Pacific Journal of Mathematics

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MICHAEL REITER

Volume 280 No. 2 February 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 280, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.280.455

TOPOLOGICAL ASPECTS OF HOLOMORPHIC MAPPINGS OF HYPERQUADRICS FROM $\mathbb{C}^2$ TO $\mathbb{C}^3$

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In this article we deduce some topological results concerning holomorphic mappings of hyperquadrics under biholomorphic equivalence. We study the class $\mathcal{F}$ of so-called nondegenerate and transversal holomorphic mappings locally sending the sphere in $\mathbb{C}^2$ to a Levi-nondegenerate hyperquadric in $\mathbb{C}^3$, which contains the most interesting mappings. We show that from a topological point of view there is a major difference when the target is the sphere or the hyperquadric with signature (2, 1). In the first case, $\mathcal{F}$ modulo the group of automorphisms is discrete, in contrast to the second case, where this property fails to hold. Furthermore, we study some basic properties such as freeness and properness of the action on $\mathcal{F}$ of automorphisms fixing a given point to obtain a structural result for a particularly interesting subset of $\mathcal{F}$.

1. Introduction and results

We study holomorphic mappings between the sphere $\mathbb{S}^2 \subset \mathbb{C}^2$ and the hyperquadric $\mathbb{S}^3_\varepsilon \subset \mathbb{C}^3$, which for $\varepsilon = \pm 1$ is given by

$$\mathbb{S}^3_\pm := \left\{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid |z_1|^2 + |z_2|^2 \pm |z_3|^2 = 1\right\},$$

so that $\mathbb{S}^3_+ = \mathbb{S}^3$ is the sphere in $\mathbb{C}^3$. Faran [1982] classified holomorphic mappings between spheres in $\mathbb{C}^2$ and $\mathbb{C}^3$ and Lebl [2011] classified mappings sending $\mathbb{S}^2$ to $\mathbb{S}^3_-$. In [Reiter 2015] we give a new CR-geometric approach to reprove Faran's and Lebl's results in a unified manner. Let us introduce the following equivalence relation. For $k = 1, 2$ let $H_k : U_k \to \mathbb{C}^3$ be a holomorphic mapping where U_k is an open and connected neighborhood of $p_k \in \mathbb{S}^2$ and $H_k(U_k \cap \mathbb{S}^2) \subset \mathbb{S}^3_\varepsilon$. We say H_1 is *equivalent* to H_2 if there exist automorphisms ϕ of $\mathbb{S}^2$ and ϕ' of $\mathbb{S}^3_\varepsilon$ such that $H_2 = \phi' \circ H_1 \circ \phi^{-1}$.

The author was supported by the FWF, projects Y377 and I382, and QNRF, project NPRP 7-511-1-098.
MSC2010: 32H02, 32V30, 57S05, 57S25, 58D19.
Keywords: holomorphic mappings, hyperquadrics, nondegeneracy, transversality, automorphism groups, group actions, quotient space, principal fiber bundle.

Theorem 1.1 [Reiter 2015, Theorem 1.3]. *Given $p \in \mathbb{S}^2$, let $U \subset \mathbb{C}^2$ be an open and connected neighborhood of p and $H : U \to \mathbb{C}^3$ a nonconstant holomorphic mapping satisfying $H(U \cap \mathbb{S}^2) \subset \mathbb{S}^3_\varepsilon$. Then H is equivalent to exactly one of the following maps*:

(i) $H_1^\varepsilon(z, w) = (z, w, 0)$,

(ii) $H_2^\varepsilon(z, w) = \Bigl(z^2, \dfrac{(1-\varepsilon+z(1+\varepsilon))w}{\sqrt{2}}, w^2\Bigr)$,

(iii) $H_3^\varepsilon(z, w) = \Bigl(z, \dfrac{(1-\varepsilon+z^2(1+\varepsilon))w}{2z}, \dfrac{(1-\varepsilon+z(1+\varepsilon))w^2}{2z}\Bigr)$,

(iv) $H_4^\varepsilon(z, w) = \dfrac{\bigl(4z^3, (3(1-\varepsilon)+(1+3\varepsilon)w^2)w, \sqrt{3}(1-\varepsilon+2(1+\varepsilon)w+(1-\varepsilon)w^2)z\bigr)}{1+3\varepsilon+3(1-\varepsilon)w^2}$.

Additionally, for $\varepsilon = -1$, we have

(v) $H_5(z, w) = \Bigl(\dfrac{(2+\sqrt{2}z)z}{1+\sqrt{2}z+w}, w, \dfrac{(1+\sqrt{2}z-w)z}{1+\sqrt{2}z+w}\Bigr)$,

(vi) $H_6(z, w) = \dfrac{\bigl((1-w)z, 1+w-w^2, (1+w)z\bigr)}{1-w-w^2}$,

(vii) $H_7(z, w) = \bigl(1, h(z, w), h(z, w)\bigr)$ *for some nonconstant holomorphic function $h : U \to \mathbb{C}$.*

In fact, we study holomorphic mappings between the Heisenberg hypersurface $\mathbb{H}^2 \subset \mathbb{C}^2$ and $\mathbb{H}^3_\varepsilon$, where $\mathbb{H}^3_+ = \mathbb{H}^3$ is the Heisenberg hypersurface in $\mathbb{C}^3$. The hypersurfaces $\mathbb{H}^2$ and $\mathbb{H}^3_\varepsilon$ are biholomorphic to $\mathbb{S}^2$ and $\mathbb{S}^3_\varepsilon$ respectively, except one point, and are given by

$$\mathbb{H}^2 = \bigl\{(z, w) \in \mathbb{C}^2 \mid \operatorname{Im} w = |z|^2\bigr\},$$
$$\mathbb{H}^3_\varepsilon = \bigl\{(z_1', z_2', w') \in \mathbb{C}^3 \mid \operatorname{Im} w' = |z_1'|^2 + \varepsilon |z_2'|^2\bigr\}.$$

We denote by $\mathcal{F}$ the class of germs of 2-nondegenerate transversal mappings sending a small piece of $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$. $\mathcal{F}$ is introduced in more detail in Definition 2.5 below. This is, in some sense, the most natural and interesting class of mappings when studying holomorphic mappings between $\mathbb{H}^2$ to $\mathbb{H}^3_\varepsilon$. From [Reiter 2015] we know that $\mathcal{F}$ consists of mappings belonging to the orbits of the maps listed in (ii)–(vi) of Theorem 1.1 with respect to the equivalence relation of automorphisms introduced above, after composing with an appropriate Cayley transform.

For a germ of a real-analytic CR-submanifold (M, p) of $\mathbb{C}^N$, we write $\operatorname{Aut}_p(M, p)$ for germs of real-analytic CR-diffeomorphisms fixing p, which we refer to as *isotropies* of (M, p). Let us denote by $G_0 := \operatorname{Aut}_0(\mathbb{H}^2, 0) \times \operatorname{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ the direct product of the groups of isotropies of $(\mathbb{H}^2, 0)$ and $(\mathbb{H}^3_\varepsilon, 0)$, which we introduce in Definition 2.3 below in more detail.

After showing that $\pi : \mathcal{F} \to \mathcal{F} / G_0$ is continuous, we obtain the following results.

Theorem 1.2. *The quotient topology τ_Q on $\mathcal{F}/G_0$ coincides with the induced topology τ_J of $\mathcal{F}$, which carries the topology induced by the jet space $J_0^3(\mathbb{H}^2, \mathbb{H}^3_\varepsilon)$.*

In the next theorem we equip $\mathcal{F}$ with the topology τ_J induced by the jet space $J_0^3(\mathbb{H}^2, \mathbb{H}^3_\varepsilon)$, the automorphism groups carry the topology of the 2-jet group (see [Baouendi et al. 1997] for more details), and the quotient space X of $\mathcal{F}$ with respect to the equivalence relation of Theorem 1.1 carries the quotient topology. For more details on the different topologies we use, we refer to Section 2 below.

Theorem 1.3. *The quotient space X of $\mathcal{F}$ with respect to the equivalence relation of automorphisms of $\mathbb{H}^2$ and $\mathbb{H}^3_\varepsilon$ is discrete for $\varepsilon = +1$ and not Hausdorff for $\varepsilon = -1$.*

The above result was not known before and shows one major difference between holomorphic mappings from the sphere in $\mathbb{C}^2$ to the sphere in $\mathbb{C}^3$ and to the hyperquadric with signature (2, 1) in $\mathbb{C}^3$. Furthermore, we study the action of G_0 on $\mathcal{F}$ given by $G_0 \times \mathcal{F} \to \mathcal{F}$, $(\phi, \phi', H) \mapsto \phi' \circ H \circ \phi^{-1}$. The action is called *proper* if the associated map $(\phi, \phi', H) \mapsto (H, \phi' \circ H \circ \phi^{-1})$ is a proper map, such that the following result holds:

Theorem 1.4. *The mapping $N : G_0 \times \mathcal{F} \to \mathcal{F}$ given by $N(\phi, \phi', H) := \phi' \circ H \circ \phi^{-1}$ is a proper action.*

We write $\mathfrak{F} \subset \mathcal{F}$ for the set of maps which have trivial stabilizers given below in Lemma 3.1. Based on the above result we obtain the following theorem concerning the real-analytic structure of $\mathfrak{F}$, where $\Pi : \mathfrak{F} \to \mathfrak{N}$ denotes the normalization map induced by the mapping N, and $\mathfrak{N}$ denotes a particular set of representatives of the quotient $\mathfrak{F}/G_0$ defined in Lemma 3.1 below.

Theorem 1.5. *If $\varepsilon = +1$ then $\Pi : \mathfrak{F} \to \mathfrak{F}/G_0$ is a real-analytic principal fiber bundle with structure group G_0. If $\varepsilon = -1$ then $\mathfrak{F}$ is locally mapped to $G_0 \times \mathfrak{N}$ via local real-analytic diffeomorphisms. In particular, $\mathfrak{F}$ is not a smooth manifold.*

Note that the second part of Theorem 1.5 stands in contrast to the case of maps in $\mathrm{Aut}_p(M, p)$. Assuming some nondegeneracy conditions for certain germs of real-analytic CR-submanifolds (M, p), such as Levi-nondegeneracy, it is known that $\mathrm{Aut}_p(M, p)$ admits a manifold structure (see [Baouendi et al. 1997; 1999; 2004; Kowalski 2005; Kim and Zaitsev 2005; Lamel and Mir 2007; Lamel et al. 2008; Juhlin and Lamel 2013]). To prove Theorem 1.5 we use a real-analytic version of the so-called local slice theorem for free and proper actions. For proper smooth actions of noncompact Lie groups the first proof of the local slice theorem was given in [Palais 1961, 2.2.2 Proposition]. In the real-analytic setting a global slice theorem was proved by [Heinzner et al. 1996, Section VI] and [Illman and Kankaanrinta 2000, Theorem 0.6].

