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LEFSCHETZ PENCILS AND FINITELY PRESENTED GROUPS

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From the works of Gompf and Donaldson, it is known that every finitely presented group can be realized as the fundamental group of the total space of a Lefschetz pencil. We give an alternative proof of this fact by providing the monodromy explicitly. In the proof, we give an alternative construction of the monodromy of Gurtas' fibration and a lift of that to the mapping class group of a surface with two boundary components.

1. Introduction

There exist Lefschetz pencils (fibrations over S^2 with (-1)-sections) whose total spaces have a prescribed fundamental group. This follows as a corollary of the results of Gompf [1995], who showed that every finitely presented group is realized as the fundamental group of some closed symplectic 4-manifold, and of Donaldson [1999], who showed that every closed symplectic 4-manifold admits a Lefschetz pencil. Note that since we obtain a Lefschetz fibration with (-1)-sections by blowing up the base locus of a Lefschetz pencil, and blowing up has no effect on the fundamental groups of 4-manifolds, the above claim for Lefschetz fibrations with (-1)-sections follows. Conversely, a 4-manifold admitting a Lefschetz pencil (fibration with fiber genus greater than one) is symplectic (cf. [Gompf and Stipsicz 1999]).

Let Σ_g^b be a compact oriented surface of genus g with b boundary components $\delta_1, \dots, \delta_b$, and let Mod_g^b be the mapping class group of Σ_g^b. We denote by t_c the right-handed Dehn twist along a simple closed curve c in Σ_g^b. Then a relation $\prod_{j=1}^{b} t_{\delta_j} = \prod_{i=1}^{m} t_{v_i}$ provides a genus-g Lefschetz pencil/fibration with b base points/(-1)-sections. Conversely, given any Lefschetz pencil (fibration with (-1)-sections), we obtain such a relation. However, the relations corresponding to the above Lefschetz pencils/fibrations constructed based on the results of [Gompf 1995] and [Donaldson 1999] are implicit. Our purpose is to provide the relation of such a genus-g Lefschetz pencil explicitly, so this gives an alternative proof of the above corollary using mapping class group arguments. To state our main result, we need to introduce some notation.

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Keywords: Lefschetz pencil, Lefschetz fibration, fundamental group, mapping class group.

Definition 1.1. Let $\Gamma = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_k \rangle$ be a finitely presented group with n generators and k relations. For $w \in \Gamma$, we define $l(w)$, called the *syllable length* of w, to be

$$l(w) = \min\{s \mid w = x_{i_1}^{m_1} x_{i_2}^{m_2} \cdots x_{i_s}^{m_s} \text{ for } 1 \le i_j \le n \text{ and } m_j \in \mathbb{Z}\}.$$

Define $l = \max\{l(r_i) \mid 1 \le i \le k\}$. If $k = 0$, we define $l = 1$ (note that l depends on the presentation and that our definition of l differs from that of [Korkmaz 2009]). We always assume that the relators r_i are cyclically reduced.

In Section 5A, we give a relation $t_{\delta_1} t_{\delta_2} = W_2^g(1, \psi_k)$ in Mod_g^2 using certain substitution techniques, where $W_2^g(1, \psi_k)$ is a product of right-handed Dehn twists. Our main result is the following:

Theorem 1.2. *If $k \ge 1$ (resp. $k = 0$), then, for $g \ge 4(n + l - 1) + k$ (resp. $g \ge 4n + 2$), there exists a genus-g Lefschetz pencil/fibration with two base points/(-1)-sections on a closed symplectic 4-manifold X such that $t_{\delta_1} t_{\delta_2} = W_2^g(1, \psi_k)$ is the corresponding relation and $\pi_1(X)$ is isomorphic to Γ.*

Theorem 1.2 gives an upper bound for the minimum g, denoted by $g_P(\Gamma)$, for which there exists a genus-g Lefschetz pencil on X such that $\pi_1(X)$ is isomorphic to Γ. We describe it in Section 8. To give a better upper bound on $g_P(\Gamma)$, we construct a lift of Gurtas' positive relator (see [Gurtas 2004]), denoted by θ^2, to Mod_g^2 in Section 6 by combining a lift of a hyperelliptic involution and the relation given in [Korkmaz 2009] to Mod_g^2. On the other hand, Gurtas showed that the positive word θ^2 given in [Gurtas 2004] is a positive relator by checking the images of certain cycles on Σ_g under θ. In this sense, our construction of the monodromy of Gurtas' fibration is different from that in [Gurtas 2004].

Here, we explain why we focus on Lefschetz fibrations with (-1)-sections. A section of a Lefschetz fibration over S^2 plays important roles in the total space. The existence of a section σ of a Lefschetz fibration $f : X \to S^2$ with a fiber F is required to compute the fundamental group of X and to decide whether X is spin or not (see [Gompf and Stipsicz 1999; Stipsicz 2001b]). In addition, the complement of a regular neighborhood of $F \cup \sigma$ is a Stein filling of its boundary equipped with the induced tight contact structure (see [Akbulut and Ozbagci 2002; Etnyre and Honda 2002; Loi and Piergallini 2001]). Especially, a (-1)-section is important in Lefschetz fibrations in the following senses.

(i) Blowing up of the base locus of a Lefschetz pencil yields a Lefschetz fibration with (-1)-sections. Conversely, we can obtain a Lefschetz pencil by blowing down of (-1)-sections of a Lefschetz fibration.

(ii) From given Lefschetz fibrations, we can construct a new Lefschetz fibration by fiber summing them. If a Lefschetz fibration admits a (-1)-section, it cannot be decomposed as any nontrivial fiber sum (see [Stipsicz 2001a; Smith 2001]).

For these reasons, we can regard Lefschetz fibrations with (-1)-sections as "fundamental" and "prime" ones.

Note that we can express Gompf's result in terms of Lefschetz fibrations over S^2. The article [Amorós et al. 2000] gave a construction of Lefschetz fibrations whose total spaces have a given fundamental group without using Donaldson's result. However, their monodromies are implicit. The explicit monodromies of such fibrations were given by Korkmaz [2009]. Akhmedov and Ozbagci [2013] gave a new construction of such fibrations, and the first author [Kobayashi 2015] improved the result of [Korkmaz 2009]. For technical reasons, the fibrations in [Korkmaz 2009; Akhmedov and Ozbagci 2013; Kobayashi 2015] have no (-1)-sections (see Section 8), so we would like to emphasize that our result is different from the above four results.

Here is an outline of this paper. In Section 2, we fix notation. In Section 3, we introduce a substitution technique and the relation constructed by Korkmaz. Section 4 reviews some standard facts on Lefschetz fibrations and pencils. In Section 5, we prove the main results. In Section 6, we give an alternative construction of the monodromy of Gurtas' fibration and provide a lift of that to the mapping class group of a surface with two boundary components. In Section 7, we introduce the construction of a loop which is needed for the proof of Theorem 1.2. In Section 8, we give an upper bound of $g_P(\Gamma)$ and some remarks.

2. Notation

Let Σ_g be the closed oriented surface of genus g standardly embedded in 3-space as shown in Figure 1. We use the symbols $a_1, b_1, \ldots, a_g, b_g$ to denote the standard generators of the fundamental group $\pi_1(\Sigma_g)$ of Σ_g. For a and b in $\pi_1(\Sigma_g)$, the notation ab means that we first apply a then b.

Let $c_0, c_1, c_2, \ldots, c_g, a_0, a_{g+1}, a'_0, a'_{g+1}$ be the simple loops in Σ_g depicted in Figure 1. Note that in $\pi_1(\Sigma_g)$, up to conjugation,

(1) $$c_i = b_i^{-1} \cdots b_1^{-1}(a_1 b_1 a_1^{-1}) \cdots (a_i b_i a_i^{-1}) \quad \text{for each } 1 \le i \le g;$$

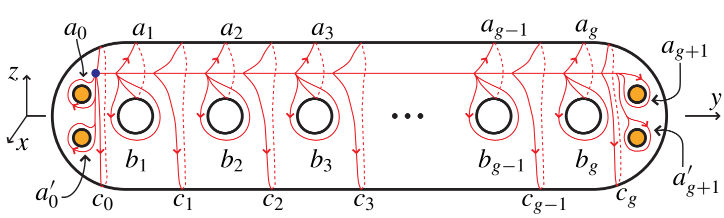

Figure 1. Generators a_j, b_j of the fundamental group and loops c_j, a'_0, a'_{g+1}.

as well as

(2) $$c_0 = c_g = 1,$$

(3) $$a_0 = a_{g+1} = a'_0 = a'_{g+1} = 1.$$

Then the fundamental group $\pi_1(\Sigma_g)$ has the presentation

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \mid c_g \rangle.$$

Let $B_0, B_1, B_2, \ldots, B_g, a'_1, \ldots, a'_g$ be the simple closed curves in Σ_g shown in Figure 2. Suppose that $g = 2r$. Then it is easy to check that, up to conjugation, the following equalities hold in $\pi_1(\Sigma_g)$:

(4) $$B_{2k-1} = a_k b_k b_{k+1} \cdots b_{g+1-k} c_{g+1-k} a_{g+1-k} \quad \text{for } 1 \le k \le r;$$

(5) $$B_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{g-k} c_{g-k} a_{g+1-k} \quad \text{for } 0 \le k \le r;$$

(6) $$a'_{k+1} = c_k a_{k+1} \quad \text{for } 0 \le k \le g-1.$$

If $g = 2r+1$, then B_{2k-1} satisfies the equality (4) for $1 \le k \le r+1$.

Let $A_1, \ldots, A_{2g+1}$ be the simple closed curves on Σ_g shown in Figure 3. It is easily seen that, up to conjugation, the following equalities hold in $\pi_1(\Sigma_g)$:

(7) $$A_{2k} = b_k \quad \text{for } 1 \le k \le g;$$

(8) $$A_{2k+1} = a_k a_{k+1}^{-1} \quad \text{for } 0 \le k \le g.$$

Moreover, when we denote by $D_0, D_1, D_2, \ldots, D_{2h_1}$ and E_{h_1} the simple closed curves on Σ_g indicated in Figure 3, it is immediate that, up to conjugation, the

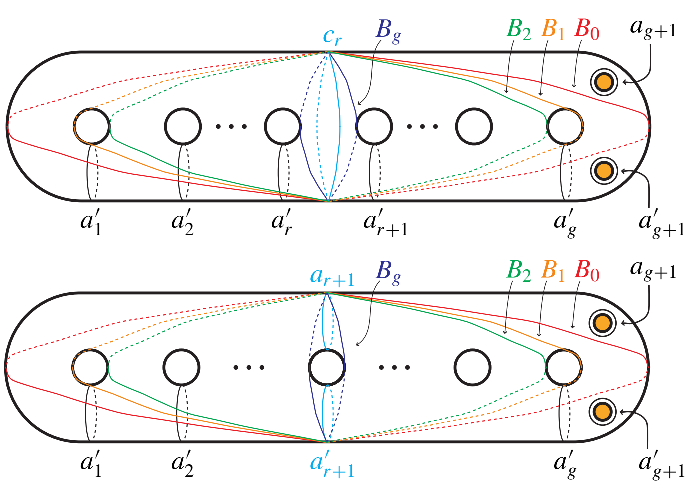

Figure 2. The curves $B_0, B_1, B_2, \ldots, B_g, a'_1, \ldots, a'_g$.

