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LIOUVILLE TYPE THEOREMS FOR THE p-HARMONIC FUNCTIONS ON CERTAIN MANIFOLDS

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We show that for a certain range of $p > n$, the Dirichlet problem at infinity is unsolvable for the p-Laplace equation for any nonconstant continuous boundary data on an n-dimensional Cartan–Hadamard manifold constructed from a complete noncompact shrinking gradient Ricci soliton. Using the steady gradient Ricci soliton, we find an incomplete Riemannian metric on $\mathbb{R}^2$ with positive Gauss curvature such that every positive p-harmonic function must be constant for $p \geq 4$.

1. Introduction

In this article, we study two questions about the p-Laplace equation on Riemannian manifolds. The first one is the solvability of the Dirichlet problem at infinity on a negatively curved complete noncompact manifold, and the second one is the Liouville property for positive solutions on $\mathbb{R}^2$ equipped with an incomplete metric with positive Gauss curvature. In both cases, the n-dimensional manifold M under consideration is equipped with a Riemannian metric $e^{2f/(p-n)}g$ where (M, g, f) is a complete gradient Ricci soliton which is shrinking for the first case and steady for the second case.

On a Riemannian manifold, for a constant $p > 1$, a function v in $W^{1,p}_{\text{loc}} \cap L^{\infty}_{\text{loc}}$ is p-harmonic if it is a weak solution to the p-Laplacian equation

$$\operatorname{div}(|\nabla v|^{p-2}\nabla v) = 0. \tag{1-1}$$

It is known that p-harmonic functions are in $C^{1,\alpha}$ (see [Tolksdorf 1984] and the references therein).

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The behavior of harmonic and, more generally, p-harmonic functions depends on the sign of the curvature of the manifold in an essential way. Therefore, we must treat negatively curved and nonnegatively curved manifolds separately.

A Cartan–Hadamard manifold is a complete simply connected Riemannian manifold with nonpositive sectional curvature everywhere. It is well-known that a Cartan–Hadamard manifold M can be compactified by attaching a sphere $M(\infty)$ at infinity. In the cone topology, the compactification is homeomorphic to a closed Euclidean n-ball [Eberlein and O'Neill 1973]. The Dirichlet problem at infinity for p-harmonic functions is to solve the p-Laplace equation (1-1) on M such that v agrees with a given continuous function φ on $M(\infty)$. For $p = 2$, the Dirichlet problem at infinity for harmonic functions is solvable if there are suitable lower and upper bounds for the sectional curvature [Anderson 1983; Anderson and Schoen 1985; Choi 1984; Hsu 2003; Sullivan 1983]. Ancona [1994] constructed an example showing that the Dirichlet problem is unsolvable if only a negative constant upper bound is imposed. For $p \in (1, \infty)$, the Dirichlet problem at infinity is solvable under similar curvature assumptions like those in the case $p = 2$; in particular, it is solvable if the sectional curvature is bounded by

$$-r^{2\alpha-4-\epsilon} \le K \le -\frac{\alpha(\alpha-1)}{r^2} \tag{1-2}$$

near $M(\infty)$ where $\epsilon > 0$ and $\alpha > 1$, where r is the distance to a fixed point, and for $p \in (1, 1+(n-1)\alpha)$ [Holopainen 2002; Holopainen and Vähäkangas 2007; Pansu 1989].

Our first result is to show the unsolvability of the Dirichlet problem at infinity on certain Cartan–Hadamard manifolds constructed from shrinking gradient Ricci solitons, for a certain range of $p > n$. In particular, the unsolvability holds for the shrinking Gaussian soliton $(\mathbb{R}^n, dx^2, |x|^2/4)$ for every $p > n$. It is interesting to observe that the sectional curvature of the complete negatively curved metric $e^{|x|^2/(2(p-n))}dx^2$ is not bounded above by $-\alpha(\alpha-1)/r^2$, for any constant $\alpha > 1$, at certain sections for sufficiently large r (see remark on page 319). This indicates the upper bound in (1-2) is sharp in some sense for the solvability of the Dirichlet problem at infinity.

Theorem 1.1. *Suppose that (M, g, f) is a simply connected n-dimensional complete noncompact shrinking gradient Ricci soliton whose sectional curvatures are bounded above by a constant K_0 with $0 < K_0 < 1/(2(n-1))$. Then the Dirichlet problem at infinity for the p-Laplace equation on $(M, e^{2f/(p-n)}g)$ is unsolvable for any nonconstant continuous boundary value φ and $n < p < \frac{1}{K_0} + 2 - n$.*

The proof relies on a Liouville type property (Proposition 2.1) for positive solutions to the p-Laplace equation on $(M, e^{-2f/(n-p)}g)$ for every $p > 1$, where Cao and Zhou's [2010] estimates on f and on the volume growth for gradient shrinking Ricci solitons are crucial as they imply that e^{-f} is integrable on (M, g). The advantage for considering the range $p > n$ is that, under the conformal change of metric, it yields a complete metric $\tilde{g}$ and it guarantees the negativity of the curvature of $\tilde{g}$ under the curvature assumption $K \le K_0$, while one does not have such flexibility for $p = 2$.