We organize this paper as follows. We introduce the necessary notations, tools and results in Section 2. In the following sections we study properties of the action of the group of isotropies on $\mathcal{F}$ and in Section 5 we investigate the connectedness of $\mathcal{F}$ and discreteness of the quotient space. Using these results, in Section 7 we obtain some structural and topological information of $\mathfrak{F}$ and $\mathfrak{F}/G_0$. Finally, in Section 8 we study different normal forms with respect to isotropies. This article is based on the author's thesis [Reiter 2014] at the University of Vienna. Some computations are carried out with *Mathematica 7.0.1.0* [Wolfram 2008].

2. Preliminaries

Definition 2.1. We fix coordinates $(z, w) = (z_1, \dots, z_n, w) \in \mathbb{C}^{n+1}$. For a germ $h : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}, 0)$ of a holomorphic function $h(z, w) = \sum_{\alpha,\beta} a_{\alpha\beta} z^\alpha w^\beta$, we write $\bar{h}(\bar{z}, \bar{w}) := \overline{h(z, w)} = \sum_{\alpha,\beta} \bar{a}_{\alpha\beta} \bar{z}^\alpha \bar{w}^\beta$ for the complex conjugate of h. Derivatives of h with respect to z or w are denoted by

$$h_{z^\alpha w^\beta}(0) := \frac{\partial^{|\alpha|+|\beta|} h}{\partial z^\alpha \partial w^\beta}(0).$$

For $n \geq 1$ and a germ of a map $H : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^{n'+1}, 0)$ with components $H = \big(f_1, \dots, f_{n'}, g\big)$, we write $H_{z^\alpha w^\beta}(0) = \big(f_{1 z^\alpha w^\beta}(0), \dots, f_{n' z^\alpha w^\beta}(0), g_{z^\alpha w^\beta}(0)\big)$.

Classes of maps, automorphisms and equivalence relations.

Definition 2.2. We write $\mathcal{H}(p; p') := \{H : (\mathbb{C}^N, p) \to (\mathbb{C}^{N'}, p') \mid H \text{ holomorphic}\}$ for the *set of germs of holomorphic mappings from* $(\mathbb{C}^N, p)$ *to* $(\mathbb{C}^{N'}, p')$. For germs of real-analytic hypersurfaces $(M, p) \subset \mathbb{C}^N$ and $(M', p') \subset \mathbb{C}^{N'}$, we denote by

$$\mathcal{H}(M, p; M', p') := \{H \in \mathcal{H}(p; p') \mid H(M \cap U) \subset M', \text{ for } U \text{ a neighborhood of } p\},$$

the *set of germs of holomorphic mappings from* (M, p) *to* (M', p').

Definition 2.3. (i) We denote the collection of germs of locally real-analytic CR-diffeomorphisms of (M, p) by

$$\operatorname{Aut}(M, p) := \big\{H : (\mathbb{C}^N, p) \to \mathbb{C}^N \mid H \text{ holomorphic}, H(M) \subset M, \det(H'(p)) \neq 0\big\}$$

and the *group of isotropies of* (M, p) *fixing* p by

$$\operatorname{Aut}_p(M, p) := \{H \in \operatorname{Aut}(M, p) \mid H(p) = p\}.$$

We write $G_0 := \operatorname{Aut}_0(\mathbb{H}^2, 0) \times \operatorname{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ and refer to elements of G_0 as *isotropies of* $(\mathbb{H}^2, 0)$ *and* $(\mathbb{H}^3_\varepsilon, 0)$.

(ii) We write $\mathbb{R}^+ := \{x \in \mathbb{R} \mid x > 0\}$ for the positive real numbers, denote the unit circle in $\mathbb{C}$ by $\mathbb{S}^1 := \{e^{it} \mid 0 \leq t < 2\pi\}$ and set $\Gamma := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathbb{C}$. For an element $\sigma_\gamma \in \operatorname{Aut}_0(\mathbb{H}^2, 0)$ we denote $\gamma = (\lambda, r, u, c) \in \Gamma$ and write

$$\sigma_\gamma(z, w) := \frac{(\lambda u(z + cw), \lambda^2 w)}{1 - 2i\bar{c}z + (r - i|c|^2)w}. \tag{2-1}$$

(iii) We define, for $\theta = \pm 1$ if $\varepsilon = -1$ and $\theta = +1$ if $\varepsilon = +1$,

$$\mathcal{S}^2_{\varepsilon,\theta} := \left\{ a' = (a'_1, a'_2) \in \mathbb{C}^2 \mid |a'_1|^2 + \varepsilon |a'_2|^2 = \theta \right\}, \tag{2-2}$$

and let

$$U' := \begin{pmatrix} u'a'_1 & -\varepsilon u' a'_2 \\ \bar{a}'_2 & \bar{a}'_1 \end{pmatrix}, \quad u' \in \mathbb{S}^1,\ a' = (a'_1, a'_2) \in \mathcal{S}^2_{\varepsilon,\theta}. \tag{2-3}$$

We set $\Gamma' := \mathbb{R}^+ \times \mathbb{R} \times \mathbb{S}^1 \times \mathcal{S}^2_{\varepsilon,\theta} \times \mathbb{C}^2$ to denote elements $\sigma'_{\gamma'} \in \mathrm{Aut}_0(\mathbb{H}^3_\varepsilon, 0)$ via $\gamma' = (\lambda', r', u', a', c') \in \Gamma'$, where $c' = (c'_1, c'_2)$:

$$\sigma'_{\gamma'}(z', w') := \frac{(\lambda' U'\,{}^t(z' + c'w'), \theta \lambda'^2 w')}{1 - 2\mathrm{i}(\bar{c}'_1 z'_1 + \varepsilon \bar{c}'_2 z'_2) + \big(r' - \mathrm{i}(|c'_1|^2 + \varepsilon |c'_2|^2)\big) w'}. \tag{2-4}$$

(iv) We call elements of $\Gamma \times \Gamma'$ *standard parameters*. If the standard parameters $(\gamma, \gamma') \in \Gamma \times \Gamma'$ are chosen such that $(\sigma_\gamma, \sigma'_{\gamma'}) = (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$, we say the standard parameters are *trivial*.

Definition 2.4. For $G, H \in \mathcal{H}(M, p; M', p')$, we define an equivalence relation

$$G \sim H :\Leftrightarrow \exists (\phi, \phi') \in \mathrm{Aut}_p(M, p) \times \mathrm{Aut}_{p'}(M', p') : G = \phi' \circ H \circ \phi^{-1}.$$

The equivalence classes in $\mathcal{H}(M, p; M', p')/\sim$ are denoted by

$$[F] := \{ G \in \mathcal{H}(M, p; M', p') \mid G \sim F \}.$$

In the case where $(p, p') = (0, 0)$ and $(M, M') = (\mathbb{H}^2, \mathbb{H}^3_\varepsilon)$, we call the above relation *isotropic equivalence* and write $O_0(H)$ for the orbit of a map H, called the *isotropic orbit of H*.

The class $\mathcal{F}$, the normal form $\mathcal{N}$ and its classification. In [Reiter 2015] we introduced the following class of mappings, which are 2-nondegenerate and transversal. These mappings represent the immersive maps, which are not equivalent to the linear embedding (see [Reiter 2015, Proposition 2.16]).

Definition 2.5. For a neighborhood $U \subset \mathbb{C}^2$ of 0, define $\mathcal{F}(U)$ to be the set of holomorphic mappings $H = (f_1, f_2, g)$, with $H(U \cap \mathbb{H}^2) \subset \mathbb{H}^3_\varepsilon$, which satisfy $H(0) = 0$, $f_{1z}(0) f_{2z^2}(0) - f_{2z}(0) f_{1z^2}(0) \neq 0$ and $g_w(0) > 0$. Define $\mathcal{F}$ to be the set of germs H, such that $H \in \mathcal{F}(U)$ for some neighborhood $U \subset \mathbb{C}^2$ of 0.

Proposition 2.6 [Reiter 2015, Proposition 3.1]. *Let $H \in \mathcal{F}$. Then there exist isotropies $(\sigma, \sigma') \in G_0$ such that $\widehat{H} := \sigma' \circ H \circ \sigma^{-1}$ satisfies $\widehat{H}(0) = 0$ and the following conditions*:

(i) $\widehat{H}_z(0) = (1, 0, 0)$, (iii) $\hat{f}_{2z^2}(0) = 2$, (vi) $\mathrm{Re}\big(\hat{g}_{w^2}(0)\big) = 0$,

(ii) $\widehat{H}_w(0) = (0, 0, 1)$, (iv) $\hat{f}_{2zw}(0) = 0$, (vii) $\mathrm{Re}\big(\hat{f}_{2z^2w}(0)\big) = 0$.

(v) $\hat{f}_{1w^2}(0) = |\hat{f}_{1w^2}(0)| \geq 0$, *A holomorphic mapping of $\mathcal{F}$ satisfying the above conditions is called a normalized mapping. The set of normalized mappings is denoted by $\mathcal{N}$.*

Remark 2.7. A mapping $H \in \mathcal{N}$ necessarily satisfies the following conditions (see [Reiter 2015, Remark 3.4]):

(i) $H(0) = (0, 0, 0)$,

(ii) $H_z(0) = (1, 0, 0)$,

(iii) $H_w(0) = (0, 0, 1)$,

(iv) $H_{z^2}(0) = (0, 2, 0)$,

(v) $H_{zw}(0) = \big(\frac{\mathrm{i}\varepsilon}{2}, 0, 0\big)$,

(vi) $H_{w^2}(0) = (|f_{1w^2}(0)|, f_{2w^2}(0), 0)$,

(vii) $H_{z^2w}(0) = \big(4\mathrm{i}|f_{1w^2}(0)|, \mathrm{i}\,\mathrm{Im}(f_{2z^2w}(0)), 0\big)$.

We classify all mappings belonging to $\mathcal{N} \simeq \mathcal{F}/G_0$ in [Reiter 2015].

