Pacific Journal of Mathematics

MONOTONICITY FORMULAE AND VANISHING THEOREMS

JINTANG LI

Volume 281 No. 1 March 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 281, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.281.125

MONOTONICITY FORMULAE AND VANISHING THEOREMS

JINTANG LI

We study Cartan–Hadamard manifolds with pinching conditions. Using the stress-energy tensor, we establish some monotonicity formulae for vector bundle-valued *p*-forms and pluriharmonic maps between Kähler manifolds. Some vanishing theorems follow immediately from the monotonicity formulae under suitable growth conditions on the energy of *p*-forms and pluriharmonic maps.

1. Introduction

Harmonic maps between Riemannian manifolds are defined as the critical points of energy functionals. They are important in both classical and modern differential geometry. As is well-known, any harmonic map $\phi : R^n \to S^m$ with finite energy must be constant [Garber et al. 1979]. This result has been generalized by Sealey [1982] to harmonic maps from a space form of nonpositive sectional curvature to any Riemannian manifold with finite energy. In 1980, Baird and Eells [1981] introduced and studied the stress-energy tensor for maps between Riemannian manifolds. Sealey [$\geq$ 2016] introduced the stress-energy tensor for vector bundle-valued p-forms and established some vanishing theorems for L^2 harmonic p-forms. The stress-energy tensors have become a useful tool for investigating the energy behavior of vector bundle-valued p-forms in various problems. Dong and Lin [2014] introduced the notion of J-invariant p-forms on Kähler manifolds. They established a monotonicity formula by means of the stress-energy tensor. Using this monotonicity formula they proved the following vanishing theorem for vector bundle-valued J-invariant p-forms satisfying the conservation law:

Theorem A. *Let M be a complex n-dimensional $(n \geq 3)$ complete Kähler manifold with radial curvature K_r satisfying $-a^2 \leq K_r \leq -b^2 < 0$ with $a \geq b > 0$ and $(2n-1)b - 2pa \geq 0$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle*

This research is supported by the Natural Science Foundation of Fujian Province of China (No. 2014J01022) and the Fundamental Research Funds for the Central Universities (No. 20720150008).
MSC2010: primary 53C43; secondary 53C55.
Keywords: monotonicity formulae, Kähler manifolds, p-forms, pluriharmonic maps.

over (M, g). If $\omega \in A^p(\xi)$ is J-invariant and satisfies the conservation law, that is, $\operatorname{div} S_\omega = 0$, *then*

$$\frac{1}{r_1^C}\int_{B_{r_1}(x_0)} |\omega|^2\,dv \le \frac{1}{r_2^C}\int_{B_{r_2}(x_0)} |\omega|^2\,dv$$

for any $0 < r_1 < r_2$, where $C = 2n - 2pa/b$ and $B_r(x_0) \subseteq M$ is a geodesic ball of radius r centered at x_0 in M. In particular, if

$$\frac{1}{r^C}\int_{B_r(x_0)} |\omega|^2\,dv \to 0 \quad as\ r \to +\infty,$$

then $\omega = 0$.

We shall establish a monotonicity formula for vector bundle-valued J-invariant p-forms satisfying the conservation law by means of the stress-energy tensor too. Using this monotonicity formula we can improve Theorem A as follows:

Theorem 1. *Let M be a complex n-dimensional $(n \ge 3)$ complete Kähler manifold with radial curvature K_r satisfying $-a^2 \le K_r \le -b^2 < 0$ with $a \ge b > 0$ and $(n-1)b - (p-1)a \ge 0$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). If $\omega \in A^p(\xi)$ is J-invariant and satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *then*

$$\frac{1}{\sinh^C(ar_1)}\int_{B_{r_1}(x_0)} \cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]\,dv \\ \le \frac{1}{\sinh^C(ar_2)}\int_{B_{r_2}(x_0)} \cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]\,dv$$

for any $0 < r_1 < r_2$, where $C = [2(n-2)b - 2(p-2)a]/a$. In particular, if

$$\frac{\int_{B_r(x_0)} |\omega|^2\,dv}{e^{a(C-1)r}} \to 0 \quad as\ r \to +\infty,$$

then $\omega = 0$. (See Section 2 for the definition of $i_{\partial/\partial r}\omega$.)

For the case of Cartan–Hadamard manifolds with some pinching conditions, Xin [1986] established a monotonicity formula for vector bundle-valued p-forms satisfying the conservation law by means of the stress-energy tensor. Using this monotonicity formula, Xin proved the following vanishing theorem:

Theorem B. *Let M be an n-dimensional $(n \ge 3)$ complete Riemannian manifold with radial curvature K_r satisfying $-a^2 \le K_r \le -b^2 < 0$ with $a \ge b > 0$ and $(n-1)b - 2pa \ge 0$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). If $\omega \in A^p(\xi)$ satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *and*

$$\frac{1}{r^C}\int_{B_r(x_0)} |\omega|^2\,dv \to 0 \quad as\ r \to +\infty,$$

where $C = n - 2pa/b$, then $\omega = 0$.