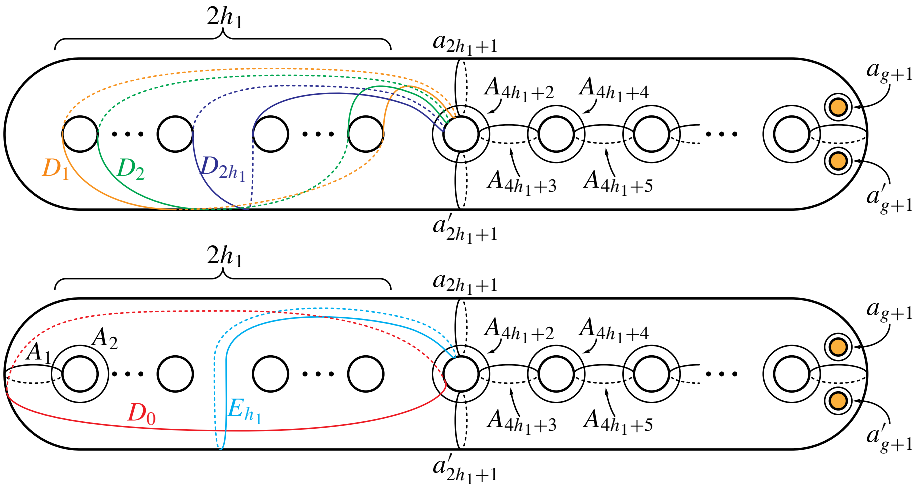

Figure 3. The curves $A_1, A_2, \ldots, A_{2g+1}, D_0, D_1, \ldots, D_{2h_1}$ and E_{h_1}.

following equalities hold in $\pi_1(\Sigma_g)$:

(9) $$D_0 = b_1 b_2 \cdots b_{2h_1} a_{2h_1+1}^{-1};$$

(10) $$D_{2k-1} = a_k b_k b_{k+1} \cdots b_{2h_1+1-k} c_{2h_1+1-k} a_{2h_1+1-k} a_{2h_1+1}^{-1} \quad \text{for } 1 \le k \le h_1;$$

(11) $$D_{2k} = a_k b_{k+1} b_{k+2} \cdots b_{2h_1-k} c_{2h_1-k} a_{2h_1+1-k} a_{2h_1+1}^{-1} \quad \text{for } 1 \le k \le h_1;$$

(12) $$E_{h_1} = c_{h_1} a_{2h_1+1}.$$

Note that we can modify Σ_g and $D_0, D_1, D_2, \ldots, D_{2h_1}, E_{h_1}$ by isotopy as in Figure 4.

Throughout this paper, we use the same symbol for a loop and its homotopy class. Similarly, we use the same symbol for a diffeomorphism and its isotopy class, or a simple closed curve and its isotopy class. A simple loop and a simple closed curve will even be denoted by the same symbol. It will cause no confusion as it will be clear from the context which one we mean.

3. Mapping class groups

3A. ***Substitution techniques.*** Let Σ_g^b be a compact oriented surface of genus g with b boundary components. The *mapping class group* of Σ_g^b, which we denote by Mod_g^b, is the group of isotopy classes of orientation preserving self-diffeomorphisms of Σ_g^b. We assume that diffeomorphisms and isotopies fix the points of the boundary. To simplify notation, we write $\Sigma_g = \Sigma_g^0$ and $\mathrm{Mod}_g = \mathrm{Mod}_g^0$. For ϕ_1 and ϕ_2 in Mod_g^b, the notation $\phi_1\phi_2$ means that we first apply ϕ_2 then ϕ_1 (Our notation differs from that of [Korkmaz 2009].) Let t_c be the Dehn twist about

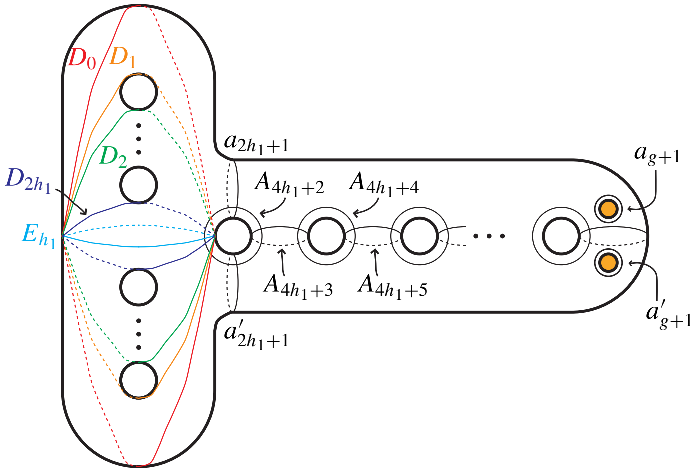

Figure 4. Modified surface Σ_g and modified curves $D_0, D_1, \ldots, D_{2h_1}$ and E_{h_1}.

a simple closed curve c in Σ_g^b. Note that $t_{\phi(c)} = \phi t_c \phi^{-1}$ for an element ϕ in Mod_g^b and $t_c t_d = t_d t_c$ if c is disjoint from d.

Definition 3.1. A word $\varrho := t_{c_1} t_{c_2} \cdots t_{c_n}$ in Mod_g is called a *positive relator* if ϱ satisfies $\varrho = 1$.

We introduce a primary technique to construct new products of right-handed Dehn twists in Mod_g^b from old ones.

Definition 3.2. Let ϕ be an element in Mod_g^b. Write

$$W = t_{c_1} t_{c_2} \cdots t_{c_k}, \qquad W^{\phi} = t_{\phi(c_1)} t_{\phi(c_2)} \cdots t_{\phi(c_k)}, \qquad V = t_{d_1} t_{d_2} \cdots t_{d_l}.$$

If the relation $V = W$ holds in Mod_g^b and $\phi(d_i) = d_i$ for all i, then by $t_{\phi(c)} = \phi t_c \phi^{-1}$ we obtain the relation

$$V = W^{\phi}.$$

in Mod_g^b. Let ϱ be a product of right-handed Dehn twists which includes V as a subword:

$$\varrho := U_1 \cdot V \cdot U_2,$$

where U_1 and U_2 are products of right-handed Dehn twists. Then we get a new product $\varsigma(\phi)$ of right-handed Dehn twists

$$\varsigma(\phi) := U_1 \cdot W^{\phi} \cdot U_2,$$

and $\varsigma(\phi)$ is said to be obtained by applying a W^{ϕ}*-substitution* of V to ϱ.

Remark 3.3. Fuller introduced the above operation for $\phi = \mathrm{id}$. Auroux [2006b; 2006a] introduced the operation to obtain $\varsigma(\phi)$ from $\varsigma(\mathrm{id})$, called a "partial conjugation" by ϕ. In a previous paper, we call the operation in Definition 3.2 a "twisted substitution". As B. Ozbagci and R. I. Baykur kindly pointed out to us, the twisted substitution is a combination of these two operations.

3B. *The word W_2^g.* In this section, we introduce a word W_2^g in Mod_g^2. We denote by Σ_g^2 the surface of genus g with two boundary components obtained from Σ_g by removing two disjoint open disks bounded by a_{g+1} and a'_{g+1} (cf. Figure 1 and 2), so a_{g+1} and a'_{g+1} are the boundary curves of Σ_g^2. Set

$$W_2^g := \begin{cases} (t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_{c_r})^2 & \text{if } g = 2r, \\ \left(t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_{a_{r+1}}^2 t_{a'_{r+1}}^2\right)^2 & \text{if } g = 2r+1. \end{cases}$$

Korkmaz [2009] gave the following relation:

Lemma 3.4 [Korkmaz 2009]. *We have* $t_{a_{g+1}}t_{a'_{g+1}} = W_2^g$ *in* Mod_g^2.

Although Korkmaz does not prove Lemma 3.4, we can prove it by applying the same argument as in Section 2 of [Korkmaz 2001]. In Section 6A, we give a very short outline of the proof. Since the simple closed curves a_{g+1} and a'_{g+1} are null-homotopic in Σ_g, it follows that $t_{a_{g+1}} = t_{a'_{g+1}} = 1$ in Mod_g. Therefore, the word W_2^g in Mod_g is a positive relator. This positive relator for $g = 2$ was discovered by Matsumoto [1996], and its generalization was constructed independently by Cadavid [1998] and Korkmaz [2001].

4. Lefschetz pencils and fibrations

We recall the definition and basic properties of Lefschetz pencils and fibrations. More details can be found in [Gompf and Stipsicz 1999].

Definition 4.1. Let X be a closed, connected, oriented smooth 4-manifold, and let $B = \{b_1, \ldots, b_m\}$ and $C = \{p_1, \ldots, p_n\}$ be finite, disjoint subsets of X.

Let $f : X \setminus B \to S^2$ be a smooth map satisfying the following three conditions:

(a) For each $b_i \in B$, called the *base point*, there are orientation-preserving complex coordinate charts on which f is of the form $f(z_1, z_2) = z_1/z_2$.

(b) C is the set of critical points of f, and for each p_i and $f(p_i)$, there are complex local coordinate charts agreeing with the orientations of X and S^2 on which f is of the form $f(z_1, z_2) = z_1 z_2$.

(c) For $q \in S^2 - f(C)$, the set $f^{-1}(q) \cup B \subset X$ is diffeomorphic to Σ_g.

Then f is called a genus-g *Lefschetz pencil* if B is a nonempty set, and f is called a genus-g *Lefschetz fibration* if B is the empty set.

The set B is called the *base locus*, and for each $q \in S^2$, the set $f(q)^{-1} \cup B$ is called the *fiber* of f. We assume that f is injective on C and that f is relatively minimal (i.e., no fiber contains a sphere with self-intersection number -1). A fiber containing a critical point is called a *singular fiber*. Each singular fiber is obtained by collapsing a simple closed curve, called the *vanishing cycle*, in the regular fiber to a point.

Once we fix an identification of Σ_g with the fiber over a base point of $S^2 - f(C)$, we can characterize the Lefschetz fibration $f : X \to S^2$ by its *monodromy representation* $\pi_1(S^2 - f(C)) \to \mathrm{Mod}_g$. Note that in this paper, this map is an antihomomorphism. Let $\gamma_1, \dots, \gamma_n$ be an ordered system of generating loops for $\pi_1(S^2 - f(C))$, such that each γ_i encircles only $f(p_i)$ and $\gamma_1\gamma_2 \cdots \gamma_n$ is homotopically trivial. Thus, since the monodromy of the fibration along each of the loops γ_i is a right-handed Dehn twist along the corresponding vanishing cycle, the monodromy of f comprises a positive relator

$$t_{v_n} \cdots t_{v_2} t_{v_1} = 1 \in \mathrm{Mod}_g,$$

where the v_i are the corresponding vanishing cycles of the singular fibers. Conversely, for any positive relator $\varrho \in \mathrm{Mod}_g$, we can construct a genus-g Lefschetz fibration over S^2 whose monodromy is ϱ. Therefore, we denote a genus-g Lefschetz fibration associated to a positive relator ϱ in Mod_g by $f_\varrho : X_\varrho \to S^2$.

Definition 4.2. For a Lefschetz fibration $f : X \to S^2$, a map $\sigma : S^2 \to X$ is called a *k-section* of f if $f \circ \sigma = \mathrm{id}_{S^2}$ and the self-intersection number of the homology class $[\sigma(S^2)]$ in $H_2(X; \mathbb{Z})$ is equal to k.

When a Lefschetz fibration $X \to S^2$ admits a section, we can compute the fundamental group of X as follows.

Lemma 4.3 (cf. [Gompf and Stipsicz 1999]). *Let ϱ be a positive relator given by $t_{v_n} \cdots t_{v_2} t_{v_1} = 1$ in Mod_g. Suppose that a genus-g Lefschetz fibration $f_\varrho : X_\varrho \to S^2$ admits a section σ. Then the fundamental group $\pi_1(X_\varrho)$ is isomorphic to the quotient of $\pi_1(\Sigma_g)$ by the normal subgroup generated by $v_1, \dots, v_n$.*

From the definitions of Lefschetz fibrations and pencils, blowing up all points of $B = \{q_1, \dots, q_b\}$ of a genus-g Lefschetz pencil yields a genus-g Lefschetz fibration with b disjoint (-1)-sections. Let $\delta_1, \delta_2, \dots, \delta_b$ be b boundary curves of Σ_g^b. Then a lift of a positive relator ϱ in Mod_g, namely $t_{v_n} \cdots t_{v_2} t_{v_1} = 1$, to Mod_g^b as

$$t_{v'_n} \cdots t_{v'_2} t_{v'_1} = t_{\delta_1} t_{\delta_2} \cdots t_{\delta_b}$$

shows the existence of b disjoint (-1)-sections of f_ϱ. Here, v'_i is a simple closed curve mapped to v_i under $\Sigma_g^b \to \Sigma_g$. Conversely, such a relation determines a genus-g Lefschetz fibration with m disjoint (-1)-sections and a genus-g Lefschetz pencil by blowing these sections down.

