is then called a *polar vector* of the complex line. The *complex reflection* with polar vector $\boldsymbol{c}$ is defined to be

$$I_{\boldsymbol{c}}(\boldsymbol{z}) = -\boldsymbol{z} + \frac{2\langle \boldsymbol{z}, \boldsymbol{c}\rangle}{\langle \boldsymbol{c}, \boldsymbol{c}\rangle}\boldsymbol{c}.$$

Bisectors and Dirichlet domains. We will consider subgroups of SU(2, 1) acting on $\boldsymbol{H}^2_{\mathbb{C}}$ and we want to show they are discrete. We will do this by constructing a fundamental polyhedron and using the Poincaré polyhedron theorem. There are no totally geodesic real hypersurfaces in $\boldsymbol{H}^2_{\mathbb{C}}$ and so we must choose hypersurfaces for the sides of our polyhedra. We choose to work with bisectors. A *bisector* in $\boldsymbol{H}^2_{\mathbb{C}}$ is the locus of points equidistant (with respect to the Bergman metric) from a given pair of points in $\boldsymbol{H}^2_{\mathbb{C}}$. Suppose that these points are u and v. Choose lifts $\boldsymbol{u} = (u_1, u_2, u_3)^t$ and $\boldsymbol{v} = (v_1, v_2, v_3)^t$ to V_- so that $\langle \boldsymbol{u}, \boldsymbol{u}\rangle = \langle \boldsymbol{v}, \boldsymbol{v}\rangle$. Then the bisector equidistant from u and v is

$$\begin{aligned}\mathcal{B} = \mathcal{B}(u, v) &= \{(z, w) \in \boldsymbol{H}^2_{\mathbb{C}} : \rho((z, w), u) = \rho((z, w), v)\}\\ &= \{(z, w) \in \boldsymbol{H}^2_{\mathbb{C}} : |z\bar{u}_1 + w\bar{u}_2 - \bar{u}_3| = |z\bar{v}_1 + w\bar{v}_2 - \bar{v}_3|\}.\end{aligned}$$

Suppose that we are given three points u, v_1 and v_2 in $\boldsymbol{H}^2_{\mathbb{C}}$. If the three corresponding vectors $\boldsymbol{u}$, $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$ in V_- form a basis for $\mathbb{C}^{2,1}$ then the intersection $\mathcal{B}(u, v_1) \cap \mathcal{B}(u, v_2)$ is called a Giraud disc. This is a particularly nice type of bisector intersection (see [Deraux et al. 2015, Section 2.5]).

Suppose that Γ is a discrete subgroup of PU(2, 1). Let u be a point of $\boldsymbol{H}^2_{\mathbb{C}}$ and write Γ_u for the stabiliser of u in Γ (that is, the subgroup of Γ comprising all elements fixing u). Then the *Dirichlet domain* $D_u(\Gamma)$ for Γ with centre u is defined to be

$$D_u(\Gamma) = \{v \in \boldsymbol{H}^2_{\mathbb{C}} : \rho(v, u) < \rho(v, A(u)) \text{ for all } A \in \Gamma - \Gamma_u\}.$$

Dirichlet domains for certain cyclic groups are particularly simple.

Proposition 2.1. *Let A be a regular elliptic element of* PU(2, 1) *of order* 3. *Then for any point u not fixed by A, the Dirichlet domain $D_u(\langle A\rangle)$ for the cyclic group $\langle A\rangle$ with centre u has exactly two sides.*

Proof. Since there are only two nontrivial elements in $\langle A\rangle$, neither of which fix u, the Dirichlet domain $D_u(\langle A\rangle)$ is

$$D_u(\langle A\rangle) = \{v \in \boldsymbol{H}^2_{\mathbb{C}} : \rho(v, u) < \rho(v, A(u)),\ \rho(v, u) < \rho(v, A^{-1}(u))\}.$$

Its images under A and A^{-1} are

$$A(D_u(\langle A\rangle)) = \{v : \rho(v, A(u)) < \rho(v, u),\ \rho(v, A(u)) < \rho(v, A^{-1}(u))\},$$
$$A^{-1}(D_u(\langle A\rangle)) = \{v : \rho(v, A^{-1}(u)) < \rho(v, u),\ \rho(v, A^{-1}(u)) < \rho(v, A(u))\}.$$

By considering the minimum of $\rho(v, u)$, $\rho(v, A(u))$, $\rho(v, A^{-1}(u))$ as v varies over $\boldsymbol{H}^2_{\mathbb{C}}$, it is clear these three domains are disjoint and their closures cover $\boldsymbol{H}^2_{\mathbb{C}}$. $\square$

Proposition 2.2 [Phillips 1992]. *Let $A \in \mathrm{SU}(2, 1)$ have real trace which is at least 3. Then for any $u \in \boldsymbol{H}^2_{\mathbb{C}}$, the bisectors $\mathcal{B}(u, A(u))$ and $\mathcal{B}(u, A^{-1}(u))$ are disjoint. Thus, the Dirichlet domain $D_u(\langle A\rangle)$ has exactly two sides.*

The Poincaré polyhedron theorem. Our goal is to construct the Dirichlet domain for a complex hyperbolic representation Γ of the (3, 3, n) triangle group with centre the fixed point of an order-n elliptic map. If we use the definition of Dirichlet domain, then we need to check infinitely many inequalities. Thus, we need to use another method. This method is to construct a candidate Dirichlet domain and then use the Poincaré polyhedron theorem.

The main tool we use to show discreteness is the Poincaré polyhedron theorem. The version of this theorem that we use is for polyhedra D with a finite stabiliser; see [Mostow 1980, Theorem 6.3.2] or [Deraux et al. 2015, Theorem 3.2]. Rather than give a general statement of this theorem, we will state it in the particular case we are interested in, namely Dirichlet polyhedra for reflection groups.