We shall establish a monotonicity formula for vector bundle-valued p-forms satisfying the conservation law by means of the stress-energy tensor. Using this monotonicity formula we can improve Theorem B as follows:

Theorem 2. *Let M be an n-dimensional $(n \geq 3)$ complete Riemannian manifold with radial curvature K_r satisfying $-a^2 \leq K_r \leq -b^2 < 0$ with $a \geq b > 0$ and $(n-1)b-(2p-1)a \geq 0$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). If $\omega \in A^p(\xi)$ satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *then*

$$\frac{1}{\sinh^C(ar_1)} \int_{B_{r_1}(x_0)} \cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]\,dv \\ \leq \frac{1}{\sinh^C(ar_2)} \int_{B_{r_2}(x_0)} \cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]\,dv$$

for any $0 < r_1 < r_2$, where $C = [(n-2)b-(2p-2)a]/a$ and $B_r(x_0) \subseteq M$ is a geodesic ball of radius r centered at x_0 in M. In particular, if

$$\frac{\int_{B_r(x_0)} |\omega|^2\,dv}{e^{a(C-1)r}} \to 0 \quad \textit{as } r \to +\infty,$$

then $\omega = 0$.

Siu [1980] introduced and studied pluriharmonic maps from a compact Kähler manifold to a Kähler manifold. When the domain of such a map is complete, Dong [2013] proved the following:

Theorem C. *Let M be a complex n-dimensional $(n \geq 2)$ complete Kähler manifold with radial curvature K_r satisfying $K_r \leq -b^2 < 0$ with $b > 0$. Suppose $\phi : M \to N$ is either a pluriharmonic map between Kähler manifolds or a harmonic map into a Kähler manifold with strongly seminegative curvature. Then*

$$\frac{\int_{B_{r_1}(x_0)} |\bar\partial\phi|^2\,dv}{r_1^{2(n-1)}} \leq \frac{\int_{B_{r_2}(x_0)} |\bar\partial\phi|^2\,dv}{r_2^{2(n-1)}} \quad \textit{and} \quad \frac{\int_{B_{r_1}(x_0)} |\partial\phi|^2\,dv}{r_1^{2(n-1)}} \leq \frac{\int_{B_{r_2}(x_0)} |\partial\phi|^2\,dv}{r_2^{2(n-1)}}$$

for any $0 < r_1 < r_2$. In particular, if

$$\frac{\int_{B_r(x_0)} |d\phi|^2\,dv}{r^{(2n-2)}} \to 0 \quad \textit{as } r \to +\infty,$$

then ϕ is constant.

We can also improve Theorem C as follows:

Theorem 3. *Let M be a complex n-dimensional $(n \geq 2)$ complete Kähler manifold with radial curvature K_r satisfying $K_r \leq -b^2 < 0$ with $b > 0$. Suppose $\phi : M \to N$ is either a pluriharmonic map between Kähler manifolds or a harmonic map into a*

Kähler manifold with strongly seminegative curvature. Then

$$\frac{\int_{B_{r_1}(x_0)} \cosh(br)|\bar{\partial}\phi|^2\,dv}{\sinh^{2(n-1)}(br_1)} \le \frac{\int_{B_{r_2}(x_0)} \cosh(br)|\bar{\partial}\phi|^2\,dv}{\sinh^{2(n-1)}(br_2)}$$

and

$$\frac{\int_{B_{r_1}(x_0)} \cosh(br)|\partial\phi|^2\,dv}{\sinh^{2(n-1)}(br_1)} \le \frac{\int_{B_{r_2}(x_0)} \cosh(br)|\partial\phi|^2\,dv}{\sinh^{2(n-1)}(br_2)}$$

for any $0 < r_1 < r_2$. *In particular, if*

$$\frac{\int_{B_r(x_0)} |d\phi|^2\,dv}{e^{(2n-3)br}} \to 0 \qquad \text{as } r \to +\infty,$$

then ϕ *is constant.*

2. Preliminaries

Let (M, g) be an n-dimensional complete Riemannian manifold. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). Let $A^p(\xi) = \Gamma(\Lambda^p T^*M \otimes E)$ be the space of smooth p-forms on M with values in the vector bundle $\xi : E \to M$. For $\omega \in A^p(\xi)$, we define the energy functional of ω by

