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**GALOIS THEORY, FUNCTIONAL
LINDEMANN–WEIERSTRASS, AND MANIN MAPS**

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We prove several new results of Ax–Lindemann type for semiabelian varieties over the algebraic closure K of $\mathbb{C}(t)$, making heavy use of the Galois theory of logarithmic differential equations. Using related techniques, we also give a generalization of the theorem of the kernel for abelian varieties over K . This paper is a continuation of earlier work by Bertrand and Pillay (2010), as well as an elaboration on the methods of Galois descent introduced by Bertrand (2009, 2011) in the case of abelian varieties.

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1. Introduction

This paper has three related themes, the common feature being differential Galois theory and its applications.

Firstly, given a semiabelian variety B over the algebraic closure K of $\mathbb{C}(t)$, a K -rational point a of the Lie algebra LG of its universal vectorial extension $G = \widetilde{B}$, and a solution $y \in G(K^{\text{diff}})$ of the logarithmic differential equation

$$\partial \ell n_G(y) = a, \quad a \in LG(K),$$

we want to describe $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp)$ in terms of gauge transformations over K itself. Here K_G^\sharp is the differential field generated over K by solutions of $\partial \ell n_G(-) = 0$ in K^{diff} . Introducing this field as base presents both advantages and difficulties. On the one hand, it allows us to use the differential Galois theory developed by

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Pillay [1998; 1997; 2004], thereby replacing the study of transcendence degrees by the computation of a Galois group. On the other hand, we have only a partial knowledge of the extension K_G^\sharp/K . However, it was observed by Bertrand [2009; 2011] that in the case of an abelian variety, what we do know essentially suffices to perform a Galois descent from K_G^\sharp to the field K of the desired gauge transformation. In Sections 2B and 3 of the present paper, we extend this principle to semiabelian varieties B whose toric part is \mathbb{G}_m , and give a definitive description of $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp)$ when B is an abelian variety.

The main application we have in mind of these Galois-theoretic results forms the second theme of our paper, and concerns Lindemann–Weierstrass statements for the semiabelian variety B over K , by which we mean the description of the transcendence degree of $\exp_B(x)$ where x is a K -rational point of the Lie algebra LB of B . The problem is covered in the above setting by choosing as data

$$a := \partial_{LG}(\tilde{x}) \in \partial_{LG}(LG(K)),$$

where \tilde{x} is an arbitrary K -rational lift of x to $G = \tilde{B}$. This study was initiated in our joint paper [2010], where the Galois approach was mentioned, but only under the hypothesis that $K_G^\sharp = K$, described as K -largeness of G . There are natural conjectures in analogy with the well-known “constant” case (where B is over \mathbb{C}), although as pointed out in [Bertrand and Pillay 2010], there are also counterexamples provided by nonconstant extensions of a constant elliptic curve by the multiplicative group. In Sections 2C and 4 of this paper, we extend the main result of [Bertrand and Pillay 2010] to the base K_G^\sharp , but assuming the toric part of B is at most 1-dimensional. Furthermore, we give in this case a full solution of the Lindemann–Weierstrass statement when the abelian quotient of B is also 1-dimensional. This uses results from [Bertrand et al. 2013] which deal with the “logarithmic” case. In this direction, we will also formulate an “Ax–Schanuel” type conjecture for abelian varieties over K .

The third theme of the paper concerns the “theorem of the kernel”, which we generalize in Sections 2D and 5 by proving that linear independence with respect to $\text{End}(A)$ of points y_1, \dots, y_n in $A(K)$ implies linear independence of

$$\mu_A(y_1), \dots, \mu_A(y_n)$$

with respect to \mathbb{C} (this answers a question posed to us by Hrushovski). Here A is an abelian variety over $K = \mathbb{C}(t)^{\text{alg}}$ with \mathbb{C} -trace 0 and μ_A is the differential-algebraic Manin map. However, we will give an example showing that its \mathbb{C} -linear extension $\mu_A \otimes 1$ on $A(K) \otimes_{\mathbb{Z}} \mathbb{C}$ is not always injective. In contrast, we observe that the \mathbb{C} -linear extension $M_{K,A} \otimes 1$ of the classical (differential-arithmetic) Manin map $M_{K,A}$ is always injective. Differential Galois theory and the logarithmic case of nonconstant Ax–Schanuel are involved in the proofs.

2. Statements of results

2A. Preliminaries on logarithmic equations. We give here a quick background to the basic notions and objects so as to be able to state our main results in the next subsections. The remaining Sections 3, 4, and 5 of the paper are devoted to the proofs. We refer the reader to [Bertrand and Pillay 2010] for more details including differential algebraic preliminaries.

We fix a differential field (K, ∂) of characteristic 0 whose field of constants C_K is algebraically closed (the reader will lose nothing by taking $C_K = \mathbb{C}$). We usually assume that K is algebraically closed, and denote by K^{diff} the differential closure of K . We let \mathcal{U} denote a universal differential field containing K , with constant field \mathcal{C} . If X is an algebraic variety over K we will identify X with its set $X(\mathcal{U})$ of \mathcal{U} points.

We start with algebraic ∂ -groups, which provide the habitat of the (generalized) differential Galois theory of [Pillay 1998; 1997; 2004] discussed later on. A (connected) *algebraic ∂ -group* over K is a (connected) algebraic group G over K together with a lifting D of the derivation ∂ of K to a derivation of the structure sheaf \mathcal{O}_G which respects the group structure. The derivation D may be identified with a section s , in the category of algebraic groups, of the projection map $T_\partial(G) \rightarrow G$, where $T_\partial(G)$ denotes the twisted tangent bundle of G . This $T_\partial(G)$ is a (connected) algebraic group over K , which is a torsor under the tangent bundle TG , and is locally defined by equations

$$\sum_{i=1}^n \frac{\partial P}{\partial x_i}(\bar{x})u_i + P^\partial(\bar{x}) = 0,$$

for polynomials P in the ideal of G , where P^∂ is obtained by applying the derivation ∂ of K to the coefficients of P . Notice for later use that for any differential extension L/K , there is a group homomorphism $G(L) \rightarrow T_\partial G(L)$, which is given in coordinates by $(x_1, \dots, x_n) \mapsto (x_1, \dots, x_n, \partial x_1, \dots, \partial x_n)$ and will be denoted by ∂ .

We write the algebraic ∂ -group as (G, D) or (G, s) . Not every algebraic group over K has a ∂ -structure. But when G is defined over the constants C_K of K , there is a privileged ∂ -structure s_0 on G which is precisely the 0-section of $TG = T_\partial G$. Given an algebraic ∂ -group (G, s) over K we obtain an associated “logarithmic derivative” $\partial \ln_{G,s}(-)$ from G to the Lie algebra LG of G defined by $\partial \ln_{G,s}(y) = \partial(y)s(y)^{-1}$, where the product is computed in the algebraic group $T_\partial(G)$. This is a differential rational crossed homomorphism from G onto LG (at the level of \mathcal{U} -points or points in a differentially closed field) defined over K . Its kernel $\ker(\partial \ln_{G,s})$ is a differential algebraic subgroup of G which we denote $(G, s)^\partial$, or simply G^∂ when the context is clear. Now s equips the Lie algebra LG of G with its own structure of a ∂ -group (in this case a ∂ -module) which we call ∂_{LG} (depending on (G, s)) and again the kernel is denoted $(LG)^\partial$.

In the case where G is defined over C_K and $s = s_0$, the map $\partial \ell n_{G,s}$ is precisely Kolchin's logarithmic derivative, taking $y \in G$ to $\partial(y)y^{-1}$. In general, as soon as s is understood, we will abbreviate $\partial \ell n_{G,s}$ by $\partial \ell n_G$.

By a *logarithmic differential equation* over K on the algebraic ∂ -group (G, s) , we mean a differential equation $\partial \ell n_{G,s}(y) = a$ for some $a \in LG(K)$. When $G = \mathrm{GL}_n$ and $s = s_0$ this is the equation for a fundamental system of solutions of a linear differential equation $Y' = aY$ in vector form. And more generally, for G an algebraic group over C_K and $s = s_0$, this is a logarithmic differential equation on G over K in the sense of Kolchin. There is a well-known Galois theory here. In the given differential closure K^{diff} of K , any two solutions y_1, y_2 of $\partial \ell n_G(-) = a$ in $G(K^{\mathrm{diff}})$ differ by an element in the kernel G^∂ of $\partial \ell n_G(-)$. But $G^\partial(K^{\mathrm{diff}})$ is precisely $G(C_K)$. Hence $K(y_1) = K(y_2)$. In particular, $\mathrm{tr.deg}(K(y)/K)$ is the same for all solutions y in K^{diff} . Moreover, $\mathrm{Aut}(K(y)/K)$ has the structure of an algebraic subgroup of $G(C_K)$: for any $\sigma \in \mathrm{Aut}(K(y)/K)$, let $\rho_\sigma \in G(C_K)$ be such that $\sigma(y) = y\rho_\sigma$. Then the map taking σ to ρ_σ is an isomorphism between $\mathrm{Aut}(K(y)/K)$ and an algebraic subgroup $H(C_K)$ of $G(C_K)$, which we call the differential Galois group of $K(y)/K$. This depends on the choice of solution y , but another choice yields a conjugate of H . Of course when G is commutative, H is independent of the choice of y . In any case $\mathrm{tr.deg}(K(y)/K) = \dim(H)$, so computing the differential Galois group gives us a transcendence estimate.

Continuing with this Kolchin situation, we have the following well-known fact, whose proof we present in the setting of the more general situation considered in [Fact 2.2\(i\)](#).

Fact 2.1 (for G/C_K). *Suppose K algebraically closed. Then, $\mathrm{tr.deg}(K(y)/K)$ is the dimension of a minimal connected algebraic subgroup H of G , defined over C_K , such that for some $g \in G(K)$, $gag^{-1} + \partial \ell n_G(g) \in LH(K)$. Moreover, $H(C_K)$ is the differential Galois group of $K(y)/K$.*

Proof. Let H be a connected algebraic subgroup of G , defined over C_K such that $H^\partial(K^{\mathrm{diff}}) = H(C_K)$ is the differential Galois group of $K(y)$ over K . Now the $H^\partial(K^{\mathrm{diff}})$ -orbit of y is defined over K in the differential algebraic sense, so the H -orbit of y is defined over K in the differential algebraic sense. A result of Kolchin on constrained cohomology (see Proposition 3.2 of [\[Pillay 1998\]](#), or Theorem 2.2 of [\[Bertrand 2011\]](#)) implies that this orbit has a K -rational point g^{-1} . So, there exists $z^{-1} \in H$ such that $g^{-1} = yz^{-1}$, and $z = gy$, which satisfies $K(y) = K(z)$, is a solution of $\partial \ell n_G(-) = a'$ where $a' = gag^{-1} + \partial \ell n_G(g)$. \square

(Such a map $LG(K) \rightarrow LG(K)$ taking $a \in LG(K)$ to $gag^{-1} + \partial \ell n_G(g)$ for some $g \in G(K)$ is called a gauge transformation.)

Now in the case of an arbitrary algebraic ∂ -group (G, s) over K , and logarithmic differential equation $\partial \ell n_{G,s}(-) = a$ over K , two solutions y_1, y_2 in $G(K^{\mathrm{diff}})$ differ

by an element of $(G, s)^\partial(K^{\text{diff}})$ which in general may not be contained in $G(K)$. (For instance, if $(G = \mathbb{G}_a, s)$ is the ∂ -module attached to $\partial y - y = 0$, and $a = 1 - t$, then $y_1 = t$ is rational over $K = \mathbb{C}(t)$, while $y_2 = t + e^t$ is transcendental over K .) So to obtain both a transcendence statement independent of the choice of solution, as well as a Galois theory, we should work over $K_{G,s}^\sharp$ which is the (automatically differential) field generated by K and $(G, s)^\partial(K^{\text{diff}})$. This field may be viewed as a field of “new constants”, and its algebraic closure in K^{diff} will be denoted by $K_{G,s}^{\sharp, \text{alg}}$. As with $\partial \ln_G$ and G^∂ , we will abbreviate $K_{G,s}^\sharp$ as K_G^\sharp , or even K^\sharp , when the context is clear, and similarly for its algebraic closure.

Fixing a solution $y \in G(K^{\text{diff}})$ of $\partial \ln_G(-) = a$, for $\sigma \in \text{Aut}(K^\sharp(y)/K^\sharp)$ we have $\sigma(y) = y\rho_\sigma$ for unique $\rho_\sigma \in G^\partial(K^{\text{diff}}) = G^\partial(K^\sharp) \subseteq G(K^\sharp)$, and again the map $\sigma \mapsto \rho_\sigma$ defines an isomorphism between $\text{Aut}(K^\sharp(y)/K^\sharp)$ and $(H, s)^\partial(K^{\text{diff}})$ for an algebraic ∂ -subgroup H of (G, s) , ostensibly defined over K^\sharp . The ∂ -group H (or more properly H^∂ , or $H^\partial(K^\sharp)$) is called the (*differential*) *Galois group* of $K^\sharp(y)$ over K^\sharp , and when G is commutative does not depend on the choice of y , just on the data $a \in LG(K)$ of the logarithmic equation, and in fact only on the image of a in the cokernel $LG(K)/\partial \ln_G G(K)$ of $\partial \ln_G$. Again $\text{tr.deg}(K^\sharp(y)/K^\sharp) = \dim(H)$. In any case, [Fact 2.1](#) extends to this context with essentially the same proof. This can also be extracted from Proposition 3.4 of [\[Pillay 1998\]](#) and the setup of [\[Pillay 2004\]](#). For the commutative case (part (ii) below) see [\[Bertrand 2011, Theorem 3.2\]](#). Note that in the present paper, it is this [Fact 2.2\(ii\)](#) we will use. Going to the algebraic closure of K^\sharp as in [Fact 2.2\(i\)](#) would force us to consider profinite groups, for which our descent arguments may not work.