Let u be a point in $\boldsymbol{H}^2_{\mathbb{C}}$ and let Υ be a finite subgroup of PU(2, 1) fixing u. Let $A_1, \dots, A_n$ be a finite collection of involutions in PU(2, 1) (so A_i^2 is the identity for each i). Suppose that no A_i fixes u. Suppose that the group Υ preserves this collection of involutions under conjugation. That is, for each A_i with $1 \le i \le n$ and each $P \in \Upsilon$, we suppose that $PA_iP^{-1} = A_j$ for some $1 \le j \le n$. Let $\mathcal{B}_i = \mathcal{B}(u, A_i(u))$ be the bisector equidistant from u and $A_i(u)$. If $P \in \Upsilon$ satisfies $PA_iP^{-1} = A_j$ then $PA_i(u) = A_j(u)$ (since $P(u) = u$) and so P maps $\mathcal{B}_i$ to $\mathcal{B}_j$. We define D to be the component of $\boldsymbol{H}^2_{\mathbb{C}} - \bigcup_{i=1}^n \mathcal{B}_i$ containing u, and we suppose that there are points from each of the $\mathcal{B}_i$ on the boundary of D (that is, the $\mathcal{B}_i$ are not nested). This construction makes D open. Note that, by construction, Υ maps D to itself.

For each $1 \le i \le n$, let $s_i = \mathcal{B}_i \cap \overline{D}$. We call s_i a *side* of D. Such a side can be given a cell structure based on how it intersects other sides. We suppose that the involutions A_i for $1 \le i \le n$ satisfy the following conditions, and so form a *side pairing* of D:

(1) For each $1 \le i \le n$, the involution A_i sends s_i to itself, preserving the cell structure. The relation $A_i^2 = \mathrm{id}$ is called a *reflection relation*.

(2) For each $1 \le i \le n$, we have $\overline{D} \cap A_i(\overline{D}) = s_i$ and $D \cap A(D) = \varnothing$.

(3) If v is a point in s_i and in no other side (that is, v lies in the relative interior of s_i) then there is an open neighbourhood U_v of v lying in $\overline{D} \cup A_i(\overline{D})$.

Note that, unlike the case of reflection groups in constant curvature, A_i does not fix s_i pointwise. Therefore, we could have subdivided s_i into two sets (each of dimension 3) that are interchanged by A_i. In practice this would cause unnecessary complication.

Suppose that s_i and s_j are two sides with nonempty intersection. Their intersection $r = s_i \cap s_j$ is called a *ridge* of D. Since A_i preserves the cell structure of s_i, we see that $A_i(r) = s_i \cap s_k$ is another ridge of D. Applying A_k gives another ridge in s_k. Continuing in this way gives a *ridge cycle*

$$(r_1, s_{i_0}, s_{i_1}) \xrightarrow{A_{i_1}} (r_2, s_{i_1}, s_{i_2}) \xrightarrow{A_{i_2}} (r_3, s_{i_2}, s_{i_3}) \cdots .$$

Here $(r_j, s_{i_{j-1}}, s_{i_j})$ is an ordered triple with $r_j = s_{i_{j-1}} \cap s_{i_j}$. Since there are finitely many Υ orbits of r_1, eventually we find a ridge $r_{m+1} = s_{i_m} \cap s_{i_{m+1}}$ so that the corresponding ordered triple satisfies

$$(r_{m+1}, s_{i_m}, s_{i_{m+1}}) \xrightarrow{P} (r_1, s_{i_0}, s_{i_1})$$

for some $P \in \Upsilon$. We call $T_1 = PA_{i_m} \cdots A_{i_1}$ the *cycle transformation* associated to r_1. It means that the ridge cycle starts at (r_1, s_{i_0}, s_{i_1}) and ends to itself by T_1. Clearly T_1 maps r_1 to itself. Of course, T_1 may not act as the identity on r_1 and even if it does, it may not act as the identity on $\boldsymbol{H}^2_{\mathbb{C}}$. Nevertheless, we suppose T_1 has finite order ℓ. The relation $T_i^{\ell} = \mathrm{id}$ is called a *cycle relation*.

In the example we are interested in, the ridge cycle is

$$(r_1, s_{i_0}, s_{i_1}) \xrightarrow{A_{i_1}} (r_2, s_{i_1}, s_{i_2}) \xrightarrow{P} (r_1, s_{i_0}, s_{i_1})$$

and, in fact, $s_{i_2} = s_{i_0}$ and so $r_2 = r_1$. Moreover, P is an involution with $P(r_1) = r_1$ and $P(s_{i_1}) = s_{i_0}$. Hence the cycle transformation is $T_1 = PA_{i_1}$, which happens to have order 3. Thus, the cycle relation is $T_1^3 = (PA_{i_1})^3 = \mathrm{id}$.

We suppose that D satisfies the *cycle condition* which means that copies of D tessellate a neighbourhood for each ridge r. Furthermore, the relevant copies of D are its preimages under suffix subwords of T^{ℓ}. The full statement is explained in [Deraux et al. 2015]. For brevity, we state this condition only in the special case we are interested in. Let r be a ridge and let $T = PA_i$ be its cycle transformation with cycle relation $(PA_i)^3 = \mathrm{id}$. Let $\mathcal{C} = \{\mathrm{id}, PA_i, (PA_i)^2\}$. Then the cycle condition states that

(1) $$r = \bigcap_{C \in \mathcal{C}} C^{-1}(\bar{D}).$$

(2) If $C_1, C_2 \in \mathcal{C}$ with $C_1 \neq C_2$ then $C_1^{-1}(D) \cap C_2^{-1}(D) = \varnothing$.

(3) If v is a point in r and in no other ridge (that is, v lies in the relative interior of r) then there is an open neighbourhood U_v of v with

$$U_v \subset \bigcup_{C \in \mathcal{C}} C^{-1}(\bar{D}).$$

It means that there are exactly three copies of D along each ridge r, which are D, $T(D)$ and $T^2(D)$. Observe that the stabiliser of r is generated by A_i and P. Hence it is a dihedral group of order 6. Since A_i, P and PA_iP^{-1} preserve one of the three copies and interchange the other two, the stabiliser preserves the three copies of D.

Finally, if two sides of D are asymptotic at a point v of $\partial \boldsymbol{H}^2_{\mathbb{C}}$ then there is a horoball H_v so that H_v intersects $\bar{D}$ only in facets of D containing v and H_v is preserved by the stabiliser of v in Γ. We say that H_v is a *consistent horoball* at v. In particular, if v is a fixed point of a parabolic element of Γ then there exists a consistent horoball at v.