$$E(\omega) = \int_M \tfrac{1}{2}|\omega|^2\,dv_g.$$

The stress-energy tensor associated with $E(\omega)$ is defined by

$$S_\omega(X, Y) = \tfrac{1}{2}|\omega|^2 g(X, Y) - (\omega \odot \omega)(X, Y), \tag{2-1}$$

where $\omega \odot \omega$ denotes the 2-tensor

$$(\omega \odot \omega)(X, Y) = \langle i_X\omega, i_Y\omega\rangle.$$

Here $\langle\cdot,\cdot\rangle$ is the induced inner product on $A^{p-1}(\xi)$ and $i_X\omega$ is the interior multiplication by the vector field X given by

$$(i_X\omega)(Y_1, \dots, Y_{p-1}) = \omega(X, Y_1, \dots, Y_{p-1})$$

for $\omega \in A^p(\xi)$ and any vector fields $Y_1, \dots, Y_{p-1}$ on M.

Let D be any bounded domain of M with C^1 boundary. We have the integral formula [Dong 2013]

$$\int_{\partial D} S_\omega(X, \nu)\,dv = \int_D \{\langle S_\omega, \nabla\theta_X\rangle + (\operatorname{div} S_\omega)(X)\}\,dv, \tag{2-2}$$

where ν is the unit normal vector field along ∂D in D, and θ_X is the dual 1-form of X and $\nabla\theta_X$ is given by

$$(\nabla\theta_X)(Y, Z) = g(\nabla_Y X, Z). \tag{2-3}$$

Proposition 2.1 [Greene and Wu 1979]. *Let (M, g) be a complete Riemannian manifold with a pole x_0 and let r be the distance function relative to x_0. Denote by K_r the radial curvature of M. If $-a^2 \le K_r \le -b^2 < 0$, where $a \ge b > 0$, then*

$$b\coth(br)[g - dr\otimes dr] \le \mathrm{Hess}(r) \le a\coth(ar)[g - dr\otimes dr],$$

where $\mathrm{Hess}(r)$ *is the Hessian of the distance function r.*

3. Monotonicity formulae for Kähler manifolds

A Hermitian metric on a complex manifold M is a Riemannian metric g such that $g(JX, JY) = g(X, Y)$ for any $X, Y \in TM$, where J denotes the complex structure of M. We say that (M, g) is a Kähler manifold if $\nabla J = 0$, where ∇ is the Levi-Civita connection of g. A p-form $\omega \in A^p(\xi)$ is called J-invariant if $(\omega\odot\omega)(JX, JY) = (\omega\odot\omega)(X, Y)$. Now we consider J-invariant p-forms on Kähler manifolds and can prove the following:

Theorem 3.1. *Let M be a complex n-dimensional $(n \ge 3)$ complete Kähler manifold with radial curvature K_r satisfying $-a^2 \le K_r \le -b^2 < 0$ with $a \ge b > 0$ and $(n-1)b \ge (p-1)a$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). If $\omega \in A^p(\xi)$ is J-invariant and satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *then*

$$\frac{1}{\sinh^C(ar_1)}\int_{B_{r_1}(x_0)}\cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]\,dv \le \frac{1}{\sinh^C(ar_2)}\int_{B_{r_2}(x_0)}\cosh(ar)\big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]\,dv$$

for any $0 < r_1 < r_2$, where $C = [2(n-2)b - 2(p-2)a]/a$ and $B_r(x_0) \subseteq M$ is a geodesic ball of radius r centered at x_0 in M.

Proof. If $X = \mathrm{grad}(\psi)$ is the gradient of a smooth function ψ on M, then $\theta_X = d\psi$ and $\nabla\theta_X = \mathrm{Hess}(\psi)$. Let $\psi = \cosh(ar)$. It is easy to see that

$$\mathrm{Hess}(\cosh(ar)) = a^2\cosh(ar)\,dr\otimes dr + a\sinh(ar)\,\mathrm{Hess}(r). \tag{3-1}$$

Let $\{e_i, Je_i\}$ with $e_n = \partial/\partial r$ be an orthonormal frame field around $x_0 \in M$. Then, for $\omega \in A^p(\xi)$, we have

$$\begin{aligned}|\omega|^2 &= \frac{1}{p}\bigg[(\omega\odot\omega)\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) + (\omega\odot\omega)\Big(J\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\Big) \\ &\qquad + \sum_{\lambda=1}^{n-1}(\omega\odot\omega)(e_\lambda, e_\lambda) + \sum_{\lambda=1}^{n-1}(\omega\odot\omega)(Je_\lambda, Je_\lambda)\bigg] \\ &= \frac{2}{p}\bigg\{|i_{\partial/\partial r}\omega|^2 + \sum_{\lambda=1}^{n-1}(\omega\odot\omega)(e_\lambda, e_\lambda)\bigg\}.\end{aligned} \tag{3-2}$$

