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# $E$-POLYNOMIAL OF THE SL(3, $\mathbb{C}$)-CHARACTER VARIETY OF FREE GROUPS

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**We compute the $E$-polynomial of the character variety of representations of a rank $r$ free group in SL(3, $\mathbb{C}$). Expanding upon techniques of Logares, Muñoz and Newstead (*Rev. Mat. Complut.* 26:2 (2013), 635–703), we stratify the space of representations and compute the $E$-polynomial of each geometrically described stratum using fibrations. Consequently, we also determine the $E$-polynomial of its smooth, singular, and abelian loci and the corresponding Euler characteristic in each case. Along the way, we give a new proof of results of Cavazos and Lawton (*Int. J. Math.* 25:6 (2014), 1450058).**

## 1. Introduction

Let $\Gamma$ be a finitely generated group, and let $G$ be a complex reductive algebraic group. The space of $G$-representations is

$$\mathcal{R}(\Gamma, G) = \{\rho : \Gamma \to G \mid \rho \text{ is a group morphism}\}.$$

Writing a presentation $\Gamma = \langle x_1, \ldots, x_n | R_1, \ldots, R_s\rangle$, we have that $\rho \in \mathcal{R}(\Gamma, G)$ is determined by the images $A_i = \rho(x_i)$, $1 \le i \le n$. Hence we can write $\rho = (A_1, \ldots, A_n)$. These matrices are subject to the relations $R_j(A_1, \ldots, A_n) = \mathrm{Id}$, $1 \le j \le s$. Hence

$$\mathcal{R}(\Gamma, G) \cong \{(A_1, \ldots, A_n) \in G^n \mid R_1(A_1, \ldots, A_n) = \cdots = R_s(A_1, \ldots, A_n) = \mathrm{Id}\}$$

is an affine algebraic set, since $G$ is algebraic.

There is an action of $G$ by conjugation on $\mathcal{R}(\Gamma, G)$, which is equivalent to the action of $PG = G/Z(G)$, where $Z(G)$ is the center of $G$, since the center acts trivially. The *$G$-character variety of* $\Gamma$ is the GIT quotient

$$\mathcal{M}(\Gamma, G) = \mathcal{R}(\Gamma, G)/\!\!/G,$$

which is an affine algebraic set by construction. Note that if we write $X := \mathcal{R}(\Gamma, G) = \mathrm{Spec}(S)$, then $X/\!\!/G = \mathrm{Spec}(S^G)$.

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Every element $g \in \Gamma$ determines a character $\chi_g : X \to \mathbb{C}$, $\chi_g(\rho) = \mathrm{tr}(\rho(g))$, with respect to an embedding $G \hookrightarrow \mathrm{GL}(n, \mathbb{C})$. These regular functions $\chi_g \in S$ are invariant by conjugation, and hence $\chi_g \in S^G$. Consider the algebra of characters

$$T = \mathbb{C}[\chi_g \mid g \in \Gamma] \subset S^G,$$

and let $\chi(\Gamma, G) = \mathrm{Spec}(T)$. There is a well-defined surjective map $\mathcal{M}(\Gamma, G) \to \chi(\Gamma, G)$, which is an isomorphism when $G = \mathrm{SL}(n, \mathbb{C})$ among other examples; see [Sikora 2013].

In this paper we are interested in the character variety for the free group on $r$ elements $\Gamma = F_r$ and for the group $G = \mathrm{SL}(3, \mathbb{C})$. We compute the $E$-polynomial (also known as Hodge–Deligne polynomial) of $\mathcal{M}(F_r, \mathrm{SL}(3, \mathbb{C}))$. The $E$-polynomial of $\mathcal{M}(F_r, \mathrm{SL}(2, \mathbb{C}))$ has been computed in [Cavazos and Lawton 2014] by arithmetic methods (using the Weil conjectures). Recently, in [Mozgovoy and Reineke 2015], the $E$-polynomials of $\mathcal{M}(F_r, \mathrm{PGL}(n, \mathbb{C}))$ have also been computed by arithmetic methods, where the result is given in the form of a generating function.

Here we use a geometric technique, introduced in [Logares et al. 2013], to compute $E$-polynomials of character varieties. This consists of stratifying the space of representations geometrically, and computing the $E$-polynomials of each stratum using the behavior of $E$-polynomials with fibrations. This technique is used in [Logares et al. 2013] for the case of $\Gamma = \pi_1(X)$ for a surface $X$ of genus $g = 1, 2$ and $G = \mathrm{SL}(2, \mathbb{C})$ (and also with one puncture, fixing the holonomy around the puncture). The case of $g = 3$ is worked out in [Martínez and Muñoz 2015a], the case of $g \geq 4$ in [Martínez and Muñoz 2015b], and the case of $g = 1$ with two punctures appears in [Logares and Muñoz 2014]. To implement this geometric technique for character varieties for $\mathrm{SL}(n, \mathbb{C})$, for $n \geq 3$, we need to introduce the *equivariant Hodge–Deligne polynomial* with respect to a finite group action on an affine variety. This will be useful for studying character varieties of surface groups in $\mathrm{SL}(n, \mathbb{C})$, $n \geq 3$.

We start by recovering the $E$-polynomials $e(\mathcal{M}(F_r, \mathrm{SL}(2, \mathbb{C})))$ of [Cavazos and Lawton 2014] and $e(\mathcal{M}(F_r, \mathrm{PGL}(2, \mathbb{C})))$ of [Mozgovoy and Reineke 2015], verifying that they are equal. Then we move to rank 3 to compute $e(\mathcal{M}(F_r, \mathrm{SL}(3, \mathbb{C})))$ and $e(\mathcal{M}(F_r, \mathrm{PGL}(3, \mathbb{C})))$. They turn out to be equal again. The latter one coincides, as expected, with the polynomial obtained in [Mozgovoy and Reineke 2015].

Unlike the methods used to obtain $e(\mathcal{M}(F_r, \mathrm{PGL}(3, \mathbb{C})))$ in [Mozgovoy and Reineke 2015], our method provides an explicit geometric description of, and the $E$-polynomial for, each stratum. By results in [Florentino and Lawton 2012] this additional information determines the $E$-polynomial of the smooth and singular loci of $\mathcal{M}(F_r, \mathrm{SL}(3, \mathbb{C}))$, and by [Florentino and Lawton 2014] also determines the $E$-polynomial of the abelian character variety $\mathcal{M}(\mathbb{Z}^r, \mathrm{SL}(3, \mathbb{C}))$.

Our main theorem is thus:

**Theorem 1.** *The $E$-polynomials $e(\mathcal{M}(F_r, \mathrm{SL}(3,\mathbb{C})))$ and $e(\mathcal{M}(F_r, \mathrm{PGL}(3,\mathbb{C})))$ are both equal to*

$$\begin{aligned}(q^8-q^6-q^5+q^3)^{r-1}+(q-1)^{2r-2}(q^{3r-3}-q^r)\\ +\tfrac{1}{6}(q-1)^{2r-2}q(q+1)+\tfrac{1}{2}(q^2-1)^{r-1}q(q-1)+\tfrac{1}{3}(q^2+q+1)^{r-1}q(q+1)\\ -(q-1)^{r-1}q^{r-1}(q^2-1)^{r-1}(2q^{2r-2}-q).\end{aligned}$$

From the definition of the $E$-polynomial of a variety $X$, the classical Euler characteristic is given by $\chi(X)=e(X;1,1)$. Consequently, we deduce:

**Corollary 2.** *Let $r\geq 2$. Then $\mathcal{M}(F_r, \mathrm{SL}(3,\mathbb{C}))$, $\mathcal{M}(F_r, \mathrm{PGL}(3,\mathbb{C}))$, and* (by [Florentino and Lawton 2009]) *$\mathcal{M}(F_r, \mathrm{SU}(3))$, have Euler characteristic given by $2\cdot 3^{r-2}$. The Euler characteristic of $\mathcal{M}(\mathbb{Z}^r, \mathrm{SL}(3,\mathbb{C}))$, and* (by [Florentino and Lawton 2014]) *also $\mathcal{M}(\mathbb{Z}^r, \mathrm{SU}(3))$, is given $3^{r-2}$.*

## 2. Hodge structures and $E$-polynomials

Our main goal is to compute the $E$-polynomial (Hodge–Deligne polynomial) of the $\mathrm{SL}(3,\mathbb{C})$-character variety of a free group. We will follow the methods in [Logares et al. 2013], so we collect some basic results from [loc. cit.] in this section.

We start by reviewing the definition of the Hodge–Deligne polynomial. A pure Hodge structure of weight $k$ consists of a finite dimensional complex vector space $H$ with a real structure, and a decomposition $H=\bigoplus_{k=p+q}H^{p,q}$ such that $H^{q,p}=\overline{H^{p,q}}$, the bar meaning complex conjugation on $H$. A Hodge structure of weight $k$ gives rise to the so-called Hodge filtration, which is a descending filtration $F^p=\bigoplus_{s\geq p}H^{s,k-s}$. We define $\mathrm{Gr}^p_F(H):=F^p/F^{p+1}=H^{p,k-p}$.

A mixed Hodge structure consists of a finite dimensional complex vector space $H$ with a real structure, an ascending (weight) filtration $\cdots\subset W_{k-1}\subset W_k\subset\cdots\subset H$ (defined over $\mathbb{R}$) and a descending (Hodge) filtration $F$ such that $F$ induces a pure Hodge structure of weight $k$ on each $\mathrm{Gr}^W_k(H)=W_k/W_{k-1}$. We define $H^{p,q}:=\mathrm{Gr}^p_F\,\mathrm{Gr}^W_{p+q}(H)$ and write $h^{p,q}$ for the *Hodge number* $h^{p,q}:=\dim H^{p,q}$.

Let $Z$ be any quasiprojective algebraic variety (possibly nonsmooth or noncompact). The cohomology groups $H^k(Z)$ and the cohomology groups with compact support $H^k_c(Z)$ are endowed with mixed Hodge structures [Deligne 1971; 1974]. We define the *Hodge numbers* of $Z$ by

$$h^{k,p,q}_c(Z)=h^{p,q}(H^k_c(Z))=\dim\mathrm{Gr}^p_F\,\mathrm{Gr}^W_{p+q}\,H^k_c(Z).$$

The Hodge–Deligne polynomial, or $E$-polynomial, is defined as

$$e(Z)=e(Z)(u,v):=\sum_{p,q,k}(-1)^k h^{k,p,q}_c(Z)u^pv^q.$$

The key property of Hodge–Deligne polynomials that permits their calculation is that they are additive for stratifications of $Z$. If $Z$ is a complex algebraic variety and $Z = \bigsqcup_{i=1}^{n} Z_i$, where all $Z_i$ are locally closed in $Z$, then

$$e(Z) = \sum_{i=1}^{n} e(Z_i).$$

Also, by [Logares et al. 2013, Remark 2.5], if $G \to X \to B$ is a principal fiber bundle with $G$ a connected algebraic group, then $e(X) = e(G)e(B)$. In general we shall use this as $e(X/G) = e(X)/e(G)$ when $B = X/G$. In particular, if $Z$ is a $G$-space, and there is a subspace $B \subset Z$ such that $B \times G \to Z$ is surjective and it is an $H$-homogeneous space for a connected subgroup $H \subset G$, then

$$e(Z) = e(B)e(G)/e(H). \tag{1}$$

**Definition 3.** Let $X$ be a complex quasiprojective variety on which a finite group $F$ acts. Then $F$ also acts on the cohomology $H_c^*(X)$ respecting the mixed Hodge structure. So $[H_c^*(X)] \in R(F)$, the representation ring of $F$. The *equivariant Hodge–Deligne polynomial* is defined as

$$e_F(X) = \sum_{p,q,k} (-1)^k [H_c^{k,p,q}(X)]\, u^p v^q \in R(F)[u, v].$$

Note that the map $\dim : R(F) \to \mathbb{Z}$ gives $\dim(e_F(X)) = e(X)$.

