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OVER CERTAIN UNBOUNDED DOMAINS
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Given an unbounded domain Ω of a Hadamard manifold M , it makes sense to consider the problem of finding minimal graphs with prescribed continuous data on its cone topology boundary, i.e., on its ordinary boundary together with its asymptotic boundary. In this article it is proved that under the hypothesis that the sectional curvature of M is ≤ -1 , this Dirichlet problem is solvable if Ω satisfies a certain convexity condition at infinity and if $\partial\Omega$ is mean convex. We also prove that mean convexity of $\partial\Omega$ is a necessary condition, extending to unbounded domains some results that are valid on bounded ones.

1. Introduction

The classical theorem of Jenkins and Serrin on minimal graphs theory, following the works of Bernstein [1910], Haar [1927], Radó [1930] and Finn [1965], states the following.

Theorem 1 [Jenkins and Serrin 1968, Theorem 1]. *Let $D \subset \mathbb{R}^n$ be a bounded domain whose boundary is of class C^2 . Then the Dirichlet problem for the minimal surface equation in D is well posed for C^2 boundary data if and only if the mean curvature of ∂D is everywhere nonnegative.*

In the last four decades, several works considered problems related to Theorem 1 in distinct directions. Some of them are listed below together with some references.

- Unbounded domains of \mathbb{R}^2 : [Hwang 1988; Collin and Krust 1991; Sá Earp and Rosenberg 1989; Ripoll and Tomi 2014; Krust 1989; Kuwert 1993; Kutev and Tomi 1998].
- Bounded domains of a Hadamard manifold M : [Folha and Rosenberg 2012; Mazet et al. 2011; Aiolfi et al. 2016].

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- Asymptotic Dirichlet problems on Hadamard manifolds: [do Espírito Santo et al. 2010; Ripoll and Telichevesky 2015; Gálvez and Rosenberg 2010; Castéras et al. 2013].
- Replace the ambient space \mathbb{R}^{n+1} by the hyperbolic spaces \mathbb{H}^{n+1} [Barbosa and Sá Earp 1998; Guio and Sá Earp 2005; López 2001; Nitsche 2002] or other ambient spaces with a Killing field satisfying certain hypotheses [Alías and Dajczer 2007; Dajczer et al. 2008; 2013]. In this setting it is natural to consider CMC Killing graphs and there is an extensive bibliography on it.

The purpose of this article is to prove that similar existence and nonexistence results remain valid if in [Theorem 1](#), \mathbb{R}^n is replaced by a Hadamard manifold M with sectional curvature $K_M \leq -1$ and the domain D is unbounded and “strictly convex at infinity” (see [Definition 4](#)).

Classically, Dirichlet problems on unbounded domains are considered in \mathbb{R}^n without prescribed values at infinity. In fact, sometimes the behavior at infinity of bounded solutions is determined by their boundary values. For instance, in \mathbb{R}^2 it is a consequence of [Theorem 2](#) of [Collin and Krust 1991], which states that if u and v are distinct solutions of the Dirichlet problem in an unbounded domain $U \subset \mathbb{R}^2$ which coincide on ∂U , then $\sup |u - v|$ must have at least logarithmic growth. However, since the manifolds that we consider in this work have sectional curvature $K_M \leq -1$, it turns out that the asymptotic boundary of unbounded domains may be “good enough” to prescribe continuous data on them. It therefore makes sense to consider the generalized Dirichlet problem for the minimal hypersurface equation, [Problem 2](#), described in the sequel. In order to state it, let us introduce some useful notations that are not standard.

Throughout this paper M denotes an m -dimensional Hadamard manifold, $m \geq 2$, with sectional curvature K_M satisfying $K_M \leq -1$. The asymptotic boundary $\partial_\infty M$ of M is defined by the set of equivalence classes of geodesic rays that stay at finite distance for all time, and it is possible to compactify M by adding $\partial_\infty M$ to it. $\bar{M} := M \cup \partial_\infty M$ carries the so-called cone topology (see [Eberlein and O’Neill 1973]), which makes it canonically homeomorphic to a closed ball. If $U \subset \bar{M}$ is any set, we denote by $\bar{U}^{\text{ct}} \subset \bar{M}$ and $\partial^{\text{ct}} U \subset \bar{M}$ its closure and boundary in terms of the cone topology; we also use the notation $\partial_\infty U := \partial^{\text{ct}} U \cap \partial_\infty M$.