Fact 2.2 (for G/K). *Let y be a solution of $\partial \ln_{G,s}(-) = a$ in $G(K^{\text{diff}})$, and let $K^\sharp = K(G^\partial)$, with algebraic closure $K^{\sharp, \text{alg}}$. Then the following hold:*

- (i) *The transcendence degree $\text{tr.deg}(K^\sharp(y)/K^\sharp)$ is the dimension of a minimal connected algebraic ∂ -subgroup H of G , which is defined over $K^{\sharp, \text{alg}}$ such that $gag^{-1} + \partial \ln_{G,s}(g) \in LH(K^{\sharp, \text{alg}})$ for some $g \in G(K^{\sharp, \text{alg}})$. And $H^\partial(K^{\sharp, \text{alg}})$ is the differential Galois group of $K^{\sharp, \text{alg}}(y)/K^{\sharp, \text{alg}}$.*
- (ii) *Suppose that G is commutative. Then the identity component of the differential Galois group of $K^\sharp(y)/K^\sharp$ is $H^\partial(K^\sharp)$, where H is the smallest algebraic ∂ -subgroup of G defined over K^\sharp such that $a \in LH + \mathbb{Q} \cdot \partial \ln_{G,s} G(K^\sharp)$.*

Remark. We point out that when G is commutative, then in [Facts 2.1](#) and [2.2](#), the Galois group, say \tilde{H} , of $K^\sharp(y)/K^\sharp$ is a unique subgroup of G , so its identity component H must indeed be the smallest algebraic subgroup of G with the required properties (see also [\[Bertrand 2011, Section 3.1\]](#)). Of course, \tilde{H} is automatically connected in [Fact 2.2\(i\)](#), where the base $K^{\sharp, \text{alg}}$ is algebraically closed, but as just announced, our proofs in [Section 3](#) will be based on [2.2\(ii\)](#). Now, in this commutative case, the map $\sigma \mapsto \rho_\sigma$ described above depends \mathbb{Z} -linearly on a . So, if

$N = [\tilde{H} : H]$ denotes the number of connected components of \tilde{H} , then replacing a by Na turns the Galois group into a connected algebraic group, without modifying K^\sharp nor $\text{tr.deg}(K^\sharp(y)/K^\sharp) = \text{tr.deg}(K^\sharp(Ny)/K^\sharp)$. Therefore, in the computations of Galois groups later on, we will tacitly replace y by Ny and determine the connected component H of \tilde{H} . But it turns out that in our main [Conjecture 2.3](#) and in all its cases under study here, we can then assume that y itself lies in H . Indeed, y appears only via its class modulo $G(K)$, and in particular, modulo its torsion subgroup (recall that K is algebraically closed). So, once we have proven that Ny lies in H , then a translate y' of y by an N -torsion point will lie in H . Replacing y by y' does not modify the Galois group \tilde{H} of $K^\sharp(y)$ over K^\sharp , so we may assume that y lies in H , in which case \tilde{H} coincides with H , and will in the end always be connected.¹

2B. Galois-theoretic results. The question which we deal with in this paper is when and whether in [Fact 2.2](#), it suffices to consider H defined over K and $g \in G(K)$. In fact it is not hard to see that the Galois group is defined over K , but the second point is problematic. The case where (G, s) is a ∂ -module, namely G is a vector space V , and the logarithmic derivative $\partial \ln_{G,s}(y)$ has the form $\nabla_V(y) = \partial y - By$ for some $n \times n$ matrix B over K , was considered in [[Bertrand 2001](#)], and shown to provide counterexamples, unless the ∂ -module (V, ∇_V) is semisimple. The rough idea is that the Galois group $\text{Gal}(K_V^\sharp/K)$ of ∇_V is then reductive, allowing an argument of Galois descent from K_V^\sharp to K to construct a K -rational gauge transformation g . The argument was extended in [[Bertrand 2009; 2011](#)] to ∂ -groups (G, s) attached to abelian varieties, which by Poincaré reducibility are in a sense again semisimple.

We will here focus on the *almost semiabelian* case namely certain ∂ -groups attached to semiabelian varieties, which provide the main source of nonsemisimple situations. If B is a semiabelian variety over K , then \tilde{B} , the universal vectorial extension of B , is a (commutative) algebraic group over K which has a *unique* algebraic ∂ -group structure. Let U be any unipotent algebraic ∂ -subgroup of \tilde{B} . Then \tilde{B}/U , which by [[Bertrand and Pillay 2010](#), Lemma 3.4] also has a unique ∂ -group structure, is what we mean by an almost semiabelian ∂ -group over K . When B is an abelian variety A we call \tilde{A}/U an almost abelian algebraic ∂ -group over K . If G is an almost semiabelian algebraic ∂ -group over K , then because the ∂ -group structure s on G is unique, the abbreviation K_G^\sharp for $K_{G,s}^\sharp$ is now unambiguous. Under these conditions, we make the following conjecture.

¹We take the opportunity of this remark to mention two errata in [[Bertrand 2011](#)]: in the proof of its Theorem 3.2, replace “of finite index” by “with quotient of finite exponent”; in the proof of Theorem 4.4, use the reduction process described above to justify that the Galois group is indeed connected.

Conjecture 2.3. *Let G be an almost semiabelian ∂ -group over $K = \mathbb{C}(t)^{\text{alg}}$. Let $a \in LG(K)$ and $y \in G(K^{\text{diff}})$ be such that $\partial \ln_G(y) = a$. Then $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp)$ is the dimension of the smallest algebraic ∂ -subgroup H of G defined over K such that $a \in LH + \partial \ln_G(G(K))$, i.e., $a + \partial \ln_G(g) \in LH(K)$ for some $g \in G(K)$; H is, equivalently, the smallest algebraic ∂ -subgroup of G , defined over K , such that $y \in H + G(K) + G^\partial(K^{\text{diff}})$. Moreover $H^\partial(K^{\text{diff}})$ is the Galois group of $K_G^\sharp(y)$ over K_G^\sharp .*

The conjecture can be restated to say that there is a smallest algebraic ∂ -subgroup H of (G, s) defined over K such that $a \in LH + \partial \ln_G(G(K))$ and it coincides with the Galois group of $K_G^\sharp(y)$ over K_G^\sharp . In comparison with [Fact 2.2\(ii\)](#), notice that since K is algebraically closed, $\partial \ln_G(G(K))$ is already a \mathbb{Q} -vector space, so we do not need to tensor with \mathbb{Q} in the condition on a .

A corollary of [Conjecture 2.3](#) is the following special *generic case*, where an additional assumption on nondegeneracy is made on a .

Conjecture 2.4. *Let G be an almost semiabelian ∂ -group over $K = \mathbb{C}(t)^{\text{alg}}$, and let $a \in LG(K)$ and $y \in G(K^{\text{diff}})$ satisfy the equation $\partial \ln_G(y) = a$. Assume that $a \notin LH + \partial \ln_G(G(K))$ for any proper algebraic ∂ -subgroup H of G , defined over K (equivalently, $y \notin H + G(K) + G^\partial(K^{\text{diff}})$ for any proper algebraic ∂ -subgroup of G defined over K). Then $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp) = \dim(G)$.*

We will prove the following results in the direction of [Conjectures 2.3](#) and (the weaker) [2.4](#).

Proposition 2.5. *[Conjecture 2.3](#) holds when G is “almost abelian”.*

The truth of the weaker [Conjecture 2.4](#) in the almost abelian case is already established in [[Bertrand 2009](#), Section 8.1(i)]. This reference does not address [Conjecture 2.3](#) itself, even if in this case, the ingredients for its proof are there (see also [[Bertrand 2011](#)]). So we take the liberty to give a reasonably self-contained proof of [Proposition 2.5](#) in [Section 3](#).

As announced above, one of the main points of the Galois-theoretic part of this paper is to try to extend [Proposition 2.5](#) to the almost semiabelian case. Due to technical complications, which will be discussed later, we restrict our attention to the simplest possible extension of the almost abelian case, namely where the toric part of the semiabelian variety is 1-dimensional, and also we sometimes just consider the generic case. For simplicity we will state and prove our results for an almost semiabelian G of the form \widetilde{B} for B semiabelian. So, the next theorem gives [Conjecture 2.4](#) for an extension by \mathbb{G}_m of the universal vectorial extension of an abelian variety.

Theorem 2.6. *Suppose that B is a semiabelian variety over $K = \mathbb{C}(t)^{\text{alg}}$ with toric part of dimension ≤ 1 . Let $G = \widetilde{B}$, $a \in LG(K)$ and $y \in G(K^{\text{diff}})$ be a solution of*

$\partial \ln_G(-) = a$. Suppose that for no proper algebraic ∂ -subgroup H of G defined over K is $y \in H + G(K)$. Then $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp) = \dim(G)$ and $G^\partial(K^{\text{diff}})$ is the differential Galois group.

Note that the above hypothesis “ $y \notin H + G(K)$ for any proper algebraic ∂ -subgroup of G over K ” is formally weaker than “ $y \notin H + G(K) + G^\partial(K^{\text{diff}})$ for any proper algebraic ∂ -subgroup of G over K ”, but nevertheless suffices, as shown by the proof of [Theorem 2.6](#) in [Section 3B](#). More specifically, assume that $G = \tilde{A}$ for a simple abelian variety A/K , that A is traceless (i.e., that there is no nonzero morphism from an abelian variety defined over \mathbb{C} to A), that the maximal unipotent ∂ -subgroup U_A of \tilde{A} vanishes, and that $a = 0 \in L\tilde{A}(K)$. [Theorem 2.6](#) then implies that any $y \in \tilde{A}^\partial(K^{\text{diff}})$ is actually defined over K , so $K_A^\sharp = K$. As in [[Bertrand 2009; 2011](#)], this property of K -largeness of \tilde{A} (when $U_A = 0$) is in fact one of the main ingredients in the proof of [Theorem 2.6](#). As explained in [[Marker and Pillay 1997](#)] it is based on the strong minimality of \tilde{A}^∂ (see [[Hrushovski and Sokolović 1994](#)]) in the context above. But it has recently been noted in [[Benoist et al. 2014](#)] that this K -largeness property can be seen rather more directly, using only the simplicity of A .

Our last Galois-theoretic result requires the *semiconstant* notions introduced in [[Bertrand and Pillay 2010](#)], although our notation will be a slight modification of the notation there. First, a connected algebraic group G over K is said to be constant if G is isomorphic (as an algebraic group) to an algebraic group defined over \mathbb{C} (equivalently, G arises via base change from an algebraic group $G_{\mathbb{C}}$ over \mathbb{C}). For G an algebraic group over K , G_0 will denote the largest (connected) constant algebraic subgroup of G . We will concentrate on the case $G = \tilde{B}$ for a semiabelian variety B over K , with $0 \rightarrow T \rightarrow B \rightarrow A \rightarrow 0$ the canonical exact sequence, where T is the maximal linear algebraic subgroup of B (which is an algebraic torus) and A is an abelian variety. So now A_0, B_0 denote the constant parts of A, B , respectively. The inverse image of A_0 in B will be called the semiconstant part of B and will now be denoted by B_{sc} . We call B semiconstant if $B = B_{\text{sc}}$, which is equivalent to requiring that $A = A_0$, and moreover allows the possibility that $B = B_0$ is constant. (Of course, when B is constant, \tilde{B} , which is also constant, obviously satisfies [Conjecture 2.3](#), in view of [Fact 2.1](#).)

Theorem 2.7. *Suppose that $K = \mathbb{C}(t)^{\text{alg}}$ and that $B = B_{\text{sc}}$ is a semiconstant semiabelian variety over K with toric part of dimension ≤ 1 . Then [Conjecture 2.3](#) holds for $G = \tilde{B}$.*

2C. Lindemann–Weierstrass via Galois theory. We are now ready to describe the impact of the previous Galois-theoretic results on Ax–Lindemann problems, where $a = \partial_{LG}(\tilde{x}) \in \partial_{LG}(LG(K))$.

Firstly, from [Theorem 2.6](#) we will deduce directly the main result of [\[Bertrand and Pillay 2010, Theorem 1.4\]](#), when B is semiabelian with toric part at most \mathbb{G}_m , but now with transcendence degree computed over K_B^{\sharp} .

Corollary 2.8. *Let B be a semiabelian variety over $K = \mathbb{C}(t)^{\text{alg}}$ such that the toric part of B is of dimension ≤ 1 and $B_{\text{sc}} = B_0$ (i.e., the semiconstant part B_{sc} of B is constant). Let $x \in LB(K)$, and lift x to $\tilde{x} \in L\tilde{B}(K)$. Assume that*

$$(*) \quad \begin{aligned} &\text{for no proper algebraic subgroup } H \text{ of } \tilde{B} \text{ defined over } K \text{ is} \\ &\tilde{x} \in LH(K) + (L\tilde{B})^{\partial}(K), \end{aligned}$$

which under the current assumptions is equivalent to demanding that for no proper semiabelian subvariety H of B is $x \in LH(K) + LB_0(\mathbb{C})$. Then

(i) any solution $\tilde{y} \in B^{(\mathcal{U})}$ of $\partial \ell n_{\tilde{B}}(-) = \partial_{L\tilde{B}}(\tilde{x})$ satisfies

$$\text{tr.deg}(K_B^{\sharp}(\tilde{y})/K_B^{\sharp}) = \dim(\tilde{B});$$

(ii) in particular, $y := \exp_B(x)$ satisfies $\text{tr.deg}(K_B^{\sharp}(y)/K_B^{\sharp}) = \dim(B)$, i.e., is a generic point over K_B^{\sharp} of B .

See [\[Bertrand and Pillay 2010\]](#) for the analytic description of $\exp_B(x)$ in (ii) above. In particular $\exp_B(x)$ can be viewed as a point of $B^{(\mathcal{U})}$. We recall briefly the argument. Consider B as the generic fiber of a family $\mathbf{B} \rightarrow S$ of complex semiabelian varieties over a complex curve S , and x as a rational section $x : S \rightarrow L\mathbf{B}$ of the corresponding family of Lie algebras. Fix a small disc U in S such that $x : U \rightarrow L\mathbf{B}$ is holomorphic, and let $\exp(x) = y : U \rightarrow \mathbf{B}$ be the holomorphic section obtained by composing with the exponential map in the fibers. So y lives in the differential field of meromorphic functions on U , which contains K , and can thus be embedded over K in the universal differentially closed field \mathcal{U} . So talking about $\text{tr.deg}(K_B^{\sharp}(y)/K_B^{\sharp})$ makes sense.

Let us comment on the methods. In [\[Bertrand and Pillay 2010\]](#) an essential use was made of the so-called “socle theorem” (see Section 4.1 of [\[Bertrand and Pillay 2010\]](#) for a discussion of this expression) in order to prove Theorem 1.4 there. As recalled in the introduction, a differential Galois-theoretic approach was also mentioned [\[Bertrand and Pillay 2010, Section 6\]](#), but could be worked out only when \tilde{B} is K -large. In the current paper, we dispose of this hypothesis, and obtain a stronger result, namely over K_B^{\sharp} , but for the time being at the expense of restricting the toric part of B .

