Pacific Journal of Mathematics

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Volume 282 No. 2 June 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 282, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.282.341

ON THE UNIFORM SQUEEZING PROPERTY OF BOUNDED CONVEX DOMAINS IN $\mathbb{C}^n$

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We prove that the bounded convex domains and the C^2-smoothly bounded strongly pseudoconvex domains in $\mathbb{C}^n$ admit the uniform squeezing property. Moreover, we prove by the scaling method that the squeezing function approaches 1 near the strongly pseudoconvex boundary points.

1. Introduction

The notion of *holomorphic homogeneous regular*, or equivalently, *uniformly squeezing* for complex manifolds has been introduced in [Liu et al. 2004; 2005] and [Yeung 2009], respectively. This concept is essential for the estimation of several invariant metrics. See the above cited papers for details.

Let Ω be a complex manifold of dimension n. The *squeezing function* $\sigma_\Omega : \Omega \to \mathbb{R}$ of Ω is defined in [Deng et al. 2012] as follows. For each $p \in \Omega$, let

$$\mathcal{F}(p, \Omega) := \{f : \Omega \to \mathbb{B}^n : f \text{ is 1-1 holomorphic, } f(p) = 0\},$$

where

- $\mathbb{B}^n(p; r) = \{z \in \mathbb{C}^n : \|z - p\| < r\}$, and
- $\mathbb{B}^n = \mathbb{B}^n(\mathbf{0}; 1) = \mathbb{B}^n((0, \dots, 0); 1)$.

Then define

$$\sigma_\Omega(p) = \sup\{r : \mathbb{B}^n(\mathbf{0}; r) \subset f(\Omega) \text{ for some } f \in \mathcal{F}(p, \Omega)\}.$$

Furthermore, the *squeezing constant* $\hat{\sigma}_\Omega$ for Ω is defined by

$$\hat{\sigma}_\Omega := \inf_{p \in \Omega} \sigma_\Omega(p).$$

Definition ([Liu et al. 2004; 2005; Yeung 2009]). A complex manifold Ω is called *holomorphic homogeneous regular* (HHR), or equivalently *uniformly squeezing*, if $\hat{\sigma}_\Omega > 0$.

MSC2010: primary 32F45; secondary 32A25.
Keywords: squeezing function, convex domains, scaling method.

Notice that the HHR property is preserved by biholomorphisms. The squeezing function and squeezing constants are also biholomorphic invariants [Deng et al. 2012].

These concepts have been developed for the study of completeness and other geometric properties such as the metric equivalence of the invariant metrics, including the Carathéodory, Kobayashi–Royden, Teichmüller, Bergman, and Kähler–Einstein metrics. It is obvious that the examples of HHR manifolds include bounded homogeneous domains. In case the manifold is biholomorphic to a bounded domain and the holomorphic automorphism orbits accumulate at every boundary point, such as in the case of the Bers embedding of the Teichmüller space, again the HHR property holds. A somewhat less obvious example are the bounded strongly convex domains (as the majority of them do not possess any holomorphic automorphisms except the identity map), proved by S.-K. Yeung [2009]. But there, some of the most standard examples, such as the bounded convex domains and the bounded strongly pseudoconvex domains, were left untouched.

Indeed, the starting point of this article is to show:

Theorem 1.1. *All bounded convex domains in* $\mathbb{C}^n$ $(n \geq 1)$ *are HHR.*

Note that we do not assume any additional conditions such as boundary smoothness or "finite type" in the sense of D'Angelo in the above theorem. Nevertheless, the concept of squeezing function σ_Ω defined above plays an important role, and moreover, it appeals to us that further investigations of this function would be worthwhile. One immediate observation is that if $\sigma_\Omega(p) = 1$ for some $p \in \Omega$, then Ω is biholomorphic to the unit open ball. In light of studies on the asymptotic behavior of several invariant metrics of strongly pseudoconvex domains, perhaps it is natural to ask, for a bounded strongly pseudoconvex domain Ω in $\mathbb{C}^n$, whether $\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1$ holds for every boundary point $p \in \partial\Omega$.

It was proved in [Deng et al. 2015] that the HHR property holds for all bounded strongly pseudoconvex domains, using an improvement of the method in [Fridman and Ma 1995]. In the present paper, by using a different approach — the scaling method — we will prove:

Theorem 1.2. *If* Ω *is a bounded domain in* $\mathbb{C}^n$ *with a* C^2 *strongly convex boundary, then* $\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1$ *for every* $p \in \partial\Omega$.

Actually, we have a more general conclusion in Theorem 3.1, which implies Theorem 1.2. The question posed above follows quickly from Theorem 3.1 and the following remarkable theorem of Diederich, Fornaess, and Wold [2014].

Theorem 1.3 [Diederich et al. 2014, Theorem 1.1]. *Let* $\Omega \subset \mathbb{C}^n$ *be a bounded domain which is locally convexifiable and has finite type* $2k$ *near a point* $p \in \partial\Omega$. *Assume further that* $\partial\Omega$ *is* C^∞*-smooth near* p, *and that* $\overline{\Omega}$ *has a Stein neighborhood* *basis. Then there exists a holomorphic embedding* $f : \overline{\Omega} \to \overline{B}^n_k$, *where*

$$B^n_k = \{z \in \mathbb{C}^2 : |z_n|^2 + |z'|^{2k} < 1\},$$

such that $f(p) = (0, \dots, 0, 1)$ *and* $\{z \in \overline{\Omega} : f(z) \in \partial B^n_k\} = \{p\}$.

In particular, if $\partial\Omega$ *is strongly pseudoconvex near* p (*i.e.*, $k = 1$), *it is enough to assume that* $\partial\Omega$ *is* C^2*-smooth near* p.

We mention here that the proof of Theorem 1.2 is of interest in its own right, and also clarifies and simplifies some previously known theorems. These are mentioned in the final section.

2. Bounded convex domains are HHR manifolds

The aim of this section is to establish Theorem 1.1 stated above. Not only does this theorem cover the case left untreated in [Yeung 2009], but our method is different (see also [Deng et al. 2012] on this matter). Our method uses a version of the "scaling method in several complex variables" initiated by S. Pinchuk [1991]. In fact, we use the version presented in [Kim 1992], modified for the purpose of studying the asymptotic boundary behavior of holomorphic invariants.

