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ESTIMATES OF THE GAPS BETWEEN CONSECUTIVE EIGENVALUES OF LAPLACIAN

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For the eigenvalue problem of the Dirichlet Laplacian on a bounded domain in Euclidean space $\mathbb{R}^n$, we obtain estimates for the upper bounds of the gaps between consecutive eigenvalues which are the best possible in terms of the orders of the eigenvalues. Therefore, it is reasonable to conjecture that this type of estimate also holds for the eigenvalue problem on a Riemannian manifold. We give some particular examples.

1. Introduction

Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M with boundary (possible empty). Then the Dirichlet eigenvalue problem of the Laplacian on Ω is given by

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1-1}$$

where Δ is the Laplacian on M. It is well known that the spectrum of (1-1) has the real and purely discrete eigenvalues

$$0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \nearrow \infty, \tag{1-2}$$

where each λ_i has finite multiplicity and is repeated according to its multiplicity. The corresponding orthonormal basis of real eigenfunctions will be denoted $\{u_j\}_{j=1}^{\infty}$. We go forward under the assumption that $L^2(\Omega)$ represents the real Hilbert space of real-valued L^2 functions on Ω. We put $\lambda_0 = 0$ if $\partial\Omega = \varnothing$.

An important aspect of estimating higher eigenvalues is to estimate as precisely as possible the gaps between consecutive eigenvalues of (1-1). In this regard, we will review some important results on the estimates of eigenvalue problem (1-1).

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For the upper bound of the gap between consecutive eigenvalues of (1-1), when Ω is a bounded domain in a 2-dimensional Euclidean space $\mathbb{R}^2$, Payne, Pólya and Weinberger (see [Payne et al. 1955; 1956]) proved

$$\lambda_{k+1} - \lambda_k \le \frac{2}{k} \sum_{i=1}^{k} \lambda_i. \tag{1-3}$$

C. J. Thompson [1969] extended (1-3) to the n-dimensional case and obtained

$$\lambda_{k+1} - \lambda_k \le \frac{4}{nk} \sum_{i=1}^{k} \lambda_i. \tag{1-4}$$

Hile and Protter [1980] improved (1-4) to

$$\sum_{i=1}^{k} \frac{\lambda_i}{\lambda_{k+1} - \lambda_i} \ge \frac{nk}{4}. \tag{1-5}$$

Yang (see [Yang 1991] and more recently [Cheng and Yang 2007]) has obtained a sharp inequality:

$$\sum_{i=1}^{k} (\lambda_{k+1} - \lambda_i)\Big(\lambda_{k+1} - \Big(1 + \frac{4}{n}\Big)\lambda_i\Big) \le 0. \tag{1-6}$$

From (1-6), one can infer

$$\lambda_{k+1} \le \frac{1}{k}\Big(1 + \frac{4}{n}\Big) \sum_{i=1}^{k} \lambda_i. \tag{1-7}$$

The inequalities (1-6) and (1-7) are called Yang's first inequality and second inequality, respectively (see [Ashbaugh 1999; 2002; Ashbaugh and Benguria 1996; Harrell and Stubbe 1997]). Also we note that Ashbaugh and Benguria gave an optimal estimate for $k = 1$ (see [Ashbaugh and Benguria 1991; 1992a; 1992b]). From Chebyshev's inequality, it is easy to prove that

$$\text{(1-6)} \Longrightarrow \text{(1-7)} \Longrightarrow \text{(1-5)} \Longrightarrow \text{(1-4)}.$$

From (1-6), Cheng and Yang [2005] obtained

$$\lambda_{k+1} - \lambda_k \le 2\bigg(\Big(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i\Big)^2 - \Big(1 + \frac{4}{n}\Big)\frac{1}{k}\sum_{i=1}^{k}\Big(\lambda_i - \frac{1}{k}\sum_{j=1}^{k}\lambda_j\Big)^2\bigg)^{\frac{1}{2}}. \tag{1-8}$$

Cheng and Yang [2007], using their recursive formula, obtained

$$\lambda_{k+1} \le C_0(n) k^{2/n} \lambda_1, \tag{1-9}$$

where $C_0(n) \leq 1 + 4/n$ is a constant. From Weyl's asymptotic formula (see [Weyl 1912]), we know that the upper bound (1-9) is best possible in terms of the order on k.

For a complete Riemannian manifold M, from Nash's theorem [1956], there exists an isometric immersion

$$\psi : M \to \mathbb{R}^N,$$

where $\mathbb{R}^N$ is Euclidean space. The mean curvature of the immersion ψ is denoted by H and $|H|$ denotes its norm. Define

$$\Phi = \{\psi \mid \psi \text{ is an isometric immersion from } M \text{ into Euclidean space}\}.$$

When Ω is a bounded domain of a complete Riemannian manifold M, isometrically immersed into a Euclidean space $\mathbb{R}^N$, Cheng and the first author [Chen and Cheng 2008] (see also [El Soufi et al. 2009; Harrell 2007]) obtained

$$\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)^2 \leq \frac{4}{n}\sum_{i=1}^{k}(\lambda_{k+1}-\lambda_i)\big(\lambda_i + \tfrac{1}{4}n^2H_0^2\big), \tag{1-10}$$

where

$$H_0^2 = \inf_{\psi\in\Phi}\sup_{\Omega}|H|^2. \tag{1-11}$$

In the same paper, using the recursive formula in [Cheng and Yang 2007], Cheng and Chen also deduced

$$\lambda_{k+1} + \tfrac{1}{4}n^2H_0^2 \leq C_0(n)k^{2/n}\big(\lambda_1 + \tfrac{1}{4}n^2H_0^2\big), \tag{1-12}$$

where H_0^2 and $C_0(n)$ are given by (1-11) and (1-9), respectively.