The Poincaré polyhedron theorem states:

Theorem 2.3 [Mostow 1980, Theorem 6.3.2; Deraux et al. 2015, Theorem 3.2]. *Suppose that D is a polyhedron on $\boldsymbol{H}^2_{\mathbb{C}}$ with sides contained in bisectors together with a side pairing. Let $\Upsilon < \mathrm{PU}(2,1)$ be a discrete group of automorphisms of D. Let Γ be the group generated by Υ and the side pairing maps. Suppose that the cycle condition holds at all ridges of D and that there is a consistent horoball at all points (if any) where sides of D are asymptotic. Then*:

(1) *Γ is discrete.*

(2) *The images of D under the cosets of Υ in Γ tessellate $\boldsymbol{H}^2_{\mathbb{C}}$.*

(3) *A fundamental domain for Γ may be obtained by intersecting D with a fundamental domain for Υ.*

(4) *A presentation for Γ is given as follows. The generators are a generating set for Υ together with all side pairing maps. The relations are generated by all relations in Υ, all reflection relations and all cycle relations.*

3. The generators

Consider complex reflections I_1 and I_2 in $\mathrm{SU}(2,1)$ so that I_1I_2 has order n and fixes the origin o. Writing $c=\cos(\pi/n)$ and $s=\sin(\pi/n)$, we may choose I_1 and I_2 to be

$$I_1=\begin{bmatrix}-c & s & 0\\ s & c & 0\\ 0 & 0 & -1\end{bmatrix},\quad I_2=\begin{bmatrix}-c & -s & 0\\ -s & c & 0\\ 0 & 0 & -1\end{bmatrix}. \tag{3-1}$$

Note that polar vectors of I_1 and I_2 are

$$\boldsymbol{n}_1=\begin{bmatrix}s\\ 1+c\\ 0\end{bmatrix},\quad \boldsymbol{n}_2=\begin{bmatrix}-s\\ 1+c\\ 0\end{bmatrix}.$$

We want to find I_3 so that I_1I_3 and I_2I_3 both have order 3. Conjugating by a diagonal map $\mathrm{diag}(e^{i\psi},e^{i\psi},e^{-2i\psi})$ if necessary, we may suppose that the polar

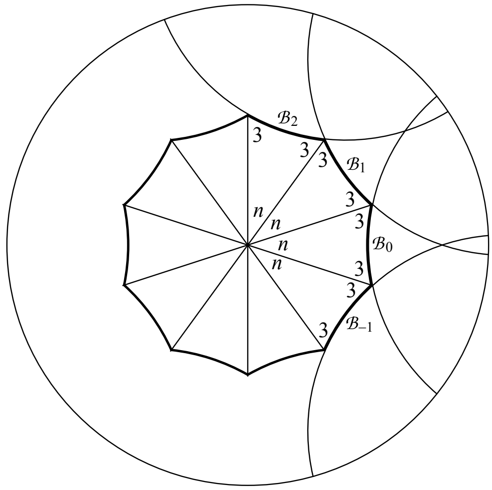

Figure 1. The $2n$-gon in the hyperbolic plane made up of $2n$ copies of a $(3, 3, n)$ triangle.

vector of I_3 is

$$\boldsymbol{n}_3 = \begin{bmatrix} a \\ be^{i\theta} \\ d-1 \end{bmatrix},$$

where a, b, d are nonnegative real numbers satisfying $a^2+b^2-(d-1)^2 = 2(d-1)$, that is, $a^2+b^2-d^2 = -1$. Furthermore, complex conjugating if necessary, we may always assume $\theta \in [0, \pi]$. Then

$$I_3 = \begin{bmatrix} -1+a^2/(d-1) & abe^{-i\theta}/(d-1) & -a \\ abe^{i\theta}/(d-1) & -1+b^2/(d-1) & -be^{i\theta} \\ a & be^{-i\theta} & -d \end{bmatrix}. \tag{3-2}$$

It is easy to check that I_3 lies in $\mathrm{SU}(2, 1)$, has order 2 and polar vector $\boldsymbol{n}_3$.

Lemma 3.1. *Let I_1, I_2 and I_3 be given by* (3-1) *and* (3-2)*. If I_1I_3 and I_2I_3 have order* 3 *then $\theta = \pi/2$ and*

$$c(a^2-b^2) = d(d-1). \tag{3-3}$$

Proof. The condition that I_1I_3 and I_2I_3 have order 3 is equivalent to $\mathrm{tr}(I_1I_3) = \mathrm{tr}(I_2I_3) = 0$. That is,

$$\frac{-c(a^2-b^2)+2sab\cos\theta}{d-1} + d = \frac{-c(a^2-b^2)-2sab\cos\theta}{d-1} + d = 0.$$

The result follows directly. □

From now on, we write $\theta = \pi/2$ in (3-2). Since we know $a^2 + b^2 = d^2 - 1$ and $a^2 - b^2 = d(d-1)/c$, we immediately have

$$a^2 = (d-1)(1+d+d/c)/2, \quad b^2 = (d-1)(1+d-d/c)/2. \tag{3-4}$$

Corollary 3.2. *Let*

$$\iota : \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} \longmapsto \begin{bmatrix} \bar{z}_1 \\ -\bar{z}_2 \\ \bar{z}_3 \end{bmatrix}.$$

Then ι has order 2 *and*

$$\iota I_1 \iota = I_2, \quad \iota I_2 \iota = I_1, \quad \iota I_3 \iota = I_3.$$

Proof. It is easy to see that ι^2 is the identity. A simple calculation shows $\iota(\boldsymbol{n}_1) = \boldsymbol{n}_2$ and $\iota(\boldsymbol{n}_3) = \boldsymbol{n}_3$, using $e^{i\theta} = i$. □

Lemma 3.3. *The group $\langle I_1, I_2, I_3 \rangle$ is determined up to conjugacy by the variable d, which lies in the interval $1 < d \le c/(1-c)$. Moreover, $\langle I_1, I_2, I_3 \rangle$ lies in* SO(2, 1) *when $d = c/(1-c)$.*