Problem 2. Let $\Omega \subset M$ be a C^2 domain of M . Given $\varphi \in C(\partial^{\text{ct}} \Omega)$, find a minimal graph over Ω that attains φ on its boundary, or, equivalently, find a solution of the Dirichlet problem

$$\begin{cases} u \in C^2(\Omega) \cap C\bar{\Omega}^{\text{ct}}, \\ \mathcal{M}(u) := \operatorname{div} \left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}} \right) = 0 \quad \text{in } \Omega, \\ u|_{\partial^{\text{ct}} \Omega} = \varphi. \end{cases}$$

Concerning the existence, perhaps the main difficulty dealing with unbounded domains is the nonexistence of natural barriers. In general, barriers are constructed using distance functions to a point or to the boundary of the domain, which cannot be adapted directly to points at infinity. Here the geometry of M at infinity plays an important role. For instance, the hyperbolic spaces \mathbb{H}^n have “good geometry” at infinity by the existence of hyperspheres separating points at infinity and having their principal curvatures with the correct sign. The natural way to generalize this fact in order to use the Hessian comparison theorem and adapt barriers of \mathbb{H}^n to other Hadamard manifolds is given by the *strict convexity condition (SC condition) at infinity*, introduced in [Ripoll and Telichevesky 2015]. In that work it is proved that [Problem 2](#) is solvable for $\Omega = M$ (in this case, it is called the asymptotic Dirichlet problem) and any continuous boundary data if M satisfies the SC condition described below.

Definition 3. A Hadamard manifold M is said to satisfy the *strict convexity condition at infinity* if for all $x \in \partial_\infty M$ and all relatively open subsets $\Gamma \subset \partial_\infty M$ with $x \in \Gamma$, there exists an open set $V \subset M$ of class C^2 such that x is an interior point of $\partial_\infty V$ (with respect to the induced topology), $\partial_\infty V \subset \Gamma$ and $M \setminus V$ is convex.

At this point, it should be mentioned that under the hypothesis $K_M \leq -1$, the SC condition is always satisfied by 2-dimensional manifolds, by the rotationally symmetric ones and by those manifolds with controlled decay on sectional curvature (exponential decay) (see also [Ripoll and Telichevesky 2015]). We also should mention that under the same assumption on K_M , the SC condition is equivalent to the *convex conic neighborhood condition* presented by H. Choi [1984] in the study of the asymptotic Dirichlet problem with respect to Laplace’s operator on Cartan–Hadamard manifolds; the equivalence is a consequence of a lemma of A. Borbély [1998b, Lemma 1]. In fact, both Dirichlet problems are closely related and may be studied together (see also [Ripoll and Telichevesky 2015]).

Contrasting with the existence results under the SC condition, we cite a counterexample constructed by I. Holopainen and J. Ripoll [2015]. In this work the authors present a Hadamard manifold with $K_M \leq -1$ that does not admit a solution to the asymptotic Dirichlet problem for the minimal hypersurface equation for any continuous $\varphi \in C(\partial_\infty M)$, although there are bounded nonconstant minimal graphs globally defined on M (see Theorem 1.1 of [Holopainen and Ripoll 2015]). This counterexample proves that the condition $K_M \leq -1$ is not sufficient to solve [Problem 2](#) with any continuous boundary data.

Taking into account all these facts, the following definition is natural.

Definition 4. A domain $\Omega \subset \bar{M}$ is *strictly convex at infinity* if for any $x \in \partial_\infty \Omega$ and any relatively open neighborhood $\Gamma \subset \partial^{\text{ct}} \Omega$ of x , there exists an open neighborhood $V = V(x, \Gamma, \Omega) \subset \Omega$ of x such that $\bar{V} \cap \partial^{\text{ct}} \Omega \subset \Gamma$ and all the principal curvatures of $\partial V \cap \Omega$, oriented in the direction of $\Omega \setminus V$, are nonnegative.

Notice that when $\Omega = M$, this definition coincides with the SC condition. With [Definition 4](#) it is now possible to state our main existence result.

Theorem 5. *Let $\Omega \subset M$ be a mean convex domain (with respect to the inward orientation) that is strictly convex at infinity. Then [Problem 2](#) is solvable for any continuous boundary data.*

Returning our attention to [Theorem 1](#), when $\Omega \subset \mathbb{R}^n$ is bounded, the mean convexity is a necessary condition to the solvability of [Problem 2](#) for any continuous φ . The second part of this article is dedicated to proving that mean convexity is also necessary in M if we deal with unbounded domains and require boundedness of solutions. In [Section 3](#) we present some necessary lemmata and the proof of the following nonexistence result.