5. Proof of Theorem 1.2

For a finitely presented group $\Gamma = \langle x_1, x_2, \dots, x_n \mid r_1, r_2, \dots, r_k \rangle$ with n generators and k relators, let $l = \max\{l(r_i) \mid 1 \le i \le k\}$, where $l(r_i)$ is the syllable length of r_i. In this section, we denote by h_1 and h_2 two integers satisfying $h_1 \ge n + l - 1$ and $2(h_2 - 1) \ge k$, respectively.

5A. ***Construction of a word*** $\boldsymbol{W_2^g(1, \psi_i)}$**.** In this subsection, we construct a key relation in Mod_g^2.

Let us consider Σ_g^2 obtained from Σ_g by removing two disjoint open disks surrounded by a_{g+1} and a'_{g+1} (see Section 2 and Figures 1–3). Write $r = 2h_1 + h_2 - 1$ and $g = 2r$ or $2r + 1$. For $h_2 - 1 \ge 1$, we set

$$X = t_{A_{4h_1+2}} t_{A_{4h_1+3}} \cdots t_{A_{2r}},$$
$$\overline{X} = t_{A_{2r}} \cdots t_{A_{4h_1+3}} t_{A_{4h_1+2}},$$
$$Y = (t_{D_0} t_{D_1} \cdots t_{D_{2h_1}})^2.$$

Moreover, we define words V_1 and V_2 to be

$$V_1 = t_{E_{h_1}} X t_{a_r} t_{a'_r} \overline{X} t_{E_{h_1}} t_{a'_r} \overline{X} Y X t_{a'_r},$$
$$V_2 = t_{E_{h_1}} X t_{a_r} t_{a'_r} \overline{X} t_{E_{h_1}} t_{A_{2r+1}} \overline{X} Y X t_{A_{2r+1}}.$$

Then we obtain the relations in the following proposition.

Proposition 5.1. *We have* $t_{c_r} = V_1$ *and* $t_{a_{r+1}} t_{a'_{r+1}} = V_2$ *in* Mod_g^2.

We postpone the proof of Proposition 5.1 until Section 6 (see Proposition 6.1).

Let $h_1 \ge n + l - 1$ and $2(h_2 - 1) \ge k$. The next proposition is needed to prove Theorem 1.2.

Proposition 5.2. *Let F_n be the subgroup of $\pi_1(\Sigma_g)$ generated by the generators $a_1, \dots, a_n$, i.e., F_n is a free group of rank n. Let $r_1, \dots, r_k$ be k elements in F_n represented as words in $a_1, \dots, a_n$. Let $l = \max_{1 \le i \le k}\{l(r_i)\}$, where $l(r_i)$ is the syllable length of r_i. Then there are simple loops $R_1, \dots, R_k$ in Σ_g (see Figure 5) with the property that, for $4h_1 + 2 \le j \le 4h_1 + 2h_2 - 2$ and $1 \le i \le k$,*

- (a) *R_i is disjoint from $A_{2h_1+1}, \dots, A_{4h_1}, c_{2h_1+h_2-1} (= c_r)$.*
- (b) *R_1 intersects $a_{2h_1+h_2-1}$ at one point and does not intersect A_j for any j.*
- (c) *R_i intersects $A_{4h_1+2h_2-i}$ at one point and intersects neither $a_{2h_1+h_2-1}$ nor A_j for any $j \ne 4h_1 + 2h_2 - i$ and $i \ge 2$.*
- (d) *$\Phi([R_i]) = r_i$, where $[R_i] \in \pi_1(\Sigma_g)$ is the homotopy class of the loop R_i, and $\Phi : \pi_1(\Sigma_g) \to \pi_1(\Sigma_n)$ is the map defined by $\Phi(a_m) = a_m$ for $1 \le m \le n$ and $\Phi(\alpha) = 1$ for $\alpha \in \{a_{n+1}, \dots, a_g, b_1, \dots, b_g\}$.*

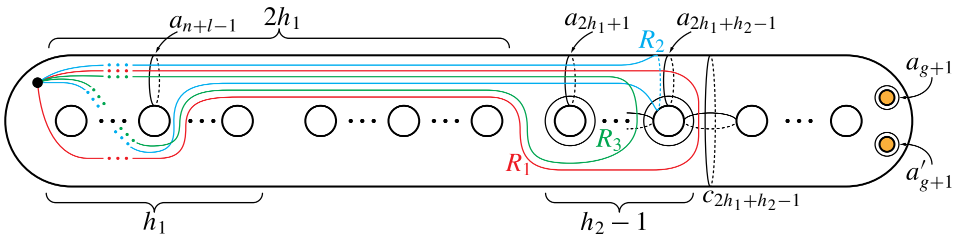

Figure 5. Curves $R_1, \dots, R_k$ in Σ_g.

In Section 7, we prove Proposition 5.2 by constructing simple loops $R_1, \dots, R_k$ explicitly. We also consider the loops $R_1, \dots, R_k$ as simple loops on Σ_g^2 by removing two disjoint open disks surrounded by a_{g+1}, a'_{g+1} from Σ_g (see Figure 5).

For $i = 0, 1, \dots, k$, we define an element ψ_i in Mod_g^2 to be

$$\psi_0 = t_{a_{h_1}} t_{b_{h_1+1}} t_{b_{h_1+2}} \cdots t_{b_{2h_1}},$$
$$\psi_i = t_{R_{k+1-i}} t_{R_{k+2-i}} \cdots t_{R_k} \psi_0,$$

where the R_i are the loops on Σ_g^2 described above. From Proposition 5.2, for each i, we see that $\psi_i(c_r) = c_r$ if $g = 2r$, while $\psi_1(a_{r+1}) = a_{r+1}$ and $\psi_1(a'_{r+1}) = a'_{r+1}$ if $g = 2r + 1$.

If $g = 2r$, then we can find two t_{c_r} in the word W_2^g. By Proposition 5.1, we can apply V_1^{id}-substitution for one t_{c_r} and $V_1^{\psi_i}$-substitution for the other.

If $g = 2r + 1$, then since $t^2_{a_{r+1}} t^2_{a'_{r+1}} = (t_{a_{r+1}} t_{a'_{r+1}})^2$, we can find four $t_{a_{r+1}} t_{a'_{r+1}}$ in the word W_2^g. By Proposition 5.1, we can apply V_2^{id}-substitution for one $t_{a_{r+1}} t_{a'_{r+1}}$ and $V_2^{\psi_i}$-substitution for the other.

If we set

$$W_2^g(1, \psi_i) := (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} V_1)(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} V_1^{\psi_i})$$

if $g = 2r$, and

$$W_2^g(1, \psi_i) := (t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_{a_{r+1}} t_{a'_{r+1}} V_2)(t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_{a_{r+1}} t_{a'_{r+1}} V_2^{\psi_i})$$

if $g = 2r + 1$, then we get the next lemma.

Lemma 5.3. *We have* $t_{a_{g+1}} t_{a'_{g+1}} = W_2^g(1, \psi_i)$ *in* Mod_g^2.

Since $t_{a_{g+1}} = 1$ and $t_{a'_{g+1}} = 1$ in Mod_g, the word $W_2^g(1, \psi_i)$ in Mod_g is a positive relator. Therefore, we obtain a genus-g Lefschetz fibration $f_{W_2^g(1,\psi_i)}$ with two disjoint (-1)-sections (and genus-g Lefschetz pencil with two base points corresponding to $W_2^g(1, \psi_i)$). Then, we have the following results which we prove in Section 5B and in Section 5C.

Theorem 5.4. *Suppose that $k = 0$. We denote by F_n a free group of rank n. If $g \geq 2(2n+1)$, then we have*

$$\pi_1(X_{W_2^g(1,\psi_0)}) \cong F_n.$$

Theorem 5.5. *Suppose that $k > 0$. If $g \geq 4(n+l-1)+k$, then we have*

$$\pi_1(X_{W_2^g(1,\psi_k)}) \cong \Gamma.$$

Combining Theorem 5.4 and 5.5, we obtain Theorem 1.2.

5B. ***Proof of Theorem 5.4.*** In this section, we prove Theorem 5.4. We begin with a lemma.

Lemma 5.6. *Let $r = 2h_1 + h_2 - 1$. Let $\langle S \rangle$ be the normal closure of the elements of the set S of simple closed curves on Σ_g defined by*

$$S = \{B_0, B_1, \dots, B_g, D_0, D_1, \dots, D_{2h_1}, E_{h_1}, A_{4h_1+2}, \dots, A_{2r}, a_r, a'_r\}$$

if $g = 2r$, and by

$$S = \{B_0, B_1, \dots, B_g, a_{r+1}, a'_{r+1}, D_0, D_1, \dots, D_{2h_1}, E_{h_1}, A_{4h_1+2}, \dots, A_{2r+1}, a_r, a'_r\}$$

if $g = 2r+1$. Then $\pi_1(\Sigma_g)/\langle S \rangle$ has a presentation with generators $a_1, b_1, \dots, a_g, b_g$ and with relations

$$\begin{aligned} a_i a_{g+1-i} = b_i a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} &= 1 \quad \text{for } 1 \leq i \leq r, \\ a_{2h_1+k} = b_{2h_1+k} &= 1 \quad \text{for } 1 \leq k \leq h_2 - 1, \\ a_j a_{2h_1+1-j} = b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} &= 1 \quad \text{for } 1 \leq j \leq h_1, \\ c_{h_1} &= 1 \end{aligned}$$

if $g = 2r$, and

$$\begin{aligned} a_i a_{g+1-i} = b_i a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} &= 1 \quad \text{for } 1 \leq i \leq r, \\ a_{2h_1+k} = b_{2h_1+k} &= 1 \quad \text{for } 1 \leq k \leq h_2 - 1, \\ a_j a_{2h_1+1-j} = b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} &= 1 \quad \text{for } 1 \leq j \leq h_1, \\ a_{r+1} = c_{h_1} &= 1 \end{aligned}$$

if $g = 2r+1$.

Proof. Suppose that $g = 2r$. From the equalities (4) and (5) in Section 2, in $\pi_1(\Sigma_g)/\langle S \rangle$ we have

$$a_i a_{g+1-i} = 1. \tag{13}$$

This gives

$$1 = B_{2i-1} = b_i b_{i+1} \cdots b_{g+1-i} c_{g+1-i} \quad \text{for } 1 \le i \le r,$$
$$1 = B_{2i} = b_{i+1} b_{i+2} \cdots b_{g-i} c_{g-i} \quad \text{for } 1 \le i \le r$$

in $\pi_1(\Sigma_g)/\langle S\rangle$. From these two equalities, we have $b_i c_{g-i}^{-1} b_{g+1-i} c_{g+1-i} = 1$ for each $1 \le i \le r$ and

(14) $$c_r = 1.$$

Note that $c_{g+1-i} = b_{g+1-i}^{-1} c_{g-i} (a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1})$ from the equality (1). Therefore, by $b_i c_{g-i}^{-1} b_{g+1-i} c_{g+1-i} = 1$, we obtain

(15) $$b_k a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} = 1.$$

From $a_r = 1$, $A_l = 1$ for $4h_1 + 2 \le l \le 2r$ and the equalities (7) and (8), we obtain

(16) $$a_{2h_1+k} = b_{2h_1+k} = 1$$

for $1 \le k \le h_2 - 1$. From $a'_r = 1$ and the equalities (6), (14), (1) and (16), we have

(17) $$c_{r-1} = c_{2h_1} = 1.$$

By $a_{2h_1+1} = 1$, $c_{2h_1} = 1$ and the equalities (9), (10) and (11), an argument similar to the proofs of the relations (13) and (15) gives

(18) $$a_j a_{2h_1+1-j} = b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} = 1 \quad \text{and} \quad c_{h_1} = 1$$

for $1 \le j \le 2h_1$.