Proof. We have conjugated so that I_1 and I_2 have the form (3-1), and I_3 has the form (3-2) with $\theta = \pi/2$. After this conjugation, the only remaining parameters are the nonnegative real numbers a, b and d. Using (3-4) these are completely determined by d. Moreover, again using (3-4) we see that a^2 and b^2 are nonnegative if and only if $d \ge 1$ and $d \le c/(1-c)$. We cannot have $d = 1$ or else $\boldsymbol{n}_3$ is the zero vector. Thus $1 < d \le c/(1-c)$. Finally, when $d = c/(1-c)$, we have $b = 0$ and the entries of I_3 are all real. □

Lemma 3.4. *Let I_1, I_2 and I_3 be given by* (3-1) *and* (3-2). *Suppose $I_1 I_3$ and $I_2 I_3$ have order* 3. *Then $I_1 I_3 I_2 I_3$ is elliptic if and only if $d < 3/(4s^2)$.*

Proof. Calculating directly, we see that

$$\begin{aligned} \operatorname{tr}(I_1 I_3 I_2 I_3) &= \frac{c^2(a^2-b^2)^2}{(d-1)^2} + \frac{2(c^2-s^2)(d-1-a^2-b^2)}{d-1} - 2c(a^2-b^2) + d^2 \\ &= 4s^2 d. \end{aligned}$$

(We could have derived this using the formulae in [Pratoussevitch 2005].) The condition that $I_1 I_3 I_2 I_3$ is elliptic is equivalent to $3 > \operatorname{tr}(I_1 I_3 I_2 I_3) = 4s^2 d$. □

Thus, our parameter space for $\langle I_1, I_2, I_3 \rangle$ with $I_1 I_3 I_2 I_3$ nonelliptic is given by

$$\frac{3}{4s^2} \le d \le \frac{c}{1-c}. \tag{3-5}$$

Note that the condition $n > 3$ implies both $3/(4s^2) > 1$ and $c/(1-c) > 1$. For example, when $n = 4$ we have $c = s = 1/\sqrt{2}$ and our range becomes

$$3/2 \le d \le \sqrt{2} + 1.$$

4. The bisectors

We define a polyhedron D bounded by sides contained in $2n$ bisectors.

Definition 4.1. For $k \in \mathbb{Z}$, define the involution $A_k \in \langle I_1, I_2, I_3\rangle$ as follows:

(1) If $k = 2m$ is an even integer then $A_k = (I_2 I_1)^{k/2} I_3 (I_1 I_2)^{k/2}$.

(2) If $k = 2m+1$ is an odd integer then $A_k = (I_2 I_1)^{(k-1)/2} I_2 I_3 I_2 (I_1 I_2)^{(k-1)/2}$.

Let o be the fixed point of $I_1 I_2$ in $\boldsymbol{H}^2_{\mathbb{C}}$. For all integers k, the bisector $\mathcal{B}_k$ is defined to be the bisector equidistant from o and $A_k(o)$. Note that in both cases $A_{k+2n} = A_k$ and so $\mathcal{B}_{k+2n} = \mathcal{B}_k$. This gives $2n$ bisectors $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ and we may take the index k mod $2n$.

The following lemma follows immediately from the definition.

Lemma 4.2. *Let $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ be as defined in Definition 4.1. Then for each k mod $2n$ and each m mod n:*

(1) *The map $(I_2 I_1)^m$ sends $\mathcal{B}_k$ to $\mathcal{B}_{2m+k}$.*

(2) *The map $(I_2 I_1)^m I_2$ sends $\mathcal{B}_k$ to $\mathcal{B}_{2m+1-k}$. In particular, the map $(I_2 I_1)^k I_2$ sends $\mathcal{B}_k$ to $\mathcal{B}_{k+1}$.*

(3) *The antiholomorphic involution ι defined in Corollary 3.2 sends $\mathcal{B}_k$ to $\mathcal{B}_{-k}$. In particular, the map $(I_2 I_1)^m I_2 \iota$ sends $\mathcal{B}_k$ to $\mathcal{B}_{2m+1+k}$.*

The main result of this section is that the combinatorial configuration of the bisectors does not change as d decreases from $c/(1-c)$ to $3/(4s^2)$. More precisely:

Theorem 4.3. *Let $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ be as defined in Definition 4.1. Suppose that $3/(4s^2) \le d \le c/(1-c)$. Then, taking the indices mod $2n$, for each k:*

(1) *The bisector $\mathcal{B}_k$ intersects $\mathcal{B}_{k\pm 1}$ in a Giraud disc. This Giraud disc is preserved by $A_k A_{k\pm 1}$, which has order 3.*

(2) *The intersection of $\mathcal{B}_k$ with $\mathcal{B}_{k\pm 2}$ is contained in the halfspace bounded by $\mathcal{B}_{k\pm 1}$ not containing o.*

(3) *The bisector $\mathcal{B}_k$ does not intersect $\mathcal{B}_{k\pm \ell}$ for $3 \le \ell \le n$. Moreover, the boundaries of these bisectors are disjoint except for when $\ell = 3$ and $d = 3/(4s^2)$, in which case the boundaries intersect in a single point, which is a parabolic fixed point.*

As a corollary to this theorem, we can use the Poincaré polyhedron theorem to prove the "if" part of Theorem 1.6.

Corollary 4.4. *Let A_{-n+1} to A_n and $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$ be as in Theorem 4.3. Suppose that $3/(4s^2) \le d \le c/(1-c)$. Let D be the polyhedron in $\boldsymbol{H}^2_{\mathbb{C}}$ containing o and bounded by $\mathcal{B}_{-n+1}$ to $\mathcal{B}_n$. Then the maps A_{-n+1} to A_n form a side paring for D that satisfies the conditions of the Poincaré polyhedron theorem, Theorem 2.3. In particular, $\langle I_1, I_2, I_3\rangle$ is a discrete and faithful representation of $\Delta_{3,3,n}$.*

Proof. Since A_k is an involution, it is clear that the $\{A_k\}$ form a side pairing for D. Now consider the ridge $r_k = \mathcal{B}_k \cap \mathcal{B}_{k+1}$. Applying either of the side pairing maps A_k or A_{k+1} sends this ridge to itself. We then apply $P_k = (I_2 I_1)^k I_2$ to obtain the cycle transformation $P_k A_k$. When k is even,

$$P_k A_k = (I_2 I_1)^k I_2 (I_2 I_1)^{k/2} I_3 (I_1 I_2)^{k/2} = (I_2 I_1)^{k/2} I_2 I_3 (I_1 I_2)^{k/2},$$

and when k is odd,

$$\begin{aligned} P_k A_k &= (I_2 I_1)^k I_2 (I_2 I_1)^{(k-1)/2} I_2 I_3 I_2 (I_1 I_2)^{(k-1)/2} \\ &= (I_2 I_1)^{(k+1)/2} I_3 I_1 (I_1 I_2)^{(k+1)/2}. \end{aligned}$$

In both cases, $P_k A_k$ is equal to $A_k A_{k+1}$, which has order 3. There is a neighbourhood U_k of the ridge r_k for which the intersection of U_k with D is the same as its intersection with the Dirichlet domain for $\langle P_k A_k \rangle$. Therefore, we have local tessellation around all the ridges of D using the argument of Proposition 2.1.