For instance, for an action of $\mathbb{Z}_2$, there are two irreducible representations $T$, $N$, where $T$ is the trivial representation, and $N$ is the nontrivial representation. Then $e_{\mathbb{Z}_2}(X) = aT + bN$. Clearly

$$e(X) = a + b, \quad e(X/\mathbb{Z}_2) = a.$$

In the notation of [Logares et al. 2013, Section 2], $a = e(X)^+$, $b = e(X)^-$. Note that if $X$, $X'$ are spaces with $\mathbb{Z}_2$-actions, then writing

$$e_{\mathbb{Z}_2}(X) = aT + bN \quad \text{and} \quad e_{\mathbb{Z}_2}(X') = a'T + b'N,$$

we have $e_{\mathbb{Z}_2}(X \times X') = (aa' + bb')T + (ab' + ba')N$ and so

$$e((X \times X')/\mathbb{Z}_2) = aa' + bb' = e(X)^+ e(X')^+ + e(X)^- e(X')^-. \tag{2}$$

When $h_c^{k,p,q} = 0$ for $p \neq q$, the polynomial $e(Z)$ depends only on the product $uv$. This will happen in all the cases that we shall investigate here. In this situation, it is conventional to use the variable $q = uv$. If this happens, we say that the variety is *of balanced type*. For instance, $e(\mathbb{C}^n) = q^n$.

## 3. $E$-polynomial of the SL(2, $\mathbb{C}$)-character variety of free groups

Let $F_r$ denote the free group on $r$ generators. Then the space of representations of $F_r$ in the group SL(2, $\mathbb{C}$) is

$$\mathcal{R}_{r,2} = \operatorname{Hom}(F_r, \mathrm{SL}(2, \mathbb{C})) = \{(A_1, \ldots, A_r) \mid A_i \in \mathrm{SL}(2, \mathbb{C})\} = \mathrm{SL}(2, \mathbb{C})^r.$$

The group PGL(2, $\mathbb{C}$) acts on $\mathcal{R}_{r,2}$ by simultaneous conjugation of all matrices, and the character variety is defined as the GIT quotient

$$\mathcal{M}_{r,2} = \mathcal{R}_{r,2} /\!\!/ \mathrm{PGL}(2, \mathbb{C}).$$

We aim to compute the $E$-polynomial of $\mathcal{M}_{r,2}$ using the methods developed in [Logares et al. 2013] and to recover the results of [Cavazos and Lawton 2014]. We have the following sets:

- Reducible representations $\mathcal{R}^{\mathrm{red}}_{r,2} \subset \mathcal{R}_{r,2}$ and the corresponding set $\mathcal{M}^{\mathrm{red}}_{r,2} \subset \mathcal{M}_{r,2}$ of characters of reducible representations. A representation $\rho = (A_1, \ldots, A_r)$ is reducible if and only if all $A_i$ share at least one eigenvector.
- Irreducible representations $\mathcal{R}^{\mathrm{irr}}_{r,2} \subset \mathcal{R}_{r,2}$ and the corresponding set $\mathcal{M}^{\mathrm{irr}}_{r,2} \subset \mathcal{M}_{r,2}$ of characters of irreducible representations. This is the complement of $\mathcal{R}^{\mathrm{red}}_{r,2}$. It consists of the representations $\rho$ such that PGL(2, $\mathbb{C}$) acts freely on $\rho$, and the orbit PGL(2, $\mathbb{C}$) $\cdot\, \rho$ is closed. Therefore $\mathcal{M}^{\mathrm{irr}}_{r,2} = \mathcal{R}^{\mathrm{irr}}_{r,2} / \mathrm{PGL}(2, \mathbb{C})$.

**3.1.** ***The reducible locus.*** Let us start by computing $e(\mathcal{M}^{\mathrm{red}}_{r,2})$. For a reducible representation, we have a basis of $\mathbb{C}^2$ in which

$$\rho = \left( \begin{pmatrix} \lambda_1 & * \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & * \\ 0 & \lambda_2^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_r & * \\ 0 & \lambda_r^{-1} \end{pmatrix} \right).$$

The associated point is determined by $(\lambda_1, \ldots, \lambda_r) \in (\mathbb{C}^*)^r$, modulo $(\lambda_1, \ldots, \lambda_r) \sim (\lambda_1^{-1}, \ldots, \lambda_r^{-1})$. Note that the action of $\lambda \mapsto \lambda^{-1}$ on $X = \mathbb{C}^*$ has $e(X)^+ = q$ and $e(X)^- = -1$. Writing $X_i = \mathbb{C}^*$, $i = 1, \ldots, r$, we have that

$$\begin{aligned} e(X_1 \times \cdots \times X_r)^+ &= \sum_{\epsilon \in A} \prod_{i=1}^{r} e(X_i)^{\epsilon_i} \\ &= q^r + \binom{r}{2} q^{r-2} + \binom{r}{4} q^{r-4} + \cdots + \binom{r}{2[r/2]} q^{r-2[r/2]} \\ &= \tfrac{1}{2}((q+1)^r + (q-1)^r), \end{aligned}$$

where $A = \{(\epsilon_1, \ldots, \epsilon_r) \in (\pm 1)^r \mid \prod \epsilon_i = +1\}$. Also

$$\begin{aligned} e(X_1 \times \cdots \times X_r)^- &= e(X_1 \times \cdots \times X_r) - e(X_1 \times \cdots \times X_r)^+ \\ &= (q-1)^r - \tfrac{1}{2}((q+1)^r + (q-1)^r) \\ &= \tfrac{1}{2}((q-1)^r - (q+1)^r). \end{aligned}$$

Also note that $e(\mathcal{M}_{r,2}^{\mathrm{red}}) = e((X_1 \times \cdots \times X_r)/\mathbb{Z}_2) = e(X_1 \times \cdots \times X_r)^+$.

**3.2.** ***The reducible representations.*** Now we move to the computation of $e(\mathcal{R}_{r,2}^{\mathrm{red}})$. We stratify the space as $\mathcal{R}_{r,2}^{\mathrm{red}} = R_0 \cup R_1 \cup R_2 \cup R_3$, where:

- $R_0$ consists of $(A_1, \ldots, A_r) = (\pm\,\mathrm{Id}, \ldots, \pm\,\mathrm{Id})$. So $e(R_0) = 2^r$.
- $R_1$ consists of

$$\rho \sim \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_r^{-1} \end{pmatrix} \right),$$

that is, abelian representations (all matrices are diagonalizable with respect to the same basis). Here $(\lambda_1, \ldots, \lambda_r) \neq (\pm 1, \ldots, \pm 1)$. Therefore this space is parametrized by

$$\big(\mathrm{PGL}(2, \mathbb{C})/D \times \big((\mathbb{C}^*)^r - \{(\pm 1, \ldots, \pm 1)\}\big)\big)/\mathbb{Z}_2,$$

where $D$ is the space of diagonal matrices. We know that $e(\mathrm{PGL}(2, \mathbb{C})/D)^+ = q^2$, $e(\mathrm{PGL}(2, \mathbb{C})/D)^- = q$ by [Logares et al. 2013, Proposition 3.2]. For $B = (\mathbb{C}^*)^r - \{(\pm 1, \ldots, \pm 1)\}$, we have $e(B)^+ = \frac{1}{2}((q+1)^r + (q-1)^r) - 2^r$ and $e(B)^- = \frac{1}{2}((q-1)^r - (q+1)^r)$, by our computation above. Therefore

$$\begin{aligned} e(R_1) &= e(\mathrm{PGL}(2, \mathbb{C})/D)^+ e(B)^+ + e(\mathrm{PGL}(2, \mathbb{C})/D)^- e(B)^- \\ &= q^2 \tfrac{1}{2}((q+1)^r + (q-1)^r - 2^r) + q \tfrac{1}{2}((q-1)^r - (q+1)^r) \\ &= \tfrac{1}{2}(q^2 - q)(q+1)^r + \tfrac{1}{2}(q^2 + q)(q-1)^r - q^2 2^r\,. \end{aligned}$$

- $R_2$ consists of

$$\rho \sim \left( \begin{pmatrix} \pm 1 & a_1 \\ 0 & \pm 1 \end{pmatrix}, \begin{pmatrix} \pm 1 & a_2 \\ 0 & \pm 1 \end{pmatrix}, \ldots, \begin{pmatrix} \pm 1 & a_r \\ 0 & \pm 1 \end{pmatrix} \right),$$

where $(a_1, \ldots, a_r) \in \mathbb{C}^r - \{0\}$. Let $B_2$ be the space of representations as above with respect to the canonical basis. Therefore, there is a canonical surjective map $B_2 \times \mathrm{PGL}(2, \mathbb{C}) \twoheadrightarrow R_2$. The fibers of this map are given by $H_2 = \left\{ \left(\begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix}\right) \right\} \cong \mathbb{C}^* \times \mathbb{C}$. That is, $H_2 \to B_2 \times \mathrm{PGL}(2, \mathbb{C}) \to R_2$ is a fibration to which we apply Formula (1) to obtain

$$e(R_2) = \frac{e(B_2) e(\mathrm{PGL}(2, \mathbb{C}))}{e(H_2)} = \frac{2^r (q^r - 1)(q^3 - q)}{q(q-1)} = 2^r (q^r - 1)(q+1).$$

- $R_3$ consists of

$$\rho \sim \left( \begin{pmatrix} \lambda_1 & b_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & b_2 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \ldots, \begin{pmatrix} \lambda_r & b_r \\ 0 & \lambda_r^{-1} \end{pmatrix} \right),$$

where $\lambda_i \in \mathbb{C}^*$, $(\lambda_1, \ldots, \lambda_r) \neq (\pm 1, \ldots, \pm 1)$. Here, $(b_1, \ldots, b_r) \in \mathbb{C}^r$ and the upper diagonal matrices $\left(\begin{smallmatrix} 1 & y \\ 0 & 1 \end{smallmatrix}\right)$ transform

$$(b_1, \ldots, b_r) \mapsto (b_1 + y(\lambda_1 - \lambda_1^{-1}), \ldots, b_r + y(\lambda_r - \lambda_r^{-1})).$$

As $(\lambda_1, \dots, \lambda_r) \neq (\pm 1, \dots, \pm 1)$, this action is nontrivial. Note that $(b_1, \dots, b_r)$ does not live in the line spanned by $(\lambda_1 - \lambda_1^{-1}, \dots, \lambda_r - \lambda_r^{-1})$. There is a fibration $H_3 \to B_3 \times \mathrm{PGL}(2, \mathbb{C}) \to R_3$ where $H_3 = \left\{ \left( \begin{smallmatrix} a & b \\ 0 & a^{-1} \end{smallmatrix} \right) \right\} \cong \mathbb{C}^* \times \mathbb{C}$. Thus

$$\begin{aligned} e(R_3) &= (q^r - q)((q-1)^r - 2^r) e(\mathrm{PGL}(2, \mathbb{C}))/q(q-1) \\ &= \frac{q^{r-1}-1}{q-1}((q-1)^r - 2^r)(q^3 - q) \\ &= (q^{r-1} - 1)(q-1)^{r-1}(q^3 - q) - 2^r \frac{q^{r-1}-1}{q-1}(q^3 - q). \end{aligned}$$

Now we add all the subsets together:

$$\begin{aligned} e(\mathcal{R}_{r,2}^{\mathrm{red}}) &= e(R_0) + e(R_1) + e(R_2) + e(R_3) \\ &= \tfrac{1}{2}(q^2 - q)(q+1)^r + \tfrac{1}{2}(q^2 + q)(q-1)^r \\ &\quad + (q^{r-1} - 1)(q-1)^{r-1}(q^3 - q). \end{aligned}$$

**3.3.** ***The irreducible locus.*** Recall that $\mathcal{R}_{r,2}^{\mathrm{irr}} = \mathrm{SL}(2, \mathbb{C})^r - \mathcal{R}_{r,2}^{\mathrm{red}}$, so

$$\begin{aligned} e(\mathcal{R}_{r,2}^{\mathrm{irr}}) = (q^3 - q)^r &- \tfrac{1}{2}(q^2 - q)(q+1)^r - \tfrac{1}{2}(q^2 + q)(q-1)^r \\ &- (q^{r-1} - 1)(q-1)^{r-1}(q^3 - q), \end{aligned}$$

and

$$\begin{aligned} e(\mathcal{M}_{r,2}^{\mathrm{irr}}) &= \frac{e(\mathcal{R}_{r,2}^{\mathrm{irr}})}{q^3 - q} \\ &= (q^3 - q)^{r-1} - \tfrac{1}{2}(q+1)^{r-1} - \tfrac{1}{2}(q-1)^{r-1} - (q^{r-1} - 1)(q-1)^{r-1}. \end{aligned}$$

Finally,

$$\begin{aligned} e(\mathcal{M}_{r,2}) &= e(\mathcal{M}_{r,2}^{\mathrm{irr}}) + e(\mathcal{M}_{r,2}^{\mathrm{red}}) = e(\mathcal{M}_{r,2}^{\mathrm{irr}}) + \tfrac{1}{2}((q+1)^r + (q-1)^r) \\ &= (q^3 - q)^{r-1} + \tfrac{1}{2}q(q+1)^{r-1} + \tfrac{1}{2}q(q-1)^{r-1} - q^{r-1}(q-1)^{r-1}. \end{aligned}$$

This agrees with [Cavazos and Lawton 2014].