Theorem 6. *Let $\Omega \subset \bar{M}$ be a domain and suppose that there exists $y \in \partial\Omega$ such that the mean curvature of $\partial\Omega$ at y (with respect to the inward orientation) satisfies $H(y) < 0$. Then there exists a continuous function $\varphi : \partial^{\text{ct}}\Omega \rightarrow \mathbb{R}$ such that [Problem 2](#) is not solvable.*

The construction of φ depends on two basic ingredients. First of all, on the local aspect concerning the negativity of the mean curvature $H(y)$, it is essential to guarantee that $\varphi(y)$ is bounded by values of the solution on a small sphere centered at y , say, on $S_r(y) \cap \Omega$. The second essential ingredient is the existence of a bounded barrier in $\Omega \setminus B_r(y)$ with some special properties. Similar results outside \mathbb{R}^n were proved on bounded domains considering barriers dependent on the diameter of Ω , as in [[Nitsche 2002](#)]. Our main improvement is dropping the dependence on the size of the domain.

Combining the results of [Theorems 5 and 6](#), we get:

Theorem 7. *Let $\Omega \subset M$ be a domain that is strictly convex at infinity. Then the Dirichlet problem ([Problem 2](#)) is solvable for any continuous boundary data if and only if Ω is mean convex.*

It remains an open question what happens if we assume that Ω is not strictly convex at infinity. We conjecture that in this case it is also possible to construct a continuous function on $\partial^{\text{ct}}\Omega$ for which the Dirichlet problem is not solvable, and therefore strict convexity at infinity is also a necessary condition. Since it deals with nonexistence of solutions in arbitrarily large domains, [Theorem 6](#) may have an important role in the study of this conjecture.

To finish, we should mention that there is a large gap between the behavior of K_M at infinity in the cases where [Theorem 5](#) is true and in the ones where it is false. It also remains unknown if there exists some sharp condition on K_M that assures solvability of [Problem 2](#) for any continuous boundary data.

2. Existence result

This section is dedicated to proving [Theorem 5](#). We start with a very important tool, the comparison principle for unbounded domains. It plays an important role in both existence and uniqueness. For now, we just need to work with functions that extend continuously to the asymptotic boundary, however in [Section 3](#) we treat a larger class of functions, as stated above.

Proposition 8 (comparison principle for unbounded domains). *Let $U \subset M$ be an unbounded domain of M . If $u, v \in C^2(U)$ are such that $\mathcal{M}(v) \leq \mathcal{M}(u)$ on U with $\limsup_{p \rightarrow x} u \leq \liminf_{p \rightarrow x} v$ for all $x \in \partial^{\text{ct}}U$, then $u \leq v$ in U .*

Proof. Choose $o \in M$. Let $\varepsilon > 0$ be an arbitrary constant. Using the basis of the cone topology of \bar{M} , we obtain that for all $x \in \partial_\infty U$, there is an open truncated cone $N_x := T_o(x, \alpha_x, R_x)$ (that is, the image of a truncated cone of opening angle α_x and radius R_x by the exponential map of a point o) such that $u < v + \varepsilon$ on N_x . Since $\partial_\infty U$ is compact, there exists uniform R such that $u < v + \varepsilon$ on $U \setminus B_R(o)$. In addition, notice that the hypothesis implies that $u \leq v$ on ∂U . Therefore we have $u \leq v + \varepsilon$ on $\partial(U \cap B_R(o))$, which implies, by the comparison principle on bounded domains, that $u \leq v + \varepsilon$ on $U \cap B_R(o)$, and hence the last inequality holds on U . Since ε is arbitrary, the proof is complete. \square

We now prove [Theorem 5](#) using Perron's method.

A function $\Sigma \in C^0(\bar{\Omega}^{\text{ct}})$ is called a *supersolution* for \mathcal{M} if, given a bounded subdomain $U \subset \Omega$, if $u \in C^2(U) \cap C^0(\bar{U})$ is a solution of $\mathcal{M} = 0$ in U , the condition $u|_{\partial U} \leq \Sigma|_{\partial U}$ implies that $u \leq \Sigma|_U$. A *subsolution* is defined by replacing \leq by \geq .