All the other sides of D are disjoint, apart from when $d = 3/(4s^2)$, in which case $\mathcal{B}_k$ and $\mathcal{B}_{k\pm 3}$ are asymptotic at a point of $\partial \boldsymbol{H}^2_{\mathbb{C}}$. This point is a parabolic fixed point, as required.

Finally, each side yields the reflection relation A_k^2, which is conjugate to I_3^2. The cycle relations give $(P_k A_k)^3$, which are conjugate to $(I_2 I_3)^3$ when k is even and $(I_3 I_1)^3$ when k is odd. In addition we have the relations from $\Upsilon = \langle I_1, I_2 \rangle$, which are I_1^2, I_2^2 and $(I_1 I_2)^n$. From the Poincaré theorem, all other relations may be deduced from these. Thus $\langle I_1, I_2, I_3 \rangle$ is a faithful representation of $\Delta_{3,3,n}$. $\square$

Write $c_k = \cos(k\pi/n)$ and $s_k = \sin(k\pi/n)$. Then

$$(I_2 I_1)^m = \begin{bmatrix} c_{2m} & -s_{2m} & 0 \\ s_{2m} & c_{2m} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad (I_2 I_1)^m I_2 \begin{bmatrix} -c_{2m+1} & -s_{2m+1} & 0 \\ -s_{2m+1} & c_{2m+1} & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

We have

$$(I_2 I_1)^m I_3(o) = \begin{bmatrix} -c_{2m} a + s_{2m} bi \\ -s_{2m} a - c_{2m} bi \\ -d \end{bmatrix}, \quad (I_1 I_2)^m I_3(o) = \begin{bmatrix} -c_{2m} a - s_{2m} bi \\ s_{2m} a - c_{2m} bi \\ -d \end{bmatrix}.$$

Also

$$(I_2 I_1)^m I_2 I_3(o) = \begin{bmatrix} c_{2m+1} a + s_{2m+1} bi \\ s_{2m+1} a - c_{2m+1} bi \\ d \end{bmatrix}, \quad (I_1 I_2)^m I_1 I_3(o) = \begin{bmatrix} c_{2m+1} a - s_{2m+1} bi \\ -s_{2m+1} a - c_{2m+1} bi \\ d \end{bmatrix}.$$

We begin by proving Theorem 4.3(1).

Proposition 4.5. *For each $-n+1 \le k \le n$, the bisectors $\mathcal{B}_k$ and $\mathcal{B}_{k\pm 1}$ (with indices taken mod $2n$) intersect in $\boldsymbol{H}^2_{\mathbb{C}}$ in a Giraud disc. This Giraud disc is preserved by $(I_2 I_1)^{k/2}(I_2 I_3)(I_1 I_2)^{k/2}$ when k is even and $(I_2 I_1)^{(k+1)/2}(I_3 I_1)(I_1 I_2)^{(k+1)/2}$ when k is odd.*

Proof. Using Lemma 4.2 we need only consider $k = 0$ and $k = 1$. The bisectors $\mathcal{B}_0$ and $\mathcal{B}_1$ are equidistant from o and from $I_3(o) = I_3I_2(o)$ and from $I_2I_3(o)$ respectively. Observe that I_2I_3 does not fix o. Since the map I_2I_3 has order 3, the Dirichlet domain with centre o for the cyclic group $\langle I_2I_3\rangle$ only contains faces contained in these two bisectors. The intersection is a Giraud disc invariant under powers of I_2I_3 by construction. □

Next we prove Theorem 4.3(3) in the case where $\ell = 2m + 1$ is odd.

Proposition 4.6. *Suppose that* $3/(4s^2) \le d \le c/(1-c)$. *For each* $-n+1 \le k \le n$ *and* $1 \le m \le (n-1)/2$, *the bisectors* $\mathcal{B}_k$ *and* $\mathcal{B}_{k\pm(2m+1)}$ (*with indices taken mod* $2n$) *do not intersect in* $\boldsymbol{H}^2_{\mathbb{C}}$. *Moreover*, *their closures intersect on* $\partial\boldsymbol{H}^2_{\mathbb{C}}$ *if and only if* $d = 3/(4s^2)$ *and* $m = 1$. *In the latter case*, *the closures intersect in a unique point*, *which is a parabolic fixed point.*

Proof. Using Lemma 4.2 we need only consider $\mathcal{B}_0$ and $\mathcal{B}_{2m+1}$. These bisectors are equidistant from o and $I_3(o) = I_3I_2(I_1I_2)^m(o)$ and from $(I_2I_1)^m I_2I_3(o)$ respectively. Consider the Dirichlet domain with centre o for the cyclic group $\langle (I_2I_1)^m I_2I_3\rangle$. We claim that this Dirichlet domain has exactly two sides and these sides are disjoint. To do so, we use Phillips' theorem, Proposition 2.2.

A brief calculation shows that

$$\operatorname{tr}\big((I_2I_1)^m I_2I_3\big) = -c_{2m+1}\frac{a^2-b^2}{d-1} + d = \frac{d(c-c_{2m+1})}{c} = \frac{2ds_{m+1}s_m}{c}.$$

When $1 \le m \le (n-1)/2$, we have

$$s_ms_{m+1} \ge ss_2 = 2s^2c$$

with equality if and only if $m = 1$. Therefore,

$$\operatorname{tr}\big((I_2I_1)^m I_2I_3\big) = 2ds_{m+1}s_m/c \ge 4ds^2$$

with equality if and only if $m = 1$. Hence, when $4ds^2 \ge 3$, we have $(I_2I_1)^m I_2I_3$ is nonelliptic with real trace, and is loxodromic unless $m = 1$ and $d = 3/(4s^2)$. By Phillips' theorem we see that any Dirichlet domain for $\langle (I_2I_1)^m I_2I_3\rangle$ has two faces and these faces do not intersect in $\boldsymbol{H}^2_{\mathbb{C}}$.

