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We prove that under some conditions every nonparametric capillary surface which has a central fan (of radial limits at a point $\mathcal{O}$) can be perturbed with respect to the contact angle and the perturbed surfaces continue to have central fans. In particular, any nonparametric capillary surface which is symmetric with respect to a vertical plane through $\mathcal{O}$ and has a central fan may be perturbed (with respect to the contact angle) in a nonsymmetric manner and the resulting capillary surfaces will not be symmetric with respect to the vertical plane but will continue to have central fans.

1. Introduction

Let Ω be a bounded open set in $\mathbb{R}^2$ with locally Lipschitz boundary $\partial\Omega$ such that a point $\mathcal{O}$ lies on $\partial\Omega$, $\partial\Omega\setminus\{\mathcal{O}\}$ is a C^2 curve and there exist distinct rays $l^\pm$ starting at $\mathcal{O}$ such that $\partial\Omega$ is tangent to $l^+\cup l^-$ at $\mathcal{O}$. By rotating and translating the domain, we may assume $\mathcal{O}=(0,0)$, $l^+=\{r(\cos\alpha,\sin\alpha):r\geq 0\}$, $l^-=\{r(\cos\alpha,-\sin\alpha):r\geq 0\}$ and

$$\Omega\cap B(\mathcal{O},\delta)=\{r(\cos\theta,\sin\theta):0<r<\delta,\ \theta^-(r)<\theta<\theta^+(r)\}$$

for some $\alpha\in(0,\pi)$, $\delta>0$ and functions $\theta^\pm\in C^0([0,\delta))$ which satisfy $\theta^-<\theta^+$, $\theta^-(0)=-\alpha$ and $\theta^+(0)=\alpha$; here $B(\mathcal{O},\delta)$ is the open ball in $\mathbb{R}^2$ centered at $\mathcal{O}$ of radius δ. We will assume this description of Ω holds throughout this paper.

Let γ be a measurable function mapping $\partial\Omega$ into $[0,\pi]$ and $f\in C^2(\Omega)\cap L^\infty(\Omega)$ be a (bounded) variational solution of the nonparametric capillary surface problem of finding a function $u\in C^2(\Omega)$ such that

$$\operatorname{div}(Tu)=\kappa u+\lambda \quad \text{in } \Omega, \tag{1}$$

$$Tu\cdot\nu=\cos\gamma \quad \text{a.e. on } \partial\Omega, \tag{2}$$

where

$$Tu=\left(\frac{D_1u}{\sqrt{1+|Du|^2}},\frac{D_2u}{\sqrt{1+|Du|^2}}\right),$$

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κ and λ are constants and ν is the outer unit normal to $\partial\Omega$. We will assume $\kappa > 0$ and therefore, by vertical translation, assume $\lambda = 0$. (Since $\kappa > 0$, f is unique.)

Lancaster and Siegel [1996] proved that if γ is bounded away from 0 and π near $\mathcal{O}$, then the radial limit of f at $\mathcal{O}$ in the direction θ,

$$Rf(\theta) \overset{\text{def}}{=} \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

exists for each $\theta \in [-\alpha, \alpha]$, Rf belongs to $C^0([-\alpha, \alpha])$, $Rf(-\alpha)$ is the limiting height at $\mathcal{O}$ of the trace of f on $\partial^-\Omega = \partial\Omega \cap \{y < 0\}$ and $Rf(\alpha)$ is the limiting height at $\mathcal{O}$ of the trace of f on $\partial^+\Omega = \partial\Omega \cap \{y > 0\}$. In particular, when $\alpha > \frac{\pi}{2}$, so that $\partial\Omega$ has a nonconvex (or reentrant) corner at $\mathcal{O}$, and f is discontinuous at $\mathcal{O}$, the conclusion of Theorem 1 of [Lancaster and Siegel 1996] is that the radial limits of f behave in one of the following ways:

(i) There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ and is strictly increasing or strictly decreasing on $[\alpha_1, \alpha_2]$. Label these case (I) and case (D), respectively.

(ii) There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $[-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha]$ and is either strictly increasing on $[\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2]$ or strictly decreasing on $[\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2]$. Label these case (ID) and case (DI), respectively.

In addition, if the limits

$$\gamma_1 = \lim_{\partial^+\Omega \ni (x,y) \to \mathcal{O}} \gamma(x, y) \quad \text{and} \quad \gamma_2 = \lim_{\partial^-\Omega \ni (x,y) \to \mathcal{O}} \gamma(x, y) \tag{3}$$

both exist, then [Lancaster 2010; 2012; Lancaster and Siegel 1996] imply that α_2 equals $\alpha - \gamma_1$ in cases (I) and (DI) and $\alpha + \gamma_1 - \pi$ in cases (D) and (ID) while α_1 equals $-\alpha + \gamma_2$ in cases (D) and (DI) and $\pi - \alpha - \gamma_2$ in cases (I) and (ID).