Proof of Theorem 1.1. We proceed in 5 steps.

Step 1. *Setup*. Let Ω be a bounded convex domain in $\mathbb{C}^n$. Suppose that Ω is not HHR. Then there exists a sequence $\{q_j\}$ in Ω converging to a boundary point, say $q \in \partial\Omega$, such that

$$\lim_{j\to\infty} \sigma_\Omega(q_j) = 0.$$

Needless to say, it suffices to show that such a sequence cannot exist.

Step 2. *The j-th orthonormal frame*. Let $\langle\, , \rangle$ represent the standard Hermitian inner product of $\mathbb{C}^n$, and let $\|v\| = \sqrt{\langle v, v\rangle}$. For every $q \in \mathbb{C}^n$ and complex linear subspace V of $\mathbb{C}^n$, denote by

$$B^V(q, r) = \{p \in \mathbb{C}^n : p - q \in V \text{ and } \|p - q\| < r\}.$$

Now let $q \in \Omega$ and define the positive number $\lambda(q, V)$ by

$$\lambda(q, V) = \max\{r > 0 : B^V(q, r) \subset \Omega\}.$$

This number is finite for each (q, V), whenever $\dim V > 0$, since Ω is Kobayashi hyperbolic.

Fix the index j momentarily. Then we choose an orthonormal basis for $\mathbb{C}^n$, with respect to the standard Hermitian inner product $\langle\, , \rangle$. First consider

$$\lambda^1_j := \lambda(q_j, \mathbb{C}^n).$$

There exists $q_j^{1*} \in \partial\Omega$ such that $\|q_j^{1*} - q_j\| = \lambda_j^1$. Let

$$e_j^1 = \frac{q_j^{1*} - q_j}{\|q_j^{1*} - q_j\|}.$$

Then consider the complex span $\mathrm{Span}_{\mathbb{C}}\{e_j^1\}$, and let V^1 be its orthogonal complement in $\mathbb{C}^n$. Take

$$\lambda_j^2 := \lambda(q_j, V^1)$$

and $q_j^{2*} \in \partial\Omega$ such that $q_j^{2*} - q_j \in V^1$ and $\|q_j^{2*} - q_j\| = \lambda_j^2$. Then let

$$e_j^2 := \frac{q_j^{2*} - q_j}{\|q_j^{2*} - q_j\|}.$$

With $e_j^1, e_j^2, \ldots, e_j^{\ell*}$ and $\lambda_j^1, \lambda_j^2, \ldots, \lambda_j^\ell$ chosen, the next elements $e_j^{\ell+1*}$ and $\lambda_j^{\ell+1}$ are selected as follows. Denote by V^ℓ the complex orthogonal complement of $\mathrm{Span}_{\mathbb{C}}\{e_j^1, e_j^2, \ldots, e_j^\ell\}$. Then

$$\lambda_j^{\ell+1} := \lambda(q_j, V^\ell)$$

and $q_j^{\ell+1*} \in \partial\Omega$ are such that $q_j^{\ell+1*} - q_j \in V^\ell$ and $\|q_j^{\ell+1*} - q_j\| = \lambda_j^{\ell+1}$. Let

$$e_j^{\ell+1} := \frac{q_j^{\ell+1*} - q_j}{\|q_j^{\ell+1*} - q_j\|}.$$

By induction, this process yields an orthonormal set $e_j^1, \ldots, e_j^n$ for $\mathbb{C}^n$ and the positive numbers $\lambda_j^1, \ldots, \lambda_j^n$.

Step 3. *Stretching complex linear maps*. Let $\hat{e}^1, \ldots, \hat{e}^n$ denote the standard orthonormal basis for $\mathbb{C}^n$, i.e.,

$$\hat{e}^1 = (1, 0, \ldots, 0),\ \hat{e}^2 = (0, 1, 0, \ldots, 0),\ \ldots,\ \hat{e}^n = (0, \ldots, 0, 1).$$

Define the *stretching linear map* $L_j : \mathbb{C}^n \to \mathbb{C}^n$ by

$$L_j(z) = \sum_{k=1}^n \frac{\langle z - q_j, e_j^k \rangle}{\lambda_j^k}\, \hat{e}^k$$

for every $z \in \mathbb{C}^n$. Note that for each j, L_j maps Ω biholomorphically onto its image.

Step 4. *Supporting hyperplanes*. Notice that

$$L_j(q_j) = \mathbf{0} = (0, \ldots, 0),\ L_j(q_j^{1*}) = \hat{e}^1,\ \ldots,\ L_j(q_j^{n*}) = \hat{e}^n.$$

We shall consider the supporting hyperplanes, say Π_j^k for $k = 1, \ldots, n$, of $L_j(\Omega)$ at points $L_j(q_j^{k*})$ for $k = 1, \ldots, n$, respectively.

Substep 4.1. The supporting hyperplane Π_j^1. As noted above, $L_j(q_j^{1*}) = \hat{e}^1$. Due to the choice of q_j^{1*}, the supporting hyperplane of Ω at q_j^{1*} must also support the sphere tangent to the boundary $\partial\Omega$. Consequently the supporting hyperplane Π_j^1 of $L_j(\Omega)$ must support a smooth surface (an ellipsoid) tangent to $L_j(\partial\Omega)$ at $\hat{e}^1$. Thus, the equation for this hyperplane Π_j^1 is

$$\operatorname{Re}(z_1 - 1) = 0$$

(independently of j, being perpendicular to $\hat{e}^1$). We also note that

$$L_j(\Omega) \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re} z_1 < 1\}.$$

Substep 4.2. The rest of the supporting hyperplanes Π_j^k*, for* $k \geq 2$. First consider the case $k = 2$. Then the supporting hyperplane Π_j^2 passes through $L_j(q_j^{2*}) = \hat{e}^2$. Since the restriction of Ω to V^1 contains the sphere in V^1 tangent to the restriction of $\partial\Omega$ at the point $\hat{e}^2$, the supporting hyperplane Π_j^2 restricted to $L_j(V^1)$ takes the equation $\{(z_2, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(z_2 - 1) = 0\}$. Hence

$$\Pi_j^2 = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(a_j^{2,1} z_1 + a_j^{2,2}(z_2 - 1)) = 0\}$$

for some $\left(a_j^{2,1}, a_j^{2,1}\right) \in \mathbb{C}^2$ with $\left|a_j^{2,1}\right|^2 + \left|a_j^{2,2}\right|^2 = 1$ and $a_j^{2,2} > 0$. We also have that

$$L_j(\Omega) \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(a_j^{2,1} z_1 + a_j^{2,2}(z_2 - 1)) < 0\}.$$