From (1-10), we can get the gaps between the consecutive eigenvalues of the Laplacian:

$$\lambda_{k+1}-\lambda_k \leq 2\left(\left(\frac{2}{n}\frac{1}{k}\sum_{i=1}^{k}\lambda_i + \tfrac{1}{2}nH_0^2\right)^2 - \left(1+\frac{4}{n}\right)\frac{1}{k}\sum_{i=1}^{k}\left(\lambda_i - \frac{1}{k}\sum_{j=1}^{k}\lambda_j\right)^2\right)^{\frac{1}{2}}. \tag{1-13}$$

Remark 1.1. When Ω is an n-dimensional compact homogeneous Riemannian manifold, a compact minimal submanifold without boundary and a connected bounded domain in the standard unit sphere $\mathbb{S}^N(1)$, and a connected bounded domain and a compact complex hypersurface without boundary of the complex projective space $\mathbb{CP}^n(4)$ with holomorphic sectional curvature 4, many mathematicians have studied the universal inequalities for eigenvalues and the difference of the consecutive eigenvalues (see [Cheng and Yang 2005; 2006; 2009; Harrell 1993;

Harrell and Michel 1994; Harrell and Stubbe 1997; Li 1980; Yang and Yau 1980; Leung 1991; Sun et al. 2008; Chen et al. 2012]).

Remark 1.2. Another problem is the lower bound of the gap between the first two eigenvalues. In general, there exists the famous fundamental gap conjecture for the Dirichlet eigenvalue problem of the Schrödinger operator (see [Ashbaugh and Benguria 1989; van den Berg 1983; Singer et al. 1985; Yau 1986; Yu and Zhong 1986] and the references therein). The fundamental gap conjecture was solved by B. Andrews and J. Clutterbuck [2011].

From (1-8) and (1-13), it is not difficult to see that both Yang's estimate for the gap between consecutive eigenvalues of (1-1) implicit in [Yang 1991] and the estimate from [Chen and Cheng 2008] are on the order of $k^{3/(2n)}$. However, by the calculation of the gap between the consecutive eigenvalues of $\mathbb{S}^n$ with the standard metric and Weyl's asymptotic formula, the order of the upper bound of this gap is $k^{1/n}$. Therefore, we make a conjecture:

Conjecture 1.3. *Let Ω be a bounded domain in an n-dimensional complete Riemannian manifold M. For the Dirichlet problem* (1-1), *the upper bound for the gap between consecutive eigenvalues of the Laplacian should be*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{1/n}, \quad k > 1, \tag{1-14}$$

where $C_{n,\Omega}$ is a constant dependent on Ω itself and the dimension n.

Remark 1.4. The famous Payne–Pólya–Weinberger conjecture (see [Payne et al. 1955; 1956; Thompson 1969; Ashbaugh and Benguria 1993a; 1993b]) states that, when $M = \mathbb{R}^n$, for the Dirichlet eigenvalue problem (1-1), one should have

$$\frac{\lambda_{k+1}}{\lambda_k} \leq \frac{\lambda_2}{\lambda_1}\bigg|_{\mathbb{B}^n} = \left(\frac{j_{n/2,1}}{j_{n/2-1,1}}\right)^2, \tag{1-15}$$

where $\mathbb{B}^n$ is the n-dimensional unit ball in $\mathbb{R}^n$, and $j_{p,k}$ is the k-th positive zero of the Bessel function $J_p(t)$. From Weyl's asymptotic formula and (1-15), the order of the upper bound of the consecutive eigenvalues of (1-1) is $k^{2/n}$. Therefore, Conjecture 1.3 reflects the distribution of eigenvalues from another point of view. From the order of the upper bound of the gap between the consecutive eigenvalues of $\mathbb{S}^n$, the estimate in (1-14) is best possible in terms of the order on k.

In the following, the constants $C_{n,\Omega}$ are allowed to be different in different cases.

When Ω is a bounded domain in $\mathbb{R}^n$, for the Dirichlet eigenvalue problem (1-1), we give an affirmative answer to Conjecture 1.3.

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain in Euclidean space $\mathbb{R}^n$ and λ_k be the k-th* ($k > 1$) *eigenvalue of the Dirichlet eigenvalue problem* (1-1). *Then we have*

$$\lambda_{k+1} - \lambda_k \leq C_{n,\Omega} k^{1/n}, \tag{1-16}$$

where $C_{n,\Omega} = 4\lambda_1\sqrt{C_0(n)/n}$ *and* $C_0(n)$ *is given by* (1-9).

It is reasonable to conjecture that this type of estimate also holds on a Riemannian manifold. We give some particular examples as follows.

Corollary 1.6. *Let* $\Omega \subset \mathbb{H}^n(-1)$ *be a bounded domain in hyperbolic space* $\mathbb{H}^n(-1)$, *and* λ_k *be the k-th* ($k > 1$) *eigenvalue of the Dirichlet eigenvalue problem* (1-1). *Then we have*

$$\lambda_{k+1} - \lambda_k \le C_{n,\Omega} k^{1/n}, \tag{1-17}$$

where $C_{n,\Omega}$ *depends on* Ω *and the dimension n and is given by*

$$C_{n,\Omega} = 4\Big(C_0(n)\big(\lambda_1 - \tfrac{1}{4}(n-1)^2\big)\big(\lambda_1 + \tfrac{1}{4}n^2 H_0^2\big)\Big)^{1/2}, \tag{1-18}$$

where $C_0(n)$ *and* H_0^2 *are the same as in* (1-12).