## 4. $E$-polynomial of the PGL(2, $\mathbb{C}$)-character variety of free groups

Let us compute the $E$-polynomial of $\mathcal{M}(F_r, \mathrm{PGL}(2, \mathbb{C}))$. The space of representations will be denoted

$$\overline{\mathcal{R}}_{r,2} = \mathrm{Hom}(F_r, \mathrm{PGL}(2, \mathbb{C})) = \{(A_1, \dots, A_r) \mid A_i \in \mathrm{PGL}(2, \mathbb{C})\} = \mathrm{PGL}(2, \mathbb{C})^r.$$

Note that $\mathrm{PGL}(2, \mathbb{C}) = \mathrm{SL}(2, \mathbb{C})/\{\pm \mathrm{Id}\}$, so $\overline{\mathcal{R}}_{r,2} = \mathcal{R}_{r,2}/\{(\pm \mathrm{Id}, \dots, \pm \mathrm{Id})\}$. The character variety is

$$\overline{\mathcal{M}}_{r,2} = \overline{\mathcal{R}}_{r,2} /\!\!/ \mathrm{PGL}(2, \mathbb{C}).$$

We denote by $\overline{\mathcal{R}}^{\mathrm{red}}_{r,2}$ and $\overline{\mathcal{R}}^{\mathrm{irr}}_{r,2}$ the subsets of reducible and irreducible representations, respectively, of $\overline{\mathcal{R}}_{r,2}$. We denote by $\overline{\mathcal{M}}^{\mathrm{red}}_{r,2}$ and $\overline{\mathcal{M}}^{\mathrm{irr}}_{r,2}$ the corresponding spaces in $\overline{\mathcal{M}}_{r,2}$.

***The reducible locus.*** We first compute $e(\overline{\mathcal{M}}^{\mathrm{red}}_{r,2})$. A reducible representation in $\overline{\mathcal{M}}^{\mathrm{red}}_{r,2}$ is determined by the eigenvalues $(\lambda_1, \dots, \lambda_r) \in (\mathbb{C}^*)^r$, modulo $\lambda_i \sim -\lambda_i$, $1 \le i \le r$, and $(\lambda_1, \dots, \lambda_r) \sim (\lambda_1^{-1}, \dots, \lambda_r^{-1})$. So it is determined by $(\lambda_1^2, \dots, \lambda_r^2) \in (\mathbb{C}^*)^r$, modulo $(\lambda_1^2, \dots, \lambda_r^2) \sim (\lambda_1^{-2}, \dots, \lambda_r^{-2})$. This space is isomorphic to the one in Section 3.1, so $e(\overline{\mathcal{M}}^{\mathrm{red}}_{r,2}) = \frac{1}{2}((q+1)^r + (q-1)^r)$.

***The reducible representations.*** Now we compute $e(\overline{\mathcal{R}}^{\mathrm{red}}_{r,2})$. We stratify it as

$$\overline{\mathcal{R}}^{\mathrm{red}}_{r,2} = \bar{R}_0 \cup \bar{R}_1 \cup \bar{R}_2 \cup \bar{R}_3,$$

where:

- $\bar{R}_0$ consists of one point $(A_1, \dots, A_r) = (\mathrm{Id}, \dots, \mathrm{Id})$. So $e(R_0) = 1$.
- $\bar{R}_1$ consists of

$$\rho \sim \left( \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & 0 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \lambda_r & 0 \\ 0 & \lambda_r^{-1} \end{pmatrix} \right),$$

where the eigenvalues are determined by $(\lambda_1^2, \dots, \lambda_r^2) \neq (1, \dots, 1)$. This space is parametrized by $\big(\mathrm{PGL}(2,\mathbb{C})/D \times ((\mathbb{C}^*)^r - \{(1,\dots,1)\})\big)/\mathbb{Z}_2$, where $D$ is the space of diagonal matrices. Using that $e(\mathrm{PGL}(2,\mathbb{C})/D)^+ = q^2$, $e(\mathrm{PGL}(2,\mathbb{C})/D)^- = q$, and $e(B)^+ = \frac{1}{2}((q+1)^r + (q-1)^r) - 1$, $e(B)^- = \frac{1}{2}((q-1)^r - (q+1)^r)$, for $B = ((\mathbb{C}^*)^r - \{(1,\dots,1)\})$, we have

$$\begin{aligned} e(\bar{R}_1) &= e(\mathrm{PGL}(2,\mathbb{C})/D)^+ e(B)^+ + e(\mathrm{PGL}(2,\mathbb{C})/D)^- e(B)^- \\ &= q^2 \tfrac{1}{2}((q+1)^r + (q-1)^r - 1) + q\tfrac{1}{2}((q-1)^r - (q+1)^r) \\ &= \tfrac{1}{2}(q^2-q)(q+1)^r + \tfrac{1}{2}(q^2+q)(q-1)^r - q^2. \end{aligned}$$

- $\bar{R}_2$ consists of

$$\rho \sim \left( \begin{pmatrix} 1 & a_1 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & a_2 \\ 0 & 1 \end{pmatrix}, \dots, \begin{pmatrix} 1 & a_r \\ 0 & 1 \end{pmatrix} \right),$$

where $(a_1, \dots, a_r) \in \mathbb{C}^r - \{0\}$. Then

$$e(\bar{R}_2) = e(R_2)/2^r = (q^r - 1)(q+1).$$

- $\bar{R}_3$ consists of

$$\rho \sim \left( \begin{pmatrix} \lambda_1 & b_1 \\ 0 & \lambda_1^{-1} \end{pmatrix}, \begin{pmatrix} \lambda_2 & b_2 \\ 0 & \lambda_2^{-1} \end{pmatrix}, \dots, \begin{pmatrix} \lambda_r & b_r \\ 0 & \lambda_r^{-1} \end{pmatrix} \right),$$

where $\lambda_i \in \mathbb{C}^*$, $(\lambda_1^2, \dots, \lambda_r^2) \neq (1, \dots, 1)$. Here

$$(b_1, \dots, b_r) \in \mathbb{C}^r - \langle(\lambda_1 - \lambda_1^{-1}, \dots, \lambda_r - \lambda_r^{-1})\rangle.$$

There is a fibration $H_3 \to B_3 \times \mathrm{PGL}(2, \mathbb{C}) \to \bar{R}_3$ where $B_3$ parametrizes $(\lambda_1^2, \dots, \lambda_r^2)$ and $(b_1, \dots, b_r)$, and $H_3 = \left\{\left(\begin{smallmatrix} a & b \\ 0 & a^{-1}\end{smallmatrix}\right)\right\} \cong \mathbb{C}^* \times \mathbb{C}$. Then

$$\begin{aligned} e(\bar{R}_3) &= (q^r - q)((q-1)^r - 1)e(\mathrm{PGL}(2, \mathbb{C}))/q(q-1) \\ &= \frac{q^{r-1} - 1}{q-1}((q-1)^r - 1)(q^3 - q). \end{aligned}$$

Now we add all subsets together to obtain:

$$\begin{aligned} e(\bar{\mathcal{R}}_{r,2}^{\mathrm{red}}) &= e(\bar{R}_0) + e(\bar{R}_1) + e(\bar{R}_2) + e(\bar{R}_3) \\ &= \tfrac{1}{2}(q^2 - q)(q+1)^r + \tfrac{1}{2}(q^2 + q)(q-1)^r + (q^{r-1} - 1)(q-1)^{r-1}(q^3 - q) \\ &= e(\mathcal{R}_{r,2}^{\mathrm{red}}). \end{aligned}$$

***The irreducible locus.*** Clearly, as $e(\mathrm{SL}(2, \mathbb{C})) = q^3 - q = e(\mathrm{PGL}(2, \mathbb{C}))$ and $e(\bar{\mathcal{R}}_{r,2}^{\mathrm{red}}) = e(\mathcal{R}_{r,2}^{\mathrm{red}})$, we have that $e(\bar{\mathcal{R}}_{r,2}^{\mathrm{irr}}) = e(\mathcal{R}_{r,2}^{\mathrm{irr}})$. Therefore $e(\bar{\mathcal{M}}_{r,2}^{\mathrm{irr}}) = e(\mathcal{M}_{r,2}^{\mathrm{irr}})$. Finally, since $e(\bar{\mathcal{M}}_{r,2}^{\mathrm{red}}) = e(\mathcal{M}_{r,2}^{\mathrm{red}})$, we have that

$$\begin{aligned} e(\bar{\mathcal{M}}_{r,2}) &= e(\mathcal{M}_{r,2}) \\ &= (q^3 - q)^{r-1} + \tfrac{1}{2}q(q+1)^{r-1} + \tfrac{1}{2}q(q-1)^{r-1} - q^{r-1}(q-1)^{r-1}. \end{aligned}$$

## 5. $E$-polynomial of the SL(3, $\mathbb{C}$)-character variety for $F_1$

Having given a new geometric derivation of the $E$-polynomial for $\mathcal{M}_{r,2}$ and $\bar{\mathcal{M}}_{r,2}$, in the next sections we work out the $E$-polynomial of $\mathcal{M}_{r,3}$ and $\bar{\mathcal{M}}_{r,3}$ in a similar fashion.

However, in this section we first address the $r = 1$ case. Although it is easy to see that $\mathcal{M}_{r,n} \cong \mathbb{C}^{n-1}$ via the coefficients of the characteristic polynomial, and hence $e(\mathcal{M}_{r,n}) = q^{n-1}$, this case will motivate the more complicated stratification, and the use of the *equivariant* $E$-polynomial, needed to compute the general $E$-polynomials for $\mathcal{M}_{r,3}$ and $\bar{\mathcal{M}}_{r,3}$ when $r \geq 2$.

We begin with the $E$-polynomials for GL(3, $\mathbb{C}$), SL(3, $\mathbb{C}$), and PGL(3, $\mathbb{C}$). Like in the previous sections, we then stratify Hom($F_1$, SL(3, $\mathbb{C}$)) by orbit type and compute the $E$-polynomial for each strata.

**Lemma 4.** $e(\mathrm{SL}(3, \mathbb{C})) = e(\mathrm{PGL}(3, \mathbb{C})) = (q^3 - 1)(q^3 - q)q^2 = q^8 - q^6 - q^5 + q^3$.

*Proof.* Consider $\mathbb{C}^n$, and let $V_k$ be the Stiefel manifold of $k$ linearly independent vectors in $\mathbb{C}^n$. Then, there is a (Zariski locally trivial) fibration $\mathbb{C}^n - \mathbb{C}^{k-1} \to V_k \to V_{k-1}$. Therefore $e(V_k) = \prod_{i=0}^{k-1}(q^n - q^i)$. So $e(\mathrm{GL}(n, \mathbb{C})) = e(V_n) = \prod_{i=0}^{n-1}(q^n - q^i)$.