However, the integration argument in the proof of Proposition 2.1 is no longer valid for steady gradient Ricci solitons due to different behavior of f (typically f tends to $-\infty$ along a sequence of points x_k that go to infinity [Munteanu and Sesum 2013; Wu 2013]). Alternatively, a powerful way to prove Liouville type theorems for positive harmonic functions on complete manifolds with nonnegative Ricci curvature is via Yau's gradient estimate [1975]. The p-harmonic version of Yau's estimate is established by Wang and Zhang [2011] (see [Sung and Wang 2014] for a sharp form of the estimate). For a positive p-harmonic function u in the conformally changed metric $\tilde{g} = e^{-2f/(n-p)}g$, we first derive a maximum principle for $|\nabla \log u|$ for steady (or shrinking) gradient Ricci solitons, via a Bochner type formula. However, the required assumption on Ricci curvature for the gradient estimates cannot hold globally for steady gradient Ricci solitons if $\dim M > 2$ because it would imply that the scalar curvature of g possesses a positive constant lower bound. But this is impossible as shown in [Munteanu and Sesum 2013; Wu 2013]. In dimension 2, we can combine the maximum principle (Proposition 3.3) and the gradient estimate to prove a Liouville type result on the 2-plane with a positively curved *incomplete* metric.

Theorem 1.2. *Let $(\mathbb{R}^2, g, f)$ be Hamilton's cigar soliton. Then there does not exist any nonconstant positive p-harmonic function on $(\mathbb{R}^2, \tilde{g})$ for $p \ge 4$.*

Harmonic functions on the complete gradient Ricci solitons have been studied by Munteanu and Sesum [2013] and Munteanu and Wang [2012] with applications to the geometry and topology of the solitons. Moser [2007] observed an interesting connection between the inverse mean curvature flow formulated as level sets in $\mathbb{R}^n$ and 1-harmonic functions. Kotschwar and Ni [2009] generalize this to Riemannian ambient manifolds. There is also recent work on gradient estimates for weighted p-harmonic functions and the first p-eigenfunctions [Dung and Dat 2015].

2. The Dirichlet problem at infinity

In this section, the triple (M, g, f) is assumed to be a complete noncompact shrinking gradient Ricci soliton. We first establish the following Liouville property for positive p-harmonic functions for $p > 1$ with no additional curvature assumption.

An n-dimensional Riemannian manifold (M, g) is a gradient Ricci soliton if

$$\mathrm{Ric} + \nabla\nabla f + \varepsilon g = 0 \tag{2-1}$$

for some smooth function f and $\varepsilon = -\frac{1}{2}, 0, \frac{1}{2}$. Corresponding to the three values of ε, the gradient Ricci soliton (M, g, f) is shrinking, steady, or expanding [Chow et al. 2006; Hamilton 1995].

Proposition 2.1. *Let (M, g, f) be a complete noncompact gradient shrinking Ricci soliton. Then there is no nonconstant positive p-harmonic function on $(M, e^{-2f/(n-p)}g)$ for $p > 1$.*

Proof. Since u is a p-harmonic function on $(M, \tilde{g})$ where $\tilde{g} = e^{-2f/(n-p)}g$,

$$\operatorname{div}_{\tilde{g}}\big(|\widetilde{\nabla} w|_{\tilde{g}}^{p-2}\widetilde{\nabla} w\big) = |\widetilde{\nabla} w|_{\tilde{g}}^{p} \tag{2-2}$$

holds for $w = -(p-1)\log u$. For any smooth cut-off function $\phi \in C_0^\infty(M)$, in the complete metric g, we require

$$\begin{cases} \phi = 1 & \text{on } B_{x_0}(\rho, g), \\ \phi = 0 & \text{on } M \setminus B_{x_0}(2\rho, g), \\ 0 \le \phi \le 1 & \text{on } M, \\ |\nabla\phi|^2 \le C/\rho^2 & \text{on } M. \end{cases}$$

Here $B_{x_0}(r, g)$ stands for the geodesic ball centered at x_0 with radius r in the metric g in M. Multiplying (2-2) by ϕ^2, then integrating and applying Stokes' theorem, we have

$$\begin{aligned} \int_M |\widetilde{\nabla} w|_{\tilde{g}}^{p}\, \phi^2\, d\mu_{\tilde{g}} &= -2\int_M \phi |\widetilde{\nabla} w|_{\tilde{g}}^{p-2}\, \widetilde{\nabla} w \widetilde{\nabla}\phi\, d\mu_{\tilde{g}} \\ &\le 2\Big(\int_M \phi^2 |\widetilde{\nabla} w|_{\tilde{g}}^{p}\, d\mu_{\tilde{g}}\Big)^{(p-1)/p}\Big(\int_M \phi^2 |\widetilde{\nabla}\phi|_{\tilde{g}}^{p}\, d\mu_{\tilde{g}}\Big)^{1/p} \end{aligned}$$

by the Cauchy–Schwarz inequality ($p > 1$). Therefore, we have

$$\int_M \phi^2 |\widetilde{\nabla} w|_{\tilde{g}}^{p}\, d\mu_{\tilde{g}} \le 2^p \int_M \phi^2 |\widetilde{\nabla}\phi|_{\tilde{g}}^{p}\, d\mu_{\tilde{g}}.$$

Converting back to the metric g, we are led to

$$\int_M \phi^2 |\nabla w|^p e^{-f}\, d\mu_g \le 2^p \int_M \phi^2 |\nabla\phi|^p e^{-f}\, d\mu_g. \tag{2-3}$$

By Theorem 1.1 in [Cao and Zhou 2010], the potential function f for a shrinking gradient Ricci soliton satisfies the pointwise estimate

$$\tfrac{1}{4}(r(x) - c)^2 \le f(x) \le \tfrac{1}{4}(r(x) + c)^2 \tag{2-4}$$

for $x \in M \setminus B_{x_0}(1, g)$, where $r(x)$ is the distance from x to a fixed point x_0 in M and c is a positive constant.