In fact, by the comparison theorem for the distance function in a Riemannian manifold, we have:

Corollary 1.7. *Let M be an n-dimensional* ($n \ge 3$) *simply connected complete noncompact Riemannian manifold with sectional curvature* Sec *satisfying*

$$-a^2 \le \mathrm{Sec} \le -b^2,$$

where a and b are constants with $0 \le b \le a$. *Let* $\Omega \subset M$ *be a bounded domain of M and* λ_k *be the k-th* ($k > 1$) *eigenvalue of* (1-1). *Then we have*

$$\lambda_{k+1} - \lambda_k \le C_{n,\Omega} k^{1/n}, \tag{1-19}$$

where $C_{n,\Omega}$ *depends on* Ω *and the dimension n and is given by*

$$C_{n,\Omega} = 4\Big(C_0(n)\big(\lambda_1 - \tfrac{1}{4}(n-1)^2 b^2 + \tfrac{1}{4}(a^2 - b^2)\big)\big(\lambda_1 + \tfrac{1}{4}n^2 H_0^2\big)\Big)^{1/2}, \tag{1-20}$$

where $C_0(n)$ *and* H_0^2 *are the same as in* (1-12).

2. Preliminaries

In this section, we first recall some basic concepts and a theorem of Chapter 10 in [Kolmogorov and Fomin 1960], and then we prove a theorem which will be used in the next section.

Define

$$\mathcal{H}^\infty = \left\{ x = (x_j)_{j=1}^\infty \;\middle|\; x_j \in \mathbb{R},\ \Big(\sum_{j=1}^\infty x_j^2\Big)^{\frac{1}{2}} < +\infty \right\}$$

and

$$\mathcal{H}^2 = \{x = (x_1, x_2) \mid x_1, x_2 \in \mathbb{R},\ (x_1^2 + x_2^2)^{1/2} < +\infty\}.$$

The inner product $\langle \cdot, \cdot \rangle_\infty$ on $\mathcal{H}^\infty$ is defined by

$$\langle x, y\rangle_\infty = \sum_{j=1}^{\infty} x_j y_j, \quad \forall\, x = (x_j)_{j=1}^{\infty},\ y = (y_j)_{j=1}^{\infty}.$$

The inner product $\langle \cdot, \cdot \rangle_2$ on $\mathcal{H}^2$ can be defined similarly. Obviously, both $\mathcal{H}^\infty$ and $\mathcal{H}^2$ are Hilbert spaces. The dual space of $\mathcal{H}^2$ is denoted by $(\mathcal{H}^2)^*$. It is well known that $(\mathcal{H}^2)^*$ is isomorphic to $\mathcal{H}^2$ itself.

In order to prove our theorem, we need the following Lagrange multiplier theorem for real Banach spaces (see Chapter 10, Section 3, paragraph 3 in [Kolmogorov and Fomin 1960] or page 270 in [Zeidler 1995]).

Theorem 2.1. *Let X and Y be real Banach spaces. Assume that $F : x_0 \in U \subset X \to \mathbb{R}$ and $\Phi : x_0 \in U \subset X \to Y$ are continuously Fréchet differentiable on an open neighborhood of x_0, where $x_0 \in \Phi^{-1}(0) = \{x \in U \mid \Phi(x) = 0 \in Y\}$. If the set $\{\Phi'(x_0)(x) \in Y \mid x \in X\}$ is closed and x_0 is an extremum (maximum or minimum) of F on $\Phi^{-1}(0)$, then there exists $\lambda_0 \in \mathbb{R}$ and a linear functional $y^* \in Y^*$, where*

$$\lambda_0^2 + \|y^*\|^2 \neq 0,$$

such that

$$\lambda_0 F'(x_0) + (\Phi'(x_0))^*(y^*) = 0. \tag{2-1}$$

Moreover, if $\{\Phi'(x_0)(x) \in Y \mid x \in X\} = Y$, then we can take $\lambda_0 = 1$.

Theorem 2.2. *Assume that $\{\mu_j\}_{j=1}^{\infty}$ is a nondecreasing sequence, i.e.,*

$$0 < \mu_1 \le \mu_2 \le \cdots \le \mu_k \le \cdots \nearrow \infty,$$

where each μ_i has finite multiplicity m_i and is repeated according to its multiplicity.

Define

$$\begin{aligned} B &= \sum_{j=1}^{\infty} x_j^2 > 0, \\ A &= \sum_{j=1}^{\infty} \mu_j^2 x_j^2, \quad x = (x_j)_{j=1}^{\infty} \in \mathcal{H}^\infty. \end{aligned} \tag{2-2}$$

If $x_{m_1} \neq 0$ and $\sum_{j=1}^{\infty} \mu_j x_j^2 < \sqrt{AB}$, under the conditions in (2-2), we have

$$\sum_{j=1}^{\infty} \mu_j x_j^2 \le \frac{A + \mu_{m_1}\mu_{m_1+1} B}{\mu_{m_1} + \mu_{m_1+1}}. \tag{2-3}$$

Proof. First, assume that $\{\mu_j\}_{j=1}^{\infty}$ is a strictly increasing sequence, i.e.,

$$0 < \mu_1 < \mu_2 < \cdots < \mu_k < \cdots \nearrow \infty.$$

Suppose

$$F(x) = \sum_{j=1}^{\infty} \mu_j x_j^2,$$

$$\Psi(x) = \left(\sum_{j=1}^{\infty} x_j^2 - B,\ \sum_{j=1}^{\infty} \mu_j^2 x_j^2 - A \right) \in \mathcal{H}^2, \quad x \in \mathcal{H}^{\infty}.$$