When $B = A$ is an abelian variety, one obtains a stronger statement than [Corollary 2.8](#). This is Theorem 4.4 of [\[Bertrand 2011\]](#), which for the sake of completeness we restate, and will deduce from [Proposition 2.5](#) in [Section 4A](#).

Corollary 2.9. *Let A be an abelian variety over $K = \mathbb{C}(t)^{\text{alg}}$. Let $x \in LA(K)$, and let B be the smallest abelian subvariety of A such that $x \in LB(K) + LA_0(\mathbb{C})$. Let*

$\tilde{x} \in \tilde{L}\tilde{A}(K)$ be a lift of x and let $\tilde{y} \in \tilde{A}(\mathcal{U})$ be such that $\partial \ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x})$. Then \tilde{B}^\sharp is the Galois group of $K_{\tilde{A}}^\sharp(\tilde{y})$ over $K_{\tilde{A}}^\sharp$, so

- (i) $\text{tr.deg}(K_{\tilde{A}}^\sharp(\tilde{y})/K_{\tilde{A}}^\sharp) = \dim(\tilde{B}) = 2 \dim(B)$, and in particular,
- (ii) $y := \exp_A(x)$ satisfies $\text{tr.deg}(K_A^\sharp(y)/K_A^\sharp) = \dim(B)$.

We now return to the semiabelian context. [Corollary 2.8](#) is not true without the assumption that the semiconstant part of B is constant. The simplest possible counterexample is given in Section 5.3 of [\[Bertrand and Pillay 2010\]](#): B is a nonconstant extension of a constant elliptic curve E_0 by \mathbb{G}_m , with judicious choices of x and \tilde{x} . Moreover \tilde{x} will satisfy assumption [\(*\)](#) in [Corollary 2.8](#), but $\text{tr.deg}(K(\tilde{y})/K) \leq 1$, which is strictly smaller than $\dim(\tilde{B}) = 3$. We will use [Theorems 2.6](#) and [2.7](#) as well as material from [\[Bertrand et al. 2013\]](#) to give a full account of this situation (now over K_B^\sharp , of course), and more generally, for all semiabelian surfaces B/K , as follows:

Corollary 2.10. *Let B be an extension over $K = \mathbb{C}(t)^{\text{alg}}$ of an elliptic curve E/K by \mathbb{G}_m . Let $x \in LB(K)$ satisfy*

- [\(*\)](#) *for any proper algebraic subgroup H of B , $x \notin LH + LB_0(\mathbb{C})$.*

Let $\tilde{x} \in \tilde{L}\tilde{B}(K)$ be a lift of x , let \bar{x} be its projection to $LE(K)$, and let $\tilde{y} \in \tilde{B}(\mathcal{U})$ be such that $\partial \ell n_{\tilde{B}}(\tilde{y}) = \tilde{x}$. Then $\text{tr.deg}(K_B^\sharp(\tilde{y})/K_B^\sharp) = 3$, unless $\bar{x} \in LE_0(\mathbb{C})$, in which case $\text{tr.deg}(K_B^\sharp(\tilde{y})/K_B^\sharp)$ is precisely 1.

Here, E_0 is the constant part of E . Notice that in view of [\(*\)](#), E must descend to \mathbb{C} and B must be nonconstant (hence not isotrivial) if x projects to $LE_0(\mathbb{C})$.

2D. Manin maps. We finally discuss the results on the Manin maps attached to abelian varieties. The expression “Manin map” covers at least two maps. The original one was introduced by Manin [\[1963\]](#) (see also [\[Coleman 1990\]](#)), and is discussed at the end of this section. Here we are mainly concerned with the model-theoretic or *differential algebraic Manin map* (see [\[Buium and Cassidy 1999, Section 2.5; Pillay 1997\]](#)). We identify our algebraic, differential algebraic groups with their sets of points in a universal differential field \mathcal{U} (or alternatively, points in a differential closure of whatever differential field of definition we work over). So for now let K be a differential field, and A an abelian variety over K . A has a smallest Zariski-dense differential algebraic (definable in \mathcal{U}) subgroup A^\sharp , which can also be described as the smallest definable subgroup of A containing the torsion. The definable group A/A^\sharp embeds definably in a commutative unipotent algebraic group (i.e., a vector group) by results of Buium, and results of Cassidy on differential algebraic vector groups yield a (noncanonical) differential algebraic isomorphism between A/A^\sharp and \mathbb{G}_a^n where $n = \dim(A)$. This differential algebraic isomorphism is defined over K , and we call it the Manin homomorphism.

There is a somewhat more intrinsic account of this isomorphism. Let \tilde{A} be the universal vectorial extension of A as discussed above, equipped with its unique algebraic ∂ -group structure, and let W_A be the unipotent part of \tilde{A} . We have the surjective differential algebraic homomorphism $\partial \ell n_{\tilde{A}} : \tilde{A} \rightarrow L\tilde{A}$. Note that if $\tilde{y} \in \tilde{A}$ lifts $y \in A$, then the image of \tilde{y} under $\partial \ell n_{\tilde{A}}$ modulo the subgroup $\partial \ell n_{\tilde{A}}(W_A)$ depends only on y . This gives a surjective differential algebraic homomorphism from A to $L\tilde{A}/\partial \ell n(W_A)$, which is defined over K , and which we call μ_A .

Remark 2.11. *Any abelian variety A/K satisfies $\ker(\mu_A) = A^\sharp$.*

Proof. Let U_A be the maximal algebraic subgroup of W_A which is a ∂ -subgroup of \tilde{A} . Then \tilde{A}/U_A has the structure of an algebraic ∂ -group, and as explained in [Bertrand and Pillay 2010], the canonical map $\pi : \tilde{A} \rightarrow A$ induces an isomorphism between $(\tilde{A}/U_A)^\partial$ and A^\sharp . As (by functoriality) $(\tilde{A})^\partial$ maps onto $(\tilde{A}/U_A)^\partial$, the map $\pi : \tilde{A} \rightarrow A$ also induces a surjective map $(\tilde{A})^\partial \rightarrow A^\sharp$. Now, as the image of μ_A is torsion-free, $\ker(\mu_A)$ contains A^\sharp . On the other hand, if $y \in \ker(\mu_A)$ and $\tilde{y} \in \tilde{A}$ lifts y , then there is $z \in W_A$ such that $\partial \ell n_{\tilde{A}}(\tilde{y}) = \partial \ell n_{\tilde{A}}(z)$. So $\partial \ell n_{\tilde{A}}(\tilde{y} - z) = 0$ and $\pi(\tilde{y} - z) = y$, hence $y \in A^\sharp$. □

Hence we call μ_A the (differential algebraic) Manin map. The target space embeds in an algebraic vector group and thus has the structure of a \mathcal{C} -vector space which is unique (any definable isomorphism between two commutative unipotent differential algebraic groups is an isomorphism of \mathcal{C} -vector spaces).

Now assume that $K = \mathbb{C}(t)^{\text{alg}}$ and that A is an abelian variety over K with \mathbb{C} -trace $A_0 = 0$. Then the “model-theoretic/differential algebraic theorem of the kernel” is (see Corollary K.3 of [Bertrand and Pillay 2010]):

Fact 2.12 ($K = \mathbb{C}(t)^{\text{alg}}$, A/K traceless). *The kernel $\ker(\mu_A) \cap A(K)$ is precisely the subgroup $\text{Tor}(A)$ of torsion points of A .*

In Section 5 we generalize Fact 2.12 by proving:

Theorem 2.13 ($K = \mathbb{C}(t)^{\text{alg}}$, A/K traceless). *Let $y_1, \dots, y_n \in A(K)$. Suppose that $a_1, \dots, a_n \in \mathbb{C}$ are not all 0, and that $a_1\mu_A(y_1) + \dots + a_n\mu_A(y_n) = 0$ in $L\tilde{A}(K)/\partial \ell n_{\tilde{A}}(W_A)$. Then y_1, \dots, y_n are linearly dependent over $\text{End}(A)$.*

Note that on reducing to a simple abelian variety, Fact 2.12 is the special case of Theorem 2.13 when $n = 1$. Hrushovski asked whether the conclusion of the theorem can be strengthened to the linear dependence of y_1, \dots, y_n over \mathbb{Z} . Namely, is the extension $\mu_A \otimes 1$ of μ_A to $A(K) \otimes_{\mathbb{Z}} \mathbb{C}$ injective? An example of André (see [Bertrand and Pillay 2010, p. 504; Lange and Birkenhake 1992, Chapter 9 §6]) of a traceless abelian variety A with $U_A \neq W_A$ yields a counterexample:

Proposition 2.14. *There exist*

- a simple traceless 4-dimensional abelian variety A over $K = \mathbb{C}(t)^{\text{alg}}$, such that $\text{End}(A)$ is an order in a CM number field F of degree 4 over \mathbb{Q} ;
- four points y_1, \dots, y_4 in $A(K)$ which are linearly dependent over $\text{End}(A)$, but linearly independent over \mathbb{Z} ; and
- four complex numbers a_1, \dots, a_4 , not all zero;

such that $a_1\mu_A(y_1) + \dots + a_4\mu_A(y_4) = 0$.

In fact, for $i = 1, \dots, 4$, we will construct lifts $\tilde{y}_i \in \tilde{A}(K)$ of the points y_i , and solutions $\tilde{x}_i \in L\tilde{A}(K^{\text{diff}})$ to the equations $\nabla(\tilde{x}_i) = \partial \ell n_{\tilde{A}} \tilde{y}_i$ (where we have set $\nabla := \nabla_{L\tilde{A}} = \partial_{L\tilde{A}}$, with $\nabla|_{LW_A} = \partial \ell n_{\tilde{A}}|_{W_A}$ in the identification $W_A = LW_A$), and will find a nontrivial relation

$$(\mathfrak{R}) \quad a_1\tilde{x}_1 + \dots + a_4\tilde{x}_4 := u \in U_A(K^{\text{diff}}).$$

Since U_A is a ∇ -submodule of $L\tilde{A}$, this implies that $a_1\partial \ell n_{\tilde{A}} \tilde{y}_1 + \dots + a_4\partial \ell n_{\tilde{A}} \tilde{y}_4$ lies in U_A . And since $U_A \subseteq W_A$, this in turn shows that

$$a_1\mu_A(y_1) + \dots + a_4\mu_A(y_4) = 0 \quad \text{in } L\tilde{A}/\partial \ell n_{\tilde{A}}(W_A),$$

contradicting the injectivity of $\mu_A \otimes 1$.

We conclude with a remark on the more classical *differential arithmetic* Manin map $M_{K,A}$, where the stronger version is true. Again A is an abelian variety over $K = \mathbb{C}(t)^{\text{alg}}$ with \mathbb{C} -trace 0. As above, we let ∇ denote $\partial_{L\tilde{A}} : L\tilde{A} \rightarrow L\tilde{A}$. The map $M_{K,A}$ is then the homomorphism from $A(K)$ to $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$, which attaches to a point $y \in A(K)$ the class $M_{K,A}(y)$ of $\partial \ell n_{\tilde{A}}(\tilde{y})$ in $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$, for any lift \tilde{y} of y to $\tilde{A}(K)$. This class is independent of the lift, since $\partial \ell n_{\tilde{A}}$ and $\partial_{L\tilde{A}}$ coincide on $W_A = LW_A$. Again $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$ is a \mathbb{C} -vector space. The initial theorem of Manin (see [Coleman 1990]) says that $\ker(M_{K,A}) = \text{Tor}(A) + A_0(\mathbb{C})$, so in the traceless case the kernel is precisely $\text{Tor}(A)$.

Proposition 2.15 ($K = \mathbb{C}(t)^{\text{alg}}$, A/K traceless). *The \mathbb{C} -linear extension*

$$M_{K,A} \otimes 1 : A(K) \otimes_{\mathbb{Z}} \mathbb{C} \rightarrow L\tilde{A}(K)/\nabla(L\tilde{A}(K))$$

is injective.

3. Computation of Galois groups

Here we prove the Galois-theoretic statements [Proposition 2.5](#) and [Theorems 2.6](#) and [2.7](#) stated in [Section 2B](#). We assume throughout that $K = \mathbb{C}(t)^{\text{alg}}$.

3A. The abelian case. Let us first set up the notation. Let A be an abelian variety over K , and let A_0 be its \mathbb{C} -trace, which we view as a subgroup of A defined over \mathbb{C} . Let \tilde{A} be the universal vectorial extension of A . We have the short exact sequence $0 \rightarrow W_A \rightarrow \tilde{A} \rightarrow A \rightarrow 0$. Let U_A denote the (unique) maximal ∂ -subgroup of \tilde{A} contained in W_A . By Remark 7.2 of [Bertrand 2009], we have:

Fact 3.1.
$$\tilde{A}^\partial(K^{\text{diff}}) = \tilde{A}_0(\mathbb{C}) + \text{Tor}(\tilde{A}) + U_A^\partial(K^{\text{diff}}).$$

Let us briefly remark that the ingredients behind Fact 3.1 include Chai’s theorem (see [Chai 1991] and Appendix K of [Bertrand and Pillay 2010]), as well as the strong minimality of A^\sharp when A is simple and traceless from [Hrushovski and Sokolović 1994]. As already pointed out in connection with K -largeness, the reference to [Hrushovski and Sokolović 1994] can be replaced by the easier arguments from [Benoist et al. 2014]. Let K_A^\sharp be the (automatically differential) field generated over K by $\tilde{A}^\partial(K^{\text{diff}})$, and likewise with $K_{U_A}^\sharp$ for $(U_A)^\partial(K^{\text{diff}})$. So by Fact 3.1, $K_A^\sharp = K_{U_A}^\sharp$. Also, as recalled at the beginning of Section 8 of [Bertrand 2009], we have:

Remark 3.2. $K_{U_A}^\sharp$ is a Picard–Vessiot extension of K whose Galois group (a linear algebraic group over \mathbb{C}) is semisimple.

Proof of Proposition 2.5. Here, G is an almost abelian ∂ -group over K . We first treat the case where $G = \tilde{A}$.

Let $y \in G(K^{\text{diff}})$ be such that $a = \partial \ell n_G(y)$ lies in $LG(K)$. Note that in the setup of Conjecture 2.3, y could very well be an element of U_A , for instance when $a \in LU_A \simeq U_A$, so in a sense we are moving outside the almost abelian context. In any case, let H be a minimal ∂ -subgroup of G defined over K such that $y \in H + G(K) + G^\partial(K^{\text{diff}})$. Since $G(K)$ contains all the torsion points, H is automatically connected. We will prove that $H^\partial(K^{\text{diff}})$ is the differential Galois group of $K^\sharp(y)$ over K^\sharp where $K^\sharp = K_G^\sharp$. We recall from the remark after Fact 2.2 on the commutative case that we can and do assume that this Galois group is connected. Also, these statements imply that H is actually the smallest ∂ -subgroup of G over K such that $y \in H + G(K) + G^\partial(K^{\text{diff}})$, as required.