Therefore, by (2-3) and (2-4),

$$\begin{aligned}\int_{B(x_0,\rho)} |\nabla w|^p e^{-(r+c)^2/4}\,d\mu_g &\le \int_M \phi^2 |\nabla w|^p e^{-f}\,d\mu_g \\ &\le \frac{2^p C e^{-(\rho-c)^2/4}}{\rho^p} \int_{B_{x_0}(2\rho,g)\setminus B_{x_0}(\rho,g)} d\mu_g \\ &\le \frac{2^p C e^{-(\rho-c)^2/4}}{\rho^p}\rho^n\end{aligned}$$

where the last inequality follows from the volume growth estimate (Theorem 1.2 in [Cao and Zhou 2010]) on shrinking gradient Ricci solitons:

$$\mathrm{Vol}(B_{x_0}(\rho, g)) \le C\rho^n$$

for sufficiently large ρ and uniform constant C. Now letting $\rho \to \infty$, we conclude $|\nabla w| \equiv 0$ on M, so u is a constant. □

Next, we show that $(M, \tilde{g})$ can be turned into a negatively curved manifold under suitable assumptions on p and the sectional curvature K of (M, g).

Proposition 2.2. *Let (M, g, f) be a simply connected n-dimensional complete noncompact shrinking gradient Ricci soliton whose sectional curvature is bounded above by a constant K_0 with $0 < K_0 < 1/(2(n-1))$. Then $(M, e^{-2f/(n-p)}g)$ is a Cartan–Hadamard manifold for $n < p \le \frac{1}{K_0} + 2 - n$.*

Proof. When $p > n$, the metric $\tilde{g} = e^{-2f/(n-p)}g$ is complete since

$$-\frac{2f(x)}{n-p} = \frac{2f(x)}{p-n} \ge \frac{(r-c)^2}{2(p-n)}$$

by [Cao and Zhou 2010] and completeness of g.

We use the conventions in [Chow et al. 2006] for curvatures. The Riemann curvature tensor is written as

$$R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k} = R_{ijk}^l \frac{\partial}{\partial x^l}$$

$$R_{ijkl} = \Big\langle R\Big(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\Big)\frac{\partial}{\partial x^k}, \frac{\partial}{\partial x^l}\Big\rangle$$

and if $\partial/\partial x^1, \ldots, \partial/\partial x^n$ is orthonormal at $x_0 \in M$, then the sectional curvature of the plane P_{ij} spanned by $\partial/\partial x^i, \partial/\partial x^j$ at x_0 is

$$K(P_{ij}) = R_{ijji}$$

and the Ricci curvature at x_0 is

$$R_{jk} = \sum_{i=1}^n R_{ijk}^i.$$

Under the conformal change of metric $\tilde{g} = e^{2f/(p-n)}g$, the sectional curvature at x_0 becomes

$$\tag{2-5} \begin{aligned} \widetilde{K}(P_{ij}) &= \frac{\tilde{g}\big(\widetilde{R}^s_{ijj}\frac{\partial}{\partial x^s}, \frac{\partial}{\partial x^i}\big)}{\tilde{g}_{ii}\tilde{g}_{jj} - \tilde{g}^2_{ij}} \\ &= e^{4f/(n-p)}\widetilde{R}_{ijji} \\ &= e^{4f/(n-p)} \cdot e^{2f/(p-n)}\bigg(R_{ijji} - \frac{f_{ii}+f_{jj}}{p-n} - \frac{|\nabla f|^2 - f_i^2 - f_j^2}{(p-n)^2}\bigg) \\ &= e^{2f/(n-p)}\bigg(K(P_{ij}) - \frac{f_{ii}+f_{jj}}{p-n} - \frac{|\nabla f|^2 - f_i^2 - f_j^2}{(p-n)^2}\bigg) \end{aligned}$$

(see p. 27 in [Chow et al. 2006]). On the gradient shrinking Ricci soliton, we therefore have

$$\widetilde{K}(P_{ij}) \le e^{2f/(n-p)}\Big(K(P_{ij}) + \frac{R_{ii}+R_{jj}-1}{p-n}\Big)$$

by using the defining equation for shrinking gradient Ricci solitons and dropping the last term above that is nonpositive for $i \neq j$.