Now there is a (Zariski locally trivial) fibration $\mathbb{C}^* \to \mathrm{GL}(n,\mathbb{C}) \to \mathrm{PGL}(n,\mathbb{C})$, hence $e(\mathrm{PGL}(n,\mathbb{C})) = e(\mathrm{GL}(n,\mathbb{C}))/(q-1) = q^{n-1}\prod_{i=0}^{n-2}(q^n - q^i)$.

For $\mathrm{SL}(n,\mathbb{C})$, the choice of $(v_1,\dots,v_{n-1}) \in V_{n-1}$ determines an affine hyperplane

$$\{v \in \mathbb{C}^n \mid \det(v_1,\dots,v_{n-1},v) = 1\}.$$

This gives a (Zariski locally trivial) affine bundle $\mathbb{C}^{n-1} \to \mathrm{SL}(n,\mathbb{C}) \to V_{n-1}$, and hence $e(\mathrm{SL}(n,\mathbb{C})) = q^{n-1}\prod_{i=0}^{n-2}(q^n - q^i)$. □

Now let us consider the representations of $F_1$ to $\mathrm{SL}(3,\mathbb{C})$. This is equivalent to studying the conjugation action of $\mathrm{PGL}(3,\mathbb{C})$ on $X := \mathrm{SL}(3,\mathbb{C})$. For this action, there are 6 strata types. In the following list, we write down all 6 strata, but include the computation of their $E$-polynomials for only the first 5. This is because the computation is apparent from the geometric description of each stratum alone in those cases.

- $X_0$ is formed by matrices of type

$$\begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 0 \\ 0 & 0 & \xi \end{pmatrix}.$$

Here $\xi^3 = 1$, so $X_0$ consists of 3 points and $e(X_0) = 3$.

- $X_1$ is formed by matrices of type

$$\begin{pmatrix} \xi & 0 & 0 \\ 0 & \xi & 1 \\ 0 & 0 & \xi \end{pmatrix}.$$

Here $\xi^3 = 1$, so $\xi$ admits 3 values. The stabilizer of this matrix is

$$U_1 = \left\{ \begin{pmatrix} \mu^{-2} & 0 & b \\ a & \mu & c \\ 0 & 0 & \mu \end{pmatrix} \right\} \cong \mathbb{C}^* \times \mathbb{C}^3.$$

So

$$\begin{aligned} e(X_1) &= 3e(\mathrm{PGL}(3,\mathbb{C})/U_1) \\ &= 3(q^3-1)(q^3-q)q^2/q^3(q-1) = 3q^4 + 3q^3 - 3q - 3. \end{aligned}$$

- $X_2$ is formed by matrices of type

$$\begin{pmatrix} \xi & 1 & 0 \\ 0 & \xi & 1 \\ 0 & 0 & \xi \end{pmatrix}.$$

Here $\xi^3 = 1$, so $\xi$ admits 3 values. The stabilizer of this matrix is

$$U_2 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix} \right\} \cong \mathbb{C}^2.$$

So

$$\begin{aligned} e(X_1) &= 3e(\mathrm{PGL}(3, \mathbb{C})/U_2) \\ &= 3(q^3 - 1)(q^3 - q)q^2/q^2 = 3q^6 - 3q^4 - 3q^3 + 3q. \end{aligned}$$

- $X_3$ is formed by matrices of type

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

where $\lambda \in \mathbb{C}^* - \{\xi | \xi^3 = 1\}$. The stabilizer of this matrix is

$$U_3 = \left\{ \begin{pmatrix} A & 0 \\ 0 & (\det A)^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}(2, \mathbb{C}) \right\} \cong \mathrm{GL}(2, \mathbb{C}).$$

So

$$\begin{aligned} e(X_3) &= (q - 4)e(\mathrm{PGL}(3, \mathbb{C})/U_3) \\ &= (q - 4)(q^3 - 1)(q^3 - q)q^2/(q^2 - 1)(q^2 - q) = q^5 - 3q^4 - 3q^3 - 4q^2. \end{aligned}$$

- $X_4$ is formed by matrices of type

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda^{-2} \end{pmatrix},$$

where $\lambda \in \mathbb{C}^* - \{\xi \mid \xi^3 = 1\}$. The stabilizer of this matrix is

$$U_4 = \left\{ \begin{pmatrix} \mu & b & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \mu^{-2} \end{pmatrix} \right\} \cong \mathbb{C}^* \times \mathbb{C}.$$

So

$$\begin{aligned} e(X_4) &= (q - 4)e(\mathrm{PGL}(3, \mathbb{C})/U_4) \\ &= (q - 4)(q^3 - 1)(q^3 - q)q^2/q(q - 1) \\ &= q^7 - 3q^6 - 4q^5 - q^4 + 3q^3 + 4q^2. \end{aligned}$$

- $X_5$ is formed by matrices of type

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \gamma \end{pmatrix},$$

where $\lambda, \mu, \gamma \in \mathbb{C}^*$ are different and $\lambda\mu\gamma = 1$. The stabilizer is isomorphic to the diagonal matrices $D \cong \mathbb{C}^* \times \mathbb{C}^*$. The parameter space is

$$B = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda \neq \mu^{-2}, \mu \neq \lambda^{-2}, \mu \neq \lambda\}.$$

The map $\mathrm{PGL}(3, \mathbb{C})/D \times B \to X_5$ is a 6:1 cover. Moreover,

$$X_5 \cong (\mathrm{PGL}(3, \mathbb{C})/D \times B)/\Sigma_3,$$

where the symmetric group $\Sigma_3$ acts on $\mathrm{PGL}(3, \mathbb{C})$ by permuting the columns and acts on the triple $(\lambda, \mu, \gamma = \lambda^{-1}\mu^{-1})$ by permuting the entries.

We now compute $e(X_5)$ using the *equivariant* $E$-polynomial. Consider the finite group $F = \Sigma_3$. The representation ring $R(F)$ is generated by three irreducible representations:

- $T$ is the (one-dimensional) trivial representation.
- $S$ is the sign representation. This is one-dimensional and given by the sign map $\Sigma_3 \to \{\pm 1\} \subset \mathrm{GL}(1, \mathbb{C})$.
- $V$ is the two-dimensional representation given as follows. Take $St = \mathbb{C}^3$ the standard 3-dimensional representation. This is generated by $e_1, e_2, e_3$ and $\Sigma_3$ acts by permuting the elements of the basis. Then $T = \langle e_1 + e_2 + e_3\rangle$ and we can decompose $St = T \oplus V$.

The representation ring $R(\Sigma_3)$ has a multiplicative structure given by: $T \otimes T = T$, $T \otimes S = S$, $T \otimes V = V$, $S \otimes S = T$, $S \otimes V = V$, $V \otimes V = T \oplus S \oplus V$.

**Lemma 5.**

$$e_{\Sigma_3}(B) = (q^2 - q + 1)T + S - 2(q - 2)V.$$
$$e_{\Sigma_3}(\mathrm{PGL}(3, \mathbb{C})/D) = q^6 T + q^3 S + (q^5 + q^4)V.$$

*Proof.* Write $e_{\Sigma_3}(X) = aT + bS + cV$, for a quasiprojective variety $X$ with a $\Sigma_3$-action. Then $a = e(X/\Sigma_3)$. If we consider the cycle $(1, 2)$ and the subgroup $H = \langle(1, 2)\rangle$, there is a map $R(F) \to R(H)$ which sends $T \mapsto T$, $S \mapsto N$ and $V \mapsto T + N$. Then $e_H(X) = aT + bN + c(T + N) = (a+c)T + (b+c)N$. Therefore, $a + c = e(X/H)$. As $e(X) = a + b + 2c$, we can compute $a, b, c$ by knowing these $E$-polynomials.

For $B = \{(\lambda, \mu) \in (\mathbb{C}^*)^2 \mid \lambda \neq \mu^{-2}, \mu \neq \lambda^{-2}, \mu \neq \lambda\}$, the three curves $\lambda = \mu^{-2}$, $\mu = \lambda^{-2}$, $\mu = \lambda$ intersect at the three points $\{(\xi, \xi) \mid \xi^3 = 1\}$. Hence $e(B) = (q - 1)^2 - 3(q - 4) - 3 = q^2 - 5q + 10$.

Now $\Sigma_3$ acts on $(\lambda, \mu, \gamma)$ and the quotient space is parametrized by $s = \lambda+\mu+\gamma$, $t = \lambda\mu + \lambda\gamma + \mu\gamma$ and $p = \lambda\mu\gamma = 1$, that is, by $(s, t) \in \mathbb{C}^2$. We have to remove the cases $s = \lambda + \lambda^{-2} + \lambda$, $t = \lambda^{-1} + \lambda^2 + \lambda^{-1}$. This defines a rational curve in $\mathbb{C}^2$. It has two points at infinity. The map $\lambda \mapsto (2\lambda + \lambda^{-2}, 2\lambda^{-1} + \lambda^2)$ is an embedding. Therefore $e(B/\Sigma_3) = q^2 - (q - 1) = q^2 - q + 1$.

The action by $H$ permutes $(\lambda, \mu)$, hence the quotient is parametrized by $s' = \lambda+\mu$, $p' = \lambda\mu \neq 0$. We have to remove the cases $s' = \lambda + \lambda^{-2}$, $p' = \lambda^{-1}$, that is, $s' = (p')^{-1} + (p')^2$; and $s' = 2\lambda$, $p' = \lambda^2$, i.e., $4p' = (s')^2$. They intersect at three points. Then $e(B/H) = q(q-1) - 2(q-1) + 3 = q^2 - 3q + 5$.

Thus

$$e_{\Sigma_3}(B) = (q^2 - q + 1)T + S - 2(q-2)V.$$

For $C = \mathrm{PGL}(3, \mathbb{C})/D$, the space $C$ consists of points in $(\mathbb{P}^2)^3 - \Delta$, where $\Delta$ is the diagonal (triples of coplanar points). Certainly, a matrix in $\mathrm{GL}(3, \mathbb{C})$ can be written as $(v_1, v_2, v_3)$, where $v_1, v_2, v_3$ are linearly independent vectors. Taking a quotient by the diagonal matrices corresponds to the vectors up to a scalar: $[v_1], [v_2], [v_3]$. Therefore, $e(C) = (q^3-1)(q^3-q)q^2/(q-1)^2 = q^6+2q^5+2q^4+q^3$.

The group $\Sigma_3$ acts by permuting the vectors, so $C/\Sigma_3 = \mathrm{Sym}^3\,\mathbb{P}^2 - \overline{\Delta}$, where $\overline{\Delta}$ consists of linearly dependent triples $([v_1], [v_2], [v_3])$. If they are equal, the set has $e(\mathbb{P}^2) = q^2+q+1$. If they are collinear, there is a fibration with fiber $\mathrm{Sym}^3(\mathbb{P}^1) - \Delta$ and base $(\mathbb{P}^2)^\vee$. This has $E$-polynomial $(1+q+q^2+q^3-1-q)(1+q+q^2) = q^5 + 2q^4 + 2q^3 + q^2$. Also $e(\mathrm{Sym}^3\,\mathbb{P}^2) = q^6 + q^5 + 2q^4 + 2q^3 + 2q^2 + q + 1$. Therefore

$$e(C/\Sigma_3) = q^6+q^5+2q^4+2q^3+2q^2+q+1-(q^5+2q^4+2q^3+q^2+q^2+q+1) = q^6.$$

The group $H$ acts by permuting the first two vectors, so $C/H = \mathrm{Sym}^2\,\mathbb{P}^2 \times \mathbb{P}^2 - \overline{\Delta}'$, where $\overline{\Delta}'$ consists of linearly dependent triples $([v_1], [v_2], [v_3])$. If $[v_1] = [v_2]$, we have the $E$-polynomial $(q^2+q+1)(q^2+q+1) = q^4+2q^3+3q^2+2q+1$. If $[v_1] \neq [v_2]$, they lie in $\mathrm{Sym}^2\,\mathbb{P}^2 - \Delta$ and we have the $E$-polynomial

$$\big(q^4+q^3+2q^2+q+1-(q^2+q+1)\big)(q+1) = q^5+2q^4+2q^3+q^2.$$

Also $e(\mathrm{Sym}^2\,\mathbb{P}^2 \times \mathbb{P}^2) = (q^4+q^3+2q^2+q+1)(q^2+q+1)$, so

$$\begin{aligned} e(C/H) &= q^6+2q^5+4q^4+4q^3+4q^2+2q+1 \\ &\quad - (q^5+2q^4+2q^3+q^2+q^4+2q^3+3q^2+2q+1) \\ &= q^6+q^5+q^4. \end{aligned}$$

This produces the polynomial

$$e_{\Sigma_3}(C) = q^6T + q^3S + (q^5+q^4)V. \qquad \square$$

**Remark 6.** If we consider $B' = \{(\lambda, \mu) \in (\mathbb{C}^*)^2\}$, then the proof of Lemma 5 says that $e_{\Sigma_3}(B') = q^2T + S - qV$.

