RICCI TENSOR OF REAL HYPERSURFACES

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Let $M$ be a real hypersurface of a complex space form $M^n(c)$, $c \neq 0$, and suppose that the structure vector field $\xi$ is an eigen vector field of the Ricci tensor $S$, which satisfies $S\xi = \beta \xi$ where $\beta$ is a function. We show that if $(\nabla_X S)Y$ is proportional to $\xi$ for any vector fields $X$ and $Y$ orthogonal to $\xi$, then $M$ is a Hopf hypersurface, and if it is perpendicular to $\xi$, then $M$ is a ruled real hypersurface.

1. Introduction

Takagi [1973] gave a classification of the homogeneous real hypersurface (see also [Takagi 1975a; 1975b]). As a consequence of this result, the structure vector $\xi$ of any homogeneous real hypersurface in $\mathbb{C}P^n$ is principal. If $\xi$ satisfies this property, then $M$ is said to be a Hopf hypersurface. When the ambient manifold is a complex hyperbolic space, Lohnherr [1998] (see also [Lohnherr and Reckziegel 1999]) discovered a homogeneous ruled real hypersurface in $\mathbb{C}H^n$ that is not a Hopf hypersurface, and further examples were given (see [Berndt and Brück 2001]). The classification theorem for homogeneous real hypersurfaces in $\mathbb{C}H^n$, $n \geq 2$, was given by Berndt and Tamaru [2007].

When a real hypersurface is Hopf, fundamental formulas are simple. So many classification theorems are given under that assumption (see, for example, [Niebergall and Ryan 1997]). Kimura [1986] has given a classification of Hopf hypersurfaces of $\mathbb{C}P^n$, $n \geq 2$, with constant principal curvatures. He showed that a real hypersurface in $\mathbb{C}P^n$ with constant principal curvatures is a Hopf hypersurface if and only if it is an open part of a homogeneous real hypersurface. A classification theorem for Hopf hypersurfaces with constant principal curvatures in $\mathbb{C}H^n$, $n \geq 2$, was given by Berndt [1989].

On the other hand, the Ricci tensor of the real hypersurfaces is an interesting subject. It is well known that any real hypersurface of $M^n(c)$, $c \neq 0$, is not Einstein. If the Ricci tensor $S$ is of the form $S(X, Y) = ag(X, Y) + b\eta(X)\eta(Y)$, then the real hypersurface is said to be pseudo-Einstein. The classification theorems for pseudo-Einstein real hypersurfaces in a complex space form $M^n(c)$ have been

MSC2010: primary 53C40; secondary 53C55, 53C25.
Keywords: real hypersurface, Ricci tensor.
completed [Cecil and Ryan 1982; Kim and Ryan 2008; Kon 1979; Montiel 1985].
Ki [1989] showed that there are no real hypersurfaces with parallel Ricci tensor,
$\nabla S = 0$, in $M^n(c), n \geq 3$. Several conditions that weaken the condition $\nabla S = 0$
have been studied (see [Ki et al. 1990; Suh 1990]).

We focus on the Ricci tensor $S$ and consider a condition $S\xi = \beta\xi$, where $\beta$
is a function. We note that this condition contains not only Hopf hypersurfaces,
$A\xi = \alpha\xi$, but also some non-Hopf hypersurfaces. For example, ruled hypersurfaces, which
are an important example of non-Hopf hypersurfaces, also satisfy $S\xi = \beta\xi$. Under
this assumption, we study some Hopf hypersurfaces and ruled real hypersurfaces
according to the direction of a covariant differentiation of $S$.

Our main result is the following theorem:

**Theorem 1.1.** Let $M$ be a connected real hypersurface of $M^n(c), c \neq 0$, and
suppose that the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta\xi$ for some function $\beta$.

1. If $(\nabla_X S)Y$ is proportional to the structure vector field $\xi$ for any vector fields
   $X$ and $Y$ orthogonal to $\xi$, then $M$ is a Hopf hypersurface.
2. If $(\nabla_X S)Y$ is perpendicular to the structure vector field $\xi$ for any vector fields
   $X$ and $Y$ orthogonal to the structure vector field $\xi$, then $M$ is a ruled real
   hypersurface.

When $n = 2$, the author gave a corresponding result in [Kon 2014].

2. Preliminaries

Let $M^n(c)$ denote the complex space form of complex dimension $n$ (real dimen-
sion $2n$) with constant holomorphic sectional curvature $4c$. We denote by $J$
the almost complex structure of $M^n(c)$. The Hermitian metric of $M^n(c)$ is denoted by $G$.

Let $M$ be a real $(2n - 1)$-dimensional hypersurface immersed in $M^n(c)$. Through-
out this paper, we suppose that $M$ is connected. We denote by $g$ the Riemannian
metric induced on $M$ from $G$. We take the unit normal vector field $N$ of $M$ in $M^n(c)$.
For any vector field $X$ tangent to $M$, we define $\phi, \eta$ and $\xi$ by

$$JX = \phi X + \eta(X)N, \quad JN = -\xi,$$

where $\phi X$ is the tangential part of $JX$, $\phi$ is a tensor field of type (1,1), $\eta$ is a
1-form, and $\xi$ is the unit vector field on $M$. We call $\xi$ the structure vector field. Then

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\phi X) = 0$$

for any vector field $X$ tangent to $M$. Moreover, we have

$$g(\phi X, Y) + g(X, \phi Y) = 0, \quad \eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y).$$

Thus $(\phi, \xi, \eta, g)$ defines an almost contact metric structure on $M$. 
We denote by $\tilde{\nabla}$ the operator of covariant differentiation in $M^n(c)$, and by $\nabla$ the operator of covariant differentiation in $M$ determined by the induced metric. Then the Gauss and Weingarten formulas are given respectively by

$$\tilde{\nabla}_X Y = \nabla_X Y + g(AX, Y)N, \quad \tilde{\nabla}_X N = -AX,$$

for any vector fields $X$ and $Y$ tangent to $M$.

For the contact metric structure on $M$, we have

$$\nabla_X \xi = \phi AX, \quad (\nabla_X \phi)Y = \eta(Y)AX - g(AX, Y)\xi.$$  

We call $A$ the shape operator of $M$. If the shape operator $A$ of $M$ satisfies $A\xi = \alpha\xi$ for some function $\alpha$, then $M$ is called a Hopf hypersurface. By the Codazzi equation, we have the following result (see [Maeda 1976]).

**Proposition A.** Let $M$ be a Hopf hypersurface in $M^n(c)$, $n \geq 2$. If $X \perp \xi$ and $AX = \lambda X$, then $\alpha = g(A\xi, \xi)$ is constant and

$$(2\lambda - \alpha)A\phi X = (\lambda\alpha + 2c)\phi X.$$  

We offer an important example of a non-Hopf hypersurface. Take a regular curve $\gamma$ in $M^n(c)$ with tangent vector field $X$. At each point of $\gamma$ there is a unique complex projective or hyperbolic hyperplane cutting $\gamma$ so as to be orthogonal to $X$ and $JX$. The union of these hyperplanes is called a ruled real hypersurface (see [Kimura and Maeda 1989; Lohnherr and Reckziegel 1999; Niebergall and Ryan 1997]).

We remark that the shape operator $A$ is $\eta$-parallel if it satisfies $g((\nabla_X A)Y, Z) = 0$ for any $X, Y$ and $Z$ orthogonal to $\xi$.