Let H_1^∂ be the Galois group of $K^\sharp(y)$ over K^\sharp with H_1 a ∂ -subgroup of G which on the face of it is defined over K^\sharp . So, H_1 is a connected ∂ -subgroup of H , and we aim to show that $H = H_1$.

Claim. H_1 is defined over K as an algebraic group.

Proof. It is enough to show that H_1^∂ is defined over K as a differential algebraic group. This is a very basic model-theoretic argument, but may be a bit surprising at the algebraic-geometric level, as $K^\sharp(y)$ need not be a “differential Galois extension” of K in any of the usual meanings. We use the fact that any definable (with parameters) set in the differentially closed field K^{diff} which is $\text{Aut}(K^{\text{diff}}/K)$ -invariant, is definable over K . This follows from model-theoretic homogeneity of K^{diff}

over K as well as elimination of imaginaries in DCF_0 . Now $H_1^\partial(K^{\text{diff}})$ is the set of $g \in G^\partial(K^{\text{diff}})$ such that $y_1 g$ and y_1 have the same type over K^\sharp for some (any) $y_1 \in G(K^{\text{diff}})$ such that $\partial \ell n_G(y_1) = a$. As $a \in LG(K)$ and K^\sharp is setwise invariant under $\text{Aut}(K^{\text{diff}}/K)$, it follows that $H_1^\partial(K^{\text{diff}})$ is also $\text{Aut}(K^{\text{diff}}/K)$ -invariant, and so defined over K . This proves the claim.

Note that since one of its translates by $G(K)$ lies in H , we may assume that $y \in H$, whereby $\partial \ell n_G(y) = a \in LH(K)$.

Let B be the image of H in A , and B_1 the image of H_1 in A . So $B_1 \leq B$ are abelian subvarieties of A . Let V be the maximal unipotent ∂ -subgroup of H , and V_1 the maximal unipotent subgroup of H_1 . So $V_1 \leq V$, and using the assumptions and the claim, everything is defined over K . Note also that the surjective homomorphism $H \rightarrow B$ induces an isomorphism between H/V and \tilde{B}/U_B (where as above U_B denotes the maximal unipotent ∂ -subgroup of \tilde{B}), and likewise for H_1/V_1 and the quotient of \tilde{B}_1 by its maximal unipotent ∂ -subgroup.

Case I. $B = B_1$.

Then by the previous paragraph, we have a canonical isomorphism ι (of ∂ -groups) between H/H_1 and V/V_1 , defined over K , so there is no harm in identifying them, although we need to remember where they came from. Let us denote V/V_1 by \bar{V} , a unipotent ∂ -group. This isomorphism respects the logarithmic derivatives in the obvious sense. Let \bar{y} denote the image of y in H/H_1 . So $\partial \ell n_{H/H_1}(\bar{y}) = \bar{a}$ where \bar{a} is the image of a in $L(H/H_1)(K)$. Via ι we identify \bar{y} with a point in $\bar{V}(K^\sharp)$ and \bar{a} with $\partial \ell n_{\bar{V}}(\bar{y}) \in L(\bar{V})(K)$.

By [Remark 3.2](#) we identify $\text{Aut}(K^\sharp/K)$ with a group $J(\mathbb{C})$ where J is a semi-simple algebraic group. We have a natural action of $J(\mathbb{C})$ on $\bar{V}^\partial(K^{\text{diff}}) = \bar{V}^\partial(K^\sharp)$. Now the latter is a \mathbb{C} -vector space, and this action can be checked to be a (rational) representation of $J(\mathbb{C})$. On the other hand, for $\sigma \in J(\mathbb{C})$, $\sigma(\bar{y})$ (which is well-defined since \bar{y} is K^\sharp -rational) is also a solution of $\partial \ell n_{\bar{V}}(-) = \bar{a}$, hence $\sigma(\bar{y}) - \bar{y} \in \bar{V}^\partial(K^{\text{diff}})$. The map taking σ to $\sigma(\bar{y}) - \bar{y}$ is then a cocycle c from $J(\mathbb{C})$ to $V^\partial(K^{\text{diff}})$ which is a morphism of algebraic varieties. Now the corresponding $H^1(J(\mathbb{C}), \bar{V}^\partial(K^{\text{diff}}))$ is trivial as it equals $\text{Ext}_{J(\mathbb{C})}(1, \bar{V}^\partial(K^{\text{diff}}))$, the group of isomorphism classes of extensions of the trivial representation of $J(\mathbb{C})$ by $\bar{V}^\partial(K^{\text{diff}})$. But $J(\mathbb{C})$ is semi-simple, so reductive, whereby every rational representation is completely reducible (see pp. 26 and 27 of [\[Mumford and Fogarty 1982\]](#), and [\[Bertrand 2001\]](#) for Picard–Vessiot applications, which actually cover the case when a lies in LU_A). Putting everything together, the original cocycle is trivial. Therefore there is $\bar{z} \in \bar{V}^\partial(K^\sharp)$ such that $\sigma(\bar{y}) - \bar{y} = \sigma(\bar{z}) - \bar{z}$ for all $\sigma \in J(\mathbb{C})$. So $\sigma(\bar{y} - \bar{z}) = \bar{y} - \bar{z}$ for all σ . Hence $\bar{y} - \bar{z} \in (H/H_1)(K)$. Lift \bar{z} to a point $z \in H^\partial(K^{\text{diff}})$. So $\overline{y - z} \in \bar{V}(K)$. As K is algebraically closed, there is $d \in H(K)$ such that $y - z + d \in H_1$. This contradicts the minimal choice of H , unless $H = H_1$. So the proof is complete in [Case I](#).

Case II. B_1 is a proper subgroup of B .

Consider the group $H_1 \cdot V$ a ∂ -subgroup of H , defined over K , which also projects onto B_1 . It is now easy to extend $H_1 \cdot V$ to a ∂ -subgroup H_2 of H over K such that H/H_2 is canonically isomorphic to \overline{B}_2 , where B_2 is a simple abelian variety, and \overline{B}_2 denotes the quotient of \widetilde{B}_2 by its maximal unipotent subgroup. Now let \bar{y} denote $y/H_2 \in H/H_2$. Hence $\partial \ell n_{\overline{B}_2}(\bar{y}) = \bar{a} \in L(\overline{B}_2)(K)$. As $H_1 \subseteq H_2$, $\bar{y} \in \overline{B}_2(K^\sharp)$. Now we have two cases. If B_2 descends to \mathbb{C} , then \bar{y} generates a strongly normal extension of K whose Galois group is a connected algebraic subgroup of $B_2(\mathbb{C})$. As this Galois group will be a homomorphic image of the linear (in fact semisimple) complex algebraic group $\text{Aut}(K^\sharp/K)$, we have a contradiction unless \bar{y} is K -rational. On the other hand, if B_2 does not descend to \mathbb{C} , then by [Fact 2.2\(ii\)](#) \bar{y} generates over K a (generalized) differential Galois extension of K with Galois group contained in $\overline{B}_2^\partial(K^{\text{diff}})$, which again will be a homomorphic image of a complex semisimple linear algebraic group (cf. [\[Bertrand 2009, 8.2\(i\)\]](#)). We get a contradiction by various possible means (for example as in [Remarque 8.2 of \[Bertrand 2009\]](#)) unless \bar{y} is K -rational. So either way we are forced into $\bar{y} \in (H/H_2)(K)$. But then, as K is algebraically closed, $y - d \in H_2$ for some $d \in H(K)$, again a contradiction. So **Case II** is impossible. This concludes the proof of [Proposition 2.5](#) when $G = \widetilde{A}$.

Finally, consider a general almost abelian ∂ -group G , given as a quotient of \widetilde{A} by a unipotent ∂ -subgroup $U \subset U_A$ defined over K . Taking the quotient by $U^\partial(K^{\text{diff}})$ of the decomposition of $\widetilde{A}^\partial(K^{\text{diff}})$ given by [Fact 3.1](#), we obtain a similar decomposition for $G^\partial(K^{\text{diff}})$. Therefore $K_G^\sharp = K((U_A/U)^\partial)$ is also a Picard–Vessiot extension of K , and we deduce from [Remark 3.2](#) that its Galois group is again semisimple. The various cases of the previous proof therefore also apply to the quotient $G = \widetilde{A}/U$, and [Proposition 2.5](#) holds for any almost abelian ∂ -group. \square

3B. The semiabelian case. We now aim towards proofs of [Theorems 2.6](#) and [2.7](#). Here, $G = \widetilde{B}$ for B a semiabelian variety over K , equipped with its unique algebraic ∂ -group structure.

We have:

- $0 \rightarrow T \rightarrow B \rightarrow A \rightarrow 0$, where T is an algebraic torus and A an abelian variety, all over K ,
- $G = \widetilde{B} = B \times_A \widetilde{A}$, where \widetilde{A} is the universal vectorial extension of A , and
- $0 \rightarrow T \rightarrow G \rightarrow \widetilde{A} \rightarrow 0$.

We use the same notation for A as at the beginning of this section, namely

$$0 \longrightarrow W_A \longrightarrow \widetilde{A} \longrightarrow A \longrightarrow 0.$$

We denote by A_0 the \mathbb{C} -trace of A (so up to isogeny we can write A as a product $A_0 \times A_1$, all defined over K , where A_1 has \mathbb{C} -trace 0), and by U_A the maximal

∂ -subgroup of \tilde{A} contained in W_A . So U_A is a unipotent subgroup of G , though not necessarily one of its ∂ -subgroups. Finally, we have the exact sequence

$$0 \longrightarrow T^\partial \longrightarrow G^\partial \xrightarrow{\pi} \tilde{A}^\partial \longrightarrow 0.$$

Note that $T^\partial = T(\mathbb{C})$. Let K_G^\sharp be the (differential) field generated over K by $G^\partial(K^{\text{diff}})$. We have already noted above that K_G^\sharp equals $K_{U_A}^\sharp$. So $K_{U_A}^\sharp < K_G^\sharp$, and we deduce from the last exact sequence above the following:

Remark 3.3. $G^\partial(K^{\text{diff}})$ is the union of the $\pi^{-1}(b)$ for $b \in \tilde{A}^\partial$, each $\pi^{-1}(b)$ being a coset of $T(\mathbb{C})$ defined over $K_{U_A}^\sharp$. Hence K_G^\sharp is (generated by) a union of Picard–Vessiot extensions over $K_{U_A}^\sharp$, each with Galois group contained in $T(\mathbb{C})$.

Proof of Theorem 2.6. Bearing in mind Proposition 2.5 we may assume that $T = \mathbb{G}_m$. We have $a \in LG(K)$ and $y \in G(K^{\text{diff}})$ such that $\partial \ell n_G(y) = a$ and $y \notin H + G(K)$ for any proper ∂ -subgroup H of G . The latter is a little weaker than the condition that $a \notin LH(K) + \partial \ell n_G(G(K))$ for any proper H , but (thanks to Fact 3.1) will suffice for the special case we are dealing with.

Fix a solution y of $\partial \ell n_G(-) = a$ in $G(K^{\text{diff}})$ and let $H^\partial(K^{\text{diff}})$ be the differential Galois group of $K_G^\sharp(y)$ over K_G^\sharp . As said after Fact 2.2, there is no harm in assuming that H is connected. So H is a connected ∂ -subgroup of G , defined over K_G^\sharp .

As in the proof of the claim in the proof of Proposition 2.5, we have:

Claim 1. H (equivalently H^∂) is defined over K .

We assume for a contradiction that $H \neq G$.

Case I. H maps onto a proper (∂ -)subgroup of \tilde{A} .

This is similar to Case II in the proof of Proposition 2.5 above. Some additional complications come from the structure of K_G^\sharp . We will make use of Remark 3.3 all the time.

As \tilde{A} is an essential extension of A by W_A , it follows that we can find a connected ∂ -subgroup H_1 of G containing H and defined over K such that the surjection $G \rightarrow \tilde{A}$ induces an isomorphism between G/H_1 and \overline{A}_2 , where A_2 is a simple abelian subvariety of A (over K of course) and \overline{A}_2 is the quotient of \tilde{A}_2 by its maximal unipotent ∂ -subgroup. Let η and α be such that the quotient map taking G to \overline{A}_2 takes y to η and also induces a surjection $LG \rightarrow L(\overline{A}_2)$ which takes a to α .

As $\eta = y/H_1$ and $H \subseteq H_1$, we see that η is fixed by $\text{Aut}(K_G^\sharp(y)/K_G^\sharp)$, establishing the following:

Claim 2. We have $\eta \in \overline{A}_2(K_G^\sharp)$.

On the other hand, η is a solution of the logarithmic differential equation $\partial \ell n_{\overline{A}_2}(-) = \alpha$ over K . By K -largeness of \overline{A}_2 , we have $K_{\overline{A}_2}^\sharp = K$, hence $K(\eta)$ is a differential Galois extension of K whose Galois group is either trivial (in which case $\eta \in \overline{A}_2(K)$), or equal to $\overline{A}_2^\partial(K^{\text{diff}})$, in view of the strong minimality of \overline{A}_2^∂ .

Claim 3. We have $\eta \in \overline{A_2}(K)$.

Proof. Suppose not. We first claim that η is independent from $K_{U_A}^\sharp$ over K (in the sense of differential fields). Indeed, the Galois theory would otherwise give us some proper definable subgroup in the product of $\overline{A_2}^\partial(K^{\text{diff}})$ by the Galois group of $K_{U_A}^\sharp$ over K (or equivalently, these two groups would share a nontrivial definable quotient). As the latter is a complex semisimple algebraic group (Remark 3.2), we get a contradiction. Alternatively, we could proceed as in Remarque 8.2 of [Bertrand 2009].

So the Galois group of $K_{U_A}^\sharp(\eta)$ over $K_{U_A}^\sharp$ is $\overline{A_2}^\partial(K^{\text{diff}})$. As there are no nontrivial definable subgroups of $\overline{A_2}(K^{\text{diff}}) \times \mathbb{G}_m(\mathbb{C})^n$, we see that η is independent of K_G^\sharp over $K_{U_A}^\sharp$, contradicting Claim 2.