It follows from (3-1), (3-2) and Proposition 2.1 that

$$
\begin{aligned}
(3\text{-}3)\quad & \langle S_\omega, \nabla\theta_X\rangle \\
&= \tfrac{1}{2}|\omega|^2\langle g, \mathrm{Hess}(\mathrm{ch}(ar))\rangle - \langle(\omega\odot\omega), \mathrm{Hess}(\mathrm{ch}(ar))\rangle \\
&= \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\Big\{\sum_\lambda a\sinh(ar)\,\mathrm{Hess}(r)(e_\lambda, e_\lambda) \\
&\qquad + \sum_\lambda a\sinh(ar)\,\mathrm{Hess}(r)(Je_\lambda, Je_\lambda) \\
&\qquad - (p-1)a^2\cosh(ar) - pa\sinh(ar)\,\mathrm{Hess}(r)\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) \\
&\qquad - (p-1)a\sinh(ar)\,\mathrm{Hess}(r)\Big(J\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\Big)\Big\} \\
&\quad + \sum_\lambda \tfrac{1}{p}(\omega\odot\omega)(e_\lambda, e_\lambda)\Big\{a^2\cosh(ar) + a\sinh(ar)\,\mathrm{Hess}(r)\Big(J\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\Big) \\
&\qquad + \sum_\mu a\sinh(ar)\,\mathrm{Hess}(r)(e_\mu, e_\mu) + \sum_\mu a\sinh(ar)\,\mathrm{Hess}(r)(Je_\mu, Je_\mu) \\
&\qquad - p\,\mathrm{Hess}(\cosh(ar))(e_\lambda, e_\lambda) - p\,\mathrm{Hess}(\cosh(ar))(Je_\lambda, Je_\lambda)\Big\} \\
&\geq \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\{2(n-1)ab\sinh(ar)\coth(br) - 2(p-1)a^2\cosh(ar)\} \\
&\quad + \sum_\lambda \tfrac{1}{p}(\omega\odot\omega)(e_\lambda, e_\lambda)\{2(n-2)ab\sinh(ar)\coth(br) \\
&\qquad - 2(p-2)a^2\cosh(ar)\} \\
&\geq \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2 a\cosh(ar)\{2(n-1)b - 2(p-1)a\} \\
&\quad + \sum_\lambda \tfrac{1}{p}(\omega\odot\omega)(e_\lambda, e_\lambda)a\cosh(ar)\{2(n-2)b - 2(p-2)a\} \\
&\geq \sum_\lambda \tfrac{1}{p}(\omega\odot\omega)(e_\lambda, e_\lambda)a\cosh(ar)\{2(n-2)b - 2(p-2)a\}.
\end{aligned}
$$

On the other hand, we have

$$
\begin{aligned}
(3\text{-}4)\qquad S_\omega\Big(X, \frac{\partial}{\partial r}\Big) &= \tfrac{1}{2}|\omega|^2 a\sinh(ar) - a\sinh(ar)(\omega\odot\omega)\Big(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Big) \\
&\leq \big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]a\sinh(ar).
\end{aligned}
$$

Substituting (3-3) and (3-4) into (2-2), we obtain

$$
\begin{aligned}
(3\text{-}5)\quad & \int_{\partial B_r(x_0)} \big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big]a\sinh(ar)\,ds \\
&\qquad \geq \int_{B_r(x_0)} \big[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2\big][2(n-2)b - 2(p-2)a]a\cosh(ar)\,dv.
\end{aligned}
$$

It can be seen from (3-5) that

$$\frac{\cosh(ar)\int_{\partial B_r(x_0)}\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]ds}{\int_{B_r(x_0)}\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]\cosh(ar)\,dv}\geq\frac{aC\cosh(ar)}{\sinh(ar)},\tag{3-6}$$

where $C=[2(n-2)b-2(p-2)a]/a$.

Thus we obtain from (3-6)

$$\frac{d}{dr}\ln\left\{\int_{B_r(x_0)}\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]\cosh(ar)\,dv\right\}\geq\frac{d}{dr}\{C\ln[\sinh(ar)]\}.\tag{3-7}$$

Integrating (3-7) over $[r_1,r_2]$, we have

$$\begin{aligned}\ln\int_{B_{r_2}(x_0)}&\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]\cosh(ar)\,dv\\&-\ln\int_{B_{r_1}(x_0)}\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]\cosh(ar)\,dv\\&\geq C\ln[\sinh(ar_2)]-C\ln[\sinh(ar_1)].\end{aligned}\tag{3-8}$$ □

Now we can deduce the following vanishing theorem from the above monotonicity formula.