We denote by $R$ the Riemannian curvature tensor field of $M$. Then the equation of Gauss is given by

$$R(X, Y)Z = c\{g(Y, Z)X - g(X, Z)Y + g(\phi Y, Z)\phi X - g(\phi X, Z)\phi Y - 2g(\phi X, Y)\phi Z\} + g(AY, Z)AX - g(AX, Z)AY,$$

and the equation of Codazzi by

$$(\nabla_X A)Y - (\nabla_Y A)X = c\{\eta(X)\phi Y - \eta(Y)\phi X - 2g(\phi X, Y)\xi\}.$$  

From the equation of Gauss, the Ricci tensor $S$ of $M$ is given by

1. $g(SX, Y) = (2n + 1)cg(X, Y) - 3c\eta(X)\eta(Y) + \text{tr} Ag(AX, Y) - g(AX, AY),$  

where $\text{tr} A$ is the trace of $A$. Taking a covariant differentiation, we have

2. $g((\nabla_X S)Y, Z) = -3cg(Y, \phi AX)\eta(Z) - 3c\phi AX, Z)\eta(Y) + (X\text{tr} A)g(AY, Z) + \text{tr} Ag((\nabla_X A)Y, Z) - g((\nabla_X A)AY, Z) - g((\nabla_X A)Y, AZ).$
Now we develop some lemmas needed to prove our main theorem. Suppose \( n \geq 3 \).

**Lemma 2.1.** Let \( M \) be a real hypersurface in a complex space form \( M^n(c) \), \( n \geq 3 \), \( c \neq 0 \). If there exists an orthonormal frame \( \{ \xi, e_1, \ldots, e_{2n-2} \} \) on a sufficiently small neighborhood \( N \) of \( x \in M \) such that the shape operator \( A \) can be represented as

\[
A = \begin{pmatrix}
\alpha & h_1 & 0 & \cdots & 0 \\
h_1 & a_1 & & & \\
0 & a_2 & & & \\
& & \ddots & & \\
0 & & & \cdots & 0
\end{pmatrix},
\]

then we have

\[
\begin{align*}
(3) & \quad (a_j - a_k) g(\nabla_{e_i} e_j, e_k) - (a_i - a_k) g(\nabla_{e_j} e_i, e_k) = 0, \\
(4) & \quad (a_j - a_1) g(\nabla_{e_i} e_j, e_1) - (a_i - a_1) g(\nabla_{e_j} e_i, e_1) = h_1 (a_i + a_j) g(e_i, \phi e_j), \\
(5) & \quad h_1 g(\nabla_{e_i} e_j, e_1) - h_1 g(\nabla_{e_j} e_i, e_1) = \{2c - 2a_i a_j + \alpha (a_i + a_j)\} g(\phi e_i, e_j), \\
(6) & \quad (e_j a_i) = (a_j - a_i) g(\nabla_{e_i} e_j, e_i), \\
(7) & \quad (e_1 a_i) = (a_1 - a_i) g(\nabla_{e_i} e_1, e_i), \\
(8) & \quad (a_1 - a_j) g(\nabla_{e_i} e_1, e_j) + (a_j - a_i) g(\nabla_{e_i} e_j, e_j) = a_i h_1 g(e_i, \phi e_j), \\
(9) & \quad (e_i h_1) = \{2c - 2a_1 a_i + \alpha (a_i + a_1)\} g(e_i, \phi e_1) - h_1 g(\nabla_{e_i} e_1, e_1), \\
(10) & \quad (e_i a_1) = h_1 (2a_i + a_1) g(e_i, \phi e_1) + (a_i - a_1) g(\nabla_{e_i} e_1, e_1), \\
(11) & \quad (\xi a_i) = h_1 g(\nabla_{e_i} e_1, e_i), \\
(12) & \quad h_1 g(\nabla_{e_i} e_1, e_j) + (a_j - a_i) g(\nabla_{\xi} e_j, e_j) = (c + a_i \alpha - a_i a_j) g(e_i, \phi e_j), \\
(13) & \quad (e_i h_1) = (c + a_i \alpha - a_1 a_i + h_1^2) g(e_i, \phi e_1) + (a_i - a_1) g(\nabla_{\xi} e_i, e_1), \\
(14) & \quad (e_i \alpha) = h_1 (\alpha - 3 a_i) g(e_i, \phi e_1) - h_1 g(\nabla_{\xi} e_i, e_1), \\
(15) & \quad (e_1 h_1) = (\xi a_1), \\
(16) & \quad (e_1 \alpha) = (\xi h_1), \\
(17) & \quad (a_1 - a_i) g(\nabla_{\xi} e_1, e_i) - h_1 g(\nabla_{e_i} e_1, e_i) = (c + a_i \alpha - a_1 a_i - h_1^2) g(e_i, \phi e_1),
\end{align*}
\]

for any \( i, j \geq 2, i \neq j \).

**Proof.** By the equation of Codazzi, we have

\[ g((\nabla_{e_i} A) e_1 - (\nabla_{e_1} A) e_i, e_j) = 0, \]

where \( i, j = 2, \ldots, 2n - 2 \). On the other hand, we have

\[
g((\nabla_{e_i} A) e_1 - (\nabla_{e_1} A) e_i, e_j) \\
= g(\nabla_{e_i} (A e_1) - A \nabla_{e_i} e_1 - \nabla_{e_1} (A e_i) + A \nabla_{e_1} e_i, e_j) \\
= (a_1 - a_j) g(\nabla_{e_i} e_1, e_j) + (a_j - a_i) g(\nabla_{e_1} e_i, e_j) + a_i h_1 g(\phi e_i, e_j).
\]
Thus we obtain (8). We obtain the other results through similar computations. □

We remark that these equations hold in the case that $M$ is a Hopf hypersurface, i.e., $h_1 = 0$. When $n = 2$, we showed the corresponding result in [Kon 2014].

We define the subspace $L_x \subset T_x(M)$ as the smallest subspace that contains $\xi$ and is invariant under the shape operator $A$. Then $M$ is Hopf if and only if $L_x$ is one-dimensional at each point $x$.

**Lemma 2.2.** Let $M$ be a real hypersurface of $M^n(c)$. If the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta\xi$ for some function $\beta$, then $\dim L_x \leq 2$ at each point $x$ of $M^n(c)$.

**Proof.** By (1), we have

$$0 = g(S\xi, Y) = -g(A^2\xi, Y)$$

for any $Y$ orthogonal to $\xi$ and $A\xi$. So $A^2\xi$ is spanned by $\xi$ and $A\xi$. Thus we see that $\dim L_x \leq 2$. □

Suppose that $M$ is not a Hopf hypersurface and that $S\xi = \beta\xi$. By Lemma 2.2, we can take an orthonormal frame $\{\xi, e_1, \ldots, e_2n-2\}$, locally, such that $A$ is of the form

$$A = \begin{pmatrix} \alpha & h_1 & 0 \\ h_1 & a_2 & \phantom{.} \\ 0 & \phantom{.} & a_{2n-2} \end{pmatrix},$$

where $h_1 = g(Ae_1, \xi)$, $a_i = g(Ae_i, e_i)$ for $i = 1, \ldots, 2n-2$, $g(Ae_i, e_j) = 0$ for $i \neq j$ and $\alpha = g(A\xi, \xi)$. By (1), we obtain

$$S\xi = (2n-2)c\xi + (\text{tr} \ A)(h_1 e_1 + \alpha\xi) - A(h_1 e_1 + \alpha\xi)$$

$$= (\text{tr} \ A - \alpha - a_1)h_1 e_1 + ((2n-2)c + (\text{tr} \ A)\alpha - h_1^2 - \alpha^2)\xi = \beta\xi.$$ 