Now suppose $e_{\Sigma_3}(X) = aT + bS + cV$ and $e_{\Sigma_3}(X') = a'T + b'S + c'V$. Then

$$e_{\Sigma_3}(X \times X') = (aa'+bb'+cc')T + (ab'+ba'+cc')S + (ac'+ca'+bc'+cb'+cc')V,$$

and hence

$$e((X \times X')/\Sigma_3) = aa' + bb' + cc' \,. \tag{3}$$

We finally obtain the $E$-polynomial for the sixth strata $X_5$:

$$\begin{aligned} e(X_5) &= e((B \times C)/\Sigma_3) \\ &= (q^2 - q + 1)q^6 + q^3 - 2(q-2)(q^5 + q^4) \\ &= q^8 - q^7 - q^6 + 2q^5 + 4q^4 + q^3 \,. \end{aligned}$$

Now we add the strata together:

$$e(X_0)+e(X_1)+e(X_2)+e(X_3)+e(X_4)+e(X_5) = q^8 - q^6 - q^5 + q^3 = e(\mathrm{SL}(3,\mathbb{C})),$$

as expected.

**Remark 7.** All elements of $X = \mathrm{SL}(3,\mathbb{C})$ are reducible. The semisimple ones are given by diagonal matrices with entries $\lambda, \mu, \gamma$ with $\lambda\mu\gamma = 1$. So they are parametrized by $s = \lambda+\mu+\gamma$, $t = \lambda\mu+\lambda\gamma+\mu\gamma = \lambda^{-1}+\gamma^{-1}+\mu^{-1}$, for $(s,t) \in \mathbb{C}^2$. Hence $e(\mathcal{M}_{1,3}) = q^2$, as noted at the beginning of this section.

## 6. $E$-polynomials of character varieties for $F_r$, $r > 1$, and SL(3, $\mathbb{C}$)

In this section we prove (most of) our main theorem (Theorem 1) by computing the $E$-polynomial for $\mathcal{M}_{r,3}$; the rest of Theorem 1 is proved in Section 7. The computation is similar to the computation in Section 3 except the stratification is more complicated and the *equivariant* $E$-polynomial is needed, as was demonstrated in Section 5.

Indeed, we want to study the space of representations

$$\begin{aligned} \mathcal{R}_{r,3} &= \mathrm{Hom}(F_r, \mathrm{SL}(3,\mathbb{C})) = \{\rho : F_r \to \mathrm{SL}(3,\mathbb{C})\} \\ &= \{(A_1, \dots, A_r) \mid A_i \in \mathrm{SL}(3,\mathbb{C})\} = \mathrm{SL}(3,\mathbb{C})^r \end{aligned}$$

and the corresponding character variety

$$\mathcal{M}_{r,3} = \mathrm{Hom}(F_r, \mathrm{SL}(3,\mathbb{C}))/\!\!/ \mathrm{PGL}(3,\mathbb{C}).$$

Much of the algebraic structure of $\mathcal{M}_{r,3}$ has been worked out in [Lawton 2007; 2008; 2010].

Let us start by computing the $E$-polynomial of the space of reducible representations $\mathcal{R}^{\mathrm{red}}_{r,3} \subset \mathrm{Hom}(F_r, \mathrm{SL}(3,\mathbb{C}))$.

We now list the stratification and the computation of the $E$-polynomial for each stratum for $\mathcal{R}^{\mathrm{red}}_{r,3}$.

(i) $R_0 = R_{01} \cup R_{02}$. $R_{01}$ is formed by representations $\rho = (A_1, \dots, A_r)$ which have a common eigenvector and such that the quotient representation is irreducible, that is,

$$A_i = \begin{pmatrix} \lambda_i^{-2} & b_i & c_i \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\lambda_i B_i$}} \\ 0 & & \end{pmatrix},$$

where $(B_1, \dots, B_r) \in \mathcal{R}_{2,r}^{\mathrm{irr}}$. Let $B_{01}$ be the space of representations of such form with respect to the standard basis. The stabilizer of $B_{01}$ (i.e., the set $H_{01} \subset \mathrm{PGL}(3, \mathbb{C})$ sending $B_{01}$ to itself) is

$$H_{01} = \left\{ \begin{pmatrix} (\det B)^{-1} & a & b \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$B$}} \\ 0 & & \end{pmatrix} \right\} \cong \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2.$$

This means that there is a fibration $H_{01} \to B_{01} \times \mathrm{PGL}(3, \mathbb{C}) \to R_{01}$. Hence

$$e(R_{01}) = (q-1)^r q^{2r} e(\mathcal{R}_{2,r}^{\mathrm{irr}}) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{q^2 e(\mathrm{GL}(2, \mathbb{C}))}.$$

$R_{02}$ is formed by representations $\rho = (A_1, \dots, A_r)$ which have a common two-dimensional space and upon which it acts irreducibly, that is,

$$A_i = \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$\lambda_i B_i$}} & 0 \\ & & 0 \\ b_i & c_i & \lambda_i^{-2} \end{pmatrix},$$

where $(B_1, \dots, B_r) \in \mathcal{R}_{2,r}^{\mathrm{irr}}$. The stabilizer is now

$$H_{02} = \left\{ \begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{$B$}} & 0 \\ & & 0 \\ a & b & (\det B)^{-1} \end{pmatrix} \right\} \cong \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2.$$

Hence

$$e(R_{02}) = (q-1)^r q^{2r} e(\mathcal{R}_{2,r}^{\mathrm{irr}}) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{q^2 e(\mathrm{GL}(2, \mathbb{C}))}.$$

The intersection $R_{01} \cap R_{02}$ consists of those representations with $b_i = c_i = 0$, which have stabilizer $\mathrm{GL}(2, \mathbb{C})$, hence

$$e(R_{01} \cap R_{02}) = (q-1)^r e(\mathcal{R}_{2,r}^{\mathrm{irr}}) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{e(\mathrm{GL}(2, \mathbb{C}))}.$$

Finally $e(R_0) = e(R_{01}) + e(R_{02}) - e(R_{01} \cap R_{02}) = 2e(R_{01}) - e(R_{01} \cap R_{02})$. Note that the remaining representations have a full invariant flag.

(ii) $R_1$ is formed by representations $\rho = (A_1, \dots, A_r)$ such that the eigenvalues of all $A_i$ are equal (and hence cubic roots of unity). This consists of the following substrata:

- $R_{11}$ consisting of matrices

$$A_i = \begin{pmatrix} \xi_i & 0 & 0 \\ 0 & \xi_i & 0 \\ 0 & 0 & \xi_i \end{pmatrix},$$

where $\xi_i^3 = 1$. So $e(R_{11}) = 3^r$.

- $R_{12}$ formed by matrices of type

$$A_i = \begin{pmatrix} \xi_i & 0 & 0 \\ 0 & \xi_i & a_i \\ 0 & 0 & \xi_i \end{pmatrix},$$

with $\xi_i^3 = 1$ and $(a_1, \dots, a_r) \neq 0$. Then the stabilizer is

$$H_{12} = \left\{ \begin{pmatrix} \mu^{-1}\gamma^{-1} & 0 & b \\ a & \mu & c \\ 0 & 0 & \gamma \end{pmatrix} \right\} \cong (\mathbb{C}^*)^2 \times \mathbb{C}^3.$$

So

$$e(R_{12}) = 3^r (q^r - 1) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q-1)^2 q^3}.$$

- $R_{13}$ formed by matrices of type

$$A_i = \begin{pmatrix} \xi_i & 0 & a_i \\ 0 & \xi_i & b_i \\ 0 & 0 & \xi_i \end{pmatrix},$$

with $\xi_i^3 = 1$ and $(a_1, \dots, a_r)$, $(b_1, \dots, b_r)$ linearly independent. Note that when they are linearly dependent, one may arrange a basis so that it belongs to the stratum $R_{12}$. Then the stabilizer is

$$H_{13} = \left\{ \begin{pmatrix} A & & b \\ & & c \\ 0 & 0 & (\det A)^{-1} \end{pmatrix} \right\} \cong \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2.$$

Hence

$$e(R_{13}) = 3^r (q^r - 1)(q^r - q) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q^2-1)(q^2-q)q^2}.$$

- $R_{14}$ formed by matrices of type

$$A_i = \begin{pmatrix} \xi_i & a_i & b_i \\ 0 & \xi_i & 0 \\ 0 & 0 & \xi_i \end{pmatrix},$$

with $\xi_i^3 = 1$ and $(a_1, \ldots, a_r)$, $(b_1, \ldots, b_r)$ linearly independent. Note again that when they are linearly dependent, one may arrange a basis so that it belongs to the stratum $R_{12}$. Then the stabilizer is

$$H_{14} = \left\{ \begin{pmatrix} (\det A)^{-1} & b \; c \\ 0 & \multirow{2}{*}{$A$} \\ 0 & \end{pmatrix} \right\} \cong \mathrm{GL}(2, \mathbb{C}) \times \mathbb{C}^2.$$

Hence

$$e(R_{14}) = 3^r (q^r - 1)(q^r - q) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q^2 - 1)(q^2 - q)q^2}.$$

- $R_{15}$ formed by matrices of type

$$A_i = \begin{pmatrix} \xi_i & a_i & b_i \\ 0 & \xi_i & c_i \\ 0 & 0 & \xi_i \end{pmatrix},$$

with $\xi_i^3 = 1$ and $(a_1, \ldots, a_r)$, $(c_1, \ldots, c_r)$ are both nonzero (if one of them is zero, then we are back in the case $R_{13}$). Then the stabilizer is

$$H_{15} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right\}.$$

Hence

$$e(R_{14}) = 3^r (q^r - 1)^2 q^r \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q - 1)^2 q^3}.$$

All together, we have

$$e(R_1) = 3^r \big(1 + (1 + q + q^2)(q^{3r+1} + q^{3r} - 2q^{2r+1} + q - 1)\big).$$

(iii) $R_2$ is formed by matrices with eigenvalues $(\lambda_i, \lambda_i, \mu_i)$. Let $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$, with $\boldsymbol{\lambda} - \boldsymbol{\mu} \neq \mathbf{0}$. Note that $\mu_i = \lambda_i^{-2}$, so the parameter space has $E$-polynomial $(q - 1)^r - 3^r$. We have the following substrata:

- $R_{21}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \mu_i \end{pmatrix}.$$

The stabilizer is $P(\mathrm{GL}(2,\mathbb{C})\times\mathbb{C}^*)\cong \mathrm{GL}(2,\mathbb{C})$, so

$$e(R_{21})=((q-1)^r-3^r)\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q^2-1)(q^2-q)}\,.$$

- $R_{22}$ consists of representations of type

$$A_i=\begin{pmatrix}\lambda_i & a_i & b_i\\ 0 & \lambda_i & c_i\\ 0 & 0 & \mu_i\end{pmatrix},$$

with $\boldsymbol{a}\neq\boldsymbol{0}$, $\boldsymbol{b}=(b_1,\dots,b_r)$, $\boldsymbol{c}=(c_1,\dots,c_r)\in\mathbb{C}^r$. The stabilizer is

$$H_{22}=\left\{\begin{pmatrix}a & b & c\\ 0 & d & e\\ 0 & 0 & f\end{pmatrix}\right\}.$$