Let S_φ be defined by

$$S_\varphi := \{v \in C^0(\bar{\Omega}^{\text{ct}}) \mid v \text{ is a subsolution for } \mathcal{M} \text{ with } v|_{\partial^{\text{ct}}\Omega} \leq \varphi\}.$$

By [Proposition 8](#), $v_0 \equiv \min \varphi \in S_\varphi$, which implies that $S_\varphi \neq \emptyset$, and $w \equiv \max \varphi$ is such that $v \leq w$ for all $v \in S_\varphi$. These facts imply that $u : \Omega \rightarrow \mathbb{R}$ given by

$$(1) \quad u(x) := \sup\{v(x) \mid v \in S_\varphi\}$$

is well-defined, and we shall prove that under the hypotheses of [Theorem 5](#), we have $u \in C^\infty(\Omega) \cap C(\bar{\Omega}^{\text{ct}})$, $\mathcal{M}(u) = 0$ and $u|_{\partial^{\text{ct}}\Omega} = \varphi$.

We first prove that $u \in C^\infty(\Omega)$ and $\mathcal{M}(u) = 0$. Given $x \in \Omega$, let $r = r_x > 0$ be sufficiently small such that the open geodesic ball of center x and radius r satisfies $B_r(x) \subset \Omega$ and furthermore r satisfies the inequality

$$\frac{(n-1)^2}{n} \coth^2 r \geq - \inf_{B_r(x)} \text{Ric}_M.$$

Such $r > 0$ exists because $\coth r \rightarrow +\infty$ as $r \rightarrow 0^+$ and Ric_M is of course bounded in bounded sets containing x .

The cylinder $\partial B_r(x) \times \mathbb{R} \subset M \times \mathbb{R}$ has mean curvature $\geq \frac{n-1}{n} \coth r$ (pointing inward) as a consequence of the Hessian comparison theorem, and therefore this choice of r implies, by Theorem 2 of [Dajczer et al. 2013], the existence of minimal graphs in $B_r(x)$ extending continuously to any prescribed continuous boundary data on $\partial B_r(x)$, and this is an essential fact when we use Perron’s method.

Consider a sequence $(v_m)_m \subset S_\varphi$ such that $\lim_m v_m(x) = u(x)$. Theorem 2 of [Dajczer et al. 2013] again implies that for each $m \in \mathbb{N}$ there exists a solution $w_{m,x} \in C^\infty(B_r(x)) \cap C(\overline{B_r(x)})$ of $\mathcal{M} = 0$ such that $w_{m,x}|_{\partial B_r(x)} = v_m|_{\partial B_r(x)}$. The interior gradient estimate given by Theorem 1 of [Dajczer et al. 2013] implies that $(w_{m,x})_m$ contains a subsequence converging uniformly on compact subsets of $B_r(x)$ to a solution $w_x \in C^\infty(B_r(x))$ of $\mathcal{M} = 0$. As in [Gilbarg and Trudinger 1998, Section 2.8], one may prove that $w_x = u|_{B_r(x)}$, which implies that $u \in C^\infty(\Omega)$ and $\mathcal{M} = 0$. This is done by taking the limit of minimal lifts $u_m \in S_\varphi$ of each v_m defined by

$$u_m(y) := \begin{cases} v_m(y) & \text{if } y \in \Omega \setminus B_r(x), \\ w_{m,x}(y) & \text{if } y \in B_r(x). \end{cases}$$

We now need to prove that u extends continuously to the desired boundary data on $\partial^{\text{ct}}\Omega$. Since $\partial\Omega$ is mean convex, standard arguments guarantee that the solution assumes the desired data on $\partial\Omega$. To conclude the proof it is necessary to guarantee that it also extends continuously to $\partial_\infty\Omega$, hence in the following we construct barriers at infinity.

Given $x \in \partial^{\text{ct}}\Omega$ and an open subset V such that $x \in \partial^{\text{ct}}V \cap \partial^{\text{ct}}\Omega$, we call an *upper barrier for \mathcal{M} relative to x and V with height C* a function $\Sigma \in C(\Omega)$ such that

- (i) Σ is a supersolution for \mathcal{M} ;
- (ii) $\Sigma \geq 0$ and $\lim_{p \in \Omega, p \rightarrow x} \Sigma(p) = 0$, the limit with respect to the cone topology;
- (iii) $\Sigma_{\Omega \setminus V} \geq C$.