By Claim 3, the coset of y modulo H_1 is defined over K (differential algebraically), so as in the proof of Fact 2.1, as K is algebraically closed there is $y_1 \in G(K)$ in the same coset of H_1 as y . So $y \in H_1 + G(K)$, contradicting the assumptions. Thus Case I is complete.

Case II. H projects onto \tilde{A} .

Our assumption that H is a proper subgroup of G and that the toric part is \mathbb{G}_m implies that (up to isogeny) G splits as $T \times H = T \times \tilde{A}$. This case is essentially dealt with in [Bertrand 2009], but nevertheless we continue with the proof. We identify G/H with T . So $y/H = d \in T$ and the image a_0 of a under the projection $G \rightarrow T$ is in $LT(K)$. As $H^\partial(K^{\text{diff}})$ is the Galois group of $K_G^\sharp(y)$ over K_G^\sharp , we see that $y \in T(K_G^\sharp)$. Now $K(d)$ is a Picard–Vessiot extension of K with Galois group a subgroup of $\mathbb{G}_m(\mathbb{C})$. Moreover, since G splits as $T \times \tilde{A}$, we have $G^\partial = T^\partial \times \tilde{A}^\partial$. Hence by Fact 3.1, $K_G^\sharp = K_{\tilde{A}}^\sharp$, and by Remark 3.2, it is a Picard–Vessiot extension of K whose Galois group is a semisimple algebraic group in the constants. We deduce from the Galois theory that d is independent from K_G^\sharp over K , and hence $d \in T(K)$. So the coset of y modulo H has a representative $y_1 \in G(K)$ and $y \in H + G(K)$, contradicting our assumption. This concludes Case II and the proof of Theorem 2.6. \square

Proof of Theorem 2.7. $G = \tilde{B}$ for $B = B_{\text{sc}}$ a semiconstant semiabelian variety over K and we may assume it has toric part \mathbb{G}_m . So although the toric part is still \mathbb{G}_m , both the hypothesis and conclusion of Theorem 2.7 are stronger than in Theorem 2.6.

We have $0 \rightarrow \mathbb{G}_m \rightarrow B \rightarrow A$ where $A = A_0$ is over \mathbb{C} . Hence \tilde{A} is also over \mathbb{C} and we have $0 \rightarrow \mathbb{G}_m \rightarrow \tilde{B} \rightarrow \tilde{A} \rightarrow 0$, and $G = \tilde{B}$. As $\tilde{A}^\partial = \tilde{A}(\mathbb{C}) \subseteq \tilde{A}(K)$, we see:

Fact 3.4. $G^\partial(K^{\text{diff}})$ is a union of cosets of $\mathbb{G}_m(\mathbb{C})$, each defined over K .

We are given a logarithmic differential equation $\partial \ln_G(-) = a \in LG(K)$ and solution $y \in G(K^{\text{diff}})$. We let H be a minimal connected ∂ -subgroup of G , defined

over K , such that $a \in LH + \partial \ell n_G(G(K))$, or equivalently, $y \in H + G(K) + G^\partial(K^{\text{diff}})$. We want to prove that $H^\partial(K^{\text{diff}})$ is the Galois group of $K_G^\sharp(y)$ over K_G^\sharp .

By [Theorem 2.6](#), we may assume that $H \neq G$. Note that after translating y by an element of $G(K)$ plus an element of $G^\partial(K^{\text{diff}})$, we can assume that $y \in H$. If H is trivial then everything is clear.

We go through the cases.

Case I. $H = \mathbb{G}_m$.

Then by [Fact 2.1](#), $K(y)$ is a Picard–Vessiot extension of K , with Galois group $\mathbb{G}_m(\mathbb{C})$, and all that remains to be proved is that y is algebraically independent from K^\sharp over K . Let $z_1, \dots, z_n \in G^\partial(K^{\text{diff}})$, and we want to show that y is independent from z_1, \dots, z_n over K (in the sense of DCF_0). By [Fact 3.4](#), $K(z_1, \dots, z_n)$ is a Picard–Vessiot extension of K and we can assume the Galois group is $\mathbb{G}_m^n(\mathbb{C})$. Suppose towards a contradiction that $\text{tr.deg}(K(y, z_1, \dots, z_n)/K) < n + 1$, and so must equal n . Hence the differential Galois group of $K(y, z_1, \dots, z_n)/K$ is of the form $L(\mathbb{C})$ where L is the algebraic subgroup of \mathbb{G}_m^{n+1} defined by $x^k x_1^{k_1} \cdots x_n^{k_n} = 1$ for k, k_i integers such that $k \neq 0$ and not all $k_i = 0$. It easily follows that in additive notation, $ky + k_1 z_1 + \cdots + k_n z_n \in G(K)$. So ky is of the form $z + g$ for $z \in G^\partial(K^{\text{diff}})$ and $g \in G(K)$. Let $z' \in G^\partial$ and $g' \in G(K)$ be such that $kz' = z$ and $kg' = g$. Then $k(y - (z' + g')) = 0$, so $y - (z' + g)$ is a torsion point of G and hence also in G^∂ . We conclude that $y \in G^\partial(K^{\text{diff}}) + G(K)$, contradicting our assumptions on y . This concludes the proof in [Case I](#).

Case II. H projects onto \tilde{A} .

So our assumption that $G \neq H$ implies that up to isogeny G is $T \times \tilde{A}$, and so defined over \mathbb{C} . Now everything follows from [Fact 2.1](#).

Case III. Otherwise.

This is more or less a combination of the previous cases. To begin, suppose H is disjoint from T (up to a finite set). So $H \leq \tilde{A}$ is a constant group, and by [Fact 2.1](#), $H^\partial(K^{\text{diff}}) = H(\mathbb{C})$ is the Galois group of $K(y)$ over K . By [Fact 3.4](#) the Galois theory tells us that y is independent from K_G^\sharp over K , so $H(\mathbb{C})$ is the Galois group of $K^\sharp(y)$ over K^\sharp as required.

So we may assume that $T \leq H$. Let $H_1 \leq H$ be the differential Galois group of $K_G^\sharp(y)$ over K_G^\sharp , and we suppose for a contradiction that $H_1 \neq H$. As in the proof of [Proposition 2.5](#), H_1 is defined over K . By the remark after [Fact 2.2](#), we can assume that H_1 is connected.

Case III(a). H_1 is a complement of T in H (in the usual sense that $H_1 \times T \rightarrow H$ is an isogeny).

So $y/H_1 \in T(K_G^\sharp)$. Let $y_1 = y/H_1$. If $y_1 \notin T(K)$, then $K(y_1)$ is a Picard–Vessiot extension of K with Galois group $\mathbb{G}_m(\mathbb{C})$. The proof in [Case I](#) above shows that

$y_1 \in G^\partial(K^{\text{diff}}) + G(K)$, whereby $y \in H_1 + G^\partial(K^{\text{diff}}) + G(K)$, contradicting the minimality assumptions on H .

Case III(b). $H_1 + T$ is a proper subgroup of H .

Note that since we are assuming $H_1 \neq H$, the negation of **Case III(a)** forces **Case III(b)** to hold. Let $H_2 = H_1 + T$, so H/H_2 is a constant group, say H_3 , which is a vectorial extension of an abelian variety. Then $y_2 = y/H_2 \in H_3(K_G^\sharp)$, and $K(y_2)$ is a Picard–Vessiot extension of K with Galois group a subgroup of $H_3(\mathbb{C})$. **Fact 3.4** and the Galois theory imply that $y_2 \in H_3(K)$. Hence $y \in H_2 + G(K)$, contradicting the minimality of H again.

This completes the proof of **Theorem 2.7**. □

3C. Discussion on nongeneric cases. We complete this section with a discussion of some complications arising when one would like to drop either the genericity assumption in **Theorem 2.6**, or the restriction on the toric part in both **Theorems 2.6** and **2.7**.

Let us first give an example which will have to be considered if we drop the genericity assumption in **Theorem 2.6**, and give some positive information as well as identify some technical complications. Let A be a simple abelian variety over K which has \mathbb{C} -trace 0 and such that $U_A \neq 0$. (Note that such an example appears below in **Section 5B** connected with Manin map issues.) Let B be a nonsplit extension of A by \mathbb{G}_m , and let $G = \widetilde{B}$. We have $\pi : G \rightarrow \widetilde{A}$ with kernel \mathbb{G}_m , and let H be $\pi^{-1}(U_A)$, a ∂ -subgroup of G . Let $a \in LH(K)$ and $y \in H(K^{\text{diff}})$ with $d\ell n_H(y) = a$. We have to compute $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp)$. **Conjecture 2.3** predicts that it is the dimension of the smallest algebraic ∂ -subgroup H_1 of H such that $y \in H_1 + G(K) + G^\partial(K^{\text{diff}})$.

Lemma 3.5. *With the above notation, suppose $y \notin H_1 + G(K) + G^\partial(K^{\text{diff}})$ for any proper algebraic ∂ -subgroup H_1 of H over K . Then $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp) = \dim(H)$ (and H is the Galois group).*

Proof. Let z and α be the images of y and a , respectively, under the maps $H \rightarrow U_A$ and $LH \rightarrow L(U_A) = U_A$ induced by $\pi : G \rightarrow \widetilde{A}$. So $\partial \ell n_{\widetilde{A}}(z) = \alpha$ with $\alpha \in L\widetilde{A}(K)$.

Claim. We have $z \notin U + \widetilde{A}(K) + \widetilde{A}^\partial(K^{\text{diff}})$ for any proper algebraic ∂ -subgroup U of U_A over K .

Proof of claim. Suppose otherwise. Then lifting suitable $z_2 \in \widetilde{A}(K)$ and $z_3 \in \widetilde{A}(K^{\text{diff}})$ to $y_2 \in G(K)$ and $y_3 \in G^\partial(K^{\text{diff}})$, respectively, we see that $y - (y_2 + y_3) \in \pi^{-1}(U)$, a proper algebraic ∂ -subgroup of H , a contradiction.

As in **Case I** in the proof of **Proposition 2.5** above, we may now conclude that $\text{tr.deg}(K_A^\sharp(z)/K_A^\sharp) = \dim(U_A)$, and U_A is the Galois group. Now K_G^\sharp is a union of Picard–Vessiot extensions of $K_A^\sharp = K_{U_A}^\sharp$, each with Galois group \mathbb{G}_m (by

Remark 3.3), so the Galois theory tells us that z is independent from K_G^\sharp over $K_{\tilde{A}}^\sharp$. Hence the differential Galois group of $K_G^\sharp(z)$ over K_G^\sharp is U_A^∂ . But then the Galois group of $K_G^\sharp(y)$ over K_G^\sharp will be the group of ∂ -points of a ∂ -subgroup of H which projects onto U_A . The only possibility is H itself, because otherwise H splits as $\mathbb{G}_m \times U_A$ as a ∂ -group, which contradicts (v) of Section 2 of [Bertrand 2009]. This completes the proof. \square

Essentially the same argument applies if we replace H by the preimage under π of some nontrivial ∂ -subgroup of U_A . So this shows that the scenario described right before **Lemma 3.5** reduces to the case where $a \in LT$ where T is the toric part \mathbb{G}_m (of both G and H), and we may assume $y \in T(K^{\text{diff}})$. We would like to show (in analogy with **Lemma 3.5**) that if $y \notin G(K) + G^\partial(K^{\text{diff}})$ then $\text{tr.deg}(K_G^\sharp(y)/K_G^\sharp) = 1$. Of course already $K(y)$ is a Picard–Vessiot extension of K with Galois group $T(\mathbb{C})$, and we have to prove that y is independent from K_G^\sharp over K . One deduces from the Galois theory that y is independent from $K_{U_A}^\sharp$ over K . It remains to show that for any $z_1, \dots, z_n \in G^\partial(K^{\text{diff}})$, y is independent from z_1, \dots, z_n over $K_{U_A}^\sharp$. If not, the discussion in **Case I** of the proof of **Theorem 2.7** gives that $y = z + g$ for some $z \in G^\partial(K^{\text{diff}})$ and $g \in G(K_{U_A}^\sharp)$, but an additional argument seems necessary to yield a contradiction.

Similar and other issues arise when we want to drop the restriction on the toric part. For example in **Case II** in the proof of **Theorem 2.6**, we can no longer deduce the splitting of G as $T \times \tilde{A}$. And in the proof of **Theorem 2.7**, both the analogues of **Case I** ($H = T$) and **Case II** (H projects on to \tilde{A}) present technical difficulties.

4. Lindemann–Weierstrass

We here prove Corollaries 2.8, 2.9, and 2.10.

4A. General results.

Proof of Corollary 2.8. We first prove (i). Write G for \tilde{B} . Let $\tilde{x} \in LG(K)$ be a lift of x and $\tilde{y} \in G^{(0)}$ a solution of $\partial \ln_G(-) = \tilde{x}$. We refer to Section 1.2 and Lemma 4.2 of [Bertrand and Pillay 2010] for a discussion of the equivalence of the hypotheses

$$x \notin LH(K) + LB_0(\mathbb{C}) \text{ for any proper semiabelian subvariety } H \text{ of } B,$$

and

$$(*) \quad \tilde{x} \notin LH(K) + (LG)^\partial(K) \text{ for any proper algebraic subgroup } H \text{ of } G \text{ over } K.$$

Let $a = \partial_{LG}(\tilde{x})$. So \tilde{y} is a solution of the logarithmic differential equation (over K) $\partial \ln_G(-) = a$. We want to show that $\text{tr.deg}(K_G^\sharp(\tilde{y})/K_G^\sharp) = \dim(G)$. If not, we may assume that $\tilde{y} \in G(K^{\text{diff}})$, and so by **Theorem 2.6**, $\tilde{y} \in H + G(K)$ for some proper

connected algebraic ∂ -subgroup H of G defined over K . Extend H to a maximal proper connected ∂ -subgroup H_1 of G defined over K . Then G/H_1 is either

- (a) \mathbb{G}_m , or
- (b) a simple abelian variety A_0 over \mathbb{C} , or
- (c) the quotient of \tilde{A}_1 by a maximal unipotent ∂ -subgroup, where A_1 is a simple abelian variety over K with \mathbb{C} -trace 0.

Let x', y' be the images of \tilde{x}, \tilde{y} under the map $G \rightarrow G/H_1$ and induced map $LG \rightarrow L(G/H_1)$. So both x' and y' are K -rational. Moreover the hypothesis $(*)$ is preserved in G/H_1 (by our assumptions on G and Lemma 4.2(ii) of [Bertrand and Pillay 2010]). As $\partial \ell n_{G/H_1}(y') = \partial_{L(G/H_1)}(x')$, we have a contradiction in each of the cases (a), (b), and (c) listed above, by virtue of the truth of Ax–Lindemann in the constant case, as well as Manin–Chai (Proposition 4.4 in [Bertrand and Pillay 2010]).