Hence

$$e(R_{22})=((q-1)^r-3^r)(q^r-1)q^{2r}\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2q^3}\,.$$

- $R_{23}$ consists of representations of type

$$A_i=\begin{pmatrix}\lambda_i & 0 & 0\\ 0 & \lambda_i & a_i\\ 0 & 0 & \mu_i\end{pmatrix},$$

with $\boldsymbol{a}\notin\langle\boldsymbol{\lambda}-\boldsymbol{\mu}\rangle$. If $\boldsymbol{a}=x(\boldsymbol{\lambda}-\boldsymbol{\mu})$, $x\in\mathbb{C}$, then we can arrange a basis so that this belongs to the stratum $R_{21}$. The stabilizer is

$$H_{23}=\left\{\begin{pmatrix}a & 0 & 0\\ b & c & d\\ 0 & 0 & e\end{pmatrix}\right\}.$$

So

$$e(R_{23})=((q-1)^r-3^r)(q^r-q)\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2q^2}\,.$$

- $R_{24}$ consists of representations of type

$$A_i=\begin{pmatrix}\lambda_i & 0 & a_i\\ 0 & \lambda_i & b_i\\ 0 & 0 & \mu_i\end{pmatrix},$$

with $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{\lambda}-\boldsymbol{\mu}$ linearly independent (if they where linearly dependent, one can arrange a basis so that we go back to case $R_{23}$). The stabilizer is

$$H_{24}=\left\{\begin{pmatrix} \multicolumn{2}{c}{\multirow{2}{*}{A}} & a \\ & & b \\ 0 & 0 & (\det A)^{-1} \end{pmatrix}\right\}.$$

Hence

$$e(R_{24})=((q-1)^r-3^r)(q^r-q)(q^r-q^2)\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q^2-1)(q^2-q)q^2}\,.$$

- $R_{25}$ consists of representations of type

$$A_i=\begin{pmatrix} \mu_i & a_i & b_i \\ 0 & \lambda_i & c_i \\ 0 & 0 & \lambda_i \end{pmatrix},$$

with $\boldsymbol{a}\notin\langle\boldsymbol{\lambda}-\boldsymbol{\mu}\rangle$, $\boldsymbol{c}\neq\boldsymbol{0}$. The stabilizer is

$$H_{25}=\left\{\begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix}\right\}.$$

Hence

$$e(R_{25})=((q-1)^r-3^r)(q^r-1)(q^r-q)q^r\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2q^3}\,.$$

- $R_{26}$ consists of representations of type

$$A_i=\begin{pmatrix} \mu_i & 0 & b_i \\ 0 & \lambda_i & c_i \\ 0 & 0 & \lambda_i \end{pmatrix},$$

with $\boldsymbol{b}\notin\langle\boldsymbol{\lambda}-\boldsymbol{\mu}\rangle$, $\boldsymbol{c}\neq\boldsymbol{0}$. (If $\boldsymbol{b}$ is a multiple of $\boldsymbol{\lambda}-\boldsymbol{\mu}$, then we can arrange with a suitable basis that $\boldsymbol{b}=0$, and this belongs to the substrata $R_{22}$). The stabilizer is

$$H_{26}=\left\{\begin{pmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix}\right\}.$$

Hence

$$e(R_{26})=((q-1)^r-3^r)(q^r-1)(q^r-q)\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2q^2}\,.$$

- $R_{27}$ consists of representations of type

$$A_i = \begin{pmatrix} \mu_i & a_i & 0 \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{pmatrix},$$

with $\boldsymbol{a} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\mu} \rangle$. The stabilizer is

$$H_{27} = \left\{ \begin{pmatrix} a & b & 0 \\ 0 & c & 0 \\ 0 & d & e \end{pmatrix} \right\}.$$

Hence

$$e(R_{27}) = ((q-1)^r - 3^r)(q^r - q) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q-1)^2 q^2}.$$

- $R_{28}$ consists of representations of type

$$A_i = \begin{pmatrix} \mu_i & a_i & b_i \\ 0 & \lambda_i & 0 \\ 0 & 0 & \lambda_i \end{pmatrix},$$

with $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{\lambda} - \boldsymbol{\mu}$ linearly independent (otherwise we can reduce to the case $R_{27}$). The stabilizer is

$$H_{28} = \left\{ \begin{pmatrix} (\det A)^{-1} & b \quad c \\ 0 & \multirow{2}{*}{$A$} \\ 0 & \end{pmatrix} \right\}.$$

Hence

$$e(R_{28}) = ((q-1)^r - 3^r)(q^r - q)(q^r - q^2) \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q^2 - q)(q^2 - 1) q^2}.$$

- $R_{29}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & a_i & b_i \\ 0 & \mu_i & c_i \\ 0 & 0 & \lambda_i \end{pmatrix},$$

with $\boldsymbol{a}, \boldsymbol{c} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\mu} \rangle$. The stabilizer is

$$H_{29} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right\}.$$

Hence

$$e(R_{29}) = ((q-1)^r - 3^r)(q^r - q)^2 q^r \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q-1)^2 q^3}.$$

All together, we have

$$e(R_2) = ((q-1)^r - 3^r)(q^2+q+1)(3q^{3r+1} + 3q^{3r} - 2q^{2r+2} - 4q^{2r+1} + q^3).$$

(iv) $R_3$ is formed by matrices with eigenvalues $(\lambda_i, \mu_i, \gamma_i)$ where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, and $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ are distinct. Note that $\lambda_i \mu_i \gamma_i = 1$ for all $1 \le i \le r$. The base $B_r$ parametrizing $(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma})$ has $E$-polynomial $e(B_r) = (q-1)^{2r} - 3(q-1)^r + 2 \cdot 3^r$.

- $R_{31}$ consists of representations of type
$$A_i = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \mu_i & 0 \\ 0 & 0 & \gamma_i \end{pmatrix}.$$
Then the stabilizer is $D \times \Sigma_3$, where $D$ is the diagonal matrices. So we have to compute the $E$-polynomial of the quotient $R_{31} = (\mathrm{PGL}(3, \mathbb{C})/D \times B_r)/\Sigma_3$. We start by computing $e_{\Sigma_3}(B_r)$. Let $B'_r = \{(\boldsymbol{\lambda}, \boldsymbol{\mu}, \boldsymbol{\gamma}) \in (\mathbb{C}^*)^{3r} \mid \boldsymbol{\lambda}\boldsymbol{\mu}\boldsymbol{\gamma} = (1, \dots, 1)\}$. This is $B'_r = (B')^r$, in the notation of Remark 6. Then
$$e_{\Sigma_3}(B'_r) = e_{\Sigma_3}(B')^r = (q^2 T + S - qV)^r.$$
Using the properties $T \otimes T = T$, $T \otimes S = S$, $T \otimes V = V$, $S \otimes S = T$, $S \otimes V = V$, $V \otimes V = T \oplus S \oplus V$, it is easy to see that $V^b = a_b V + a_{b-1}(T+S)$, where $a_b = a_{b-1} + 2a_{b-2}$, with $a_0 = 0$, $a_1 = 1$. This recurrence solves as $a_b = (2^b - (-1)^b)/3$. Therefore

$$\begin{aligned}
e_{\Sigma_3}(B'_r) &= (q^2T + S - qV)^r \\
&= \sum \frac{r!}{(r-a-b)!a!b!} q^{2(r-a-b)} S^a (-q)^b V^b \\
&= \sum \frac{r!}{(r-a)!a!} q^{2(r-a)} S^a \\
&\quad + \sum_{b>0} \frac{r!}{(r-a-b)!a!b!} q^{2(r-a-b)} S^a (-q)^b V^b \\
&= \tfrac{1}{2}((q^2+1)^r + (q^2-1)^r)T + \tfrac{1}{2}((q^2+1)^r - (q^2-1)^r)S \\
&\quad + \sum_{b>0} \frac{r!}{(r-a-b)!a!b!} q^{2(r-a-b)} S^a (-q)^b \\
&\qquad \times \left(\tfrac{1}{3}(2^b - (-1)^b)V + \tfrac{1}{3}(2^{b-1} - (-1)^{b-1})(T+S)\right) \\
&= \tfrac{1}{2}\big((q^2+1)^r + (q^2-1)^r\big)T + \tfrac{1}{2}\big((q^2+1)^r - (q^2-1)^r\big)S \\
&\quad + \tfrac{1}{3}\big((q^2-2q+1)^r - (q^2+q+1)^r\big)V \\
&\quad + \tfrac{1}{3}\big(\tfrac{1}{2}(q^2-2q+1)^r + (q^2+q+1)^r - \tfrac{3}{2}(q^2+1)^r\big)(T+S)
\end{aligned} \tag{4}$$

$$
\begin{aligned}
&= \left(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\right)T \\
&\quad + \left(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\right)S \\
&\quad + \tfrac{1}{3}\left((q-1)^{2r} - (q^2+q+1)^r\right)V.
\end{aligned}
$$

Now we have to look at the part that we removed:

$$C_r = \{(\boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}^{-2}) \mid \boldsymbol{\lambda} \in (\mathbb{C}^*)^r\} \cup \{(\boldsymbol{\lambda}, \boldsymbol{\lambda}^{-2}, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in (\mathbb{C}^*)^r\} \cup \{(\boldsymbol{\lambda}^{-2}, \boldsymbol{\lambda}, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in (\mathbb{C}^*)^r\}.$$

Then $e(C_r) = 3(q-1)^r - 2\cdot 3^r$. The quotient $C_r/\Sigma_3 \cong (\mathbb{C}^*)^r$, so $e(C_r/\Sigma_3) = (q-1)^r$. And for $H = \langle (1,2) \rangle$ we have $C_r/H \cong \{(\boldsymbol{\lambda}, \boldsymbol{\lambda}, \boldsymbol{\lambda}^{-2}) \mid \boldsymbol{\lambda} \in (\mathbb{C}^*)^r\} \cup \{(\boldsymbol{\lambda}, \boldsymbol{\lambda}^{-2}, \boldsymbol{\lambda}) \mid \boldsymbol{\lambda} \in (\mathbb{C}^*)^r\}$, so $e(C_r/H) = 2(q-1)^r - 3^r$. Hence,

$$e_{\Sigma_3}(C_r) = (q-1)^r T + ((q-1)^r - 3^r)V.$$

For $B_r = B_r' - C_r$, we have

$$
\begin{aligned}
e_{\Sigma_3}(B_r) &= \left(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r\right)T \\
&\quad + \left(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\right)S \\
&\quad + \left(\tfrac{1}{3}(q-1)^{2r} - \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r + 3^r\right)V.
\end{aligned}
\tag{5}
$$

Hence Formula (3) and Lemma 5 imply

$$
\begin{aligned}
e(R_{31}) &= aa' + bb' + cc' \\
&= \left(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r\right)q^6 \\
&\quad + \left(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\right)q^3 \\
&\quad + \left(\tfrac{1}{3}(q-1)^{2r} - \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r + 3^r\right)(q^5+q^4).
\end{aligned}
$$

- $R_{32}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & 0 & 0 \\ 0 & \mu_i & a_i \\ 0 & 0 & \gamma_i \end{pmatrix},$$

with $\boldsymbol{a} \notin \langle \boldsymbol{\mu} - \boldsymbol{\gamma} \rangle$. The stabilizer is

$$H_{32} = \left\{ \begin{pmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & 0 & d \end{pmatrix} \right\}.$$

Hence,

$$e(R_{32}) = ((q-1)^{2r} - 3(q-1)^r + 2\cdot 3^r)(q^r - q)\frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2 q}\,.$$

- $R_{33}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & 0 & a_i \\ 0 & \mu_i & b_i \\ 0 & 0 & \gamma_i \end{pmatrix},$$

with $\boldsymbol{a} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\gamma} \rangle$, $\boldsymbol{b} \notin \langle \boldsymbol{\mu} - \boldsymbol{\gamma} \rangle$. The stabilizer is

$$H_{33} = \left\{ \begin{pmatrix} a & 0 & b \\ 0 & c & d \\ 0 & 0 & e \end{pmatrix} \right\} \times \mathbb{Z}_2,$$

where $\mathbb{Z}_2$ permutes the eigenvalues $\lambda_i$, $\mu_i$. Therefore,

$$R_{33} = \big(B_r \times (\mathbb{C}^r - \mathbb{C})^2 \times (\mathrm{PGL}(3,\mathbb{C})/H_{33})\big)/\mathbb{Z}_2.$$