From the equalities (13), (14), (15), (16), (17) and (18), we see that $\pi_1(\Sigma_g)/\langle S\rangle$ has a presentation with generators $a_1, b_1, \dots, a_g, b_g$ and with relations

$$a_i a_{g+1-i} = b_i a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} = 1 \quad \text{for } 1 \le i \le r,$$
$$a_{2h_1+k} = b_{2h_1+k} = 1 \quad \text{for } 1 \le k \le h_2 - 1,$$
$$a_j a_{2h_1+1-j} = b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} = 1 \quad \text{for } 1 \le j \le h_1,$$
$$c_g = c_r = c_{r-1} = c_{2h_1} = c_{h_1} = 1.$$

Then by the equalities (1), (16) and (18), we can delete from the above the relations $c_g = c_r = c_{r-1} = c_{2h_1} = 1$. This is our claim.

Suppose now that $g = 2r + 1$. Since $a_{r+1} = a'_{r+1} = 1$ and $a'_{r+1} = c_r a_{r+1}$, we have $c_r = 1$. A similar argument as in the case $g = 2r$ shows that $\pi_1(\Sigma_g)/\langle S\rangle$ has the desired presentation. This completes the proof. □

We can now prove Theorem 5.4.

Proof of Theorem 5.4. Let $h_1 \geq n$ and $h_2 - 1 \geq 1$. For simplicity of notation, we write G instead of $\pi_1(X_{W_2^g(1,\psi_0)})$.

Suppose that $g = 2(2h_1 + h_2 - 1)$ and let $r = 2h_1 + h_2 - 1$. Note that G has a presentation with generators $a_1, b_1, \ldots, a_g, b_g$ and with relations

$$\begin{aligned} c_g &= 1, \\ B_i &= 1 \quad \text{for } 0 \leq i \leq g, \\ a_r = a'_r = E_{h_1} &= 1, \\ D_j = A_k &= 1 \quad \text{for } 0 \leq j \leq 2h_1,\ 4h_1 + 2 \leq k \leq 4h_1 + 2h_2 - 2, \\ \psi_0(a_r) = \psi_0(a'_r) = \psi_0(E_{h_1}) &= 1, \\ \psi_0(D_j) = \psi_0(A_k) &= 1 \quad \text{for } 0 \leq j \leq 2h_1,\ 4h_1 + 2 \leq k \leq 4h_1 + 2h_2 - 2. \end{aligned}$$

It is easily seen that, up to conjugation, we have the equalities

$$\begin{aligned} \psi_0(D_0) &= a_{h_1} \cdots a_{n+2} a_{n+1} D_0, \\ \psi_0(D_{2l-1}) &= b^{-1}_{2h_1-l+1} a_{h_1} \cdots a_{n+2} a_{n+1} D_{2l-1} \quad \text{for } 1 \leq l \leq n, \\ \psi_0(D_{2l}) &= b^{-1}_{2h_1-l+1} a_{h_1} \cdots a_{n+2} a_{n+1} D_{2l} \quad \text{for } 1 \leq l \leq n \end{aligned}$$

in $\pi_1(\Sigma_g)$. Thus, by $D_0 = \psi_0(D_0) = D_j = \psi_0(D_j) = 1$ for $1 \leq j \leq 2h_1$, we obtain

$$b_{2h_1-l+1} = 1 \quad \text{for } 1 \leq l \leq n.$$

Similarly, we have the following equalities (up to conjugation) in $\pi_1(\Sigma_g)$:

$$\begin{aligned} \psi_0(D_{2l-1}) &= b^{-1}_{2h_1-l+1} a_{h_1} \cdots a_{l+1} a_l D_{2l-1} \quad \text{for } n+1 \leq l \leq r-1, \\ \psi_0(D_{2l}) &= b^{-1}_{2h_1-l+1} a_{h_1} \cdots a_{l+2} a_{l+1} D_{2l-1} \quad \text{for } n+1 \leq l \leq r-1, \\ \psi_0(D_{2h_1-1}) &= b^{-1}_{h_1+1} a_{h_1} D_{2h_1-1}, \\ \psi_0(D_{2h_1}) &= b^{-1}_{h_1+1} B_{2h_1}. \end{aligned}$$

By $D_j = 1$ for $1 \leq j \leq 2h_1$ and $\psi_0(D_{2l-1}) = \psi_0(D_{2l}) = 1$ for $n+1 \leq l \leq h_1$, we obtain

$$a_l = 1 \quad \text{for } n+1 \leq l \leq h_1.$$

Moreover, by $\psi_0(D_{2l}) = \psi_0(D_{2l+1}) = \psi_0(D_{2h_1}) = 1$ for $n+1 \leq l \leq h_1 - 1$, we have

$$b_{2h_1-l+1} = 1 \quad \text{for } n+1 \leq l \leq h_1.$$

Here, since $\psi_0(a_r) = a_r$, $\psi_0(a'_r) = a'_r$, $\psi_0(E_{h_1}) = E_{h_1}$ and $\psi_0(A_k) = A_k$ in $\pi_1(\Sigma_g)$ for each $4h_1 + 2 \leq k \leq 4h_1 + 2h_2 - 2$, we can delete the relations $\psi_0(a_r) = 1$, $\psi_0(a'_r) = 1$, $\psi_0(E_{h_1}) = 1$ and $\psi_0(A_K) = 1$ from the above presentation of G.

From the above arguments and Lemma 5.6, we see that G has a presentation with generators $a_1, b_1, \dots, a_g, b_g$ and with relations

$$\begin{aligned} a_i a_{g+1-i} &= b_i a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} && \text{for } 1 \le i \le r, \\ a_{2h_1+k} &= b_{2h_1+k} = 1 && \text{for } 1 \le k \le h_2 - 1, \\ a_j a_{2h_1+1-j} &= b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} = 1 && \text{for } 1 \le j \le h_1, \\ c_{h_1} &= 1, \\ a_{n+1} &= a_{n+2} = \cdots = a_{h_1} = 1, \\ b_{h_1} &= b_{h_1+1} = \cdots = b_{2h_1} = 1. \end{aligned}$$

It is easily shown that this is a presentation of the free group of rank n with free basis $a_1, \dots, a_n$, that is, G is isomorphic to F_n.

The proof for $g = 2r + 1$ is similar. This completes the proof of Theorem 5.4. □

5C. *Proof of Theorem 5.5.* We now prove Theorem 5.5. The proof is inspired by [Korkmaz 2009] and that of Proposition 13 in [Akhmedov and Ozbagci 2013]. For simplicity, we write G' instead of $\pi_1(X_{W_2^g(1,\psi_1)})$.

Proof of Theorem 5.5. Suppose that $g = 2(2h_1 + h_2 - 1)$. Since R_1 intersects $a_{2h_1+h_2-1}$ at one point and does not intersect A_j for $j = 4h_1+2, \dots, 4h_1+2h_2-2$, and $a_{2h_1+h_2-1}$ is disjoint from $a_{n+1}, \dots, a_{h_1}, b_{h_1+1}, \dots, b_{2h_1}$ and $R_2, \dots, R_k$, we see that in $\pi_1(\Sigma_g)$, up to conjugation,

$$\psi_k(a_{2h_1+h_2-1}) = t_{R_1}(a_{2h_1+h_2-1}) = a_{2h_1+h_2-1} R_1^{\epsilon},$$

where ϵ is equal to 1 or -1. Since $a_{2h_1+h_2-1} = 1$ in G', we may replace the relator $\psi_k(a_{2h_1+h_2-1}) = 1$ by $R_1 = 1$.

Let c be an element of the set of the vanishing cycles of $f_{W_2^g(1,\psi_k)}$. If R_1 is disjoint from $\psi_{k-1}(c)$, then we have $\psi_k(c) = t_{R_1}(\psi_{k-1}(c)) = \psi_{k-1}(c)$. If R_1 intersects $\psi_{k-1}(c)$ at t points, then it is easily seen that there are elements $x_1, \dots, x_{t+1}$ in $\pi_1(\Sigma_g)$ such that $\psi_{k-1}(c) = x_1 x_2 \cdots x_{t+1}$ and that

$$t_{R_1}(\psi_{k-1}(c)) = x_1 R_1^{\zeta_1} x_2 R_1^{\zeta_2} \cdots x_t R_1^{\zeta_t} x_{t+1}$$

(up to conjugacy), where each ζ_s is equal to 1 or -1. From $R_1 = 1$, we obtain $\psi_k(c) = t_{R_1}(\psi_{k-1}(c)) = \psi_{k-1}(c)$ in G'. Therefore, we may replace the relator $\psi_k(c) = 1$ by $\psi_{k-1}(c) = 1$.

By repeating this argument for each $i = k-1, \dots, 1$, we see that we may replace the relators $\psi_k(A_{4h_1+2h_2-(k+1-i)}) = 1$ and $\psi_k(c) = 1$ by $R_{k+1-i} = 1$ and $\psi_0(c) = 1$, respectively. In particular, since for each $j = 4h_1+2, \dots, 4h_1+2h_2-2$, $a_{2h_1+h_2-1} = 1$ and $A_j = 1$ in G' and $a_{2h_1+h_2-1} = \psi_0(a_{2h_1+h_2-1})$ and $A_j = \psi_0(A_j)$ in $\pi_1(\Sigma_g)$ (up to conjugation), we can delete the relators $\psi_k(a_{2h_1+h_2-1}) = 1$ and

$\psi_k(A_j) = 1$. Therefore, from the proof of Theorem 5.4, we see that G' has a presentation with generators $a_1, b_1, \dots, a_g, b_g$ and with relations

$$\begin{aligned} a_i a_{g+1-i} &= b_i a_{g+1-i} b_{g+1-i} a_{g+1-i}^{-1} && \text{for } 1 \le i \le r, \\ a_{2h_1+k} &= b_{2h_1+k} = 1 && \text{for } 1 \le k \le h_2 - 1, \\ a_j a_{2h_1+1-j} &= b_j a_{2h_1+1-j} b_{2h_1+1-j} a_{2h_1+1-j}^{-1} = 1 && \text{for } 1 \le j \le h_1, \\ c_{h_1} &= 1, \\ a_{n+1} &= a_{n+2} = \cdots = a_{h_1} = 1, \\ b_{h_1} &= b_{h_1+1} = \cdots = b_{2h_1} = 1, \\ R_1 &= R_2 = \cdots = R_k = 1. \end{aligned}$$

We note that the element $[R_i] \in \pi_1(\Sigma_g)$ is contained in the subgroup generated by $a_1, b_1, \dots, a_{h_1}, b_{h_1}$ and $a_{2h_1+1}, b_{2h_1+1}, \dots, a_{2h_1+h_2-1}, b_{2h_1+h_2-1}$. Since from this presentation, we see that $a_s = 1$ for $s = n+1, \dots, h_1, 2h_1+1, \dots, 2h_1+h_2-1$ and $b_j = 1$ for $j = 1, \dots, h_1, 2h_1+1, \dots, 2h_1+h_2-1$, we get a word representing the element r_i by Proposition 5.2. Therefore, G' is isomorphic to Γ.

A similar argument works for $g = 2(2h_1 + h_2 - 1) + 1$. Since $f_{W_2^g(1,\psi_k)}$ has at least two disjoint (-1)-sections, by blowing down one of them we obtain the required genus-g Lefschetz pencil. This completes the proof of Theorem 5.5 and therefore, as discussed in Section 5A, also of Theorem 1.2. □

6. Construction of a lift of Gurtas' positive relator

In this section, we prove Proposition 5.1 and give a lift to Mod_g^2 of the positive relator in Mod_g given by Gurtas [2004].

6A. ***Outline of the proof of Lemma 3.4.*** We now give an outline of the proof of Lemma 3.4, which is needed to prove Proposition 5.1.