Let $x_0 = (a_j)_{j=1}^{\infty}$ be an extremum of $F(x)$ on $\Phi^{-1}(0)$. Since $\forall\, h = (h_j)_{j=1}^{\infty} \in \mathcal{H}^{\infty}$,

$$F'(x_0)h = 2 \sum_{j=1}^{\infty} \mu_j x_j h_j,$$

$$\Psi'(x_0)h = \left(2 \sum_{j=1}^{\infty} x_j h_j,\ 2 \sum_{j=1}^{\infty} \mu_j^2 x_j h_j \right),$$

and

$$\Psi'(x_0)(\mathcal{H}^{\infty}) = \mathcal{H}^2,$$

there exists $y^* \in (\mathcal{H}^2)^*$ such that

$$F'(x_0)h + (\Psi'(x_0))^*(y^*)h = 0. \tag{2-4}$$

Since $\mathcal{H}^2 = (\mathcal{H}^2)^*$, we can use some unique vector $(\mu, \lambda) \in \mathcal{H}^2$ to rewrite (2-4) as

$$\sum_{j=1}^{\infty} \mu_j a_j h_j + \mu \sum_{j=1}^{\infty} a_j h_j + \lambda \sum_{j=1}^{\infty} \mu_j^2 a_j h_j = 0. \tag{2-5}$$

Choosing

$$h_j = \delta_{jk}, \quad j = 1, 2, \ldots,$$

from (2-5), we obtain a system of equations

$$\mu_k a_k + \mu a_k + \lambda \mu_k^2 a_k = 0, \quad k = 1, 2, \ldots. \tag{2-6}$$

Since $\{\mu_k\}$ is a strictly increasing sequence, and there are only two varieties μ and λ, there are only two cases for x_0.

Case 1. There is only one $a_k \neq 0$, whether $k = 1$ or not. In this case, the critical value of $F(x)$ is given by

$$F(x_0) = \sqrt{AB},$$

which contradicts the assumption of the theorem.

Case 2. There are only two nonzero components of x_0, say a_k and a_l (without loss of generality, set $k < l$). In this case, we have

$$A = \mu_k^2 a_k^2 + \mu_l^2 a_l^2, \quad B = a_k^2 + a_l^2. \tag{2-7}$$

From (2-7), we have

$$F(x_0) = \frac{A + \mu_k \mu_l B}{\mu_k + \mu_l}.$$

Since

$$A = \mu_k^2 a_k^2 + \mu_l^2 a_l^2 > \mu_k^2 (a_k^2 + a_l^2) = \mu_k^2 B,$$

we have

$$\mu_k < \sqrt{A/B}. \tag{2-8}$$

Similarly, we can also deduce

$$\mu_l > \sqrt{A/B}. \tag{2-9}$$

Hence, we have

$$F(x_0) - \sqrt{AB} = \frac{B\big(\mu_k - \sqrt{A/B}\big)\big(\mu_l - \sqrt{A/B}\big)}{\mu_k + \mu_l} < 0. \tag{2-10}$$

Since $\{\mu_i\}$ is a strictly increasing sequence, for μ_k fixed, from (2-8) and (2-9), we know that the right side of (2-10) is strictly decreasing in μ_l, i.e.,

$$\frac{B\big(\mu_k - \sqrt{A/B}\big)\big(\mu_{k+1} - \sqrt{A/B}\big)}{\mu_k + \mu_{k+1}} > \frac{B\big(\mu_k - \sqrt{A/B}\big)\big(\mu_{k+2} - \sqrt{A/B}\big)}{\mu_k + \mu_{k+2}} > \cdots .$$

Hence, we know that

$$\frac{A + \mu_k \mu_{k+1} B}{\mu_k + \mu_{k+1}}, \quad k = 1, 2, \ldots,$$

are local maximal values of $F(x)$.

Since $x_{m_1} = x_1 \neq 0$, k must be equal to $m_1 = 1$ only. Finally, we have the global maximum of $F(x)$

$$\frac{A + \mu_1 \mu_2 B}{\mu_1 + \mu_2}.$$

Second, assume that $\{\mu_j\}_{j=1}^{\infty}$ is an increasing sequence, i.e.,

$$0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_k \leq \cdots \nearrow \infty,$$

where each μ_i has finite multiplicity m_i and is repeated according to its multiplicity.

Replacing (2-7) by

$$\begin{aligned} A &= m_k \mu_k^2 a_k^2 + m_l \mu_l^2 a_l^2, \\ B &= m_k a_k^2 + m_l a_l^2, \end{aligned}$$

and following the above steps almost word for word, we deduce that the local maximal value of $F(x)$ is

$$\frac{A + \mu_{m_k} \mu_{m_k+1} B}{\mu_{m_k} + \mu_{m_k+1}}$$

and

$$\mu_{m_k} < \sqrt{A/B}, \quad \mu_{m_k+1} > \sqrt{A/B}.$$

Since $x_{m_1} \neq 0$, m_k must be equal to m_1 and the local maximal value of $F(x)$ is the global maximum. Since

$$\frac{A + \mu_{m_1} \mu_{m_1+1} B}{\mu_{m_1} + \mu_{m_1+1}} - \sqrt{AB} = \frac{B\big(\mu_{m_1} - \sqrt{A/B}\big)\big(\mu_{m_k+1} - \sqrt{A/B}\big)}{\mu_{m_1} + \mu_{m_1+1}} < 0,$$

we can obtain (2-3). □

3. Proofs of main results

In this section, we will give the proof of Theorem 1.5. In order to prove our main results, we need the following key lemma and related corollaries of Theorem 2.2.