Theorem 2.8 [Reiter 2015, Theorem 4.1]. *The set $\mathcal{N}$ consists of the following mappings, where $s \geq 0$:*

$$
\begin{aligned}
G_1^\varepsilon(z, w) &:= \big(2z(2 + \mathrm{i}\varepsilon w), 4z^2, 4w\big)/(4 - w^2),\\
G_{2,s}^\varepsilon(z, w) &:= \big(4z - 4\varepsilon s z^2 + \mathrm{i}(\varepsilon - s^2)zw + sw^2,\\
&\qquad 4z^2 + s^2w^2, w(4 - 4\varepsilon s z - \mathrm{i}(\varepsilon + s^2)w)\big)\\
&\qquad /\big(4 - 4\varepsilon s z - \mathrm{i}(\varepsilon + s^2)w - 2\mathrm{i}szw - \varepsilon s^2 w^2\big),\\
G_{3,s}^\varepsilon(z, w) &:= \big(256\varepsilon z + 96\mathrm{i}zw + 64\varepsilon s w^2 + 64z^3 + 64\mathrm{i}\varepsilon s z^2 w\\
&\qquad - 3(3\varepsilon - 16s^2)zw^2 + 4\mathrm{i}sw^3,\\
&\qquad 256\varepsilon z^2 - 16w^2 + 256sz^3 + 16\mathrm{i}z^2w - 16\varepsilon szw^2 - \mathrm{i}\varepsilon w^3,\\
&\qquad w(256\varepsilon - 32\mathrm{i}w + 64z^2 - 64\mathrm{i}\varepsilon szw - (\varepsilon + 16s^2)w^2)\big)\\
&\qquad /\big(256\varepsilon - 32\mathrm{i}w + 64z^2 - 192\mathrm{i}\varepsilon szw - (17\varepsilon + 144s^2)w^2\\
&\qquad + 32\mathrm{i}\varepsilon z^2 w + 24szw^2 + \mathrm{i}w^3\big).
\end{aligned}
$$

Each mapping in $\mathcal{N}$ is not isotropically equivalent to any different mapping in $\mathcal{N}$.

For $\varepsilon = \pm 1$, Figure 1 depicts $\mathcal{N}$ in the parameter space according to Theorem 2.8 (see [Reiter 2015, §4] for more details).

Associated topologies. We deal with the following topologies (see, e.g., [Baouendi et al. 1997]).

Definition 2.9. For $K \subset \mathbb{C}^N$ a compact neighborhood of $p \in \mathbb{C}^N$, we denote by $\mathcal{H}_K(p; p')$ the space of holomorphic mappings, defined in a neighborhood of K, which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$ equipped with the uniform norm on K. We equip $\mathcal{H}(p; p')$ with the inductive limit topology with respect to $\mathcal{H}_K(p; p')$, where K is some compact neighborhood of p in $\mathbb{C}^N$. Then for $H, H_n \in \mathcal{H}(p; p')$, we say that H_n converges to H if there exists $K \subset \mathbb{C}^N$ a compact neighborhood of p such that

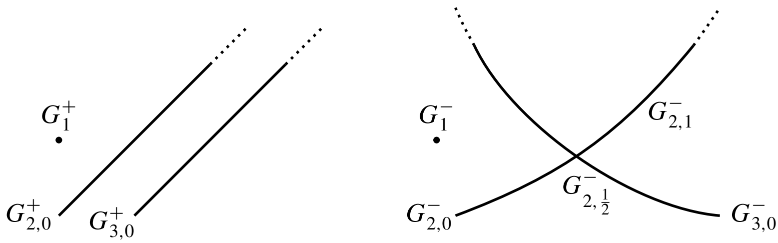

Figure 1. $\mathcal{N}$ for $\varepsilon = \pm 1$ in the parameter space.

each H_n is holomorphic in a neighborhood of K and H_n converges uniformly to H on K. For $\mathcal{H}(M, p; M', p') \subset \mathcal{H}(p; p')$, we consider the induced topology of $\mathcal{H}(p; p')$ denoted by τ_C.

Definition 2.10. Let $Z \in \mathbb{C}^N$ be coordinates in $\mathbb{C}^N$, $H : \mathbb{C}^N \to \mathbb{C}^{N'}$ a holomorphic mapping defined at $p \in \mathbb{C}^N$ and $\alpha \in \mathbb{N}^N$. We denote by $j_p^k H$ the *k-jet of H at p* defined as

$$j_p^k H := \Big(\frac{\partial^{|\alpha|} H}{\partial Z^\alpha}(p) : |\alpha| \le k\Big),$$

and by $J_{p,p'}^k$ the collection of all k-jets at p of germs of mappings from $(\mathbb{C}^N, p)$ to $(\mathbb{C}^{N'}, p')$. We set $J_p^k := J_{p,p}^k$ and denote the topology for $J_{p,p'}^k$ by τ_J, which we refer to as the *topology of the jet space*. Let $(M, p) \subset (\mathbb{C}^N, p)$ and $(M', p') \subset (\mathbb{C}^{N'}, p')$ be germs of submanifolds. For $k \in \mathbb{N}$ we denote by $J_q^k(M, p; M', p')$ the *space of k-jets of* $\mathcal{H}(M, p; M', p')$ *at* q. We also define $J_q^k(M, p) := J_q^k(M, p; M, p)$ and $J_0^k(M; M') := J_0^k(M, 0; M', 0)$. We denote by $G_p^k(M, p) \subset J_p^k(M, p)$ the *space of k-jets of* $\mathrm{Aut}_p(M, p)$ *at* p.

Note that $J_p^k(M, p; M', p') \subset J_{p,p'}^k$. We identify $J_{p,p'}^k$ with the space of germs of holomorphic polynomial mappings, up to degree k, from $\mathbb{C}^N$ to $\mathbb{C}^{N'}$, which map $p \in \mathbb{C}^N$ to $p' \in \mathbb{C}^{N'}$. Thus $J_{p,p'}^k$ can be identified with some $\mathbb{C}^K$, where $K := N'\binom{N+k}{N}$, such that the topology τ_J for $J_{p,p'}^k$ is induced by the natural topology of $\mathbb{C}^K$.

Definition 2.11. We say $\mathcal{K} \subset \mathcal{H}(M, p; M', p')$ admits a *jet parametrization for* $\mathcal{K}$ *of order* k if there exists a mapping $\Psi : \mathbb{C}^N \times \mathbb{C}^K \supset U \to \mathbb{C}^{N'}$, with $K = N'\binom{N+k}{N}$, from above and U an open neighborhood of $\{p\} \times J_p^k(M, p; M', p')$, which is holomorphic in the first N variables and real-analytic in the remaining K variables, such that $F(Z) = \Psi(Z, j_p^k F)$ for all $F \in \mathcal{K}$.

If $\mathcal{K} \subset \mathcal{H}(M, p; M', p')$ admits a jet parametrization of some order k, then $\tau_C = \tau_J$, which follows from the real-analyticity in the last K variables. We need the following jet determination result which is an immediate consequence of the normalization and classification of maps in $\mathcal{F}$.

Corollary 2.12 [Reiter 2015, Corollary 4.8]. *Let $U \subset \mathbb{C}^2$ be a neighborhood of* 0 *and $H : U \to \mathbb{C}^3$ a holomorphic mapping. We denote the components of H by $H = (f, g) = (f_1, f_2, g)$ and write $j_0(H) := \{j_0^2(H), f_{z^2w}(0)\}$. If for $H_1, H_2 \in \mathcal{F}$ the coefficients belonging to $j_0(H_1)$ and $j_0(H_2)$ coincide, then we have $H_1 \equiv H_2$.*

Remark 2.13. Based on [Lamel 2001, Proposition 25, Corollary 26–27] we obtain a jet parametrization of order 4 for $\mathcal{K} = \mathcal{F}$ in [Reiter 2015, Lemma 4.3], and by Corollary 2.12 we have that $K = K_0 := 15$. Using Theorem 2.8 and the notation from Corollary 2.12, we identify $\mathcal{F}$ with a subset $\mathfrak{J} \subset \mathbb{C}^{K_0}$ given by $\mathfrak{J} := \{j_0(H) \mid H \in \mathcal{F}\}$, and the topology we use in the sequel for $\mathcal{F}$ is τ_J.

Definition 2.14. Let X be a topological space, Y a set and $q : X \to Y$ a surjective mapping. We call the topology on Y induced by q the *quotient topology τ_Q on Y*, where a set $U \subset Y$ is open in Y if $q^{-1}(U)$ is open in X.

3. The isotropic stabilizer and freeness of the group action on $\mathfrak{F}$

Lemma 3.1. *Set $\mathfrak{N} := \mathcal{N} \setminus \{G_1^\varepsilon, G_{2,0}^\varepsilon, G_{3,0}^\varepsilon\}$ and $\mathfrak{F} := \bigcup_{H\in\mathfrak{N}} O_0(H)$. The isotropic stabilizer* $\mathrm{stab}_0(H) := \{(\phi, \phi') \in G_0 \mid \phi' \circ H \circ \phi^{-1} = H\}$ *of H is trivial for $H \in \mathfrak{N}$. Furthermore, we have that* $\mathrm{stab}_0(G_1^\varepsilon) = \mathrm{stab}_0(G_{2,0}^\varepsilon)$ *is homeomorphic to $\mathbb{S}^1$ and* $\mathrm{stab}_0(G_{3,0}^\varepsilon)$ *is homeomorphic to $\mathbb{Z}_2$.*

Proof. Let $H = (f, g) = (f_1, f_2, g) \in \mathcal{N}$ satisfy the conditions in Remark 2.7. We write $s := 2|f_{1w^2}(0)| \geq 0$, $x := f_{2w^2}(0) \in \mathbb{C}$ and $y := \mathrm{Im}(f_{2z^2w}(0)) \in \mathbb{R}$. By Corollary 2.12 we only need to consider coefficients in $j_0(H)$. We let $(\sigma, \sigma') \in G_0$ with the notation from (2-1), (2-3) and (2-4), and consider the equation

$$\sigma' \circ H \circ \sigma^{-1} = H, \tag{3-1}$$

where we parametrize σ^{-1} as in (2-1). The coefficients of order 1, which are $f_z(0)$ and $H_w(0)$, are given by

$$U'^{\,t}(u\lambda\lambda', 0) = (1, 0) \quad \text{and} \quad (U'^{\,t}(uc + \lambda c_1', \lambda c_2'), \theta\lambda\lambda') = (0, 0, 1).$$

These equations imply $\theta = +1$, $\lambda' = 1/\lambda$, $a_2' = c_2' = 0$, $a_1' = 1/(uu')$ and $c_1' = -uc/\lambda$. Assuming these standard parameters we consider the coefficients of order 2, which are $f_{z^2}(0)$, $H_{zw}(0)$ and $H_{w^2}(0)$, given by

$$\big(0, 2u'u^3\lambda\big) = (0, 2), \tag{3-2}$$

$$\big(-r - \lambda^2 r' + i\varepsilon\lambda^2/2, 2u'u^3\lambda c, 0\big) = (i\varepsilon/2, 0, 0), \tag{3-3}$$

$$\big(\lambda^2(\lambda s + i\varepsilon uc)/u, uu'\lambda(\lambda^2 x + 2u^2c^2), -2(r + \lambda^2 r')\big) = (s, x, 0). \tag{3-4}$$

The second component of (3-3) implies $c = 0$. If we assume this value for c we obtain for the third order terms $f_{z^2w}(0)$ the equation

$$\big(2\mathrm{i}u\lambda^3 s, u'u^3\lambda(-4r - 2\lambda^2 r' + \mathrm{i}\lambda^2 y)\big) = (4\mathrm{i}s, \mathrm{i}y). \tag{3-5}$$

The second component of (3-2) shows $\lambda = 1$. Furthermore we obtain from the third component of (3-4) that $r' = -r$ and since from the second component of (3-2) we get $u'u^3 = 1$, which uniquely determines u', we obtain from the second component of (3-5) that $r = 0$. The remaining equation from the first component of (3-4), which comes from the coefficient $f_{1w^2}(0)$, is $s/u = s$. If $s > 0$ we obtain that $u = 1$ and hence all standard parameters are trivial, which proves the first claim of the lemma.