For $k \in \{3, \dots, n\}$, one deduces inductively that the supporting hyperplane Π_j^k passes through the point $\hat{e}^k$, and that

$$\Pi_j^k = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(a_j^{k,1} z_1 + \cdots + a_j^{k,k-1} z_{k-1} + a_j^{k,k}(z_k - 1)) = 0\},$$

with $a_j^{k,k} > 0$ and $\sum_{\ell=1}^k \left|a_j^{k,\ell}\right|^2 = 1$. Also,

$$L_j(\Omega) \subset \{(z_1, \dots, z_n) \in \mathbb{C}^n : \operatorname{Re}(a_j^{k,1} z_1 + \cdots + a_j^{k,k-1} z_{k-1} + a_j^{k,k}(z_k - 1)) < 0\}.$$

Substep 4.3. Polygonal envelopes. We add this small substep for convenience. From the discussion so far in this step, we have the j-th polygonal envelope (of $L_j(\Omega)$)

$$\begin{aligned}\Sigma_j := \{(z_1, \dots, z_n) \in \mathbb{C}^n :\ & \operatorname{Re} z_1 < 1, \\ & \operatorname{Re}(a_j^{2,1} z_1 + a_j^{2,2}(z_2 - 1)) < 0, \\ & \vdots \\ & \operatorname{Re}(a_j^{n,1} z_1 + \cdots + a_j^{n,n-1} z_{n-1} + a_j^{n,n}(z_n - 1)) < 0\}.\end{aligned}$$

Step 5. *Bounded realization.* Notice that, for every $k \in \{1, \dots, n\}$, the disc

$$D_j^k := \{z = (z_1, \dots, z_n) \in \mathbb{C}^n : \|z - q_j\| < \lambda_j^k \text{ and } \forall \ell \neq k, \langle z - q_j, e_j^\ell \rangle = 0\}$$

is contained in Ω. Hence, every $L_j(\Omega)$ contains the discs $D^k := \{\zeta \hat{e}^k : \zeta \in \mathbb{C}, |\zeta| < 1\}$ for every $k = 1, \dots, n$. Since Ω is convex and L_j is linear, $L_j(\Omega)$ is also convex. Therefore, the "unit acorn"

$$A := \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_1| + \cdots + |z_n| < 1\}$$

is contained in $L_j(\Omega)$. This restricts the unit normal vectors

$$n_j^k := (a_j^{k,1}, \dots, a_j^{k,k}, 0, \dots, 0) \in \mathbb{C}^n$$

for every $k = 2, \dots, n$. Namely, there is a positive constant $\delta > 0$ independent of j and k such that $a_j^{k,k} \geq \delta$ for every j, k.

Now taking a subsequence (of q_j), we may assume that the sequence of unit vectors $\{n_j^k\}_{j=1}^{\infty}$ converges for every $k \in \{2, \dots, n\}$. Let us write

$$\lim_{j\to\infty} n_j^k = n^k = (a^{k,1}, \dots, a^{k,k}, 0, \dots, 0)$$

for each $k = 1, 2, \dots, n$.

Consider the maps

$$B_j(z_1, \dots, z_n) = (\zeta_1, \dots, \zeta_n)$$

defined by

$$\zeta_1 = z_1,\ \zeta_2 = a_j^{2,1} z_1 + a_j^{2,2} z_2,\ \dots,\ \zeta_n = a_j^{n,1} z_1 + \cdots + a_j^{n,n} z_n.$$

Then it follows that

$$\begin{aligned} B_j \circ L_j(\Omega) &\subset B_j(\Sigma_j) \\ &= \{(\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n : \operatorname{Re}\zeta_1 < 1, \operatorname{Re}\zeta_2 < a_j^{2,2}, \dots, \operatorname{Re}\zeta_n < a_j^{n,n}\}. \end{aligned}$$

Now, for each j, we consider the Cayley transformation

$$\Phi_j(z_1, \dots, z_n) = \left(\frac{z_1}{2 - z_1}, \frac{z_2}{2a_j^{2,2} - z_2}, \dots, \frac{z_n}{2a_j^{n,n} - z_n}\right).$$

Then $\Phi_j \circ B_j(\Sigma_j) \subset D^n$, where D^n denotes the unit polydisc in $\mathbb{C}^n$ centered at the origin. Also, there exists a positive constant $\delta' \in (0, \delta)$ such that $\Phi_j \circ B_j(\Sigma_j) \subset D^n$ contains the ball of radius δ' centered at the origin $\mathbf{0}$.

Since $\Phi_j \circ B_j \circ L_j(q_j) = (0, \dots, 0)$ for every j, we now conclude that the squeezing function satisfies

$$\sigma_\Omega(q_j) \geq \frac{\delta'}{\sqrt{n}}.$$

This estimate, which holds for every sequence q_j approaching the boundary, yields the desired contradiction at last. Thus the proof is complete. $\square$

3. Boundary behavior of the squeezing function on strongly convex domains

Definition. Let Ω be a domain in $\mathbb{C}^n$. A boundary point $p \in \partial\Omega$ is said to be *spherically extreme* if

(1) the boundary $\partial\Omega$ is C^2-smooth in an open neighborhood of p, and

(2) there exists a ball $\mathbb{B}^n(c(p); R)$ in $\mathbb{C}^n$ of some radius R, centered at some point $c(p)$ such that $\Omega \subset \mathbb{B}^n(c(p); R)$ and $p \in \partial\Omega \cap \partial\mathbb{B}^n(c(p); R)$.

The main goal of this section is to establish the following theorem.

Theorem 3.1. *If a domain Ω in $\mathbb{C}^n$ admits a spherically extreme boundary point p in a neighborhood of which the boundary $\partial\Omega$ is C^2-smooth, then*

$$\lim_{\Omega \ni q \to p} \sigma_\Omega(q) = 1.$$

Since every boundary point of a C^2 strongly convex bounded domain is spherically extreme, this theorem implies Theorem 1.2. The rest of this section is devoted to the proof of Theorem 3.1.