The intervals in $[-\alpha, \alpha]$ on which Rf is constant are called "fans" in, for example, [Lancaster 1985]; specifically, $[-\alpha, \alpha_1]$ and $[\alpha_2, \alpha]$ are called "side fans" and, if it exists, $[\alpha_L, \alpha_L + \pi]$ is called a "central fan". When Ω and γ are symmetric with respect to the x-axis, we have $Rf(\alpha) = Rf(-\alpha)$ and, if $\alpha > \frac{\pi}{2}$, $\alpha_L = -\frac{\pi}{2}$ and $\alpha_R = \frac{\pi}{2}$. (If $\kappa < 0$ in (1), we would need to explicitly assume $f(x, y) = f(x, -y)$ for $(x, y) \in \Omega$.) If the fans touch or overlap (e.g., $\gamma_1 + \gamma_2 \ge 2\alpha - \pi$ in a situation where case (DI) would hold), then f is continuous at $\mathcal{O}$.

Let Ω be a bounded domain in $\mathbb{R}^2$ which is symmetric with respect to the x-axis and has a reentrant corner of size $2\alpha > \pi$ at the origin $\mathcal{O}$. Let $\gamma : \partial\Omega \to (0, \pi)$ also be symmetric with respect to the x-axis such that the limits in (3) exist and $\gamma_1 = \gamma_2 < \frac{\pi}{2}$. As in Example 2 of [Lancaster and Siegel 1996], it follows that the solution f of (1)–(2) with the domain Ω and contact angle γ above is continuous at $\mathcal{O}$ if and only if $\gamma_1 \geq \frac{\pi}{2} - \alpha$ and the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ have a central fan if $\gamma_1 < \frac{\pi}{2} - \alpha$. Danzhu Shi and Robert Finn [2004] considered the borderline case in which $\gamma_1 = \alpha - \frac{\pi}{2}$, so that f is continuous at $\mathcal{O}$. By perturbing the domain (using "an asymmetric domain perturbation that is in an asymptotic sense arbitrarily small"), they showed that the solution of the perturbed capillary problem is discontinuous at $\mathcal{O}$. (They convert the behavior of the radial limit function from a constant in Example 2 to case (I) in the perturbed problem.)

Consider a similar (symmetric) situation with a constant contact angle γ which satisfies $\gamma < \alpha - \frac{\pi}{2}$, so that the solution f of (1)–(2) with the (symmetric) domain Ω and contact angle γ is discontinuous at $\mathcal{O}$, the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ have a central fan and case (DI) holds. Applying the procedure of Finn and Shi, one makes a suitable, nonsymmetric (with respect to the x-axis) perturbation of Ω outside a neighborhood of $\mathcal{O}$ and obtains a new solution $\tilde{f}$ of (1)–(2) in the perturbed domain $\widetilde{\Omega}$, and one then shows that $\tilde{f}$ is discontinuous at $\mathcal{O}$ and the radial limits $R\tilde{f}(\theta)$ have no central fan (i.e., case (I) holds); the size of the domain perturbation required to achieve this depends on the size of $\alpha - \frac{\pi}{2} - \gamma$.

We might view their example and procedure as a perturbation of the contact angle in a fixed domain $\widehat{\Omega}$ as follows. Let $\widehat{\Omega}$ be the largest open subset of $\Omega \cap \widetilde{\Omega}$ which is symmetric with respect to the x-axis. Let $\hat{\nu}$ denote the exterior unit normal to $\widehat{\Omega}$ at points of $\partial\widehat{\Omega}$ where it exists. Define (variable) contact angles $\lambda, \tilde{\lambda} : \partial\widehat{\Omega} \to [0, \pi]$ as follows:

- On $\partial\widehat{\Omega} \cap \partial\Omega$, set $\lambda = \gamma$.
- On $\partial\widehat{\Omega} \cap \partial\widetilde{\Omega}$, set $\tilde{\lambda} = \gamma$.
- On $\partial\widehat{\Omega} \cap \Omega$, set $\lambda = Tf \cdot \hat{\nu}$ when $\hat{\nu}$ is defined; recall that $f \in C^2(\Omega)$.
- On $\partial\widehat{\Omega} \cap \widetilde{\Omega}$, set $\tilde{\lambda} = T\tilde{f} \cdot \hat{\nu}$ when $\hat{\nu}$ is defined; recall that $\tilde{f} \in C^2(\widetilde{\Omega})$.

Using the procedure given in [Shi and Finn 2004], notice that $\hat{\nu}$ exists at all but a finite number of points and so λ and $\tilde{\lambda}$ are defined almost everywhere on $\partial\widehat{\Omega}$. From Theorem 5.1 of [Finn 1986], we see that f and $\tilde{f}$ are the solutions of (1)–(2) with domain $\widehat{\Omega}$ and contact angles λ and $\tilde{\lambda}$ respectively. We may therefore view $\tilde{\lambda}$ as a perturbation of the (symmetric) contact angle λ and, when $\gamma < \alpha - \frac{\pi}{2}$, this perturbation $\tilde{\lambda}$ destroys the central fan. In this paper, we establish the stability of central fans with respect to sufficiently small perturbations of the contact angle γ, leaving the domain Ω fixed; this implies that $\tilde{\lambda}$ is a "large" perturbation of λ. We shall prove the following result.