By (5), we have that

$$\begin{aligned} e_H(B_r) = {} & \big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - 2(q-1)^r + 3^r\big)T \\ & + \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - (q-1)^r + 3^r\big)N, \end{aligned}$$

since under $H \subset \Sigma_3$, we have $T \mapsto T$, $S \mapsto N$, $V \mapsto T + N$. For the second factor, $e((\mathbb{C}^r - \mathbb{C})^2) = (q^r - q)^2$ and $e(\mathrm{Sym}^2(\mathbb{C}^r - \mathbb{C})) = q^{2r} - q^{r+1}$, so

$$e_H((\mathbb{C}^r - \mathbb{C})^2) = (q^{2r} - q^{r+1})T + (q^2 - q^{r+1})N.$$

Finally, $\mathrm{PGL}(3,\mathbb{C})/H_{33} \cong \mathbb{P}^2 \times \mathbb{P}^2 - \Delta$, by considering the first two columns of the matrix, where $\Delta$ is the diagonal. As $e(\mathbb{P}^2 \times \mathbb{P}^2 - \Delta) = (1+q+q^2)(q+q^2)$ and $e(\mathrm{Sym}^2\, \mathbb{P}^2 - \overline{\Delta}) = q^4 + q^3 + q^2$, we have

$$e_H(\mathrm{PGL}(3,\mathbb{C})/H_{33}) = (q^4 + q^3 + q^2)T + (q^3 + q^2 + q)N.$$

Hence,

$$\begin{aligned} e(R_{33}) = {} & \big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - 2(q-1)^r + 3^r\big) \\ & \times \big((q^{2r} - q^{r+1})(q^4 + q^3 + q^2) + (q^2 - q^{r+1})(q^3 + q^2 + q)\big) \\ & + \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - (q-1)^r + 3^r\big) \\ & \times \big((q^{2r} - q^{r+1})(q^3 + q^2 + q) + (q^2 - q^{r+1})(q^4 + q^3 + q^2)\big). \end{aligned}$$

- $R_{34}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & a_i & b_i \\ 0 & \mu_i & 0 \\ 0 & 0 & \gamma_i \end{pmatrix},$$

with $\boldsymbol{a} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\mu} \rangle$, $\boldsymbol{b} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\gamma} \rangle$. The stabilizer is

$$H_{34} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & 0 \\ 0 & 0 & e \end{pmatrix} \right\}.$$

The computations are analogous to the case of $R_{33}$, so $e(R_{33}) = e(R_{34})$.

- $R_{35}$ consists of representations of type

$$A_i = \begin{pmatrix} \lambda_i & a_i & b_i \\ 0 & \mu_i & c_i \\ 0 & 0 & \gamma_i \end{pmatrix},$$

with $\boldsymbol{a} \notin \langle \boldsymbol{\lambda} - \boldsymbol{\mu} \rangle$, $\boldsymbol{c} \notin \langle \boldsymbol{\mu} - \boldsymbol{\gamma} \rangle$. The stabilizer is

$$H_{35} = \left\{ \begin{pmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{pmatrix} \right\}.$$

Hence

$$e(R_{35}) = ((q-1)^{2r} - 3(q-1)^r + 2 \cdot 3^r)(q^r - q)^2 q^r \frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q-1)^2 q^3}.$$

All together, we have

$$\begin{aligned} e(R_3) = {} & \big(2 \cdot 3^r - 3(q-1)^r + (q-1)^{2r}\big) \\ & \qquad \times (q+1)(q^2+q+1)(q^r-q)(q^2+q^{2r}-q^{1+r}) \\ & + \big(2 \cdot 3^r - 2(q-1)^r + (q-1)^{2r} - (q^2-1)^r\big) \\ & \qquad \times q(q^2+q+1)(q^r-q)(q^r-q^2) \\ & + \big(2 \cdot 3^r - 4(q-1)^r + (q-1)^{2r} + (q^2-1)^r\big) \\ & \qquad \times q^2(q^2+q+1)(q^r-1)(q^r-q) \\ & + \tfrac{1}{6} q^3 \big((q-1)^{2r} - 3(q^2-1)^r + 2(q^2+q+1)^r \\ & \qquad\qquad + 2q(q+1)\big(3^{r+1} - 3(q-1)^r + (q-1)^{2r} - (q^2+q+1)^r\big)\big) \\ & + \tfrac{1}{6} q^6 \big(-6(q-1)^r + (q-1)^{2r} + 3(q^2-1)^r + 2(q^2+q+1)^r\big). \end{aligned}$$

Therefore,

$$\begin{aligned} e(\mathcal{R}^{\mathrm{red}}_{r,3}) = {} & \tfrac{1}{3}(q^2+q+1)^r (q-1)^2 q^3 (q+1) \\ & + (q^2+q+1)(2q^{2r}-q^2)(q-1)^{2r} q^r (q+1)^r \\ & - \tfrac{1}{3}(q-1)^{2r}(q+1)(q^2+q+1)(3q^{3r} - 3q^{r+2} + q^3), \end{aligned}$$

and so,

$$e(\mathcal{R}_{r,3}^{\mathrm{irr}}) = e(\mathcal{R}_{r,3}) - e(\mathcal{R}_{r,3}^{\mathrm{red}}) = e(\mathrm{SL}(3, \mathbb{C}))^r - e(\mathcal{R}_{r,3}^{\mathrm{red}}),$$

and consequently,

$$e(\mathcal{M}_{r,3}^{\mathrm{irr}}) = e(\mathcal{R}_{r,3}^{\mathrm{irr}})/e(\mathrm{PGL}(3, \mathbb{C})) = e(\mathrm{SL}(3, \mathbb{C}))^{r-1} - e(\mathcal{R}_{r,3}^{\mathrm{red}})/e(\mathrm{SL}(3, \mathbb{C})).$$

***E-polynomial of the moduli of reducible representations.*** To compute $e(\mathcal{M}_{r,3})$, it remains to compute the moduli space of reducible representations $\mathcal{M}_{r,3}^{\mathrm{red}}$. This is formed by two strata:

(i) $M_0$ formed by semisimple representations which split into irreducible representations of ranks 1 and 2, that is, of the form:

$$A_i = \begin{pmatrix} \lambda_i^{-2} & 0 \quad 0 \\ 0 & \\ 0 & \lambda_i B_i \end{pmatrix},$$

where $(B_1, \dots, B_r) \in \mathcal{M}_{r,2}^{\mathrm{irr}}$. So $e(M_0) = (q-1)^r e(\mathcal{M}_{r,2}^{\mathrm{irr}})$.

(ii) $M_1$ formed by semisimple representations which split into three irreducible representations of rank 1. These are given by eigenvalues $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r)$, $\boldsymbol{\mu} = (\mu_1, \dots, \mu_r)$, $\boldsymbol{\gamma} = (\gamma_1, \dots, \gamma_r)$ in $(\mathbb{C}^*)^r$, where $\lambda_i \mu_i \gamma_i = 1$ for all $1 \le i \le r$. This is the space $B_r'$ whose $E$-polynomial has been computed in (4). Thus

$$e(M_1) = e(B_r'/\Sigma_3) = \tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r.$$

Finally, $e(\mathcal{M}_{r,3}^{\mathrm{red}}) = e(M_0) + e(M_1)$, and adding up everything we get

$$\begin{aligned} e(\mathcal{M}_{r,3}) &= e(\mathcal{M}_{r,3}^{\mathrm{irr}}) + e(\mathcal{M}_{r,3}^{\mathrm{red}}) \\ &= (q^8 - q^6 - q^5 + q^3)^{r-1} + (q-1)^{2r-2}(q^{3r-3} - q^r) \\ &\quad + \tfrac{1}{6}(q-1)^{2r-2} q(q+1) + \tfrac{1}{2}(q^2-1)^{r-1} q(q-1) \\ &\quad + \tfrac{1}{3}(q^2+q+1)^{r-1} q(q+1) \\ &\quad - (q-1)^{r-1} q^{r-1} (q^2-1)^{r-1} (2q^{2r-2} - q). \end{aligned}$$

This completes the main part of the proof of Theorem 1. It remains to show that $e(\mathcal{M}_{r,3}) = e(\overline{\mathcal{M}}_{r,3})$, which we do in the following section.

**Remark 8.** By [Florentino and Lawton 2012], the singular locus of $\mathcal{M}_{r,3}$ is exactly the reducible locus (and so the smooth locus is its complement). Therefore, the above computation of $M_0$ and $M_1$ gives the $E$-polynomial of the singular locus of $\mathcal{M}_{r,3}$. Likewise, $e(\mathcal{M}_{r,3}^{\mathrm{irr}})$ is the $E$-polynomial of the smooth locus of $\mathcal{M}_{r,3}$. Moreover, by [Florentino and Lawton 2014], the abelian character variety $\mathcal{M}(\mathbb{Z}^r, \mathrm{SL}(3, \mathbb{C}))$ is exactly the diagonalizable representations in $\mathcal{M}_{r,3}$. The above computation of $M_1$ gives the $E$-polynomial of $\mathcal{M}(\mathbb{Z}^r, \mathrm{SL}(3,\mathbb{C}))$. In each case, setting $q = 1$ gives the Euler characteristic of the corresponding space.

## 7. $E$-polynomials of character varieties for $F_r$, $r > 1$, and PGL(3, $\mathbb{C}$)

In this final section, we focus on the space of representations

$$\begin{aligned}\overline{\mathcal{R}}_{r,3} &= \mathrm{Hom}(F_r, \mathrm{PGL}(3,\mathbb{C})) = \{\rho : F_r \to \mathrm{PGL}(3,\mathbb{C})\}\\ &= \{(A_1, \dots, A_r) \mid A_i \in \mathrm{PGL}(3,\mathbb{C})\} = \mathrm{PGL}(3,\mathbb{C})^r\end{aligned}$$

and the character variety

$$\overline{\mathcal{M}}_{r,3} = \mathrm{Hom}(F_r, \mathrm{PGL}(3,\mathbb{C})) /\!\!/ \mathrm{PGL}(3,\mathbb{C}).$$

Let $\zeta = e^{2\pi\sqrt{-1}/3}$, and let $\mathbb{Z}_3 = \{1, \zeta, \zeta^2\}$ be the space of cubic roots of unity. Then $\mathrm{PGL}(3,\mathbb{C}) = \mathrm{SL}(3,\mathbb{C})/\mathbb{Z}_3$,

$$\overline{\mathcal{R}}_{r,3} = \mathcal{R}_{r,3}/(\mathbb{Z}_3)^r, \quad \text{and} \quad \overline{\mathcal{M}}_{r,3} = \mathcal{M}_{r,3}/(\mathbb{Z}_3)^r,$$

where $(\zeta^{a_1}, \dots, \zeta^{a_r})$ acts as $(A_1, \dots, A_r) \mapsto (\zeta^{a_1}A_1, \dots, \zeta^{a_r}A_r)$. Clearly $\overline{\mathcal{R}}^{\mathrm{red}}_{r,3} = \mathcal{R}^{\mathrm{red}}_{r,3}/(\mathbb{Z}_3)^r$ and $\overline{\mathcal{R}}^{\mathrm{irr}}_{r,3} = \mathcal{R}^{\mathrm{irr}}_{r,3}/(\mathbb{Z}_3)^r$.

We know from Lemma 4 that $e(\mathrm{PGL}(3,\mathbb{C})) = e(\mathrm{SL}(3,\mathbb{C}))$. Let us see now that $e(\overline{\mathcal{R}}^{\mathrm{red}}_{r,3}) = e(\mathcal{R}^{\mathrm{red}}_{r,3})$. We stratify $\overline{\mathcal{R}}^{\mathrm{red}}_{r,3} = \bar{R}_0 \sqcup \bar{R}_1 \sqcup \bar{R}_2 \sqcup \bar{R}_3$, where $\bar{R}_i = R_i/(\mathbb{Z}_3)^r$ and the $R_i$, $i = 0, 1, 2, 3$, have been defined in Section 6.

