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ON THE NUMBER OF LINES IN THE LIMIT SET FOR DISCRETE SUBGROUPS OF PSL(3, $\mathbb{C}$)

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Given a discrete subgroup $G \subset \mathrm{PSL}(3, \mathbb{C})$, acting on the complex projective plane, $\mathbb{P}^2_{\mathbb{C}}$, in the canonical way, we list all possible values for the number of complex projective lines and for the maximum number of complex projective lines lying in the complement of each of: the equicontinuity set of G, the Kulkarni discontinuity region of G, and maximal open subsets of $\mathbb{P}^2_{\mathbb{C}}$ on which G acts properly discontinuously.

1. Introduction

A classical result in the theory of Kleinian groups states that the limit set of an infinite Kleinian group consists of one, two, or uncountably many points. If the number of points in the limit set is smaller or equal to two then the Kleinian group is called *elementary*. On the other hand, if the number of points in the limit set is greater than two then the group is called *nonelementary* and its limit set is a perfect set.

In this paper, we prove an analogous result for complex Kleinian groups acting on $\mathbb{P}^2_{\mathbb{C}}$. We recall that $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a complex Kleinian group, whenever there exists a G-invariant nonempty open set $U \subset \mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously.

There is no standard definition of limit set in the theory of complex Kleinian groups, and we use the following three notions of limit set for a complex Kleinian group: The Myrberg limit set $\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G)$ (see Section 3A), the Kulkarni limit set $\mathbb{P}^2_{\mathbb{C}} \setminus \Omega(G)$ (see Section 3B), and the complement of a maximal G-invariant open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. In what follows, we denote by $U_{\max}(G)$ any maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously.

The main results in this paper are the following:

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Theorem 1.1. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is an infinite discrete subgroup and U is equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$, *then the number of complex projective lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is equal to* 1, 2, 3, *or* ∞. *Moreover, if there are infinitely many complex projective lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$, then there exists a perfect set of complex projective lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Theorem 1.2. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is an infinite discrete subgroup and U is equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$, *then the maximum number of complex projective lines in general position contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is equal to* 1, 2, 3, 4, *or* ∞.

We begin our exposition with a brief section on projective geometry. The material in this section is standard. Also, we set the notation we use throughout the paper.

In Section 3, we recall the definitions of equicontinuity set, Myrberg limit set, Kulkarni limit set, and Kulkarni discontinuity region. Also, we include some useful results such as Theorem 3.4 and Proposition 3.6. Finally, we recall the definition of complex Kleinian group.

In Section 4, we use Segre's embedding to prove that the set of effective lines is closed in $(\mathbb{P}^2_{\mathbb{C}})^*$. Consequently, the union of all effective lines for a discrete group $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a closed set of $\mathbb{P}^2_{\mathbb{C}}$ and this union is equal to the complement of the equicontinuity set of G, except in one case; see Corollary 4.5. The existence of loxodromic elements, whenever the limit set contains at least three lines in general position, is proved in Proposition 4.10.

In Section 5, we include all results needed to prove the main Theorem 1.1. In order to give a sketch of the proof of this theorem, and for the reader's convenience, we use the notation $\lambda(U)$ and $\mu(U)$ to denote the number of complex projective lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ and the maximum number of complex projective lines in general position contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$, respectively. A sketch of the proof of Theorem 1.1 is as follows:

Since G is an infinite group, $\mu(U) \geq 1$ (see the proof of [Cano et al. 2013, Proposition 3.3.4]).

If $\mu(U) \leq 3$ and $\lambda(U) < \infty$, then $\lambda(U) = \mu(U)$ (see Propositions 5.4 and 5.6).

If $1 < \mu(U) \leq 3$ and $\lambda(U) = \infty$, then there exists a perfect set of lines contained in the complement of U (see Proposition 5.7).

If $\mu(U) \geq 4$, then the complement of U is the union of a perfect set of lines (see Proposition 5.15).

In Section 6, we prove Theorem 1.2. The sketch of the proof is the following: We assume that $4 < \mu(U) < \infty$ and we find precisely two points called *vertices* such that each one of these points lies in infinitely many lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ (see Proposition 6.5). Moreover, if the line ℓ is contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ and does not pass through one of these vertices then its orbit contains infinitely many lines. Since $\mu(U) < \infty$, we obtain another vertex, contradicting Proposition 6.5.

The last section contains examples showing all distinct possible values that $\lambda(U)$ and $\mu(U)$ can take.

2. Preliminaries and notation

We recall that the complex projective plane $\mathbb{P}^2_{\mathbb{C}}$ is defined as

$$\mathbb{P}^2_{\mathbb{C}} := (\mathbb{C}^3 \setminus \{\mathbf{0}\}) / \mathbb{C}^*,$$

where $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ acts on $\mathbb{C}^3 \setminus \{\mathbf{0}\}$ by the usual scalar multiplication. This is a compact connected complex 2-dimensional manifold. Let $[\,] : \mathbb{C}^3 \setminus \{\mathbf{0}\} \to \mathbb{P}^2_{\mathbb{C}}$ be the quotient map. If $\beta = \{e_1, e_2, e_3\}$ is the standard basis of $\mathbb{C}^3$, we write $[e_j] = e_j$, for $j = 1, 2, 3$, and if $\boldsymbol{z} = (z_1, z_2, z_3) \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ then we write $[\boldsymbol{z}] = [z_1 : z_2 : z_3]$. Also, $\ell \subset \mathbb{P}^2_{\mathbb{C}}$ is said to be a complex line if $[\ell]^{-1} \cup \{\mathbf{0}\}$ is a complex linear subspace of dimension 2. Given two distinct points $[\boldsymbol{z}], [\boldsymbol{w}] \in \mathbb{P}^2_{\mathbb{C}}$, there is a unique complex projective line passing through $[\boldsymbol{z}]$ and $[\boldsymbol{w}]$; such a complex projective line is called a *line*, for short, and it is denoted by $\overleftrightarrow{[\boldsymbol{z}], [\boldsymbol{w}]}$. Consider the action of $\mathbb{C}^*$ on GL(3, $\mathbb{C}$) given by the usual scalar multiplication. Then

$$\mathrm{PGL}(3, \mathbb{C}) = \mathrm{GL}(3, \mathbb{C}) / \mathbb{C}^*$$

is a Lie group whose elements are called projective transformations. Now let $[\![\,]\!] : \mathrm{GL}(3, \mathbb{C}) \to \mathrm{PGL}(3, \mathbb{C})$ be the quotient map, $g \in \mathrm{PGL}(3, \mathbb{C})$ and $\boldsymbol{g} \in \mathrm{GL}(3, \mathbb{C})$, we say that $\boldsymbol{g}$ is a lift of g if $[\![\boldsymbol{g}]\!] = g$. One can show that PGL(3, $\mathbb{C}$) is a Lie group which acts transitively, effectively, and by biholomorphisms on $\mathbb{P}^2_{\mathbb{C}}$ via $[\![\boldsymbol{g}]\!]([\boldsymbol{w}]) = [\boldsymbol{g}(\boldsymbol{w})]$, where $\boldsymbol{w} \in \mathbb{C}^3 \setminus \{\mathbf{0}\}$ and $\boldsymbol{g} \in \mathrm{GL}(3, \mathbb{C})$.

We could have considered the action of the cube roots of unity $\{1, \omega, \omega^2\} \subset \mathbb{C}^*$ on SL(3, $\mathbb{C}$) given by the usual scalar multiplication, in which case

$$\mathrm{PSL}(3, \mathbb{C}) = \mathrm{SL}(3, \mathbb{C}) / \{1, \omega, \omega^2\} \cong \mathrm{PGL}(3, \mathbb{C}).$$

We denote by $\mathrm{M}_{3\times3}(\mathbb{C})$ the space of all 3×3 matrices with entries in $\mathbb{C}$ equipped with the standard topology. The quotient space

$$\mathrm{SP}(3, \mathbb{C}) := (\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}) / \mathbb{C}^*$$

is called the space of *pseudoprojective maps of* $\mathbb{P}^2_{\mathbb{C}}$ and it is naturally identified with the projective space $\mathbb{P}^8_{\mathbb{C}}$. Since GL(3, $\mathbb{C}$) is an open, dense, $\mathbb{C}^*$-invariant set of $\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$, we obtain that the space of pseudoprojective maps of $\mathbb{P}^2_{\mathbb{C}}$ is a compactification of PGL(3, $\mathbb{C}$) (or PSL(3, $\mathbb{C}$)). As in the case of projective maps, if $\boldsymbol{s}$ is an element in $\mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$, then $[\boldsymbol{s}]$ denotes the equivalence class of the matrix $\boldsymbol{s}$ in the space of pseudoprojective maps of $\mathbb{P}^2_{\mathbb{C}}$. Also, we say that $\boldsymbol{s} \in \mathrm{M}_{3\times3}(\mathbb{C}) \setminus \{\mathbf{0}\}$ is a lift of the pseudoprojective map S whenever $[\boldsymbol{s}] = S$.

Let S be an element in $(\mathrm{M}_{3\times 3}(\mathbb{C})\setminus\{\mathbf{0}\})/\mathbb{C}^*$ and $\boldsymbol{s}$ a lift to $\mathrm{M}_{3\times 3}(\mathbb{C})\setminus\{\mathbf{0}\}$ of S. The matrix $\boldsymbol{s}$ induces a nonzero linear transformation $s:\mathbb{C}^3\to\mathbb{C}^3$, which is not necessarily invertible. Let $\mathrm{Ker}(s)\subsetneq\mathbb{C}^3$ be its kernel and let $\mathrm{Ker}(S)$ denote its projectivization to $\mathbb{P}^2_{\mathbb{C}}$, taking into account that $\mathrm{Ker}(S):=\varnothing$ whenever $\mathrm{Ker}(s)=\{(0,0,0)\}$.

3. Discontinuous actions on $\mathbb{P}^2_{\mathbb{C}}$

Definition 3.1. Let $G\subset\mathrm{PSL}(3,\mathbb{C})$ be a discrete group. We say that G acts *properly and discontinuously* on the open nonempty G-invariant set $U\subset\mathbb{P}^2_{\mathbb{C}}$ if and only if, for each pair of compact subsets C, $D\subset U$, the set

$$\{g\in G: g(C)\cap D\neq\varnothing\}$$

is *finite*.

3A. *The equicontinuity set.*

Definition 3.2. The *equicontinuity set* for a family $\mathcal{F}$ of endomorphisms of $\mathbb{P}^2_{\mathbb{C}}$, denoted $\mathrm{Eq}(\mathcal{F})$ is defined as the set of points $z\in\mathbb{P}^2_{\mathbb{C}}$ for which there is an open neighborhood U of z such that $\{f|_U : f\in\mathcal{F}\}$ is a normal family.

Definition 3.3. Let $G\subset\mathrm{PSL}(3,\mathbb{C})$ be a discrete group. If

$$G'=\{S \text{ is a pseudoprojective map of } \mathbb{P}^2_{\mathbb{C}} : S \text{ is a cluster point of } G\};$$

then the *Myrberg limit set* [1925] is defined as the set

$$\Lambda_{\mathrm{Myr}}(G)=\bigcup_{S\in G'}\mathrm{Ker}(S).$$

Myrberg [1925] shows that G acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}}\setminus\Lambda_{\mathrm{Myr}}(G)$.

Theorem 3.4 [Barrera et al. 2011a]. *If* $G\subset\mathrm{PSL}(3,\mathbb{C})$ *is a discrete group*, *then*:

(i) *The group G acts properly and discontinuously on* $\mathrm{Eq}(G)$.

(ii) *The equicontinuity set of G satisfies*:

$$\mathrm{Eq}(G)=\mathbb{P}^2_{\mathbb{C}}\setminus\Lambda_{\mathrm{Myr}}(G)$$

(iii) *If U is an open G-invariant subset such that* $\mathbb{P}^2_{\mathbb{C}}\setminus U$ *contains at least three complex lines in general position*, *then* $U\subset\mathrm{Eq}(G)$.