So we see that

$$\text{tr} \ A = \alpha + a_1, \quad a_2 + \cdots + a_{2n-2} = 0.$$ 

Moreover, (1) implies that the Ricci tensor $S$ can be represented as

$$S = \begin{pmatrix} \beta & 0 \\ \lambda_1 & \phantom{.} \\ 0 & \lambda_{2n-2} \end{pmatrix},$$

where $\beta$ and $\lambda_i$ satisfy

$$\beta = (2n-2)c + (\alpha a_1 - h_1^2), \quad \lambda_1 = (2n + 1)c + (\alpha a_1 - h_1^2),$$

$$\lambda_j = (2n + 1)c + \text{tr} \ A \cdot a_j - a_j^2, \quad j = 2, \ldots, 2n - 2.$$
3. Real hypersurfaces with η-parallel Ricci tensor

In this section, we consider the additional condition that the Ricci operator $S$ is $η$-parallel, that is,

$$g((\nabla_X S) Y, Z) = 0$$

for any vector fields $X$, $Y$ and $Z$ orthogonal to $ξ$. This is equivalent to the condition that $(\nabla_X S) Y$ is proportional to $ξ$ [Suh 1990].

**Theorem 3.1.** Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, with $η$-parallel Ricci tensor. If the Ricci tensor $S$ of $M$ satisfies $Sξ = βξ$ for some function $β$, then $M$ is a Hopf hypersurface.

Before proving Theorem 3.1, we need the following lemma.

**Lemma 3.2.** Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, with $η$-parallel Ricci tensor. If the Ricci tensor $S$ of $M$ satisfies $Sξ = βξ$ for some function $β$, then we have

$$g((R(W, X) S) Y, Z) = -g(Sφ AX, Z)g(φ AW, Y) - g(Sφ AX, Y)g(φ AW, Z)$$

$$+ g(Sφ AW, Z)g(φ AX, Y) + g(Sφ AW, Y)g(φ AX, Z)$$

$$- g((\nabla_ξ S) Y, Z)g((φ A + Aφ) X, W)$$

for any $X$, $Y$, $Z$ and $W$ orthogonal to $ξ$.

**Proof.** Since $S$ is $η$-parallel, we have

$$g((R(W, X) S) Y, Z)$$

$$= g(R(W, X) SY, Z) - g(R(W, X) Y, SZ)$$

$$= g(∇W ∇X SY - ∇X ∇W SY - [W, X] SY, Z)$$

$$- g(∇W ∇X Y - ∇X ∇W Y - [W, X] Y, SZ)$$

$$= -g((∇X S) Y, ∇W Z) + g(∇W (S∇X Y), Z) + g((∇W S) Y, ∇X Z)$$

$$- g((∇X (S∇W Y), Z) - g((∇[W, X] S) Y, Z) - g(∇W ∇Y, SZ)$$

$$+ g(∇X ∇W Y, SZ)$$

$$= -g((∇X S) Y, ξ)g(ξ, ∇W Z) + g((∇W S) ∇X Y, Z)$$

$$+ g((∇W S) Y, ξ)g(ξ, ∇X Z) - g((∇X S) ∇W Y, Z)$$

$$- g((∇ξ S) Y, Z)g(ξ, [W, X])$$

$$= -g(Sφ AX, Y)g(φ AW, Z) + g(Sφ AW, Z)g(φ AX, Y)$$

$$+ g(Sφ AW, Y)g(φ AX, Z) - g(Sφ AX, Z)g(φ AW, Y)$$

$$- g((∇ξ S) Y, Z)g((φ A + Aφ) X, W).$$

From Lemma 3.2 we obtain the following:
Lemma 3.3. Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$, with $\eta$-parallel Ricci tensor. Suppose that the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta \xi$ for some function $\beta$. If $SY = \lambda Y$ and if $Y$ is orthogonal to $\xi$, then we have

$$g((\nabla_\xi S)Y, Y)g((\phi A + A\phi)X, W) = 0$$

for any $X$, $Y$ and $W$ orthogonal to $\xi$.

Proof of Theorem 3.1.

In the following, we suppose that $M$ is not a Hopf hypersurface. We work in an open set where $h_1 \neq 0$.

Case (I): First we consider the case $g((\nabla_\xi S)Y, Y) = 0$.

Lemma 3.4. $\beta, \lambda_1, \ldots, \lambda_{2n-2}$ are constant.

Proof. Since the Ricci tensor $S$ is $\eta$-parallel and since $g((\nabla_\xi S)Y, Y) = 0$, we have

$$0 = g((\nabla_Z S)Y, Y) = g(\nabla_Z SY, Y) - g(S\nabla_Z Y, Y) = Z\lambda$$

for any tangent vector field $Z$. So we see that $\lambda_1, \ldots, \lambda_{2n-2}$ are constant. On the other hand, since $\beta = \lambda_1 - 3c$, we see that $\beta$ is also constant.

Lemma 3.5. If $\lambda_i \neq \lambda_j$, $i, j = 1, \ldots, 2n - 2$, then we have $g(\nabla_X e_i, e_j) = 0$ for any $X$ orthogonal to $\xi$.

Proof. Since we have $Se_i = \lambda_i e_i$ and $Se_j = \lambda_j e_j$ and since $S$ is $\eta$-parallel, we obtain

$$0 = g((\nabla_X S)e_i, e_j) = (\lambda_i - \lambda_j)g(\nabla_X e_i, e_j).$$

If $\lambda_1 = \cdots = \lambda_{2n-2} = \lambda$, then $M$ is pseudo-Einstein, i.e., $SX = \lambda X + (\beta - \lambda)\eta(X)\xi$, and so it is a Hopf hypersurface (see [Kon 1979]).

Suppose that $M$ is non-Hopf and that there exist $\lambda_t$ and $\lambda_j$, $t, j \geq 2$, satisfying $\lambda_1 \neq \lambda_t$ and $\lambda_t \neq \lambda_j$. By Lemma 3.5,

$$g(\nabla_j \nabla_t e_t, e_j) = -g(\nabla_j e_t, \nabla_t e_j)$$

$$= -g(\nabla_j e_t, \xi) \delta_j e_j - \sum_k g(\nabla_t e_t, e_k)g(e_k, \nabla_t e_j)$$

$$= -g(e_t, \phi A e_t)g(\phi A e_j, e_j) = 0,$$

$$g(\nabla_t \nabla_j e_t, e_j) = -g(\nabla_j e_t, e_t) = -g(\nabla_j e_t, \xi)g(\delta_j e_j, e_t)$$

$$= -g(\delta_t e_t, \phi A e_j)g(\phi A e_t, e_j) = -a_j a_t g(\phi e_j, e_t) g(\phi e_t, e_j).$$

On the other hand, from (8),

$$(a_1 - a_t)g(\nabla_j e_1, e_t) + (a_t - a_j)g(\nabla_j e_1, e_t) + a_j h_1 g(\phi e_j, e_t) = 0.$$

From Lemma 3.5, we have $g(\nabla_j e_1, e_t) = 0$, $g(\nabla_j e_1, e_t) = 0$. Since $h_1 \neq 0$,

$$a_j g(\phi e_j, e_t) = 0,$$
from which we obtain
\[ g(\nabla_{e_i} \nabla_{e_j} e_t, e_j) = 0. \]