Outline of the proof of Lemma 3.4. We define $\Delta_0 = \overline{\Delta}_0 = 1$. Moreover, for each $k = 1, \dots, 2g+1$, we define Δ_k and $\overline{\Delta}_k$ to be the words

$$\Delta_k = t_{A_1} t_{A_2} \cdots t_{A_k} \quad \text{and} \quad \overline{\Delta}_k = t_{A_k} \cdots t_{A_2} t_{A_1}.$$

For each $k = 0, 1, \dots, g$, the words β_k and β are defined by

$$\beta_k = \overline{\Delta}_k \Delta_{2g+1-k} \Delta_{2g-k}^{-1} \overline{\Delta}_k^{-1} \quad \text{and} \quad \beta = \overline{\Delta}_g^{g+1}.$$

Then by applying the argument from Section 2 of [Korkmaz 2001] with σ_i (which is the standard generator of the braid group B_{2g+2} on $2g+2$ strings) replaced by t_{A_i}, we have the relation

$$\beta_0 \beta_1 \beta_2 \cdots \beta_g \beta^2 = \Delta_{2g+1} \Delta_{2g} \cdots \Delta_3 \Delta_2 \Delta_1. \tag{19}$$

It is easy to check that $\overline{\Delta}_k \Delta_{2g-k}(A_{2g+1-k}) = B_k$. This gives

$$t_{B_k} = (\overline{\Delta}_k \Delta_{2g-k}) t_{A_{2g+1-k}} (\overline{\Delta}_k \Delta_{2g-k})^{-1} = \overline{\Delta}_k \Delta_{2g+1-k} \Delta_{2g-k}^{-1} \overline{\Delta}_k^{-1} = \beta_k.$$

Therefore, from the relation (19), we have

$$t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} (\overline{\Delta}_g)^{2g+2} = \Delta_{2g+1} \Delta_{2g} \cdots \Delta_3 \Delta_2 \Delta_1.$$

Using the chain relations $\overline{\Delta}_g^{2g+2} = t_{c_r}$ when $g = 2r$ and $\overline{\Delta}_g^{g+1} = t_{a_{r+1}} t_{a'_{r+1}}$ when $g = 2r+1$, we have

$$\text{(20)} \qquad \Delta_{2g+1} \Delta_{2g} \cdots \Delta_3 \Delta_2 \Delta_1 = \begin{cases} t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_{c_r} & \text{for } g = 2r, \\ t_{B_0} t_{B_1} t_{B_2} \cdots t_{B_g} t_{a_{r+1}} t_{a'_{r+1}} & \text{for } g = 2r+1. \end{cases}$$

If we prove that $t_{a_{g+1}} t_{a'_{g+1}} = (\Delta_{2g+1} \Delta_{2g} \cdots \Delta_3 \Delta_2 \Delta_1)^2$ in Mod_g^2, the assertion follows. Note that *the chain relation* $\Delta_{2g+1}^{2g+2} = t_{a_{g+1}} t_{a'_{g+1}}$, and $t_{A_k} \Delta_m = \Delta_m t_{A_{k-1}}$ if $1 < k \leq m$ (see [Korkmaz 2001, Lemma 2.1(a)]), hold in Mod_g^2. Then we have

$$\begin{aligned}
\Delta_{2g+1}^{2g+2} &= \Delta_{2g+1} \Delta_{2g} t_{A_{2g+1}} \Delta_{2g+1} \Delta_{2g+1}^{2g-1} \\
&= \Delta_{2g+1} \Delta_{2g} \Delta_{2g+1} t_{A_{2g}} \Delta_{2g+1}^{2g-1} \\
&= \Delta_{2g+1} \Delta_{2g} \Delta_{2g-1} (t_{A_{2g}} t_{A_{2g+1}}) t_{A_{2g}} \Delta_{2g+1}^{2g-1} \\
&= \Delta_{2g+1} \Delta_{2g} \Delta_{2g-1} \Delta_{2g+1} (t_{A_{2g-1}} t_{A_{2g}}) t_{A_{2g-1}} \Delta_{2g+1}^{2g-2} \\
&= \Delta_{2g+1} \Delta_{2g} \Delta_{2g-1} \Delta_{2g-2} (t_{A_{2g-1}} t_{A_{2g}} t_{A_{2g+1}}) (t_{A_{2g-1}} t_{A_{2g}}) t_{A_{2g-1}} \Delta_{2g+1}^{2g-2} \\
&\;\;\vdots \\
&= \Delta_{2g+1} \Delta_{2g} \cdots \Delta_1 (t_{A_2} t_{A_3} \cdots t_{A_{2g+1}}) (t_{A_2} t_{A_3} \cdots t_{A_{2g}}) \cdots (t_{A_2} t_{A_3}) t_{A_2} \Delta_{2g+1} \\
&= \Delta_{2g+1} \Delta_{2g} \cdots \Delta_1 \Delta_{2g+1} \Delta_{2g} \cdots \Delta_1,
\end{aligned}$$

and the proof is complete. □

6B. *Proof of Proposition 5.1.* In this section, we prove Proposition 6.1 instead of Proposition 5.1. Note that if we set $g = r$ in the notation of Proposition 6.1 and consider an embedding $\Sigma_r^2 \hookrightarrow \Sigma_g^2$ (resp. $\Sigma_r^1 \hookrightarrow \Sigma_g^2$) mapping (a_{r+1}, a'_{r+1}) (resp. a_{r+1}) in Proposition 6.1 to (a_{r+1}, a'_{r+1}) (resp. c_r) in Proposition 5.1, then we get Proposition 5.1. Therefore, it is sufficient to prove Proposition 6.1.

Proposition 6.1. *Let Σ_g^2 (resp. Σ_g^1) be the compact oriented surface of genus g with two boundary components (resp. one boundary component) obtained from Σ_g by removing two disjoint open disks (resp. one open disk). Let $a_{g+1}, a'_{g+1} = c_g a_{g+1}$ (resp. a_{g+1}) be the boundary curves of Σ_g^2 (resp. the boundary curve of Σ_g^1). Then* *the relations*

$$\text{(21)}\quad t_{a_{g+1}}t_{a'_{g+1}} = t_{E_{h_1}}t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{a_g}t_{a'_g}t_{A_{2g}}\cdots t_{A_{4h_1+2}}t_{E_{h_1}} \cdot t_{A_{2g+1}}t_{A_{2g}}\cdots t_{A_{4h_1+2}}\cdot(t_{D_0}t_{D_1}\cdots t_{D_{2h_1}})^2\cdot t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{A_{2g+1}},$$

$$\text{(22)}\quad t_{a_{g+1}} = t_{E_{h_1}}t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{a_g}t_{a'_g}t_{A_{2g}}\cdots t_{A_{4h_1+2}}t_{E_{h_1}} \cdot t_{a'_g}t_{A_{2g}}\cdots t_{A_{4h_1+2}}\cdot(t_{D_0}t_{D_1}\cdots t_{D_{2h_1}})^2\cdot t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{a'_g}$$

hold in Mod_g^2 *and* Mod_g^1, *respectively.*

In order to prove Proposition 6.1, we prepare Lemma 6.2 and Proposition 6.3.

Lemma 6.2. *Suppose that* $g = 2r$. *In the notation of Lemma 3.4, let* c'_r *be the separating simple closed curve defined by* $a_{g+1}(b_{r+1}\cdots b_g)a'_{g+1}(b_{r+1}\cdots b_g)^{-1}c_r$ (*cf. Figure* 6(*a*)). *We modify* Σ_g^2 *and* $B_0, \ldots, B_g, c_r, c'_r$ *by isotopy as shown in Figure* 6(*b*) *and* (*c*). *Then in* Mod_g^2, *the following relation holds*:

$$t_{a_{g+1}}t_{a'_{g+1}} = t_{c_r}t_{c'_r}(t_{B_0}t_{B_1}\cdots t_{B_g})^2.$$

Proof. It is easily seen that for each $i = 1, \ldots, g$, we have

$$\Delta_{2g+1}\cdots\Delta_2\Delta_1(A_i) = A_{2g+2-i}.$$

This gives the relation

$$\Delta_{2g+1}\cdots\Delta_2\Delta_1 t_{A_i} = t_{A_{2g+i}}\Delta_{2g+1}\cdots\Delta_2\Delta_1$$

for each $i = 1, \ldots, 2r$. Therefore, we have

$$\Delta_{2g+1}\cdots\Delta_2\Delta_1(\overline{\Delta}_g)^{-(2g+2)} = (t_{A_{g+2}}\cdots t_{A_{2g+1}})^{-(2g+2)}\Delta_{2g+1}\cdots\Delta_2\Delta_1.$$

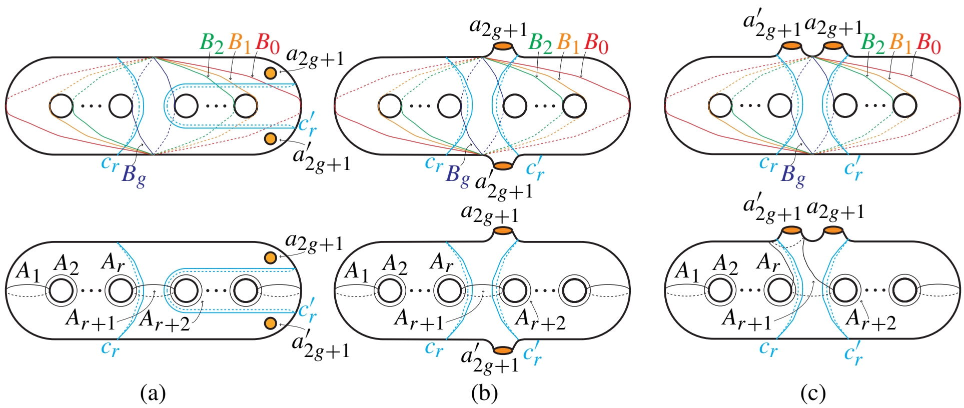

Figure 6. Modified surface Σ_g^2 and curves $B_0, \ldots, B_g, c_r, c'_r$.

Since

$$t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}(\overline{\Delta}_g)^{2g+2}=\Delta_{2g+1}\cdots\Delta_2\Delta_1\ (=t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_{c_r})$$

from the proof of Lemma 3.4, we have

$$(t_{A_{g+2}}\cdots t_{A_{2g+1}})^{2g+2}t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}=\Delta_{2g+1}\cdots\Delta_2\Delta_1$$
$$(=t_{B_0}t_{B_1}t_{B_2}\cdots t_{B_g}t_{c_r}).$$

By the chain relation, we obtain $t_{c'_r}=(t_{A_{g+2}}\cdots t_{A_{2g+1}})^{2g+2}$. Therefore,

$$t_{a_{g+1}}t_{a'_{g+1}}=t_{c'_r}t_{B_0}t_{B_1}\cdots t_{B_g}\cdot t_{B_0}t_{B_1}\cdots t_{B_g}t_{c_r}$$

follows by Lemma 3.4. By conjugation by t_{c_r}, we have

$$t_{a_{g+1}}t_{a'_{g+1}}=t_{c_r}t_{c'_r}(t_{B_0}t_{B_1}\cdots t_{B_g})^2. \qquad \square$$

Proposition 6.3 was shown by Hamada [≥ 2016] based on the argument of [Tanaka 2012]. Its statement concerns a'_0, a null-homotopic simple closed curve in Σ_g defined by $a'_0=c_0a_0$.

Proposition 6.3 [Hamada ≥ 2016]. *Let Σ_g^4 be the compact oriented surface of genus g with four boundary components obtained from Σ_g by removing four disjoint open disks surrounded by a_0, a'_0, a_{g+1} and a'_{g+1}. Then the following relation in* Mod_g^4 *holds*:

$$t_{a_0}t_{a'_0}t_{a_{g+1}}t_{a'_{g+1}}=t_{A_{2g+1}}\cdots t_{A_2}t_{a_1}t_{a'_1}t_{A_2}\cdots t_{A_{2g+1}}\cdot t_{A_1}\cdots t_{A_{2g}}t_{a_g}t_{a'_g}t_{A_{2g}}\cdots t_{A_1}.$$

Proof. The proof is by induction on the genus.

Suppose that $g=1$. The *four-holed torus relation*,

$$t_{a_0}t_{a'_0}t_{a_2}t_{a'_2}=(t_{A_1}t_{A_3}t_{A_2}t_{a_1}t_{a'_1}t_{A_2})^2,$$

was constructed by Korkmaz and Ozbagci [2008, Section 3.4]. Since a_0, a'_0, a_2, a'_2 are disjoint from A_1 and A_1 is disjoint from A_3, by conjugation by t_{A_1}, we have

$$t_{a_0}t_{a'_0}t_{a_2}t_{a'_2}=t_{A_3}t_{A_2}t_{a_1}t_{a'_1}t_{A_2}t_{A_1}\cdot t_{A_3}t_{A_2}t_{a_1}t_{a'_1}t_{A_2}t_{A_1}$$
$$=t_{A_3}t_{A_2}t_{a_1}t_{a'_1}t_{A_2}t_{A_3}\cdot t_{A_1}t_{A_2}t_{a_1}t_{a'_1}t_{A_2}t_{A_1}.$$

Hence, the conclusion of the proposition holds for $g=1$.