3B. *The Kulkarni discontinuity region.*

Definition 3.5 [Kulkarni 1978]. If $G\subset\mathrm{PSL}(3,\mathbb{C})$ is a group, then:

- The set $L_0(G)$ is the closure of the set of points in $\mathbb{P}^2_{\mathbb{C}}$ with infinite isotropy group.
- The set $L_1(G)$ is the closure of the set of cluster points of the orbit Gz, where z runs over $\mathbb{P}^2_{\mathbb{C}}\setminus L_0(G)$.

- The set $L_2(G)$ is the closure of the set of cluster points of the family of compact sets $\{g(K) : g \in G\}$, where K runs over all the compact subsets of $\mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G))$.

The *Kulkarni limit set* of G is defined as

$$\Lambda_{\mathrm{Kul}}(G) = L_0(G) \cup L_1(G) \cup L_2(G).$$

The *Kulkarni discontinuity region* of G is defined as

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \Lambda_{\mathrm{Kul}}(G).$$

Kulkarni [1978] proves that G acts properly and discontinuously on the set $\Omega(G)$. However, $\Omega(G)$ is not necessarily the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. It is proved in [Barrera et al. 2011a] that $\mathrm{Eq}(G) \subset \Omega(G)$ whenever $G \subset \mathrm{PSL}(3, \mathbb{C})$ is discrete.

Proposition 3.6. *If H is a finite index subgroup of $G \subset \mathrm{PSL}(3, \mathbb{C})$, then*

(i) $L_0(H) = L_0(G)$,

(ii) $L_1(H) = L_1(G)$,

(iii) $L_2(H) = L_2(G)$,

(iv) $\Lambda_{\mathrm{Kul}}(H) = \Lambda_{\mathrm{Kul}}(G)$ *and* $\Omega(H) = \Omega(G)$.

Proof. Let us assume that $m = [G : H]$ and

$$G = \bigcup_{i=1}^{m} H\gamma_i.$$

(i) It is not hard to see that $L_0(H) \subset L_0(G)$. Now, if $x \in \mathbb{P}^2_{\mathbb{C}}$ and $|\mathrm{Isot}(x, G)| = \infty$, then there exists a sequence of distinct elements $(g_n) \subset G$ such that

$$g_n(x) = x \quad \text{for all } n \in \mathbb{N}.$$

We can assume there exists $1 \le i_0 \le m$ such that

$$g_n = h_n \gamma_{i_0} \quad \text{for all } n \in \mathbb{N}.$$

Hence, $(\tilde{h}_n) \subset H$, where $\tilde{h}_n = h_n h_1^{-1}$, is a sequence of distinct elements in H such that $\tilde{h}_n(x) = x$ for all $n \in \mathbb{N}$. Therefore $x \in L_0(H)$.

(ii) It is not hard to check that $L_1(H) \subset L_1(G)$. Conversely, if $(g_n) \subset G$ is a sequence of distinct elements and $x \in \mathbb{P}^2_{\mathbb{C}} \setminus L_0(G) = \mathbb{P}^2_{\mathbb{C}} \setminus L_0(H)$, such that

$$g_n(x) \to z \quad \text{as } n \to \infty,$$

then we can assume that

$$g_n(x) = h_n(\gamma_{i_0}(x)) \to z \quad \text{as } n \to \infty,$$

where $\gamma_{i_0}(x) \in \mathbb{P}^2_{\mathbb{C}} \setminus L_0(G) = \mathbb{P}^2_{\mathbb{C}} \setminus L_0(H)$. It follows that $z \in L_1(H)$.

(iii) It is not hard to check that $L_2(H) \subset L_2(G)$. Conversely, let us assume z is a cluster point of the family

$$\{g_n(K) : n \in \mathbb{N}\},$$

where $(g_n) \subset G$ is a sequence of distinct elements and $K \subset \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G)) = \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(H) \cup L_1(H))$ is a compact set. We can assume there exists $1 \le i_0 \le m$ such that

$$g_n = h_n \gamma_{i_0} \quad \text{for all } n \in \mathbb{N}.$$

It follows that z is a cluster point of the family

$$\{h_n(\gamma_{i_0}(K)) : n \in \mathbb{N}\},$$

where $(h_n) \subset H$ is a sequence of distinct elements and $\gamma_{i_0}(K) \subset \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(G) \cup L_1(G)) = \mathbb{P}^2_{\mathbb{C}} \setminus (L_0(H) \cup L_1(H))$ is a compact set. Therefore $z \in L_2(H)$.

(iv) It follows from (i), (ii), and (iii). □

Definition 3.7 [Cano et al. 2013]. We say that $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a *complex Kleinian group* if there exists a G-invariant nonempty open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously.

4. Some useful results

Definition 4.1. We say ℓ is an *effective line* for the discrete group $G \subset \mathrm{PSL}(3, \mathbb{C})$ if there exists a pseudoprojective transformation $S \in G'$ such that $\ell = \mathrm{Ker}(S)$. The set of effective lines for G is denoted by $\mathcal{E}(G)$, or simply $\mathcal{E}$ when there is no danger of confusion.

Proposition 4.2. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete group then $\mathcal{E}$ is a closed subset of $(\mathbb{P}^2_{\mathbb{C}})^*$, where $(\mathbb{P}^2_{\mathbb{C}})^*$ denotes the space of complex projective lines in $\mathbb{P}^2_{\mathbb{C}}$.*

Proof. We assume that (ℓ_n) is a sequence in $\mathcal{E}$ such that $\ell_n \to \ell$ as $n \to \infty$. For each $n \in \mathbb{N}$, there exists $S_n \in G' \subset \mathrm{SP}(3, \mathbb{C})$ such that $\ell_n = \mathrm{Ker}(S_n)$. Since $\mathrm{SP}(3, \mathbb{C})$ is compact, we can assume that $S_n \to S \in \mathrm{SP}(3, \mathbb{C})$ as $n \to \infty$. Moreover, $S \in G'$ because G' is closed.

In order to prove that $\ell = \mathrm{Ker}(S)$ we use the Segre embedding:

$$\psi : \mathbb{P}^2_{\mathbb{C}} \times \mathbb{P}^2_{\mathbb{C}} \to \mathrm{SP}(3, \mathbb{C})$$
$$\psi([\boldsymbol{v}], [\boldsymbol{t}]) = \left[\begin{pmatrix} t_1 \boldsymbol{v} \\ t_2 \boldsymbol{v} \\ t_3 \boldsymbol{v} \end{pmatrix}\right] = \left[\begin{pmatrix} v_1 \boldsymbol{t} & v_2 \boldsymbol{t} & v_3 \boldsymbol{t} \end{pmatrix}\right],$$

where

$$\boldsymbol{v} = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix} \quad \text{and} \quad \boldsymbol{t} = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix}.$$

We notice that the image of ψ is precisely the set of pseudoprojective transformations in SP(3, $\mathbb{C}$) whose kernel is equal to one line. In fact, $[\boldsymbol{v}]$ can be identified with $\operatorname{Ker}(\psi([\boldsymbol{v}],[\boldsymbol{t}]))$. Since ψ is continuous, it follows that $\psi(\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}})$ is compact in SP(3, $\mathbb{C}$), so it is closed. Therefore, $\operatorname{Ker}(S)$ is equal to one line.

Set $\psi^{-1}(S_n)=([\boldsymbol{v}_n],[\boldsymbol{t}_n])$, for each $n\in\mathbb{N}$, and $\psi^{-1}(S)=([\boldsymbol{v}],[\boldsymbol{t}])$. Since $\psi^{-1}:\psi(\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}})\to\mathbb{P}^2_{\mathbb{C}}\times\mathbb{P}^2_{\mathbb{C}}$ is continuous, it follows that $\psi^{-1}(S_n)\to\psi^{-1}(S)$ as $n\to\infty$. Therefore $[\boldsymbol{v}_n]\to[\boldsymbol{v}]$ as $n\to\infty$. In other words, $\ell_n\to\operatorname{Ker}(S)$. □

Corollary 4.3. *If $G\subset\mathrm{PSL}(3,\mathbb{C})$ is a discrete subgroup then*

$$\bigcup_{\ell\in\mathcal{E}}\ell\subset\mathbb{P}^2_{\mathbb{C}}$$

is a closed set.

Proof. If (x_n) is a sequence of points in $\bigcup_{\ell\in\mathcal{E}}\ell$ such that $x_n\to x$ as $n\to\infty$, then for each $n\in\mathbb{N}$ there exists $\ell_n\in\mathcal{E}$ such that $x_n\in\ell_n$. Since $(\mathbb{P}^2_{\mathbb{C}})^*$ is compact and $\mathcal{E}$ is closed, we can assume that $\ell_n\to\ell\in\mathcal{E}$ as $n\to\infty$. It follows that $x\in\ell$ and

$$x\in\bigcup_{\ell\in\mathcal{E}}\ell.$$ □

The following lemma is a generalization of a classical result in Kleinian groups theory. See, for example, [Maskit 1988, Proposition II.C.6].

Lemma 4.4. *If $g\in\mathrm{PSL}(3,\mathbb{C})$ is a complex homothety such that $\Lambda_{\mathrm{Kul}}(g)=\ell\cup\{p\}$ and $h\in\mathrm{PSL}(3,\mathbb{C})$ is a transformation such that $h(\ell)=\ell$ and $h(p)\neq p$ then the subgroup $\langle g,h\rangle\subset\mathrm{PSL}(3,\mathbb{C})$ is not discrete.*

Proof. We can assume that $\ell=\overleftrightarrow{e_1,e_2}$ and $p=e_3$. Then

$$\boldsymbol{g}=\begin{pmatrix}a&0&0\\0&a&0\\0&0&a^{-2}\end{pmatrix},\quad\text{where } 0<|a|<1 \text{ and } \boldsymbol{h}=\begin{pmatrix}\boldsymbol{h}_{11}&\boldsymbol{h}_{12}&\boldsymbol{h}_{13}\\\boldsymbol{h}_{21}&\boldsymbol{h}_{22}&\boldsymbol{h}_{23}\\0&0&\boldsymbol{h}_{33}\end{pmatrix}$$

are lifts of g and h respectively. Since $h(p)\neq p$, either $\boldsymbol{h}_{13}\neq 0$ or $\boldsymbol{h}_{23}\neq 0$.

By straightforward computations $[g^n,h]$ is induced by the matrix:

$$\begin{pmatrix}1&0&(a^{3n}-1)\boldsymbol{h}_{13}/\boldsymbol{h}_{33}\\0&1&(a^{3n}-1)\boldsymbol{h}_{23}/\boldsymbol{h}_{33}\\0&0&1\end{pmatrix}.$$

It follows that the sequence of distinct elements $[g^n,h]\in\langle g,h\rangle$ converges to the transformation in PSL(3, $\mathbb{C}$) induced by the matrix

$$\begin{pmatrix}1&0&-\boldsymbol{h}_{13}/\boldsymbol{h}_{33}\\0&1&-\boldsymbol{h}_{23}/\boldsymbol{h}_{33}\\0&0&1\end{pmatrix}.$$

Hence, $\langle g,h\rangle$ is not discrete. □

Corollary 4.5. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete subgroup then $\Lambda_{\mathrm{Myr}}(G)$ and $\bigcup_{\ell \in \mathcal{E}} \ell$ are equal except in the case when $\Lambda_{\mathrm{Myr}}(G)$ is equal to a disjoint union of one line and one point.*

Proof. Clearly $\bigcup_{\ell \in \mathcal{E}} \ell \subset \Lambda_{\mathrm{Myr}}(G)$.