Moreover, we have
\[
g(\nabla_{[e_j,e_i]} e_t, e_j) = g(\nabla_\xi e_t, e_j) [e_j, e_i] = 0.\]

Using (12), we see that
\[
(c + a_j \alpha - a_j a_t) g(\phi e_j, e_t) + h_1 g(\nabla_{e_j} e_1, e_t) + (a_t - a_j) g(\nabla_{e_j} e_t, e_t) = 0.
\]

From these equations, we obtain
\[
 c g(\phi e_j, e_t)^2 + a_t g(\phi e_j, e_t) g(\nabla_\xi e_j, e_t) = 0.
\]

Hence we have
\[
g(\nabla_{[e_j,e_t]} e_t, e_j) = -c g(\phi e_j, e_t)^2.
\]

Therefore,
\[
g(R(e_j, e_t)e_t, e_j) = c g(\phi e_j, e_t)^2.
\]

On the other hand, the equation of Gauss implies
\[
g(R(e_j, e_t)e_t, e_j) = c + 3 c g(\phi e_j, e_t)^2 + a_t a_j.
\]

From these equations, we have
\[
c(1 + 2 g(\phi e_j, e_t)^2) + a_t a_j = 0.
\]

Sine \( c \neq 0 \), we see that \( a_t \neq 0 \) and \( a_j \neq 0 \). Thus \( g(\phi e_j, e_t) = 0 \) and \( c + a_t a_j = 0 \). So we can represent \( A \) as
\[
A = \begin{pmatrix}
\alpha & h_1 \\

h_1 & a_t \\
\vdots & \ddots \\
\end{pmatrix}
\]
by setting \( a = a_j, \ b = a_t \) and taking a suitable permutation of \( \{e_2, \ldots, e_{2n-2}\} \).
Suppose there exist \( j \) and \( t \) such that \( g(\phi e_j, e_1) \neq 0 \) and \( g(\phi e_t, e_1) \neq 0 \). Then \( \phi e_j \) and \( \phi e_t \) satisfy

\[
\phi e_j = \sum_k g(\phi e_j, e_k)e_k + g(\phi e_j, e_1)e_1, \quad A e_k = a e_k, \\
\phi e_t = \sum_l g(\phi e_t, e_l)e_l + g(\phi e_t, e_1)e_1, \quad A e_l = b e_l.
\]

So we have

\[
0 = g(\phi e_j, \phi e_t) = g(\phi e_j, e_1)g(\phi e_t, e_1),
\]

from which we see that \( g(\phi e_j, e_1) = 0 \) or \( g(\phi e_t, e_1) = 0 \), and hence \( A \phi e_1 = a \phi e_1 \) or \( A \phi e_1 = b \phi e_1 \).

When \( A \phi e_1 = a \phi e_1 \), we have \( A \phi e_t = b \phi e_t \). By (4),

\[
(b - a_1)g(\nabla_{e_i} \phi e_t, e_1) - (b - a_1)g(\nabla_{\phi e_t} e_i, e_1) + 2h_1 b g(\phi e_t, \phi e_t) = 0.
\]

Thus we obtain \( b = 0 \), which contradicts \( c + ab = 0 \) and \( c \neq 0 \). By a similar computation, the case \( A \phi e_1 = b \phi e_1 \) does not occur.

Next we consider the case \( \lambda_2 = \cdots = \lambda_{2n-2} \neq \lambda_1 \). We set \( \lambda = \lambda_j, j = 2, \ldots, 2n-2 \). From Lemma 3.5, we have \( g(\nabla_X e_1, e_i) = 0, i \geq 2 \), for any \( X \) orthogonal to \( \xi \).

By (4) and (5),

\[
h_1(a_i + a_j)g(\phi e_i, e_j) = 0, \quad (2c - 2a_ia_j + \alpha(a_i + a_j))g(\phi e_i, e_j) = 0.
\]

Since \( a_j \) satisfies

\[
\lambda = (2n + 1)c + \text{tr } A \cdot a_j - a_j^2,
\]

we can represent \( A \) as

\[
A = \begin{pmatrix}
a & \alpha & h_1 & h_1 & a_1 \\
h_1 & h_1 & a & \alpha & h_1 \\
a & \alpha & h_1 & h_1 & a_1 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
2a & b & a & \alpha & h_1 \\
b & a & \alpha & h_1 & h_1 \\
\end{pmatrix}
\]

by taking a suitable permutation of \( \{e_2, \ldots, e_{2n-2}\} \).

There exist \( i \) and \( j \) satisfying \( g(\phi e_i, e_j) \neq 0 \). Therefore, using \( h_1 \neq 0 \),

\[
a_i + a_j = 0, \quad 2c - 2a_ia_j + \alpha(a_i + a_j) = 0.
\]

We notice that \( \text{tr } A = a_1 + \alpha \) and \( \sum_{j=2}^{2n-2} a_j = ka + lb = 0 \), where \( k \) and \( l \) are the multiplicities of \( a \) and \( b \), respectively.
When \( a_i = a_j = a \), then we have \( a_i + a_j = 2a = 0 \). Combining this with the above equations, we obtain \( b = 0 \) and \( c = 0 \). This is a contradiction. Similarly, the case \( a_i = a_j = b \) does not occur.

Next, when \( a_i = a , a_j = b \) and \( a = b = 0 \) and \( c = 0 \). This is a contradiction.

Finally we consider the case \( a_i = a , a_j = b \) and \( a = b \neq b \). Then we have \( a = −b \neq 0 \). Since \( ka + lb = 0 \), we obtain \( k = l \). This contradicts the fact that \( M \) is an odd-dimensional real hypersurface.

Case (II): Next we consider the case

\[
(18) \quad g((\phi A + A\phi)X, W) = 0
\]

for any \( X \) and \( W \) orthogonal to \( \xi \).

Since \( \{\xi, \phi e_1, \ldots, \phi e_{2n-2}\} \) is an orthonormal basis of the tangent space, we have

\[
\begin{align*}
\text{tr} A &= g(A\xi, \xi) + \sum_{i=1}^{2n-2} g(A\phi e_i, \phi e_i) \\
&= \alpha - \sum_{i=1}^{2n-2} g(A\phi e_i, \phi e_i) = \alpha - \sum_{i=1}^{2n-2} g(A e_i, e_i).
\end{align*}
\]

Since \( \text{tr} A = \alpha + \sum_{i=1}^{2n-2} g(A e_i, e_i) \), we obtain \( \sum_{i=1}^{2n-2} g(A e_i, e_i) = 0 \) and \( \text{tr} A = \alpha \). On the other hand, from \( \text{tr} A = a_1 + \alpha \), we have \( a_1 = 0 \). Substituting \( X = e_1 \) in (18), we see that \( g(A\phi e_1, W) = 0 \) for any \( W \) orthogonal to \( \xi \). Since

\[
g(A\phi e_1, \xi) = g(\phi e_1, A\xi) = 0,
\]

we have \( A\phi e_1 = 0 \). Without loss of generality, we can set \( \phi e_1 = e_2 \). From (13) and (17), we obtain

\[
(19) \quad (e_2 h_1) = c + h_1^2,
\]

\[
(20) \quad (c - h_1^2) + h_1 g(\nabla e_1 e_2, e_1) = 0.
\]

On the other hand, since \( S \) is \( \eta \)-parallel, putting \( X = Y = e_1 \) and \( Z = e_2 \) into (2), we have

\[
0 = \text{tr} g((\nabla e_1 A)e_1, e_2) - g((\nabla e_1 A)e_1, e_2) = h_1^2 g(e_1, \nabla e_1 e_2).
\]

Since \( h_1 \neq 0 \), we have \( g(\nabla e_1 e_2, e_1) = 0 \). Combining this with (20), we see that \( h_1^2 = c \). This contradicts (19), finishing the proof.