From the assumption on K_0 and $p > n$, it follows that

$$\begin{aligned} K(P_{ij}) + \frac{R_{ii}+R_{jj}-1}{p-n} &= K(P_{ij}) + \frac{\sum_{s\neq i} K(P_{is}) + \sum_{s\neq j} K(P_{sj}) - 1}{p-n} \\ &\le \Big(1 + \frac{2(n-1)}{p-n}\Big)K_0 - \frac{1}{p-n} \\ &\le \frac{1}{p-n}((p+n-2)K_0 - 1). \end{aligned}$$

Therefore, we conclude that the sectional curvature $\widetilde{K}$ of $(M, e^{2f/(p-n)}g)$ is nonpositive since $p+n-2 \le \frac{1}{K_0}$. □

Proof of Theorem 1.1. Suppose there is a solution u to the Dirichlet problem at infinity and $u = \varphi$ on $M(\infty)$ for some nonconstant function $\varphi \in C^0(M(\infty))$. Then u is continuous on $M \cup M(\infty)$, hence it is bounded. Then $u - \inf_M u + 1$ is a positive solution to the p-Laplace equation on $(M, \tilde{g})$, therefore it must be constant from Proposition 2.1. Thus, u is constant on M and φ must be constant on $M(\infty)$. The contradiction concludes the proof. □

When $\mathbb{R}^n$ is viewed as a shrinking gradient Ricci soliton with $f(x) = |x|^2/4$, we can take $K_0 = 0$ and obtain the following corollary.

Corollary 2.3. *The Dirichlet problem at infinity for the p-Laplace equation is unsolvable on $(\mathbb{R}^n, e^{|x|^2/(2(p-n))}dx^2)$ for every $p > n$.*

Remark. The sectional curvature of $\tilde{g} = e^{2|x|^2/(4(p-n))}dx^2$ can be computed from (2-5):

$$\widetilde{K}(P_{ij})(x) = -e^{-|x|^2/(2(p-n))}\Big(\frac{1}{p-n} + \frac{|x|^2-(x^i)^2-(x^j)^2}{4(p-n)^2}\Big)$$

where $P_{ij}(x)$ is the plane spanned by $\{\partial/\partial x^i, \partial/\partial x^j\}$ at $x \in \mathbb{R}^n$. The Riemannian distance from x to the origin is

$$r(x) = \int_0^{|x|} e^{s^2/(4(p-n))}\, ds.$$

If we take $x = (0, \dots, 0, x^i, 0, \dots, 0)$, then $|x|^2 - (x^i)^2 - (x^j)^2 = 0$ and

$$\begin{aligned}\lim_{|x|\to\infty} -\widetilde{K}(P_{ij}(x))r^2(x) &= \lim_{|x|\to\infty} \frac{\big(\int_0^{|x|} e^{s^2/(4(p-n))}\, ds\big)^2}{(p-n)e^{|x|^2/(2(p-n))}} \\ &= \frac{1}{p-n}\Big(\lim_{|x|\to\infty} \frac{2(p-n)}{|x|}\Big)^2 = 0\end{aligned}$$

by l'Hôpital's rule. This in particular shows that there does not exist a constant $\alpha > 1$ for which

$$K(x) \le -\frac{\alpha(\alpha-1)}{r^2(x)}$$

for all sections at x for large $r(x)$.

3. A Liouville theorem on $\mathbb{R}^2$ with an incomplete metric with positive curvature

In this section, we consider the p-Laplace equation weighted by a smooth function f on a manifold (M, g), which is equivalent to the p-Laplace equation on $(M, e^{-2f/(n-p)}g)$, and derive a Bochner formula for its solutions. Specialized to the shrinking or steady gradient Ricci solitons, the Bochner formula yields a maximum principle, and this is applied to Hamilton's cigar soliton.

A Bochner type formula for the weighted p-Laplace equation. Let g be a Riemannian metric on an n-dimensional manifold M, and let f be a smooth real-valued function on M. Consider the equation

$$\operatorname{div}(|\nabla u|^{p-2}\nabla u) - |\nabla u|^{p-2}\langle\nabla f, \nabla u\rangle = 0 \tag{3-1}$$

on M. This equation has a variational structure; in fact, it is the Euler–Lagrange equation of the weighted p-energy functional

$$E_{p,f}(u) = \int_M |\nabla u|^p e^{-f}\, d\mu_g.$$

We call (3-1) the f-weighted p-Laplacian equation on (M, g).

Proposition 3.1. *Under a conformal change $\tilde{g} = e^{-2f/(n-p)}g$, u is a solution to* (3-1) *on (M, g) if and only if u is a solution to the p-Laplace equation* (1-1) *on $(M, \tilde{g})$.*

Proof. We write ∇ for ∇_g and $\widetilde{\nabla}$ for $\nabla_{\tilde{g}}$. For any $\varphi \in C_0^\infty(M)$,

$$\begin{aligned}
\int_M \langle \widetilde{\nabla}\varphi, |\widetilde{\nabla}u|_{\tilde{g}}^{p-2}\widetilde{\nabla}u\rangle_{\tilde{g}}\, d\mu_{\tilde{g}} \\
&= \int_M |\widetilde{\nabla}u|_{\tilde{g}}^{p-2}\langle \widetilde{\nabla}\varphi, \widetilde{\nabla}u\rangle_{\tilde{g}}\, d\mu_{\tilde{g}} \\
&= \int_M \bigl(e^{(p-2)f/(n-p)}|\nabla u|_g^{p-2}\bigr)e^{2f/(n-p)}\langle \nabla\varphi, \nabla u\rangle_g\, e^{-nf/(n-p)}\, d\mu_g \\
&= \int_M \langle \nabla\varphi, |\nabla u|_g^{p-2}\nabla u\rangle_g\, e^{-f}\, d\mu_g.
\end{aligned}$$