If $\Lambda_{\mathrm{Myr}}(G)$ is equal to one line, then G contains a parabolic element, hence $\Lambda_{\mathrm{Myr}}(G) \subset \bigcup_{\ell \in \mathcal{E}} \ell$.

If we assume that $\Lambda_{\mathrm{Myr}}(G)$ is not equal to one line and $x \in \Lambda_{\mathrm{Myr}}(G) \setminus \bigcup_{\ell \in \mathcal{E}} \ell$ then $\{x\} = \mathrm{Ker}(S)$ for some $S \in G'$. It follows that $\mathrm{Im}(S)$ is an effective line by [Barrera et al. 2011a, Lemma 3.2(ii)]. Thus, x does not lie on the line $\mathrm{Im}(S)$. Hence, G contains a complex homothety with an isolated fixed point not lying in the closed set $\bigcup_{\ell \in \mathcal{E}} \ell$. To see this, consider a "round" closed neighborhood U of x disjoint from the closed set $\bigcup_{\ell \in \mathcal{E}} \ell$. Since $S \in G'$, there exists a sequence of distinct elements $g_n \in G$ such that $g_n \to S$ uniformly on compact subsets of $\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S)$. Now, by [loc. cit.] there exists a subsequence of g_n denoted the same, such that $g_n^{-1}(\,\cdot\,) \to x$ as $n \to \infty$ uniformly on compact subsets of $\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Im}(S)$. For n large enough, g_n^{-1} sends U into its interior. It follows that g_n is loxodromic, with a fixed point in the interior of U. This g_n is necessarily a complex homothety, because otherwise U would intersect $\bigcup_{\ell \in \mathcal{E}} \ell$.

If $\bigcup_{\ell \in \mathcal{E}} \ell$ is not equal to one line then there is an effective line, ℓ_0, different from the fixed line of the complex homothety. We reach a contradiction because we can iterate ℓ_0 with respect to the complex homothety and obtain that its isolated fixed point is in the closed set $\bigcup_{\ell \in \mathcal{E}} \ell$.

If $\bigcup_{\ell \in \mathcal{E}} \ell$ is equal to one line then, by hypothesis, there exists points $y \neq x$ such that $y \notin \bigcup_{\ell \in \mathcal{E}} \ell$. It follows that there exist two distinct complex homotheties with one common fixed line, so G is not discrete by Lemma 4.4. □

Notation 4.6. Let $U \subset \mathbb{P}^2_{\mathbb{C}}$ be an open set.

- The *number of lines* contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is denoted by $\lambda(U)$.
- The *maximum number of lines in general position* contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is denoted by $\mu(U)$.

There are examples of discrete groups $G \subset \mathrm{PU}(2, 1) \subset \mathrm{PSL}(3, \mathbb{C})$ such that

$$\Omega(G) = \mathrm{Eq}(G) = \mathbb{H}^2_{\mathbb{C}}.$$

Thus, in this case, the set

$$\bigcup_{\ell \in \mathcal{E}} \ell = \mathbb{P}^2_{\mathbb{C}} \setminus \mathbb{H}^2_{\mathbb{C}}$$

is big enough as to contain infinitely many lines which are not effective lines. On the other hand, we have the following:

Remark 4.7. If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a discrete subgroup, then:

(i) $\mu(\mathrm{Eq}(G)) = 1$ if and only if there is only one effective line for G.

(ii) If we assume that $\Lambda_{\mathrm{Myr}}(G) \neq \mathbb{P}^2_{\mathbb{C}}$ then the maximum number of effective lines for G in general position is equal to two if and only if $\mu(\mathrm{Eq}(G)) = 2$. (It could happen that $\Lambda_{\mathrm{Myr}}(G) = \mathbb{P}^2_{\mathbb{C}}$ but the maximum number of effective lines in general position is equal to two, for example in the double suspension of a Picard group. See Example (iii) in Section 7B.)

(iii) If we assume that $\Lambda_{\mathrm{Myr}}(G) \neq \mathbb{P}^2_{\mathbb{C}}$ then the maximum number of effective lines for G in general position is equal to three if and only if $\mu(\mathrm{Eq}(G)) = 3$. (It could happen that $\Lambda_{\mathrm{Myr}}(G) = \mathbb{P}^2_{\mathbb{C}}$ but the maximum number of effective lines in general position is equal to three, for example, in the suspension of a Picard group extended by an infinite group. See Example (ii) in Section 7C)

Lemma 4.8. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a subgroup, and there exists $S \in G' \subset \mathrm{SP}(3, \mathbb{C})$ such that* $\mathrm{Ker}(S)$ *is a line and* $\mathrm{Im}(S) \notin \mathrm{Ker}(S)$, *then G contains a loxodromic element.*

Proof. There exists a sequence of distinct elements $g_n \in G$ such that $g_n \to S$ as $n \to \infty$, uniformly on compact subsets of $\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Ker}(S)$. In particular, if $V \subset \mathbb{P}^2_{\mathbb{C}}$ is an open "ball" containing the point $\mathrm{Im}(S)$, such that $\overline{V} \cap \mathrm{Ker}(S) = \varnothing$ then there exists $N > 0$ such that $n \geq N$ implies that $g_n(\overline{V}) \subset V$. Therefore, g_n is loxodromic for every $n \geq N$; see [Navarrete 2008, Definition 6.1]. □

Lemma 4.9. *If $G \subset \mathrm{PSL}(3, \mathbb{C})$ is a subgroup and there exists $S, T \in G' \subset \mathrm{SP}(3, \mathbb{C})$ such that* $\mathrm{Ker}(S)$ *and* $\mathrm{Ker}(T)$ *are lines*, $\mathrm{Im}(T) \notin \mathrm{Ker}(S)$ *and* $\mathrm{Im}(S) \notin \mathrm{Ker}(T)$, *then G contains a loxodromic element.*

Proof. Let (g_n) and (h_n) be sequences of distinct elements in G such that $g_n \to S$ and $h_n \to T$ as $n \to \infty$. Then the sequence $f_n := g_n \circ h_n$ of elements of G satisfies that $f_n \to S \circ T$ as $n \to \infty$ and $\mathrm{Im}(S \circ T) = \mathrm{Im}(S) \notin \mathrm{Ker}(T) = \mathrm{Ker}(S \circ T)$. It follows from Lemma 4.8 that G contains a loxodromic element. □

Proposition 4.10. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $U \neq \varnothing$ be equal to* $\mathrm{Eq}(G)$, $\Omega(G)$ *or* $U_{\max}(G)$. *If $\mu(U) \geq 3$ then G contains a loxodromic element*

Proof. The hypothesis $\mu(U) \geq 3$ and Theorem 3.4(iii) imply that $U = \mathrm{Eq}(G)$.

By Corollary 4.5, $\Lambda_{\mathrm{Myr}}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G) \neq \mathbb{P}^2_{\mathbb{C}}$ is the union of effective lines for G. Therefore, there exist three pseudoprojective maps $S_1, S_2, S_3 \in G' \subset \mathrm{SP}(3, \mathbb{C})$ such that $\mathrm{Ker}(S_1)$, $\mathrm{Ker}(S_2)$, $\mathrm{Ker}(S_3)$ are three lines in general position. If $\mathrm{Im}(S_j) \notin \mathrm{Ker}(S_j)$ for some $1 \leq j \leq 3$ then Lemma 4.8 implies that G contains a loxodromic element. Hence, we can assume that

$$\mathrm{Im}(S_1) \in \mathrm{Ker}(S_1), \qquad \mathrm{Im}(S_2) \in \mathrm{Ker}(S_2), \qquad \mathrm{Im}(S_3) \in \mathrm{Ker}(S_3).$$

In this case, it is not hard to check that there exists $i \neq j$, $1 \leq i, j \leq 3$, such that $\mathrm{Im}(S_i) \notin \mathrm{Ker}(S_j)$ and $\mathrm{Im}(S_j) \notin \mathrm{Ker}(S_i)$. By Lemma 4.9, there exists a loxodromic element. $\square$

5. Counting lines

Definition 5.1. If p is a point and ℓ is a line such that $p \notin \ell$, then there is a *projection* from $\mathbb{P}^2_{\mathbb{C}} \setminus \{p\}$ to ℓ, denoted by

$$\pi = \pi_{p,\ell} : \mathbb{P}^2_{\mathbb{C}} \setminus \{p\} \to \ell,$$
$$\pi(z) = \overleftrightarrow{z, p} \cap \ell.$$

Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a group, and $p \in \mathbb{P}^2_{\mathbb{C}}$ a point such that $Gp = p$, then there is a group morphism given by

$$\Pi = \Pi_{p,\ell} : G \to \mathrm{Bihol}(\ell),$$
$$\Pi(g)(x) = \pi(g(x)).$$

Lemma 5.2. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup. If $V \subset \mathbb{P}^2_{\mathbb{C}}$ is an open G-invariant set such that $\mu(V) = 2$, then there is a point $p \in \mathbb{P}^2_{\mathbb{C}} \setminus V$ such that $Gp = p$.*

Proof. Let $\mathcal{L} = \{\ell \in (\mathbb{P}^2_{\mathbb{C}})^* : \ell \subset \mathbb{P}^2_{\mathbb{C}} \setminus V\}$. Since $\mu(V) = 2$, it follows that $\bigcap_{\ell \in \mathcal{L}} \ell$ is equal to one point denoted p. If $g \in G$ then $g(\mathcal{L}) = \mathcal{L}$, so

$$g(p) = g\left(\bigcap_{\ell \in \mathcal{L}} \ell\right) = \bigcap_{\ell \in \mathcal{L}} g(\ell) = \bigcap_{\ell \in \mathcal{L}} \ell = p. \qquad \square$$

Lemma 5.3. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $V \subset \mathbb{P}^2_{\mathbb{C}}$ a G-invariant open set such that $L_0(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus V$. If $3 \leq \lambda(V) < \infty$ and $\mu(V) = 2$, then:*

(i) *If ℓ is any line not containing the G fixed point, p, then $\Pi_{p,\ell}(G)$ is finite.*

(ii) *The normal subgroup* $\mathrm{Ker}(\Pi)$ *has finite index in G.*

(iii) *There exists $h_0 \in \mathrm{PSL}(3, \mathbb{C})$ such that every element in $h_0 \mathrm{Ker}(\Pi)(h_0)^{-1}$ of infinite order has a lift to* $\mathrm{SL}(3, \mathbb{C})$ *of the form*

$$\begin{pmatrix} 1 & b & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

(iv) *If h_0 is as in (iii), then the set A consisting of all $a^3 \in \mathbb{C}^*$, such that there exists $g \in h_0 \mathrm{Ker}(\Pi) h_0^{-1}$ with a lift of the form:*

$$\begin{pmatrix} a^{-2} & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix},$$

is a finite subgroup of $\mathbb{C}^*$.

(v) *There is a line* ℓ_0 *such that* $\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \ell_0$. *Moreover,* $L_0(G) = \ell_0$.

(vi) *The Kulkarni discontinuity region* $\Omega(G)$ *is equal to* $\mathbb{P}^2_{\mathbb{C}} \setminus \ell_0 = \mathrm{Eq}(G)$.

Proof. Set $\mathcal{L} = \{\ell \in (\mathbb{P}^2_{\mathbb{C}})^* : \ell \subset \mathbb{P}^2_{\mathbb{C}} \setminus V\}$ and $n_0 = |\mathcal{L}|$. Since $\mu(V) = 2$ then every line in $\mathcal{L}$ passes through a point denoted by p, and $Gp = p$.

(i) Since $\{g(\ell) : g \in G,\ \ell \in \mathcal{L}\} = \mathcal{L}$, it follows that $F := \pi\left(\bigcup_{\ell\in\mathcal{L}}(\ell \setminus \{p\})\right)$ is a $\Pi(G)$-invariant set whose cardinality is $n_0 \geq 3$. Thus

$$\Gamma = \bigcap_{x\in F} \mathrm{Isot}(x, \Pi(G))$$

is a normal subgroup of $\Pi(G)$ with finite index. Moreover, every element in F is fixed by Γ. Since F contains more than three elements we conclude that $\Gamma = \{\mathrm{Id}\}$. Therefore $\Pi(G)$ is finite.