Next we assume, inductively, that the relation holds in Mod_{g-1}^4. Since then a_0, a'_0, a_g, a'_g are disjoint from $A_1,\ldots,A_{2g-1}$, we have the relation

$$t_{a_0}t_{a'_0}t_{a_g}t_{a'_g}=t_{A_{2g-2}}\cdots t_{A_1}\cdot t_{A_{2g-1}}\cdots t_{A_2}t_{a_1}t_{a'_1}t_{A_2}\cdots t_{A_{2g-1}}\cdot t_{A_1}\cdots t_{A_{2g-2}}t_{a_{g-1}}t'_{a_{g-1}}$$

in Mod_g^4 by conjugation by $t_{A_{2g-2}}\cdots t_{A_1}$. Since $a_{g-1}, a'_{g-1}, a_{g+1}, a'_{g+1}$ are disjoint from $A_{2g-1}, A_{2g}, A_{2g+1}, a_g, a'_g$, by the four-holed torus relation

$$t_{a_{g-1}}t_{a'_{g-1}}t_{a_{g+1}}t_{a'_{g+1}}=(t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}t_{a_g}t_{a'_g}t_{A_{2g}})^2$$

and conjugation by $t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}$, we have the relation

$$t_{a_g}^{-1}t_{a'_g}^{-1}t_{a_{g+1}}t_{a'_{g+1}} = t_{a'_{g-1}}^{-1}t_{a_{g-1}}^{-1}t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}t_{a_g}t_{a'_g}t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}.$$

By combining these relations, we have

$$t_{a_0}t_{a'_0}t_{a_{g+1}}t_{a'_{g+1}} = t_{A_{2g-2}}\cdots t_{A_1}\cdot t_{A_{2g-1}}\cdots t_{A_2}t_{a_1}t_{a'_1}t_{A_2}\cdots t_{A_{2g-1}}\cdot t_{A_1}\cdots t_{A_{2g-2}} \\ \cdot t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}\cdot t_{a_g}t_{a'_g}t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}.$$

Note that $A_1, \ldots, A_{2g+1}$ are disjoint from $a_0, a'_0, a_{g+1}, a'_{g+1}$. Moreover, A_{2g} and A_{2g+1} are disjoint from $A_1, \ldots, A_{2g-2}$ and $A_1, \ldots, A_{2g-1}$, respectively. Therefore, by conjugation by $t_{A_{2g-2}}\cdots t_{A_1}$ and $t_{A_{2g+1}}t_{A_{2g}}$, we have

$$\begin{aligned} t_{a_0}t_{a'_0}t_{a_{g+1}}t_{a'_{g+1}} &= t_{A_{2g-2}}\cdots t_{A_1}\cdot t_{A_{2g-1}}\cdots t_{A_2}t_{a_1}t_{a'_1}t_{A_2}\cdots t_{A_{2g-1}}\cdot t_{A_1}\cdots t_{A_{2g-2}} \\ &\qquad \cdot t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}}\cdot t_{a_g}t_{a'_g}t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g+1}}t_{A_{2g}} \\ &= t_{A_{2g+1}}t_{A_{2g}}\cdot t_{A_{2g-1}}\cdots t_{A_2}t_{a_1}t_{a'_1}t_{A_2}\cdots t_{A_{2g-1}}\cdot t_{A_{2g}}t_{A_{2g+1}}\cdot t_{A_1}\cdots t_{A_{2g-2}} \\ &\qquad \cdot t_{A_{2g-1}}t_{A_{2g}}\cdot t_{a_g}t_{a'_g}t_{A_{2g}}t_{A_{2g-1}}t_{A_{2g-2}}\cdots t_{A_1}. \end{aligned}$$

This completes the proof of Proposition 6.3. □

We now prove Proposition 6.1.

Proof of Proposition 6.1. Let c'_{h_1} be the separating simple closed curve as shown in Figure 7. By Lemma 6.2 and Proposition 6.3, we have

$$\begin{aligned} t_{a_{h_1+1}}t_{a'_{h_1+1}} &= t_{c_{h_1}}t_{c'_{h_1}}(t_{D_0}t_{D_1}\cdots t_{D_{2h_1}})^2, \\ t_{c_{h_1}}t_{c'_{h_1}}t_{a_{g+1}}t_{a'_{g+1}} &= t_{a_g}t_{A_{2g}}\cdots t_{A_{4h_1+2}}t_{E_{h_1}}t_{E_{h_1}}t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{a'_g} \\ &\qquad \cdot t_{A_{2g+1}}\cdots t_{A_{4h_1+2}}t_{a_{h_1+1}}t_{a'_{h_1+1}}t_{A_{4h_1+2}}\cdots t_{A_{2g+1}}. \end{aligned}$$

Since c_{h_1} and c'_{h_1} are disjoint from $A_{2h_1+2}, \ldots, A_{2g}, E_{h_1}, a_{h_1+1}, a'_{h_1+1}$, it follows that

$$\begin{aligned} t_{c'_{h_1}}^{-1}t_{c_{h_1}}^{-1}\cdot t_{a_{h_1+1}}t_{a'_{h_1+1}} &= (t_{D_0}t_{D_1}\cdots t_{D_{2h_1}})^2, \\ t_{a_{g+1}}t_{a'_{g+1}} &= t_{a_g}t_{A_{2g}}\cdots t_{A_{4h_1+2}}t_{E_{h_1}}t_{E_{h_1}}t_{A_{4h_1+2}}\cdots t_{A_{2g}}t_{a'_g} \\ &\qquad \cdot t_{A_{2g+1}}\cdots t_{A_{4h_1+2}}\cdot t_{c'_{h_1}}^{-1}t_{c_{h_1}}^{-1}\cdot t_{a_{h_1+1}}t_{a'_{h_1+1}}\cdot t_{A_{4h_1+2}}\cdots t_{A_{2g+1}}. \end{aligned}$$

Combining these relations gives the relation (21) in Proposition 6.1.

In Σ_g^1, A_{2g+1} is homotopic to a'_g, and (22) follows, completing the proof. □

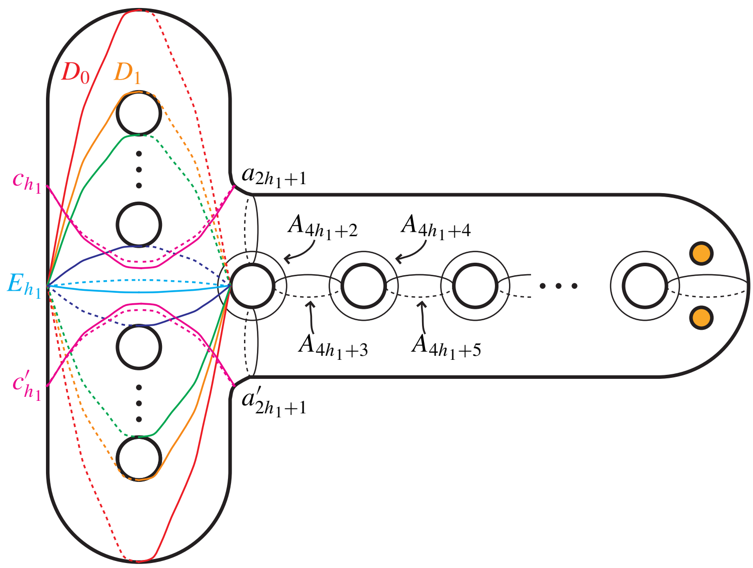

Figure 7. The curve c'_{h_1} on Σ_g^2.

6C. ***A lift of Gurtas' positive relator.*** Since a_{g+1} and a'_{g+1} are null-homotopic in Σ_g, we have $t_{a_{g+1}} = t_{a'_{g+1}} = 1$ in Mod_g, so the relation in Proposition 6.1 is a positive relator in Mod_g. Then we note that A_{2g+1} and a'_g are homotopic to a_g. On the other hand, Gurtas [2004] gave the positive relator

$$(t_{A_{4h_1+2}} \cdots t_{A_{2g}} t_{a_g} t_{a_g} t_{A_{2g}} \cdots t_{A_{4h_1+2}} t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} t_{E_{h_1}})^2 = 1.$$

in Mod_g. Using the following theorem of Kas [1980] and Matsumoto [1996], we show that the relation in Proposition 6.1 gives a lift of Gurtas' positive relator in Mod_g to Mod_g^2.

Theorem 6.4 [Kas 1980; Matsumoto 1996]. *If $g \geq 2$, then the isomorphism class of a Lefschetz fibration is determined by a positive relator modulo simultaneous conjugations*

$$t_{v_n} \cdots t_{v_2} t_{v_1} \sim t_{\phi(v_n)} \cdots t_{\phi(v_2)} t_{\phi(v_1)} \quad \textit{for any } \phi \in \Gamma_g$$

and elementary transformations

$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_i} t_{t_{v_i}^{-1}(v_{i+1})} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1},$$
$$t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{v_i} t_{v_{i-1}} t_{v_{i-2}} \cdots t_{v_1} \sim t_{v_n} \cdots t_{v_{i+2}} t_{v_{i+1}} t_{t_{v_i}(v_{i-1})} t_{v_i} t_{v_{i-2}} \cdots t_{v_1}.$$

The aim of this section is to prove the following proposition. This proposition applied to Proposition 6.1 gives the above mentioned lift.

Proposition 6.5. *In* Mod_g, *the following relation holds*:

$$t_{E_{h_1}} t_{A_{4h_1+2}} \cdots t_{A_{2g}} t_{a_g} t_{a_g} t_{A_{2g}} \cdots t_{A_{4h_1+2}} t_{E_{h_1}} \cdot t_{a_g} t_{A_{2g}} \cdots t_{A_{4h_1+2}} \cdot (t_{D_0} t_{D_1} \cdots t_{D_{2h_1}})^2 \cdot t_{A_{4h_1+2}} \cdots t_{A_{2g}} t_{a_g} \sim (t_{A_{4h_1+2}} \cdots t_{A_{2g}} t_{a_g} t_{a_g} t_{A_{2g}} \cdots t_{A_{4h_1+2}} t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} t_{E_{h_1}})^2.$$

In order to prove this, we need a lemma.

Lemma 6.6. *We deform* Σ_g^2 *as shown in Figure 8*(*a*) *and* (*b*). *Let* E *and* E' *be the simple closed curves in* Σ_g^2 *as in Figure 8*(*a*) *and* (*b*), *and let* a *be the arc connecting the boundary components of* Σ_g^2 *as in the figure. Then*

(23) $$t_{B_0} t_{B_1} \cdots t_{B_g}(E) = E',$$

(24) $$t_{B_0} t_{B_1} \cdots t_{B_g} t_E(a) = t_{a_{g+1}} t_{a'_{g+1}}(a).$$

Proof. From the equality (20), we see that

$$t_{B_0} t_{B_1} \cdots t_{B_g} = \Delta_{2g+1} \cdots \Delta_2 \Delta_1 t_{c_r}^{-1}.$$

By drawing corresponding curves and applying the corresponding Dehn twist, we find that

$$\Delta_{2g+1} \cdots \Delta_2 \Delta_1 t_{c_r}^{-1}(E) = E' \quad \text{and} \quad \Delta_{2g+1} \cdots \Delta_2 \Delta_1 t_{c_r}^{-1} t_E(a) = t_{a_{g+1}} t_{a'_{g+1}}(a).$$

This proves the lemma. □

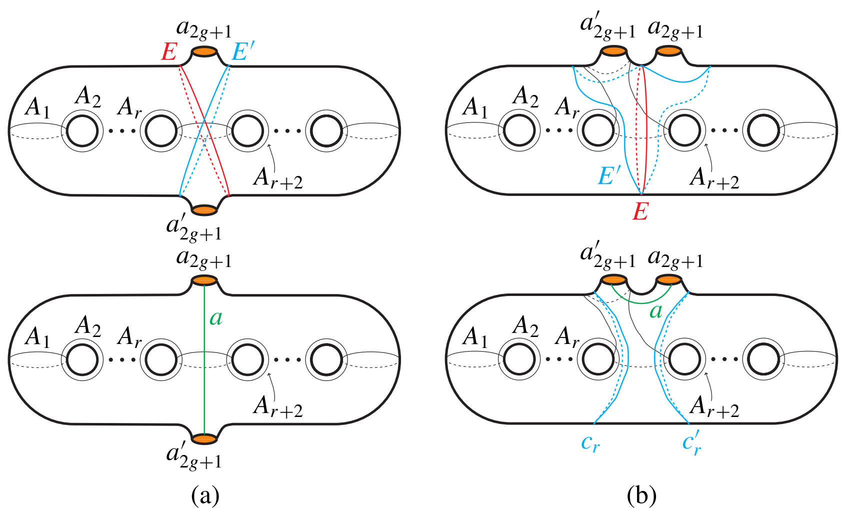

Figure 8. The curves E, E' and the arc a.