Lower barriers are defined analogously.

A point $x \in \partial^{\text{ct}}\Omega$ is said to be regular (with respect to the mean curvature operator \mathcal{M}) if it satisfies the following property: given $C > 0$ and a relatively open subset $\Gamma \subset \partial^{\text{ct}}\Omega$ with $x \in \Gamma$, there exist an open set $V \subset \Omega$ such that x is an interior point of $\overline{V} \cap \partial^{\text{ct}}\Omega$ (with respect to the topology induced on the boundary), with $\overline{V} \cap \partial^{\text{ct}}\Omega \subset \Gamma$, and an upper barrier $\Sigma : \Omega \rightarrow \mathbb{R}$ relative to x and V with height C .

The following lemma is analogous to Theorem 2.7 of [Choi 1984], but we present the proof for the sake of completeness.

Lemma 9. *The function u given by (1) extends continuously to φ to each regular point $x \in \partial_\infty\Omega$.*

Proof. Given $x \in \partial_\infty\Omega$ and $\varepsilon > 0$, let $\Gamma \subset \partial^{\text{ct}}\Omega$ be such that $|\varphi - \varphi(x)| < \varepsilon/2$ in Γ . Let $\Sigma : \Omega \rightarrow \mathbb{R}$ be an upper barrier relative to x and V with height $C = \max |\varphi|$,

where V is given by the definition of regularity. It follows that $w := \Sigma + \varphi(x) + \varepsilon$ is a supersolution for \mathcal{M} . By the choice of Γ , it holds that $w > \varphi$ on Γ and since $w|_{\partial^{\text{ct}}\Omega \setminus \Gamma} \geq \max|\varphi|$, it of course satisfies $w \geq \varphi$ on $\partial^{\text{ct}}\Omega \setminus \Gamma$. Therefore $v \leq w$ for all $v \in S_\varphi$, which implies that

$$\lim_{p \in \Omega, p \rightarrow x} u(p) \leq \lim_{p \in \Omega, p \rightarrow x} w(p) = \varphi(x) + \varepsilon.$$

On the other hand, notice that $v_0 := \varphi(x) - \varepsilon - \Sigma$ belongs to S_φ and therefore $u \geq v_0$ in Ω , which implies that

$$\lim_{p \in \Omega, p \rightarrow x} u(p) \geq \lim_{p \in \Omega, p \rightarrow x} v_0(p) = \varphi(x) - \varepsilon.$$

Since ε is arbitrary, we have $\varphi(x) \leq \lim_{p \in \Omega, p \rightarrow x} u(p) \leq \varphi(x)$. \square

To finish, we now prove regularity at the points of $\partial_\infty\Omega$.

Proposition 10. *Let $\Omega \subset M$ be a domain that is strictly convex at infinity. Then \mathcal{M} is regular at each point of $\partial_\infty\Omega$.*

Proof. Let $x \in \partial_\infty\Omega$ and let $\Gamma \subset \partial^{\text{ct}}\Omega$ be a relatively open neighborhood of x . By hypothesis, there exists an open neighborhood $V \subset \Omega$ of x such that $\bar{V} \cap \partial^{\text{ct}}\Omega \subset \Gamma$ and $\partial V \cap \Omega$ has nonnegative principal curvatures.

Let $s : V \rightarrow \mathbb{R}$ be the distance function to $\partial V \cap \Omega$. Since $K_M \leq -1$ and all principal curvatures of $\partial V \cap \Omega$ are nonnegative, we have that the Laplacian of s satisfies

$$(2) \quad \Delta s \geq (n-1) \tanh s$$

(see, for instance, Theorem 4.3 of [Choi 1984]).

Define $g : (0, +\infty) \rightarrow \mathbb{R}$ by

$$g(s) := \int_s^{+\infty} \frac{dt}{\sqrt{\cosh^{2(n-1)} t - 1}}.$$

Notice that g is well-defined and $\lim_{s \rightarrow 0^+} g(s) = +\infty$, $\lim_{s \rightarrow +\infty} g(s) = 0$. Define now $w : V \rightarrow \mathbb{R}$ by $w(p) := g(s(p))$. A straightforward computation gives

$$\mathcal{M}(w) = (n-1) \cosh^{n-1} s \sinh s + (1-n) \cosh^{1-n} s \Delta s$$

and hence the estimate $\Delta s \geq \tanh s$ leads to $\mathcal{M}(w) \leq 0$.