Theorem 1. *Suppose Ω is a bounded open domain in $\mathbb{R}^2$ which has a reentrant corner at $\mathcal{O}$ of size 2α with $\alpha \in \left(\frac{\pi}{2}, \pi\right)$ as described above. Suppose also that there is a finite set $A = \{P_1, \dots, P_m\} \subset \partial\Omega$ with $m \geq 1$ and $P_1 = \mathcal{O}$ such that $\partial\Omega \setminus A$ is a C^4 curve (if $m = 1$) or a finite disjoint union of C^4 curves (if $m > 1$). Let*

$\gamma \in C^{1,\beta}(\partial\Omega \setminus A)$, for some $\beta \in (0,1)$, satisfy $\delta_0 \le \gamma \le \pi - \delta_0$ for some $\delta_0 > 0$ such that the limits

$$\gamma_1 = \lim_{\partial^+\Omega \ni (x,y) \to \mathcal{O}} \gamma(x,y) \quad \text{and} \quad \gamma_2 = \lim_{\partial^-\Omega \ni (x,y) \to \mathcal{O}} \gamma(x,y)$$

both exist. Suppose there exists $f \in C^2(\Omega) \cap L^\infty(\Omega)$ which satisfies (1)–(2) and is discontinuous at $\mathcal{O}$ and the radial limit function of f at $\mathcal{O}$, $Rf(\cdot)$, behaves as in case (ID) or case (DI).

Then there exist functions $\omega^\pm : \partial\Omega \to [0,\pi]$ with $0 \le \omega^+ \le \gamma \le \omega^- \le \pi$ on $\partial\Omega$ and $\omega^+ < \gamma < \omega^-$ on $\partial\Omega \setminus A$ such that if $\sigma : \partial\Omega \to (0,\pi)$ with $\omega^+ \le \sigma \le \omega^-$ a.e. on $\partial\Omega$ and $\delta_1 \le \sigma \le \pi - \delta_1$ for some $\delta_1 \in (0, \delta_0)$, then the radial limit function Rh of the solution $h \in C^2(\Omega)$ of (1)–(2) with γ replaced by σ in (2) has the same type of behavior (i.e., case (ID) or case (DI) holds) as does Rf. In particular, the radial limits of h have a central fan.

The following corollary shows that Example 2 of [Lancaster and Siegel 1996] can be perturbed (with respect to the contact angle) and that the resulting nonsymmetric nonparametric capillary surfaces will have central fans.

Corollary 2. *Let Ω be an open, connected, bounded Lipschitz domain which is symmetric with respect to the x-axis such that $\mathcal{O} = (0,0) \in \partial\Omega$, $\partial\Omega \setminus \{\mathcal{O}\}$ is a C^4 curve and Ω has a corner at $\mathcal{O}$ with opening angle $2\alpha > \pi$. Suppose $\gamma : \partial\Omega \setminus \{\mathcal{O}\} \to (0,\pi)$ is a $C^{1,\beta}$ map which satisfies $\gamma(x,-y) = \gamma(x,y)$ for $(x,y) \in \partial\Omega$ and for which the limit*

$$\lim_{\partial\Omega \ni (x,y) \to \mathcal{O}} \gamma(x,y) = \gamma_0,$$

exists and $0 < \gamma_0 < \alpha - \frac{\pi}{2}$. Let $f \in C^2(\Omega) \cap C^{1,\beta}(\overline{\Omega} \setminus \{\mathcal{O}\})$ of (1)–(2). Then f is discontinuous at $\mathcal{O}$, the radial limit function Rf behaves as in case (DI) and there exist functions $\omega^\pm : \partial\Omega \to [0,\pi]$ with $0 \le \omega^+ \le \gamma \le \omega^- \le \pi$ on $\partial\Omega$ and $\omega^+ < \gamma < \omega^-$ on $\partial\Omega \setminus A$ such that if $\sigma : \partial\Omega \to (0,\pi)$ with $\omega^+ \le \sigma \le \omega^-$ a.e. on $\partial\Omega$ and $\delta_1 \le \sigma \le \pi - \delta_1$ for some $\delta_1 \in (0,\delta_0)$, then the radial limit function Rh of the solution $h \in C^2(\Omega)$ of (1)–(2) with γ replaced by σ in (2) is discontinuous at $\mathcal{O}$ and behaves as in case (DI).

We do not address the stability of the continuity at $\mathcal{O}$ of a solution f of (1)–(2) but we note that the procedure in [Shi and Finn 2004], as stated, would not establish the discontinuity at $\mathcal{O}$ of f for arbitrarily small perturbations of the domain (in the asymptotic sense of Shi and Finn) when $\gamma_1 = \gamma_2 > \alpha - \frac{\pi}{2}$.

2. Some lemmas

Lemma 3. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a $C^{2,\beta}$ curve for some $\beta \in (0,1)$. Let*

$\gamma \in L^\infty(\partial\Omega)$ satisfy $\delta \le \gamma \le \pi - \delta$ a.e. on $\partial\Omega$ for some $\delta > 0$ and $\gamma \in C^{1,\beta}(\Gamma)$. Suppose there exists $f \in C^2(\Omega) \cap L^\infty(\Omega)$ which satisfies

$$\operatorname{div}(Tu) = \kappa u \quad \text{in } \Omega \tag{4}$$

and

$$Tu \cdot \nu = \cos\gamma \quad \text{on } \Gamma. \tag{5}$$

Then $f \in C^{2,\beta}(\Omega \cup \Gamma)$.