Proof of Proposition 6.5. For simplicity of notation, we write

$$\tau := t_{A_{4h_1+2}} \cdots t_{A_{2g}} t_{a_g} \quad \text{and} \quad \overline{\tau} := t_{a_g} t_{A_{2g}} \cdots t_{A_{4h_1+2}}.$$

Note that for each $i = 2h_1 + 2, \dots, 2g$, we find that

$$t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}}(A_i) = A_i \quad \text{and} \quad t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}}(a_g) = a_g.$$

This gives

$$t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot t_{A_i} \sim t_{A_i} \cdot t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \quad \text{and} \quad t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot t_{a_g} \sim t_{a_g} \cdot t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}},$$

so we obtain the relation

$$t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot \tau \sim \tau \cdot t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}}.$$

Therefore, applying elementary transformations (including cyclic permutations) gives

$$t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot \overline{\tau} (t_{D_0} t_{D_1} \cdots t_{D_{2h_1}})^2 \cdot \tau \sim t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot \tau \cdot \overline{\tau} (t_{D_0} t_{D_1} \cdots t_{D_{2h_1}})^2. \tag{25}$$

Since by drawing corresponding curves, applying the corresponding Dehn twist and (24) in Lemma 6.6, we have

$$(\tau \overline{\tau})^{-1}(E_{h_1}) = t_{a_{2h_1+1}} t_{a'_{2h_1+1}}(E_{h_1}) = t_{D_0} t_{D_1} \cdots t_{D_{2h_1}}(E_{h_1}),$$

we thus obtain

$$\tau \overline{\tau} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} \cdot t_{E_{h_1}} \sim t_{E_{h_1}} \cdot \tau \overline{\tau} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}}.$$

Therefore, by using this relation, we have

$$\begin{aligned} t_{E_{h_1}} \tau \overline{\tau} t_{E_{h_1}} \cdot \tau \overline{\tau} \cdot (t_{D_0} t_{D_1} \cdots t_{D_{2h_1}})^2 \\ \sim t_{E_{h_1}} \tau \overline{\tau} \cdot \tau \overline{\tau} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} \cdot t_{E_{h_1}} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}}. \end{aligned} \tag{26}$$

By drawing corresponding curves, applying the corresponding Dehn twist and (23) in Lemma 6.6, we obtain

$$(\tau \overline{\tau})^{-1}(A_{4h_1+2}) = t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} t_{E_{h_1}}(A_{4h_1+2}).$$

Therefore, we have

$$\tau \overline{\tau} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} t_{E_{h_1}} \cdot t_{A_{4h_1+2}} \sim t_{A_{4h_1+2}} \cdot \tau \overline{\tau} \cdot t_{D_0} t_{D_1} \cdots t_{D_{2h_1}} t_{E_{h_1}}.$$

Note that for each $i = 4h_1 + 3, \dots, 2g$, we find that

$$\tau \overline{\tau}(A_i) = A_i \quad \text{and} \quad \tau \overline{\tau}(a_g) = a_g.$$

Moreover, since $A_{4h_1+3}, \dots, A_{2g}$ and a_g are disjoint from $D_0, \dots, D_{2h_1}, E_{h_1}$, we therefore obtain, for each $i = 2h_1+3, \dots, 2g$,

$$\tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}\cdot t_{A_i} \sim t_{A_i}\cdot t\tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}},$$
$$\tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}\cdot t_{a_g} \sim t_{a_g}\cdot \tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}.$$

This gives

$$\tau\bar{\tau}\cdot\tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}} \sim \tau\bar{\tau}t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}\cdot\tau\bar{\tau}.$$

From this relation, applying elementary transformations (including cyclic permutations) gives

$$(27)\quad t_{E_{h_1}}\tau\bar{\tau}\cdot\tau\bar{\tau}\cdot t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}\cdot t_{D_0}t_{D_1}\cdots t_{D_{2h_1}} \sim \tau\bar{\tau}\cdot t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}t_{E_{h_1}}\cdot\tau\bar{\tau}\cdot t_{D_0}t_{D_1}\cdots t_{D_{2h_1}}\cdot t_{E_{h_1}}.$$

Proposition 6.5 follows from the relations (25)–(27). □

7. Construction of simple loops $R_1, \dots, R_k$

In this section, we prove Proposition 5.2. This was based on Korkmaz's work [2009] and the argument in [Akhmedov and Ozbagci 2013]. In Proposition 4.3 of [Korkmaz 2009], he defined l as $l = l(r_1) + \cdots + l(r_k)$. However, in this paper, it is sufficient to consider l as $l = \max_{1\le i\le k}\{l(r_i)\}$. Before providing the simple loops in Σ_g in Proposition 5.2, we need the following proposition about simple loops $R_1, \dots, R_k$ in Σ_{n+l-1}.

Proposition 7.1. *Let F_n be the subgroup of $\pi_1(\Sigma_n)$ generated by $a_1, \dots, a_n$, i.e., F_n is a free group of rank n. Let $r_1, \dots, r_k$ be k arbitrary elements in F_n represented as words in $a_1, \dots, a_n$. Let $l = \max_{1\le i\le k}\{l(r_i)\}$, where $l(r_i)$ is the syllable length of r_i. Then there are simple loops $R_1, \dots, R_k$ in Σ_{n+l-1} with the property that for each $1 \le i \le k$:*

(a) *R_i is freely homotopic to a simple closed curve which intersects a_{n+l-1} transversely at only one point.*

(b) *$\Phi([R_i]) = r_i$, where $[R_i] \in \pi_1(\Sigma_{n+l-1})$ is the homotopy class of R_i, and $\Phi : \pi_1(\Sigma_{n+l-1}) \to \pi_1(\Sigma_n)$ is the map defined by $\Phi(a_j) = a_j$ for $1 \le j \le n$ and $\Phi(\alpha) = 1$ for $\alpha \in \{a_{n+1}, \dots, a_{n+l-1}, b_1, \dots, b_{n+l-1}\}$.*

Proof. Let us consider the surface Σ_n embedded in $\mathbb{R}^3$ as shown in Figure 1 such that for each $1 \le j \le n$, a simple closed curve b'_j in Σ_n which is isotopic to b_j lies on the plane $x = 0$. Write $r_i = a_{i_1}^{m_1}\cdots a_{i_d}^{m_d}$, where $d = l(r_i)$ is the syllable length of r_i. We denote by ξ a constant such that the base point lies in the plane $z = \xi$. Let L be an arc in Σ_n which lies in the half plane $\{z = \xi\} \cap \{x \ge 0\}$.

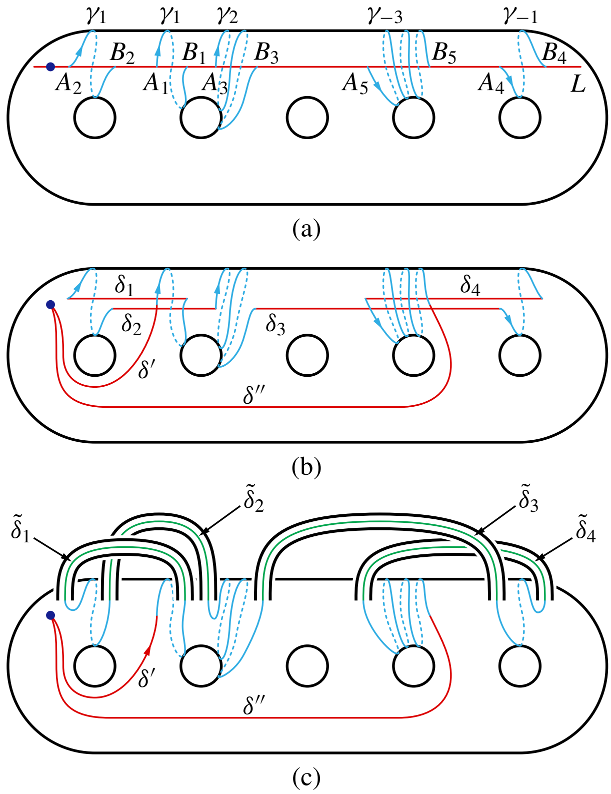

Figure 9. Construction of R_i on Σ_{n+d-1} for $r_i = a_2 a_1 a_2^2 a_5^{-1} a_4^{-3}$ and for $n = 5$.

For $1 \leq t \leq d$, let α_t be a loop in Σ_n which is isotopic to a_{i_t}. If $j_s = j_{s'}$ for some $s < s'$, then we assume that $\alpha_{s'}$ is to the right of α_s and that $\alpha_{s'}$ is disjoint from α_s. Here, right means the positive direction of the y-axis. Let A_t (resp. B_t) be points on L lying to the left (resp. right) of α_t such that there are no A_s (resp. B_s) between α_t and A_t (resp. B_t).

Let $\gamma_{m_t} = t_{\alpha_t}^{-m_t}(\zeta_t)$, where ζ_t is the subarc of L from the point A_j to the point B_j. For each $1 \leq j \leq d-1$, let δ_j denote the subarc of L from the point B_j to the point A_{j+1}. Then we can define an arc β in Σ_n connecting A_1 to B_d to be

$$\beta = \gamma_{m_1} \star \delta_1 \star \gamma_{m_2} \star \delta_2 \star \cdots \star \delta_{d-1} \star \gamma_{m_d},$$

where $\gamma \star \delta$ denotes an arc γ followed by an arc δ. Let δ_0 be the subarc of L from the base point to A_1, and δ_d the subarc from B_d to the base point. Then $\delta_0 \star \beta \star \delta_d$ represents r_i (cf. Figure 9(a)). After perturbing β slightly, we assume that $\delta_1, \ldots, \delta_{d-1}$ are pairwise disjoint and lie parallel to the plane $x = 0$. Note that all self-intersection points of $\delta_0 \star \beta \star \delta_d$ lie on $\delta_0 \cup \delta_1 \cup \cdots \cup \delta_d$.

Let δ' and δ'' be arcs from the base point to A_1 and from B_d to the base point, respectively, which are disjoint from $\alpha_1, \alpha_2, \dots, \alpha_d$ and $b'_1, b'_2, \dots, b'_n$ and lie in the space $\{z \leq \xi\}$. Suppose that the interiors of δ', δ'' and β are pairwise disjoint. Then the loop $\delta' \star \beta \star \delta''$ represents

$$b_1 b_2 \cdots b_{i_1 - 1} r_i b_{i_d}^{-1} \cdots b_2^{-1} b_1^{-1}$$

in $\pi_1(\Sigma_n)$ (cf. Figure 9(b)).