If $s=0$, then $H \in \{G_1^\varepsilon, G_{2,0}^\varepsilon, G_{3,0}^\varepsilon\}$, since these maps are precisely those satisfying $f_{1w^2}(0) = 0$ in the list of mappings from Theorem 2.8. It is easy to check that the isotropic stabilizers of the maps G_1^ε and $G_{2,0}^\varepsilon$ are generated by the isotropies $(\sigma(z, w), \sigma'(z_1', z_2', w')) = (uz, w, z_1'/u, z_2'/u^2, w')$ with $|u| = 1$. If we consider $G_{3,0}^\varepsilon$ in (3-1), then we obtain that $(\sigma(z, w), \sigma'(z_1', z_2', w')) = (\delta z, w, \delta z_1', z_2', w')$, where $\delta = \pm 1$, are the only elements of $\mathrm{stab}_0(G_{3,0}^\varepsilon)$, which proves the last claim of the lemma. □

Proposition 3.2. *The map $N : G_0 \times \mathfrak{F} \to \mathfrak{F}$ given by $N(\phi, \phi', H) := \phi' \circ H \circ \phi^{-1}$ is a free action.*

Proof. Lemma 3.1 shows that N restricted to $\mathfrak{N}$ is a free action. We assume the general case $H \in \mathfrak{F}$ and consider the equation $\phi' \circ H \circ \phi^{-1} = H$ for $(\phi, \phi') \in G_0$. We write $H = \widehat{\phi'} \circ \widehat{H} \circ \widehat{\phi}^{-1}$, where $\widehat{H} \in \mathfrak{N}$ and $(\widehat{\phi}, \widehat{\phi'}) \in G_0$ are unique according to Lemma 3.1. After setting $(\psi, \psi') = (\widehat{\phi}^{-1} \circ \phi \circ \widehat{\phi}, \widehat{\phi'}^{-1} \circ \phi' \circ \widehat{\phi'})$, we rewrite $\phi' \circ H \circ \phi^{-1} = H$ as $\psi' \circ \widehat{H} \circ \psi^{-1} = \widehat{H}$. Since each map in $\mathfrak{N}$ admits a trivial stabilizer, we obtain that $(\psi, \psi') = (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$ and the freeness of the action. □

4. Continuity of the normalization map

Remark 4.1. For $F : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ a germ of a holomorphic mapping, for which we assume that $F \in \mathcal{F}$ and the jet $j_0(F) \subset j_0^3 F$ is of the form as in Remark 2.7, we write $F = (f^1, f^2, f^3)$ for the components and denote derivatives of F at 0 by $f_{\ell m}^k := f_{z^\ell w^m}^k(0)$. We set $\Delta(F) := f_{10}^1 f_{20}^2 - f_{20}^1 f_{10}^2$.

Lemma 4.2. *For $n \in \mathbb{N}$, we let $(\phi_n, \phi_n') \in G_0$ and $H_n, H \in \mathcal{F}$ be such that $\phi_n' \circ H_n \circ \phi_n^{-1} \to H$ as $n \to \infty$, where $\mathcal{F}$ is equipped with the topology τ_J. If we assume $H_n, H \in \mathcal{N}$, then $H_n \to H$, and if we assume $H_n, H \in \mathfrak{N}$, then $(\phi_n, \phi_n') \to (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$ as $n \to \infty$.*

Proof. We assume that $H_n = (h_n^1, h_n^2, h_n^3)$ and $H = (h^1, h^2, h^3)$ are given as in Remark 4.1, where the coefficients of H_n depend on $n \in \mathbb{N}$. Let $s_n := 2|h_{n02}^1| \geq 0$,

$x_n := h^2_{n02} \in \mathbb{C}$ and $y_n := \mathrm{Im}\big(h^2_{n21}\big)$. To each $(\phi_n, \phi'_n) \in G_0$ we associate the standard parameters $(\gamma_n, \gamma'_n) \in \Gamma \times \Gamma'$, where we use the notation for the parametrization of G_0 from (2-1) and (2-4). According to Theorem 2.8, H_n depends on $s_n \geq 0$. Let us denote $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+_0$ and write $\xi_n = (\gamma_n, \gamma'_n, s_n) \in \Xi$. We define $\Psi_n := \phi'_n \circ H_n \circ \phi_n^{-1}$, which depends on $\xi_n \in \Xi$. For components of Ψ_n, we write $\Psi_n = (\psi^1_n, \psi^2_n, \psi^3_n)$ and $\psi_n = (\psi^1_n, \psi^2_n)$. Limits are always considered when $n \to \infty$.

We start with the first order terms of Ψ_n. We let U'_n be the 2×2-matrix from (2-3) with entries u'_n, a'_{1n} and a'_{2n} instead of u', a'_1 and a'_2, so that we have

$$\psi_{nz}(0) = \lambda_n \lambda'_n U'^{\,t}_n (u_n, 0), \tag{4-1}$$

$$\Psi_{nw}(0) = \lambda_n \lambda'_n \big(U'^{\,t}_n (u_n c_n + \lambda_n c'_{1n}, \lambda_n c'_{2n}), \theta_n \lambda_n \lambda'_n\big). \tag{4-2}$$

Since $\psi^3_{nw}(0) \to 1$ we obtain $\theta_n = +1$, $\lambda_n \lambda'_n \to 1$. This implies that $u_n u'_n a'_{1n} \to 1$ and $a'_{2n} \to 0$, considering $\psi_{nz}(0) \to (1, 0)$ in (4-1). Because $a'_n = (a'_{1n}, a'_{2n}) \in \mathcal{S}^2_{\varepsilon,\theta}$ from (2-2), we have $|a'_{1n}| \to 1$. If we consider the first two components in (4-2), we obtain from $\psi_{nw}(0) \to (0, 0)$ and $(|a'_{1n}|, |a'_{2n}|) \to (1, 0)$ that $u_n c_n + \lambda_n c'_{1n} \to 0$ and $c'_{2n} \to 0$.

Next we consider the second order terms of Ψ_n to obtain

$$\psi_{nz^2}(0) = 2u_n \lambda_n \lambda'_n U'^{\,t}_n \big(2\mathrm{i}(\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}), u_n \lambda_n\big), \tag{4-3}$$

where the left-hand side of (4-3), $\psi_{nz^2}(0)$, must converge to $(0, 2)$. After applying U'^{-1}_n we rewrite the second component of (4-3) as

$$2u^2_n \lambda^2_n \lambda'_n = a'_{1n}\big(-\bar{a}'_{2n} \psi^1_{nz^2}(0)/(u'_n a'_{1n}) + \psi^2_{nz^2}(0)\big). \tag{4-4}$$

Since $(|a'_{1n}|, |a'_{2n}|) \to (1, 0)$, the absolute value of the right-hand side of (4-4) converges to 2. Taking the absolute value of the left-hand side of (4-4) implies $\lambda_n \to 1$, which together with $\lambda_n \lambda'_n \to 1$ shows $\lambda'_n \to 1$. Next we consider

$$\psi_{nzw}(0) = \tfrac{\mathrm{i}}{2} \lambda_n \lambda'_n U'^{\,t}_n \big(T_1(\gamma_n, \gamma'_n), 4\lambda_n (c'_{2n}(\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}) - \mathrm{i} u^2_n c_n)\big), \tag{4-5}$$

where the real-analytic function $T_1 : \Gamma \times \Gamma' \to \mathbb{C}$ does not depend on $a'_n \in \mathcal{S}^2_{\varepsilon,\theta}$ and u'_n. The left-hand side of (4-5) has to converge to $(\mathrm{i}\varepsilon/2, 0)$ and we rewrite the second component of (4-5) as

$$\begin{aligned} 4\lambda_n \big(c'_{2n}(\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}) - \mathrm{i} u^2_n c_n\big) \\ = -2\mathrm{i}\big(-\bar{a}'_{2n} \psi^1_{nzw}(0) + u'_n a'_{1n} \psi^2_{nzw}(0)\big)/(\lambda_n \lambda'_n u'_n). \end{aligned} \tag{4-6}$$

Taking the limit, we know, since $(|a'_{1n}|, |a'_{2n}|) \to (1, 0)$ and $(\lambda_n, \lambda'_n) \to (1, 1)$, that the right-hand side of (4-6) converges to 0 and if we also use $u_n c_n + \lambda_n c'_{1n} \to 0$ we obtain that $c_n \to 0$, such that $c'_{1n} \to 0$.

