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NIMA ANVARI

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Let X_0 denote a compact, simply connected, smooth 4-manifold with boundary the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$ and with even negative definite E_8 intersection form. We obtain constraints on the rotation data if a free $\mathbb{Z}/p$-action on $\Sigma(2, 3, 5)$ extends to a smooth, homologically trivial action on X_0 with isolated fixed points, for any odd prime $p \geq 7$. The approach is to study the equivariant Yang–Mills instanton-one moduli space for cylindrical-end 4-manifolds. As an application we show that a smooth, homologically trivial $\mathbb{Z}/7$-action on $\#^8 S^2 \times S^2$ with isolated fixed points does not equivariantly split along a free action on $\Sigma(2, 3, 5)$.

1. Introduction

It is well known that the Poincaré homology 3-sphere $\Sigma(2, 3, 5)$ can be realized as the boundary of a smooth, compact, simply connected 4-manifold X_0 obtained by plumbing disk bundles over 2-spheres along the E_8 graph. Let π denote a cyclic group of prime order $p \geq 7$; then the Poincaré homology sphere admits a free π-action contained in the circle action that gives it the structure of a Seifert fibered manifold. In this paper, we obtain constraints for this action to extend smoothly and homologically trivially to X_0 with isolated fixed points. An action is *homologically trivial* if it induces the identity on integral homology $H_2(X_0, \mathbb{Z})$ and in this case the action necessarily has fixed sets in the interior of X_0.

When studying symmetries of 4-manifolds, typically there are known linear actions and one would like to understand how closely a general smooth group action resembles the linear models. In our case, the linear actions are obtained by plumbing equivariantly; in fact, the circle action on $\Sigma(2, 3, 5)$ extends to X_0. These actions, however, always contain a fixed 2-sphere, namely the central node in the E_8 graph. We ask if there can be any smooth extension with only isolated fixed points. We have local tangential representations at each fixed point described by rotation numbers, a pair (a, b) of nonzero integers modulo p, well-defined up to order and simultaneous change of sign. For a fixed generator t of π, the local

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representation is given by $t \cdot (z_1, z_2) = (t^a z_1, t^b z_2)$. The constraints are in the form of congruence relations satisfied by the rotation numbers.

Theorem A. *Let X_0 denote a compact, simply connected smooth 4-manifold with boundary $\Sigma(2,3,5)$ and negative definite intersection form E_8. For any prime $p > 5$, if a free $\mathbb{Z}/p$-action on $\Sigma(2,3,5)$ extends to a smooth, homologically trivial action on X_0, then the rotation data of the isolated fixed points are (a, b) such that $a + b \equiv \pm 1 \pmod p$ or $a + b \equiv \pm 7 \pmod p$.*

Remark 1.1. Note the action is automatically homologically trivial for $p > 7$, since the π-action on X_0 gives rise to an integral representation on $H_2(X_0; \mathbb{Z})$ and a decomposition (see [Curtis and Reiner 1962, p. 508; Edmonds 1989, p. 111])

$$H_2(X_0; \mathbb{Z}) = \mathbb{Z}[\pi]^r \oplus \mathbb{Z}^t \oplus \mathbb{Z}[\zeta_p]^c$$

as $\mathbb{Z}[\pi]$-modules with multiplicities $r, t, c \geq 0$ and $b_2(X_0) = rp + t + (p-1)c$. When $p > b_2(X_0) + 1$ we must have $r, c = 0$ and $t = b_2(X_0)$. When $p = 7$ the action need not be homologically trivial (see [Quebbemann 1981, Example 3.10, p. 168]), as the splitting $H_2(X_0; \mathbb{Z}) = \mathbb{Z}[\pi] \oplus \mathbb{Z}$ may occur (c must be even by [Edmonds 1989, Proposition 2.4(i)]; see also the algebraic result in [Hambleton and Riehm 1978, Proposition 10(c)]). For homologically trivial actions, the fixed set X_0^π consists of isolated points and 2-spheres, and the Lefschetz fixed point formula gives the Euler characteristic $\chi(X_0^\pi) = 9$.

The necessary conditions for a smooth extension from Theorem A can be checked against the Atiyah–Patodi–Singer G-signature formula for manifolds with boundary; see [Atiyah et al. 1975b]. This leads to the following rigidity result.

Theorem B. *Let X_0 denote a compact, simply connected, smooth 4-manifold with boundary $\Sigma(2,3,5)$ and negative definite E_8 intersection form. A free $\mathbb{Z}/7$-action on $\Sigma(2,3,5)$ does not extend to a smooth, homologically trivial action on X_0 with fixed set consisting of only isolated fixed points.*

As a consequence, we have the following corollary regarding equivariant embedding of the Poincaré homology sphere in $\#^8 S^2 \times S^2$.

Corollary C. *The 4-manifold $X = \#^8 S^2 \times S^2$, with a smooth, homologically trivial $\mathbb{Z}/7$-action with only isolated fixed points, does not contain an equivariant embedding of $\Sigma(2,3,5)$ with a free action.*

Remark 1.2. If $\Sigma(2,3,5)$ embeds smoothly in $\#^8 S^2 \times S^2$, it separates X into two smooth, spin 4-manifolds with boundary, each with even intersection form. By van der Blij's lemma and the nontriviality of the Rokhlin invariant of $\Sigma(2,3,5)$, each side must have signature divisible by 8 and since $b_2(X) = 16$, each side must have definite intersection form. The additivity of the signature shows that they must have opposite sign. So any embedding of $\Sigma(2,3,5)$ in $\#^8 S^2 \times S^2$ decomposes X as $X_0 \cup_{\Sigma(2,3,5)} X_1$ with intersection form $Q_X = E_8 \oplus -E_8$.

In the next section we summarize the results of equivariant Yang–Mills moduli spaces that are needed in the proof of Theorem A. In the equivariant setting, a crucial technical component is provided by equivariant transversality results, as developed by Hambleton and Lee [1992] and based on Bierstone's theory of equivariant general position [1977]. This provides a suitable perturbation theory that gives the moduli spaces the structure of a Whitney stratified space.