Let $D_1, D'_1, \dots, D_{d-1}, D'_{d-1}$ be pairwise disjoint disks in Σ_n such that for each $1 \leq t \leq d-1$, $\mathrm{Int}(D_t)$ and $\mathrm{Int}(D'_t)$ are disjoint from δ', β and δ'', and $A_t \in \partial D_t$ and $B_t \in \partial D'_t$. We remove $2d-2$ open disks $\mathrm{Int}(D_t)$ and $\mathrm{Int}(D'_t)$ from Σ_n. Then for each $1 \leq t \leq d-1$, by attaching an annulus, denote by $\mathcal{A}_t$, to the surface

$$\Sigma_n \setminus \bigcup_{t=1}^{d-1} \bigl(\mathrm{Int}(D_t) \cup \mathrm{Int}(D'_t)\bigr)$$

along ∂D_t and $\partial D'_t$, we obtain the closed oriented surface

$$\left(\Sigma_n \setminus \bigcup_{t=1}^{d-1} \bigl(\mathrm{Int}(D_t) \cup \mathrm{Int}(D'_t)\bigr)\right) \cap \left(\bigcup_{t=1}^{d-1} \mathcal{A}_t\right)$$

of genus $n+d-1$, denoted by Σ_{n+d-1}. An orientation on Σ_{n+d-1} is given by the orientation on Σ_n.

We define a loop R_i in Σ_{n+d-1} as follows. For each $1 \leq t \leq d-1$, let $\tilde{\delta}_t$ be a simple arc in $\mathcal{A}_t$ from the point B_t to the point A_{t+1} such that $\tilde{\delta}_t$ lies parallel to the plane $x = 0$. Then by "replacing" δ_t in $\delta' \star \beta \star \delta''$ by $\tilde{\delta}_t$, we obtain the loop

$$R = \delta' \star \gamma_{m_1} \star \tilde{\delta}_1 \star \gamma_{m_2} \star \tilde{\delta}_2 \star \cdots \star \tilde{\delta}_{d-1} \star \gamma_{m_d} \star \delta''.$$

In particular, R_i is simple in Σ_{n+d-1} (cf. Figure 9(c)).

Note that from construction, $\tilde{\delta}_t \star \delta_t$ is a simple closed curve in Σ_{n+d-1}. If we collapse each $\mathcal{A}_t$ onto the arc δ_t, then we obtain a map $\Sigma_{n+d-1} \to \Sigma_n$. The induced map $\pi_1(\Sigma_{n+d-1}) \to \pi_1(\Sigma_n)$ takes $[R]$ to

$$b_1 b_2 \cdots b_{i_1 - 1} r_i b_{i_d}^{-1} \cdots b_2^{-1} b_1^{-1},$$

which in turn is mapped to r_i under the map $\pi_1(\Sigma_n) \to \pi_1(\Sigma_n)$ sending a_j to a_j and b_j to 1 for all j.

Let $h = n + l - 1$, where $l = \max_{1 \leq i \leq k}\{l(r_i)\}$. For each $1 \leq i \leq k$, we now construct a loop R_i in Σ_h as follows. First, by sliding $\mathcal{A}_1, \dots, \mathcal{A}_{l(r_i)-1}$, we deform the surface $\Sigma_{n+l(r_i)-1}$ into the standard position as shown in Figure 1 in such a way that the simple loop $\tilde{\delta}_t \star \delta_t$ becomes isotopic to b_{n+t} and the boundary curves of $\mathcal{A}_t$ become isotopic to a_{n+t} (cf. Figure 10(a), (b) and (c)). If $l(r_j) = l$ for some j, then we see that the simple closed curve a_h intersects R_j transversely at one point.

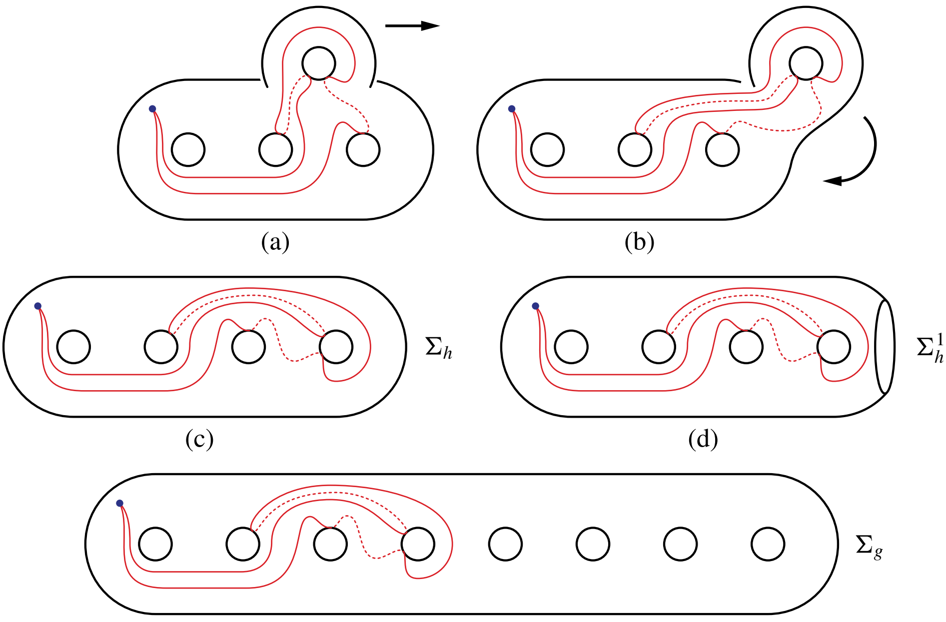

Figure 10. Construction of R_i for $r_i = a_3^{-1}a_2^{-1}$ in the case $n = 3$ and $g = 8$.

Therefore, we assume that $l(r_i) < l$. Next, we remove a small open disk from the deformed surface near $a_{n+l(r_i)-1}$ and disjoint from R_i (cf. Figure 10(d)). Thus, we obtain a surface of genus $n + l(r_i) - 1$ with one boundary component, denoted by $\Sigma^1_{n+l(r_i)-1}$. We embed $\Sigma^1_{n+l(r_i)-1}$ into the standard surface Σ_h in such a way that for each $1 \leq t \leq n + l(r_i) - 1$, simple loops a_t, b_t in $\Sigma^1_{n+l(r_i)-1}$ correspond to the simple loops a_t, b_t in Σ_h (cf. Figure 10(e)). Finally, we replace R_i with a simple representative of $[R_i]\big((b_1b_2\cdots b_{h-1})(b_1b_2\cdots b_h)^{-1}\big)^{\epsilon}$, where $\epsilon = \pm 1$ (cf. Figure 10(d)). Then we see that the resulting simple loop R_i intersects a_h transversely at one point.

From the above construction, $\Phi : \pi_1(\Sigma_h) \to \pi_1(\Sigma_n)$ maps $[R_i]$ to r_i for each $i = 1, \dots, k$. This gives the required simple loops $R_1, \dots, R_k$. □

Proof of Proposition 5.2. Consider a surface Σ_{n+l-1} and the loops $R_1, \dots, R_k$ constructed in Proposition 7.1. We remove a small open disk from Σ_{n+l-1} near a_{n+l-1} and disjoint from all R_i (cf. Figure 11(a)). Denote by Σ^1_{n+l-1} the resulting surface of genus $n+l-1$ with one boundary component. We embed Σ^1_{n+l-1} into the standard surface Σ_g in such a way that for each $1 \leq t \leq n+l-1$, simple loops a_t, b_t in Σ^1_{n+l-1} correspond to the simple loops a_t, b_t in Σ_g (cf. Figure 11(b)). Then we can modify $R_1, \dots, R_k$ so that each R_i $(i = 1, \dots, k)$ satisfies the property of Proposition 5.2

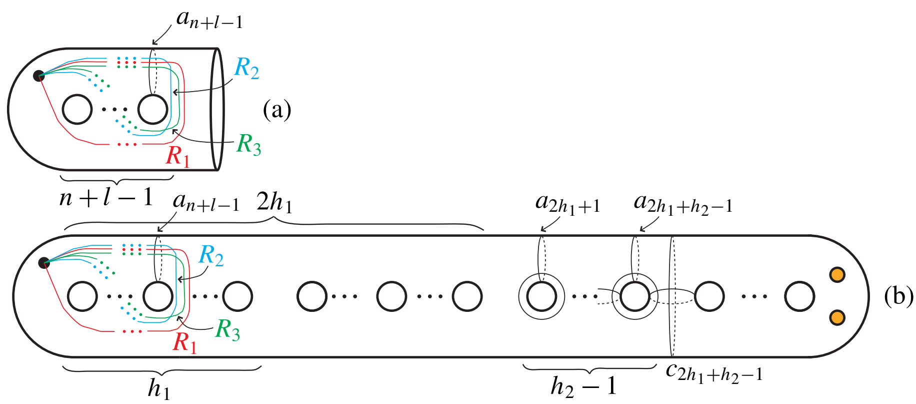

Figure 11. Modified curves $R_1, \dots, R_k$ in Σ_g.

by replacing R_i with a simple representative of $[R_i](b_{2h_1+1}b_{2h_2+2}\cdots b_{2h_1+h_2-i})^{\epsilon}$ if i is odd, and $[R_i]a^{\epsilon}_{2h_1+h_2-i}$ if i is even, where $\epsilon = \pm 1$ (cf. Figure 5). Therefore, we obtain the required simple loops $R_1, \dots, R_k$. □

8. Remarks

The results of [Gompf 1995; Donaldson 1999; Gompf and Stipsicz 1999] mentioned in the introduction naturally raise the following two basic questions, which remain open.

Question 8.1 (cf. [Korkmaz and Stipsicz 2009]). Given a symplectic 4-manifold, what is the minimal genus g for which it has a genus-g Lefschetz pencil?

Question 8.2. Given a finitely presented group Γ, what is the minimal genus, denoted by $g_P(\Gamma)$, for which there exists a genus-g Lefschetz pencil on a symplectic 4-manifold with fundamental group Γ?

Although these two questions remain open, for Question 8.2, we can give an upper bound for $g_P(\Gamma)$ as a corollary of Theorem 1.2.

Corollary 8.3. *We have* $g_P(\Gamma) \leq 4(n+l-1)+k$ *for* $k \geq 1$, *and* $g_P(F_n) \leq 4n+2$.

However, this upper bound for $g_P(\Gamma)$ may not be sharp. In fact, since $\mathbb{CP}^2$ admits a genus-0 Lefschetz pencil, $g_P(\Gamma) = 0$ if Γ is the trivial group. When we replace the relations in Proposition 5.1 and the map ψ_k in Section 5A by another relation and map, we can improve the upper bound of $g_P(\Gamma)$. For example, for every positive integer n, the article [Hamada et al. $\geq$ 2016] gave a genus-g Lefschetz pencil on a 4-manifold X_n such that $\pi_1(X_n) \cong \mathbb{Z} \oplus \mathbb{Z}_n$ for every $g \geq 4$ using a similar construction to this paper. Therefore, $g_P(\mathbb{Z} \oplus \mathbb{Z}_m) \leq 4$.

We expect that by a combination of substitution techniques and partial conjugation techniques, we could obtain results for Lefschetz fibrations with (-1)-sections analogous to those obtained by fiber sum operations. The articles [Ozbagci and Stipsicz 2000; Korkmaz 2001; Monden 2014] gave examples of nonholomorphic Lefschetz fibrations by fiber sum operations (and lantern substitutions). By a similar technique to this paper (and a lantern substitution), two kinds of nonholomorphic Lefschetz fibrations with (-1)-sections were constructed in [Hamada et al. $\geq$ 2016]. One is a Lefschetz fibration with noncomplex total space, and the other is a Lefschetz fibration violating the "slope inequality".

Finally, we explain why the Lefschetz fibrations constructed in [Korkmaz 2009; Akhmedov and Ozbagci 2013; Kobayashi 2015] do not have (-1)-sections. In [Korkmaz 2009; Kobayashi 2015], twisted fiber sum operations were adopted, and the fibrations in [Akhmedov and Ozbagci 2013] were obtained by performing Luttinger surgeries and knot surgeries on the symplectic sum of certain symplectic 4-manifolds. The fiber sum of Lefschetz fibrations has no (-1)-sections (see [Stipsicz 2001a], and also [Smith 2001]). In particular, the symplectic sum of symplectic 4-manifolds is minimal, that is, it does not contain any (-1)-spheres (see [Usher 2006], and also [Sato 2006; Baykur 2015]), and Luttinger surgery and knot surgery preserve minimality of symplectic 4-manifolds from the result of [Usher 2006]. Therefore, we see that the fibrations in [Korkmaz 2009; Akhmedov and Ozbagci 2013; Kobayashi 2015] do not have any (-1)-sections.

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