This shows that any weak solution to (3-1) on (M, g) is also a weak solution to (1-1) on $(M, \tilde{g})$ and vice versa. □

Suppose $u(x, t)$ is a positive solution of (3-1). Define

$$\begin{aligned}
w &= -(p-1)\log u, \\
h &= |\nabla w|^2.
\end{aligned}$$

We consider the symmetric $n \times n$ matrix

$$A = \mathrm{id} + (p-2)\frac{\nabla w \otimes \nabla w}{h}.$$

Note that A is well defined whenever $h > 0$ and is positive definite for $p > 1$. Arising from the linearized operator of the nonlinear p-harmonic equations, this matrix was first introduced in [Moser 2007] and was used in [Kotschwar and Ni 2009; Wang and Zhang 2011] to study positive p-harmonic functions.

For the f-weighted p-Laplace equation (3-1), the linearized operator is

$$\mathcal{L}(\psi) = \mathrm{div}\bigl(h^{\frac{p}{2}-1}A(\nabla\psi)\bigr) - h^{\frac{p}{2}-1}\langle \nabla f, A(\nabla\psi)\rangle - ph^{\frac{p}{2}-1}\langle \nabla w, \nabla\psi\rangle$$

for smooth functions ψ on M, and the following Bochner type formula holds.

Proposition 3.2. *Let u be a positive smooth solution to* (3-1) *in an open subset U in M and assume $h > 0$ on U. Then*

$$\begin{aligned}
\text{(3-2)}\quad & \mathrm{div}\bigl(h^{\frac{p}{2}-1}A(\nabla h)\bigr) - h^{\frac{p}{2}-1}\langle \nabla f, A(\nabla h)\rangle - ph^{\frac{p}{2}-1}\langle \nabla w, \nabla h\rangle \\
&= \bigl(\tfrac{p}{2}-1\bigr)|\nabla h|^2 h^{\frac{p}{2}-2} + 2h^{\frac{p}{2}-1}\bigl(|\nabla\nabla w|^2 + \mathrm{Ric}(\nabla w, \nabla w) + \nabla\nabla f(\nabla w, \nabla w)\bigr).
\end{aligned}$$

Proof. Using (3-1), we first observe

$$\begin{aligned}\text{(3-3)}\quad \operatorname{div}(|\nabla w|^{p-2}\nabla w) - |\nabla w|^p &= -(p-1)^{p-1}\operatorname{div}\Big(\frac{|\nabla u|^{p-2}\nabla u}{u^{p-1}}\Big) - (p-1)^p\frac{|\nabla u|^p}{u^p}\\ &= -(p-1)^{p-1}\frac{|\nabla u|^{p-2}\langle\nabla f, \nabla u\rangle}{u^{p-1}}\\ &= |\nabla w|^{p-2}\langle\nabla f, \nabla w\rangle.\end{aligned}$$

Then we calculate directly

$$\begin{aligned}&\operatorname{div}\big(h^{\frac{p}{2}-1}A(\nabla h)\big)\\ &\quad= \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}|\nabla h|^2 + h^{\frac{p}{2}-1}\Delta h + \big(\tfrac{p}{2}-2\big)(p-2)h^{\frac{p}{2}-3}\langle\nabla w, \nabla h\rangle^2\\ &\qquad + (p-2)h^{\frac{p}{2}-2}\langle\nabla w, \nabla h\rangle\Delta w + (p-2)h^{\frac{p}{2}-2}\langle\nabla\langle\nabla w, \nabla h\rangle, \nabla w\rangle.\end{aligned}$$

Using the standard Bochner type formula for $h = |\nabla w|^2$, namely

$$\Delta h = 2|\nabla\nabla w|^2 + 2\operatorname{Ric}(\nabla w, \nabla w) + 2\langle\nabla\Delta w, \nabla w\rangle,$$

we have

$$\begin{aligned}\text{(3-4)}\quad &\operatorname{div}\big(h^{\frac{p}{2}-1}A(\nabla h)\big)\\ &= \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}|\nabla h|^2 + 2h^{\frac{p}{2}-1}\big(|\nabla\nabla w|^2 + \operatorname{Ric}(\nabla w, \nabla w) + \langle\nabla\Delta w, \nabla w\rangle\big)\\ &\quad + \big(\tfrac{p}{2}-2\big)(p-2)h^{\frac{p}{2}-3}\langle\nabla w, \nabla h\rangle^2 + (p-2)h^{\frac{p}{2}-2}\langle\nabla w, \nabla h\rangle\Delta w\\ &\quad + (p-2)h^{\frac{p}{2}-2}\langle\nabla\langle\nabla w, \nabla h\rangle, \nabla w\rangle.\end{aligned}$$

Rewrite (3-3) by using $h = |\nabla w|^2$ as

$$\text{(3-5)}\qquad h^{\frac{p}{2}-1}\Delta w + \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}\langle\nabla h, \nabla w\rangle - h^{\frac{p}{2}} = h^{\frac{p}{2}-1}\langle\nabla f, \nabla w\rangle.$$