(ii) Immediate as in [Bertrand and Pillay 2010]: choosing $\tilde{y} = \exp_G(\tilde{x})$, then $\exp_B(y)$ is the projection of \tilde{y} on B . □

Proof of Corollary 2.9. This is like the proof of Corollary 2.8. So $x \in LA(K)$. Let $\tilde{x} \in L\tilde{A}(K)$ lift x and let $\tilde{y} \in \tilde{A}(K^{\text{diff}})$ be such that $\partial \ell n_{\tilde{A}}(\tilde{y}) = \partial_{L\tilde{A}}(\tilde{x}) = a$, say. Let B be a minimal abelian subvariety of A such that $x \in LB(K) + LA_0(\mathbb{C})$, and we want to prove that $\text{tr.deg}(K_{\tilde{A}}^{\sharp}(\tilde{y})/K_{\tilde{A}}^{\sharp}) = \dim(\tilde{B})$.

Claim. We may assume that $x \in LB(K)$, $\tilde{x} \in L\tilde{B}(K)$, and $\tilde{y} \in \tilde{B}(K^{\text{diff}})$.

Proof of claim. Let $x = x_1 + c$ for $x_1 \in LB$ and $c \in LA_0(\mathbb{C})$. Let $\tilde{x}_1 \in L\tilde{B}(K)$ be a lift of x_1 and $\tilde{c} \in L\tilde{A}_0(\mathbb{C})$ be a lift of c . Finally let $\tilde{y}_1 \in \tilde{B}(K^{\text{diff}})$ be such that $\partial \ell n_{\tilde{A}}(\tilde{y}_1) = \partial_{L\tilde{A}}(\tilde{x}_1) = a_1$, say. As $\tilde{x}_1 + \tilde{c}$ projects onto x , it differs from \tilde{x} by an element $z \in LW(K)$. Now $\partial_{L\tilde{A}}(z) = \partial \ell n_{\tilde{A}}(z)$. So

$$a = \partial_{L\tilde{A}}(\tilde{x}) = \partial_{L\tilde{A}}(\tilde{x}_1 + \tilde{c} + z) = \partial_{L\tilde{A}}(\tilde{x}_1) + \partial \ell n_{\tilde{A}}(z) = a_1 + \partial \ell n_{\tilde{A}}(z).$$

Hence $\partial \ell n(\tilde{y}_1 + z) = a$, and so $\tilde{y}_1 + z$ differs from \tilde{y} by an element of \tilde{A}^{∂} . Hence $\text{tr.deg}(K_{\tilde{A}}^{\sharp}(\tilde{y}_1)/K_{\tilde{A}}^{\sharp}) = \text{tr.deg}(K_{\tilde{A}}^{\sharp}(\tilde{y}_1)/K_{\tilde{A}}^{\sharp})$. Moreover the same hypothesis remains true of x_1 (namely B is minimal such that $x_1 \in LB + LA_0(\mathbb{C})$). So we can replace x, \tilde{x}, \tilde{y} by $x_1, \tilde{x}_1, \tilde{y}_1$.

As recalled in the proof of Corollary 2.8 (see Corollary H.5 of [Bertrand and Pillay 2010]), the condition that $x \notin B_1(K) + LA_0(\mathbb{C})$ for any proper abelian subvariety B_1 of B is equivalent to

$$(*) \quad \tilde{x} \notin LH(K) + (L\tilde{A})^{\partial}(K) \text{ for any proper algebraic subgroup } H \text{ of } \tilde{B} \text{ defined over } K.$$

Now we can use the Galois-theoretic result Proposition 2.5, namely the truth of Conjecture 2.3 for \tilde{A} , as above. That is, if to obtain a contradiction we suppose $\text{tr.deg}(K_{\tilde{A}}^{\sharp}(\tilde{y})/K_{\tilde{A}}^{\sharp}) < \dim(\tilde{B})$, then $\tilde{y} \in H + \tilde{A}(K) + (\tilde{A})^{\partial}(K^{\text{diff}})$ for some proper

connected algebraic ∂ -subgroup of \widetilde{B} , defined over K , and moreover H^∂ is the differential Galois group of $K_{\widetilde{A}}^\sharp(\widetilde{y})/K_{\widetilde{A}}^\sharp$. As at the end of the proof of [Corollary 2.8](#) above, we get a contradiction by choosing H_1 to be a maximal proper connected algebraic ∂ -subgroup of \widetilde{A} containing H and defined over K . This concludes the proof of [Corollary 2.9](#). \square

4B. Semiabelian surfaces. We first recall the counterexample from Section 5.3 of [[Bertrand and Pillay 2010](#)]. This example shows that in [Corollary 2.8](#), we cannot drop the assumption that the semiconstant part is constant. We go through it again briefly. Let B over K be a nonconstant extension of a constant elliptic curve $E = E_0$ by \mathbb{G}_m , and let $G = \widetilde{B}$. Let $\tilde{x} \in LG(K)$ map onto a point \check{x} in $L\widetilde{E}(\mathbb{C})$ which itself maps onto a nonzero point \bar{x} of $LE(\mathbb{C})$. As pointed out in [[Bertrand and Pillay 2010](#)], we have $(LG)^\partial(K) = (L\mathbb{G}_m)(\mathbb{C})$, whereby \tilde{x} satisfies the hypothesis $(*)$ from [Corollary 2.8](#): $\tilde{x} \notin LH(K) + (LG)^\partial(K)$ for any proper algebraic subgroup H of G . Let $a = \partial_{LG}(\tilde{x}) \in LG(K)$, and $\tilde{y} \in G(K^{\text{diff}})$ such that $\partial \ln_G(\tilde{y}) = a$. Then as the image of a in $L\widetilde{E}$ is 0, \tilde{y} projects onto a point of $\widetilde{E}(\mathbb{C})$, and hence \tilde{y} is in a coset of \mathbb{G}_m defined over K , whereby $\text{tr.deg}(K(\tilde{y})/K) \leq 1$, so a fortiori the same is true with K_G^\sharp in place of K . A consequence of [Corollary 2.10](#), in fact the main part of its proof, is that with the above choice of \tilde{x} , we have $\text{tr.deg}(K_G^\sharp(\tilde{y})/K_G^\sharp) = 1$ (as announced in [[Bertrand et al. 2013](#), Footnote 5]).

Proof of [Corollary 2.10](#). Let us fix notation: B is a semiabelian variety over K with toric part \mathbb{G}_m and abelian quotient a not necessarily constant elliptic curve E/K , with constant part E_0 ; G denotes the universal vectorial extension \widetilde{B} of B and \widetilde{E} the universal vectorial extension of E . For $x \in LB(K)$, \tilde{x} denotes a lift of x to a point of $LG(K)$, \bar{x} denotes the projection of x to $LE(K)$, and \check{x} denotes the projection of \tilde{x} to $L\widetilde{E}(K)$.

Recall the hypothesis $(*)$ in [Corollary 2.10](#): $x \notin LH + LB_0(\mathbb{C})$ for any proper algebraic subgroup H of B . As pointed out after the statement of [Corollary 2.10](#), under this hypothesis, the condition $\bar{x} \in LE_0(\mathbb{C})$ can occur only if B is semiconstant and not constant. Indeed, if B were not semiconstant then $E_0 = 0$, so $x \in L\mathbb{G}_m$, contradicting the hypothesis on x . And if B were constant then $B = B_0$ and \bar{x} would have a lift in $LB_0(\mathbb{C})$, whereby $x \in L\mathbb{G}_m + LB_0(\mathbb{C})$, contradicting the hypothesis.

Now if the semiconstant part of B is constant, then we can simply quote [Corollary 2.8](#), bearing in mind the paragraph above which rules out the possibility that $\bar{x} \in LE_0(\mathbb{C})$. So we will assume that $B_{\text{sc}} \neq B_0$, namely $E = E_0$ and $B_0 = \mathbb{G}_m$.

Case I. We have $\bar{x} \in LE(\mathbb{C}) (= LE_0(\mathbb{C}))$ as $E = E_0$.

This is where the bulk of the work goes. We first check that we are essentially in the situation of the ‘‘counterexample’’ mentioned above. The argument is a bit like in the proof of the claim in [Corollary 2.9](#). Note that $\bar{x} \neq 0$ by hypothesis $(*)$. Let \check{x}' be a lift of \bar{x} to a point in $L\widetilde{E}(\mathbb{C})$ (noting that \widetilde{E} is also over \mathbb{C}). Then $\check{x}' = \check{x} - \beta$ for some

$\beta \in L\mathbb{G}_a(K)$. Let $\tilde{x}' = \tilde{x} - \beta$. Let $a' = \partial_{LG}(\tilde{x}')$. Then (as $\partial_{LG}(\beta) = \partial \ln_G(\beta)$, under the usual identifications) $a' = a + \partial \ln_G(\beta)$, and if $\tilde{y}' \in G$ is such that $\partial \ln_G(\tilde{y}') = a'$ then $\partial \ln_G(\tilde{y}' - \beta) = a$. As $\beta \in G(K)$, $\text{tr.deg}(K_G^\sharp(\tilde{y}')/K_G^\sharp) = \text{tr.deg}(K_G^\sharp(\tilde{y})/K_G^\sharp)$.

The end result is that we can assume that $\tilde{x} \in LG(K)$ maps onto $\tilde{x}' \in L\tilde{E}(\mathbb{C})$ which in turn maps on to our nonzero $\bar{x} \in LE(\mathbb{C})$, precisely the situation in the example above from Section 5.1 of [Bertrand and Pillay 2010]. So to deal with **Case I**, we need to prove:

Claim 1. We have $\text{tr.deg}(K_G^\sharp(\tilde{y})/K_G^\sharp) = 1$.

Proof of Claim 1. Remember that a denotes $\partial_{LG}(\tilde{x})$. Now by [Theorem 2.7](#), it suffices to prove that $a \notin \partial \ln_G(G(K))$.

We assume for a contradiction that there is $\tilde{s} \in G(K)$ such that

$$(\dagger) \quad a = \partial_{LG}(\tilde{x}) = \partial \ln_G(\tilde{s}).$$

This is the semiabelian analogue of a Manin kernel statement, which can probably be studied directly, but we will deduce the contradiction from [Bertrand et al. 2013]. Let $\tilde{x}_1 = \log_G(\tilde{s})$ be a solution given by complex analysis to the linear inhomogeneous equation $\partial_{LG}(-) = \partial \ln_G(\tilde{s})$. Namely, with notations as in the appendix to [Bertrand and Pillay 2010] (generalizing those given after [Corollary 2.8](#) above), a local analytic section of $LG^{\text{an}}/S^{\text{an}}$ such that $\exp_G(\tilde{x}_1) = \tilde{s}$. Let $\xi \in (LG)^\partial$ be $\tilde{x} - \tilde{x}_1$. Then ξ lives in a differential field (of meromorphic functions on some disc in S) which extends K and has the same constants as K , namely \mathbb{C} . As ξ is the solution of a linear homogeneous differential equation over K , it follows that ξ lives in $(LG)^\partial(K^{\text{diff}})$. Hence, as $\tilde{x} \in LG(K)$, this implies that $\tilde{x}_1 \in LG(K_{LG}^\sharp)$ where K_{LG}^\sharp is the differential field generated over K by $(LG)^\partial(K^{\text{diff}})$.

Now from Section 5.1 of [Bertrand et al. 2013], K_{LG}^\sharp coincides with the “field of periods” F_q attached to the point $q \in \hat{E}(K)$ which parametrizes the extension B of E by \mathbb{G}_m . Hence from (\dagger) we conclude that $F_q(\log_G(\tilde{s})) = F_q$.

Let $s \in B(K)$ be the projection of \tilde{s} , and $p \in E(K)$ the projection of s . By the discussion in Section 5.1 of [Bertrand et al. 2013], $F_{pq}(\log_B(s)) = F_q(\log_G(\tilde{s}))$. Therefore, $F_q = F_{pq} = F_{pq}(\log_B(s))$.

Now as $\tilde{x} \in LG(K)$ maps onto the constant point $\check{x} \in L\tilde{E}(\mathbb{C})$, so also \tilde{s} maps onto a constant point $\check{p} \in \tilde{E}(\mathbb{C})$ and hence $p \in E(\mathbb{C})$. So we are in **Case (SC2)** of the proof of the Main Lemma of [Bertrand et al. 2013, Section 6], namely p constant while q nonconstant. The conclusion of **(SC2)** is that $\log_B(s)$ is transcendental over F_{pq} if p is nontorsion. So the previous equality forces $p \in E(\mathbb{C})$ to be torsion.

Let $\tilde{s}_{\text{tor}} \in G(K)$ be a torsion point lifting p , hence $\tilde{s} - \tilde{s}_{\text{tor}}$ is a K -point of the kernel of the surjection $G \rightarrow E$. Thus $\tilde{s} = \tilde{s}_{\text{tor}} + \delta + \beta$ where $\beta \in \mathbb{G}_a(K)$ and $\delta \in \mathbb{G}_m(K)$. Taking logs, putting again $\xi = \tilde{x} - \tilde{x}_1$, and using that $\log_G(-)$ restricted to $\mathbb{G}_a(K)$ is the identity, we see that $\tilde{x} = \xi + \log_G(\tilde{s}_{\text{tor}}) + \log_G(\delta) + \beta = \xi' + \log_{\mathbb{G}_m}(\delta) + \beta$

where $\xi' \in (LG)^\partial$. It follows that $\ell = \log_{\mathbb{G}_m}(\delta) \in K_G^\sharp = F_q$. But by Lemma 1 of [Bertrand et al. 2013] (proof of Main Lemma in isotrivial case, but reversing roles of p and q), such ℓ is transcendental over F_q unless δ is constant.

Hence $\delta \in \mathbb{G}_m(\mathbb{C})$, whereby $\log_{\mathbb{G}_m}(\delta) \in L\mathbb{G}_m(\mathbb{C})$ so is in $(LG)^\partial(K^{\text{diff}})$, and we conclude that $\tilde{x} - \beta \in (LG)^\partial(K^{\text{diff}})$. As also $\tilde{x} - \beta \in LG(K)$, from Claim III in Section 5.3 of [Bertrand and Pillay 2010] (alternatively, using the fact that $K_{LG}^\sharp = F_q$ has transcendence degree 2 over K), we conclude that $\tilde{x} - \beta \in L\mathbb{G}_m(\mathbb{C})$ whereby $\tilde{x} \in L\mathbb{G}_a(K) + L\mathbb{G}_m(\mathbb{C})$, contradicting that x projects onto a nonzero element of LE . This contradiction completes the proof of Claim 1 and hence of Case I of Corollary 2.10.