Lemma 3.1. *For the Dirichlet eigenvalue problem* (1-1), *let u_k be the orthonormal eigenfunction corresponding to the k-th eigenvalue λ_k, i.e.,*

$$\begin{cases} \Delta u_k = -\lambda_k u_k & \text{in } \Omega, \\ u_k = 0 & \text{on } \partial\Omega, \\ \int_\Omega u_i u_j = \delta_{ij}. \end{cases}$$

Then for any complex-valued function $g \in C^3(\Omega) \cap C^2(\overline{\Omega})$ such that $g u_i$ is not the $\mathbb{C}$-linear combination of

$$u_1, \ldots, u_{k+1},$$

and such that

$$a_{k+1} = \int_\Omega g u_i u_{k+1} \neq 0,$$

with $\lambda_i < \lambda_{k+1} < \lambda_{k+2}$, $k, i \in \mathbb{Z}^+$, $i \geq 1$, we have

$$\begin{aligned} \text{(3-1)} \quad \big((\lambda_{k+1} - \lambda_i) + (\lambda_{k+2} - \lambda_i)\big) &\int_\Omega |\nabla g|^2 u_i^2 \\ &\leq \int_\Omega |2\nabla g \cdot \nabla u_i + u_i \Delta g|^2 + (\lambda_{k+1} - \lambda_i)(\lambda_{k+2} - \lambda_i) \int_\Omega |g u_i|^2. \end{aligned}$$

Proof. Define

$$a_{ij} = \int_\Omega g u_i u_j,$$
$$b_{ij} = \int_\Omega \big(\nabla u_i \cdot \nabla g + \tfrac{1}{2} u_i \Delta g\big) u_j,$$

where ∇ denotes the gradient operator. Obviously,

$$a_{ij} = a_{ji}. \tag{3-2}$$

Then, from Stokes' theorem, we get

$$\begin{aligned}\lambda_j a_{ij} &= \int_\Omega g u_i(-\Delta u_j)\\ &= -\int_\Omega (u_i \Delta g + g\Delta u_i + 2\nabla g \cdot \nabla u_i) u_j\\ &= \lambda_i \int_\Omega g u_i u_j - 2\int_\Omega \big(\nabla u_i \cdot \nabla g + \tfrac{1}{2} u_i \Delta g\big) u_j,\end{aligned}$$

i.e.,

$$2b_{ij} = (\lambda_i - \lambda_j) a_{ij}. \tag{3-3}$$

From Stokes' theorem, we have

$$\int_\Omega |\nabla g|^2 u_i^2 = -2\int_\Omega g u_i \big(\nabla \bar{g} \cdot \nabla u_i + \tfrac{1}{2} u_i \Delta \bar{g}\big). \tag{3-4}$$

Since $\{u_k\}_{k=1}^\infty$ consists of a complete orthonormal basis of $L^2(\Omega)$, by the definition of a_{ij} and b_{ij}, from (3-3), (3-4) and Parseval's identity, we obtain

$$\int_\Omega |g u_i|^2 = \sum_{j=1}^\infty |a_{ij}|^2, \tag{3-5}$$

$$\int_\Omega |\nabla g|^2 u_i^2 = 2\sum_{j=1}^\infty a_{ij}\overline{b_{ij}} = \sum_{j=1}^\infty (\lambda_j - \lambda_i)|a_{ij}|^2, \tag{3-6}$$

$$\int_\Omega |2\nabla \bar{g} \cdot \nabla u_i + u_i \Delta \bar{g}|^2 = 4\sum_{j=1}^\infty |b_{ij}|^2 = \sum_{j=1}^\infty (\lambda_j - \lambda_i)^2 |a_{ij}|^2. \tag{3-7}$$

From the Cauchy–Schwarz inequality, we have

$$\Big(\sum_{j=k+1}^\infty (\lambda_j - \lambda_i)|a_{ij}|^2\Big)^2 \le \sum_{j=k+1}^\infty (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \sum_{j=k+1}^\infty |a_{ij}|^2. \tag{3-8}$$

From (3-5), (3-6), (3-7) and (3-8), we can deduce

$$\left(\int_\Omega |\nabla g|^2 u_i^2 - \sum_{j=1}^{k} (\lambda_j - \lambda_i)|a_{ij}|^2\right)^2 \le \left(\int_\Omega |g u_i|^2 - \sum_{j=1}^{k} |a_{ij}|^2\right)\left(\int_\Omega |2\nabla g \cdot \nabla u_i + u_i \Delta g|^2 - \sum_{j=1}^{k} (\lambda_j - \lambda_i)^2 |a_{ij}|^2\right). \tag{3-9}$$

Define

$$\widetilde{B}(i) = \int_\Omega |g u_i|^2 - \sum_{j=1}^{k} |a_{ij}|^2 = \sum_{j=k+1}^{\infty} |a_{ij}|^2 > 0, \quad \text{since } \int_\Omega g u_i u_{k+1} \neq 0,$$

$$\widetilde{A}(i) = \int_\Omega |2\nabla g \cdot \nabla u_i + u_i \Delta g|^2 - \sum_{j=1}^{k} (\lambda_j - \lambda_i)^2 |a_{ij}|^2 = \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i)^2 |a_{ij}|^2 \geq 0,$$

$$\widetilde{C}(i) = \int_\Omega |\nabla g|^2 u_i^2 - \sum_{j=1}^{k} (\lambda_j - \lambda_i)|a_{ij}|^2 = \sum_{j=k+1}^{\infty} (\lambda_j - \lambda_i)|a_{ij}|^2.$$