We remark that w is a solution if $M = \mathbb{H}^n$ and V is a totally geodesic hypersphere.

To finish with the proof, define the supersolution $\Sigma \in C^0(\bar{\Omega})$ by

$$\Sigma(p) = \begin{cases} \min\{w(p), C\} & \text{if } p \in V, \\ C & \text{if } p \in \bar{\Omega} \setminus V, \end{cases}$$

which is of course an upper barrier relative to x and V with height C , and hence the proof is complete. \square

3. Nonexistence result

We now prove that mean convexity of $\partial\Omega$ is a necessary condition, as stated in [Theorem 6](#). We start with the next classical lemma, proved by Jenkins and Serrin [\[1968\]](#) in the case where the domain is bounded and contained on \mathbb{R}^n .

Lemma 11. *Let $U \subset M$ be an open domain and Γ a relatively C^1 open subset of ∂U . If $u \in C(\bar{U}) \cap C^2(U \cup \Gamma)$ and $w \in C(\bar{U}) \cap C^2(U)$ satisfy*

$$(3) \quad \mathcal{M}(w) < \mathcal{M}(u) \quad \text{in } U,$$

$$(4) \quad u \leq w \quad \text{on } \partial U \setminus \Gamma, \text{ and}$$

$$(5) \quad \frac{\partial w}{\partial \nu} = -\infty \quad \text{in } \Gamma,$$

where ν is the inner normal vector to ∂U , then $u \leq w$ in U .

Proof. If $u \leq w$ on Γ , the result is a consequence of the comparison principle. Suppose, towards a contradiction, that there exists $y \in \text{Int } \Gamma$ such that

$$d := \max_{\Gamma} (u - w) = u(y) - w(y) > 0.$$

Then $u \leq w + d$ on ∂U , and hence by the comparison principle we have $u \leq w + d$ in U . Therefore

$$\frac{\partial}{\partial \nu} (u - w)(y) \leq 0 \Rightarrow \frac{\partial}{\partial \nu} (u)(y) \leq -\infty,$$

contradicting the fact that $u \in C^2(U \cup \Gamma)$. □

Lemma 12. *Let $\Omega \subset M$ be an open C^2 domain (possibly unbounded) with mean curvature (with respect to the inner normal) $H : \partial\Omega \rightarrow \mathbb{R}$. Suppose that there exist $y \in \partial\Omega$ such that $H(y) < 0$. Then there exists $s > 0$ depending only on the local geometry of Ω near y and $C > 0$ depending only on $H(y)$ such that if $u \in C^2(\bar{\Omega}) \cap C(\bar{\Omega}^{\text{ct}})$ satisfy $\mathcal{M}(u) = 0$ in Ω , then*

$$u(y) \leq C + \sup_{\partial B_s(y) \cap \Omega} u.$$

Proof. Let $d : \tilde{\Omega} \rightarrow \mathbb{R}$ be given by $d(x) = \text{dist}(x, \partial\Omega)$, where $\tilde{\Omega} \subset \Omega$ is the open subset where d is smooth. Since $H(y) < 0$, it holds that $\Delta d(y) = -H(y) > 0$. Since $\partial\Omega$ is C^2 , there exists $s > 0$ such that $B_s(y) \cap \Omega \subset \tilde{\Omega}$ and

$$\Delta d(x) > -\frac{H(y)}{2} =: \epsilon, \quad \forall x \in B_s(y) \cap \Omega.$$

This is the required s .

We claim that if $x \in B_s(y) \cap \Omega$, then $u(x) \leq C + \sup_{\partial B_s(y) \cap \Omega} u$. To prove it, let Γ_x be the level set of d that contains x and Ω_x be the set enclosed by Γ_x and $\partial B_s(y)$, that is, $\Omega_x := \{p \in B_s(y) \mid d(p) > d(x)\}$.