Case II. The point $\bar{x} \in LE(K) \setminus LE(\mathbb{C})$ is a nonconstant point of $LE(K) = LE_0(K)$.

Let $\tilde{y} \in G(K^{\text{diff}})$ be such that $\partial \ell n_G(\tilde{y}) = a = \partial_{LG}(\tilde{x})$. Let \check{y} be the projection of \tilde{y} to \tilde{E} . Hence $\partial \ell n_{\tilde{E}}(\check{y}) = \partial_{L\tilde{A}}(\check{x})$ (remembering that \check{x} is the projection of \tilde{x} to $L\tilde{E}$). Noting that \check{x} lifts $\bar{x} \in LE(K)$, and using our case hypothesis, we can apply Corollary 2.9 to E to conclude that $\text{tr.deg}(K(\check{y})/K) = 2$ with Galois group $\tilde{E}^\partial(K^{\text{diff}}) = \tilde{E}(\mathbb{C})$. (In fact as E is constant this is already part of the Ax–Kolchin framework and appears in [Bertrand 2008].)

Claim 2. We have $\text{tr.deg}(K_G^\sharp(\check{y})/K_G^\sharp) = 2$.

Proof of Claim 2. Fact 3.4 applies to the current situation, showing that K_G^\sharp is a directed union of Picard–Vessiot extensions of K each with Galois group some product of $\mathbb{G}_m^n(\mathbb{C})$'s. As there are no proper algebraic subgroups of $\tilde{E}(\mathbb{C}) \times \mathbb{G}_m^n(\mathbb{C})$ projecting onto each factor, it follows from the Galois theory that \check{y} is independent from K_G^\sharp over K , yielding Claim 2.

Now $K_G^\sharp(\check{y})/K_G^\sharp$ is a differential Galois extension with Galois group of the form $H^\partial(K^{\text{diff}})$ where H is a connected algebraic ∂ -subgroup of G . So H^∂ projects onto the (differential) Galois group of $K_G^\sharp(\check{y})$ over K_G^\sharp , which by Claim 2 is $\tilde{E}^\partial(K^{\text{diff}})$. In particular, H projects onto \tilde{E} . If H is a proper subgroup of G , then projecting H and \tilde{E} to B and E , respectively, shows that B splits (up to isogeny), so $B = B_0$ is constant, contradicting the current assumptions. Hence the (differential) Galois group of $K_G^\sharp(\check{y})$ over K_G^\sharp is $G^\partial(K^{\text{diff}})$, whereby $\text{tr.deg}(K_G^\sharp(\check{y})/K_G^\sharp)$ is 3. This concludes the proof of Corollary 2.10. \square

4C. An Ax–Schanuel conjecture. As a conclusion to the first two themes of the paper, we may say that both at the Galois-theoretic level and for Lindemann–Weierstrass, we have obtained rather definitive results for families of abelian varieties, and working over a suitable base K^\sharp . There remain open questions for families of semiabelian varieties, such as Conjecture 2.3, as well as dropping the restriction on the toric part in Theorems 2.6 and 2.7 and Corollaries 2.8 and 2.10. It also remains to formulate a qualitative description of $\text{tr.deg}(K^\sharp(\exp_B(x))/K^\sharp)$

where B is a semiabelian variety over K of dimension > 2 , and $x \in LB(K)$, under the nondegeneracy hypothesis that $x \notin LH + LB_0(\mathbb{C})$ for any proper semiabelian subvariety H of B .

Before turning to our third theme, it seems fitting to propose a more general *Ax–Schanuel* conjecture for families of abelian varieties:

Conjecture 4.1. *Let A be an abelian variety over $K = \mathbb{C}(S)$ for a curve S/\mathbb{C} , and let F be the field of meromorphic functions on some disc in S . Let K^\sharp now denote K_{LA}^\sharp (which contains $K_{\tilde{A}}^\sharp$). Let \tilde{x}, \tilde{y} be F -rational points of $L\tilde{A}, \tilde{A}$, respectively, such that $\exp_{\tilde{A}}(\tilde{x}) = \tilde{y}$, and let y be the projection of \tilde{y} on A . Assume that $y \notin H + A_0(\mathbb{C})$ for any proper algebraic subgroup H of A . Then $\text{tr.deg}(K^\sharp(\tilde{x}, \tilde{y})/K^\sharp) \geq \dim(\tilde{A})$.*

We point out that the assumption concerns y , and not the projection x of \tilde{x} to LA . Indeed, the conclusion would in general not hold true under the weaker hypothesis that $x \notin LH + LA_0(\mathbb{C})$ for any proper abelian subvariety H of A . As a counterexample, take for A a simple nonconstant abelian variety over K , and for \tilde{x} a nonzero period of $L\tilde{A}$. Then $x \neq 0$ satisfies the hypothesis above and \tilde{x} is defined over $K^\sharp = K_{LA}^\sharp$, but $\tilde{y} = \exp_{\tilde{A}}(\tilde{x}) = 0$, so $\text{tr.deg}(K^\sharp(\tilde{x}, \tilde{y})/K^\sharp) = 0$.

Finally, here is a concrete corollary of the conjecture. Let $E: y^2 = x(x-1)(x-t)$ be the universal Legendre elliptic curve over $S = \mathbb{C} \setminus \{0, 1\}$, and let $\omega_1(t), \omega_2(t)$ be a basis of the group of periods of E over some disk, so $K^\sharp = K_{LE}^\sharp$ is the field generated over $K = \mathbb{C}(t)$ by ω_1, ω_2 and their first derivatives. Let $\wp = \wp_t(z), \zeta = \zeta_t(z)$ be the standard Weierstrass functions attached to $\{\omega_1(t), \omega_2(t)\}$. For $g \geq 1$, consider $2g$ algebraic functions $\alpha_1^{(i)}(t), \alpha_2^{(i)}(t) \in K^{\text{alg}}, i = 1, \dots, g$, and assume that the vectors

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} \alpha_1^{(1)} \\ \alpha_2^{(1)} \end{pmatrix}, \dots, \begin{pmatrix} \alpha_1^{(g)} \\ \alpha_2^{(g)} \end{pmatrix}$$

are linearly independent over \mathbb{Z} . Then the $2g$ functions

$$\wp(\alpha_1^{(i)}\omega_1 + \alpha_2^{(i)}\omega_2), \zeta(\alpha_1^{(i)}\omega_1 + \alpha_2^{(i)}\omega_2), \quad i = 1, \dots, g,$$

of the variable t are algebraically independent over K^\sharp . In the language of [Bertrand et al. 2013, Section 3.3], this says in particular that a g -tuple of \mathbb{Z} -linearly independent local analytic sections of E/S with algebraic *Betti* coordinates forms a generic point of E^g/S . Such a statement is not covered by our Lindemann–Weierstrass results, which concern analytic sections with algebraic logarithms.

5. Manin maps

5A. Injectivity. We here prove Theorem 2.13 and Proposition 2.15. Both statements will follow fairly quickly from Fact 5.1 below, which is Theorem 4.3 of [Bertrand 2011] and relies on the strongest version of “Manin–Chai”, namely formula (2*) from Section 4.1 of [Bertrand 2011]. We should mention that a more

direct proof of [Proposition 2.15](#) can be extracted from the proof of [Proposition J.2](#) (Manin–Coleman) in [\[Bertrand and Pillay 2010\]](#). But we will stick with the current proof below, as it provides a good introduction to the counterexample in [Section 5B](#).

We set up some notation: K is $\mathbb{C}(t)^{\text{alg}}$ as usual, A is an abelian variety over K , and A_0 is the \mathbb{C} -trace of A . For $y \in \tilde{A}(K)$, we let \bar{y} be its image in $A(K)$. Let $b = \partial \ell n_{\tilde{A}}(y)$. We consider the differential system in unknown x :

$$\nabla_{L\tilde{A}}(x) = b,$$

where we write $\nabla_{L\tilde{A}}$ for $\partial_{L\tilde{A}}$. Let $K_{L\tilde{A}}^{\sharp}$ be the differential field generated, over K , by $(L\tilde{A})^{\partial}(K^{\text{diff}})$. So for x a solution in $L\tilde{A}(K^{\text{diff}})$, the differential Galois group of $K_{L\tilde{A}}^{\sharp}(x)$ over $K_{L\tilde{A}}^{\sharp}$ pertains to Picard–Vessiot theory, and is well-defined as a \mathbb{C} -subspace of the \mathbb{C} -vector space $(L\tilde{A})^{\partial}(K^{\text{diff}})$.

Fact 5.1 ($A =$ any abelian variety over $K = \mathbb{C}(t)^{\text{alg}}$). *Let $y \in \tilde{A}(K)$. Let B be the smallest abelian subvariety of A such that a multiple of \bar{y} by a nonzero integer is in $B + A_0(\mathbb{C})$. Let x be a solution of $\nabla_{L\tilde{A}}(-) = b$ in $L\tilde{A}(K^{\text{diff}})$. Then the differential Galois group of $K_{L\tilde{A}}^{\sharp}(x)$ over $K_{L\tilde{A}}^{\sharp}$ is $(L\tilde{B})^{\partial}(K^{\text{diff}})$. In particular, $\text{tr.deg}(K_{L\tilde{A}}^{\sharp}(x)/K_{L\tilde{A}}^{\sharp}) = \dim \tilde{B} = 2 \dim B$.*

Proof of [Theorem 2.13](#). Here, the abelian variety A has \mathbb{C} -trace 0. By assumption we have $y_1, \dots, y_n \in A(K)$ and $a_1, \dots, a_n \in \mathbb{C}$ not all 0 such that

$$a_1 \mu_A(y_1) + \dots + a_n \mu_A(y_n) = 0$$

in $L\tilde{A}(K)/\partial \ell n_{\tilde{A}}(W_A)$. Lifting y_i to $\tilde{y}_i \in \tilde{A}(K)$, we derive that

$$a_1 \partial \ell n_{\tilde{A}}(\tilde{y}_1) + \dots + a_n \partial \ell n_{\tilde{A}}(\tilde{y}_n) = \partial \ell n_{\tilde{A}}(z)$$

for some $z \in W_A$. Via our identification of W_A with LW_A we write the right hand side as $\nabla_{L\tilde{A}}z$ with $z \in LW_A \subset L\tilde{A}$. Let $\tilde{x}_i \in L\tilde{A}$ be such that $\nabla_{L\tilde{A}}(\tilde{x}_i) = \partial \ell n_{\tilde{A}}(\tilde{y}_i)$. Hence $a_1 \tilde{x}_1 + \dots + a_n \tilde{x}_n - z \in (L\tilde{A})^{\partial}$, and there exists $d \in (L\tilde{A})^{\partial}$ such that

$$a_1 \tilde{x}_1 + \dots + a_n \tilde{x}_n - d = z \in LW_A.$$

Suppose for a contradiction that y_1, \dots, y_n are linearly independent with respect to $\text{End}(A)$. Then no multiple of $y = (y_1, \dots, y_n)$ by a nonzero integer lies in any proper abelian subvariety B of the traceless abelian variety $A^n = A \times \dots \times A$. By [Fact 5.1](#), we have $\text{tr.deg}(K^{\sharp}(\tilde{x}_1, \dots, \tilde{x}_n)/K^{\sharp}) = \dim(\tilde{A}^n)$, where we have set $K^{\sharp} := K_{L\tilde{A}^n}^{\sharp} = K_{L\tilde{A}}^{\sharp}$. So, the points $\tilde{x}_1, \dots, \tilde{x}_n$ of $L\tilde{A}$ are generic and independent over K^{\sharp} . Hence, because a_1, \dots, a_n are in \mathbb{C} and therefore K^{\sharp} , it follows that $a_1 \tilde{x}_1 + \dots + a_n \tilde{x}_n$ is a generic point of $L\tilde{A}$ over K^{\sharp} . And as d is a K^{\sharp} -rational point of $(L\tilde{A})^{\partial}$, also $a_1 \tilde{x}_1 + \dots + a_n \tilde{x}_n - d = z$ is a generic point of $L\tilde{A}$ over K^{\sharp} ,

so cannot lie in its strict subspace LW_A . This contradiction concludes the proof of [Theorem 2.13](#). \square

Proof of Proposition 2.15. We use the same notation as at the end of [Section 2D](#), and recall that A is traceless. Furthermore, the functoriality of $M_{K,A}$ in A allows us to assume that A is a simple abelian variety.

Step I. We show, as in the proof of [Theorem 2.13](#), that if $M_{K,A}(y_1), \dots, M_{K,A}(y_n)$ are \mathbb{C} -linearly dependent, then y_1, \dots, y_n are $\text{End}(A)$ -linearly dependent. Indeed, assume that $a_i \in \mathbb{C}$ are not all 0 and that $a_1 M_{K,A}(y_1) + \dots + a_n M_{K,A}(y_n) = 0$ in the target space $L\tilde{A}(K)/\nabla(L\tilde{A}(K))$. Lifting y_i to $\tilde{y}_i \in \tilde{A}(K)$, we derive that

$$a_1 \partial \ell n_{\tilde{A}}(\tilde{y}_1) + \dots + a_n \partial \ell n_{\tilde{A}}(\tilde{y}_n) \in \nabla(L\tilde{A}(K)).$$

Letting $\tilde{x}_i \in L\tilde{A}(K^{\text{diff}})$ be such that $\nabla \tilde{x}_i = \partial \ell n_{\tilde{A}}(\tilde{y}_i)$, we obtain a K -rational point $z \in L\tilde{A}(K)$ such that

$$a_1 \tilde{x}_1 + \dots + a_n \tilde{x}_n - z := d \in (L\tilde{A})^\partial(K^{\text{diff}}).$$

Taking $K^\sharp := K_{L\tilde{A}}^\sharp$ as in the proof of [Theorem 2.13](#), we get

$$\text{tr.deg}(K^\sharp(\tilde{x}_1, \dots, \tilde{x}_n)/K^\sharp) < \dim(\tilde{A}^n).$$

Hence by [Fact 5.1](#), some integral multiple of (y_1, \dots, y_n) lies in a proper abelian subvariety of A^n , whereby y_1, \dots, y_n are $\text{End}(A)$ -linearly dependent.