We remark that Suh [1990] and Maeda [2013] classified Hopf hypersurfaces of nonflat complex space forms with \( \eta \)-parallel Ricci tensor.
4. Ruled real hypersurfaces

In the previous sections, under the condition that the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta \xi$, we gave sufficient conditions for $M$ to be a Hopf hypersurface with respect to the covariant derivative of the Ricci tensor of $S$. The purpose of this section is to give a condition on the Ricci tensor for $M$ to be a ruled real hypersurface.

**Theorem 4.1.** Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$. If the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta \xi$ for some function $\beta$ and if $g((\nabla_X S)Y, \xi) = 0$ for any vector fields $X$ and $Y$ orthogonal to $\xi$, then $M$ is a ruled real hypersurface.

**Proof.** To prove Theorem 4.1, we need the following proposition:

**Proposition 4.2.** Let $M$ be a real hypersurface of $M^n(c)$, $c \neq 0$. If the Ricci tensor $S$ of $M$ satisfies $S\xi = \beta \xi$ for some function $\beta$ and if $g((\nabla_X S)Y, \xi) = 0$ for any vector fields $X$ and $Y$ orthogonal to $\xi$, then $M$ is not Hopf.

**Proof.** Suppose that $M$ is a Hopf hypersurface. Then we have $A\xi = \alpha \xi$, and hence $S\xi = \beta \xi$. We note that $\alpha$ is constant. Therefore, we have

$$g((\nabla_X S)Y, \xi) = g((\nabla_X S)\xi, Y) - g(S\phi AX, Y) = \beta g(\phi AX, Y) - g(\phi AX, SY)$$

for any $X$ and $Y$ orthogonal to $\xi$. We take an orthonormal basis $\{\xi, e_1, \ldots, e_{2n-2}\}$ that satisfies $e_{2i} = \phi e_{2i-1}$, $i = 1, \ldots, n-1$, and set $Ae_t = a_t e_t$, $t = 1, \ldots, 2n-2$. Then we have $A\phi e_t = \bar{a}_t \phi e_t$ since $M$ is Hopf. Then the Ricci operator $S$ satisfies $S\xi = \beta \xi$ and $Se_t = \lambda_t e_t$, $t = 1, \ldots, 2n-2$, where

$$\beta = (2n-1)c + \text{tr } A \cdot \alpha - \alpha^2, \quad \lambda_t = (2n+1)c + \text{tr } A \cdot a_t - a_t^2.$$ 

Thus we obtain

$$0 = (\beta - \lambda_t)g(\phi AX, e_t) = -(\beta - \lambda_t)g(X, A\phi e_t)$$

for any $X$ orthogonal to $\xi$. Since $A\xi = \alpha \xi$, we have $g(A\phi e_t, \xi) = 0$. From these equations, we have:

**Lemma 4.3.** If $\beta \neq \lambda_t$, then $A\phi e_t = 0$, that is, $\bar{a}_t = 0$.

We suppose $\beta \neq \lambda_t$. Then, from (1), we have

$$\bar{\lambda}_t = g(S\phi e_t, \phi e_t) = (2n+1)c.$$ 

Using Proposition A and $c \neq 0$, we have $\alpha \neq 0$ and

$$a_t = -\frac{2c}{\alpha}.$$
If $\beta \neq \lambda_t$ and $\beta \neq \bar{\lambda}_t = g(\phi e_t, \phi e_t)$, then we have $a_t = \tilde{a}_t = 0$. This is a contradiction. Thus we obtain:

**Lemma 4.4.** If $\beta \neq \lambda_t$, then $\beta = \bar{\lambda}_t = (2n + 1)c$.

Since $M$ is not Einstein, there exists a $t$ such that $\beta \neq \lambda_t$. So we see that $\lambda_t$ satisfies $\beta = \lambda_t = \bar{\lambda}_t$ or $\beta = \bar{\lambda}_t = \lambda_t$.

When $\beta = \lambda_t = \bar{\lambda}_t$, since $\beta = (2n + 1)c$, we have

$$0 = a_t(\text{tr } A - a_t).$$

So we obtain $a_t = 0$ or $a_t = \text{tr } A$. If $a_t = 0$, then $\tilde{a}_t = -2c/\alpha$. There exists an $s$ that satisfies $\lambda_s \neq \beta$, and hence $a_s = -2c/\alpha$. Thus we have

$$\beta \neq \lambda_s = (2n + 1)c + \text{tr } A \left( -\frac{2c}{\alpha} \right) - \left( -\frac{2c}{\alpha} \right)^2.$$ 

Thus $\bar{\lambda}_t = \lambda_s \neq \beta$. This is a contradiction. So we see that $a_t = \text{tr } A \neq 0$. In the following, we set $a = a_t = \text{tr } A$. Since $a_t = \tilde{a}_t = \text{tr } A$, we have

$$(2a - \alpha)a = (\alpha a + 2c).$$

Thus $a$ satisfies $a^2 - \alpha a - c = 0$, and hence $a$ turns to be constant. In the following, we set $a_1 = -2c/\alpha$ and $\tilde{a}_1 = a_2 = 0$.

Next we compute $g(R(e_1, e_2)e_2, e_1)$. By the equation of Gauss,

$$g(R(e_1, e_2)e_2, e_1) = g(R(e_1, \phi e_1)\phi e_1, e_1) = 4c.$$ 

Using (7), $a_1 g(\nabla e_2 e_1, e_2) = 0$. Since $a_1 \neq 0$, we have $g(\nabla e_2 e_2, e_1) = 0$. Moreover,

$$g(\nabla e_2 e_2, e_2) = 0, \quad g(\nabla e_2 e_2, \xi) = -g(e_2, \phi Ae_2) = 0.$$ 

When $k \geq 3$, by (6),

$$a_k g(\nabla e_2 e_2, e_k) = 0.$$ 

When $a_k \neq 0$, we have $g(\nabla e_2 e_2, e_k) = 0$. By (10), $g(\nabla e_1 e_1, e_2) = 0$. Moreover,

$$g(\nabla e_1 e_1, e_1) = 0, \quad g(\nabla e_1 e_1, \xi) = 0.$$ 

Since $k \geq 3$, by (10) and the fact that $a_1$ is constant,

$$(a_1 - a_k) g(\nabla e_1 e_k, e_1) = 0.$$ 

By $a_1 \neq 0$, if $a_k = 0$, then $g(\nabla e_1 e_1, e_k) = 0$. Thus we have

$$\sum_{k=1}^{2n-2} g(\nabla e_1 e_1) g(e_k, \nabla e_2 e_2) = 0.$$
Thus we have
\[ g(\nabla_{e_1} \nabla_{e_2} e_2, e_1) = e_1 g(\nabla_{e_2} e_2, e_1) - g(\nabla_{e_2} e_2, \nabla_{e_1} e_1) \]
\[ = - \sum_k g(\nabla_{e_2} e_2, e_k)g(e_k, \nabla_{e_1} e_1) = 0, \]
and

\[ g(\nabla_{e_1} \nabla_{e_2} e_2, e_1) = e_2 g(\nabla_{e_1} e_2, e_1) - g(\nabla_{e_1} e_2, \nabla_{e_2} e_1) = -g(\nabla_{e_1} \phi e_1, \nabla_{e_2} e_1) \]
\[ = g(\nabla_{e_1} e_1, \phi \nabla_{e_2} e_1) = g(\nabla_{e_1} e_1, \nabla_{e_2} e_2) = 0, \]

so we have
\[ g(\nabla_{e_1} e_1, \nabla_{e_2} e_2) = c. \]