Proof. The proof proceeds in seven steps.

Step 1. *Sphere envelopes.* Let Ω be a bounded domain in $\mathbb{C}^n$ with a boundary point $p \in \partial\Omega$ such that

(i) $\partial\Omega \cap \mathbb{B}^n(p; r_0)$ is C^2-smooth for some $r_0 > 0$, and

(ii) p is a spherically extreme boundary point of Ω.

Then there exist positive constants r_1, r_2, and R with $r_0 > r_1 > r_2$ such that every $q \in \Omega \cap \mathbb{B}^n(p; r_2)$ admits points $b(q) \in \partial\Omega \cap \mathbb{B}^n(p; r_1)$ and $c(q) \in \mathbb{C}^n$ satisfying the conditions

(iii) $\|q - b(q)\| < \|q - z\|$ for any $z \in \partial\Omega - \{b(q)\}$, and

(iv) $\|c(q) - b(q)\| = R$ and $\Omega \subset \mathbb{B}^n(c(q); R)$.

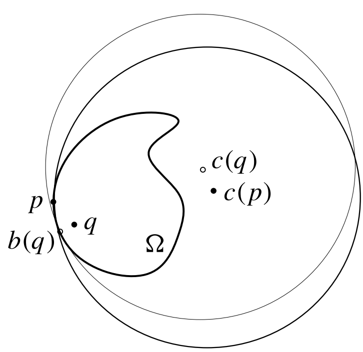

Figure 1. Sphere envelopes.

See Figure 1. Notice that (iii) says that $b(q)$ is the unique boundary point that is the closest to q, and that the constant R in (iv) is independent of the choice of $q \in \mathbb{B}^n(p; r_2)$.

Step 2. *Centering.* In the following we shall use the familiar notation

$$z = (z_1, \ldots, z_n), \qquad z' = (z_2, \ldots, z_n), \qquad u = \operatorname{Re} z_1, \qquad v = \operatorname{Im} z_1. \tag{3-1}$$

For each $q \in \Omega \cap \mathbb{B}^n(p; r_2)$, choose a unitary transform U_q of $\mathbb{C}^n$ such that the map $A_q(z) := U_q(z - b(q))$, depicted in Figure 2, satisfies the following conditions:

$$A_q(q) = (\lambda_q, 0, \ldots, 0) \tag{3-2}$$

for some $\lambda_q > 0$, and

$$A_q(\Omega) \subset \mathbb{B}^n((R, 0, \ldots, 0); R) = \{z \in \mathbb{C}^n : |z_1 - R|^2 + \|z'\|^2 < R^2\}. \tag{3-3}$$

Then there exists a positive constant $r_3 < r_2$ such that

$$z \in A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; r_3) \iff \|z\| < r_3 \text{ and } 2u > H_{b(q)}(z') + \mathcal{K}_{b(q)}(v, z') + \mathcal{R}_{b(q)}(v, z'), \tag{3-4}$$

where

- $H_{b(q)}$ is a quadratic positive-definite Hermitian form such that there exists a constant $c_0 > 0$, independent of q, satisfying

$$H_{b(q)}(z') \ge c_0 \|z'\|^2, \tag{3-5}$$

  and

- there exists a constant $C > 0$, independent of $q \in \mathbb{B}^n(p; r_3) \cap \Omega$, such that

$$|\mathcal{K}_{b(q)}(v, z')| \le C(|v|^2 + |v|\|z'\|) \tag{3-6}$$

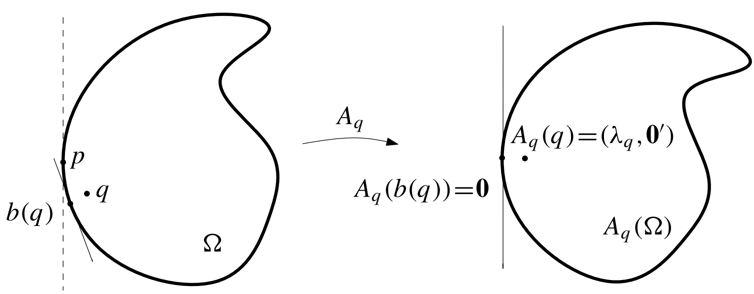

Figure 2. The centering process.

whenever $z \in \mathbb{B}^n(\mathbf{0}; r_3)$. Furthermore, we have

$$|\mathcal{R}_{b(q)}(v, z')| = o(|v|^2 + \|z'\|^2).$$

In particular, the choice of r_3 can allow us the estimate

$$|\mathcal{R}_{b(q)}(v, z')| \le \frac{c_0}{2}(|v|^2 + \|z'\|^2).$$

Notice that

$$\lim_{\Omega \ni q \to p} b(q) = p, \qquad \lim_{\Omega \ni q \to p} H_{b(q)}(z') = H_p(z'),$$

and

$$\lim_{\Omega \ni q \to p} A_q = I \text{ (the identity map).}$$

The latter limit and an inductive construction yield that for each integer $m > 2$ there exists a strictly increasing integer-valued function $k(m)$ such that

$$\mathbb{B}^n\big(\mathbf{0}; r_3/(2k(m))\big) \subset A_q\big(\mathbb{B}^n(p; r_3/k(m))\big) \subset \mathbb{B}^n(\mathbf{0}; r_3/m) \tag{3-7}$$

whenever $q \in \mathbb{B}^n\big(p; r_3/(2k(m))\big)$.

Step 3. *The Cayley transform.* The *Cayley transform* considered here is the map

$$\kappa(z) := \Big(\frac{1-z_1}{1+z_1}, \frac{\sqrt{2}z_2}{1+z_1}, \dots, \frac{\sqrt{2}z_n}{1+z_1}\Big), \tag{3-8}$$

well-defined except at points of $Z = \{z \in \mathbb{C}^n : z_1 = -1\}$. Notice that this transform maps the open unit ball $\mathbb{B}^n(\mathbf{0}; 1)$ biholomorphically onto the Siegel half-space

$$\mathcal{S}_0 := \{z \in \mathbb{C}^n : 2\,\mathrm{Re}\, z_1 > \|z'\|^2\}. \tag{3-9}$$

Moreover, $\kappa \circ \kappa = 1$ and consequently, $\kappa(\mathcal{S}_0) = \mathbb{B}^n(\mathbf{0}; 1)$. Notice also that, for $\mathbf{1} = (1, 0, \dots)$ and $-\mathbf{1} = (-1, 0, \dots)$, we have $\kappa(\mathbf{1}) = (0, \dots, 0)$, $\kappa((0, \dots, 0)) = \mathbf{1}$, $\kappa(-\mathbf{1}) = \infty$, and $\kappa(\infty) = -\mathbf{1}$.