2. The equivariant moduli space

Let X_0 denote a smooth, compact, simply connected 4-manifold with negative definite intersection form E_8 whose boundary is the Poincaré homology sphere $\partial X_0 = \Sigma(2,3,5)$. Suppose we have a cyclic group $\pi = \mathbb{Z}/p$ of odd prime order acting smoothly on X_0 which is both homologically trivial and free on the boundary. Denote by (X, g) the cylindrical-end Riemannian manifold $X = X_0 \cup \mathrm{End}(X)$ where $\mathrm{End}(X)$ is orientation preserving isometric to $\Sigma(2,3,5) \times [0,\infty)$, with g a π-invariant metric which restricts to a product metric on end. Yang–Mills moduli spaces for cylindrical-end 4-manifolds have been studied extensively; see [Taubes 1987; Morgan et al. 1994; Donaldson 2002]. We briefly sketch the main ideas here and refer the reader to the sources for details.

Consider a principal SU(2) bundle P over X. By fixing a trivialization we obtain bundle maps which cover the π-action on X. Let $\mathcal{G}(\pi) = \{\hat{t} : P \to P \mid t \in \pi\}$; then there exists an exact sequence

$$1 \to \mathcal{G} \to \mathcal{G}(\pi) \to \pi \to 1 \tag{2-1}$$

where $\mathcal{G}$ is the gauge group of P. The natural action of $\mathcal{G}(\pi)$ on the space of connections $\mathcal{A}(P)$ is given by pullback; it is well-defined modulo gauge; and thus the space $\mathcal{B}(P) = \mathcal{A}/\mathcal{G}$ of connections up to gauge transformations inherits an action of $\mathcal{G}(\pi)/\mathcal{G} = \pi$.

Recall that the Yang–Mills energy functional acts on the space of connections by

$$\mathcal{YM}(A) = -\frac{1}{8\pi^2}\int_X \mathrm{Tr}(F_A \wedge *F_A) = \frac{1}{8\pi^2}\int_X |F_A|^2 = \|F_A\|^2_{L^2}, \tag{2-2}$$

where F_A is the $\mathrm{ad}(P)$-valued curvature 2-form of the connection, and $*$ is the Hodge star operator associated to the Riemannian metric. The Hodge star-operator on X extends to an involution on bundle valued 2-forms giving rise to a splitting $\Omega^2(\mathrm{ad}\,P) = \Omega^2_+(\mathrm{ad}\,P) \oplus \Omega^2_-(\mathrm{ad}\,P)$ into self-dual and anti-self-dual (ASD) 2-forms. The L^2-finite moduli spaces are anti-self-dual connections modulo gauge with finite

Yang–Mills action:

$$\mathcal{M}(X,g)=\{[A]\in\mathcal{B}(P)\mid F_A^+=0,\ \|F_A\|_{L^2}^2<\infty\}. \tag{2-3}$$

This space is π-invariant since π acts by isometries. It is a fundamental result that g-ASD connections with finite Yang–Mills energy are asymptotic to flat connections down the cylindrical-end; see [Donaldson 2002, p. 77]. Since flat connections are π-invariant under the pullback action, this defines a π-equivariant boundary map $\partial_\infty : \mathcal{M}(X,g)\to\mathcal{R}(\Sigma(2,3,5))$ where $\mathcal{R}(\Sigma)$ denotes the representation variety of flat SU(2)-connections modulo gauge. This gives a π-invariant partition of the moduli space according to its limiting flat connection:

$$\mathcal{M}(X,g)=\bigsqcup_{\alpha\in\mathcal{R}(\Sigma)}\mathcal{M}(X,\alpha). \tag{2-4}$$

The energy of a g-ASD connection A is congruent modulo $\mathbb{Z}$ to the Chern–Simons invariant of the limiting flat connection α and so the energy takes on a discrete set of values determined by the Chern–Simons invariant and we get a further π-invariant decomposition according to energy value $\mathcal{M}(X,\alpha)=\bigsqcup_{\ell\geq 0}\mathcal{M}_\ell(X,\alpha)$ with $\ell\equiv \mathrm{CS}(\alpha) \bmod \mathbb{Z}$. The index of the δ-decay complex [Morgan et al. 1994]

$$0\to\Omega^0_{3,\delta}(X,\operatorname{ad}P)\xrightarrow{d_A}\Omega^1_{2,\delta}(X,\operatorname{ad}P)\xrightarrow{d_A^+}\Omega^2_{1,\delta,+}(X,\operatorname{ad}P)\to 0 \tag{2-5}$$

gives the formal dimension of the moduli space

$$\dim\mathcal{M}_\ell(X,\alpha)=8\ell-\tfrac{3}{2}(\chi(X)+\mathrm{Sign}(X))-\tfrac{1}{2}(h^1_\alpha+h^0_\alpha)+\tfrac{1}{2}\rho_\alpha(\Sigma), \tag{2-6}$$

where $h^i_\alpha=\dim_{\mathbb{R}} H^i(\Sigma,\operatorname{ad}\alpha)$ for $i=0,1$ and $\rho_\alpha(\Sigma)$ is the Atiyah–Patodi–Singer rho invariant [Atiyah et al. 1975a]. The corresponding dimension formula for a Floer-type moduli space on the cylinder is given by

$$\dim\mathcal{M}_\ell(\Sigma\times\mathbb{R},\alpha,\beta)=8\ell-\tfrac{1}{2}(h_\alpha+h_\beta)+\tfrac{1}{2}(\rho_\beta(\Sigma)-\rho_\alpha(\Sigma)) \tag{2-7}$$

with $h_\alpha=h^1_\alpha+h^0_\alpha$, similarly for h_β, and $\ell\equiv\mathrm{CS}(\beta)-\mathrm{CS}(\alpha) \bmod \mathbb{Z}$.

The moduli space of interest in this paper are ASD connections on X asymptotic to the trivial product connection and with unit L^2-norm. Since the intersection form of X is negative definite, the formal dimension is $\dim\mathcal{M}_1(X,\theta)=5$. This is the π-equivariant instanton-one moduli space we study to extract information about the original π-action on X.