(ii) The normal subgroup $\mathrm{Ker}(\Pi)$ has finite index because $G/\mathrm{Ker}(\Pi) \cong \Pi(G)$ is finite.

(iii) We can assume, by conjugating, that $Ge_1 = e_1$ and projection $\Pi = \Pi_{e_1, \overleftrightarrow{e_2, e_3}}$. If $g \in \mathrm{Ker}(\Pi)$ then any lift for g in $\mathrm{SL}(3, \mathbb{C})$ has the form

$$\boldsymbol{g} = \begin{pmatrix} a^{-2} & b & c \\ 0 & a & 0 \\ 0 & 0 & a \end{pmatrix}, \quad \text{where } a \in \mathbb{C}^* \text{ and } b, c \in \mathbb{C}.$$

If $\boldsymbol{g}$ is diagonalizable, then there are $\boldsymbol{v}_1, \boldsymbol{v}_2 \in \mathbb{C}^3$ such that $\{e_1, \boldsymbol{v}_1, \boldsymbol{v}_2\}$ is an eigenbasis for $\boldsymbol{g}$ whose respective eigenvalues are $\{a^{-2}, a, a\}$. Consequently, $\overleftrightarrow{v_1, v_2} \subset L_0(g) \subset L_0(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus V$ and $e_1 \notin \overleftrightarrow{v_1, v_2}$, which contradicts the hypothesis that $\mu(V) = 2$. Therefore, $\boldsymbol{g}$ is not diagonalizable, which implies that $a^{-2} = a$, so g has a lift of the form

$$\boldsymbol{g} = \begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } b, c \in \mathbb{C}. \tag{1}$$

We can assume that there is an element $g_0 \in \mathrm{Ker}(\Pi)$ such that g_0 has a lift $\boldsymbol{g}_0 \in \mathrm{SL}(3, \mathbb{C})$ given by

$$\boldsymbol{g}_0 = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{2}$$

If we assume that there is an element $g_1 \in \mathrm{Ker}(\Pi)$ which has a lift $\boldsymbol{g}_1 \in \mathrm{SL}(3,\mathbb{C})$ given as in (1) with $c \neq 0$, then for every $n \in \mathbb{N}$,

$$\boldsymbol{g}_0^n \boldsymbol{g}_1 = \begin{pmatrix} 1 & b+n & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

By straightforward computations, we see that

$$\ell_n = \overleftrightarrow{[e_1], [0:-c:b+n]} \subset L_0(g_0^n g_1) \subset L_0(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus V.$$

Moreover, $\ell_n \neq \ell_m$ whenever $n \neq m$. Thus $\mathcal{L}$ contains infinitely many lines, which contradicts the hypothesis that $\lambda(V) < \infty$.

(iv) By straightforward computations, the set A is a subgroup of $\mathbb{C}^*$. By (iii), every element in $\mathrm{Ker}(\Pi)$ is elliptic or parabolic. It follows that $A \subset \mathbb{S}^1$. Assume that A is infinite; then there is a sequence $a_n^3 \subset A$ of distinct elements such that $a_n^{1/2} \to 1$ as $n \to \infty$. For each $n \in \mathbb{N}$, let $g_n \in \mathrm{Ker}(\Pi)$ with a lift $\boldsymbol{g}_n \in \mathrm{SL}(3,\mathbb{C})$ of the form

$$\boldsymbol{g}_n = \begin{pmatrix} a_n & b_n & c_n \\ 0 & a_n^{-1/2} & 0 \\ 0 & 0 & a_n^{-1/2} \end{pmatrix}, \quad \text{where } b_n, c_n \in \mathbb{C}.$$

If $\boldsymbol{g_0}$ is as in (2), then

$$\boldsymbol{g}_n^{-1} \boldsymbol{g}_0 \boldsymbol{g}_n = \begin{pmatrix} 1 & a_n^{-3/2} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

as $n \to \infty$, a contradiction to the hypothesis that G is discrete.

(v) Let H denote the finite index subgroup of $h_0 \, \mathrm{Ker}(\Pi) h_0^{-1}$ consisting of all elements with a lift of the form

$$\begin{pmatrix} 1 & b & c \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

We can assume, by (iii), that each element of infinite order, $h \in H$ has a lift $\boldsymbol{h} \in \mathrm{SL}(3,\mathbb{C})$ which is given by

$$\boldsymbol{h} = \begin{pmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

It follows that $\mathrm{Eq}(H) = \mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_3}$. Since H has finite index in $h_0 \, \mathrm{Ker}(\Pi) h_0^{-1}$,

$$\mathrm{Eq}(h_0 G h_0^{-1}) = \mathrm{Eq}(h_0 \, \mathrm{Ker}(\Pi) h_0^{-1}) = \mathrm{Eq}(H) = \mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_3}.$$

Finally, $L_0(h_0 G h_0^{-1}) = L_0(h_0 \, \mathrm{Ker}(\Pi) h_0^{-1}) = L_0(H) = \overleftrightarrow{e_1, e_3}$.

(vi) G acts properly and discontinuously on $\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \ell_0$. Since $\mathrm{Ker}(\Pi)$ has finite index in G and every element of infinite order has canonical form as in (2), we notice that G does not contain loxoparabolic elements. It follows by [Barrera et al. 2014a, Theorem 1.2] that $\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \ell_0$. □

Proposition 5.4. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $U \subset \mathbb{P}^2_{\mathbb{C}}$ be one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 2$ and $\lambda(U) < \infty$, then $\lambda(U) = 2$.*

Proof. If U is either $\mathrm{Eq}(G)$ or $\Omega(G)$ and $2 < \lambda(U) < \infty$, then by Lemma 5.3(v) and (vi), $\mathrm{Eq}(G) = \Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \ell$ for some line ℓ. Thus, $\mu(U) = 1 = \lambda(U)$, a contradiction.

If $U = U_{\max}(G)$ and $2 < \lambda(U) < \infty$ then $L_0(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G)$ by Lemma 5.3(v). Since G acts properly and discontinuously on U, it follows that $L_0(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$. Thus, $U \subset \mathrm{Eq}(G)$, so $U = \mathrm{Eq}(G)$ is the complement of one line in $\mathbb{P}^2_{\mathbb{C}}$, a contradiction of the hypothesis that $\mu(U) = 2$. □

Lemma 5.5. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete group and $V \subset \mathbb{P}^2_{\mathbb{C}}$ be an open G-invariant set such that $L_0(G) \subset \mathbb{P}^2_{\mathbb{C}} \setminus V$. If $\mu(V) = 3$ and $3 < \lambda(V) < \infty$, then there is a line ℓ_1 and $p \in \mathbb{P}^2_{\mathbb{C}} \setminus \ell_1$ such that*

$$\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\ell_1 \cup \{p\}) = \mathbb{P}^2_{\mathbb{C}} \setminus L_0(G)$$

Proof. Let us assume that $n_0 = \lambda(V) > 3$. If we define

$$\mathcal{L} = \{\ell \in (\mathbb{P}^2_{\mathbb{C}})^* : \ell \subset \mathbb{P}^2_{\mathbb{C}} \setminus V\} = \{\ell_1, \ldots, \ell_{n_0}\},$$

then the group

$$G_0 = \bigcap_{i=1}^{n_0} \mathrm{Isot}(\ell_i, G)$$

is a finite index normal subgroup of G. Since $\mu(U) = 3$ then, conjugating by a projective transformation, we can assume that

$$\ell_1 = \overleftrightarrow{e_1, e_2}, \qquad \ell_2 = \overleftrightarrow{e_2, e_3}, \qquad \ell_3 = \overleftrightarrow{e_3, e_1}.$$

It follows that every element $g \in G_0$ has a lift $\boldsymbol{g} \in \mathrm{SL}(3, \mathbb{C})$ of the form

$$\boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}_{11} & 0 & 0 \\ 0 & \boldsymbol{g}_{22} & 0 \\ 0 & 0 & \boldsymbol{g}_{33} \end{pmatrix}, \quad \text{where } \boldsymbol{g}_{11}\boldsymbol{g}_{22}\boldsymbol{g}_{33} = 1.$$

If ℓ_j is any line in $\mathcal{L} \setminus \{\ell_1, \ell_2, \ell_3\}$ then $e_1 \in \ell_j$ or $e_2 \in \ell_j$ or $e_3 \in \ell_j$, because $\mu(V) = 3$. We assume, without loss of generality, that $e_3 \in \ell_j$ for all lines ℓ_j in $\mathcal{L} \setminus \{\ell_1, \ell_2, \ell_3\}$. Set $\Pi = \Pi_{e_3, \ell_1}$ and $\pi = \pi_{e_3, \ell_1}$, and notice that $\pi(e_1)$, $\pi(e_2)$, $\pi(\ell_j \setminus e_3)$ are three distinct fixed points in ℓ_1 for the group $\Pi(G_0)$, so $\Pi(G_0) = \{\mathrm{Id}\}$. Therefore, for each $g \in G_0$, there is a nonzero complex number $\boldsymbol{g}_{33}^2$ such that $\boldsymbol{g} \in \mathrm{SL}(3, \mathbb{C})$ given by

$$\boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}_{33} & 0 & 0 \\ 0 & \boldsymbol{g}_{33} & 0 \\ 0 & 0 & \boldsymbol{g}_{33}^{-2} \end{pmatrix}$$

is a lift of g. We conclude that

$$\mathrm{Eq}(G) = \mathrm{Eq}(G_0) = \mathbb{P}^2_{\mathbb{C}} \setminus (\ell_1 \cup \{e_1\}). \qquad \square$$

Proposition 5.6. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $U \subset \mathbb{P}^2_{\mathbb{C}}$ be one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 3$ and $\lambda(U) < \infty$ then $\lambda(U) = 3$.*

Proof. Given that $\mu(U) = 3$, Theorem 3.4(iii) implies that $U = \mathrm{Eq}(G)$. If $\lambda(U) > 3$ then applying Lemma 5.5, we obtain that $\mu(U) = 1$, a contradiction to the hypothesis that $\mu(U) = 3$. Therefore $\lambda(U) = 3$. $\square$

Proposition 5.7. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $\varnothing \neq U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to on of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) \in \{2, 3\}$ and $\lambda(U) = \infty$, then there is a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$, so there are uncountably many lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

First, we consider the case when $\mu(U) = 2$. In this case, there exists a fixed point of G corresponding to the intersection point of any two distinct lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$. We can assume that e_3 is this fixed point. Hence every element $g \in G$ is induced by a unique matrix of the form

$$\boldsymbol{g} = \begin{pmatrix} \boldsymbol{g}_{11} & \boldsymbol{g}_{12} & 0 \\ \boldsymbol{g}_{21} & \boldsymbol{g}_{22} & 0 \\ \boldsymbol{g}_{31} & \boldsymbol{g}_{32} & 1 \end{pmatrix} \tag{3}$$

If we set $\Pi = \Pi_{e_3, \overleftrightarrow{e_1, e_2}} : G \to \mathrm{Bihol}(\overleftrightarrow{e_1, e_2})$, then we consider the subcases depending on whether $\Pi(G)$ is not elementary or not discrete, elementary of two limit points, elementary of one limit point, or finite. These subcases are considered in Lemmas 5.8, 5.12, 5.13, and 5.14.

The subgroup $\mathrm{Ker}(\Pi)$ also plays an important role in the proof of Proposition 5.7 in the case when $\mu(U) = 2$, and we prove in Lemma 5.9 that $\mathrm{Ker}(\Pi)$ contains a free abelian finite index subgroup H, consisting of all elements of infinite order and the identity. Moreover, we prove in Lemma 5.10 that necessarily the rank of H is smaller or equal to 2. This result is analogous to the first Bieberbach theorem with the difference that H does not have maximal rank.