Next we compute

$$\begin{aligned}(4\text{-}7)\quad &\psi^3_{nw^2}(0)\\ &= 2\lambda_n^2\lambda_n'^2\big(-(r_n+\lambda_n^2 r_n') + \mathrm{i}\big(|c_n|^2+\varepsilon\lambda_n^2|c_{2n}'|^2+\lambda_n\bar{c}_{1n}'(2u_nc_n+\lambda_nc_{1n}')\big)\big).\end{aligned}$$

We take all of the previously obtained limits as $n\to\infty$ of the sequences $c_n'=(c_{1n}',c_{2n}')\in\mathbb{C}^2$, c_n and λ_n, λ_n'. Then since $\psi^3_{nw^2}(0)\to 0$, we have that $r_n+\lambda_n^2r_n'\to 0$. Next we compute

$$(4\text{-}8)\qquad \psi_{nw^2}(0) = \lambda_n\lambda_n' U_n'{}^{\,t}\big(\lambda_n^3 s_n + T_2(\gamma_n,\gamma_n'),\, \lambda_n^3x_n+T_3(\gamma_n,\gamma_n')\big),$$

where $T_2, T_3 : \Gamma\times\Gamma'\to\mathbb{C}$ are real-analytic functions and T_2 is given by

$$\begin{aligned}T_2(\gamma_n,\gamma_n') = 2(u_nc_n+c_{1n}'\lambda_n)\big(\mathrm{i}|c_n|^2-r_n-\lambda_n^2r_n'\big)&\\ +2\mathrm{i}\lambda_n\bar{c}_{1n}'(u_nc_n+\lambda_nc_{1n}')(2u_nc_n+\lambda_nc_{1n}')&\\ +\mathrm{i}\varepsilon\lambda_n^2\big(u_nc_n(1+2|c_{2n}'|^2)+2\lambda_nc_{1n}'|c_{2n}'|\big)&,\end{aligned}$$

so that $T_2(\gamma_n,\gamma_n')\to 0$. Then the first component of (4-8) becomes

$$(4\text{-}9)\qquad \lambda_n^3s_n+T_2(\gamma_n,\gamma_n') = \big(\bar{a}_{1n}'\psi^1_{nw^2}(0)+\varepsilon u_n'a_{2n}'\psi^2_{nw^2}(0)\big)/(\lambda_n\lambda_n'u_n').$$

Since $\big(\psi^1_{nw^2}(0),\psi^2_{nw^2}(0)\big)\to\big(2|h^1_{02}|, h^2_{02}\big)\in\mathbb{R}^+\times\mathbb{C}$, we obtain $s_n\to 2|h^1_{02}|$, and if $|h^1_{02}|\neq 0$ we have $\bar{a}_{1n}'/u_n'\to 1$. Then $u_nu_n'a_{1n}'\to 1$ implies that $u_n\to 1$ and further inspection of (4-4) gives $u_n^2/a_{1n}'\to 1$, which shows $a_{1n}'\to 1$ and $u_n'\to 1$. Note that if $|h^1_{02}|=0$ we have that $a_{1n}', u_n, u_n'\in\mathbb{S}^1$ for all $n\in\mathbb{N}$. Observe that the following considerations are independent of the value of h^1_{02}:

$$\begin{aligned}(4\text{-}10)\quad &\psi_{nz^2w}(0)\\ &=\lambda_n\lambda_n'U_n'\begin{pmatrix} -4\mathrm{i}u_n^2\lambda_n^3s_n+T_4(\gamma_n,\gamma_n')\\ -2\varepsilon u_n^2\lambda_n(2r_n+\lambda_n^2r_n')+\mathrm{i}\varepsilon u_n^2\lambda_n^3y_n+6u_n^3\lambda_n^2c_ns_n+T_5(\gamma_n,\gamma_n')\end{pmatrix},\end{aligned}$$

where $T_4, T_5:\Gamma\times\Gamma'\to\mathbb{C}$ are real-analytic functions and T_5 is given by

$$\begin{aligned}T_5(\gamma_n,\gamma_n') = 2\mathrm{i}\varepsilon\lambda_n\big(4\mathrm{i}\bar{c}_nc_{2n}'(\bar{c}_n+2u_n\lambda_n\bar{c}_{1n}')+2c_nu_n^2(5\bar{c}_n+3u_n\lambda_n\bar{c}_{1n}')&\\ +u_n^2\lambda_n^2\big(|c_{1n}'|^2+3\varepsilon|c_{2n}'|^2+4\mathrm{i}\bar{c}_{1n}'c_{2n}'\big)\big)&,\end{aligned}$$

hence $T_5(\gamma_n,\gamma_n')\to 0$. Since $\big(\psi^1_{nz^2w}(0),\psi^2_{nz^2w}(0)\big)\to\big(2\mathrm{i}|h^1_{02}|,\mathrm{i}h^2_{21}\big)\in\mathrm{i}\mathbb{R}\times\mathrm{i}\mathbb{R}$, considering the real part of the second component of (4-10) we obtain $2r_n+r_n'\to 0$, which together with $r_n+\lambda_n^2r_n'\to 0$ shows $(r_n,r_n')\to(0,0)$. To sum up, we obtain that $H_n\to H$, and moreover, if $|h^1_{02}|\neq 0$, we conclude $(\phi_n,\phi_n')\to(\mathrm{id}_{\mathbb{C}^2},\mathrm{id}_{\mathbb{C}^3})$, which completes the proof. □

Proposition 4.3. *The map $\pi:\mathcal{F}\to\mathcal{N}$ given by $\pi(H):=\phi'\circ H\circ\phi^{-1}$ for $(\phi,\phi')\in G_0$, according to Proposition 2.6, is continuous with respect to τ_J.*

Proof. Let $(H_n)_{n\in\mathbb{N}}$ be a sequence of mappings in $\mathcal{F}$ and $H\in\mathcal{F}$, such that $H_n\to H$. Assuming without loss of generality that $H\in\mathcal{N}$, we need to show $\widetilde{H}_n:=\pi(H_n)\to H$. We have $\widetilde{H}_n=\phi'_n\circ H_n\circ\phi_n^{-1}\in\mathcal{N}$, where $(\phi_n,\phi'_n)\in G_0$ are the isotropies according to Proposition 2.6. Since $H_n={\phi'}_n^{-1}\circ\widetilde{H}_n\circ\phi_n\to H$, we conclude by Lemma 4.2 that $\widetilde{H}_n\to H$. □

Using Proposition 4.3 we are able to prove Theorem 1.2.

Proof of Theorem 1.2. We show that $\pi:\mathcal{F}\to\mathcal{N}$ is a surjective, continuous and closed mapping with respect to τ_J. Surjectivity is clear from Proposition 2.6 and Theorem 2.8 and continuity we have shown in Proposition 4.3. It remains to prove that π is closed with respect to τ_J. Let $C\subset\mathcal{F}$ be a closed subset. We need to show that $\pi(C)\subset\mathcal{N}$ is a closed subset. To prove this statement we let $H_n\in\pi(C)$ for $n\in\mathbb{N}$, forming a sequence of mappings in $\mathcal{N}$ such that $H_n\to H_0$, where $H_0\in\mathcal{N}$. To show that $\pi(C)$ is closed we need to conclude that $H_0\in\pi(C)$. By Theorem 2.8 we can write $H_n=G^{\varepsilon}_{k_n,s_n}$ and $H_0=G^{\varepsilon}_{k_0,s_0}$ for $k_n,k_0\in\{2,3\}$. Note that since $H_n\to H_0$ in $\mathcal{N}$ we have $s_n\to s_0$. This implies that for any convergent sequence $G_n\in\pi^{-1}(H_n)$ the map $G_0:=\lim_{n\to\infty}G_n$ belongs to $\pi^{-1}(H_0)$. Since C is closed, an arbitrary convergent sequence $F_n\in\pi^{-1}(H_n)\cap C$ with $F_n\to F_0$ thus satisfies $F_0\in\pi^{-1}(H_0)\cap C$, which implies $H_0=\pi(F_0)\in\pi(C)$. □

5. A topological property of the quotient space of $\mathcal{F}$

Lemma 5.1. *The class $\mathcal{F}$ consists of $\frac{5+\varepsilon}{2}$ connected components.*

Proof. According to Proposition 2.6 and Proposition 4.3, we denote by $\pi:\mathcal{F}\to\mathcal{N}$ the normalization map, which is continuous with respect to τ_J. By Theorem 1.2, we equip $\mathcal{F}$ and $\mathcal{N}$ with τ_J. For $k\in\{2,3\}$, we set $C_k:=\{G^{\varepsilon}_{k,s}\mid s\geq 0\}$ and $\mathcal{N}^*:=C_2\cup C_3$. The space of standard parameters $\Gamma\times\Gamma'$ is path-connected, since as defined in Definition 2.5 for maps $H=(f_1,f_2,g)\in\mathcal{F}$, we assumed $g_w(0)>0$, which implies that for isotropies as in (2-4) we require $\theta=+1$ for $\varepsilon=\pm1$. Thus for any $H\in\mathcal{N}$ the isotropic orbit $O_0(H)$ is path-connected. First we treat the case $\varepsilon=-1$. We observe that $\mathcal{F}^*:=\bigcup_{H\in\mathcal{N}^*}O_0(H)$ is path-connected. If $\mathcal{F}$ were connected then $\pi(\mathcal{F})=\mathcal{N}$ would be connected, which is not possible, since $\mathcal{N}$ consists of 2 connected components G_1^- and $\mathcal{N}^*$. Thus $\mathcal{F}$ has 2 connected components $O_0(G_1^-)$ and $\mathcal{F}^*$. For $\varepsilon=+1$ we note that the set $O_0(C_k):=\bigcup_{H\in C_k}O_0(H)$ for $k\in\{2,3\}$ is path-connected and $\mathcal{N}$ consists of 3 connected components. Thus $\mathcal{F}$ admits at most 3 connected components. $\mathcal{F}$ is not connected since then $\pi(\mathcal{F})=\mathcal{N}$ would be connected. If $\mathcal{F}$ consists of 2 connected components $\mathcal{F}_1,\mathcal{F}_2$ such that $\mathcal{F}=\mathcal{F}_1\cup\mathcal{F}_2$, we need to distinguish several cases. Either $\mathcal{F}_1=O_0(G_1^+)\cup O_0(C_k)$, and $\mathcal{F}_2=O_0(C_\ell)$, where $k\neq\ell$ and $k,\ell\in\{2,3\}$, or $\mathcal{F}_1=O_0(C_2)\cup O_0(C_3)$ and $\mathcal{F}_2=O_0(G_1^+)$. In all cases we have, by the continuity of π, that $\pi(\mathcal{F}_1)$ is connected, which is not possible. □

Proof of Theorem 1.3. The quotient space X consists of elements denoted by $[F]$ for $F \in \mathcal{F}$. We equip X with the quotient topology such that the canonical projection $p : \mathcal{F} \to X$ is continuous. For $\varepsilon = +1$ we have $X = \{G_1^+, G_{2,0}^+, G_{3,0}^+\}$ by our classification. By Lemma 5.1 we obtain that $p^{-1}(H)$ for $H \in X$ is a connected component of $\mathcal{F}$, hence open. Thus X carries the discrete topology. To prove the statement for $\varepsilon = -1$ we write $H_0 := G_{2,1/2}^- \in \mathcal{N}$ and $H_1 := G_{3,0}^- \in \mathcal{N}$. For $k \in \{0, 1\}$, let $U_k \in X$ be an open neighborhood of $[H_k]$. Then $V_k := p^{-1}(U_k)$ is an open neighborhood of the orbit of H_k in $\mathcal{F}$. According to our classification there exists a sequence $(G_n)_{n\in\mathbb{N}}$ of mappings in $\mathcal{F}$, where each $G_n \in [H_1]$ and $G_n \to H_0$ in $\mathcal{F}$ as $n \to \infty$. Thus there exists $N \in \mathbb{N}$ such that $G_n \in V_0 \cap V_1$ for all $n \geq N$, which shows $[H_1] \in U_0 \cap U_1$ and completes the proof. □

6. Properness of the group action

Proof of Theorem 1.4. For $n \in \mathbb{N}$ we let $G_n = (g_n^1, g_n^2, g_n^3)$, $H_n' = (h'^1_n, h'^2_n, h'^3_n) \in \mathcal{F}$ with $G_n = \varphi_n' \circ H_n' \circ \varphi_n^{-1}$, where $(\varphi_n, \varphi_n') \in G_0$. Equipping J_0^3 with a suitable norm $\|\cdot\|$, we need to show that if we let $N > 1$ such that $\|j_0(G_n)\|, \|j_0(H_n')\| \leq N$ and $|g_{n01}^3|, |h'^3_{n01}| \geq 1/N$ as well as $|\Delta(G_n)|, |\Delta(H_n')| \geq 1/N$, then we have that $\{(\varphi_n, \varphi_n') \mid n \in \mathbb{N}\}$ is relatively compact in G_0. For a simplification, we write $H_n' = \phi_n' \circ H_n \circ \phi_n^{-1}$, where $H_n \in \mathcal{N}$ and $(\phi_n, \phi_n') \in G_0$ according to Proposition 2.6. Since we have shown in Proposition 4.3 that the map $\pi : \mathcal{F} \to \mathcal{N}$ is continuous, it follows that the sequence H_n is relatively compact, and we assume that each H_n satisfies all conditions we assumed for H_n'. Further we assume that H_n is given as described in Remark 2.7, where we set $s_n := 2|h_{n02}^1| \geq 0$, $x_n := h_{n02}^2 \in \mathbb{C}$ and $y_n := \operatorname{Im}(h_{n21}^2)$.