2.1. ***Uhlenbeck–Taubes compactification.*** Let us recall the compactness theorem of Uhlenbeck [Lawson 1985; Freed and Uhlenbeck 1991; Donaldson and Kronheimer 1990] for the instanton moduli spaces. Intuitively, if we are given an infinite sequence of uniformly bounded g-ASD connections without a convergent subsequence, then there exists a gauge equivalent subsequence which has a weak limit, where the limiting ASD connection has a curvature density that accumulates in integral amounts of the total energy around a finite number of points in X. For a moduli space with one unit of total energy, there can be at most one point where curvature becomes highly concentrated. Uhlenbeck compactness continues to hold in the cylindrical-end setting. After passing to a subsequence we can find a gauge equivalent sequence that converges on compact subsets, but since our manifold is noncompact there is the possibility that curvature escapes down the cylindrical-end. This corresponds to broken trajectories of flat connections for the Chern–Simons flow and leads to convergence without loss of energy; see [Morgan et al. 1994, 6.3.3; Donaldson 2002, 5.1]. Weak limits are defined as a tuple of gauge equivalence class of L^2-finite ASD connections $[A] := ([A_0], [A_1], \dots, [\theta])$ where $[A_0] \in \mathcal{M}_{\ell_0}(X, \alpha_0)$ and $[A_i] \in \mathcal{M}_{\ell_i}(\Sigma \times \mathbb{R}, \alpha_{i-1}, \alpha_i)$, α_i are flat connections on Σ and have compatible boundary values $\partial_\infty(A_i) = \partial_\infty(A_{i+1})$. The "ends" of the moduli space $\mathcal{M}_1(X, \theta)$ are parametrized by products of the form

$$\mathcal{M}_{\ell_0}(X, \alpha_0) \times \mathcal{M}_{\ell_1}(\Sigma \times \mathbb{R}, \alpha_0, \alpha_1) \times \cdots \times \mathcal{M}_{\ell_k}(\Sigma \times \mathbb{R}, \alpha_{k-1}, \theta). \tag{2-8}$$

with $\sum_i \ell_i = 1$.

We also have the analogue of the Taubes construction; see [Freed and Uhlenbeck 1991] for details and also [Buchdahl et al. 1990] for the equivariant case. Since X is negative definite, this provides an equivariant collar neighborhood in the moduli space and a partial compactification

$$\overline{\mathcal{M}}_1(X, \theta) = \mathcal{M}_1(X, \theta) \cup X \times (0, \lambda_0) \tag{2-9}$$

consisting of g-ASD connections with highly concentrated curvature. In particular, the equivariant moduli space $\mathcal{M}_1(X, \theta)$ is nonempty when the fixed set X^π is nonempty. For connections $[A] \in X \times (0, \lambda_0)$ Taubes also shows that $H^2_A = 0$ [Lawson 1985, Theorem 3.38, p. 81]; thus a neighborhood of the collar is a smooth 5-manifold and these connections are irreducible. The fixed set X^π give rise to a family of ASD connections which correspond to equivariant lifts of the π-action on X to a $\tilde{\pi} = \mathbb{Z}/2p$-action on the principal SU(2)-bundle; see [Braam and Matić 1993]. We study the π-equivariant compactification of the fixed set $\mathcal{M}_1(X, \theta)^\pi$ which originates in the Taubes collar to obtain information about the fixed set X^π.

2.2. ***Equivariant general position.*** In the nonequivariant setting, the argument in Freed and Uhlenbeck [1991] can be adapted to show that for a Baire set of metrics g which restrict to a product metric on the End(X), the moduli space $\mathcal{M}_1(X, \theta)$ is a smooth 5-dimensional manifold. In the equivariant setting we have a theorem of Cho [1991] on the existence of a Baire set of π-invariant metrics on X such that all the components of the fixed set $\mathcal{M}_1(X, \theta)^\pi$ are either empty or smooth manifolds. This π-invariant version of Freed and Uhlenbeck is also valid on cylindrical end

4-manifolds; see [Buchdahl et al. 1990]. Even though $\mathcal{M}_1(X,\theta)^\pi$ is smooth if nonempty, it may not have smooth π-invariant neighborhoods and in general the surrounding moduli space may be highly singular. Another approach would be to perturb the anti-self-duality equations at the chart level by passing to the Kuranishi model

$$\phi : H^1_A \to H^2_A \tag{2-10}$$

as in Donaldson [1983]. In the equivariant case, H^1_A and H^2_A are finite dimensional real π-representation spaces and the obstruction to the existence of an equivariant perturbation is

$$[H^1_A] - [H^2_A] \in R^+(\pi) \tag{2-11}$$

being an actual representation. Hambleton and Lee in [1992] applied the theory of equivariant general position of Bierstone [1977] to equivariant moduli spaces. For our setting, we use Wilson loop perturbations in free π-orbits of embedded circles in X. The nonequivariant case is described in [Donaldson 1987, pp. 400–401]. The perturbed section $F^+_A + \hat{\sigma}_+(A)$ is now $\mathcal{G}(\pi)$-equivariant and the perturbed moduli space inherits a π-action as before.

Since Bierstone general position is an open-dense condition, a generic equivariant perturbation of the ASD equations give the moduli spaces the structure of a Whitney stratified space, with open manifold strata and equivariant cone bundle neighborhoods; see [Bierstone 1977; Hambleton and Lee 1992] for details.

3. Proof of Theorem A

We begin with a lemma to determine the equivariant bundle structures. These are described by weights $\pm\lambda$ of the isotropy representation $\tilde{\pi} \to \mathrm{SU}(2)$ over a fixed point. Each of the fixed points $p_i \in X^\pi$ lies at the Taubes collar $X \times (0, \lambda_0)$ of the moduli space and is one end of a π-fixed arc γ_i. We would like to show that none of these arcs connect with each other in the irreducible component of the moduli space $\mathcal{M}_1(X,\theta)$.