Since $g u_i$ is not the $\mathbb{C}$-linear combination of

$$u_1, \ldots, u_{k+1},$$

there exists some $l > k+1$ such that

$$a_l = \int_\Omega g u_i u_l \neq 0.$$

Since

$$\lambda_i < \lambda_{k+1} < \lambda_{k+2} \leq \lambda_l,$$

the vector

$$(|a_{ij}|)_{j=k+1}^{\infty}$$

is not proportional to

$$\big((\lambda_j - \lambda_i)^2 |a_{ij}|\big)_{j=k+1}^{\infty}.$$

From the Cauchy–Schwarz inequality, we have

$$\widetilde{C}(i) < \sqrt{\widetilde{A}(i)\widetilde{B}(i)}. \tag{3-10}$$

Since $a_{k+1} \neq 0$, from (3-10) and Theorem 2.2, we have

$$\widetilde{C}(i) \leq \frac{\widetilde{A}(i) + (\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)\widetilde{B}(i)}{(\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)}. \tag{3-11}$$

From (3-11) and the definition of $\widetilde{A}(i)$, $\widetilde{B}(i)$ and $\widetilde{C}(i)$, we obtain

$$\begin{aligned}(3\text{-}12)\quad & \big((\lambda_{k+2}-\lambda_i)+(\lambda_{k+1}-\lambda_i)\big)\int_\Omega |\nabla g|^2 u_i^2 \\ &\le \int_\Omega |2\nabla g\cdot\nabla u_i+u_i\Delta g|^2+(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i)\int_\Omega |gu_i|^2 \\ &\qquad -\sum_{j=1}^{k}(\lambda_{k+1}-\lambda_j)(\lambda_{k+2}-\lambda_j)|a_{ij}|^2 \\ &\le \int_\Omega |2\nabla g\cdot\nabla u_i+u_i\Delta g|^2+(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i)\int_\Omega |gu_i|^2. \qquad \square\end{aligned}$$

Corollary 3.2. *Under the assumption of Lemma 3.1, for any nonconstant real-valued function $f\in C^3(\Omega)\cap C^2(\overline{\Omega})$, we have*

$$\begin{aligned}(3\text{-}13)\quad & \big((\lambda_{k+2}-\lambda_i)+(\lambda_{k+1}-\lambda_i)\big)\int_\Omega |\nabla f|^2 u_i^2 \\ &\le 2\sqrt{\big((\lambda_{k+2}-\lambda_i)(\lambda_{k+1}-\lambda_i)\big)\int_\Omega |\nabla f|^4 u^2}+\int_\Omega (2\nabla f\cdot\nabla u_i+u_i\Delta f)^2.\end{aligned}$$

Proof. Taking $g=\exp\big(\sqrt{-1}\,\alpha f\big)$, $\alpha\in\mathbb{R}\setminus\{0\}$, in (3-1), we have

$$\begin{aligned}(3\text{-}14)\quad & \alpha^2\big((\lambda_{k+1}-\lambda_i)+(\lambda_{k+2}-\lambda_i)\big)\int_\Omega |\nabla f|^2 u_i^2 \\ &\le \alpha^4\int_\Omega |\nabla f|^4 u_i^2+\alpha^2\int_\Omega |2\nabla f\cdot\nabla u_i+u_i\Delta f|^2+(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i).\end{aligned}$$

From (3-14), we have

$$\begin{aligned}(3\text{-}15)\quad & \big((\lambda_{k+1}-\lambda_i)+(\lambda_{k+2}-\lambda_i)\big)\int_\Omega |\nabla f|^2 u_i^2 \\ &\le \alpha^2\int_\Omega |\nabla f|^4 u_i^2+\frac{1}{\alpha^2}(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i)+\int_\Omega |2\nabla f\cdot\nabla u_i+u_i\Delta f|^2.\end{aligned}$$

Since the inequality (3-15) is valid for any $\alpha\neq 0$ and

$$(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i)\neq 0,\qquad \int_\Omega |\nabla f|^4 u_i^2\neq 0,$$

we can choose

$$\alpha^2=\left(\frac{(\lambda_{k+1}-\lambda_i)(\lambda_{k+2}-\lambda_i)}{\int_\Omega |\nabla f|^4 u_i^2}\right)^{\frac{1}{2}}$$

to have (3-13). $\square$

Corollary 3.3. *Under the assumption of Lemma 3.1, for any real-valued function $f \in C^3(\Omega) \cap C^2(\overline{\Omega})$ with $|\nabla f|^2 = 1$, we have*

$$(3\text{-}16)\quad (\lambda_{k+2} - \lambda_{k+1})^2 \le 16\left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2\right)\lambda_{k+2}.$$

Furthermore, we have

$$(3\text{-}17)\quad \lambda_{k+2} - \lambda_{k+1} \le 4\left(\lambda_i - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2\right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}}.$$

Proof. From Corollary 3.2 and $|\nabla f|^2 = 1$, we have

$$\big((\lambda_{k+2} - \lambda_i) + (\lambda_{k+1} - \lambda_i)\big) - 2\sqrt{(\lambda_{k+2} - \lambda_i)(\lambda_{k+1} - \lambda_i)} \le \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2,$$

that is,

$$\big(\sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i}\big)^2 \le \int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2.$$