In the proof of Proposition 2.6 given in [Reiter 2015, Proposition 3.1], we give explicit formulas for (ϕ_n, ϕ_n'), which shows that $\{(\phi_n, \phi_n') \mid n \in \mathbb{N}\}$ is bounded, since the sequence H_n' is relatively compact. We set $\psi_n := \varphi_n \circ \phi_n$ and $\psi_n' := \varphi_n' \circ \phi_n'$. Hence we need to prove that $\{(\psi_n, \psi_n') \mid n \in \mathbb{N}\}$ is bounded in G_0. If we use the parametrization of (ψ_n, ψ_n') from (2-1) and (2-4), we show that $\{(\gamma_n, \gamma_n') \mid n \in \mathbb{N}\}$ is bounded in $\Gamma \times \Gamma'$. More precisely, we need to show the boundedness of the sequences $\lambda_n, c_n, r_n, a'_{1n}, a'_{2n}, \lambda'_n, c'_n, r'_n$ in $\Gamma \times \Gamma'$. We use the equations from the proof of Lemma 4.2, where Ψ_n plays the role of G_n. We start considering the third component of (4-2), which gives $1/\sqrt{N} \leq \lambda_n \lambda_n' \leq \sqrt{N}$. Then we rewrite (4-1) to obtain, for $k = 1, 2$, that $|a'_{kn}| = |g_{n10}^k|/(\lambda_n \lambda_n') \leq N\sqrt{N}$. After rewriting the first two components of (4-2) we obtain that $|u_n c_n + \lambda_n c'_{1n}|, |\lambda_n c'_{2n}| \leq 2N^3$. Then, using (4-2) and (4-3), we compute

$$\Delta(G_n) = \left| \lambda_n \lambda_n' U_n' \begin{pmatrix} u_n & 4\mathrm{i} u_n (\bar{c}_n + u_n \lambda_n \bar{c}'_{1n}) \\ 0 & u_n \lambda_n \end{pmatrix} \right| = u_n^2 \lambda_n^3 \lambda_n'^2,$$

such that the boundedness of $\Delta(G_n)$ from below implies that $1/N^2 \leq \lambda_n \leq N^2$. This gives $1/(\sqrt{N} N^2) \leq \lambda_n' \leq \sqrt{N} N^2$, and from (4-2) we derive boundedness of the sequence $|c'_{2n}|$. Then from (4-5) we obtain that the sequence $|c_n|$ is bounded, such that (4-2) shows the boundedness of $|c'_{1n}|$.

Finally, using all the previous bounds, we get from (4-7) and the second component of (4-10) that the sequences $|r_n+\lambda_n^2 r'_n|$ and $|2r_n+\lambda_n^2 r'_n|$ are both bounded, which implies that $|r_n|$ and $|r'_n|$ are bounded from above. Thus the sequence (ψ_n, ψ'_n) is relatively compact. Since (ϕ_n, ϕ'_n) is relatively compact, this implies that (φ_n, φ'_n) is also relatively compact, completing the proof. □

7. On the real-analytic structure of $\mathfrak{F}$

Lemma 7.1. *Let $\Pi : \mathfrak{F} \to \mathfrak{N}$ be given by $\Pi(H) := \phi' \circ H \circ \phi^{-1}$, where $(\phi, \phi') \in G_0$ are the unique isotropies according to Proposition 2.6 and Lemma 3.1. For $k = 2, 3$ we write $M_{k,\varepsilon} := \{\Pi^{-1}(G^{\varepsilon}_{k,s}) \mid s > 0\}$. Then $M_{k,\varepsilon}$ is a real-analytic real submanifold of $\mathfrak{F}$ of real dimension* 16.

Proof. For fixed $k \in \{2, 3\}$, $s > 0$ and $\delta > 0$, we write $G_{\delta,s} := \{G^{\varepsilon}_{k,t} \mid t \in B_\delta(s) \cap \mathbb{R}^+\}$, where $B_\delta(s) := \{t \in \mathbb{R}^+ \mid |t - s| < \delta\}$. To prove the lemma we show that for every $s_0 \in \mathbb{R}^+$ and sufficiently small $\delta_0 > 0$, there exists a locally real-analytic parametrization for $M := \Pi^{-1}(G_{\delta_0,s_0})$. As noted in Remark 2.13, we identify $\mathcal{F}$ with the set $\mathfrak{J} \subset \mathbb{C}^{K_0}$.

Theorem 2.8 implies that for each $H \in M$ there exist $(\phi, \phi') \in G_0$, $k \in \{2, 3\}$ and $s_1 \in B_{\delta_0}(s_0) \cap \mathbb{R}^+$, such that $H = \phi' \circ G^{\varepsilon}_{k,s_1} \circ \phi^{-1}$. This fact is used to describe M locally via parametrizations as follows. For $s > 0$ sufficiently near s_0, let F_s be a mapping as in Remark 4.1, which depends real-analytically on $s := 2|f^1_{02}|$. For the remaining coefficients in $j_0(F_s)$ we write $x := f^2_{02}$ and $y := \mathrm{Im}\big(f^2_{21}\big)$, where we suppress the dependence on s notationally. We use the real version of the notation for the parametrization of G_0 as in (2-1) and (2-4). Here we denote the set of real parameters of G_0 by $\Gamma \times \Gamma'$. Let us write $\Xi := \Gamma \times \Gamma' \times \mathbb{R}^+ \subset \mathbb{R}^{N_0}$, where $N_0 := 16$. For $\xi = (\gamma, \gamma', s) \in \Xi$, we define the mapping

$$\Psi : \Xi \to \mathfrak{J}, \quad \Psi(\xi) := j_0(\phi'_{\gamma'} \circ F_s \circ \phi_\gamma^{-1}), \tag{7-1}$$

where we use the notation as in (2-1) and (2-4) for ϕ_γ and $\phi'_{\gamma'}$ respectively and suppress the dependence on ε.

We set $\check{\Psi}(z, w) := (\phi'_{\gamma'} \circ F_s \circ \phi_\gamma^{-1})(z, w)$ with components $\check{\Psi} = (\check{\psi}^1, \check{\psi}^2, \check{\psi}^3)$ and $\check{\psi} := \big(\check{\psi}^1, \check{\psi}^2\big)$. The holomorphic mapping $\check{\Psi}$ is defined in a small neighborhood $U \subset \mathbb{C}^2$ of 0 and satisfies $\check{\Psi}(\mathbb{H}^2 \cap U) \subset \mathbb{H}^3_\varepsilon$. By Theorem 2.8 and the real-analytic dependence of the isotropies on the standard parameters, which can be observed by inspecting the proof of Proposition 2.6 in [Reiter 2015, Proposition 3.1], we note that Ψ and $\check{\Psi}$ are real-analytic in $\xi \in \Xi$. We assume without loss of generality that ξ_0 is chosen in such a way that $(\phi_\gamma, \phi'_{\gamma'}) = (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$. Consequently we write $O(2)$ for terms involving standard parameters of the isotropies which vanish to second order at ξ_0, and we consider $a_1' \in \mathbb{C}$ near 1 such that we substitute $\bar{a}_1' = (1-\varepsilon|a_2'|^2)/a_1'$ and take $\theta = +1$ in Ψ, which is then given by the following expressions:

$$\begin{aligned}
\check{\psi}_z(0) &= \big(uu'\lambda\lambda' a_1', u\lambda\lambda'\bar{a}_2'\big),\\
\check{\Psi}_w(0) &= \big(u'\lambda\lambda' a_1'(uc+\lambda c_1'), \lambda^2\lambda' c_2'/a_1', \lambda^2\lambda'^2\big) + O(2),\\
\check{\psi}_{z^2}(0) &= \big(2\mathrm{i}uu'\lambda\lambda'(\mathrm{i}\varepsilon u\lambda a_2' + 2(\bar{c}+u\lambda\bar{c}_1')a_1'), 2u^2\lambda^2\lambda'/a_1'\big) + O(2),\\
\check{\Psi}_{zw}(0) &= \Big(-\frac{1}{2}uu'\lambda\lambda' a_1'(2(r+\lambda^2 r') - \mathrm{i}\varepsilon\lambda^2),\\
&\qquad\qquad u\lambda^2\lambda'\Big(\frac{\mathrm{i}\varepsilon}{2}\lambda\bar{a}_2' + \frac{2uc}{a_1'}\Big), 2\mathrm{i}\lambda^2\lambda'^2(\bar{c}+u\lambda\bar{c}_1')\Big) + O(2),\\
\check{\Psi}_{w^2}(0) &= \big(u'\lambda^3\lambda'\big(a_1'(\mathrm{i}\varepsilon uc + \lambda s) - \varepsilon\lambda a_2' x\big),\\
&\qquad\qquad \lambda^4\lambda'\big(x/a_1' + \bar{a}_2' s\big), -2\lambda^2\lambda'^2(r+\lambda^2 r')\big) + O(2),\\
\check{\psi}_{z^2 w}(0) &= \Big(-uu'\lambda^3\lambda'\big(4a_1'\big(-\mathrm{i}u\lambda s + \varepsilon(\bar{c}+u\lambda\bar{c}_1')\big) + \mathrm{i}\varepsilon u\lambda a_2' y\big),\\
&\qquad\qquad u^2\lambda^2\lambda'\big(\big(-2(2r+\lambda^2 r') + 6\varepsilon u\lambda cs + \mathrm{i}\lambda^2 y\big)/a_1' + 2\mathrm{i}\lambda^2\bar{a}_2' s\big)\Big) + O(2).
\end{aligned}$$