Lemma 3.1 [Hambleton and Lee 1995, Lemma 17]. *If a fixed point has rotation numbers (a, b) then the equivariant lift it generates in the moduli space has an isotropy representation over the fiber of this point given by $\mathbb{Z}/2p$-weights $\pm(b-a)$ and over the other fixed points $\pm(a+b)$. Moreover, the γ_i represent distinct equivariant bundle structures and are therefore disjoint in $\mathcal{M}^*_1(X,\theta)$.*

Proof. Since the Euler characteristic $\chi(\mathrm{Fix}(X_0,\pi)) = 9$, we may suppose there are at least three fixed points of the π-action p_i, say with rotation numbers (a_1, b_1), (a_2, b_2), (a_3, b_3). Suppose that γ connects p_1 and p_2; the normal bundle information is propagated along this oriented arc and gives a canceling pair of rotation numbers $(a_2, b_2) = (a_1, -b_1)$. We will use the presence of the third distinct fixed point p_3 to get a contradiction. Because the point p_1 is fixed, there is a π-invariant ball $B(p_1)$ with a linear action and an equivariant degree one map $f_1 : X \to S^4$. We can pullback the equivariant bundle structure $Q \to S^4$ via f_1 and get an equivariant bundle $(X, f_1^* Q)$ with $\mathbb{Z}/2p$-weights $\pm(b_1 - a_1)$ over p_1 and $\pm(a_1 + b_1)$ over the other fixed points [Fintushel and Lawson 1986]. Similarly, we can do this with a map f_2 about the point p_2, this gives an equivariant bundle structure $(X, f_2^* Q')$. Since these bundle structures are equivalent, the isotropy at p_3 has to agree and a comparison shows that either $2a_1 \equiv 0 \pmod p$ or $2b_1 \equiv 0 \pmod p$, in either case we get a contradiction. □

The lemma shows that the fixed arcs γ_i generated by the fixed points in X must have an end that is not a component of the Taubes collar and according to the Uhlenbeck compactness results applicable here, these arcs must lead to energy or charge splitting down the cylindrical end. We first rule out the case of a trivial splitting:

Lemma 3.2. *The one-dimensional fixed set generated by the fixed points in the Taubes boundary $X \times (0, \lambda_0)$ cannot split energy in the equivariant compactification of $\mathcal{M}_1(X, \theta)$ by $\mathcal{M}_0(X, \theta) \times \mathcal{M}_1(\theta, \theta)$.*

Proof. The idea is that $\mathcal{M}_0(X, \theta)$ has zero energy, so it leaves behind a flat equivariant bundle which identifies the isotropy over the fibers of each fixed point. Suppose that γ is a one-parameter family of π-fixed ASD connections generated at the Taubes boundary from the fixed point with rotation numbers (a, b). Then the corresponding equivariant lift has isotropy over the fiber of this fixed point with weight $\pm(b - a)$ and $\pm(a + b)$ over the other fixed points. In such an energy splitting a flat equivariant bundle identifies the isotropy over all the points, so $a + b = \pm(b - a)$ and this forces either $2a \equiv 0 \pmod p$ or $2b \equiv 0 \pmod p$. Since p is odd and (a, b) are rotation numbers for a fixed point we get a contradiction. □

A nontrivial charge splitting will involve the flat connections of $\Sigma(2, 3, 5)$, which as representations α of the fundamental group into SU(2), are determined by rotation numbers (ℓ_1, ℓ_2, ℓ_3) and for $\Sigma(2, 3, 5)$ there are only two irreducible representations [Saveliev 2002]. We record in Table 1 the necessary values for index calculations.

The energy in $\mathcal{M}(\alpha_i, \theta)$ is given by $-\operatorname{CS}(\Sigma(2, 3, 5), \alpha_i) \bmod \mathbb{Z} \in (0, 1]$; see [Fintushel and Stern 1990, Saveliev 2002, p. 101]. In an energy splitting, the moduli space has an end given by a local diffeomorphism

$$\mathcal{M}_{\ell_0}(X, \alpha_0) \times_{\alpha_0} \mathcal{M}_{\ell_1}(\alpha_0, \alpha_1) \times_{\alpha_1} \cdots \times_{\alpha_{k-1}} \mathcal{M}_{\ell_k}(\alpha_{k-1}, \theta) \to \mathcal{M}_1(X, \theta),$$

where $\{\alpha_i\}_{i=1}^{k-1}$ are irreducible flat connections on $\Sigma(2, 3, 5)$; this then leads to a

α	(ℓ_1, ℓ_2, ℓ_3)	$\mu(\alpha)$	$\rho(\alpha)/2$	$-\operatorname{CS}(\alpha) \in (0, 1]$
1	$(1, 2, 2)$	5	$-97/30$	$49/120$
2	$(1, 2, 4)$	1	$-73/30$	$1/120$

Table 1. For each flat connection α of $\Sigma(2, 3, 5)$ are listed values for the Floer μ-index modulo 8, one-half the Atiyah–Patodi–Singer ρ-invariant and minus the Chern–Simons invariant of the given flat connection [Fintushel and Stern 1990]. The values for the ρ-invariant can be computed using a flat SO(3)-cobordism to a disjoint union of lens spaces; see [Saveliev 2002, p. 144].

dimension count

$$5 = \dim \mathcal{M}_{\ell_0}(X, \alpha_0) + \sum_{i=1}^{k} \dim \mathcal{M}_{\ell_i}(\alpha_{i-1}, \alpha_i)$$

with $\alpha_k = \theta$ and, as the convergence is without loss of energy, we have the condition

$$\sum_{i=0}^{k} \ell_i = 1.$$

The dimensions modulo 8 can be determined [Floer 1988] by the formulas

$$\begin{aligned} \dim \mathcal{M}(\alpha, \beta) &\equiv \mu(\alpha) - \mu(\beta) - \dim \operatorname{Stab}(\beta) \pmod 8, \\ \dim \mathcal{M}(X, \alpha) &\equiv -\mu(\alpha) - 3 \pmod 8, \end{aligned} \tag{3-1}$$

where μ is the Floer index and $\mu(\theta) = -3$. Imposing the energy condition allows one to determine the exact geometric dimensions. Since there are only 2 irreducible flat connections on $\Sigma(2, 3, 5)$ denoted by $\alpha_1 = (1, 2, 2)$ and $\alpha_2 = (1, 2, 4)$, we have only the possibilities listed in Table 2.