Theorem 3.2. *Let M be a complex n-dimensional ($n\geq 3$) complete Kähler manifold with radial curvature K_r satisfying $-a^2\leq K_r\leq -b^2<0$ with $a\geq b>0$ and $(n-1)b\geq(p-1)a$. Let $\xi:E\to M$ be a smooth Riemannian vector bundle over (M,g). If the J-invariant p-form $\omega\in A^p(\xi)$ satisfies the conservation law, that is,* div $S_\omega=0$, *and*

$$\frac{\int_{B_r(x_0)}|\omega|^2\,dv}{e^{a(C-1)r}}\to 0\quad \text{as } r\to+\infty,$$

where $C=[2(n-2)b-2(p-2)a]/a$, then $\omega\equiv 0$.

Proof. **Case 1.** If $1\geq(n-1)b-(p-1)a\geq 0$, i.e., $C\leq 1$, it is obvious that $\omega\equiv 0$.

Case 2. If $(n-1)b-(p-1)a>1$, i.e., $C>1$, using the fact $\coth(ar)\to 1$ as $r\to+\infty$ and our condition, we have

$$\begin{aligned}\frac{1}{\sinh^C(ar_2)}\int_{B_{r_2}(x_0)}&\cosh(ar)\left[\frac{1}{2}|\omega|^2-\frac{1}{p}|i_{\partial/\partial r}\omega|^2\right]dv\\&\leq\frac{\cosh(ar_2)\int_{B_{r_2}(x_0)}\frac{1}{2}|\omega|^2\,dv}{\sinh^C(ar_2)}\\&=\frac{\int_{B_{r_2}(x_0)}\frac{1}{2}|\omega|^2\,dv}{e^{a(C-1)r_2}}\left[\frac{e^{ar_2}}{\sinh(ar_2)}\right]^{C-1}\coth(ar_2)\\&\to 0\quad\text{as } r_2\to+\infty.\end{aligned}\tag{3-9}$$

It follows from (3-9) and Theorem 3.1 that

$$\tfrac{1}{2}|\omega|^2 - \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2 = 0, \quad \text{i.e., } (\omega \odot \omega)(e_\lambda, e_\lambda) = 0. \tag{3-10}$$

Set $X = r\partial/\partial r$. It is easy to see from (3-10), (3-4), (3-3), (3-2) and (2-2) that

$$\begin{aligned}
\int_{\partial B_r(x_0)} -\frac{p-1}{p} r|i_{\partial/\partial r}\omega|^2\,ds
&= \int_{\partial B_r(x_0)} \bigl[\tfrac{r}{2}|\omega|^2 - r|i_{\partial/\partial r}\omega|^2\bigr]\,ds \\
&= \int_{\partial B_r(x_0)} S_\omega\Bigl(X, \frac{\partial}{\partial r}\Bigr)\,ds \\
&= \tfrac{1}{p}|i_{\partial/\partial r}\omega|^2 \Bigl\{\sum_\lambda r\,\mathrm{Hess}(r)(e_\lambda, e_\lambda) + \sum_\lambda r\,\mathrm{Hess}(r)\Bigl(Je_\lambda, Je_\lambda\Bigr) \\
&\qquad - (p-1) - pr\,\mathrm{Hess}(r)\Bigl(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\Bigr) - (p-1)r\,\mathrm{Hess}(r)\Bigl(J\frac{\partial}{\partial r}, J\frac{\partial}{\partial r}\Bigr)\Bigr\} \\
&\ge \int_{B_r(x_0)} \tfrac{1}{p}[2(n-1)br\coth(br) - p + 1 - (p-1)ar\coth(ar)]|i_{\partial/\partial r}\omega|^2\,dv \\
&\ge \int_{B_r(x_0)} \tfrac{1}{p}\bigl\{(n-1)br\coth(br) - p + 1 \\
&\qquad + [(n-1)br - (p-1)ar]\coth(br)\bigr\}|i_{\partial/\partial r}\omega|^2\,dv \\
&\ge \int_{B_r(x_0)} \tfrac{1}{p}[(n-1)br\coth(br) - p + 1]|i_{\partial/\partial r}\omega|^2\,dv \\
&\ge \int_{B_r(x_0)} \tfrac{1}{p}[n-p]|i_{\partial/\partial r}\omega|^2\,dv.
\end{aligned} \tag{3-11}$$

Using our condition $(n-1)b - (p-1)a \ge 0$, we get $n - p \ge 0$, which, together with (3-11) and $x\coth x > 1$ for $x > 0$, yields $|i_{\partial/\partial r}\omega|^2 = 0$. □