As a first step we show that for given $\xi_0 \in \Xi$ the Jacobian of Ψ with respect to ξ evaluated at ξ_0, denoted by $\Psi_\xi(\xi_0)$, is of full rank N_0. Instead of considering the real equations of Ψ, however, we conjugate Ψ and compute the Jacobian of the system $\Phi := (\Psi, \overline{\Psi}) \in \mathbb{C}^{2K_0}$ with respect to

$$\xi = (u, \lambda, c, r, u', a_1', a_2', \lambda', c_1', c_2', r', s; \bar{c}, \bar{a}_2', \bar{c}_1', \bar{c}_2') \in \mathbb{C}^{N_0}$$

and evaluate at

$$\xi_0 = (1, 1, 0, 0, 1, 1, 0, 1, 0, 0, 0, s_0; 0, 0, 0, 0) \in \mathbb{R}^{N_0}, \tag{7-2}$$

denoted by $\Phi_\xi(\xi_0)$. We bring the transpose of $\Phi_\xi(\xi_0)$ into echelon form, denoting the resulting matrix by $\varphi = (\varphi^1, \dots, \varphi^{N_0})$, where $\varphi^j = (\varphi_1^j, \dots, \varphi_{2K_0}^j) \in \mathbb{C}^{2K_0}$ for $1 \le j \le N_0$, such that $\operatorname{rank}\big(\Phi_\xi(\xi_0)\big) = \operatorname{rank}(\varphi)$. In the following we suppress the evaluation of Φ at ξ_0 notationally and perform elementary row operations. The matrix given by

$$\begin{aligned}
(\varphi^1, \dots, \varphi^{11}) := \Big(&\Phi_u, \Phi_{\bar{a}_2'}, \Phi_{c_1'}, \Phi_{c_2'}, \Phi_\lambda, \Phi_{\bar{c}}, \Phi_{a_1'}, \Phi_{r'}, \Phi_c, \Phi_{a_2'}, \Phi_s\Big)\\
&- \Big(0, 0, 0, \Phi_u, \Phi_u, 0, \Phi_u, 0, \Phi_{c_1'}, \frac{\mathrm{i}\varepsilon}{2}\Phi_{\bar{c}}, 0\Big),
\end{aligned}$$

is in row echelon form, with constant nonzero entries on the main diagonal. Each 0 above represents $0 \in \mathbb{C}^{2K_0}$. Next we define

$$\begin{aligned}
\varphi^{12} &:= \Phi_{\lambda'} + \frac{1}{3}\Phi_u - \Phi_\lambda - \frac{1}{3}\Phi_{a_1'} - \frac{\mathrm{i}\varepsilon}{8}\Phi_{r'} + \frac{10 s_0}{3}\Phi_s,\\
\varphi^{13} &:= \Phi_{u'} - \frac{1}{3}\Phi_u - \frac{2}{3}\Phi_{a_1'} - \frac{2}{3}\Phi_s,
\end{aligned}$$

which are of the following form, where we denote by h' derivatives of a function h depending on s with respect to s:

$$\begin{aligned}\varphi^{12} &= (0, \dots, 0, \varphi^{12}_{12}, \dots, \varphi^{12}_{2K_0})\\ &= \Big(0, \dots, 0, \frac{-2(4x-5s_0x')}{3}, 2\mathrm{i}\varepsilon, \frac{8\mathrm{i}s_0}{3}, \frac{2\mathrm{i}(3\varepsilon-3y+5s_0y')}{3}, -\frac{1}{3}, \varphi^{12}_{17}, \dots, \varphi^{12}_{2K_0}\Big)\\ \varphi^{13} &= (0, \dots, 0, \varphi^{13}_{12}, \dots, \varphi^{13}_{2K_0})\\ &= \Big(0, \dots, 0, \frac{2x-s_0x'}{3}, 0, -\frac{8\mathrm{i}s_0}{3}, -\frac{\mathrm{i}s_0y'}{3}, -\frac{2}{3}, \varphi^{13}_{17}, \dots, \varphi^{13}_{2K_0}\Big).\end{aligned}$$

Then we define $\varphi^{14} := \Phi_r - \Phi_{r'}$, $\varphi^{15} := \Phi_{\bar{c}_2'}$ and $\varphi^{16} := \Phi_{\bar{c}_1'}$, from which we compute $\varphi^{14} = -2(e_{15} + e_{2K_0})$, $\varphi^{15} = e_{19}$ and $\varphi^{16} = -2e_{24} + \mathrm{i}\varepsilon e_{26} - 12\varepsilon s e_{2K_0}$, where for $j \in \mathbb{N}$ we denote by e_j the j-th unit vector in $\mathbb{R}^{2K_0}$.

We have to consider several cases. First, in case $\varphi^{12}_{12} \neq 0$, then we consider $\tilde{\varphi}^{13} := \varphi^{13} - \varphi^{13}_{12}\varphi^{12}/\varphi^{12}_{12}$, such that $\tilde{\varphi}^{13}_{13}$ is a multiple of $-2x + s_0x'$. If $\tilde{\varphi}^{13}_{13} \neq 0$, then $\varphi = (\varphi^1, \dots, \varphi^{12}, \tilde{\varphi}^{13}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form. If $\tilde{\varphi}^{13}_{13} = 0$, then $x = Cs^2$, where $C \in \mathbb{C} \setminus \{0\}$ and we have $\tilde{\varphi}^{13}_{14} \neq 0$, which again implies that $\varphi = (\varphi^1, \dots, \varphi^{12}, \tilde{\varphi}^{13}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form.

Next we treat the case $\varphi^{12}_{12} = 0$. First we consider the trivial case. If $x = 0$, then since $s_0 > 0$, we have $x' = 0$ and so $\varphi = (\varphi^1, \dots, \varphi^{16})$ is in echelon form. Now we assume $x \neq 0$, which implies $x' \neq 0$, and solve $\varphi^{12}_{12} = 0$. The solution is given by $x = Cs^{4/5}$, where $C \in \mathbb{C} \setminus \{0\}$ and $\varphi = (\varphi^1, \dots, \varphi^{11}, \varphi^{13}, \varphi^{12}, \varphi^{14}, \varphi^{15}, \varphi^{16})$ is in echelon form.

To sum up, we conclude that in all cases the Jacobian $\Phi_\xi(\xi_0)$ of the system Φ evaluated at ξ_0 is of full rank N_0, and hence that Ψ from (7-1) is a real-analytic locally regular mapping if we choose $\delta_0 > 0$ sufficiently small in M. For Ψ to be a local parametrization of M it remains to show that for each sufficiently small neighborhood $U \subset \Xi \subset \mathbb{R}^{N_0}$ of ξ_0, there exists a neighborhood $W \subset \mathbb{C}^{K_0}$ of $\Psi(\xi_0) = F_{s_0}$, such that $\Psi(U) = W \cap M$. We have

$$\Psi(U) = \{j_0(H) \mid \exists \xi = (\gamma, \gamma', t) \in U : H = {\phi'}^{-1}_{\gamma'} \circ F_t \circ \phi_\gamma\}$$

and with the notation from the beginning of this proof for $\delta > 0$ we have

$$\begin{aligned}M &= \Pi^{-1}(F_{\delta,s_0})\\ &= \{H \in \mathfrak{F} \mid \exists (\gamma, \gamma', s) \in \Gamma \times \Gamma' \times B_\delta(s_0) \cap \mathbb{R}^+ : {\phi'}_{\gamma'} \circ H \circ \phi_\gamma^{-1} = F_s\}.\end{aligned}$$

Remark 2.13, together with the fact that for each $H \in M$ we can write $H = {\phi'}^{-1}_{\gamma'} \circ F_s \circ \phi_\gamma$, shows $\Psi(U) \subset M$. We assume that there exists a neighborhood $U \subset \Xi$ of ξ_0, such that for any neighborhood W of $\Psi(\xi_0) = F_{s_0}$ we have $\Psi(U) \neq W \cap M$. We choose open, connected neighborhoods $(W_n)_{n\in\mathbb{N}}$ of F_{s_0} with $\bigcap_n W_n = \{F_{s_0}\}$ and $\Psi(U) \neq W_n \cap M$ for all $n \in \mathbb{N}$. There exists a sequence of mappings $(H_n)_{n\in\mathbb{N}} \in \mathfrak{F}$

such that $H_n \in W_n \cap M$ and $H_n \notin \Psi(U)$. We write $H_n = {\phi'}^{-1}_{\gamma'_n} \circ F_{s_n} \circ \phi_{\gamma_n}$ and conclude by Lemma 4.2 that $(\gamma_n, \gamma'_n, s_n) \to \xi_0$ in Ξ. Thus eventually $H_n \in \Psi(U)$ for large enough $n \in \mathbb{N}$, which completes the proof of the lemma. □

We need the following theorem concerning free and proper group actions on manifolds.