Taking the gradient of both sides of (3-5) and then taking the product with ∇w, we are led to

$$\begin{aligned}\text{(3-6)}\quad &\big(\tfrac{p}{2}-1\big)\big(\tfrac{p}{2}-2\big)h^{\frac{p}{2}-3}\langle\nabla w, \nabla h\rangle^2 + \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}\langle\nabla\langle\nabla w, \nabla h\rangle, \nabla w\rangle\\ &\quad + h^{\frac{p}{2}-1}\langle\nabla\Delta w, \nabla w\rangle + \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}\langle\nabla h, \nabla w\rangle\Delta w - \tfrac{p}{2}h^{\frac{p}{2}-1}\langle\nabla h, \nabla w\rangle\\ &= \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}\langle\nabla f, \nabla w\rangle\langle\nabla h, \nabla w\rangle + h^{\frac{p}{2}-1}\langle\nabla\langle\nabla f, \nabla w\rangle, \nabla w\rangle.\end{aligned}$$

Adding (3-4) and twice (3-6) together and then simplifying, we have

$$\begin{aligned}\text{(3-7)}\quad &\operatorname{div}\big(h^{\frac{p}{2}-1}A(\nabla h)\big) - ph^{\frac{p}{2}-1}\langle\nabla h, \nabla w\rangle\\ &= \big(\tfrac{p}{2}-1\big)h^{\frac{p}{2}-2}|\nabla h|^2 + 2h^{\frac{p}{2}-1}|\nabla\nabla w|^2 + 2h^{\frac{p}{2}-1}\operatorname{Ric}(\nabla w, \nabla w)\\ &\quad + (p-2)h^{\frac{p}{2}-2}\langle\nabla f, \nabla w\rangle\langle\nabla h, \nabla w\rangle + 2h^{\frac{p}{2}-1}\langle\nabla\langle\nabla f, \nabla w\rangle, \nabla w\rangle.\end{aligned}$$

We also have

$$
\begin{aligned}
(3\text{-}8)\quad 2h^{\frac{p}{2}-1}\langle\nabla\langle\nabla f,\nabla w\rangle,\nabla w\rangle
&= 2h^{\frac{p}{2}-1}(\nabla\nabla f)(\nabla w,\nabla w)+2h^{\frac{p}{2}-1}(\nabla\nabla w)(\nabla f,\nabla w)\\
&= 2h^{\frac{p}{2}-1}(\nabla\nabla f)(\nabla w,\nabla w)+h^{\frac{p}{2}-1}\langle\nabla f,\nabla|\nabla w|^2\rangle\\
&= 2h^{\frac{p}{2}-1}(\nabla\nabla f)(\nabla w,\nabla w)+h^{\frac{p}{2}-1}\langle\nabla f,\nabla h\rangle.
\end{aligned}
$$

Moreover,

$$
\begin{aligned}
(3\text{-}9)\quad h^{\frac{p}{2}-1}\langle\nabla f,A(\nabla h)\rangle &= h^{\frac{p}{2}-1}\langle\nabla f,\nabla h\rangle+(p-2)h^{\frac{p}{2}-2}\langle\nabla f,(\nabla w\otimes\nabla w)\nabla h\rangle\\
&= h^{\frac{p}{2}-1}\langle\nabla f,\nabla h\rangle+(p-2)h^{\frac{p}{2}-2}\langle\nabla f,\nabla w\rangle\langle\nabla h,\nabla w\rangle.
\end{aligned}
$$

Now, (3-7) − (3-9) + (3-8) yields the desired result. □

A maximum principle. When the triple (M, g, f) is either shrinking or steady, Proposition 3.2 can be used to prove the following maximum principle.

Proposition 3.3. *Let u be a positive smooth solution to* (3-1) *in a bounded connected open subset U in M with smooth boundary ∂U, $p > 1$. Suppose (M, g, f) is a shrinking or steady gradient Ricci soliton. Then $|\nabla u|/u$ attains its maximum on ∂U.*

Proof. Let $h = (p-1)^2|\nabla u|^2/u^2$. Assume $\max_{\overline{U}} h > \max_{\partial U} h$. Then there exists $x_0 \in U$ such that $h(x_0) = \max_{\overline{U}} h > 0$. Since $u \in C^{1,\alpha}$ and $u > 0$, h is continuous. Let

$$V = \{x \in U : h(x) = h(x_0)\}.$$

By the continuity of h, V is a closed subset of U and V does not intersect ∂U. In fact, h is positive and hence smooth in a neighborhood of V. There exists a point $x_1 \in V$ such that for some r_0 the geodesic ball $B_{x_1}(r, g) \subset U$ is not contained in V for any $0 < r < r_0$, i.e., x_1 is a boundary point of V. By the continuity of h again, there is a geodesic ball $B_{x_1}(r_1, g)$ in U on which h is positive. Observe that

$$
\begin{aligned}
\text{RHS of (3-2)} &= \tfrac{p-2}{2}|\nabla h|^2h^{\frac{p}{2}-2}+2h^{\frac{p}{2}-1}|\nabla\nabla w|^2+2h^{\frac{p}{2}-1}(\mathrm{Ric}+\nabla\nabla f)(\nabla w,\nabla w)\\
&\ge 2h^{\frac{p}{2}-1}(\mathrm{Ric}+\nabla\nabla f)(\nabla w,\nabla w)\\
&= \begin{cases} 2h^{\frac{p}{2}-1}|\nabla w|^2 \ge 0 & \text{if } (M,g,f) \text{ is a shrinking soliton,}\\ 0 & \text{if } (M,g,f) \text{ is a steady soliton,}\end{cases}
\end{aligned}
$$