See [Finn 1986, p. 210, Note 5], or [Finn 1988], or the introduction of [Korevaar and Simon 1996], which references [Simon and Spruck 1976; Taylor 1977].

The next result uses the notation of [Korevaar and Simon 1996, Theorem 2]; in particular,

$$\frac{(\nabla g(x), -1)}{\sqrt{1 + |\nabla g(x)|^2}}$$

denotes the continuous extension of the (downward) unit normal to the graph of g when considered as a function on this graph.

Lemma 4. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a C^3 curve. Let $\phi \in L^\infty(\partial\Omega)$ be in $C^{1,\beta}(\Gamma)$ for some $\beta \in (0, 1)$. Suppose $g \in C^2(\Omega) \cap L^\infty(\Omega)$ is the variational solution of*

$$\begin{aligned} \operatorname{div}(Tu) &= \kappa u && \text{in } \Omega, \\ u &= \phi && \text{on } \partial\Omega; \end{aligned}$$

that is, g minimizes $J(\,\cdot\,)$ over $\mathrm{BV}(\Omega)$, where

$$J(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega \int_0^u \kappa t \, dt \, dx + \int_{\partial\Omega} |u - \phi| \, ds, \quad u \in \mathrm{BV}(\Omega).$$

Set

$$Q = \{(x, t) \in \Gamma \times \mathbb{R} : \min\{\phi(x), g(x)\} \le t \le \max\{\phi(x), g(x)\}\}$$

and $Q_0 = Q \setminus T$, where $T \subset \partial\Omega \times \mathbb{R}$ is the graph of ϕ, and let G be the graph of g over Ω. Then for each $x_0 \in \Gamma$, there exists a $\delta > 0$ such that $\{x \in \partial\Omega : |x - x_0| \le \delta\} \subset \Gamma$ and the following conclusions hold:

(a) *$\Pi = \{(x, t) \in Q \cup G : |x - x_0| \le \delta\}$ is a $C^{1,\sigma}$ manifold with boundary whose boundary is the union of $\{(x, \phi(x)) \in T : x \in \Gamma,\ |x - x_0| \le \delta\}$ and $\{(x, g(x)) : x \in \Omega,\ |x - x_0| = \delta\}$ for some $\sigma \in (0, 1)$.*

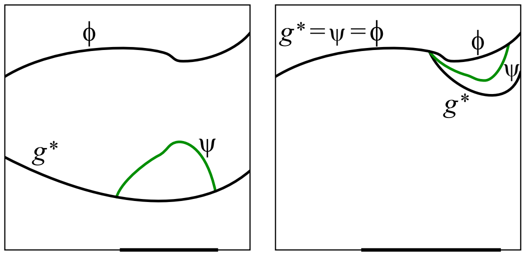
(b) *The (downward) unit normal $\vec{N}$ to Π is a continuous function and*

$$\vec{N}(x,t)=\begin{cases}\dfrac{(\nabla g(x),-1)}{\sqrt{1+|\nabla g(x)|^2}} & \text{if } x\in\Omega\cup\Gamma \text{ and } t=g(x),\\ (\nu(x),0) & \text{if } (x,t)\in Q \text{ and } g(x)\le t<\phi(x),\\ (-\nu(x),0) & \text{if } (x,t)\in Q \text{ and } \phi(x)<t\le g(x),\end{cases}$$

where ν denotes the outward unit normal to $\partial\Omega$.

Proof. Let $A=\{x\in\Gamma : g^*(x)=\phi(x)\}$, $B=\{x\in\Gamma : g^*(x)\neq\phi(x)\}$, and A_0 be the interior (in Γ) of A, where g^* is the trace of g on $\partial\Omega$; let us define $g^*(x)$ to be $\phi(x)$ if $x\in\Gamma$ and $g^*(x)$ is not otherwise defined. Using the arguments in [Elcrat and Lancaster 1986], we see that if $x_0\in A_0$, then there exists a $\delta>0$ such that $\{x\in\partial\Omega : |x-x_0|\le\delta\}\subset A_0$ and (a) and (b) hold.