Theorem 7.2 [Duistermaat and Kolk 2000, Theorem 1.11.4]. *Let $k \in \mathbb{N} \cup \{\infty, \omega\}$ be nonzero and M a C^k manifold equipped with a C^k action $G \times M \to M$, where G is a C^k Lie group. Assume that the action is free and proper. Then M/G has the unique structure of a C^k manifold of real dimension* $\dim_{\mathbb{R}} M - \dim_{\mathbb{R}} G$ *and the topology of M/G is the quotient topology τ_Q. We denote by $\varphi : M \to M/G$ the canonical projection given by $\varphi(m) = G \cdot m := \{g \cdot m \mid g \in G\}$ for $m \in M$. For every $s \in M/G$ there is an open neighborhood $S \subset M/G$ of s and a C^k diffeomorphism $\psi : \varphi^{-1}(S) \to G \times S$, $\psi : m \mapsto (\psi_1(m), \psi_2(m))$, such that for $m \in \varphi^{-1}(S)$, $g \in G$ we have $\varphi(m) = \psi_2(m)$ and $\psi(g \cdot m) = (g \cdot \psi_1(m), \psi_2(m))$. We say the triple $(\varphi, M, M/G)$ is a C^k* principal fiber bundle with structure group *G.*

Proof of Theorem 1.5. By [Baouendi et al. 1997, Corollary 1.2] the group G_0 is a totally real, closed, real-analytic submanifold of

$$G^2_0(\mathbb{H}^2, 0) \times G^2_0(\mathbb{H}^3_\varepsilon, 0) \subset J^2_0(\mathbb{H}^2, 0) \times J^2_0(\mathbb{H}^3_\varepsilon, 0).$$

Hence G_0 is a real-analytic real Lie group. With the notation of Lemma 7.1 we define for $(\gamma, \gamma') \in \Gamma \times \Gamma'$ the map $N_{\gamma,\gamma'} : M_{k,\varepsilon} \to M_{k,\varepsilon}$, $N_{\gamma,\gamma'}(H) := \phi'_{\gamma'} \circ H \circ \phi^{-1}_\gamma$, where $(\phi_\gamma, \phi'_{\gamma'}) \in G_0$ according to (2-1) and (2-4). We would like to conclude that for each fixed $(\gamma, \gamma') \in \Gamma \times \Gamma'$, the map $N_{\gamma,\gamma'}$ is real-analytic. By Remark 2.13, instead of $N_{\gamma,\gamma'}$ it suffices to consider $N'_{\gamma,\gamma'} : \mathfrak{J}_{k,\varepsilon} \to \mathfrak{J}_{k,\varepsilon}$, where $\mathfrak{J}_{k,\varepsilon} := \{j_0(H) \mid H \in M_{k,\varepsilon}\}$, and $N'_{\gamma,\gamma'}(j_0(H)) := j_0(\phi'_{\gamma'} \circ H \circ \phi^{-1}_\gamma)$ is a restriction of $N_{\gamma,\gamma'}$. By considering the components of $N'_{\gamma,\gamma'}(j_0(H))$ for $H \in M_{k,\varepsilon}$, we obtain that $N'_{\gamma,\gamma'}(j_0(H))$ is a polynomial in $j_0(H)$, thus by [Bochner and Montgomery 1945, Theorem 4] the action of G_0 on $M_{k,\varepsilon}$ is real-analytic.

By Proposition 3.2 and Theorem 1.4 the map $N : \mathfrak{F} \times G_0 \to \mathfrak{F}$ defined by $N(\phi, \phi', H) = \phi' \circ H \circ \phi^{-1}$ is a free and proper action. For $\varepsilon = +1$ we note that by Lemma 5.1 and Lemma 7.1 the set $\mathfrak{F}$ is a real-analytic manifold, thus from Theorem 7.2 the conclusion for $\varepsilon = +1$ follows.

Next we show the claim for $\varepsilon = -1$. According to Lemma 7.1, for $k = 1, 2$ we set $N_k := \{G^-_{k+1,s} \mid s > 0\}$ and $N_0 := N_1 \cap N_2 = \{G^-_{2,1/2}\}$. The corresponding preimages are denoted by $M_k := \Pi^{-1}(N_k) \subset \mathfrak{F}$, so that $M_0 := M_1 \cap M_2 = \Pi^{-1}(N_0)$. Now set $M := M_1 \cup M_2$. By Lemma 7.1 for $k = 1, 2$ we have that M_k is a real-analytic submanifold of $\mathfrak{F}$. We obtain by Theorem 7.2 that locally M_k is real-analytically diffeomorphic to $G_0 \times S_k$, where S_k is a real submanifold with $\dim_{\mathbb{R}}(S_k) = \dim_{\mathbb{R}}(M_k) - \dim_{\mathbb{R}}(G_0) = 1$, by Lemma 7.1. By Proposition 2.6 it is possible to normalize any element in S_k with unique isotropies which depend real-analytically on elements of S_k. Thus, since $\dim_{\mathbb{R}}(N_k) = 1$, we map S_k to N_k via real-analytic diffeomorphisms. We obtain that for $k = 1, 2$ there exists an open neighborhood $U_k \subset \mathfrak{F}$ of N_0 and a real-analytic diffeomorphism $\phi_k : U_k \to V_k$ such that $\phi_k(U_k \cap M_k) = (G_0 \times N_k) \cap V_k$, where V_k is an open neighborhood of $N_0' := \{\mathrm{id}\} \times N_0 \subset G_0 \times M$, with $\mathrm{id} = (\mathrm{id}_{\mathbb{C}^2}, \mathrm{id}_{\mathbb{C}^3})$. Moreover, we have that $\phi_k(U_k \cap N_k) = (\{\mathrm{id}\} \times N_k) \cap V_k$ and ϕ_k satisfies the properties given in Theorem 7.2. We define $\phi : U_0 \to V_0$, $\phi(x) := \phi_k(x)$ for $x \in U_0 \cap U_k$, where $k = 1, 2$ and $V_0 = V_1 \cup V_2$ is an open neighborhood of N_0'. Write $\widetilde{U} := U_1 \cap U_2 \cap U_0 \subset \mathfrak{F}$ for an open neighborhood of N_0. Then we have $\phi|_{\widetilde{U}} = \phi_1|_{\widetilde{U}} = \phi_2|_{\widetilde{U}}$, which implies that ϕ is a real-analytic diffeomorphism. Furthermore, since

$$\operatorname{image}(\phi_1|_{\widetilde{U}\cap M}) = \operatorname{image}(\phi_2|_{\widetilde{U}\cap M}) = (G_0 \times N_0) \cap \widetilde{V},$$

where $\widetilde{V}$ is an open neighborhood of $N_0' \subset G_0 \times M$, the mapping ϕ locally maps M_0 real-analytic diffeomorphically to $G_0 \times N_0$.

Finally the last statement follows from Theorem 7.2, since if $\mathfrak{F}$ were a smooth manifold, then by the smooth version of Theorem 7.2, the quotient $\mathfrak{N}$ would have to be a smooth manifold, which is not the case. □

8. Homeomorphic variations of normal forms

In the following we use the notation from Definition 2.4.

Definition 8.1. Let $\mathcal{H}$ be a subset of $\mathcal{H}(M, p; M', p')$. A proper subset $\mathcal{K} \subsetneq \mathcal{H}$ is called a *normal form for* $\mathcal{H}$ if for each $[F] \in \mathcal{H}/\sim$, there exists a unique representative $G \in \mathcal{K} \cap [F]$. We denote the mapping which assigns to each $H \in \mathcal{H}$ the representative $G \in \mathcal{K} \cap [H]$ as $\pi : \mathcal{H} \to \mathcal{K}$. A normal form $\mathcal{K}$ for $\mathcal{H}$ is called *admissible* if $\pi : \mathcal{H} \to \mathcal{K}$ is continuous.

The uniqueness of the representative $F \in \mathcal{K} \cap [F]$ in Definition 8.1 is not a restriction. Assume we have another representative $F \neq G \in \mathcal{K}$ in the class $[F]$, then G is equivalent to F, hence it suffices to choose exactly one element from the set of all representatives which belong to $\mathcal{K} \cap [F]$. If there exists an admissible normal form $\mathcal{K}$ for $\mathcal{H}$ we observe that in each orbit of any not necessarily admissible normal form $\mathcal{K}'$ for $\mathcal{H}$, there exists an element of $\mathcal{K}$.

Theorem 8.2. *Let $\mathcal{N}'$ be an admissible normal form for $\mathcal{F}$. Then $\mathcal{N}'$ is homeomorphic to $\mathcal{N}$, where we equip $\mathcal{N}'$ and $\mathcal{N}$ with τ_J.*

Proof. Let us denote by $\pi' : \mathcal{F} \to \mathcal{N}'$ the continuous mapping as in Definition 8.1. We note that the class $\mathcal{N}$ introduced in Proposition 2.6 is an admissible normal form for $\mathcal{F}$ as in Definition 2.5. If we equip $\mathcal{F}$ with τ_J, we obtain by Proposition 4.3 that the mapping $\pi : \mathcal{F} \to \mathcal{N}$, $H \mapsto \sigma' \circ H \circ \sigma^{-1}$ is continuous.

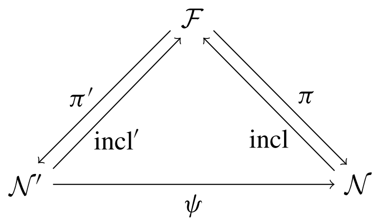

Figure 2. Diagram for admissible normal forms.

In Figure 2, the mapping $\mathrm{incl}' : \mathcal{N}' \to \mathcal{F}$ is the inclusion mapping, which is given by $\mathrm{incl}'(H) := H$ for all $H \in \mathcal{N}'$, and similarly for $\mathrm{incl} : \mathcal{N} \to \mathcal{F}$. The map $\psi : \mathcal{N}' \to \mathcal{N}$ is given by $\psi(H) := F$ for $H \in \mathcal{N}'$ and $F \in \mathcal{N} \cap [H]$. Since $\mathcal{N}'$ and $\mathcal{N}$ are normal forms, we obtain that ψ is a bijective mapping. Furthermore, since $\psi = \pi \circ \mathrm{incl}'$ and $\psi^{-1} = \pi' \circ \mathrm{incl}$ are compositions of continuous mappings, we obtain that ψ is a homeomorphism. □

Example 8.3. Beginning with $\mathcal{N}$, we can construct different admissible normal forms $\mathcal{N}'$ as follows. We fix a pair of isotropies $(\phi_0, \phi_0') \in G_0$ and consider the isotropies $(\tilde{\phi}, \tilde{\phi}') \in G_0$ from Proposition 2.6, such that $\pi : \mathcal{F} \to \mathcal{N}$ is given by $\pi(H) := \tilde{\phi}' \circ H \circ \tilde{\phi}^{-1}$, denoted by $\widehat{H}$. We define $\phi := \phi_0 \circ \tilde{\phi}$ and $\phi' := \phi_0' \circ \tilde{\phi}'$, to obtain for $F \in \mathcal{F}$ that

$$\phi' \circ F \circ \phi^{-1} = \phi_0' \circ \tilde{\phi}' \circ F \circ \tilde{\phi}^{-1} \circ \phi_0^{-1} = \phi_0' \circ \widehat{F} \circ \phi_0^{-1},$$

where $\widehat{F} \in \mathcal{N}$. We define $\mathcal{N}' := \{\phi_0' \circ \widehat{F} \circ \phi_0^{-1} \mid \widehat{F} \in \mathcal{N}\}$. As observed above π induces an admissible normal form, which implies that the mapping $\pi' : \mathcal{F} \to \mathcal{N}'$ given by $\pi'(F) := \phi' \circ F \circ \phi^{-1}$ is continuous and $\mathcal{N}'$ is an admissible normal form.

Acknowledgments

We would like to thank Bernhard Lamel, Giuseppe della Sala and Nordine Mir for their interest in this work and many discussions related to the topic of this article.

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Received November 6, 2014. Revised May 20, 2015.

MICHAEL REITER
FACULTY OF MATHEMATICS
UNIVERSITY OF VIENNA
OSKAR-MORGENSTERN-PLATZ 1
1090 VIENNA
AUSTRIA
m.reiter@univie.ac.at

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 280 No. 2 February 2016