Lemma 5.8. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $\varnothing \neq U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 2$, $\lambda(U) = \infty$, and $\Pi(G)$ is not elementary or not discrete then there is a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Proof. The action of $\Pi(G)$ over the line $\overleftrightarrow{e_1, e_2}$ represents the action of the group G on the full pencil of lines passing through the point e_3. Since $\Pi(G)$ is not elementary or not discrete, we have that for any line ℓ passing through e_3 (except for a finite number of lines), the closure of the set $\{g(\ell) : g \in G\}$ contains a perfect set of lines. Since $\lambda(U) = \infty$, there exists a line ℓ contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ such that

$$\overline{\{g(\ell) : g \in G\}} \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$$

contains a perfect set of lines. □

Lemma 5.9. *The set $H \subset \mathrm{Ker}(\Pi)$, consisting of all elements of infinite order and the identity, is a free abelian finite index normal subgroup of* $\mathrm{Ker}(\Pi)$. *Moreover, every element in H is induced by a matrix of the form*

$$\boldsymbol{h} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \boldsymbol{h}_{31} & \boldsymbol{h}_{32} & 1 \end{pmatrix}. \tag{4}$$

Hence, H is isomorphic to a discrete subgroup of $\mathbb{C}^2$.

Proof. Let us assume that $h \in \mathrm{Ker}(\Pi)$ has infinite order and it is induced by a matrix of the form

$$\boldsymbol{h} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ \boldsymbol{h}_{31} & \boldsymbol{h}_{32} & 1 \end{pmatrix}. \tag{5}$$

If $|a| \neq 1$ then h is a complex homothety and $\{e_3\} \cup \ell_h$ is the set of fixed points of h, where ℓ_h is a line not passing through e_3. It is a contradiction of the hypothesis that $\mu(U) = 2$. Hence $|a| = 1$.

If we assume that $a \neq 1$ then h is an elliptic element of infinite order, so G is not discrete, contradiction. Therefore, $a = 1$ and every element in H is induced by a matrix of the form (4). It follows immediately that H is a free abelian normal subgroup of $\mathrm{Ker}(\Pi)$.

If $H = \{\mathrm{Id}\}$ then $\mathrm{Ker}(\Pi)$ is a discrete subgroup and every element in $\mathrm{Ker}(\Pi)$ has finite order. It follows that $\mathrm{Ker}(\Pi)$ is finite.

If $H \neq \{\mathrm{Id}\}$ then there exists $h \in H$ induced by a matrix of the form (4) where $(\boldsymbol{h}_{31}, \boldsymbol{h}_{32}) \neq (0, 0)$. If $g \in \mathrm{Ker}(\Pi)$ is induced by a matrix of the form

$$\boldsymbol{g} = \begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ \boldsymbol{g}_{31} & \boldsymbol{g}_{32} & 1 \end{pmatrix}, \tag{6}$$

then $g^{-n}hg^n \in H$ is induced by a matrix of the form

$$\boldsymbol{h} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a^n \boldsymbol{h}_{31} & a^n \boldsymbol{h}_{32} & 1 \end{pmatrix}. \tag{7}$$

It follows that H is a finite index subgroup of $\mathrm{Ker}(\Pi)$. Otherwise, there exists a sequence of distinct elements of H tending to an element in $\mathrm{PSL}(3, \mathbb{C})$. □

Lemma 5.10. *Let H be as in Lemma 5.9. If H has rank at least* 3, *then H is not a complex Kleinian group.*

Proof. Let us assume that H contains the free abelian group generated by

$$\boldsymbol{h}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & b_1 & 1 \end{pmatrix}, \qquad \boldsymbol{h}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_2 & b_2 & 1 \end{pmatrix}, \qquad \boldsymbol{h}_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_3 & b_3 & 1 \end{pmatrix},$$

where $\{(a_1, b_1), (a_2, b_2), (a_3, b_3)\}$ is an $\mathbb{R}$-linearly independent set of vectors, and $\{(a_1, b_1), (a_2, b_2)\}$ is a $\mathbb{C}$-linearly independent set of vectors.

Conjugating by an element in $\mathrm{PGL}(3, \mathbb{C})$, we can assume that $(a_1, b_1) = (1, 0)$, $(a_2, b_2) = (0, 1)$, and $(a_3, b_3) = (\lambda, \mu)$, where $\lambda \notin \mathbb{R}$.

If m, n, k are integers, not all zero, then the element

$$\boldsymbol{h}_1^n \boldsymbol{h}_2^m \boldsymbol{h}_3^k = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n + k\lambda & m + k\mu & 1 \end{pmatrix}$$

has the property that

$$\mathrm{Ker}(h_1^n h_2^m h_3^k) = L_0(h_1^n h_2^m h_3^k)$$

is the complex line in $\mathbb{P}^2_{\mathbb{C}}$, passing through e_3 and defined by

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : (n + k\lambda)z_1 + (m + k\mu)z_2 = 0\}.$$

Moreover, this complex line can be identified via dual vector with the point

$$[n + k\lambda : m + k\mu : 0].$$

Now, the set

$$\{[n + k\lambda : m + k\mu : 0] : m, n, k \text{ are integers, not all zero}\}$$

is dense in the set $\{[A : B : 0] : A, B \in \mathbb{C} \text{ not both zero}\}$ which is identified with the set of all lines in $\mathbb{P}^2_{\mathbb{C}}$ passing through e_3.

Now, let us assume that H acts properly and discontinuously on the open set $U \subset \mathbb{P}^2_{\mathbb{C}}$. Since, $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is a closed set containing every line of the form $\mathrm{Ker}(h_1^n h_2^m h_3^k) = L_0(h_1^n h_2^m h_3^k)$, for $m, n, k \in \mathbb{Z}$, not all zero, it follows that $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains the complete pencil of lines passing through e_3. Therefore, $U = \varnothing$. □

Lemma 5.11. *Let H be a free abelian group of rank 2, as in Lemma 5.9, acting properly and discontinuously on the open set $U \subset \mathbb{P}^2_{\mathbb{C}}$. If H is generated by transformations h_1 and h_2 induced by the matrices*

$$\boldsymbol{h}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & b_1 & 1 \end{pmatrix}, \quad \boldsymbol{h}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_2 & b_2 & 1 \end{pmatrix},$$

and $\{(a_1, b_1), (a_2, b_2)\}$ is a basis of $\mathbb{C}^2$ then $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains a perfect set of lines.

Proof. As in the proof of Lemma 5.10, conjugating if needed, we can assume that $(a_1, b_1) = (1, 0)$ and $(a_2, b_2) = (0, 1)$.

If m, n are integers, not both zero, then the element

$$\boldsymbol{h}_1^n \boldsymbol{h}_2^m = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ n & m & 1 \end{pmatrix}$$

has the property that

$$\operatorname{Ker}(h_1^n h_2^m) = L_0(h_1^n h_2^m)$$

is the complex line in $\mathbb{P}^2_{\mathbb{C}}$, passing through e_3 and defined by

$$\{[z_1 : z_2 : z_3] \in \mathbb{P}^2_{\mathbb{C}} : n z_1 + m z_2 = 0\}.$$

Moreover, this complex line can be identified with the point

$$[n : m : 0].$$

Now, the set

$$\{[n : m : 0] : m, n \text{ are integers, not both zero }\}$$

is dense in the set of lines $\{[A : B : 0] : A, B \in \mathbb{R} \text{ not both zero }\} \cong \mathbb{P}^1_{\mathbb{R}}$.

Since, $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is a closed set containing every line of the form $\operatorname{Ker}(h_1^n h_2^m) = L_0(h_1^n h_2^m)$, for $m, n \in \mathbb{Z}$, not both zero, it follows that $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains a circle of lines passing through e_3. □

Lemma 5.12. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $\varnothing \neq U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 2$ and $\Pi(G)$ is elementary with two limit points, then $\lambda(U) = 2$ or there exists a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Proof. Since $\Pi(G)$ is elementary with two limit points, there exists a G-invariant set of two lines passing through e_3; we can assume that $\overleftrightarrow{e_1, e_3}$ and $\overleftrightarrow{e_2, e_3}$ are these two lines. Moreover, we can assume that each one of these lines is G-invariant (up to a finite index subgroup). It follows that every element in G is induced by a matrix of the form

$$\begin{pmatrix} \boldsymbol{g}_{11} & 0 & 0 \\ 0 & \boldsymbol{g}_{22} & 0 \\ \boldsymbol{g}_{31} & \boldsymbol{g}_{32} & 1 \end{pmatrix}. \tag{8}$$

By Lemma 5.10, H has rank at most 2, and we consider the cases according to the rank of this group. If the rank of H is 2 and we assume that the hypotheses of Lemma 5.11 are satisfied, then $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains a perfect set of lines. Hence, we can assume that the rank of H is 2 and H is generated by two elements h_1 and h_2, induced by the matrices

$$\boldsymbol{h}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a_1 & b_1 & 1 \end{pmatrix}, \quad \boldsymbol{h}_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \omega a_1 & \omega b_1 & 1 \end{pmatrix},$$

where $\omega \in \mathbb{C} \setminus \mathbb{R}$.

Since $\Pi(G)$ is elementary with two limit points, there exists an element $g \in G$ induced by a matrix of the form (8) where $|\boldsymbol{g}_{11}| \neq |\boldsymbol{g}_{22}|$.

Since $g^{-1}hg \in H$, it follows that

$$(\boldsymbol{g}_{11}a_1, \boldsymbol{g}_{22}b_1) = m(a_1, b_1) + n\omega(a_1, b_1), \quad \text{for some } m, n \in \mathbb{Z}. \tag{9}$$

If $a_1 \neq 0 \neq b_1$ then, by (9),

$$\boldsymbol{g}_{11} = m + n\omega = \boldsymbol{g}_{22},$$

a contradiction. Hence, $a_1 = 0$ or $b_1 = 0$.

We can assume that $a_1 = 0$; then for every $g_1, g_2 \in G$ we have $[g_1, g_2] \in H$, and by a straightforward computation, we can conjugate G so that every element in this conjugate group is induced by a matrix of the form

$$\begin{pmatrix} \boldsymbol{g}_{11} & 0 & 0 \\ 0 & \boldsymbol{g}_{22} & 0 \\ 0 & \boldsymbol{g}_{32} & 1 \end{pmatrix}.$$

Moreover, G contains a finite index abelian subgroup, G_0, generated by the elements

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & b_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \omega b_1 & 1 \end{pmatrix}, \quad \begin{pmatrix} \zeta & 0 & 0 \\ 0 & 1 & 0 \\ 0 & y & 1 \end{pmatrix},$$

where $b_1 \in \mathbb{C} \setminus \{0\}$, $\omega \in \mathbb{C} \setminus \mathbb{R}$ and $|\zeta| < 1$. It follows that $\mathrm{Eq}(G) = \mathrm{Eq}(G_0) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_3} \cup \overleftrightarrow{e_2, e_3})$. Therefore, $\lambda(\mathrm{Eq}(G)) = \lambda(\mathrm{Eq}(G_0)) = 2$.

If U is an open set where G acts properly and discontinuously, then $\overleftrightarrow{e_1, e_3} = L_0(G_0) \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$. If $\mu(U) = 2$, there exists another line ℓ passing through e_3 such that $\ell \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$. By hypothesis, $\Pi(G)$ is an elementary group of two limit points, it follows that $\overleftrightarrow{e_2, e_3} \subset \overline{G \cdot \ell} \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$. Hence,

$$U \subset \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_3} \cup \overleftrightarrow{e_2, e_3}) = \mathrm{Eq}(G). \tag{10}$$

In the case when $U = \Omega(G)$, it is known that $\mathrm{Eq}(G) \subset \Omega(G) = U$; hence

$$U = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_3} \cup \overleftrightarrow{e_2, e_3}).$$

Therefore, $\lambda(U) = 2$.