Using (4), we have
\[ g(\nabla_{e_1} e_2, e_1) = \frac{a_k - a_1}{a_1} g(\nabla_{e_2} e_1, e_k). \]

On the other hand, by (8),
\[ g(\nabla_{e_1} e_2, e_1) = \frac{a_k}{a_1} g(\nabla_{e_1} e_2, e_k). \]

So we obtain
\[ \sum_{k \geq 3} g(\nabla_{e_k} e_2, e_1)(e_k, \nabla_{e_1} e_2) - \sum_{k \geq 3} g(\nabla_{e_k} e_2, e_1)g(e_k, \nabla_{e_2} e_1) \]
\[ = \sum_{k \geq 3} (a_k - a_1) g(\nabla_{e_2} e_1, e_k)g(e_k, \nabla_{e_1} e_2) - \sum_{k \geq 3} \frac{a_k}{a_1} g(\nabla_{e_1} e_2, e_k)(e_k, \nabla_{e_2} e_1) \]
\[ = - \sum g(\nabla_{e_2} e_1, e_k)g(e_k, \nabla_{e_1} e_2) \]
\[ = - \sum g(\nabla_{e_2} e_1, \phi e_k)g(\phi e_k, \nabla_{e_1} e_2) \]
\[ = \sum g(\nabla_{e_2} e_2, e_k)g(e_k, \nabla_{e_1} e_1) = 0. \]

Thus we have
\[ g(R(e_1, e_2) e_2, e_1) = c, \]
from which we obtain \( c = 0 \). This is a contradiction. Hence we see that \( M \) is not Hopf. Thus we have proven Proposition 4.2. \( \square \)
From Proposition 4.2, if \( g((\nabla_X S)Y, \xi) = 0 \) for \( X, Y \in H \), then \( M \) is not Hopf. In the following, we suppose that \( M \) is not Hopf, that is, \( h_1 \neq 0 \). Then, by Lemma 2.2, we can take an orthonormal basis \( \{ \xi, e_1, \ldots, e_{2n-2} \} \) such that

\[
(21) \quad A\xi = \alpha \xi + h_1 e_1, \quad Ae_1 = a_1 e_1 + h_1 \xi, \quad Ae_j = a_j e_j, \quad j = 2, \ldots, 2n - 2, \quad \text{tr} A = \alpha + a_1, \quad a_2 + \cdots + a_{2n-2} = 0.
\]

Then we have

\[
\beta = g(S\xi, \xi) = (2n - 2)c + (a_1 \alpha - h_1^2),
\]

\[
\lambda_1 = g(Se_1, e_1) = (2n + 1)c + (a_1 \alpha - h_1^2),
\]

\[
\lambda_j = g(Se_j, e_j) = (2n + 1)c + \text{tr} A \cdot a_j - a_j^2, \quad j \geq 2.
\]

By the assumption, for any \( X \) and \( Y \) orthogonal to \( \xi \),

\[
0 = g((\nabla_X S)\xi, Y) = g(\nabla_X S\xi, Y) - g(S\phi AX, Y).
\]

We set \( SY = \lambda Y \). Then we have

\[
0 = (\beta - \lambda)g(\phi AX, Y).
\]

Since \( \beta \neq \lambda_1 \), we see that

\[
g(\phi AX, e_1) = -g(AX, \phi e_1) = -g(X, A\phi e_1) = 0
\]

for any \( X \in H \). We also have \( g(\xi, A\phi e_1) = 0 \). Thus we have \( A\phi e_1 = 0 \). In the following, we set \( \phi e_1 = e_2 \). Then we have

\[
0 = (\beta - \lambda_2)g(\phi Ae_1, e_2) = (-3c + a_1 \alpha - h_1^2)a_1.
\]

**Lemma 4.5.** If \( h_1 \neq 0 \), then \( a_2 = 0 \). Moreover, \( a_1 = 0 \) or \( a_1 \alpha - h_1^2 = 3c \).

Case (I): Suppose \( a_1 = 0 \).

Since \( a_1 = a_2 = 0 \), (13) implies

\[
(e_2 h_1) = c + h_1^2.
\]

If \( \beta = (2n + 1)c = \lambda_2 \), then \( h_1^2 = -3c \) and \( e_2 h_1 = 0 \). Then we have \( h_1^2 = -c \) and \( c = 0 \). This is a contradiction. So we have:

**Lemma 4.6.** If \( a_1 = 0 \), then \( \beta \neq (2n + 1)c = \lambda_2 \).

For any \( X \in H \), we see that

\[
(\beta - \lambda_k)g(\phi AX, e_k) = 0, \quad k \geq 3.
\]

If \( \beta \neq \lambda_k \), then \( g(A\phi e_k, X) = 0 \), and moreover \( g(A\phi e_k, \xi) = 0 \). This shows that \( A\phi e_k = 0 \) and that \( \phi e_k \) is a principal vector of \( A \). We set

\[
\tilde{\lambda}_k = g(S\phi e_k, \phi e_k).
\]
Since \( a_1 \alpha - h_1^2 \neq 3c \), we have \( \bar{\lambda}_k = (2n + 1)c \neq \beta \). Then, from

\[
(\beta - \bar{\lambda}_k)g(\phi AX, \phi e_k) = 0,
\]

we have \( g(AX, e) = 0 \). We also have \( g(Ae_k, \xi) = 0 \) since \( k \geq 3 \). Hence we obtain \( Ae_k = 0 \) for \( e_k \) satisfying \( \beta \neq \lambda_k \).

We next consider the case \( \beta = \lambda_j \) for some \( j \geq 3 \). If \( \beta = \lambda_j = \lambda_i \), then

\[
\beta = (2n + 1)c + \text{tr} A \cdot a_j - a_j^2 = (2n + 1)c + \text{tr} A \cdot a_i - a_i^2.
\]

Therefore, at most two \( a_j \) are different. By this equation, we have

\[
0 = (a_j - a_i)(\text{tr} A - (a_j + a_i)).
\]

If \( a_j = a_i = a \) for all \( j \) and \( i \), then (21) implies \( \sum a_j = 0 \). Thus we have all \( a_j = 0, j = 2, \ldots, 2n - 2 \). Since \( a_1 = 0, M \) is a ruled real hypersurface.

Let us suppose that two \( a_j \) are different. We set

\[
T_a = \{ X \mid AX = aX, X \in H_x \}, \quad T_b = \{ X \mid AX = bX, X \in H_x \},
\]

where \( \beta = \lambda_a = \lambda_b, a \neq b \). We notice \( \text{tr} A = a + b \). If \( a = 0 \) or \( b = 0 \), then, by (21), \( a = b = 0 \). This contradicts the assumption that \( a \neq b \). So we obtain \( a \neq 0 \) and \( b \neq 0 \). We notice that \( \text{dim} T_a + \text{dim} T_b \) is even number.