Consider ψ given by

$$(6) \quad \psi(t) = \frac{\pi}{2} - \operatorname{arcsec}(t+1).$$

Then $\psi \geq 0$, $\psi(0) = \pi/2$ and $\lim_{t \rightarrow +\infty} \psi(t) = 0$. Its first and second derivatives are given below:

$$\begin{aligned} \psi'(t) &= -\frac{1}{(t+1)\sqrt{t^2+2t}}, \\ \psi''(t) &= \frac{1}{(t+1)^2\sqrt{t^2+2t}} + \frac{1}{(t^2+2t)^{3/2}}. \end{aligned}$$

Define $w : B_s(y) \cap \Omega_x \rightarrow \mathbb{R}$ by

$$w(p) := A\psi(d(p)) + \sup_{\partial B_s(y) \cap \Omega} u,$$

where $A > 0$ is to be determined. After some computations we obtain

$$(1 + |\nabla w|^2)^{3/2} \mathcal{M}(w)(p) = A\psi''(d(p)) + (A\psi'(d(p)) + A^3\psi'(d(p))^3) \Delta d(p).$$

Using then that $\Delta d(p) > \epsilon$ and $\psi' < 0$ in the domain we are considering, we obtain

$$\begin{aligned} (1 + |\nabla w|^2)^{3/2} \mathcal{M}(w) &\leq A[\psi'' + \epsilon\psi' + \epsilon A^2\psi'^3] \\ &= A(t+1)^{-3}(t^2+2t)^{-3/2}[(t+1)(t^2+2t) + (t+1)^3 \\ &\quad - \epsilon(t+1)^2(t^2+2t) - \epsilon A^2]. \end{aligned}$$

Notice that the term in the brackets is a polynomial of degree 4 with leading coefficient $-\epsilon < 0$ and constant term $1 - \epsilon A^2$. Then it is clear that there exists $A > 0$ large enough that this polynomial is negative for all $t \geq 0$; with this choice of A we obtain that $\mathcal{M}(w) < 0$ on Ω_x .

Furthermore, by definition of w we have $w \geq u$ on $\partial B_s(y) \cap \Omega_x$ and $\partial w / \partial \nu = -\infty$ on Γ_x , which is an open C^1 portion of $\partial \Omega_x$. We also notice that $u \in C^2(\Gamma_x)$. By [Lemma 11](#), we obtain $w \geq u$ in Ω_x . Since x is arbitrary and u is continuous, it holds the desired inequality with $C = A\frac{\pi}{2}$, which concludes the proof. \square

Proposition 13. *Let M be a Hadamard manifold with sectional curvature $K_M \leq -1$. There exists universal $C > 0$ such that if Ω is a C^1 domain of M and u satisfies $\mathcal{M}(u) = 0$ in Ω , then*

$$\sup_{\partial B_s(y) \cap \Omega} u \leq C + \sup_{\partial^{\text{ct}} \Omega \setminus B_s(y)} u$$

for all $y \in \partial \Omega$ and $s > 0$ such that $\partial B_s(y) \cap \Omega$ is a nonempty connected set.

Proof. Consider $w : \Omega \setminus B_s(y) \rightarrow \mathbb{R}$ given by

$$w(x) = B\psi(r(x)) + \sup_{\partial^{\text{ct}} \Omega \setminus B_s(y)} u,$$

where ψ is given by (6), $r(x) := \text{dist}(x, \partial B_s(y))$ and B is an appropriate constant to be chosen latter. Since $K_M \leq -1$, we have by the Hessian comparison theorem that $\Delta r \geq n - 1$. Hence, mimicking the computations of the previous lemma, we obtain the same polynomial, except that we have $n - 1$ instead of ϵ and B instead of A . It is again clear that there exists B large enough that $\mathcal{M}(w) \leq 0$. We remark that such a constant does not depend on anything (except in the fact that $K_M \leq -1$) since we may choose the constant that is appropriate to the case $n = 2$ and it works on all dimensions.

We are again in the situation of the hypotheses of Lemma 11, with $U = \Omega \setminus B_s(y)$. Hence we obtain, for all $x \in \partial B_s(y) \cap \Omega$,

$$u(x) \leq \sup_{\partial^{\text{ct}} \Omega \setminus B_s(y)} u + B \frac{\pi}{2}$$

and the proof is complete. □

Proof of Theorem 6. By combining the estimates obtained in Lemma 12 and Proposition 13, we obtain the existence of a continuous function $\varphi : \partial\Omega \rightarrow \mathbb{R}$ for which Problem 2 is not solvable: it suffices to put $\varphi(y) = \pi(A + B)$, where A and B are given by the previous results, and $\varphi = 0$ on $\partial\Omega \setminus B_s(y)$, where s is given in the proof of Lemma 12. □

4. Applications

Corollary 14. *Let Ω be a domain that has only finitely many points on $\partial_\infty \Omega$. Then Problem 2 is solvable for any continuous φ if and only if Ω is mean convex.*

Proof. Notice that since $\partial_\infty M$ is compact, $\partial_\infty \Omega$ is also compact and therefore “finitely many points on $\partial_\infty \Omega$ ” is equivalent to “isolated points on $\partial_\infty \Omega$ ”. In order to apply Theorem 7, it suffices to prove that Ω is strictly convex at infinity.