In fact, when $d = 3/(4s^2)$ and $m = 1$, the bisectors $\mathcal{B}_0$ and $\mathcal{B}_3$ are asymptotic on the boundary of $\boldsymbol{H}^2_{\mathbb{C}}$ at the (parabolic) fixed point of $I_2I_1I_2I_3$. □

Proposition 4.7. (i) *Suppose* $p = [z, w, 1]^t$ *lies on* $\mathcal{B}_{2\ell} \cap \mathcal{B}_{-2\ell}$. *Then for some angles* θ, ϕ, *we have*

$$z = \frac{s_{2\ell}a(\cos\theta e^{i\phi} + d) - c_{2\ell}b\sin\theta e^{i\phi}}{c_{2\ell}s_{2\ell}(a^2-b^2)},$$

$$w = \frac{-s_{2\ell}bi(\cos\theta e^{i\phi} + d) + c_{2\ell}ai\sin\theta e^{i\phi}}{c_{2\ell}s_{2\ell}(a^2-b^2)}.$$

(ii) *Suppose* $p=[z,w,1]^t$ *lies on* $\mathcal{B}_{2\ell+1}\cap\mathcal{B}_{-2\ell-1}$. *Then for some angles* θ, ϕ, *we have*

$$z=\frac{s_{2\ell+1}a(\cos\theta e^{i\phi}+d)-c_{2\ell+1}b\sin\theta e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(a^2-b^2)},$$
$$w=\frac{s_{2\ell+1}bi(\cos\theta e^{i\phi}+d)-c_{2\ell+1}ai\sin\theta e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(a^2-b^2)}.$$

Proof. First consider the bisector intersection from (i). Then z and w satisfy

$$1=\left|z(-c_{2\ell}a+s_{2\ell}bi)+w(s_{2\ell}a+c_{2\ell}bi)+d\right|,$$
$$1=\left|z(-c_{2\ell}a-s_{2\ell}bi)+w(-s_{2\ell}a+c_{2\ell}bi)+d\right|.$$

Expanding out, adding and subtracting yields

$$1=\left|zc_{2\ell}a-wc_{2\ell}bi-d\right|^2+\left|zs_{2\ell}bi+ws_{2\ell}a\right|^2,$$
$$0=2\,\mathrm{Re}\big((zc_{2\ell}a-wc_{2\ell}bi-d)(-\bar{z}s_{2\ell}bi+\bar{w}s_{2\ell}a)\big).$$

Thus we can write

$$zc_{2\ell}a-wc_{2\ell}bi-d=\cos\theta e^{i\phi},$$
$$zs_{2\ell}bi+ws_{2\ell}a=i\sin\theta e^{i\phi}.$$

Inverting these equations yields

$$z=\frac{s_{2\ell}a(\cos\theta e^{i\phi}+d)-c_{2\ell}b\sin\theta e^{i\phi}}{c_{2\ell}s_{2\ell}(a^2-b^2)},$$
$$w=\frac{-s_{2\ell}bi(\cos\theta e^{i\phi}+d)+c_{2\ell}ai\sin\theta e^{i\phi}}{c_{2\ell}s_{2\ell}(a^2-b^2)}.$$

For the second bisector intersection, we have

$$1=\left|z(c_{2\ell+1}a+s_{2\ell+1}bi)+w(-s_{2\ell+1}a+c_{2\ell+1}bi)-d\right|^2,$$
$$1=\left|z(c_{2\ell+1}a-s_{2\ell+1}bi)+w(s_{2\ell+1}a+c_{2\ell+1}bi)-d\right|^2.$$

Expanding out, adding and subtracting yields

$$1=\left|zc_{2\ell+1}a+wc_{2\ell+1}bi-d\right|^2+\left|-zs_{2\ell+1}bi+ws_{2\ell+1}a\right|^2,$$
$$0=2\,\mathrm{Re}\big((zc_{2\ell+1}a+wc_{2\ell+1}bi-d)(\bar{z}s_{2\ell+1}bi+\bar{w}s_{2\ell+1}a)\big).$$

So once again we have

$$zc_{2\ell+1}a+wc_{2\ell+1}bi-d=\cos\theta e^{i\phi},$$
$$-zs_{2\ell+1}bi+ws_{2\ell+1}a=-i\sin\theta e^{i\phi}.$$

Thus,

$$z = \frac{s_{2\ell+1}a(\cos\theta e^{i\phi}+d) - c_{2\ell+1}b\sin\theta e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(a^2-b^2)},$$
$$w = \frac{s_{2\ell+1}bi(\cos\theta e^{i\phi}+d) - c_{2\ell+1}ai\sin\theta e^{i\phi}}{c_{2\ell+1}s_{2\ell+1}(a^2-b^2)}. \qquad \square$$

We can now prove Theorem 4.3(3) in the case where $\ell = 2m$ is even.

Proposition 4.8. *Suppose that* $3/(4s^2) \le d \le c/(1-c)$. *For each* $-n+1 \le k \le n$ *and* $2 \le m \le n/2$, *the bisectors* $\mathcal{B}_k$ *and* $\mathcal{B}_{k\pm 2m}$ (*with indices taken mod* $2n$) *do not intersect in complex hyperbolic space.*

Proof. Using Lemma 4.2, we need only consider $\mathcal{B}_m$ and $\mathcal{B}_{-m}$ where $2 \le m \le n/2$.