Suppose $x_0\in B$ such that $g^*(x_0)=z_0<\phi(x_0)$ and so (x_0,z_0) is an interior point of $\mathcal{Q}\cup G$. Standard results on the regularity of solutions of obstacle problems at interior points imply g is continuous on $(\overline{\Omega}\times\mathbb{R})\cap U$, where U is a neighborhood in $\mathbb{R}^3$ of (x_0,z_0), and, considered as a parametric surface, $(\mathcal{Q}\cup G)\cap U$ is a $C^{1,\alpha}$ surface for some $\alpha\in(0,1)$. (For example, this follows from [Simon and Spruck 1976] or [Taylor 1977], since the contact angle is zero at these interior points. Another argument follows from [Hildebrandt 1973]; by "blowing up" or dilating $\mathbb{R}^3$ about (x_0,z_0), we may assume the function $f : E\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\to\mathbb{R}$ given by $f(w,X,p,q)=|p|^2+|q|^2+\frac{1}{2}\kappa(X_3-z_0)((X-(x_0,z_0))\cdot(p\times q))$ satisfies conditions A and B of that paper in a neighborhood $U\times I$ of (x_0,z_0), where I is an open interval containing z_0, and so, for smooth Dirichlet data ψ slightly larger than z_0 near x_0 and equal to g on $\Omega\cap\partial U$, a theorem in [Hildebrandt 1973] shows there is a parametric minimizer of $\int_E f(w,z,\nabla z)\,du\,dv$ that is smooth in the interior of its domain $E=\{(u,v) : u^2+v^2<1\}$ and, from [Miranda 1964] (or [Finn 1986, p. 16, Note 10]), we see that this parametric solution is a graph $z=h(x,y)$. Since $g\le h$ on ∂U by the choice of ψ and $h\le g$ on ∂U since $\psi<\phi$ on $U\cap\partial\Omega$, we see that $g=h$ in U. In particular, g is continuous at each point of $\Gamma\cap B$ and the points of $A\setminus A_0$ are isolated.) An application of [Bourni 2011] shows that (a) and (b) hold; that is, we may choose a domain $\mathcal{V}\subset\Omega$ such that $\partial\mathcal{V}$ is a $C^{1,\alpha}$ curve in $\mathbb{R}^2$, $x_0\in\partial\mathcal{V}\subset\Gamma\cup\Omega$, $\partial\Omega\cap\overline{(\Omega\cap\partial\mathcal{V})}=\{x^{(j)} : j=1,2\}$ with $x^{(j)}\in B$ $(j=1,2)$, the closure in $\mathbb{R}^3$ of $\{(x,g(x))\in\Omega\times\mathbb{R} : x\in\partial\mathcal{V}\}$ is a $C^{1,\alpha}$ curve in $\mathbb{R}^3$ which meets $\Gamma\times\mathbb{R}$ tangentially at $(x^{(j)},g^*(x^{(j)}))$, $j=1,2$, and we can find a function $\psi:\partial\mathcal{V}\to\mathbb{R}$ whose graph is a $C^{1,\alpha}$ curve in $\mathbb{R}^3$ such that $g^*\le\psi\le\phi$ on $\partial\Omega\cap\partial\mathcal{V}$ (see Figure 1(a)) and apply the conclusion of [Bourni 2011] to see that (a) and (b) hold in a neighborhood of x_0. If $x_0\in B$ such that $g^*(x_0)=z_0>\phi(x_0)$, apply the argument above to $-g$ (with $-\phi$ as Dirichlet data).



(a) $\partial\mathcal{V}\cap\partial\Omega$ (b) $\partial\mathcal{V}\cap\partial\Omega$

Figure 1. The traces of g, ψ and ϕ.

Suppose $x_0 \in A \setminus A_0$. Notice, from the arguments above, that A_0 and B are open. There exist a domain $\mathcal{V} \subset \Omega$ and a function $\psi : \partial\mathcal{V} \to \mathbb{R}$ as above such that $x_0 \in \partial\mathcal{V} \subset \Gamma \cup \Omega$ and $\partial\Omega \cap \overline{(\Omega \cap \partial\mathcal{V})} = \{x^{(j)} : j = 1, 2\}$ with $x^{(j)} \in A_0 \cup B$ $(j = 1, 2)$; we argue as above (see, for example, Figure 1(b)). □

Lemma 5. *Let Ω be a bounded open domain in $\mathbb{R}^2$ with Lipschitz boundary and let Γ be an open subset of $\partial\Omega$ which is a C^4 curve or a finite disjoint union of C^4 curves. Let $\gamma \in L^\infty(\partial\Omega)$ satisfy $\delta \le \gamma \le \pi - \delta$ a.e. on $\partial\Omega$ for some $\delta > 0$ and $\gamma \in C^{1,\beta}(\Gamma)$ for some $\beta \in (0, 1)$. Suppose there exists $f \in C^2(\Omega) \cap L^\infty(\Omega)$ which satisfies* (4) *and* (5). *Let $\epsilon > 0$. Define $g = g_\epsilon \in \mathrm{BV}(\Omega)$ to be the minimizer over $\mathrm{BV}(\Omega)$ of $J_\epsilon(\,\cdot\,)$, where*

$$J_\epsilon(u) = \int_\Omega \sqrt{1 + |Du|^2} + \int_\Omega \int_0^u kt\,dt\,dx + \int_{\partial\Omega} |u - (f + \epsilon)|\,ds$$

for $u \in \mathrm{BV}(\Omega)$. We have:

(i) *$g \in C^2(\Omega)$ and satisfies* (4).

(ii) *The unit normal $\vec{N}$ to Π is in $C^{0,\beta}(\Omega \cup E)$ for each compact subset E of Γ and hence the contact angle*

$$\gamma_g \stackrel{\text{def}}{=} \arccos(Tg \cdot \nu) \in [0, \pi] \tag{6}$$

is well defined and continuous on Γ, where ν denotes the outward unit normal to $\partial\Omega$. In particular, $\gamma_g = 0$ on $\{x \in \Gamma : g(x) < f(x) + \epsilon\}$.

(iii) *Suppose there is a finite set $A = \{x_1, \dots, x_m\} \subset \partial\Omega$ such that $\Gamma = \partial\Omega \setminus A$. Then $f \le g \le f + \epsilon$ in Ω.*

(iv) *Suppose there is a finite set $A = \{x_1, \dots, x_m\} \subset \partial\Omega$ such that $\Gamma = \partial\Omega \setminus A$. Then $\gamma_g < \gamma$ on Γ.*

Proof. (i) The existence of g follows from Theorem 5 of [Gerhardt 1974], or Theorem 2.1 of [Giusti 1976]. The interior regularity of g follows from Theorem 3.1

of [Giusti 1976] (see also [Gerhardt 1974, p. 174; Williams 1978, Theorem 3]). The fact that g satisfies (4) is standard (e.g., [Gerhardt 1974, p. 174]).