Step 4. *Stretching.* Let $q \in \Omega \cap \mathbb{B}^n\big(p; r_3/(2k(m))\big)$. If we let m tend to infinity, then of course $A_q(q) = (\lambda_q, 0, \dots, 0)$ approaches $A_q(b(q)) = (0, \dots, 0)$, and so λ_q approaches zero. For simplicity, we write $\lambda = \lambda_q$, suppressing the q but keeping in mind that λ is still dependent upon q. Note that

$$A_q(\mathbb{B}^n(c(q); R)) = \{z \in \mathbb{C}^n : 2R \cdot \mathrm{Re}\, z_1 > \|z\|^2\}. \tag{3-10}$$

Define the *stretching map* $\Lambda_\lambda : \mathbb{C}^n \to \mathbb{C}^n$, first introduced in [Pinchuk 1991], by

$$\Lambda_\lambda(z) := \Big(\frac{z_1}{\lambda}, \frac{z_2}{\sqrt{\lambda}}, \dots, \frac{z_n}{\sqrt{\lambda}}\Big). \tag{3-11}$$

Recall (3-6). The stretching map transforms $A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; r_3/3)$ to the domain $\Lambda_\lambda\big(A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; r_3/3)\big)$ so that

$$\text{(3-12)} \quad z \in \Lambda_\lambda \circ A_q(\Omega) \cap \mathbb{B}^n\Big(\mathbf{0}; \frac{r_3}{\sqrt{\lambda}k(3)}\Big) \iff \|z\| < \frac{r_3}{\sqrt{\lambda}k(3)} \text{ and } 2u > H_{b(q)}(z') + \frac{1}{\lambda} K_{b(q)}\big(\lambda v, \sqrt{\lambda} z'\big) + \frac{1}{\lambda} \mathcal{R}_{b(q)}\big(\lambda v, \sqrt{\lambda} z'\big).$$

On the other hand, notice that

$$\Big\| \frac{1}{\lambda} K_{b(q)}\big(\lambda v, \sqrt{\lambda} z'\big) \Big\| \le C\sqrt{\lambda}\big(\sqrt{\lambda}|v|^2 + |v|\|z'\|\big)$$

and that

$$\Big\| \frac{1}{\lambda} \mathcal{R}_{b(q)}\big(\lambda v, \sqrt{\lambda} z'\big) \Big\| \le \frac{1}{\lambda} o\big((|\lambda v|^2 + \|\sqrt{\lambda} z'\|^2)\big) = \frac{1}{\lambda} o(\lambda)$$

on $\mathbb{B}^n(\mathbf{0}; \rho)$ for any fixed constant $\rho > 0$. Notice that both terms approach zero as λ tends to zero. Thus, these terms can become sufficiently small if we limit q to being contained in $\mathbb{B}^n\big(p; r_3/(2k(m))\big)$ for some sufficiently large m.

Step 5. *Set-convergence.* This step is in part heuristic; the heuristics appearing in this step, especially those which concern set convergences, are not used in the proof, strictly speaking. We include this step because they seem to help us to grasp the logical structure of the proof. On the other hand, the constructions in (3-13)–(3-15) shall be used in the remainder of the proof, especially in Step 7.

The main role of the stretching map Λ_λ, as $\lambda \searrow 0$, is to rescale the domains successively, letting them converge to the set limits.

For instance, if one considers

$$\Lambda_\lambda(A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; r_3)),$$

then one can see that $\Lambda_\lambda(\mathbb{B}^n(\mathbf{0}; r_3))$ contains $\mathbb{B}^n\big(\mathbf{0}; r_3/\sqrt{\lambda}\big)$, a very large ball which exhausts $\mathbb{C}^n$ as λ approaches zero. In the meantime, within that large ball, $\Lambda_\lambda(A_q(\Omega))$ is restricted only by the inequality

$$2u > H_{b(q)}(z') + \widetilde{K}_\lambda(v, z'),$$

where $\widetilde{K}_\lambda = o(\lambda)$ is small enough to be negligible. One can imagine that indeed the "limit domain" of this procedure should be

$$\text{(3-13)} \qquad \widehat{\Omega} := \{z \in \mathbb{C}^n : 2u > H_p(z')\}.$$

Here, of course, $H_p(z')$ is the quadratic positive-definite Hermitian form which appears in the defining inequality of Ω about the boundary point p (understood as the origin):

$$2\operatorname{Re} z_1 > H_p(z') + o(|\operatorname{Im} z_1| + \|z'\|^2).$$

Notice that

$$\kappa(\widehat{\Omega}) = \{z \in \mathbb{C}^n : |z_1|^2 + H_p(z') < 1\},$$

and hence there is a $\mathbb{C}$-linear isomorphism

$$L : \mathbb{C}^n \to \mathbb{C}^n \tag{3-14}$$

that maps $\kappa(\widehat{\Omega})$ biholomorphically onto the unit ball $\mathbb{B}^n(\mathbf{0}; 1)$ with $L(\mathbf{1}) = \mathbf{1}$.

Before moving on to the next step we remark that, since $\Omega \subset \mathbb{B}^n(c(q); R)$ whenever $q \in \mathbb{B}^n(p; r_2)$,

$$A_q(\Omega) \subset A_q(\mathbb{B}^n(c(q); R)) = \mathbb{B}^n((R, 0, \dots, 0); R).$$

This in turn implies that

$$\begin{aligned} \Lambda_\lambda \circ A_q(\Omega) &\subset \Lambda_\lambda\big(\mathbb{B}^n((R, 0, \dots, 0); R)\big) \\ &\subset \mathcal{E} := \{z \in \mathbb{C}^n : 2R \cdot \operatorname{Re} z_1 > \|z'\|^2\}. \end{aligned} \tag{3-15}$$

The last inclusion follows by (3-10).