Using Proposition 4.7 we see that an intersection point $p = [z, w, 1]^t$ of $\mathcal{B}_m$ and $\mathcal{B}_{-m}$ must satisfy

$$z = \frac{s_m a(\cos\theta e^{i\phi}+d) - c_m b\sin\theta e^{i\phi}}{c_m s_m(a^2-b^2)},$$
$$w = \pm\frac{-s_m bi(\cos\theta e^{i\phi}+d) + c_m ai\sin\theta e^{i\phi}}{c_m s_m(a^2-b^2)}.$$

We claim that $|z|^2+|w|^2 \ge 1$ and so such a point does not lie in $\boldsymbol{H}^2_{\mathbb{C}}$. We have

$$\begin{aligned}
&c_m^2 s_m^2(a^2-b^2)^2(|z|^2+|w|^2-1)\\
&\quad= \left|s_m a(\cos\theta e^{i\phi}+d) - c_m b\sin\theta e^{i\phi}\right|^2\\
&\qquad\quad + \left|-s_m bi(\cos\theta e^{i\phi}+d) + c_m ai\sin\theta e^{i\phi}\right|^2 - c_m^2 s_m^2(a^2-b^2)^2\\
&\quad= s_m^2(a^2+b^2)(\cos^2\theta + 2d\cos\theta\cos\phi + d^2)\\
&\qquad\quad - 2c_m s_m ab(2\cos\theta\sin\theta + 2d\sin\theta\cos\phi)\\
&\qquad\qquad + c_m^2(a^2+b^2)\sin^2\theta - c_m^2 s_m^2(a^2+b^2)^2 + 4c_m^2 s_m^2 a^2 b^2\\
&\quad= s_m^2(d^2-1)(\cos^2\theta + 2d\cos\theta\cos\phi + d^2)\\
&\qquad\quad - 4c_m s_m ab(\cos\theta\sin\theta + d\sin\theta\cos\phi)\\
&\qquad\qquad + c_m^2(d^2-1)\sin^2\theta - c_m^2 s_m^2(d^2-1)^2 + 4c_m^2 s_m^2 a^2 b^2\\
&\quad= \bigl(\cos\theta\sin\theta + d\sin\theta\cos\phi - 2c_m s_m ab\bigr)^2 + d^2\sin^2\theta\sin^2\phi\\
&\qquad\quad + (s_m^2(d^2-1) - \sin^2\theta)\bigl(\cos^2\theta + 2d\cos\theta\cos\phi + d^2 - c_m^2(d^2-1)\bigr)\\
&\quad\ge (s_m^2(d^2-1) - \sin^2\theta)\bigl(\cos^2\theta + 2d\cos\theta\cos\phi + d^2 - c_m^2(d^2-1)\bigr).
\end{aligned}$$

Therefore, it is sufficient to prove

$$0 < s_m^2(d^2-1) - \sin^2\theta, \tag{4-1}$$

$$0 < \cos^2\theta + 2d\cos\theta\cos\phi + d^2 - c_m^2(d^2-1). \tag{4-2}$$

In order to prove these inequalities, we need to use the lower bound on d. Using $m \geq 2$ and $d \geq 3/(4s^2)$, we have

$$(1-c_m)d \geq (1-c_2)d = 2s^2 d \geq 3/2. \tag{4-3}$$

We also use $s_m^2 = 1-c_m^2 = (1-c_m)(1+c_m)$ and $c_m \geq 0$ (the latter uses $m \leq n/2$).

First, we consider (4-1):

$$\begin{aligned} s_m^2(d^2-1)-\sin^2\theta &= \frac{1+c_m}{1-c_m}((1-c_m)d)^2 - 2 + c_m^2 + \cos^2\theta \\ &\geq ((1-c_m)d)^2 - 2 \\ &\geq 1/4, \end{aligned}$$

where the last inequality follows from (4-3). This proves (4-1).

Now consider (4-2):

$$\begin{aligned} \cos^2\theta+2d\cos\theta\cos\phi+d^2-c_m^2(d^2-1) &= \frac{(d(1-c_m)+\cos\theta\cos\phi)^2+\cos^2\theta\sin^2\phi}{1-c_m} \\ &\quad + \frac{c_m}{1-c_m}\big((d(1-c_m))^2-\cos^2\theta\big)+c_m^2 \\ &\geq \frac{c_m}{1-c_m}(9/4-\cos^2\theta) \\ &> 0. \end{aligned}$$

Again we used (4-3). This proves (4-2) and so establishes the result. □

Propositions 4.6 and 4.8 complete the proof of Theorem 4.3(3). It remains to prove Theorem 4.3(2). That is, we must consider the intersection of $\mathcal{B}_k$ and $\mathcal{B}_{k\pm 2}$.

Consider $\mathcal{B}_1 \cap \mathcal{B}_{-1}$. We claim that the fixed point of $I_3 I_1 I_2 I_3$ (that is $I_3(o)$) lies on $\mathcal{B}_1 \cap \mathcal{B}_{-1}$. The bisector $\mathcal{B}_1$ consists of all points equidistant from o and $A_1(o) = I_2 I_3 I_2(o) = I_2 I_3(o)$. We have

$$\rho\big(I_3(o),\, I_2 I_3(o)\big) = \rho\big(o,\, I_3 I_2 I_3(o)\big) = \rho\big(o,\, I_2 I_3(o)\big).$$

The first equality follows since I_3 is an isometry and the second since $I_3 I_2 I_3 = I_2 I_3 I_2$ and $I_2(o) = o$. Thus $I_3(o)$ lies on $\mathcal{B}_1$. A similar argument shows

$$\rho\big(I_3(o),\, I_1 I_3(o)\big) = \rho\big(o,\, I_1 I_3(o)\big).$$

and so $I_3(o)$ lies on $\mathcal{B}_{-1}$ as well. Thus $\mathcal{B}_1 \cap \mathcal{B}_{-1}$ is nonempty, which can be seen in Figure 1. By symmetry, this comment also applies to the intersection of $\mathcal{B}_k$ and $\mathcal{B}_{k\pm 2}$. We must show that this intersection never contributes a ridge of D.

Proposition 4.9. *Suppose that* $3/(4s^2) \leq d \leq c/(1-c)$. *For each* $-n+1 \leq k \leq n$, *all points of* $\mathcal{B}_k \cap \mathcal{B}_{k\pm 2}$ *lie in the halfspace bounded by* $\mathcal{B}_{k\pm 1}$ *not containing* o.

Proof. Using Lemma 4.2 as before, it suffices to consider $\mathcal{B}_1$ and $\mathcal{B}_{-1}$. We need to show that all points of $\mathcal{B}_1 \cap \mathcal{B}_{-1}$ lie in the halfspace closer to $I_3(o)$ than to o.