(ii) The boundary regularity of g follows from Lemma 4. On $\{x \in \Gamma : g(x) < f(x)+\epsilon\}$, we have $\vec{N}(x, g(x)) = (\nu(x), 0)$, $Tg(x) = \nu(x)$, and so

$$\gamma_g(x) = \arccos(\nu(x)\cdot\nu(x)) = \arccos(1) = 0.$$

(iii) Notice that $f, g \in C^2(\Omega)\cap C^0(\Omega\cup\Gamma)$. Set $M = \{x \in \Omega : f(x) > g(x)\}$. On $\partial M \cap \Gamma$, $g < f+\epsilon$ and so by (ii) and Lemma 4, with $\Pi = Q \cup G$, where $Q = \{(x,z) : (x,z) \in E,\ g(x) \le z < f(x)\} \in M\times\mathbb{R}$, implies that $\sup\cos\gamma_g = Tg\cdot\nu = 1$; hence $\gamma_g = 0$ on $\partial M\cap\Gamma$. Thus $f = g$ on $\Omega\cap\partial M$ and $\gamma_g = 0$ almost everywhere on $\partial\Omega\cap\partial M$ and so the general comparison principle (e.g., [Finn 1986, Theorem 5.1]) implies $f \le g$ in M; hence $M = \varnothing$.

Now let $\tau > 0$ and set $N = \{x \in \Omega : g(x) > f(x)+\epsilon+\tau\}$. Then $g = f+\epsilon+\tau$ on $\Omega\cap\partial N$ and $g > f+\epsilon$ on $\partial N\cap\Gamma$ and so Lemma 4 implies $\gamma_g = \pi$ almost everywhere on $\partial\Omega\cap\partial N$. The general comparison principle then implies $g \le f+\epsilon+\tau$ and so $N = \varnothing$. Therefore $g \le f+\epsilon+\tau$ in Ω for each $\tau > 0$ and so $g \le f+\epsilon$ in Ω.

(iv) Suppose first $x \in \Gamma$ and there is a sequence $\{y_j\}$ in Γ such that $x = \lim_{j\to\infty} y_j$ and $g(y_j) < f(y_j)+\epsilon$ for each j. Then (ii) implies $\gamma_g(y_j) = 0$ for each j and so $\gamma_g(x) = 0$. Since $\gamma \in (0,\pi)$, we see that $\gamma_g(x) = 0 < \gamma(x)$.

Suppose next that $x \in \Gamma$ and $g \ge f+\epsilon$ in $\mathcal{P}\cap\Gamma$, where $\mathcal{P}$ is a neighborhood of x in $\mathbb{R}^2$. From (iii), we see that $g = f+\epsilon$ in $\mathcal{P}\cap\Gamma$. If $\gamma_g(x) > \gamma(x)$, then $g(x - t\nu(x)) > f(x - t\nu(x)) + \epsilon$ for $t > 0$ small and this contradicts (iii). (Recall that $\nu(x)$ is the exterior unit normal to $\partial\Omega$ at x.) Thus $\gamma_g \le \gamma$ on Γ.

Finally, suppose $x \in \Gamma$, $\gamma_g(x) = \gamma(x)$ and $g = f+\epsilon$ in $\mathcal{P}\cap\partial\Omega$, where $\mathcal{P}$ is a neighborhood of x in $\mathbb{R}^2$; notice that [Heinz 1970] and Lemma 3 imply $g \in C^{2,\beta}(\mathcal{P}\cap\overline{\Omega})$. Since $g \le f+\epsilon$ in Ω and $\gamma_g(x) = \gamma(x)$, the tangent plane Π_g to $z = g$ at $(x, g(x))$ and the tangent plane Π to $z = f+\epsilon$ at $(x, g(x)) = (x, f(x)+\epsilon)$ must coincide. Now the mean curvature H_g of $z = g$ at $(x, g(x))$ is $\frac{1}{2}\kappa g(x)$ and the mean curvature H_f of $z = f+\epsilon$ at $(x, g(x))$ is $\frac{1}{2}\kappa f(x) = \frac{1}{2}(\kappa g(x) - \kappa\epsilon)$. Since $g = f+\epsilon$ in $\mathcal{P}\cap\Gamma$, the (signed) curvature of the curve $z = f(x - t\nu(x)) + \epsilon$ must be strictly less than the (signed) curvature of the curve $z = g(x - t\nu(x))$ for $t > 0$ small and so $g(x - t\nu(x)) > f(x - t\nu(x)) + \epsilon$ for $t > 0$ small, contradicting (iii). □

3. Stability of central fans

We will begin by establishing the stability of the central fans with respect to "one-sided" perturbations of γ. (See Figures 2 and 3.)