4. Monotonicity formulae for Riemannian manifolds

Theorem 4.1. *Let M be an n-dimensional ($n \ge 3$) complete Riemannian manifold with radial curvature K_r satisfying $-a^2 \le K_r \le -b^2 < 0$ with $a \ge b > 0$ and $(n-1)b - (2p-1)a \ge 0$. Let $\xi : E \to M$ be a smooth Riemannian vector bundle over (M, g). If $\omega \in A^p(\xi)$ satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *then*

$$\begin{aligned}
\frac{1}{\sinh^C(ar_1)} \int_{B_{r_1}(x_0)} \cosh(ar)\bigl[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\bigr]\,dv& \\
\le \frac{1}{\sinh^C(ar_2)} \int_{B_{r_2}(x_0)} \cosh(ar)\bigl[\tfrac{1}{2}|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\bigr]\,dv&
\end{aligned}$$

for any $0 < r_1 < r_2$, *where* $C = [(n-2)b - (2p-2)a]/a$ *and* $B_r(x_0) \subseteq M$ *is a geodesic ball of radius* r *centered at* x_0 *in* M.

Proof. Set $X = \sinh(ar)\partial/\partial r$, where $\partial/\partial r$ denotes the unit radial vector. Obviously, the unit normal vector to $\partial B_r(x_0)$ is $\partial/\partial r$. Let $\{e_\lambda, \partial/\partial r\}$ be an orthonormal frame field on $B_r(x_0)$, where $\lambda = 1, \dots, n-1$. Then we have that

$$\nabla_{\partial/\partial r} X = a\cosh(ar)\frac{\partial}{\partial r} \quad \text{and} \quad \nabla_{e_\lambda} X = \sinh(ar)\sum_\mu h_{\lambda\mu} e_\mu, \tag{4-1}$$

where the $-h_{\lambda\mu}$ are the components of the second fundamental form of $\partial B_r(x_0)$ in $B_r(x_0)$.

On the other hand, we have

$$\operatorname{Hess}(r)(e_\lambda, e_\mu) = \langle e_\lambda, \nabla_{\partial/\partial r} e_\mu\rangle = \langle e_\lambda, h_{\mu\nu} e_\nu\rangle = h_{\lambda\mu}. \tag{4-2}$$

We can choose an orthonormal frame field $\{e_\lambda\}$ on $\partial B_r(x_0)$ such that $h_{\lambda\mu} = \delta_{\lambda\mu} h_{\lambda\lambda}$. It follows from (4-1), (4-2), (2-1) and (2-3) that

$$\begin{aligned}
&\langle S_\omega, \nabla\theta_X\rangle \\
&\quad = \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\Big\{a\cosh(ar) + \sinh(ar)\sum_\lambda h_{\lambda\lambda} - 2pa\cosh(ar)\Big\} \\
&\qquad + \sum_{\lambda=1}^{n-1}\tfrac{1}{2p}(\omega\odot\omega)(e_\lambda, e_\lambda)\Big\{a\cosh(ar) + \sinh(ar)\sum_\mu h_{\nu\nu} - 2p\sinh(ar)h_{\lambda\lambda}\Big\} \\
&\quad \geq \big[\tfrac12|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]aC\cosh(ar).
\end{aligned}\tag{4-3}$$

On the other hand, we have

$$S_\omega\Big(X, \frac{\partial}{\partial r}\Big) \leq \big[\tfrac12|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]\sinh(ar). \tag{4-4}$$

Substituting (4-3) and (4-4) into (2-2), we obtain

$$\begin{aligned}
&\int_{\partial B_r(x_0)} \big[\tfrac12|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]\sinh(ar)\,ds \\
&\qquad\qquad \geq \int_{B_r(x_0)} \big[\tfrac12|\omega|^2 - \tfrac{1}{2p}|i_{\partial/\partial r}\omega|^2\big]aC\cosh(ar)\,dv.
\end{aligned}\tag{4-5}$$

The proof is completed using (4-5) along with the same arguments used in the proof of Theorem 3.1. □

Similarly, using Theorem 4.1 along with the same arguments used in the proof of Theorem 3.2, we get the following vanishing theorem:

Theorem 4.2. *Let* M *be an* n*-dimensional* $(n \geq 3)$ *complete Riemannian manifold with radial curvature* K_r *satisfying* $-a^2 \leq K_r \leq -b^2 < 0$ *with* $a \geq b > 0$ *and*

$(n-1)b-(2p-1)a \geq 0$. *Let* $\xi : E \to M$ *be a smooth Riemannian vector bundle over* (M, g). *If* $\omega \in A^p(\xi)$ *satisfies the conservation law, that is,* $\operatorname{div} S_\omega = 0$, *and*

$$\frac{\int_{B_r(x_0)} |\omega|^2 \, dv}{e^{a(C-1)r}} \to 0 \quad \text{as } r \to +\infty,$$

where $C = [(n-2)b-(2p-2)a]/a$, *then* $\omega \equiv 0$.