Let $\Sigma(b = 0; (a_1, b_1), (a_2, b_2), (a_3, b_3))$ be the Seifert invariants $\Sigma(a_1, a_2, a_3)$ and $\pi = \mathbb{Z}/p$ act as the standard action on $\Sigma(a_1, a_2, a_3)$. Then the orbit space $Q = \Sigma/\pi$ is a rational homology sphere with Seifert invariants $Q(b = 0; (a_i, \beta_i))$, where $\beta_i = pb_i$. We will need the formula for the Chern–Simons invariant of reducible flat connections on Q. Note that if we take the p-fold cover we get the trivial product connection on $\Sigma(a_1, a_2, a_3)$ and, as Chern–Simons invariants are multiplicative under finite covers, we expect an expression of the form

$$\operatorname{CS}(Q, \rho(k)) \equiv \frac{n}{p} \mod \mathbb{Z}$$

for some integer n, where $\rho : \pi_1(Q) \to U(1)$ is a reducible flat connection. The fundamental group of Q has presentation

$$\pi_1(Q) = \langle x_1, x_2, x_3, h \mid h \text{ central}, \ x_i^{a_i} h^{\beta_i} = 1, \ x_1 x_2 x_3 = 1 \rangle$$

	charge-splitting	dimension	energy
A	$\mathcal{M}(X,\alpha_1)\times\mathcal{M}(\alpha_1,\alpha_2)\times\mathcal{M}(\alpha_2,\theta)$	$0+4+1$	$71/120+2/5+1/120$
B	$\mathcal{M}(X,\alpha_1)\times\mathcal{M}(\alpha_1,\theta)$	$0+5$	$71/120+49/120$
C	$\mathcal{M}(X,\alpha_2)\times\mathcal{M}(\alpha_2,\theta)$	$4+1$	$119/120+1/120$
D	$\mathcal{M}(X,\theta)\times\mathcal{M}(\theta,\theta)$	$0+5$	$0+1$

Table 2. All possible energy splitting in the compactification of $\mathcal{M}_1(X,\theta)$. Note that the total energy in each case is 1.

with abelianization $\mathbb{Z}/p$ generated by the regular fiber h. A reducible representation sends $h\mapsto e^{2\pi ik/p}$ for some integer k and $x_i\mapsto 1$.

Theorem 3.3. *The Chern–Simons invariant satisfies* $\mathrm{CS}(Q,\rho(k))\equiv \dfrac{n_0k}{p} \pmod{\mathbb{Z}}$, *where* n_0 *is an integer such that* $n_0a_1a_2a_3\equiv k \pmod p$.

Proof. This congruence is obtained from Auckly's formula [1994] using a representation $\rho(n_0,n_1,n_2,n_3):\pi_1(Q)\to U(1)$, where n_0 satisfies $a_1a_2a_3\cdot n_0\equiv k \pmod p$ and $n_i=0$ for $i\neq 0$. The Seifert invariants satisfy

$$\sum_i \frac{\beta_i}{a_i}=\frac{p}{a_1a_2a_3} \tag{3-2}$$

and the formula for the Chern–Simons invariant of the corresponding flat connection is given in [Auckly 1994, p. 234][1] as

$$\mathrm{CS}(Q,\rho)\equiv\sum_{j=1}^{3}\frac{\rho_jn_j^2+n_j(n_0+c/2+\sum_{i=1}^3 n_i/a_i)/(b+\sum_i\beta_i/a_i)}{a_j} \\ +\frac{(n_0+c/2)(n_0+c/2+\sum_i n_i/a_i)}{b+\sum_i\beta_i/a_i} \pmod{\mathbb{Z}}$$

with $c=0$ and such that ρ_j satisfies $a_j\sigma_j-\beta_j\rho_j=1$ for some integers σ_j. This simplifies to $n_0k/p \bmod \mathbb{Z}$. □

The Chern–Simons invariants for the irreducible flat connections on Q can be computed by using an SO(3) flat cobordism to a disjoint union of lens spaces as with $\Sigma(a_1,a_2,a_3)$, but also again from [Auckly 1994, p. 232]. We now investigate whether any of the charge splittings given in Table 2 contain π-fixed ASD connections. It follows immediately from Lemma 3.2 that case D is ruled out. We now rule out the possibility of a 1-dimensional fixed set in the equivariant moduli space $(\mathcal{M}_1(X,\theta),\pi)$ splitting in case A of Table 2.

[1] Note that Auckly [1994] and Kirk–Klassen [1990] use opposite orientations on the Seifert fibered manifolds than the one used in this paper. As a result the Chern–Simons invariant differs by a sign.

Lemma 3.4. *The moduli space $\mathcal{M}_\ell(\alpha_1, \alpha_2)$ does not support π-fixed ASD connections with energy $\ell = 2/5$ for any odd prime $p \geq 7$.*

Proof. If there exists a π-fixed ASD connection with energy $\ell = 2/5$ in $\mathcal{M}_\ell^\pi(\alpha_1, \alpha_2)$ then it corresponds to an equivariant lift of the π-action to the principal bundle which leaves that connection invariant. Since a π-invariant connection descends to an SO(3) connection on the cylinder $Q \times \mathbb{R}$ where $Q = \Sigma(2, 3, 5)/\pi$ is a rational homology 3-sphere, the moduli space in the quotient must be nonempty. Let α_1' and α_2' denote the irreducible limiting flat connections on $Q \times \mathbb{R}$. The connection in the quotient has energy or Pontryagin charge $4\ell/p = 8/5p$, however, a nonempty moduli space must have energy that is congruent modulo $4\mathbb{Z}$ (see [Saveliev 2002, Remark 5.6, p. 102]) to the difference of the SO(3) Chern–Simons invariants $\mathrm{CS}(Q, \alpha_2') - \mathrm{CS}(Q, \alpha_1')$. It follows from Auckly's formula that this difference has the form $n/30$ for some integer n. But

$$\frac{n}{30} \not\equiv \frac{8}{5p} \mod 4\mathbb{Z}$$

since the former has denominator at most 30 and for the latter $p \geq 7$. It must be that the moduli space $\mathcal{M}_\ell^\pi(\alpha_1, \alpha_2)$ is empty. □