Step 6. *Auxiliary domains.* Let $\delta > 0$ be given. Consider the domains

$$\mathcal{G}_\delta := \{z \in \mathbb{C}^n : 2u > -\delta|v| + (1-\delta)H_{b(q)}(z')\}, \tag{3-16}$$

$$\mathcal{F}_\delta := \{z \in \mathbb{C}^n : 2u > \delta|v| + (1+\delta)H_{b(q)}(z')\}, \text{ and} \tag{3-17}$$

$$\mathcal{H}_q := \{z \in \mathbb{C}^n : 2u > H_{b(q)}(z')\}, \tag{3-18}$$

in addition to $\widehat{\Omega}$ and $\mathcal{E}$ introduced in (3-13) and (3-15).

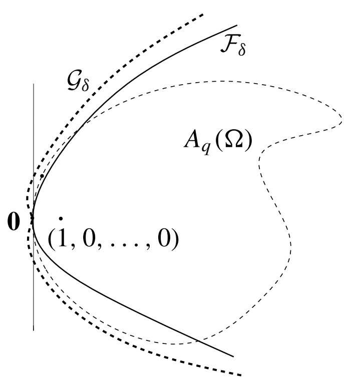

Figure 3. Auxiliary domains $\mathcal{G}_\delta$ and $\mathcal{F}_\delta$.

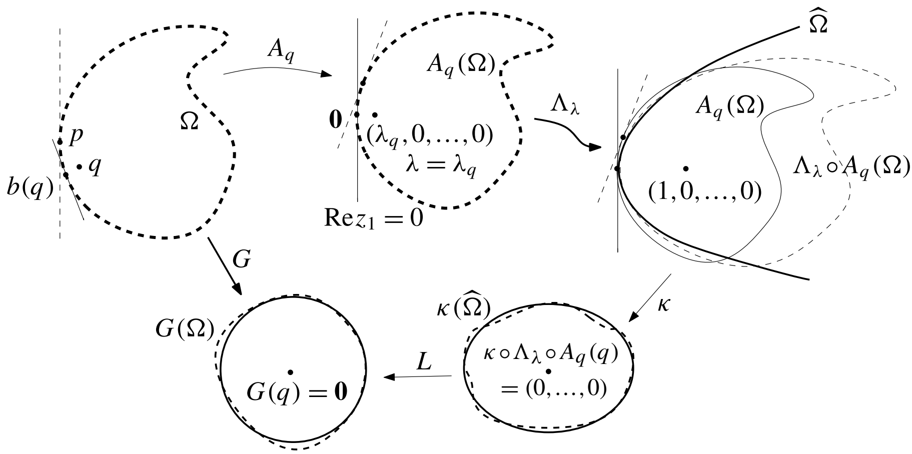

Figure 4. $G(\Omega) = L \circ \kappa \circ \Lambda_\lambda \circ A_q(\Omega)$ for $q \sim p$.

A straightforward computation checks that the image $\kappa(\mathcal{G}_\delta)$ of $\mathcal{G}_\delta$ via the Cayley transform κ introduced earlier is

$$\kappa(\mathcal{G}_\delta) = \Big\{ z \in \mathbb{C}^n : |z_1|^2 - \frac{\delta}{2}|z_1 - \bar{z}_1| + (1-\delta) H_{b(q)}(z') < 1 \Big\}. \tag{3-19}$$

Hence, there exists $\delta_0 > 0$ such that, for every δ with $0 < \delta < \delta_0$, $\kappa(\mathcal{G}_\delta)$ is a bounded domain. Notice also that this domain is arbitrarily close to the domain $\kappa(\mathcal{H}_{b(q)})$ as δ_0 becomes arbitrarily small. It follows therefore that, for every $\epsilon > 0$, there exists $\delta_0 > 0$ such that

$$L \circ \kappa(\mathcal{G}_\delta) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon) \tag{3-20}$$

whenever $0 < \delta < \delta_0$. Moreover, observe that the stretching map Λ_λ preserves all such domains as

$$\mathcal{F}_\delta, \quad \mathcal{G}_\delta, \quad \mathcal{H}_q, \quad \widehat{\Omega}, \quad \text{and} \quad \mathcal{E}.$$

Let us now define the expression

$$G(z) := L \circ \kappa \circ \Lambda_\lambda \circ A_q(z) \tag{3-21}$$

for $z \in \mathbb{C}^n - (\Lambda_\lambda \circ A_q)^{-1}(Z)$. (The set Z is defined in (3-8). Notice that this expression G depends upon $q \in \mathbb{B}^n(\mathbf{0}; r_2)$; see Figure 3 for an illustration.) In particular, this G maps Ω onto its image $G(\Omega)$ biholomorphically. See also Figure 4.

Step 7. *Proof of Theorem 3.1.* Our final goal is to establish the following claim.

Claim. *For any* ϵ *with* $0 < \epsilon < 1/2$, *there exists an integer* $m > 0$ *such that*

$$\mathbb{B}^n(\mathbf{0}; 1-\epsilon) \subset G(\Omega) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon) \tag{3-22}$$

whenever $q \in \Omega \cap \mathbb{B}^n\big(p; r_3/(2k(m))\big)$.

Since $G(q) = \mathbf{0}$, this implies that the squeezing function σ_Ω satisfies

$$\sigma_\Omega(q) \geq \frac{1-\epsilon}{1+\epsilon}.$$

Notice that this completes the proof of Theorem 3.1. Therefore it remains only to prove the claim.

Proof. Start with $\mathbb{B}^n(\mathbf{0}; 1-\epsilon)$. Notice first, by the definition of $\mathcal{F}_\delta$, that for every $\delta > 0$ there exists $m_1 > 0$ such that

$$\mathcal{F}_\delta \cap \mathbb{B}^n(\mathbf{0}; r_2/m) \subset A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; r_2/m)$$

for any $m > m_1$.

Also,

$$\kappa^{-1} \circ L^{-1}(\mathbb{B}^n(\mathbf{0}; 1-\epsilon)) \Subset \kappa^{-1} \circ L^{-1}(\mathbb{B}^n(\mathbf{0}; 1)) = \widehat{\Omega}.$$

As discussed in (3-4)–(3-7), $L \circ \kappa(\mathcal{H}_q)$ is sufficiently close to $L \circ \kappa(\widehat{\Omega})$, which is the unit ball, whenever $q \in \mathbb{B}^n\big(p; r_3/(2k(m))\big)$ and m is sufficiently large. Therefore there exists an integer $m_2 > m_1$ such that $(L \circ \kappa)^{-1}(\mathbb{B}^n(\mathbf{0}; 1-\epsilon)) \Subset \mathcal{H}_q$ whenever $q \in \mathbb{B}^n(p; r_3/m_2)$.