Theorem 6. *Let Ω be an open, connected, bounded Lipschitz domain such that $\mathcal{O} = (0,0) \in \partial\Omega$, Ω has a corner at $\mathcal{O}$ with opening angle $2\alpha > \pi$ and there is a finite set $A = \{P_1, \dots, P_m\} \subset \partial\Omega$ with $m \ge 1$ and $P_1 = \mathcal{O}$ such that $\partial\Omega \setminus A$ is*

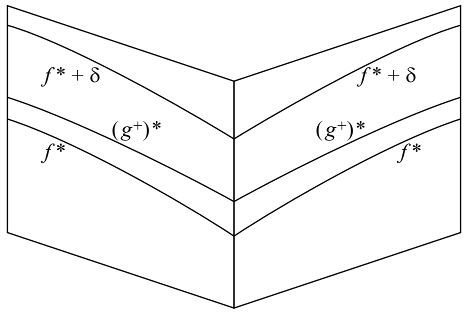


Figure 2. The traces of f, g^+ and $f + \delta$.

a C^4 curve (if $m = 1$) or a finite disjoint union of C^4 curves (if $m > 1$). Suppose $\gamma : \partial\Omega \setminus \mathcal{O} \to (0, \pi)$ is a $C^{1,\beta}$ map for which the limits

$$\lim_{\partial^+\Omega \ni (x,y) \to \mathcal{O}} \gamma(x, y) = \gamma_1, \qquad \lim_{\partial^-\Omega \ni (x,y) \to \mathcal{O}} \gamma(x, y) = \gamma_2$$

exist with $\gamma_i \in (0, \pi)$, $i = 1, 2$, and $f \in C^2(\Omega) \cap C^{1,\beta}(\overline{\Omega} \setminus \{\mathcal{O}\})$ satisfies

$$\begin{aligned} \operatorname{div}(Tf) &= \kappa f && \text{in } \Omega, \\ Tf \cdot \nu &= \cos\gamma && \text{on } \partial\Omega \setminus \{\mathcal{O}\} \end{aligned}$$

such that f is discontinuous at $\mathcal{O}$ and the radial limits $Rf(\,\cdot\,)$ of f at $\mathcal{O}$ have a central fan.

There exists a $\delta > 0$ such that if $g^+ \in \mathrm{BV}(\Omega) \cap C^2(\Omega)$ is the variational solution of the Dirichlet problem

$$\operatorname{div}(Tg) = \kappa g \quad \text{in } \Omega, \tag{7}$$

$$g = f + \delta \quad \text{on } \partial\Omega \setminus A, \tag{8}$$

and if $\omega^+ \stackrel{\text{def}}{=} \arccos(Tg^+ \cdot \nu)$ on $\partial\Omega \setminus A$, then for any function $\sigma \in L^\infty(\partial\Omega)$ satisfying

$$\omega^+ \le \sigma \le \gamma \quad \text{a.e. on } \partial\Omega \tag{9}$$

and $\delta_1 \le \sigma \le \pi - \delta_1$ for some $\delta_1 > 0$, the variational solution $h \in \mathrm{BV}(\Omega) \cap C^2(\Omega)$ of the capillary problem

$$\operatorname{div}(Th) = \kappa h \ \text{ in } \Omega, \quad Th \cdot \nu = \cos\sigma \ \text{ on } \partial\Omega \setminus A \tag{10}$$

is discontinuous at $\mathcal{O}$, the radial limits $Rh(\,\cdot\,)$ of h at $\mathcal{O}$ have a central fan and they have the same type of behavior (i.e., case (DI) or (ID)) as $Rf(\,\cdot\,)$.

Proof. Suppose first that Rf behaves as in case (DI) and so

$$Rf(0) < \min\{Rf(\alpha), Rf(-\alpha)\}.$$

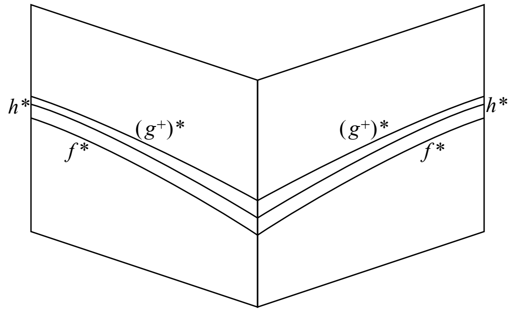


Figure 3. The traces of f, h and g^+.

Let $\delta < \min\{Rf(\alpha) - Rf(0),\ Rf(-\alpha) - Rf(0)\}$ and let g^+ be the variational solution of the Dirichlet problem (7)–(8) for this choice of δ (see Figure 2). From Lemma 5(iv), we see that $\omega^+ < \gamma$ on $\partial\Omega \setminus A$ and therefore there exist $\sigma \in L^\infty(\partial\Omega)$ which satisfy (9); let us select σ and h as in the theorem (see Figure 3). From Lemma 5(iii) and the general comparison principle, we see that $f \le h \le g^+ \le f + \delta$ in Ω and hence

$$Rf(\theta) \le Rh(\theta) \le Rg^+(\theta) \le Rf(\theta) + \delta \quad \text{for } \theta \in [-\alpha, \alpha]; \tag{11}$$

thus

$$Rh(\alpha) - Rh(0) \ge Rf(\alpha) - (Rf(0) + \delta) = Rf(\alpha) - Rf(0) - \delta > 0$$

and

$$Rh(-\alpha) - Rh(0) \ge Rf(-\alpha) - (Rf(0) + \delta) = Rf(-\alpha) - Rf(0) - \delta > 0.$$

Now we know that the radial limits of h at $\mathcal{O}$ exist and behave as in [Lancaster and Siegel 1996] (i.e., one of case (I), (D), (ID) or (DI) must hold; if, for example, case (I) held, one would have $Rf(-\alpha) < Rf(0) < Rf(\alpha)$). The calculations above show that $Rf(-\alpha) > Rf(0)$ and $Rf(\alpha) > Rf(0)$ and therefore $Rh(\cdot)$ must behave as in case (DI); hence $Rh(\cdot)$ has a central fan (see Figure 4).