5. Monotonicity formulae for pluriharmonic maps

Let M be a complex n-dimensional ($n \geq 3$) Kähler manifold. The complex structure of M gives a decomposition of TM^C into tangent vectors of types (1,0) and (0,1), i.e,

$$TM^C = T^{1,0}M \oplus T^{0,1}M.$$

Let $\phi : M \to N$ be a smooth map between Kähler manifolds. Then we have the following bundle maps:

$$\begin{aligned} \partial\phi &: T^{1,0}M \to T^{1,0}N, & \bar{\partial}\phi &: T^{0,1}M \to T^{1,0}N, \\ \partial\bar{\phi} &: T^{1,0}M \to T^{0,1}N, & \overline{\partial\phi} &: T^{0,1}M \to T^{0,1}N. \end{aligned}$$

A direct computation gives

$$|\bar{\partial}\phi|^2 = \frac{1}{4}\sum_{i=1}^{n}\bigl\{\langle d\phi(e_i), d\phi(e_i)\rangle + \langle d\phi(Je_i), d\phi(Je_i)\rangle - 2\langle d\phi(Je_i), J' d\phi(e_i)\rangle\bigr\} \tag{5-1}$$

and

$$|\partial\phi|^2 = \frac{1}{4}\sum_{i=1}^{n}\bigl\{\langle d\phi(e_i), d\phi(e_i)\rangle + \langle d\phi(Je_i), d\phi(Je_i)\rangle + 2\langle d\phi(Je_i), J' d\phi(e_i)\rangle\bigr\}, \tag{5-2}$$

where $\{e_i, Je_i\}$ is an orthonormal frame field on M, and J and J' are the complex structures of M and N, respectively.

We introduce two 1-forms $\sigma, \tau \in A^1(\phi^{-1}TN)$ given by

$$\sigma(X) = \frac{d\phi(X) + J' d\phi(JX)}{2} \quad \text{and} \quad \tau(X) = \frac{d\phi(X) - J' d\phi(JX)}{2}$$

for any $X \in TM$.

Lemma 5.1 [Dong 2013]. σ, τ *are J-invariant, and* $|\sigma|^2 = 2|\bar{\partial}\phi|^2$, $|\tau|^2 = 2|\partial\phi|^2$.

Siu [1980] introduced pluriharmonic maps. A smooth map $\phi : M \to N$ between Kähler manifolds is called pluriharmonic if $(\nabla d\phi)(X, \overline{Y}) = 0$, for all $X, Y \in T^{1,0}M$.

Lemma 5.2 [Dong 2013]. *If a map* $\phi : M \to N$ *between Kähler manifolds is pluriharmonic, then we have* $\operatorname{div} S_\sigma = \operatorname{div} S_\tau = 0$, *where* $S_\sigma = \frac{1}{2}|\sigma|^2 g - \sigma \odot \sigma$ *and* $S_\tau = \frac{1}{2}|\tau|^2 g - \tau \odot \tau$.

In this section, we will establish monotonicity formulae for pluriharmonic maps and harmonic maps.

Theorem 5.3. *Let M be a complex n-dimensional $(n \geq 2)$ complete Kähler manifold with radial curvature K_r satisfying $K_r \leq -b^2 < 0$ with $b > 0$. Suppose $\phi : M \to N$ is either a pluriharmonic map between Kähler manifolds or a harmonic map into a Kähler manifold with strongly seminegative curvature. Then*

$$\frac{\int_{B_{r_1}(x_0)} \cosh(br)|\bar{\partial}\phi|^2\, dv}{\sinh^{2(n-1)}(br_1)} \leq \frac{\int_{B_{r_2}(x_0)} \cosh(br)|\bar{\partial}\phi|^2\, dv}{\sinh^{2(n-1)}(br_2)}$$

and

$$\frac{\int_{B_{r_1}(x_0)} \cosh(br)|\partial\phi|^2\, dv}{\sinh^{2(n-1)}(br_1)} \leq \frac{\int_{B_{r_2}(x_0)} \cosh(br)|\partial\phi|^2\, dv}{\sinh^{2(n-1)}(br_2)}$$

for any $0 < r_1 < r_2$.