where for the first inequality, we argue as

$$
\begin{aligned}
4h|\nabla\nabla w|^2+(p-2)|\nabla h|^2 &\ge 4|\nabla w|^2|\nabla\nabla w|^2-|\nabla|\nabla w|^2|^2\\
&= 4|\nabla w|^2(|\nabla\nabla w|^2-|\nabla|\nabla w||^2)\\
&\ge 0
\end{aligned}
$$

by Kato's inequality and $p \geq 1$. Then it follows that the linear differential operator $\mathcal{L}$ satisfies $\mathcal{L}(h) \geq 0$ on U. Next, since A is positive definite and symmetric on $B_{x_1}(r_1, g)$, so is $h^{\frac{p}{2}-1}A$; therefore, $\mathcal{L}$ is uniformly elliptic on $B_{x_1}(r_1, g)$. By Hopf's strong maximum principle (see Theorem 3.5 in [Gilbarg and Trudinger 1998]), h must be a constant on $B_{x_1}(r_1, g)$ since it attains its maximum at the interior point x_1. But this contradicts the maximality of V as $B_{x_1}(r_1, g)$ contains points not in V. □

Gradient estimates. Let us first recall the following gradient estimate:

Theorem 3.4 [Wang and Zhang 2011]. *Let (M^n, g) be a complete Riemannian manifold with* Ric $\geq -(n-1)\kappa$ *for some positive constant κ. Assume that v is a positive p-harmonic function on the geodesic ball $B_{x_0}(R, g) \subset M$. Then*

$$\frac{|\nabla v|}{v} \leq C(p, n)\Big(\frac{1}{R} + \sqrt{\kappa}\Big)$$

on $B_{x_0}\big(\frac{R}{2}, g\big)$ for some constant $C(p, n)$.

We now prove a gradient estimate for the f-weighted p-Laplacian equation.

Proposition 3.5. *Let (M^n, g, f) be a complete gradient Ricci soliton with*

$$\Big(\frac{2-p}{n-p}\Big)\operatorname{Ric} \geq -(n-1)\kappa e^{-2f/(n-p)} g - \frac{2\varepsilon g}{n-p} - \frac{Sg}{n-p} - (df \otimes df - |\nabla f|^2 g)\frac{n-2}{(n-p)^2}, \tag{3-10}$$

where S is the scalar curvature of (M, g). Assume that u is a positive solution of equation (3-1). *Then there exists a constant $C(p, n)$ such that*

$$\frac{|\nabla u(x)|}{u(x)} \leq C(p, n)\Big(\frac{1}{R} + \sqrt{\kappa}\Big)e^{-f(x)/(n-p)}$$

for $x \in B_{x_0}\big(\frac{R}{2}, e^{-2f/(n-p)}g\big)$.

Proof. For a smooth function f, let ∇f be the gradient, Δf the Laplacian, and $\nabla\nabla f$ the Hessian with respect to g. For the conformal change of metrics $\tilde{g} = e^{-2f/(n-p)}g$, the Ricci tensors of $\tilde{g}$ and g are related by

$$\widetilde{\operatorname{Ric}} = \operatorname{Ric} - (n-2)\Big(-\frac{\nabla\nabla f}{n-p} - \frac{df \otimes df}{(n-p)^2}\Big) + \Big(-\frac{\Delta f}{n-p} - \frac{n-2}{(n-p)^2}|\nabla f|^2\Big)g \tag{3-11}$$

(see [Anderson and Schoen 1985, p. 59]).

From the gradient Ricci soliton equation (2-1), the scalar curvature S of M satisfies the two equations

$$S + \Delta f - n\varepsilon = 0, \tag{3-12}$$

$$S + |\nabla f|^2 + \varepsilon f = 0 \tag{3-13}$$

(see [Besse 1987]).

Putting (2-1) and (3-12) into (3-11), we have

$$\begin{aligned}\widetilde{\mathrm{Ric}} &= \mathrm{Ric} + (n-2)\Big(\frac{-\mathrm{Ric} - \varepsilon g}{n-p} + \frac{df \otimes df}{(n-p)^2}\Big) + \Big(\frac{S + n\varepsilon}{n-p} - \frac{n-2}{(n-p)^2}|\nabla f|^2\Big)g \\ &= \frac{2-p}{n-p}\mathrm{Ric} + \frac{2\varepsilon g}{n-p} + \frac{Sg}{n-p} + (df \otimes df - |\nabla f|^2 g)\frac{n-2}{(n-p)^2}.\end{aligned}$$