It remains to investigate the cases $\mathcal{M}_\ell(\alpha_i, \theta)$. The next proposition is more general and gives a necessary condition for $\Sigma(a_1, a_2, a_3) \times \mathbb{R}$ with an irreducible flat limit at $-\infty$ and the trivial connection θ at $+\infty$ to admit π-invariant ASD connections: the numerator in the energy must be a square integer. Since every reducible flat SU(2) connection on Q sends h to $\exp(2\pi i k/p)$ in $U(1) \subset \mathrm{SU}(2)$, the integer k is referred to as the holonomy number of the flat connection. The SO(3) holonomy number is obtained by applying the adjoint representation $\mathrm{ad} : \mathrm{SU}(2) \to \mathrm{SO}(3)$.

Proposition 3.5. *Suppose a principal* SU(2) *bundle over $\Sigma(a_1, a_2, a_3) \times \mathbb{R}$ admits π-invariant ASD connections with energy*

$$\ell \equiv \frac{e^2}{4a_1a_2a_3} \in (0, 1]$$

asymptotic to an irreducible flat connection α at $-\infty$ and the trivial connection at $+\infty$. Then this connection descends to an SO(3) *ASD connection on the quotient $Q \times \mathbb{R}$ with energy $4\ell/p$ which limits to an irreducible connection still denoted by α at $-\infty$ and a flat $U(1)$-reducible connection β at $+\infty$ which has* SO(3) *holonomy number* $\pm e \pmod p$.

Proof. Since an invariant connection descends to an SO(3) ASD connection, the moduli space in the quotient is nonempty; this again gives the relation between the SO(3) Chern–Simons invariants

$$\mathrm{CS}(Q, \beta) - \mathrm{CS}(Q, \alpha) \equiv \frac{4\ell}{p} \equiv \frac{e^2}{pa_1a_2a_3} \mod 4\mathbb{Z} \tag{3-3}$$

But the Chern–Simons invariant of the reducible connection is given by

$$\mathrm{CS}(Q, \beta(k)) \equiv \frac{n_0 k}{p}$$

for some integer n_0 such that $n_0(a_1a_2a_3) \equiv k \pmod{p}$ and where k is the SO(3) holonomy number of the representation $\beta(k)$. On the other hand,

$$\mathrm{CS}(Q, \alpha) \equiv \frac{m}{a_1a_2a_3}$$

for some integer m. Taking the difference gives the relation

$$\frac{n_0 k(a_1a_2a_3) - mp}{p(a_1a_2a_3)} \equiv \frac{e^2}{p(a_1a_2a_3)} \quad \mathrm{mod}\ 4\mathbb{Z}. \tag{3-4}$$

This implies that the numerators are congruent modulo $4p(a_1a_2a_3)\mathbb{Z}$ and

$$k^2 \equiv e^2 \pmod{p}. \tag{3-5}$$

Since $\mathbb{Z}/p$ has no zero divisors completes the proof. □

Proof of Theorem A. Suppose there exists a smooth extension to X_0 with isolated fixed points. If a fixed point of the π-action on X_0 has rotation numbers (a, b), where a, b are nonzero integers well-defined modulo p, then there is an equivariant lift corresponding to the 1-parameter family of π-fixed ASD connections in $\mathcal{M}_1^{\pi}(X, \theta)$ that it generates at the Taubes boundary. This is a $\tilde{\pi}$-action on the principal SU(2) bundle and has isotropy representation over the fixed point with weights $\pm(b - a)$ and action on

$$P|_{\mathrm{End}(X)} = \Sigma(2, 3, 5) \times [0, \infty) \times \mathrm{SU}(2)$$

is given by

$$\tilde{t} \cdot (x, s, U) = (tx, s, \phi(\tilde{t})U), \tag{3-6}$$

where $s \in \mathbb{R}$, $U \in \mathrm{SU}(2)$, and ϕ is the isotropy representation $\tilde{\pi} \to \mathrm{SU}(2)$ at ∞ with weights $\pm(a+b)$. We can mod out by the involution to get the π-equivariant adjoint SO(3)-bundle over $\Sigma(2, 3, 5) \times \mathbb{R}$ with action given by the adjoint representation sending t to $\mathrm{Diag}(1, t^{a+b})$ with $\mathbb{Z}/p = \langle t \rangle$. In the limit at $+\infty$ on $\Sigma(2, 3, 5) \times \mathbb{R}$ the trivial product connection descends to a flat reducible connection on Q whose SO(3) holonomy representation is isomorphic to the adjoint isotropy representation $\mathrm{ad}\,\phi$. Since this holonomy is either ± 1 and $\pm 7 \pmod{p}$ this completes the proof. □

The equivariant plumbing actions predict the existence of nonempty Floer type moduli spaces with fractional Yang–Mills energy; these dimensions can be computed by an index calculation using [Atiyah et al. 1975a]:

$$\dim \mathcal{M}_{4\ell/p}(Q \times \mathbb{R}, \alpha, \beta) = \tfrac{8\ell}{p} - \tfrac{1}{2}\big(h_\alpha + h_\beta\big) + \tfrac{1}{2}\big(\rho_\beta(Q) - \rho_\alpha(Q)\big). \tag{3-7}$$