In the case when $U = U_{\max}(G)$, it follows from (10) that $U = \mathrm{Eq}(G)$, and $\lambda(U) = 2$.

The case when H has rank one is analogous and we omit it. Finally, when $H = \{\mathrm{Id}\}$, the group is a finite extension of a cyclic group generated by a loxodromic element. It follows that $\mathrm{Eq}(G) = \Omega(G)$ is the complement of two lines in $\mathbb{P}^2_{\mathbb{C}}$. On the other hand, every maximal open set, U, where the action of G is properly discontinuous, is equal to the complement of one line and one point or the complement of one single line. □

Lemma 5.13. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $\varnothing \neq U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 2$, $\lambda(U) = \infty$, and $\Pi(G)$ is elementary with one limit point, then there exists a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Proof. Since $\Pi(G)$ is elementary with one fixed point, we can assume that $\overleftrightarrow{e_2, e_3}$ is a G-invariant line, and every element in G is induced by a unique matrix of the form

$$\begin{pmatrix} g_{11} & 0 & 0 \\ g_{21} & g_{11} & 0 \\ g_{31} & g_{32} & 1 \end{pmatrix}. \tag{11}$$

Now, the rank of the finite index abelian subgroup $H \subset \mathrm{Ker}(\Pi)$ (see Lemmas 5.9 and 5.10) is at most 2.

If the rank of H is at most 2 and the hypotheses of Lemma 5.11 are not satisfied, then we can assume, conjugating with the appropriate matrix, that every element of H is induced by a matrix of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ h_{31} & 0 & 1 \end{pmatrix}. \tag{12}$$

(Alternatively, it can be of the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & h_{32} & 1 \end{pmatrix},$$

but the proof is analogous.) It follows that H acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_2, e_3}$, so $\mathrm{Ker}(\Pi)$ acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_2, e_3}$.

Now, we prove that G acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_2, e_3}$. Let (g_n) be a sequence of distinct elements of G such that $g_n(C) \cap D \neq \varnothing$ for some compact subsets C, $D \subset \mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_2, e_3}$. Since $\Pi(G)$ is elementary with one limit point, we can assume that $\Pi(g_n) = \Pi(g_1)$ for every $n \in \mathbb{N}$, so $g_1^{-1} g_n \in \mathrm{Ker}(\Pi)$ for every $n \in \mathbb{N}$. It follows that

$$g_1^{-1} g_n(C) \cap g_1^{-1}(D) \neq \varnothing,$$

contradicting the fact that $\mathrm{Ker}(\Pi)$ acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_2, e_3}$.

It follows, by [Barrera et al. 2014a, Theorem 1.2], that $\lambda(U) \leq 2$, a contradiction. Hence, the rank of H is equal to 2 and hypotheses of Lemma 5.11 are satisfied. Therefore, $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains a perfect set of lines. □

Lemma 5.14. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $\varnothing \neq U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) = 2$ and $\Pi(G)$ is finite, then there exists a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Proof. Since $\Pi(G)$ is finite, it follows that $\mathrm{Ker}(\Pi)$ has finite index in G. By Lemma 5.9, H has finite index in G. Thus,

$$\Omega(G) = \Omega(H) = \mathrm{Eq}(H) = \mathrm{Eq}(G)$$

By Lemma 5.10, H has rank at most 2. First, we assume that H has rank 1. Then $\mu(\Omega(H)) = \mu(\mathrm{Eq}(H)) = 1$, and $\Omega(H)$ is a maximal open subset where G acts properly and discontinuously, contradicting the hypothesis $\mu(U) = 2$. Therefore, H has rank 2.

In the case when H does not satisfy the hypotheses of Lemma 5.11, $\Omega(H) = \mathrm{Eq}(H)$ is the complement of one line in $\mathbb{P}^2_{\mathbb{C}}$, and again $\Omega(H)$ is a maximal open set where G acts properly and discontinuously, contradicting the hypothesis $\mu(U) = 2$.

It follows that H has rank 2, and it satisfies the hypotheses of Lemma 5.11. Therefore, $\mathbb{P}^2_{\mathbb{C}} \setminus U$ contains a perfect set of lines. □

Proof of Proposition 5.7. If $\mu(U) = 2$ then the result is obtained by applying Lemmas 5.8, 5.12, 5.13, and 5.14.

Now, we consider the case when $\mu(U) = 3$ and $\lambda(U) = \infty$; then there exists a point p and a line ℓ not passing through p such that $G \cdot p = p$ and $G \cdot \ell = \ell$. We can assume that $p = e_3$ and $\ell = \overleftrightarrow{e_1, e_2}$, so every element $g \in G$ is induced by a matrix of the form

$$g = \begin{pmatrix} g_{11} & g_{12} & 0 \\ g_{21} & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix} \tag{13}$$

Since $\mu(U) = 3$, by Theorem 3.4(iii), we obtain $U \subset \mathrm{Eq}(G)$. It follows that

$$U = \mathrm{Eq}(G),$$

whenever U is a maximal open set where G acts properly and discontinuously or U is Kulkarni domain of discontinuity. Hence, it suffices to prove Proposition 5.7 for $U = \mathrm{Eq}(G)$.

If we set, as before, $\Pi = \Pi_{e_3,\overleftrightarrow{e_1,e_2}} : G \to \mathrm{Bihol}(\overleftrightarrow{e_1, e_2})$, then we consider the subcases depending on whether $\Pi(G)$ is not elementary or not discrete, elementary with two limit points, elementary with one limit point, or finite.

In the case when $\Pi(G)$ is not elementary or not discrete, one can prove that there exists a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ as in the proof of Lemma 5.8.

In the case when $\Pi(G)$ is finite, $\mathrm{Ker}(\Pi)$ is a finite index subgroup of G, then

$$\mathrm{Eq}(G) = \mathrm{Eq}(\mathrm{Ker}(\Pi)).$$

Since every element in $\mathrm{Ker}(\Pi)$ is induced by a matrix of the form

$$\begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{11} & 0 \\ 0 & 0 & g_{33} \end{pmatrix},$$

it follows that $\mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(\mathrm{Ker}(\Pi))$ is at most $\overleftrightarrow{e_1, e_2} \cup \{e_3\}$, contradicting the hypothesis. Therefore, $\Pi(G)$ cannot be finite.

If $\Pi(G)$ is Euclidean — i.e., elementary with one limit point — then there exists a finite index subgroup of G such that every element of this subgroup can be induced by a matrix of the form

$$\begin{pmatrix} 1 & g_{12} & 0 \\ 0 & 1 & 0 \\ 0 & 0 & g_{33} \end{pmatrix}.$$

It follows that $\lambda(\mathrm{Eq}(G)) \leq 2$, contradicting the hypothesis. Therefore, $\Pi(G)$ cannot be Euclidean.

If $\Pi(G)$ is a two limit points elementary group, then there exists a finite index subgroup of G such that every element of this subgroup can be induced by a matrix of the form

$$\begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}.$$

It follows that $\lambda(\mathrm{Eq}(G)) \leq 3$, contradicting the hypothesis. Therefore, $\Pi(G)$ cannot be elementary with two limit points. □

Some examples of groups as in the statement of Proposition 5.7 are given in Sections 7B and 7C.

Proposition 5.15. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and $U \subset \mathbb{P}^2_{\mathbb{C}}$ be equal to one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If $\mu(U) \geq 4$ then $\lambda(U) = \infty$. Moreover $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is the union of a perfect set of lines, so there are uncountably many lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.*

Proof. If $U = \Omega(G)$ then $U = \mathrm{Eq}(G)$, by [Barrera et al. 2011a, Theorem 3.6]. If $U = U_{\max}(G)$. Thus, $U = \mathrm{Eq}(G)$ by maximality. Hence, it suffices to prove the statement for $U = \mathrm{Eq}(G)$. If $\Lambda_{\mathrm{Myr}}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}}$ then there is nothing to prove, so we assume $\Lambda_{\mathrm{Myr}}(G) \neq \mathbb{P}^2_{\mathbb{C}}$.

Finally, we prove that $\mathcal{E}(G)$ is a perfect set. By Proposition 4.2, $\mathcal{E}(G)$ is closed. Moreover, if ℓ is an effective line for G, then $\ell = \mathrm{Ker}(S)$ for some $S \in G'$ so it follows from in [op. cit., Lemma 3.2(3)] that there is a sequence of distinct effective lines accumulating at ℓ (because the maximum number of effective lines for G is at least 4, by Remark 4.7). □

Proof of Theorem 1.1. First of all, $\lambda(U) \geq 1$ because the complement of an open subset of $\mathbb{P}^2_{\mathbb{C}}$, where the infinite discrete group $G \subset \mathrm{PSL}(3, \mathbb{C})$ acts properly and discontinuously, always contains a line.

- If $\mu(U) = 1$ then $\lambda(U) = 1$.
- If $\mu(U) = 2$ then there are two subcases depending on whether $\lambda(U) < \infty$ or $\lambda(U) = \infty$. If $\lambda(U) < \infty$ then $\lambda(U) = 2$, by Proposition 5.4. In the other case, $\lambda(U) = \infty$ and Proposition 5.7 implies that there exists a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.
- If $\mu(U) = 3$ then there are two subcases depending on whether $\lambda(U) < \infty$ or $\lambda(U) = \infty$. If $\lambda(U) < \infty$ then $\lambda(U) = 3$, by Proposition 5.6. In the other case, by Proposition 5.7, there is a perfect set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$.
- If $\mu(U) \geq 4$ then Proposition 5.15 implies that $\lambda(U) = \infty$ and $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is the union of a perfect set of lines. □

6. Proof of Theorem 1.2

Lemma 6.1. *Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be a discrete subgroup and ℓ a G-invariant line such that the action of G restricted to ℓ is trivial, then*:

(i) *If there is an element in G with infinite order and a diagonalizable lift, then G is conjugate to a subgroup of* $\mathrm{PSL}(3, \mathbb{C})$ *such that every element has a lift to* $\mathrm{SL}(3, \mathbb{C})$ *of the form*

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}, \quad \text{where } a \in \mathbb{C}^*.$$

(ii) *If G does not contain an element with infinite order and diagonalizable lift, then G is conjugate to a subgroup of* $\mathrm{PSL}(3,\mathbb{C})$ *such that every element has a lift to* $\mathrm{SL}(3,\mathbb{C})$ *of the form*

$$\begin{pmatrix} a & 0 & b \\ 0 & a & c \\ 0 & 0 & a^{-2} \end{pmatrix}, \quad \textit{where } |a| = 1.$$

Proof. (i) After conjugating with a projective transformation, we can assume that $\ell = \overleftrightarrow{e_1, e_2}$ and there exists $g_0 \in G$ with a lift $\boldsymbol{g}_0 \in \mathrm{SL}(3,\mathbb{C})$ of the form

$$\begin{pmatrix} a_0 & 0 & 0 \\ 0 & a_0 & 0 \\ 0 & 0 & a_0^{-2} \end{pmatrix}, \quad \text{where } |a_0| < 1.$$

On the other hand, each element $g \in G$ has a lift $\boldsymbol{g} \in \mathrm{SL}(3,\mathbb{C})$ of the form

$$\begin{pmatrix} \boldsymbol{g}_{11} & 0 & \boldsymbol{g}_{13} \\ 0 & \boldsymbol{g}_{11} & \boldsymbol{g}_{23} \\ 0 & 0 & \boldsymbol{g}_{11}^{-2} \end{pmatrix}. \tag{14}$$

If $\boldsymbol{g}_{13} \neq 0$ or $\boldsymbol{g}_{23} \neq 0$ for some $g \in G$, then Lemma 4.4 implies that G is not discrete. Therefore, $\boldsymbol{g}_{13} = 0 = \boldsymbol{g}_{23}$ for every $g \in G$.