Step II. Assuming that y_1, \dots, y_n are $\text{End}(A)$ -linearly dependent, given by [Step I](#), as well as the relation on the point d above with not all $a_i = 0$, we will show that the points y_i are \mathbb{Z} -linearly dependent. Equivalently we will show that if a similar relation holds with the a_i linearly independent over \mathbb{Z} , then $y = (y_1, \dots, y_n)$ is a torsion point of A^n . Let $\tilde{x} = (\tilde{x}_1, \dots, \tilde{x}_n)$. Let B be the connected component of the Zariski closure of the group $\mathbb{Z} \cdot y$ of multiples of y in A^n . By [Fact 5.1](#), the differential Galois group of $K^\sharp(\tilde{x})$ over $K^\sharp := K_{L\tilde{A}}^\sharp$ is $(L\tilde{B})^\partial$. More precisely, the set of $\sigma(\tilde{x}) - \tilde{x}$ as σ varies in $\text{Aut}_\partial(K^\sharp(\tilde{x})/K^\sharp)$ is precisely $(L\tilde{B})^\partial \subseteq (L\tilde{A}^n)^\partial$. Since z and d are defined over K^\sharp , the relation on d implies that

$$\forall (\tilde{c}_1, \dots, \tilde{c}_n) \in (L\tilde{B})^\partial, a_1 \tilde{c}_1 + \dots + a_n \tilde{c}_n = 0.$$

Let now

$$\mathcal{B} = \{ \alpha = (\alpha_1, \dots, \alpha_n) \in (\text{End}(A))^n = \text{Hom}(A, A^n) : \alpha(A) \subseteq B \subset A^n \}.$$

Claim. Assume that a_1, \dots, a_n are linearly independent over \mathbb{Z} . Then any $\alpha \in \mathcal{B}$ is identically 0.

It follows from the claim that $B = 0$ and hence some multiple of y by a nonzero integer vanishes, namely y is a torsion point of A^n . This completes the proof of [Step II](#), hence of [Proposition 2.15](#), and we are now reduced to proving the claim.

Proof of claim. Since A is simple, $\text{End}(A)$ is an order in a simple algebra D over \mathbb{Q} . For $i = 1, \dots, n$, denote by $\rho(\alpha_i)$ the \mathbb{C} -linear map induced on $(L\tilde{A})^\delta$ by the endomorphism α_i of A . So we view $(L\tilde{A})^\delta$ as a complex representation, of degree $2 \dim A$, of the \mathbb{Z} -algebra $\text{End}(A)$, or more generally, of D . Let f^2 be the dimension of D over its center F , let e be the degree of F over \mathbb{Q} and let R be a reduced representation of D , viewed as a complex representation of degree ef . As the representation ρ is defined over \mathbb{Q} (since it preserves the Betti homology), ρ is equivalent to the direct sum $R^{\oplus r}$ of $r = 2 \dim A/ef$ copies of R (cf. [Shimura and Taniyama 1961, Section 5.1]). Furthermore,

$$R : D \rightarrow \text{Mat}_f(F \otimes \mathbb{C}) \simeq (\text{Mat}_f(\mathbb{C}))^e \subset \text{Mat}_{ef}(\mathbb{C})$$

extends by \mathbb{C} -linearity to an injection $R \otimes 1 : D \otimes \mathbb{C} \simeq (\text{Mat}_f(\mathbb{C}))^e \subset \text{Mat}_{ef}(\mathbb{C})$.

Recall now that $a_1\tilde{c}_1 + \dots + a_n\tilde{c}_n = 0$ for any $(\tilde{c}_1, \dots, \tilde{c}_n)$ in $(L\tilde{B})^\delta$. Applied to the image under $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathcal{B}$ of the generic element of $(L\tilde{A})^\delta$, this relation implies that

$$a_1\rho(\alpha_1) + \dots + a_n\rho(\alpha_n) = 0 \in \text{End}_{\mathbb{C}}((L\tilde{A})^\delta).$$

So $a_1R(\alpha_1) + \dots + a_nR(\alpha_n) = 0$ in $(\text{Mat}_f(\mathbb{C}))^e$. From the injectivity of $R \otimes 1$ on $D \otimes \mathbb{C}$ and the \mathbb{Z} -linear independence of the a_i , we derive that each $\alpha_i \in D$ vanishes, hence $\alpha = 0$, proving the claim. \square

5B. A counterexample. We conclude with the promised counterexample to the injectivity of $\mu_A \otimes 1$, namely Proposition 2.14.

Construction of A . We will use Yves André’s example of a simple traceless abelian variety A over $\mathbb{C}(t)^{\text{alg}}$ with $0 \neq U_A \subsetneq W_A$ (cf. [Bertrand and Pillay 2010], just before Remark 3.10). Since $U_A \neq W_A$, this A is not constant, but we will derive this property and the simplicity of A from another argument, borrowed from [Lange and Birkenhake 1992, Chapter 9 §6].

We start with a CM field F of degree $2k$ over \mathbb{Q} , over a totally real number field F_0 of degree $k \geq 2$, and denote by $\{\sigma_1, \bar{\sigma}_1, \dots, \sigma_k, \bar{\sigma}_k\}$ the complex embeddings of F . We further fix the CM type $S := \{\sigma_1, \bar{\sigma}_1, 2\sigma_2, \dots, 2\sigma_k\}$. By [Lange and Birkenhake 1992, Chapter 9 §6], we can attach to S and to any $\tau \in \mathcal{H}$ (the Poincaré half-plane, or equivalently, the open unit disk) an abelian variety $A = A_\tau$ of dimension $g = 2k$ and an embedding of F into $\text{End}(A) \otimes \mathbb{Q}$ such that the representation r of F on W_A is given by the type S . The representation ρ of F on $L\tilde{A}$ is then $r \oplus \bar{r}$, equivalent to twice the regular representation. (The notation used by [Lange and Birkenhake 1992] here read: $e_0 = k, d = 1, m = 2, r_1 = s_1 = 1, r_2 = \dots = r_{e_0} = 2, s_2 = \dots = s_{e_0} = 0$, so, the product of the \mathcal{H}_{r_i, s_i} of [loc. cit.] is just \mathcal{H} . Also, [loc. cit.] considers the more standard “analytic” representation of F on the Lie algebra $LA = L\tilde{A}/W_A$, which is \bar{r} in our notation.)

From the bottom of [Lange and Birkenhake 1992, p. 271], one infers that the moduli space of such abelian varieties A_τ is an analytic curve \mathcal{H}/Γ . But Shimura has shown that it can be compactified to an algebraic curve \mathcal{X} (cf. [Lange and Birkenhake 1992, p. 247]). So, we can view the universal abelian variety $A_\tau = A$ of this moduli space as an abelian variety over $\mathbb{C}(\mathcal{X})$, hence as an abelian variety A over $K = \mathbb{C}(t)^{\text{alg}}$. This will be our A ; it is by construction not constant — and it is a fourfold if we take $k = 2$, as we will in what follows.

Finally, since A is the general element over \mathcal{H}/Γ , Theorem 9.1 of [Lange and Birkenhake 1992] and the hypothesis $k \geq 2$ imply that $\text{End}(A) \otimes \mathbb{Q}$ is equal to F . Therefore, A is a simple abelian variety, necessarily traceless since it is not constant. We denote by \mathbb{O} the order $\text{End}(A)$ of F .

Action of F and of ∇ on $L\tilde{A}$. For simplicity, we will now restrict to the case $k = 2$, but the general case (requiring $2k$ points) would work in exactly the same way. So, F is a totally imaginary quadratic extension of a real quadratic field F_0 , and $L\tilde{A}$ is 8-dimensional. As said in [Bertrand and Pillay 2010], and by definition of the CM-type S , the action ρ of F splits $L\tilde{A}$ into eigenspaces for its irreducible representations σ 's, as follows:

- $W_A = D_{\sigma_1} \oplus D_{\bar{\sigma}_1} \oplus P_{\sigma_2}$, where the D 's are lines and P_{σ_2} is a plane;
- LA lifts to $L\tilde{A}$ into $D'_{\sigma_1} \oplus D'_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$, with the same notation.

Since $\nabla := \nabla_{L\tilde{A}} = \partial_{L\tilde{A}}$ commutes with the action ρ of F and since A is not constant, we infer that the maximal ∂ -submodule of W_A is

$$U_A = P_{\sigma_2},$$

while $W_A + \nabla(W_A) = \Pi_{\sigma_1} \oplus U_A \oplus \Pi_{\bar{\sigma}_1}$, with the planes

$$\begin{aligned} \Pi_{\sigma_1} &= D_{\sigma_1} \oplus D'_{\sigma_1}, \\ \Pi_{\bar{\sigma}_1} &= D_{\bar{\sigma}_1} \oplus D'_{\bar{\sigma}_1}, \end{aligned}$$

each stable under ∇ (just as is $P_{\bar{\sigma}_2}$, of course). In fact, for our proof, we only need to know that $P_{\sigma_2} \subset U_A$.

Now let $\tilde{y} \in \tilde{A}(K)$ be a lift of a point $y \in A(K)$. Going into a complex analytic setting, we choose a logarithm $\tilde{x} \in L\tilde{A}(K^{\text{diff}})$ of \tilde{y} , locally analytic on a small disk in $\mathcal{X}(\mathbb{C})$. Let further $\alpha \in \mathbb{O}$, which canonically lifts to $\text{End}(\tilde{A})$. Then $\rho(\alpha)\tilde{x}$ is a logarithm of $\alpha \cdot \tilde{y} \in \tilde{A}(K)$, and therefore satisfies

$$\nabla(\rho(\alpha)\tilde{x}) = \partial \ell n_{\tilde{A}}(\alpha \cdot \tilde{y}).$$

In fact, this appeal to analysis is not necessary; the formula just says that $\partial \ell n_{\tilde{A}}$ (and ∇) commutes with the actions of \mathbb{O} . But once one \tilde{y} and one \tilde{x} are chosen, it

will be crucial, for the desired relation (\mathfrak{R}) following [Proposition 2.14](#), that we take these $\rho(\alpha)\tilde{x}$ as solutions to the equations on the \mathbb{C} -orbit of \tilde{y} .

Concretely, if

$$\tilde{x} = x_{\sigma_2} \oplus x_{\sigma_1} \oplus x_{\bar{\sigma}_1} \oplus x_{\bar{\sigma}_2}$$

is the decomposition of \tilde{x} in

$$L\tilde{A} = P_{\sigma_2} \oplus \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2},$$

then for any $\alpha \in \mathbb{C}$, we have

$$\rho(\alpha)(\tilde{x}) = \sigma_2(\alpha)x_{\sigma_2} \oplus \sigma_1(\alpha)x_{\sigma_1} \oplus \bar{\sigma}_1(\alpha)x_{\bar{\sigma}_1} \oplus \bar{\sigma}_2(\alpha)x_{\bar{\sigma}_2}.$$

Conclusion. Let $y \in A(K)$ be a nontorsion point of the simple abelian variety A , for which we choose at will a lift \tilde{y} to $\tilde{A}(K)$ and a logarithm $\tilde{x} \in L\tilde{A}(K^{\text{diff}})$. Let $\{\alpha_1, \dots, \alpha_4\}$ be an integral basis of F over \mathbb{Q} . We will consider the 4 points $y_i = \alpha_i \cdot y$ of $A(K)$, $i = 1, \dots, 4$. Since the action of \mathbb{C} on A is faithful, they are linearly independent over \mathbb{Z} . For each $i = 1, \dots, 4$, we consider the lift $\tilde{y}_i = \alpha_i \tilde{y}$ of y_i to $L\tilde{A}(K)$, and set as above $\tilde{x}_i = \rho(\alpha_i)\tilde{x}$, which satisfies $\nabla(\tilde{x}_i) = \partial \ell n_{\tilde{A}} \tilde{y}_i$.

We claim that there exist complex numbers a_1, \dots, a_4 , not all zero, such that

$$u := a_1\tilde{x}_1 + \dots + a_4\tilde{x}_4 = (a_1\rho(\alpha_1) + \dots + a_4\rho(\alpha_4))(\tilde{x}) \in U_A(K^{\text{diff}}),$$

i.e., such that in the decomposition above of $L\tilde{A} = P_{\sigma_2} \oplus \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$, the components of $u = u_{\sigma_2} \oplus u_{\sigma_1} \oplus u_{\bar{\sigma}_1} \oplus u_{\bar{\sigma}_2}$ on the last three planes vanish.

The whole point is that the complex representation $\hat{\sigma}^{\oplus 2}$ of F which ρ induces on $\Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$ is twice the representation $\hat{\sigma} := \sigma_1 \oplus \bar{\sigma}_1 \oplus \bar{\sigma}_2$ of F on \mathbb{C}^3 , and so, does not contain the full regular representation of F . More concretely, the 4 vectors $\hat{\sigma}(\alpha_1), \dots, \hat{\sigma}(\alpha_4)$ of \mathbb{C}^3 are of necessity linearly dependent over \mathbb{C} , so, there exists a nontrivial linear relation

$$a_1\hat{\sigma}(\alpha_1) + \dots + a_4\hat{\sigma}(\alpha_4) = 0 \quad \text{in } \mathbb{C}^3$$

(where the complex numbers a_i lie in the normal closure of F). Therefore, any element $\tilde{x}_{\hat{\sigma}} = (x_{\sigma_1}, x_{\bar{\sigma}_1}, x_{\bar{\sigma}_2})$ of $\Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$ satisfies

$$(a_1\hat{\sigma}^{\oplus 2}(\alpha_1) + \dots + a_4\hat{\sigma}^{\oplus 2}(\alpha_4))\tilde{x}_{\hat{\sigma}} = 0 \quad \text{in } \Pi_{\sigma_1} \oplus \Pi_{\bar{\sigma}_1} \oplus P_{\bar{\sigma}_2}$$

(viewing each $\hat{\sigma}^{\oplus 2}(\alpha_i)$ as a (6×6) diagonal matrix inside the (8×8) diagonal matrix $\rho(\alpha_i)$), i.e., the 3 plane-components $u_{\sigma_1}, u_{\bar{\sigma}_1}, u_{\bar{\sigma}_2}$ of u all vanish, and u indeed lies in P_{σ_2} , and so in U_A .

The existence of such a point $u = a_1\tilde{x}_1 + \dots + a_4\tilde{x}_4$ in $U_A(K^{\text{diff}})$ establishes relation (\mathfrak{R}) of [Section 2D](#), and concludes the proof of [Proposition 2.14](#).

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