Proof. When $\phi : M \to N$ is a pluriharmonic map between Kähler manifolds, it follows from Lemmas 5.1 and 5.2 and (3-4), in which $p = 1$ and $\omega = \sigma$, that

$$\begin{aligned}(5\text{-}3)\quad \langle S_\sigma, \nabla\theta_X\rangle &\\ &\geq |i_{\partial/\partial r}\sigma|^2 2(n-1)b^2\cosh(br) + (\sigma \odot \sigma)(e_\lambda, e_\lambda)2(n-1)b^2\cosh(br)\\ &= (n-1)b^2\cosh(br)|\sigma|^2 = 2(n-1)b^2\cosh(br)|\bar{\partial}\phi|^2.\end{aligned}$$

On the other hand, we have

$$S_\sigma(X, v) \leq b\sinh(br)|\bar{\partial}\phi|^2. \tag{5-4}$$

Substituting (5-3) and (5-4) into (2-2) yields

$$\int_{\partial B_r} b\sinh(br)|\bar{\partial}\phi|^2\, ds \geq \int_{B_r} 2(n-1)b^2\cosh(br)|\bar{\partial}\phi|^2\, dv. \tag{5-5}$$

When $\phi : M \to N$ is a harmonic map into a Kähler manifold with strongly seminegative curvature, we have $\int_{B_r}(\operatorname{div} S_\sigma)(X)\, dv = \int_{B_r}(\operatorname{div} S_\tau)(X)\, dv \geq 0$ [Dong 2013]. Then ϕ also satisfies the integral formula (5-5).

The proof is completed using (5-5) and the same arguments used in the proof of Theorem 3.1. □

Similarly, using Theorem 5.3 along with the same arguments used in the proof of Theorem 3.2, we get the following theorem:

Theorem 5.4. *Let M be a complex n-dimensional $(n \geq 2)$ complete Kähler manifold with radial curvature K_r satisfying $K_r \leq -b^2 < 0$ with $b > 0$. Suppose $\phi : M \to N$ is either a pluriharmonic map between Kähler manifolds or a harmonic map into a*

Kähler manifold with strongly seminegative curvature. If

$$\frac{\int_{B_r(x_0)} |\bar{\partial}\phi|^2\, dv}{e^{(2n-3)br}} \to 0 \quad \left(\textit{resp. } \frac{\int_{B_r(x_0)} |\partial\phi|^2\, dv}{e^{(2n-3)br}} \to 0\right) \quad \textit{as } r \to +\infty$$

then ϕ is holomorphic (*resp. antiholomorphic*). *In particular*, *if*

$$\frac{\int_{B_r(x_0)} |d\phi|^2\, dv}{e^{(2n-3)br}} \to 0 \quad \textit{as } r \to +\infty,$$

then ϕ is constant.

Acknowledgement

The author thanks the referee for valuable comments and suggestions.

References

[Baird and Eells 1981] P. Baird and J. Eells, "A conservation law for harmonic maps", pp. 1–25 in *Geometry Symposium* (Utrecht, 1980), edited by E. Looijenga et al., Lecture Notes in Math. **894**, Springer, Berlin, 1981. MR 83i:58031 Zbl 0485.58008

[Dong 2013] Y. X. Dong, "Monotonicity formulae and holomorphicity of harmonic maps between Kähler manifolds", *Proc. Lond. Math. Soc.* (3) **107**:6 (2013), 1221–1260. MR 3149846 Zbl 1295.53070

[Dong and Lin 2014] Y. X. Dong and H. Z. Lin, "Monotonicity formulae, vanishing theorems and some geometric applications", *Q. J. Math.* **65**:2 (2014), 365–397. MR 3230367 Zbl 1303.53046

[Garber et al. 1979] W.-D. Garber, S. N. M. Ruijsenaars, E. Seiler, and D. Burns, "On finite action solutions of the nonlinear σ-model", *Ann. Physics* **119**:2 (1979), 305–325. MR 80e:81059 Zbl 0412.35089

[Greene and Wu 1979] R. E. Greene and H. Wu, *Function theory on manifolds which possess a pole*, Lecture Notes in Math. **699**, Springer, Berlin, 1979. MR 81a:53002 Zbl 0414.53043

[Sealey 1982] H. C. J. Sealey, "Some conditions ensuring the vanishing of harmonic differential forms with applications to harmonic maps and Yang–Mills theory", *Math. Proc. Cambridge Philos. Soc.* **91**:3 (1982), 441–452. MR 83i:58038 Zbl 0494.58002

[Sealey $\geq$ 2016] H. C. J. Sealey, "The stress-energy tensor and vanishing of L^2 harmonic forms", preprint.

[Siu 1980] Y. T. Siu, "The complex-analyticity of harmonic maps and the strong rigidity of compact Kähler manifolds", *Ann. of Math.* (2) **112**:1 (1980), 73–111. MR 81j:53061 Zbl 0517.53058

[Xin 1986] Y. L. Xin, "Differential forms, conservation law and monotonicity formula", *Sci. Sinica Ser. A* **29**:1 (1986), 40–50. MR 87m:58046 Zbl 0616.58001

Received January 27, 2015. Revised May 25, 2015.

JINTANG LI
SCHOOL OF MATHEMATICAL SCIENCES
XIAMEN UNIVERSITY
XIAMEN, FUJIAN 361005
CHINA
dli66@xmu.edu.cn

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscribers address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 281 No. 1 March 2016