By integration by parts, we have

$$\int_\Omega (2\nabla f \cdot \nabla u_i + u_i \Delta f)^2 = 4\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \int_\Omega (\Delta f)^2 u_i^2 - 2\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2.$$

Hence, we have

$$(3\text{-}18)\quad \big(\sqrt{\lambda_{k+2} - \lambda_i} - \sqrt{\lambda_{k+1} - \lambda_i}\big)^2 \le 4\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \int_\Omega (\Delta f)^2 u_i^2 - 2\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2.$$

Multiplying (3-18) by $\big(\sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i}\big)^2$ on both sides, we can get

$$\begin{aligned}(\lambda_{k+2} - \lambda_{k+1})^2 &\le 4\left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2\right) \\ &\qquad \times \big(\sqrt{\lambda_{k+2} - \lambda_i} + \sqrt{\lambda_{k+1} - \lambda_i}\big)^2 \\ &\le 16\left(\int_\Omega (\nabla f \cdot \nabla u_i)^2 - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f) \cdot \nabla f) u_i^2\right)\lambda_{k+2},\end{aligned}$$

which is the inequality (3-16).

Since $|\nabla f|^2 = 1$, from (3-16), the Cauchy–Schwarz inequality and integration by parts, we obtain

$$\begin{aligned}(\lambda_{k+2} - \lambda_{k+1})^2 &\le 16\biggl(\int_\Omega |\nabla u_i|^2 - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f)\cdot\nabla f)u_i^2\biggr)\lambda_{k+2}\\ &= 16\biggl(\lambda_i - \frac{1}{4}\int_\Omega (\Delta f)^2 u_i^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta f)\cdot\nabla f)u_i^2\biggr)\lambda_{k+2}. \qquad \square\end{aligned}$$

Remark 3.4. If $\lambda_{k+1} = \lambda_{k+2}$, (3-17) also holds trivially. Hence, under the conditions in Corollary 3.3, when $i = 1$, (3-17) holds for any $k > 1$.

Proof of Theorem 1.5. Since the inequality (3-20) always holds for $\lambda_{k+1} = \lambda_{k+2}$, without loss of generality, we assume that $\lambda_{k+1} < \lambda_{k+2}$ in the following discussion.

Let $\{x_1, x_2, \dots, x_n\}$ be the standard coordinate functions in $\mathbb{R}^n$. Taking

$$i = 1 \qquad \text{and} \qquad f = x_l, \quad l = 1, \dots, n,$$

in (3-16) and summing over l from 1 to n, we have

$$n(\lambda_{k+2} - \lambda_{k+1})^2 \le 16\lambda_{k+2}\int_\Omega \sum_{l=1}^{n}\left(\frac{\partial u_1}{\partial x_l}\right)^2 = 16\lambda_1\lambda_{k+2}, \tag{3-19}$$

where we use $|\nabla x_l| = 1,\ l = 1, \dots, n$.

From Theorem 3.1 in [Cheng and Yang 2007] (see also (1-9)) and from (3-19), we deduce

$$\lambda_{k+2} - \lambda_{k+1} \le 4\sqrt{\frac{\lambda_1}{n}}\sqrt{\lambda_{k+2}} \le 4\lambda_1\sqrt{\frac{C_0(n)}{n}}(k+1)^{1/2} = C_{n,\Omega}(k+1)^{1/2}, \tag{3-20}$$

where $C_{n,\Omega} = 4\lambda_1\sqrt{C_0(n)/n}$ and $C_0(n)$ is given by (1-9).

Therefore, (3-20) holds for arbitrary $k > 1$. $\square$

Proof of Corollary 1.6. For convenience, we will use the upper-half-plane model of hyperbolic space, i.e.,

$$\mathbb{H}^n(-1) = \{(x_1, \dots, x_n) \in \mathbb{R}^n \mid x_n > 0\}$$

with the standard metric

$$\mathrm{d}s^2 = \frac{(\mathrm{d}x_1)^2 + \cdots + (\mathrm{d}x_n)^2}{(x_n)^2}.$$

Taking $r = \log x_n$, we have

$$\mathrm{d}s^2 = (\mathrm{d}r)^2 + \mathrm{e}^{-2r}\sum_{i=1}^{n-1}(\mathrm{d}x_i)^2.$$

Without loss of generality, we assume that $\lambda_{k+1} < \lambda_{k+2}$. Taking $f = r$ and $i = 1$ in (3-17), we have

$$\begin{aligned}\lambda_{k+2}-\lambda_{k+1} &\le 4\left(\lambda_1-\frac{1}{4}\int_\Omega(\Delta r)^2u_i^2-\frac{1}{2}\int_\Omega(\nabla(\Delta r)\cdot\nabla r)u_1^2\right)^{\frac{1}{2}}\sqrt{\lambda_{k+2}}\\ &= 4\big(\lambda_1-\tfrac{1}{4}(n-1)^2\big)^{1/2}\sqrt{\lambda_{k+2}},\end{aligned} \tag{3-21}$$

where $|\nabla r| = 1$ and $\Delta r = -(n-1)$ are used.