Let \( e_i, e_j \in T_a \). By (8) and (12),

\[
-ag(\nabla_{e_i} e_1, e_j) + ah_1 g(\phi e_i, e_j) = 0, \\
(c + a\alpha - a^2)g(\phi e_i, e_j) + h_1 g(\nabla_{e_i} e_1, e_j) = 0.
\]

From these, we obtain

\[
(c + a\alpha - a^2 + h_1^2)g(\phi e_i, e_j) = 0.
\]

If there exist \( e_i \) and \( e_j \) such that \( g(\phi e_i, e_j) \neq 0 \), then

\[
c + a\alpha - a^2 + h_1^2 = 0.
\]

On the other hand, we have

\[
\beta = (2n - 2)c - h_1^2 = (2n + 1)c + \text{tr} A \cdot a - a^2.
\]

Since \( \text{tr} A = \alpha + a_1 = \alpha \), we have

\[
3c + a\alpha - a^2 + h_1^2 = 0.
\]

Therefore, we have \( 2c = 0 \). This contradicts \( c \neq 0 \). Hence \( g(\phi e_i, e_j) = 0 \) for all \( e_i \) and \( e_j \) of \( T_a \). So we have \( \phi T_a \subset T_b \). Similarly, we also have \( \phi T_b \subset T_a \). Consequently, we see that

\[
\phi T_a = T_b, \quad \phi T_b = T_a.
\]
If \( \dim T_a = \dim T_b = 1 \), then \( \phi T_a = T_b \). We see that if \( Ae_j = ae_j \), then \( A\phi e_j = b\phi e_j \) and \( a + b = \text{tr} A \). From (21), we have \( a + b = 0 \) and \( \text{tr} A = 0 \). Therefore, we obtain

\[ \text{tr} A = \alpha = 0. \]

We will prove that there is no real hypersurface that satisfies

\[ a + b = 0, \quad \alpha = 0, \quad a_1 = 0, \quad a_2 = 0, \quad \text{tr} A = 0, \]

and also

\[ a^2 - h_1^2 = 3c. \]

By (5),

\[ (2c + 2a^2)g(\phi e_i, \phi e_i) - h_1 g(\nabla e_i, e_i, e_1) = 0. \]

On the other hand, we have

\[ g(\nabla e_i, \phi e_i) = -g(\nabla e_i, e_i, e_2). \]

By (6),

\[ (a_2 - a_i)g(\nabla e_i, e_i) - (e_2 a_i) = 0. \]

Using \( a_2 = 0 \) and \( a_i = a \), we obtain

\[ ag(\nabla e_i, e_2) = (e_2 a). \]

From this equation and \( a \neq 0 \), we have

\[ g(\nabla e_i, e_2) = \frac{(e_2 a)}{a}. \]

On the other hand,

\[ g(\nabla \phi e_i, e_1) = g(\phi \nabla e_i, e_i) = g(\nabla \phi e_i, \phi e_1). \]

By (6), we obtain

\[ (a_2 + a)g(\nabla \phi e_i, e_i) + (e_2 a) = 0, \]

and hence

\[ g(\nabla \phi e_i, e_i, e_2) = \frac{(e_2 a)}{a}. \]

Substituting these equations into (22), we get

\[ 2(c + a^2) + h_1 \frac{(e_2 a)}{a} + h_1 \frac{(e_2 a)}{a} = 0. \]

Thus we have

\[ (c + a^2) a = -h_1(e_2 a). \]

On the other hand, since \( a^2 - h_1^2 = 3c \),

\[ a(e_2 a) = h_1(e_2 h_1). \]
Since \( a_1 = a_2 = 0 \), by (13), we have
\[
e_2 h_1 = c + h_1^2,
\]
from which we obtain
\[
e_2 a = \frac{h_1}{a} (c + h_1^2).
\]
Substituting this into (23), we get
\[
(c + a^2) a = -\frac{h_1^2}{a} (c + h_1^2) = -\frac{1}{a} (a^2 - 3c)(a^2 - 2c).
\]
Thus we obtain
\[
(a^2 - c)^2 + 2c^2 = 0.
\]
So we have \( c = 0 \). This is a contradiction. Consequently, if \( a_1 = 0 \), then \( M \) is a ruled real hypersurface.

Case (II): Suppose \( a_1 \neq 0 \).

We notice that \( a_2 = 0 \) and \( \alpha a_1 h_1^2 = 3c \) by Lemma 4.5. So we have
\[
(24) \quad (Xa_1)\alpha + a_1 (X\alpha) - 2h_1 (Xh_1) = 0
\]
for any tangent vector field \( X \).

**Lemma 4.7.** \( \nabla_{e_1} e_1 \) and \( \nabla_{e_2} e_2 \) are perpendicular to \( \xi, e_1 \) and \( e_2 \).

**Proof.** By (14),
\[
(e_2 \alpha) = \alpha h_1 + h_1 g(\nabla_{\xi} e_1, e_2).
\]
By (10),
\[
(e_2 a_1) = a_1 h_1 + a_1 g(\nabla_{e_1} e_1, e_2).
\]
Substituting these into (24), we get
\[
2a_1 \alpha h_1 + \alpha a_1 g(\nabla_{e_1} e_1, e_2) + a_1 h_1 g(\nabla_{\xi} e_1, e_2) - 2h_1 (e_2 h_1) = 0.
\]
By (9) and (13),
\[
(e_2 h_1) = (2c + \alpha a_1) + h_1 g(\nabla_{e_1} e_1, e_2) = (5c + h_1^2) + h_1 g(\nabla_{e_1} e_1, e_2),
\]
\[
(e_2 h_1) = (c + h_1^2) + a_1 g(\nabla_{\xi} e_1, e_2).
\]
From these equations and (24), we have
\[
2h_1 (a_1 \alpha - h_1^2 - 3c) + (a_1 \alpha - h_1^2) g(\nabla_{e_1} e_1, e_2) = 0.
\]
Since \( a_1 \alpha - h_1^2 = 3c \), we have

\[ g(\nabla e_1 e_1, e_2) = 0. \]

By (7), \( a_1 \neq 0 \) and \( a_2 = 0 \),

\[ g(\nabla e_2 e_2, e_1) = 0. \]

Moreover, we have

\[ g(\nabla e_2 e_2, \xi) = -g(e_2, \phi A e_2) = 0, \quad g(\nabla e_1 e_1, \xi) = -g(e_1, \phi A e_1) = 0. \]

These equations prove our lemma.

\[ \square \]

**Lemma 4.8.** Suppose \( j \geq 3 \). If \( a_j = 0 \), then \( g(\nabla e_1 e_1, e_j) = 0 \). If \( a_j \neq 0 \), then \( g(\nabla e_2 e_2, e_j) = 0 \).