Since α is irreducible and β is reducible, $h_\alpha = 0$ and $h_\beta = 1$. The rho invariants for reducible flat connections are determined using [Kwasik and Lawson 1993, p. 40]:

$$\rho_\beta(Q)(l) = -\frac{2}{p}\sum_{k=1}^{p-1}\sin^2\frac{\pi kl}{p} + \frac{2}{30p}\sum_{k=1}^{p-1}\csc^2\frac{\pi k}{p}\sin^2\frac{\pi kl}{p} + \sum_{i=1}^{3}\frac{2}{pa_i}\sum_{m_1=0}^{p-1}\sum_{m_2=1}^{a_{i-1}}\cot\frac{\pi m_2}{a_i}\cot\Big(\frac{\pi m_1}{p} - \frac{\pi m_2 b_i}{a_i}\Big)\sin^2\frac{\pi m_1 l}{p}, \tag{3-8}$$

where l is the rotation number for the holonomy representation of β in SO(3). For irreducible flat connections α, the rho invariants can be calculated by an SO(3)-flat cobordism to a union of lens spaces $L(a_i, pb_i)$ using the mapping cylinder for the Seifert fibration of Q [Yu 1991], as in the case of $\Sigma(a_1, a_2, a_3)$ [Saveliev 2002, p. 144]. In this way, the linear equivariant plumbing actions imply that the moduli space $\mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_2, \beta)$ for $\ell = 1/120$ is nonempty with

$$\dim \mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_2, \beta) = \tfrac{8}{p}\big(\tfrac{1}{120}\big) - \tfrac{1}{2} + \tfrac{1}{2}\big(\rho_\beta(Q)(1) - \rho_{\alpha_2}(Q)\big) = 1. \tag{3-9}$$

If we now imagine a nonlinear smooth π-extension to X_0, we do not know if $\mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_1, \beta)$ for $\ell = 49/120$ is nonempty but we have the following formal dimension:

$$\dim \mathcal{M}_\ell(Q \times \mathbb{R}, \alpha_1, \beta) = \tfrac{8}{p}\big(\tfrac{49}{120}\big) - \tfrac{1}{2} + \tfrac{1}{2}\big(\rho_\beta(Q)(7) - \rho_{\alpha_1}(Q)\big) = 1. \tag{3-10}$$

We summarize this in the following theorem. The irreducible flat connections α_1 and α_2 on $\Sigma(2, 3, 5)$ descend to irreducible flat connections on the quotient $\Sigma(2, 3, 5)/\pi$, which we still denote by α_i.

Theorem 3.6. *Let Q denote the rational homology sphere quotient $\Sigma(2, 3, 5)/\pi$ and let $\ell = 49/120$. When the holonomy representation of the flat connection β is $\pm 7 \pmod p$, the formal dimension of the moduli space $\mathcal{M}_{4\ell/p}(Q \times \mathbb{R}, \alpha_1, \beta)$ of* SO(3)*-ASD connections on the cylinder $Q \times \mathbb{R}$ with energy $4\ell/p$ that limit to α_1 at $-\infty$ and to a reducible connection β at $+\infty$ is* 1. *Similarly, when $\ell = 1/120$ and the holonomy representation of the flat connection β is $\pm 1 \pmod p$, the formal dimension of $\mathcal{M}_{4\ell/p}(Q \times \mathbb{R}, \alpha_2, \beta)$ is* 1.

We have obtained congruence relations that give constraints on the rotation data for the fixed points of a smooth extension. The next natural step is to check these constraints against the G-signature formula and we do this in the next section.

4. *G*-signature for 4-manifolds with boundary

For smooth, closed, orientable 4-manifolds X, recall that the Hodge star operator induces an involution τ on the space of complexified sections of forms $\Omega^* =$

$\bigoplus_k C^\infty(\Lambda^k TX \otimes \mathbb{C})$, splitting it into ± 1 eigenspaces $\Omega^+ \oplus \Omega^-$. The signature operator $D^+ = d + d^*$ restricted to Ω^+ is an elliptic operator $D^+ : \Omega^+ \to \Omega^-$ whose index is the signature $\mathrm{Sign}(X) = b_2^+ - b_2^-$ of the nondegenerate quadratic form on $H^2(X; \mathbb{R})$. When a finite group G acts by orientation preserving isometries on X, the cotangent bundle, as an equivariant bundle over X, has an action that commutes with the Hodge star operator. So we obtain a G-invariant elliptic operator D^+ whose G-index is a complex virtual representation $\mathrm{Ind}_G(D^+) = H^+ - H^- \in R(G)$. The associated character, or Lefschetz number,

$$\mathrm{Sign}(X, g) = \mathrm{Tr}(g|_{H^2_+}) - \mathrm{Tr}(g|_{H^2_-})$$

is the g-signature. Note that when the action of G is homologically trivial, the g-signature coincides with the usual signature.

The g-signature can be computed from the fixed set by the Atiyah–Singer fixed point index theorem [1968]. Consider the case when G is a finite cyclic group of odd prime order p with generator $t = e^{2\pi i/p}$ and let $T_{p_i} X = \mathbb{C}^2(a_i, b_i)$ be the local tangential representation over the fixed points p_i for a homologically trivial action. Then

$$\mathrm{Sign}(X) = \sum_i \left(\frac{t^{a_i}+1}{t^{a_i}-1}\right)\left(\frac{t^{b_i}+1}{t^{b_i}-1}\right) - 4 \sum_j \frac{\alpha_j t^{c_j}}{(t^{c_j}-1)^2} \tag{4-1}$$

where, for each j, α_j is the self-intersection of the fixed 2-sphere and c_j is the rotation number on its normal bundle.

Consider the situation where X_0 denotes a compact, simply connected, smooth 4-manifold with boundary $\partial X_0 = \Sigma$ an integral homology 3-sphere. If a free action on Σ by $\mathbb{Z}/p = \langle t \rangle$ extends to a locally linear, homologically trivial action on X_0 (not necessarily free), then the G-signature theorem for manifolds with boundary is given in Atiyah–Patodi–Singer [Atiyah et al. 1975b, p. 413]:

$$\mathrm{Sign}(X_0, t) = L(X_0, t) - \eta_t(0) \tag{4-2}$$

where $L(X_0, t)$ is the collection of terms occurring in the closed manifold case and $\eta_t(0)$ is the equivariant eta invariant of Σ or the G-signature defect. This invariant depends only on the 3-manifold Σ and not on how the action extends to X_0 nor on which 4-manifold it equivariantly bounds. To see this, suppose the action on Σ extends to another 4-manifold X_1. Then consider the G-signature theorem on $X_0 \cup_\Sigma -X_1$ to see that the signature defect terms are equal.