Therefore, the curvature assumption in Proposition 3.5 implies

$$\widetilde{\mathrm{Ric}} \geq -(n-1)\kappa.$$

By Proposition 3.1, we know that u is also a positive solution to (1-1) for the metric $\tilde{g}$, hence by Theorem 3.4 we have

$$\frac{|\nabla u|_{\tilde{g}}}{u} \leq C(p,n)\Big(\frac{1}{R} + \sqrt{\kappa}\Big)$$

on $B_{x_0}\big(\frac{R}{2}, \tilde{g}\big)$. This is equivalent to

$$\frac{|\nabla u(x)|}{u(x)} \leq C(p,n)\Big(\frac{1}{R} + \sqrt{\kappa}\Big)e^{-f(x)/(n-p)}$$

for $x \in B_{x_0}\big(\frac{R}{2}, \tilde{g}\big)$. □

A Liouville type theorem for the p-Laplace equation in dimension 2. For a steady gradient Ricci soliton, the condition (3-10) on the Ricci curvature in Proposition 3.5 cannot hold globally when $n \geq 3$ because it would imply, by taking the trace, that the scalar curvature is bounded below by a positive constant, which is impossible. However, the condition (3-10) is satisfied when $n = 2$ for $p \geq 4$ or $1 < p < 2$ because

$$\mathrm{Ric} = \tfrac{1}{2}Sg \geq \tfrac{1}{p-2}Sg,$$

since $S \geq 0$ for any steady gradient Ricci soliton [Chen 2009] and $\kappa = 0$.

Note that Hamilton's cigar soliton is the unique 2-dimensional nonflat complete noncompact steady gradient Ricci soliton. The cigar soliton is $\mathbb{R}^2$ equipped with the complete metric

$$g = \frac{dx^2 + dy^2}{1 + x^2 + y^2}$$

(see [Chow et al. 2006]) and the potential function

$$f(x,y) = -\log(1 + x^2 + y^2).$$

The conformally altered metric is

$$\tilde{g} = e^{2\log(1+x^2+y^2)/(2-p)}g = (1 + x^2 + y^2)^{p/(2-p)}(dx^2 + dy^2).$$

In particular, $\tilde{g}$ is complete if $1 < p < 2$ and incomplete if $p > 2$. However, to use the gradient estimate in proving a Liouville type result, we will need $p \geq 4$. It is straightforward to compute the Gauss curvature of $\tilde{g}$:

$$\begin{aligned}\widetilde{K} &= -\frac{1}{2}(1+r^2)^{p/(p-2)}\Big(\partial^2_{rr}+\frac{1}{r}\partial_r\Big)\log(1+r^2)^{-p/(p-2)}\\ &= \frac{2p}{p-2}(1+r^2)^{(p/(p-2))-2}\\ &= \frac{2p}{p-2}(1+r^2)^{-(p-4)/(p-2)}\end{aligned}$$

which is positive and tends to 0 as $r \to \infty$ if $p > 4$. When $p = 4$, the incomplete metric $(1+x^2+y^2)^{-2}(dx^2+dy^2)$ has constant curvature $\widetilde{K} = 4$.

Theorem 3.6. *Let $(\mathbb{R}^2, g, f)$ be Hamilton's cigar soliton. Then there does not exist any nonconstant positive p-harmonic function on $(\mathbb{R}^2, \tilde{g})$ for $p \geq 4$.*

Proof. Let u be a positive solution to (3-1). For any point $x_0 \in M$, the maximum principle (Proposition 3.3) asserts

$$\frac{|\nabla u(x_0)|}{u(x_0)} \leq \max_{x\in\partial B_0(R,g)} \frac{|\nabla u(x)|}{u(x)} = \frac{|\nabla u(x_R)|}{u(x_R)}$$

for some $x_R \in \partial B_0(R, g)$ where $x_0 \in B_0(R, g)$ and $r(x_0, 0) < R$. From the discussion above, when $n = 2$ and $p \geq 4$, the Ricci curvature condition (3-10) in Proposition 3.5 is satisfied. The diameter of $(\mathbb{R}^2, \tilde{g})$ is

$$2R_0 = 2\int_0^\infty \frac{dr}{(1+r^2)^{p/(2(p-2))}} < \infty.$$

It is clear that $r(x_R, 0) \to \infty$ if and only if $\tilde{r}(x_R, 0) \to R_0$, where $\tilde{r}$ denotes the distance function for the metric $\tilde{g}$. Let

$$r_R = \int_R^\infty \frac{dr}{(1+r^2)^{p/(2(p-2))}}.$$

It follows from Proposition 3.5, applied on the ball $B_{x_R}(r_R, \tilde{g})$, that

$$\begin{aligned}\frac{|\nabla u(x_R)|}{u(x_R)} &\leq C(n, p)\Big(\frac{r_{x_R}}{2}\Big)^{-1} e^{-2\log(1+|x_R|^2)/(p-2)}\\ &= 2C(n, p)\bigg(\int_R^\infty \frac{dr}{(1+r^2)^{p/(2(p-2))}}(1+R^2)^{2/(p-2)}\bigg)^{-1}\\ &\leq 2C(n, p)\bigg((1+R^2)^{2/(p-2)}\int_R^\infty \frac{dr}{r^{p/(p-2)}}\bigg)^{-1}\\ &= 2C(n, p)\Big(\frac{p-2}{2}(1+R^2)^{2/(p-2)}R^{-2/(p-2)}\Big)^{-1}.\end{aligned}$$

Since $p > 2$, letting $R \to 0$ we conclude $|\nabla u(x_0)| = 0$, hence u is constant as x_0 is arbitrary. $\square$

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