**Proof.** By (6), we have

\[ a_j g(\nabla e_2 e_2, e_j) = 0, \quad j \geq 3. \]

If \( a_j \neq 0 \), then \( g(\nabla e_2 e_2, e_j) = 0 \) for \( j \geq 3 \). Suppose \( a_j = 0, \quad j \geq 3 \). Then, by (10), (14), (9) and (13),

\[
\begin{align*}
(e_j a_1) &= a_1 g(\nabla e_1 e_1, e_j), \\
(e_j \alpha) &= h_1 g(\nabla \xi e_1, e_j), \\
(e_j h_1) &= h_1 g(\nabla e_1 e_1, e_j), \\
(e_j h_1) &= a_1 g(\nabla \xi e_1, e_j).
\end{align*}
\]

Substituting these into (24), we get

\[
0 = (e_j a_1) \alpha + a_1 (e_j \alpha) - 2h_1 (e_j h_1) \\
= a a_1 g(\nabla e_1 e_1, e_j) + a_1 h_1 g(\nabla \xi e_1, e_j) - h_1^2 g(\nabla e_1 e_1, e_j) - h_1 a_1 g(\nabla \xi e_1, e_j) \\
= (aa_1 - h_1^2) g(\nabla e_1 e_1, e_j).
\]

Since \( a_1 \alpha - h_1^2 = 3c \), we have our lemma.

\[ \square \]

Using these lemmas, we compute \( g(R(e_1, e_2) e_2, e_1) \). We note that \( e_2 = \phi e_1 \) and \( a_2 = 0 \). First, we have

\[
g(\nabla e_1 \nabla e_2 e_2, e_1) = e_1 g(\nabla e_2 e_2, e_1) - g(\nabla e_2 e_2, \nabla e_1 e_1) \\
= -g(\nabla e_2 e_2, \xi) g(\xi, \nabla e_1 e_1) - g(\nabla e_2 e_2, e_1) g(\xi, \nabla e_1 e_1) \\
- g(\nabla e_2 e_2, e_2) g(e_1, \nabla e_1 e_1) - \sum_{k \geq 3} g(\nabla e_2 e_2, e_j) g(e_j, \nabla e_1 e_1) = 0.
\]
Next, we have
\[ g(\nabla_{e_2} \nabla_{e_1} e_2, e_1) \]
\[ = e_2 g(\nabla_{e_1} e_2, e_1) - g(\nabla_{e_1} e_2, \nabla_{e_2} e_1) \]
\[ = -g(\nabla_{e_1} e_2, \xi) g(\xi, \nabla_{e_2} e_1) - g(\nabla_{e_1} e_2, e_1) g(e_1, \nabla_{e_2} e_1) \]
\[ - g(\nabla_{e_1} e_2, \xi) g(\xi, \nabla_{e_2} e_1) - \sum_{k \geq 3} g(\nabla_{e_1} e_2, e_k) g(e_k, \nabla_{e_2} e_1) \]
\[ = - \sum_{k \geq 3} g(\nabla_{e_1} e_2, e_k) g(e_k, \nabla_{e_2} e_1) = - \sum_{k \geq 3} g(\nabla_{e_1} \phi e_1, e_k) g(\phi e_k, \phi \nabla_{e_2} e_1) \]
\[ = \sum_{k \geq 3} g(\nabla_{e_1} e_1, \phi e_k) g(\phi e_k, \nabla_{e_2} e_2) = \sum_{l \geq 3} g(\nabla_{e_1} e_1, e_l) g(e_l, \nabla_{e_2} e_2) = 0. \]

Moreover, we obtain
\[ g(\nabla_{[e_1, e_2]} e_2, e_1) = g(\nabla_{\xi} e_2, e_1) g(\xi, [e_1, e_2]) + g(\nabla_{e_1} e_2, e_1) g(e_1, [e_1, e_2]) \]
\[ + g(\nabla_{e_2} e_2, e_1) g(e_2, [e_1, e_2]) + \sum_{k \geq 3} g(\nabla_{e_k} e_2, e_1) g(e_k, [e_1, e_2]) \]
\[ = g(\nabla_{\xi} e_2, e_1) g(\xi, \nabla_{e_1} e_2) \]
\[ + \sum_{k \geq 3} (g(\nabla_{e_k} e_2, e_1) g(e_k, \nabla_{e_1} e_2) - g(\nabla_{e_k} e_2, e_1) g(e_k, \nabla_{e_2} e_1)). \]

On the other hand, by (8), when \( j \geq 3 \),
\[ a_1 g(\nabla_{e_j} e_2, e_1) - a_j g(\nabla_{e_1} e_2, e_j) = 0, \]
\[ (a_1 - a_j) g(\nabla_{e_2} e_1, e_j) + a_j g(\nabla_{e_1} e_2, e_j) = 0. \]

Thus, if \( a_1 = a_j \), then we see that \( a_j \neq 0 \) and hence \( g(\nabla_{e_1} e_2, e_j) = 0 \) since \( a_1 \neq 0 \).

Next, when \( a_1 \neq a_j \) we have
\[ g(\nabla_{e_2} e_1, e_j) = -\frac{a_j}{a_1 - a_j} g(\nabla_{e_1} e_2, e_j). \]

On the other hand,
\[ g(\nabla_{e_j} e_2, e_1) = \frac{a_j}{a_1} g(\nabla_{e_1} e_2, e_j) = -\frac{(a_1 - a_j)}{a_1} g(\nabla_{e_2} e_1, e_j). \]

So we have
\[ \sum_{k \geq 3} (g(\nabla_{e_k} e_2, e_1) g(e_k, \nabla_{e_1} e_2) - g(\nabla_{e_k} e_2, e_1) g(e_k, \nabla_{e_2} e_1) \]
\[ = - \sum_{k \geq 3} g(\nabla_{e_2} e_1, e_k) g(e_k, \nabla_{e_1} e_2) = - \sum_{k \geq 3} g(\phi \nabla_{e_2} e_1, e_k) g(\phi e_k, \nabla_{e_1} e_2) \]
\[ = \sum_{l \geq 3} g(\nabla_{e_1} e_1, e_l) g(e_l, \nabla_{e_2} e_2) = 0. \]
Thus we obtain
\[
g(\nabla_{[e_1,e_2]}e_2, e_1) = g(\nabla_\xi e_2, e_1)g(\xi, \nabla_\xi e_2)
= -g(\nabla_\xi e_2, e_1)g(\phi Ae_1, e_2) = -a_1 g(\nabla_\xi e_2, e_1),
\]
and so
\[
g(R(e_1, e_2)e_2, e_1) = a_1 g(\nabla_\xi e_2, e_1).
\]
On the other hand, by (9),
\[
-(2c + \alpha a_1) + h_1 g(\nabla_\xi e_2, e_1) + (e_2 h_1) = 0.
\]
Using Lemma 4.7 and \( a_1 \alpha - h_1^2 = 3c \), we have
\[
(e_2 h_1) = 2c + \alpha a_1 = 5c + h_1^2.
\]
By (13),
\[
-(c + h_1^2) + a_1 g(\nabla_\xi e_2, e_1) + e_2 h_1 = 0,
\]
from which we obtain
\[
a_1 g(\nabla_\xi e_2, e_1) = -4c,
\]
and so
\[
g(R(e_1, e_2)e_2, e_1) = -4c.
\]
On the other hand, the equation of Gauss implies
\[
g(R(e_1, e_2)e_2, e_1) = 4c,
\]
and hence \( c = 0 \). This is a contradiction.

Consequently, \( M \) is a ruled real hypersurface.

From (2), any ruled real hypersurface satisfies \( g((\nabla_X S)Y, \xi) = 0 \) for any \( X \) and \( Y \) orthogonal to \( \xi \), and \( S\xi = \beta \xi \) for some function \( \beta \).

From Theorems 3.1 and 4.1, we have Theorem 1.1.

References


Received May 7, 2015. Revised July 21, 2015.

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