As in (3-19), a direct computation yields

$$\kappa(\mathcal{F}_\delta) = \Big\{z \in \mathbb{C}^n : |z_1|^2 + \tfrac{1}{2}\delta|z_1 - \bar{z}_1| + (1+\delta)H_{b(q)}(z') < 1\Big\}. \tag{3-23}$$

Now, consider the set $L \circ \kappa \circ \Lambda_\lambda(\mathcal{F}_\delta)$ for each $\delta > 0$. (Recall that $\Lambda_\lambda(\mathcal{F}_\delta) = \mathcal{F}_\delta$ as remarked in the line below (3-20).) These domains increase monotonically as $\delta \searrow 0$ (since the $\mathcal{F}_\delta$'s do) in such a way that the union $\bigcup_{0<\delta<\delta_0} L \circ \kappa \circ (\mathcal{F}_\delta)$ becomes arbitrarily close to $\mathbb{B}^n(\mathbf{0}; 1)$ for m sufficiently large. Consequently there exists a constant $\delta > 0$ such that $\mathbb{B}^n(\mathbf{0}; 1-\epsilon) \Subset L \circ \kappa \circ (\mathcal{F}_\delta)$. Moreover there is an integer $m_3 > m_2$ such that

$$\Lambda_\lambda^{-1}\big(\kappa^{-1} \circ L^{-1}(\mathbb{B}^n(\mathbf{0}; 1-\epsilon)\big) \subset \mathbb{B}^n(\mathbf{0}; r_3/k(m_1)), \tag{3-24}$$

as Λ_λ^{-1} scales down the compact subsets (for $\lambda < r_3/m_2$ sufficiently small) to a small set near the origin. Hence, we have

$$\Lambda_\lambda^{-1}\big(\kappa^{-1} \circ L^{-1}(\mathbb{B}^n(\mathbf{0}; 1-\epsilon))\big) \subset \mathcal{F}_\delta \cap \mathbb{B}^n(\mathbf{0}; r_3/k(m_1)) \subset \Omega.$$

Consequently, as long as $q \in \mathbb{B}^n\big(p; r_3/(2k(m_3))\big)$,

$$\begin{aligned} \mathbb{B}^n(\mathbf{0}; 1-\epsilon) &\subset L \circ \kappa \circ \Lambda_\lambda\big(\mathcal{F}_\delta \cap \mathbb{B}^n(\mathbf{0}; r_3/k(m_1))\big) \\ &\subset L \circ \kappa \circ \Lambda_\lambda(A_q(\Omega)) \\ &= G(\Omega). \end{aligned} \tag{3-25}$$

See Figure 5.

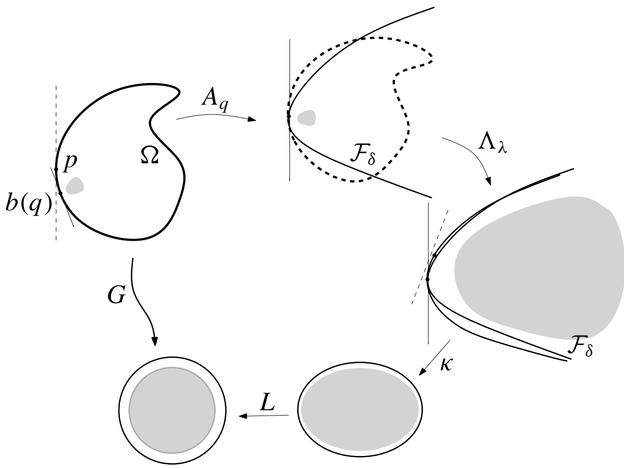

Figure 5. $\mathbb{B}^n(\mathbf{0}; 1-\epsilon) \subset G(\Omega)$.

Now we show that $G(\Omega) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon)$. Consider

$$\Omega' := \Omega - \mathbb{B}^n(p; r_2).$$

Notice that there exists an integer $\ell \gg 1$ such that

$$A_q(\Omega') \subset A_q(\Omega) - \mathbb{B}^n(\mathbf{0}; r_2/\ell) \subset \mathcal{E} - \mathbb{B}^n(\mathbf{0}; r_2/\ell). \tag{3-26}$$

Now, there is an integer $m_4 > m_3$ such that, if $m > m_4$ and $q \in \mathbb{B}^n\big(p; r_3/(2k(m))\big)$, then

$$\Lambda_\lambda(\mathcal{E} - \mathbb{B}^n(\mathbf{0}; r_2/k)) \subset \Big\{z \in \mathcal{E} : \operatorname{Re} z_1 > \frac{r_2}{r_3} \cdot \frac{m_4}{\ell}\Big\}.$$

This implies that there exists m_4 such that

$$G(\Omega') \subset L \circ \kappa\Big(\Big\{z \in \mathcal{E} : \operatorname{Re} z_1 > \frac{r_2}{r_3} \cdot \frac{m_4}{\ell}\Big\}\Big) \subset \mathbb{B}^n(-\mathbf{1}; \rho(m_4))$$

for some $\rho(m)$ which approaches zero as m tends to infinity; a direct computation with the Cayley transform and the choice of L (see (3-14)) verify this immediately. Therefore, choosing m_4 sufficiently large, we arrive at

$$G(\Omega') \subset \mathbb{B}^n(-\mathbf{1}; \epsilon), \tag{3-27}$$

as in Figure 6. For the ϵ given above, there exists δ such that

$$L \circ \kappa(\mathcal{G}_\delta) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon). \tag{3-28}$$

Fix this δ, and recall how the auxiliary domain $\mathcal{G}_\delta$ was defined in (3-16). Given any $\delta > 0$, according to (3-4)–(3-6), there exists $\rho > 0$ such that

$$A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; \rho) \subset \mathcal{G}_\delta.$$

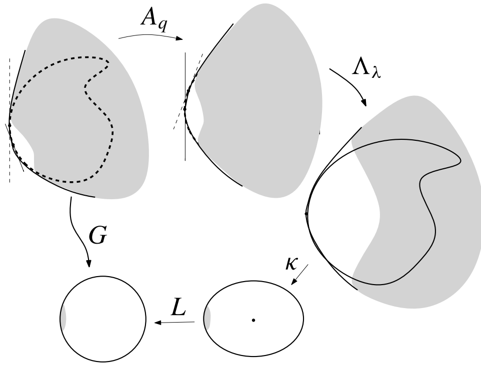

Figure 6. $G(\Omega') \subset \mathbb{B}^n(-\mathbf{1}; \epsilon)$.