By (1-12) and (3-21), we have

$$\begin{aligned}\lambda_{k+2}-\lambda_{k+1} &\le 4\big(\lambda_1-\tfrac{1}{4}(n-1)^2\big)^{1/2}\sqrt{C_0(n)\big(\lambda_1+\tfrac{1}{4}n^2H_0^2\big)(k+1)^{1/n}}\\ &= C_{n,\Omega}(k+1)^{1/n},\end{aligned} \tag{3-22}$$

where $C_{n,\Omega}$ is defined by (1-18). Therefore, we can deduce (3-22) for any $k > 1$. $\square$

4. Proof of Corollary 1.7

Assume that (M, g) is an n-dimensional complete noncompact Riemannian manifold with sectional curvature Sec satisfying $-a^2 \le \mathrm{Sec} \le -b^2$, where a and b are constants with $0 \le b \le a$. Let Ω be a bounded domain of M. For a fixed point $p \notin \overline{\Omega}$, the distance function $\rho(x)$ is defined by $\rho(x) = \mathrm{distance}(x, p)$. From $|\nabla\rho| = 1$ and Proposition 2.2 of [Schoen and Yau 1994], we have

$$\nabla\rho\cdot\nabla(\Delta\rho) = -|\mathrm{Hess}\,\rho|^2-\mathrm{Ric}(\nabla\rho,\nabla\rho). \tag{4-1}$$

Assume that $h_1, \dots, h_{n-1}$, with $0 \le h_1 \le \cdots \le h_{n-1}$, are the eigenvalues of Hess ρ. We have

$$\begin{aligned}2|\mathrm{Hess}\,\rho|^2-(\Delta\rho)^2 &= 2\sum_{i=1}^{n-1}h_i^2-\left(\sum_{i=1}^{n-1}h_i\right)^2\\ &= \sum_{i=1}^{n-1}h_i^2-\sum_{i\ne j}h_ih_j\\ &\le h_{n-1}^2+h_1h_2+\cdots+h_{n-2}h_{n-1}-\sum_{i\ne j}h_ih_j\\ &= h_{n-1}^2-h_1h_2-\cdots-h_{n-2}h_{n-1}-\sum_{\substack{i\ne j\\ i,j\le n-2}}h_ih_j\\ &\le h_{n-1}^2-(n-2)^2h_1^2.\end{aligned} \tag{4-2}$$

From the Hessian comparison theorem (see [Wu et al. 1989]), we have

$$a\frac{\cosh a\rho}{\sinh a\rho} \ge h_{n-1} \ge \cdots \ge h_1 \ge b\frac{\cosh b\rho}{\sinh b\rho}. \tag{4-3}$$

Since $n \geq 3$ and $a^2/(\sinh^2 a\rho)$ is a decreasing function of a, from (4-2) and (4-3), we have

$$
\begin{aligned}
(4\text{-}4)\quad & 2|\mathrm{Hess}\,\rho|^2 + 2\,\mathrm{Ric}(\nabla\rho,\nabla\rho) - (\Delta\rho)^2 \\
&\leq a^2 \frac{\cosh^2 a\rho}{\sinh^2 a\rho} - (n-2)^2 b^2 \frac{\cosh^2 b\rho}{\sinh^2 b\rho} - 2(n-1)b^2 \\
&= a^2 + \frac{a^2}{\sinh^2 a\rho} - (n-2)^2 b^2 - (n-2)^2 \frac{b^2}{\sinh^2 b\rho} - 2(n-1)b^2 \\
&\leq -(n-1)^2 b^2 + (a^2 - b^2) + \frac{b^2}{\sinh^2 b\rho} - (n-2)^2 \frac{b^2}{\sinh^2 b\rho} \\
&\leq -(n-1)^2 b^2 + (a^2 - b^2).
\end{aligned}
$$

Without loss of generality, we assume $\lambda_{k+1} < \lambda_{k+2}$. By taking $f = \rho$ and $i = 1$ in (3-17), we have

$$
(4\text{-}5)\qquad \lambda_{k+2} - \lambda_{k+1} \leq 4\left(\lambda_1 - \frac{1}{4}\int_\Omega (\Delta\rho)^2 u_1^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta\rho)\cdot\nabla\rho) u_1^2\right)^{\frac{1}{2}} \sqrt{\lambda_{k+2}}.
$$

From (4-1) and (4-4), we obtain

$$
\begin{aligned}
(4\text{-}6)\quad & \lambda_1 - \frac{1}{4}\int_\Omega (\Delta\rho)^2 u_1^2 - \frac{1}{2}\int_\Omega (\nabla(\Delta\rho)\cdot\nabla\rho) u_1^2 \\
&\qquad = \lambda_1 + \frac{1}{4}\int_\Omega \bigl(2|\mathrm{Hess}\,\rho|^2 + 2\,\mathrm{Ric}(\nabla\rho,\nabla\rho) - (\Delta\rho)^2\bigr) u_1^2 \\
&\qquad \leq \lambda_1 - \tfrac{1}{4}(n-1)^2 b^2 + \tfrac{1}{4}(a^2 - b^2).
\end{aligned}
$$

By (1-12), (4-5) and (4-6), we have

$$
\begin{aligned}
(4\text{-}7)\quad & \lambda_{k+2} - \lambda_{k+1} \\
&\leq 4\bigl(\lambda_1 - \tfrac{1}{4}(n-1)^2 b^2 + \tfrac{1}{4}(a^2 - b^2)\bigr)^{1/2} \sqrt{C_0(n)\bigl(\lambda_1 + \tfrac{1}{4} n^2 H_0^2\bigr)}\,(k+1)^{1/n} \\
&\leq C_{n,\Omega}(k+1)^{1/n},
\end{aligned}
$$

where $C_{n,\Omega}$ is defined by (1-20). Therefore, we can deduce (4-7) for any $k > 1$. $\square$

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