(ii) As before, we can assume that every $g \in G$ has a lift $\boldsymbol{g} \in \mathrm{SL}(3,\mathbb{C})$ as in (14). If for some $g \in G$ we assume that $|\boldsymbol{g}_{11}| \neq 1$, then $\boldsymbol{g}_{11} \neq \boldsymbol{g}_{11}^{-2}$ and $\boldsymbol{g}$ is diagonalizable, so we have a contradiction. □

Proposition 6.2. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be an infinite discrete subgroup and ℓ a G-invariant line such that G acts trivially on ℓ, then there exists a point p such that*

$$\mathrm{Eq}(G) = \ell \cup \{p\}.$$

Proof. By Lemma 6.1 we have two cases according to whether there is an element in G of infinite order with a diagonalizable lift or there is not such an element in G. In the first case, $\mathrm{Eq}(G) = \ell \cup \{p\}$ where p is the isolated fixed point of any element in G of infinite order. In the second case, $\mathrm{Eq}(G) = \ell$. □

Lemma 6.3. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete subgroup such that each element $g \in G$ has a lift $\boldsymbol{g} \in \mathrm{SL}(3,\mathbb{C})$ of the form*

$$\begin{pmatrix} \boldsymbol{g}_{11} & 0 & 0 \\ 0 & \boldsymbol{g}_{22} & \boldsymbol{g}_{23} \\ 0 & \boldsymbol{g}_{32} & \boldsymbol{g}_{33} \end{pmatrix}.$$

Then the maximum number of effective lines for G in general position is at most 3. *In particular, if* $\Lambda_{\mathrm{Myr}}(G) \neq \mathbb{P}^2_{\mathbb{C}}$, *then*

$$\mu(\mathrm{Eq}(G)) \leq 3.$$

Proof. If $(g_n) \subset G$ is a sequence of distinct elements in G such that $g_n \to S$ in $\mathrm{SP}(3, \mathbb{C})$ as $n \to \infty$, then S is induced by a matrix of the form

$$\mathbf{0} \neq s = \begin{pmatrix} s_{11} & 0 & 0 \\ 0 & s_{22} & s_{23} \\ 0 & s_{32} & s_{33} \end{pmatrix}.$$

Since G is discrete,

$$s_{11}(s_{22}s_{33} - s_{23}s_{32}) = 0.$$

Hence, $\mathrm{Ker}(S)$ is equal to the point e_1, a line passing through e_1, the line $\overleftrightarrow{e_2, e_3}$, or a point in $\overleftrightarrow{e_2, e_3}$. □

Definition 6.4. If $U \subset \mathbb{P}^2_{\mathbb{C}}$ is an open set and $\mathcal{L}$ is a set of lines in general position contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$, then we say the point $v \in \mathbb{P}^2_{\mathbb{C}} \setminus U$ is a *vertex for* U *and* $\mathcal{L}$ whenever:

- There are infinitely many lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ passing through v.
- There exist two distinct lines ℓ_1, ℓ_2 in $\mathcal{L}$ passing through v.

Proposition 6.5. *Let* $G \subset \mathrm{PSL}(3, \mathbb{C})$ *be a discrete group and* $U \subset \mathbb{P}^2_{\mathbb{C}}$ *be one of* $\mathrm{Eq}(G)$, $\Omega(G)$, *or* $U_{\max}(G)$. *If* $4 \leq \mu(U) < \infty$ *then, for each set* $\mathcal{L}$ *consisting of lines in general position contained in* $\mathbb{P}^2_{\mathbb{C}} \setminus U$ *such that* $|\mathcal{L}| = \mu(U)$:

(i) *There are precisely two vertices for* U *and* $\mathcal{L}$.

(ii) *If* ℓ *is a line not containing vertices for* U, *then the* G*-orbit of* ℓ *is infinite.*

Proof. If U is equal to $\Omega(G)$ or $U_{\max}(G)$, then the hypothesis $\mu(U) \geq 4$ and Theorem 3.4(iii) imply that $U = \mathrm{Eq}(G)$. Hence it suffices to prove the lemma in the case $U = \mathrm{Eq}(G)$.

(i) Since $|\mathcal{L}| = \mu(U) < \infty$, every line contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ passes through an intersection point of lines in $\mathcal{L}$. By Proposition 5.15, $\lambda(U) = \infty$. Hence, *there exists at least one vertex for* U *and* $\mathcal{L}$.

Given that $\mu(U) < \infty$, the set of vertices for U and $\mathcal{L}$ is finite and it is G-invariant. It follows that *the isotropy subgroup of any vertex for* U *and* $\mathcal{L}$ *has finite index in* G.

Since $\mu(U) \geq 4$, $\mathbb{P}^2_{\mathbb{C}} \setminus U$ is a union of lines (see Corollary 4.5). If we assume that there is only one vertex for U and $\mathcal{L}$ then

$$\mathbb{P}^2_{\mathbb{C}} \setminus U = \Big(\bigcup_{\ell \in \mathcal{B}} \ell\Big) \cup \Big(\bigcup_{\ell \in \mathcal{A}} \ell\Big),$$

where $\mathcal{B}$ is the closed set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ passing through the vertex, and $\mathcal{A}$ is the set of lines contained in $\mathbb{P}^2_{\mathbb{C}} \setminus U$ not passing through the vertex.

Since there is only one vertex, the G-invariant set $\mathcal{A}$ contains finitely many lines. If $\ell_0 \in \mathcal{A}$ then the subgroup $G_0 = \mathrm{Isot}(\ell_0, G)$ has finite index in G, so the open set $U_0 = \mathbb{P}^2_{\mathbb{C}} \setminus \left(\left(\bigcup_{\ell \in \mathcal{B}} \ell\right) \cup \ell_0\right)$ is G_0-invariant. It follows from Theorem 3.4 iii) that

$$U_0 \subset \mathrm{Eq}(G_0) = \mathrm{Eq}(G).$$

Thus, $3 = \mu(U_0) = \mu(\mathrm{Eq}(G)) \geq 4$, a contradiction. Therefore, *there exist at least two vertices for U and $\mathcal{L}$.*

If we assume that the vertices for U and $\mathcal{L}$ do not lie on a line, then there are three vertices of U and $\mathcal{L}$ in general position. Moreover, we can assume that $\{e_1, e_2, e_3\}$ are those vertices. Thus every element in the finite index subgroup

$$G_1 = \bigcap_{j=1}^{3} \mathrm{Isot}(e_j, G) \subset G$$

is induced by a diagonal matrix. It follows from Lemma 6.3 that

$$3 \geq \mu(\mathrm{Eq}(G_1)) = \mu(\mathrm{Eq}(G)) \geq 4,$$

a contradiction. Therefore, *the vertices for U and $\mathcal{L}$ lie in a complex line.*

If we assume that there are more than two vertices for U and $\mathcal{L}$ then there exist three distinct vertices, v_1, v_2, v_3 for U and $\mathcal{L}$ contained in a line ℓ. The finite index subgroup

$$G_1 = \bigcap_{j=1}^{3} \mathrm{Isot}(v_j, G) \subset G$$

fixes three distinct points in the line ℓ, so it acts trivially on ℓ. It follows from Proposition 6.2 that

$$\mu(\mathrm{Eq}(G_1)) = 1,$$

contradicting the fact that $\mathrm{Eq}(G_1) = \mathrm{Eq}(G)$ and $\mu(\mathrm{Eq}(G)) \geq 4$. Therefore, *there are precisely two vertices for U and $\mathcal{L}$.*

(ii) If we assume there exists a line ℓ_0 with finite G-orbit and not passing through any vertex v_1 or v_2 for U, then

$$G_2 = \mathrm{Isot}(\ell_0, G) \cap \mathrm{Isot}(v_1, G) \cap \mathrm{Isot}(v_2, G)$$

is a finite index subgroup of G fixing the points

$$v_1,\ v_2,\ \ell_0 \cap \overleftrightarrow{v_1, v_2}.$$

Thus, G_2 acts trivially on $\overleftrightarrow{v_1, v_2}$, so $\mathrm{Eq}(G) = \mathrm{Eq}(G_2)$ is the complement of the union of a line and a point (by Proposition 6.2), contradicting the hypothesis that $\mu(\mathrm{Eq}(G)) \geq 4$. □

Proof of Theorem 1.2. On the contrary, let us assume that $4 < \mu(U) < \infty$. Then there is a finite set of lines in general position, $\mathcal{L}$, such that $\ell \subset \mathbb{P}^2_{\mathbb{C}} \setminus U$ for every $\ell \in \mathcal{L}$, and $|\mathcal{L}| = \mu(U)$. By Proposition 6.5(i), there are precisely two vertices for U and $\mathcal{L}$. Let us denote by v_1 and v_2 these two vertices. Since $\mu(U) > 4$, there is a line in $\mathcal{L}$ not passing through v_1 nor v_2. By Proposition 6.5(ii), this line has infinite G-orbit, then there is another vertex for U and $\mathcal{L}$ distinct from v_1, v_2, a contradiction. Therefore, $\mu(U)$ is equal to 1, 2, 3, 4, or ∞. In Sections 7A to 7E we give examples of infinite discrete subgroups $G \subset \mathrm{PSL}(3, \mathbb{C})$ with corresponding open sets U satisfying $\mu(U) \in \{1, 2, 3, 4, \infty\}$. □

7. Examples

7A. *One line complex Kleinian groups.* (i) Suppose that G is the cyclic subgroup of $\mathrm{PSL}(3, \mathbb{C})$ generated by a complex homothety g_0 induced by a matrix of the form

$$\begin{pmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-2} \end{pmatrix}, \quad \text{where } 0 < |a| < 1.$$

Then

$$\Omega(G) = \mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \{e_3\})$$

is the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. Hence we have the table:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	1	1	1
μ	1	1	1

(ii) If $G \subset \mathrm{PGL}(3, \mathbb{C})$ is the cyclic group generated by the loxoparabolic element induced by the matrix

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad 0 < |a| < 1,$$

then

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_1, e_3}) = \mathrm{Eq}(G)$$

However, G acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_2}$; see [Barrera et al. 2014a, Example 2.3]. Moreover, $U_{\max} = \mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_2}$ is the maximal open subset of

$\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. Hence we have the table:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	2	2	1
μ	2	2	1

(iii) The abelian group G, generated by two projective transformations induced by the matrices in GL(3, ℂ)

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & a & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } 0 < |a| < 1,$$

satisfies the property that

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus \overleftrightarrow{e_1, e_2}$$

is the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$, where G acts properly and discontinuously. On the other hand,

$$\mathrm{Eq}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_1, e_3}),$$

and we have the table:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	1	2	1
μ	1	2	1

Those subgroups of PGL(3, ℂ) whose Kulkarni limit set is equal to one line are classified in [Barrera et al. 2014a, Theorem 1.1].

If $\Lambda_{\mathrm{Kul}}(G)$ is equal to one line then $\Omega(G)$ is a maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. (If an infinite subgroup $G \subset \mathrm{PGL}(3, \mathbb{C})$ acts properly and discontinuously on the open set $U \subset \mathbb{P}^2_{\mathbb{C}}$ then $\lambda(U) \geq 1$.)

Conversely, if $\mathbb{P}^2_{\mathbb{C}} \setminus \ell$ is maximal open set where G acts properly and discontinuously, then $\Lambda_{\mathrm{Kul}}(G)$ is equal to one line, except in the case when G contains a cyclic subgroup of finite index generated by a loxoparabolic element; see [op. cit., Theorem 1.2].

In the case when $\Lambda_{\mathrm{Myr}}(G)$ is equal to one line, ℓ, then G does not contain loxoparabolic elements and it acts properly and discontinuously on $\mathbb{P}^2_{\mathbb{C}} \setminus \ell$, then $\Lambda_{\mathrm{Kul}}(G) = \ell$ by the same theorem.