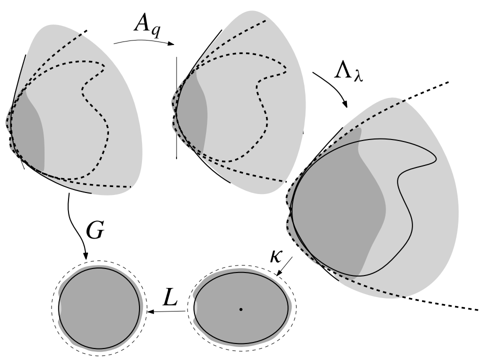

Figure 7. $G(\Omega) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon)$.

On the other hand, we can go back to (3-26) and require that $r_2/\ell < \rho/2$. Then we have

$$A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; 2r_2/\ell) \subset \mathcal{G}_\delta. \tag{3-29}$$

Since there exists an integer $m_5 > 0$ such that $A_q(\mathbb{B}^n(p; r_2/\ell)) \subset \mathbb{B}^n(\mathbf{0}; 2r_2/\ell)$, we have that

$$G(\Omega - \Omega') \subset L \circ \kappa \circ \Lambda_\lambda\big(A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; 2r_2/\ell)\big).$$

This implies

$$\begin{aligned} G(\Omega - \Omega') &\subset L \circ \kappa \circ \Lambda_\lambda\big(A_q(\Omega) \cap \mathbb{B}^n(\mathbf{0}; 2r_2/\ell)\big) \\ &\subset L \circ \kappa \circ \Lambda_\lambda(\mathcal{G}_\delta) && \text{by (3-29)} \\ &\subset L \circ \kappa(\mathcal{G}_\delta) && \text{by the sentence following (3-20)} \\ &\subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon). \end{aligned} \tag{3-30}$$

By (3-27) and (3-30), as in Figure 7 we have that

$$G(\Omega) \subset \mathbb{B}^n(\mathbf{0}; 1+\epsilon).$$

This completes the proof of the claim, and therefore of Theorem 3.1. □

4. Remarks

In this final section we present several remarks.

On the spherically extreme points. Pertaining to the question in the introduction, one of the naturally arising questions would be whether one may re-embed (the closure of) the bounded strongly pseudoconvex domain so that the preselected boundary point becomes spherically extreme. This question was answered affirmatively in by Diederich, Fornaess, and Wold in [Diederich et al. 2014]. Owing to this new result, Theorem 3.1 now implies the following.

Theorem 4.1. *If Ω is a bounded domain in $\mathbb{C}^n$ with a C^2-smooth strongly pseudoconvex boundary, then* $\lim_{\Omega\ni z\to\partial\Omega} \sigma_\Omega(z) = 1$.

On the other hand, a more ambitious goal may be to re-embed the domain using the automorphisms of $\mathbb{C}^n$ to achieve the same result. But this cannot work, as shown by the following counterexample.

Example. Consider the domain U which is the open $1/10$-tubular neighborhood of the circle $S := \{(e^{it}, 0) \in \mathbb{C}^2 : t \in \mathbb{R}\}$. This domain is strongly pseudoconvex. Let $p = (9/10, 0)$. Clearly $p \in \partial U$. If there were $\psi \in \mathrm{Aut}(\mathbb{C}^2)$ which makes $\psi(p)$ spherically extreme for $\psi(U)$, then consider the analytic disc $\Sigma := \psi(\Delta)$ where $\Delta := \{(z, 0) : |z| \le 1\}$). Since Δ crosses ∂U transversally at $\psi(p)$, Σ crosses the sphere envelope at $\psi(p)$ and extends to the exterior of the sphere. On the other hand, the boundary of Σ remains inside $\psi(U)$ and hence inside the sphere. Now let the sphere expand radially from its center, stopping at the radius beyond which it cannot intersect the holomorphic disc Σ. Then the sphere is tangent to a point to Σ at an interior point, keeping the whole disc inside the sphere. The maximum principle now implies that Σ should be entirely on the sphere. But the boundary of Σ is strictly inside the sphere, which is a contradiction. This implies that p cannot be made spherically extreme via any re-embedding by an automorphism of $\mathbb{C}^n$.

Acknowledgement: This example was obtained after a valuable discussion between Kim and Josip Globevnik. Kim would like to express his thanks to Josip Globevnik for pointing out such a possibility.

On the exhaustion theorem by Fridman–Ma. The main theorem by Buma Fridman and Daowei Ma [1995] obtained the conclusion of Theorem 3.1 in the special case $\Omega \ni q \to p$ *transversely* to the boundary $\partial\Omega$. However, that is not sufficient to prove Theorem 3.1; it is indeed necessary to consider all possible sequences approaching the boundary. Fridman and Ma [1995] did not need to consider the point sequences approaching the boundary tangentially, as their interest was only on the holomorphic exhaustion of the ball by the biholomorphic images of a bounded strongly pseudoconvex domain. On the other hand, our proof of Theorem 3.1 gives a proof to their theorem as well; we only need to use $(1+\epsilon)^{-1}G(z)$ instead of G. (Recall that G depends upon q. Letting q converge to p and ϵ tend to zero, one gets a sequence of maps that exhausts the unit ball holomorphically.)

Plane domain cases. For domains in $\mathbb{C}$, several theorems have been obtained by F. Deng, Q. Guan, and L. Zhang [Deng et al. 2012]. Theorem 3.1 obviously includes many of those results, as every boundary point of a plain domain with C^2-smooth boundary is spherically extreme.

Acknowledgements

This research is supported in part by SRC-GaiA (Center for Geometry and its Applications) and the Grant 2011-0030044 from the Ministry of Education. The research of Kim is also supported in part by National Research Foundation Grant 2011-0007831 of South Korea.

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Received October 8, 2014. Revised October 13, 2015.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 282 No. 2 June 2016