When the boundary ∂X_0 is a Seifert fibered homology sphere — thought of as a link of a complex surface singularity — it has a canonical negative definite resolution $\tilde{X}_0$ obtained by plumbing disk bundles over 2-spheres. Let $(\tilde{a}_i, \tilde{b}_i)$ denote the rotation numbers by equivariant plumbing [Fintushel 1977, p. 152] along the

E_8 graph. They are given by

$$\{(-4,5),(-3,4),(-2,3),(-2,3),(-1,2),(-1,2),(-1,2)\}.$$

The central node in the plumbing graph is a fixed 2-sphere with self-intersection number -2, and the rotation on the normal fiber is congruent to $1 \pmod p$. This gives the following formula for the equivariant eta invariant:

$$\eta_t(0) = \sum_{i=1}^{7}\left(\frac{t^{\tilde{a}_i}+1}{t^{\tilde{a}_i}-1}\right)\left(\frac{t^{\tilde{b}_i}+1}{t^{\tilde{b}_i}-1}\right) + \frac{8t}{(t-1)^2} + 8.$$

Proof of Theorem B. Suppose a free $\mathbb{Z}/7$-action on $\Sigma(2,3,5)$ extends to a smooth homologically trivial action on X_0 with fixed set consisting of only isolated fixed points with rotation data $\{(a_i,b_i)\}_{i=1}^{9}$. By Theorem A these rotation numbers must satisfy the congruence relations $a_i + b_i \equiv \pm 1$ or $0 \pmod 7$. There are three types of rotation numbers that satisfy the first constraint: $(1,5)$, $(2,4)$, $(3,3)$ and three types that satisfy the second constraint:$(1,6)$, $(2,5)$, $(3,4)$. By the G-signature theorem, the rotation numbers must satisfy

$$-8 = \sum_{i=1}^{9}\left(\frac{t^{a_i}+1}{t^{a_i}-1}\right)\left(\frac{t^{b_i}+1}{t^{b_i}-1}\right) - \eta_t(0), \tag{4-3}$$

where $\eta_t(0)$ is determined from the equivariant plumbing action. One may check this formula directly for all possible rotation data of the types listed above. There will be 2002 independent G-signature checks since repetition of rotation numbers is allowed. This number may be significantly cut down in the following way. In (4-3) we can sum over nonidentity roots of unity to obtain

$$-8(p-1) = \sum_{i=1}^{9}\operatorname{def}(p;a_i,b_i) - \sum_{\substack{t^p=1\\ t\neq 1}}\eta_{t^i}, \tag{4-4}$$

where

$$\operatorname{def}(p;a,b) = -\sum_{k=1}^{p-1}\cot\frac{\pi ak}{p}\cot\frac{\pi bk}{p}$$

are the G-signature defects. The second term above can easily be computed from the formula for η_t above and gives $\sum_{t^p=1,t\neq 1}\eta_{t^i} = 6$. The defect terms for the rotation data that satisfy the first constraint can also be computed for $p=7$ and we have $\operatorname{def}(7;1,5) = 2 = -\operatorname{def}(7;2,4)$, $\operatorname{def}(7;3,3) = -10$. Similarly for the second constraint, all the defect terms sum to 10. If n_i are the number of rotation numbers of the types listed above respectively, we have the following system of linear Diophantine equations

$$-21 = n_1 - n_2 - 5n_3 + 5(n_4 + n_5 + n_6) \tag{4-5}$$

$$9 = n_1 + n_2 + n_3 + n_4 + n_5 + n_6. \tag{4-6}$$

The second equation holds since there must be 9 isolated fixed points for a homologically trivial extension. Note that not all rotation numbers can satisfy the second constraint $a_i + b_i \equiv 0 \pmod{p}$, since the left-hand side of (4-5) would not be divisible by 5. So there must be rotation numbers of the first type. There are 12 solutions $(n_1, n_2, n_3, n_4, n_5, n_6)$ to this system in total, which may be enumerated as follows:

$$(0, 1, 6, 0, 0, 2), (0, 1, 6, 0, 1, 1), (0, 1, 6, 0, 2, 0), (0, 1, 6, 1, 0, 1),$$
$$(0, 1, 6, 1, 1, 0), (0, 1, 6, 2, 0, 0), (0, 6, 3, 0, 0, 0), (1, 2, 5, 0, 0, 1),$$
$$(1, 2, 5, 0, 1, 0), (1, 2, 5, 1, 0, 0), (2, 3, 4, 0, 0, 0), (4, 0, 5, 0, 0, 0).$$

One can check that none of these candidates satisfies the G-signature formula in (4-3). This concludes the proof since we have shown that there are no solutions to the G-signature formula that satisfy the constraints for a smooth, homologically trivial extension. □

Proof of Corollary C. Suppose that (X, π) is a smooth, homologically trivial π-action on $X = \#^8 S^2 \times S^2$ with fixed set consisting of isolated fixed points. If $\Sigma(2, 3, 5)$ with a free π-action smoothly and equivariantly embeds in (X, π), we obtain a π-equivariant decomposition $X = X_0 \cup_{\Sigma(2,3,5)} X_1$ with intersection forms $\pm E_8$ on each side by Remark 1.2. If X_0 is not simply connected then, by van Kampen's theorem, there are no nontrivial representations $\pi_1(X_0) \to SU(2)$ whose restriction to $\Sigma(2, 3, 5)$ are trivial. In particular, no additional flat connections appear in the charge splitting case D and the corollary follows from Theorems A and B. □

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NIMA ANVARI
MATHEMATICS
UNIVERSITY OF MIAMI
1365 MEMORIAL DRIVE
CORAL GABLES, FL 33146
UNITED STATES

anvarin@math.miami.edu

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chari@math.ucr.edu

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Department of Mathematics
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cooper@math.ucsb.edu

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Department of Mathematics
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