7B. ***Two line complex Kleinian groups.*** In this section we give some examples of complex Kleinian groups such that $\mu(\Omega(G)) = 2$, $\mu(\mathrm{Eq}(G)) = 2$, or $\mu(U_{\max}(G)) = 2$.

(i) If $g \in \mathrm{PGL}(3, \mathbb{C})$ is an element induced by a matrix of the form

$$g = \begin{pmatrix} g_{11} & 0 & 0 \\ 0 & g_{22} & 0 \\ 0 & 0 & g_{33} \end{pmatrix}, \quad \text{where } |g_{11}| < |g_{22}| < |g_{33}|,$$

then the cyclic group $G = \langle g \rangle$ satisfies

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3}) = \mathrm{Eq}(G).$$

On the other hand, $U_{\max}(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \{e_3\})$ is a maximal open set where G acts properly and discontinuously. Thus we have:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	2	2	1
μ	2	2	1

(ii) Let $G \subset \mathrm{PSL}(3, \mathbb{C})$ be the group induced by matrices of the form:

$$\begin{pmatrix} 1 & m & n \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } m, n \in \mathbb{Z}.$$

This group contains only parabolic elements (except for the identity) and satisfies that $U = \Omega(G) = \mathrm{Eq}(G)$ is the maximal open set where G acts properly and discontinuously. Moreover, we have:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	∞	∞	∞
μ	2	2	2

(iii) *The double suspension construction*. Given a subgroup $G \subset \mathrm{PSL}(2, \mathbb{C})$ we can construct a new group $\hat{G} \subset \mathrm{PSL}(3, \mathbb{C})$, called *the double suspension of* G, acting on $\mathbb{P}^2_{\mathbb{C}}$ in such way that the restriction of this action to the line at infinity is the action of G on $\mathbb{P}^1_{\mathbb{C}} \cong S^2$. See [Navarrete 2008; Seade and Verjovsky 2001]. The elements in $\hat{G}$ are represented by all matrices of the form

$$\begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \text{where } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2, \mathbb{C}) \text{ induces an element in } G.$$

In other words, $\hat{G}$ is a double covering of G. Moreover, when G is a classical Kleinian group with limit set $L(G)$, then $\Lambda_{\mathrm{Kul}}(\hat{G}) = \Lambda_{\mathrm{Myr}}(\hat{G})$ is equal to the complex cone with vertex e_3 and base $L(G)$ (considered as a subset of the line at infinity $\overleftrightarrow{e_1, e_2}$). In symbols,

$$\Lambda_{\mathrm{Kul}}(\hat{G}) = \Lambda_{\mathrm{Myr}}(\hat{G}) = \bigcup_{x \in L(G)} \overleftrightarrow{e_3, x}.$$

If G is a nonelementary classical Kleinian group, then $\Omega(\hat{G})$ is the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where $\hat{G}$ acts properly and discontinuously. Thus:

	$\Omega(\hat{G})$	$\mathrm{Eq}(\hat{G})$	$U_{\max}(\hat{G})$
λ	∞	∞	∞
μ	2	2	2

7C. ***Three line complex Kleinian groups.*** (i) Let G be the group generated by $A, B \in \mathrm{PSL}(3, \mathbb{C})$ where A and B are induced by the matrices

$$\boldsymbol{A} = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad \boldsymbol{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

Then

$$\Omega(G) = \mathbb{P}^2_{\mathbb{C}} \setminus (\overleftrightarrow{e_1, e_2} \cup \overleftrightarrow{e_2, e_3} \cup \overleftrightarrow{e_3, e_1}) = \mathrm{Eq}(G)$$

is a maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously; see [Barrera et al. 2011a, Example 4.3]. It follows that:

	$\Omega(G)$	$\mathrm{Eq}(G)$	$U_{\max}(G)$
λ	3	3	3
μ	3	3	3

(ii) [Cano et al. 2013, Subsection 5.5.1] If $G \subset \mathrm{PSL}(2, \mathbb{C})$ is a Kleinian group and $D \subset \mathbb{C}^*$ a discrete subgroup, then the *suspension of G extended by the group D*, denoted by $\mathrm{Susp}(G, D)$ is the group generated by the double suspension and all the elements in $\mathrm{PSL}(3, \mathbb{C})$ induced by diagonal matrices of the form:

$$\begin{pmatrix} d & 0 & 0 \\ 0 & d & 0 \\ 0 & 0 & d^{-2} \end{pmatrix}, \quad \text{where } d \in D.$$

In the case when $G \subset \mathrm{PSL}(2, \mathbb{C})$ is a nonelementary Kleinian group and $D \subset \mathbb{C}^*$ is an infinite discrete subgroup,

$$\Lambda_{\mathrm{Kul}}(\mathrm{Susp}(G, D)) = \overleftrightarrow{e_1, e_2} \cup \left(\bigcup_{x \in L(G)} \overleftrightarrow{x, e_3} \right) = \Lambda_{\mathrm{Myr}}(\mathrm{Susp}(G, D)),$$

and $\Omega(\mathrm{Susp}(G,D)) = \mathrm{Eq}(\mathrm{Susp}(G,D))$ is the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where $\mathrm{Susp}(G,D)$ acts properly discontinuously. Therefore:

	$\Omega(\mathrm{Susp}(G,D))$	$\mathrm{Eq}(\mathrm{Susp}(G,D))$	$U_{\max}(\mathrm{Susp}(G,D))$
λ	∞	∞	∞
μ	3	3	3

7D. ***Four line complex Kleinian groups.*** See [Barrera et al. 2011b].

An element $A \in \mathrm{SL}(2,\mathbb{Z})$, is called a *hyperbolic toral automorphism* if none of its eigenvalues lie on the unit circle. Any subgroup of $\mathrm{PSL}(3,\mathbb{C})$ conjugate to the group

$$G_A = \left\{ \begin{pmatrix} A^k & \boldsymbol{b} \\ \boldsymbol{0} & 1 \end{pmatrix} : \boldsymbol{b} \in M(2\times 1,\mathbb{Z}),\ k \in \mathbb{Z} \right\},$$

where $A \in \mathrm{SL}(2,\mathbb{Z})$ is a hyperbolic toral automorphism, is called a *hyperbolic toral group*.

Theorem 7.1 [Barrera et al. 2011b]. *Let $G \subset \mathrm{PSL}(3,\mathbb{C})$ be a discrete group. The maximum number of complex lines in general position contained in Kulkarni's limit set is equal to four if and only if G contains a hyperbolic toral group whose index is at most eight.*

Furthermore, it is proved that $\Omega(G) = \mathrm{Eq}(G)$. However $\Omega(G)$ is not the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously. It can be shown that there exist two open maximal sets $U^{(1)}_{\max}$, $U^{(2)}_{\max} \subset \mathbb{P}^2_{\mathbb{C}}$ where G acts properly and discontinuously.

For the reader's convenience, we give a brief outline of the proof that the group G_A has the properties mentioned above. Let $A \in \mathrm{SL}(2,\mathbb{Z})$ be a hyperbolic toral automorphism. We define

$$T = \begin{pmatrix} \boldsymbol{t} & \boldsymbol{0} \\ 0 & 1 \end{pmatrix}, \quad \text{where } \boldsymbol{t} = \begin{pmatrix} 1 & u \\ v & 1 \end{pmatrix} \text{ with } u, v \in \mathbb{R} - \mathbb{Q}.$$

Hence the group $\hat{G}_A = T G_A T^{-1}$ is equal to

$$\begin{pmatrix} \alpha^n & 0 & ky_0 + lx_0 \\ 0 & \alpha^{-n} & kx_0 + lz_0 \\ 0 & 0 & 1 \end{pmatrix}$$

where $k, l, n \in \mathbb{Z}$ and

$$x_0 = \frac{-1}{uv-1}, \quad y_0 = \frac{u}{uv-1}, \quad z_0 = \frac{v}{uv-1}.$$

By straightforward computations, the Kulkarni limit set is

$$\overleftrightarrow{e_1, e_2} \cup \bigcup_{r\in\mathbb{R}} \overleftrightarrow{e_1, [0:r:1]} \cup \overleftrightarrow{e_2, [r:0:1]}$$

It is not hard to check that the set of vertices is $\{e_1, e_2\}$ and $\mu(\hat{G}_A) = 4$, which implies that $\Omega(\hat{G}_A) = \text{Eq}(\hat{G}_A)$. The group $\hat{G}_A$ acts properly and discontinuously on $U^{(i)}_{\max}(\hat{G}_A) = \mathbb{P}^2_{\mathbb{C}} - \mathcal{C}_i$, $i = 1, 2$, where

$$\mathcal{C}_1 = \overleftrightarrow{e_1, e_2} \cup \bigcup_{r\in\mathbb{R}} \overleftrightarrow{e_1, [0 : r : 1]} \quad \text{and} \quad \mathcal{C}_2 = \overleftrightarrow{e_1, e_2} \cup \bigcup_{r\in\mathbb{R}} \overleftrightarrow{e_2, [r : 0 : 1]}$$

are maximal regions where the group $\hat{G}_A$ acts properly and discontinuously. In summary, we have the following table:

	$\Omega(G)$	$\text{Eq}(G)$	$U^{(1)}_{\max}(G)$	$U^{(2)}_{\max}(G)$
λ	∞	∞	∞	∞
μ	4	4	2	2

7E. ***Complex Kleinian groups with infinitely many lines.*** (i) If $G \subset \text{PU}(2, 1)$ then it is proved in [Navarrete 2006] that

$$\Lambda_{\text{Kul}}(G) = \bigcup_{z\in L(G)} \ell_z,$$

where $L(G) \subset \partial\mathbb{H}^2_{\mathbb{C}} = S^3$ denotes the Chen–Greenberg limit set of G considered as acting on $\mathbb{H}^2_{\mathbb{C}}$ by holomorphic isometries, and ℓ_z is the only tangent line to $\partial\mathbb{H}^2_{\mathbb{C}}$ at z. The Kulkarni region of discontinuity, $\Omega(G)$, is the maximal open subset where G acts properly and discontinuously, whenever $\lambda(\Omega(G)) > 2$. Moreover, it is proved in [Cano and Seade 2010] that

$$\Lambda_{\text{Myr}}(G) = \bigcup_{z\in L(G)} \ell_z.$$

In the case when G satisfies $L(G) = \partial\mathbb{H}^2_{\mathbb{C}}$,

$$\Omega(G) = \mathbb{H}^2_{\mathbb{C}} = \text{Eq}(G)$$

is the maximal open subset where G acts properly and discontinuously. Thus we have the table:

	$\Omega(G)$	$\text{Eq}(G)$	$U_{\max}(G)$
λ	∞	∞	∞
μ	∞	∞	∞

If $G \subset \text{PU}(2, 1)$ satisfies the property that $L(G)$ is an $\mathbb{R}$-circle, then we have the same table as above; see [Cano et al. $\geq$ 2016].

(ii) In [Barrera et al. 2014b], it is shown that a family of complex Kleinian groups $G_n \subset \text{PSL}(3, \mathbb{R})$ exists, such that for all $n \in \mathbb{N}$, G_n is a free group, not conjugate in $\text{PSL}(3, \mathbb{C})$ to any subgroup of $\text{PU}(2, 1)$. G_n has no invariant lines nor fixed points.

The Kulkarni limit set $\Lambda_{\mathrm{Kul}}(G_n)$ contains at least five complex projective lines in general position. Hence it contains infinitely many complex projective lines in general position. Moreover, $\Omega(G_n) = \mathrm{Eq}(G_n)$ is the maximal open subset of $\mathbb{P}^2_{\mathbb{C}}$, where G_n acts properly and discontinuously. Thus we have the table:

	$\Omega(G_n)$	$\mathrm{Eq}(G_n)$	$U_{\max}(G_n)$
λ	∞	∞	∞
μ	∞	∞	∞

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