Suppose that $p = [z, w, 1]^t$ lies on $\mathcal{B}_1 \cap \mathcal{B}_{-1}$. Using Proposition 4.7(ii) with $m = 0$, and using (3-3) to write $c(a^2 - b^2) = d(d-1)$, we find

$$z = \frac{sa(\cos\theta e^{i\phi} + d) - cb\sin\theta e^{i\phi}}{sd(d-1)}, \tag{4-4}$$

$$w = \frac{sbi(\cos\theta e^{i\phi} + d) - cai\sin\theta e^{i\phi}}{sd(d-1)}. \tag{4-5}$$

Note that we used (3-3) to simplify the denominator.

The point $p = [z, w, 1]^t$ lies in the halfspace closer to $I_3(o)$ than to o if and only if $1 > |za - wbi - d|$. We want to give this inequality in terms of θ, ϕ and d. Suppose z and w satisfy (4-4) and (4-5) and consider $za - wbi - d$:

$$\begin{aligned}
za - wbi - d &= \frac{sa^2(\cos\theta e^{i\phi} + d) - cab\sin\theta e^{i\phi}}{sd(d-1)} \\
&\qquad\qquad + \frac{sb^2(\cos\theta e^{i\phi} + d) - cab\sin\theta e^{i\phi}}{sd(d-1)} - d \\
&= \frac{s(a^2+b^2)\cos\theta e^{i\phi}}{sd(d-1)} - \frac{2cab\sin\theta e^{i\phi}}{sd(d-1)} + \frac{s(a^2+b^2)d}{sd(d-1)} - d \\
&= \frac{s(d^2-1)\cos\theta e^{i\phi}}{sd(d-1)} - \frac{2cab\sin\theta e^{i\phi}}{sd(d-1)} + \frac{s(d^2-1)d}{sd(d-1)} - d \\
&= \frac{(d+1)\cos\theta e^{i\phi}}{d} - \frac{\sqrt{c^2(d+1)^2 - d^2}\,\sin\theta e^{i\phi}}{sd} + 1.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|za-wbi-d|^2-1 &= \frac{(d+1)^2\cos^2\theta}{d^2} + \frac{c^2(d+1)^2\sin^2\theta}{s^2d^2} - \frac{\sin^2\theta}{s^2} \\
&\quad - \frac{2(d+1)\sqrt{c^2(d+1)^2-d^2}\,\cos\theta\sin\theta}{sd^2} \\
&\qquad + \frac{2(d+1)\cos\theta\cos\phi}{d} - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\sin\theta\cos\phi}{sd}.
\end{aligned}$$

Arguing as in the proof of Proposition 4.8, we have

$$\begin{aligned}
&|z|^2 + |w|^2 - 1 \\
&= \left|\frac{sa(\cos\theta e^{i\phi} + d) - cb\sin\theta e^{i\phi}}{sd(d-1)}\right|^2 + \left|\frac{sbi(\cos\theta e^{i\phi} + d) - cai\sin\theta e^{i\phi}}{sd(d-1)}\right|^2 - 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{s^2(a^2+b^2)|\cos\theta e^{i\phi}+d|^2}{s^2d^2(d-1)^2} + \frac{c^2(a^2+b^2)\sin^2\theta}{s^2d^2(d-1)^2} - 1 \\
&\qquad\qquad + \frac{isc(2abi)(2\cos\theta\sin\theta+2d\sin\theta\cos\phi)}{s^2d^2(d-1)^2} \\
&= \frac{(d+1)\cos^2\theta}{d^2(d-1)} + \frac{2(d+1)\cos\theta\cos\phi}{d(d-1)} + \frac{d+1}{d-1} + \frac{c^2(d+1)\sin^2\theta}{s^2d^2(d-1)} - 1 \\
&\qquad - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\cos\theta\sin\theta}{sd^2(d-1)} - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\sin\theta\cos\phi}{sd(d-1)} \\
&= \frac{2}{d-1} + \frac{(d+1)\cos^2\theta}{d^2(d-1)} + \frac{c^2(d+1)\sin^2\theta}{s^2d^2(d-1)} - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\cos\theta\sin\theta}{sd^2(d-1)} \\
&\qquad + \frac{2(d+1)\cos\theta\cos\phi}{d(d-1)} - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\sin\theta\cos\phi}{sd(d-1)}.
\end{aligned}
$$

Now we eliminate $\cos\phi$ using the equation for $|za-wbi-d|^2$ derived above:

$$
\begin{aligned}
|z|^2+|w|^2-1 &= \frac{1}{d-1}(|za-wbi-d|^2-1) + \frac{2\cos^2\theta}{d-1} + \frac{2\sin^2\theta}{d-1} \\
&\quad + \frac{(d+1)\cos^2\theta}{d^2(d-1)} + \frac{c^2(d+1)\sin^2\theta}{s^2d^2(d-1)} - \frac{2\sqrt{c^2(d+1)^2-d^2}\,\cos\theta\sin\theta}{sd^2(d-1)} \\
&\quad - \frac{(d+1)^2\cos^2\theta}{d^2(d-1)} - \frac{c^2(d+1)^2\sin^2\theta}{s^2d^2(d-1)} + \frac{\sin^2\theta}{s^2(d-1)} \\
&\quad + \frac{2(d+1)\sqrt{c^2(d+1)^2-d^2}\,\cos\theta\sin\theta}{sd^2(d-1)} \\
&= \frac{1}{d-1}(|za-wbi-d|^2-1) \\
&\quad + \frac{1}{d}\left(\cos\theta + \frac{\sqrt{c^2(d+1)^2-d^2}\,\sin\theta}{s(d-1)}\right)^2 + \frac{(4s^2d-3)\sin^2\theta}{s^2(d-1)^2}.
\end{aligned}
$$

Since the last two terms are nonnegative, all points $p=[z,\, w,\, 1]^t$ with z and w given by (4-4) and (4-5) and that satisfy $|z|^2+|w|^2<1$ must also satisfy $|za-wbi-d|<1$. Geometrically, this means that all points in $\boldsymbol{H}^2_{\mathbb{C}}$ that are on $\mathcal{B}_1\cap\mathcal{B}_{-1}$ are in the halfspace closer to $I_3(o)$ than to o. This proves the result. □

This completes the proof of Theorem 4.3.

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