We now list the strata with the computation of their $E$-polynomials:

(i) $\bar{R}_0 = \bar{R}_{01} \cup \bar{R}_{02}$, where $\bar{R}_{0j} = R_{0j}/(\mathbb{Z}_3)^r$, $j = 1, 2$. To compute $e(\bar{R}_{01})$, recall that $R_{01}$ is formed by representations $\rho = (A_1, \dots, A_r)$ with

$$A_i = \begin{pmatrix} \lambda_i^{-2} & b_i & c_i \\ 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\lambda_i B_i$}} \\ 0 & & \end{pmatrix},$$

where $(B_1, \dots, B_r) \in \mathcal{R}^{\mathrm{irr}}_{2,r}$. The action of $\zeta^{a_i}$ on $A_i$ is given by $(\lambda_i, b_i, c_i, B_i) \mapsto (\zeta^{a_i}\lambda_i, \zeta^{a_i}b_i, \zeta^{a_i}c_i, \zeta^{a_i}B_i)$. Note that $\mathbb{C}/\mathbb{Z}_3 \cong \mathbb{C}$ and $\mathbb{C}^*/\mathbb{Z}_3 \cong \mathbb{C}^*$, so the relevant cohomology is invariant. Therefore

$$e\big(((\mathbb{C}^*)^r \times \mathbb{C}^r \times \mathbb{C}^r \times \mathcal{R}^{\mathrm{irr}}_{2,r})/(\mathbb{Z}_3)^r\big) = e(\mathbb{C}^*)^r e(\mathbb{C})^r e(\mathbb{C})^r e(\mathcal{R}^{\mathrm{irr}}_{2,r}/(\mathbb{Z}_3)^r).$$

This means that $e(\bar{R}_{01}) = e(R_{01})$. Analogously $e(\bar{R}_{02}) = e(R_{02})$ and $e(\bar{R}_{01} \cap \bar{R}_{02}) = e(R_{01} \cap R_{02})$, so $e(\bar{R}_0) = e(R_0)$.

(ii) $\bar{R}_1 = R_1/(\mathbb{Z}_3)^r$. Note that $R_1$ is formed by $3^r$ copies of the same subvariety. Hence

$$e(\bar{R}_1) = \frac{e(R_1)}{3^r} = 1 + (1 + q + q^2)(q^{3r+1} + q^{3r} - 2q^{2r+1} + q - 1).$$

(iii) $\bar{R}_2 = R_2/(\mathbb{Z}_3)^r$. Recall that $R_2$ is formed by matrices with eigenvalues $(\lambda_i, \lambda_i, \mu_i)$ where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_r) \in P = (\mathbb{C}^*)^r - \{1, \zeta, \zeta^2\}^r$. Now

$$\bar{P} = P/(\mathbb{Z}_3)^r \cong (\mathbb{C}^*)^r - \{(1, 1, \dots, 1)\},$$

so $e(\bar{P}) = (q-1)^r - 1$. It is more or less straightforward to see that $\bar{R}_2$ can be stratified by $\bar{R}_{2j} = R_{2j}/(\mathbb{Z}_3)^r$, $j = 1, 2, \dots, 9$. For each $\bar{R}_{2j}$ the computation of $e(\bar{R}_{2j})$ is the same as that of $e(R_{2j})$, but replacing $e(P) = (q-1)^r - 3^r$ by $e(\bar{P}) = (q-1)^r - 1$. Hence

$$e(\bar{R}_2) = ((q-1)^r - 1)(q^2+q+1)(3q^{3r+1} + 3q^{3r} - 2q^{2r+2} - 4q^{2r+1} + q^3).$$

(iv) $\bar{R}_3 = R_3/(\mathbb{Z}_3)^r$. We follow the lines of the computation of $e(R_3)$. The base for the space of eigenvalues is $\bar{B}_r = B_r/(\mathbb{Z}_3)^r$ with $e(\bar{B}_r) = (q-1)^{2r} - 3(q-1)^r + 2$.

- Let $\bar{R}_{31} = R_{31}/(\mathbb{Z}_3)^r \cong (\mathrm{PGL}(3, \mathbb{C})/D \times \bar{B}_r)/\Sigma_3$. If $\bar{B}'_r = B'_r/(\mathbb{Z}_3)^r$, then easily $e_{\Sigma_3}(\bar{B}'_r) = e_{\Sigma_3}(\bar{B}')^r = (q^2T + S - qV)^r = e_{\Sigma_3}(B'_r)$. For $\bar{C}_r = C_r/(\mathbb{Z}_3)^r$, we have instead that $e_{\Sigma_3}(\bar{C}_r) = (q-1)^r T + ((q-1)^r - 1)V$, so $\bar{B}_r = \bar{B}'_r - \bar{C}_r$ has

$$\begin{aligned} e_{\Sigma_3}(B_r) = {} & \big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r\big)T \\ & + \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\big)S \\ & + \big(\tfrac{1}{3}(q-1)^{2r} - \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r + 1\big)V, \end{aligned}$$

  and

$$\begin{aligned} e(\bar{R}_{31}) = {} & \big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r\big)q^6 \\ & + \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{6}(q-1)^{2r} + \tfrac{1}{3}(q^2+q+1)^r\big)q^3 \\ & + \big(\tfrac{1}{3}(q-1)^{2r} - \tfrac{1}{3}(q^2+q+1)^r - (q-1)^r + 1\big)(q^5+q^4). \end{aligned}$$

- $\bar{R}_{32} = R_{32}/(\mathbb{Z}_3)^r$ has

$$e(\bar{R}_{32}) = ((q-1)^{2r} - 3(q-1)^r + 2)(q^r - q)\frac{e(\mathrm{PGL}(3, \mathbb{C}))}{(q-1)^2 q}\cdot$$

- $\bar{R}_{33} = R_{33}/(\mathbb{Z}_3)^r \cong \big(\bar{B}_r \times (\mathbb{C}^r - \mathbb{C})^2 \times (\mathrm{PGL}(3, \mathbb{C})/H_{33})\big)/\mathbb{Z}_2$, where $H = \mathbb{Z}_2$ acts by swapping the first two eigenvalues. Now

$$\begin{aligned} e_H(\bar{B}_r) = {} & \big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - 2(q-1)^r + 1\big)T \\ & + \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - (q-1)^r + 1\big)N, \end{aligned}$$

so

$$\begin{aligned} e(\bar{R}_{33}) = &\big(\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - 2(q-1)^r + 1\big) \\ &\quad \times \big((q^{2r}-q^{r+1})(q^4+q^3+q^2) + (q^2-q^{r+1})(q^3+q^2+q)\big) \\ &+ \big(-\tfrac{1}{2}(q^2-1)^r + \tfrac{1}{2}(q-1)^{2r} - (q-1)^r + 1\big) \\ &\quad \times \big((q^{2r}-q^{r+1})(q^3+q^2+q) + (q^2-q^{r+1})(q^4+q^3+q^2)\big). \end{aligned}$$

- $\bar{R}_{34} = R_{34}/(\mathbb{Z}_3)^r$ has $e(\bar{R}_{34}) = e(\bar{R}_{33})$.
- $\bar{R}_{35} = R_{35}/(\mathbb{Z}_3)^r$ has

$$e(\bar{R}_{35}) = ((q-1)^{2r} - 3(q-1)^r + 2)(q^r-q)^2 q^r \frac{e(\mathrm{PGL}(3,\mathbb{C}))}{(q-1)^2 q^3}.$$

All together, we have:

$$\begin{aligned} e(\bar{R}_3) = &\big(2 - 3(q-1)^r + (q-1)^{2r}\big)(q+1)(q^2+q+1) \\ &\quad \times (q^r-q)(q^2+q^{2r}-q^{r+1}) \\ &+ \big(2 - 2(q-1)^r + (q-1)^{2r} - (q^2-1)^r\big) \\ &\quad \times q(q^2+q+1)(q^r-q)(q^r-q^2) \\ &+ \big(2 - 4(q-1)^r + (q-1)^{2r} + (q^2-1)^r\big) \\ &\quad \times q^2(q^2+q+1)(q^r-1)(q^r-q) \\ &+ \tfrac{1}{6}q^3\big((q-1)^{2r} - 3(q^2-1)^r + 2(q^2+q+1)^r \\ &\qquad + 2q(q+1)\big(3 - 3(q-1)^r + (q-1)^{2r} - (q^2+q+1)^r\big)\big) \\ &+ \tfrac{1}{6}q^6\big(-6(q-1)^r + (q-1)^{2r} + 3(q^2-1)^r + 2(q^2+q+1)^r\big). \end{aligned}$$

Adding up all the contributions we get:

$$\begin{aligned} e(\bar{\mathcal{R}}^{\mathrm{red}}_{r,3}) &= \tfrac{1}{3}(q^2+q+1)^r (q-1)^2 q^3 (q+1) \\ &\quad + (q^2+q+1)(2q^{2r}-q^2)(q-1)^{2r} q^r (q+1)^r \\ &\quad - \tfrac{1}{3}(q-1)^{2r}(q+1)(q^2+q+1)(3q^{3r} - 3q^{r+2} + q^3) \\ &= e(\mathcal{R}^{\mathrm{red}}). \end{aligned}$$

From this $e(\bar{\mathcal{R}}^{\mathrm{irr}}_{r,3}) = e(\mathcal{R}^{\mathrm{irr}}_{r,3})$ and $e(\overline{\mathcal{M}}^{\mathrm{irr}}_{r,3}) = e(\mathcal{M}^{\mathrm{irr}}_{r,3})$.

The remaining thing to compute is $e(\overline{\mathcal{M}}^{\mathrm{red}}_{r,3})$. This is formed by two strata:

(i) $\overline{M}_0 = M_0/(\mathbb{Z}_3)^r \cong ((\mathbb{C}^*)^r \times \mathcal{M}^{\mathrm{irr}}_{r,2})/(\mathbb{Z}_3)^r$. Hence

$$e(\overline{M}_0) = (q-1)^r e(\overline{\mathcal{M}}^{\mathrm{irr}}_{r,2}) = e(M_0).$$

(ii) $\overline{M}_1 = M_1/(\mathbb{Z}_3)^r \cong ((\mathbb{C}^*)^r/(\mathbb{Z}_3)^r)/\Sigma_3 \cong (\mathbb{C}^*)^r/\Sigma_3$. So $e(\overline{M}_1) = e(M_1)$.

We get finally $e(\overline{\mathcal{M}}_{r,3}^{\mathrm{red}}) = e(\mathcal{M}_{r,3}^{\mathrm{red}})$. This concludes the proof of the equality $e(\overline{\mathcal{M}}_{r,3}) = e(\mathcal{M}_{r,3})$.

**Remark 9.** There is an arithmetic argument communicated to us by S. Mozgovoy to prove that $e(\overline{\mathcal{M}}_{r,n}) = e(\mathcal{M}_{r,n})$ for $n$ odd. It goes as follows: find infinitely many primes $p$ such that $p-1$ and $n$ are coprime (by Dirichlet's theorem on arithmetic progressions); then $\mathrm{SL}(n, \mathbb{F}_p) \to \mathrm{PGL}(n, \mathbb{F}_p)$ is bijective and one gets a bijection between corresponding character varieties over $\mathbb{F}_p$. So the count number of points of $\mathcal{M}_{r,n}$ and $\overline{\mathcal{M}}_{r,n}$ over $\mathbb{F}_p$ coincide, and hence the $E$-polynomials coincide.

However this argument cannot be used for even $n$. Despite this, the $E$-polynomials for the $\mathrm{SL}(2, \mathbb{C})$-character varieties of free groups do equal those of $\mathrm{PGL}(2, \mathbb{C})$. We expect to address the case of $\mathrm{SL}(4, \mathbb{C})$ in future work.

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