Given $x \in \partial_\infty \Omega$, let $W \subset \bar{\Omega}^{\text{ct}}$ be a relatively open neighborhood of x . We may suppose without loss of generality that x is the only point at infinity of W , otherwise we just work with any open subset of W where this property holds. Choosing $o \in M \setminus W$, we have that W is contained on some truncated cone centered at o with radius $R > 0$, and as a consequence we have $\partial W \subset M \setminus B_R(o)$. Set $V := \Omega \setminus \overline{B_R(o)}$, and it is clear that it satisfies the required conditions. □

Corollary 15. *If M satisfies the SC condition and Ω is a mean convex domain of M such that $\partial_\infty \Omega$ is composed only of open portions and isolated points, then Problem 2 is solvable in Ω . In particular, this is the case if either $\dim M = 2$ or M is rotationally symmetric, or*

$$(7) \quad \min\{K_M(\Pi) \mid \Pi \text{ is a 2-plane in } T_p M, p \in B_{R+1}(o)\} \geq -\frac{e^{2kR}}{R^{2+2\epsilon}}, \quad R \geq R^*$$

for some constants $\epsilon, R^* > 0$.

Particular cases of [Corollary 15](#) may be found in [\[Ripoll and Telichevesky 2015\]](#).

4.1. Application of the technique: Dirichlet problems for p -Laplacians. Consider now the following Dirichlet problem for the p -Laplacian operator, $p > 1$, for continuous u in the Sobolev space $W^{1,p}(\Omega)$:

$$(8) \quad \begin{cases} \Delta_p(u) := \operatorname{div}(|\nabla u|^{p-2}\nabla u) = 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = \varphi. \end{cases}$$

Concerning the case $\Omega = M$, the counterexamples constructed by A. Ancona [\[1994\]](#) and by A. Borbély [\[1998a\]](#) show that some convexity at infinity is also needed to obtain existence of solutions of asymptotic Dirichlet problems related to the Laplacian operator Δ . I. Holopainen [\[2015\]](#) constructed a counterexample for the p -Laplacian operator Δ_p . The manifolds constructed by them contain a point in $\partial_\infty M$ with the property that any open neighborhood of it has the whole manifold as the convex hull, and hence M is not strictly convex at infinity.

On the other hand, in [\[Ripoll and Telichevesky 2015\]](#) the authors proved that the SC condition is sufficient for solvability of asymptotic Dirichlet problems with respect to Δ_p . We may extend this result to our case, proving that if Ω is strictly convex at infinity, then every $x \in \partial_\infty \Omega$ is regular with respect to the operator Δ_p .

The proof is *mutatis mutandis* the same as we have done above; it is sufficient to replace \mathcal{M} by Δ_p and the function g constructed in [Proposition 10](#) by

$$g(s) := c \int_s^{+\infty} \cosh^{(1-n)/(p-1)}(t) dt,$$

where c is a sufficiently large constant ($c = 2C(\cosh 1)^{(n-1)/(p-1)}$ works).

Together with the classical theory of existence of solutions over bounded domains that satisfy the exterior sphere condition, we obtain the following result.

Theorem 16. *Let M be a Hadamard manifold with sectional $K_M \leq -1$. Let $\Omega \subset M$ be an unbounded domain that is strictly convex at infinity and that satisfies the exterior sphere condition on its finite part, namely, given $x \in \partial\Omega$, there exist a sphere contained in $M \setminus \Omega$ that is tangent to $\partial\Omega$ at x . Then (8) is solvable for any $\varphi \in C(\partial^{\text{ct}}\Omega)$.*

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MIRIAM TELICHEVESKY
 INSTITUTO DE MATEMÁTICA
 UNIVERSIDADE FEDERAL DO RIO GRANDE DO SUL
 91509-900 PORTO ALEGRE, RS
 BRAZIL
miriam.telichevesky@ufrgs.br

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balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

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