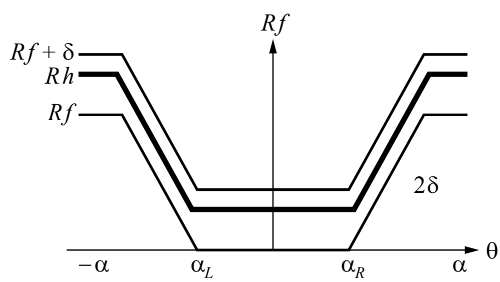


Figure 4. The radial limits of f, h and $f + \delta$.

Suppose next that Rf behaves as in case (ID) and so

$$Rf(0) > \min\{Rf(\alpha), Rf(-\alpha)\}.$$

If we let $\delta < \min\{Rf(0) - Rf(\alpha),\ Rf(0) - Rf(-\alpha)\}$, we may repeat the same argument as above and obtain $Rh(0) > Rh(-\alpha)$ and $Rh(0) > Rh(\alpha)$; hence $Rh(\,\cdot\,)$ must behave as in case (ID) and therefore has a central fan. $\square$

Remark 7. The corresponding theorem with (8) replaced by

$$g^- = f - \delta \quad \text{on } \partial\Omega \setminus A \tag{12}$$

and with $\omega^- \overset{\text{def}}{=} \arccos(Tg^- \cdot \nu)$ on $\partial\Omega \setminus A$, $\omega^- \geq \sigma \geq \gamma$ on $\partial\Omega \setminus A$, $\delta_1 \leq \sigma \leq \pi - \delta_1$ for some $\delta_1 > 0$ and h a solution of (10) yields $f - \delta \leq g^- \leq h \leq f$ in Ω and

$$Rf(\theta) \leq Rh(\theta) \leq Rf(\theta) + \delta \quad \text{for } \theta \in [-\alpha, \alpha]; \tag{13}$$

hence h is discontinuous at $\mathcal{O}$ and the radial limits $Rh(\,\cdot\,)$ of h at $\mathcal{O}$ have the same type of behavior (i.e., case (DI) or (ID)) as $Rf(\,\cdot\,)$.

Proof of Theorem 1. Suppose Rf behaves as in case (DI) or case (ID) and define $\delta = \frac{1}{2}\min\{|Rf(\alpha) - Rf(0)|, |Rf(-\alpha) - Rf(0)|\}$. Combining the arguments in Theorem 6 and Remark 7, we obtain

$$Rf(\alpha) - Rf(0) - \delta \leq Rh(\alpha) - Rh(0) \leq Rf(\alpha) - Rf(0) + \delta$$

and

$$Rf(-\alpha) - Rf(0) - \delta \leq Rh(\alpha) - Rh(0) \leq Rf(-\alpha) - Rf(0) + \delta.$$

If Rf behaves as in case (DI), we have $0 < Rh(\alpha) - Rh(0)$ and $0 < Rh(-\alpha) - Rh(0)$ and therefore Rh behaves as in case (DI). If Rf behaves as in case (ID), we have $Rh(\alpha) - Rh(0) < 0$ and $Rh(-\alpha) - Rh(0) < 0$ and therefore Rh behaves as in case (ID). $\square$

Proof of Corollary 2. Since $\gamma_0 < \alpha - \frac{\pi}{2}$, we see that $|2\gamma - \pi| > 2\pi - 2\alpha$. It follows from [Lancaster 2012] that f is discontinuous at $\mathcal{O}$. Since $f(x, y) = f(x, -y)$ for each $(x, y) \in \Omega$, the radial limits of f cannot behave as in cases (I) or (D) of Theorem 1 of [Lancaster and Siegel 1996] and therefore they have a central fan. That case (DI) holds for $Rf(\,\cdot\,)$ follows from [Lancaster 2012] or directly from the fact that $(\pi - \gamma_0) + (\pi - \gamma_0) + \pi > 4\pi - 2\alpha > 2\alpha$ means case (ID) cannot hold. The claim follows from Theorem 1. $\square$

Remark 8. It should be emphasized that the conclusion of Theorem 1 is not that "there exists a $\delta > 0$ such that if $\sigma : \partial\Omega \to [0, \pi]$ satisfies $\gamma - \delta \leq \sigma \leq \gamma + \delta$ a.e. on $\partial\Omega$, then the radial limit function Rh of the solution $h \in C^2(\Omega)$ of (1)–(2) with γ replaced by σ in (2) has the same type of behavior (i.e., case (ID) or case (DI) holds) as does Rf". The validity of such a conclusion is an interesting question which might spur further investigation.

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