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We discuss an extrinsic property of knots in a 3-subspace of the 3-sphere S^3 to characterize how the subspace is embedded in S^3. Specifically, we show that every knot in a subspace of the 3-sphere is transient if and only if the exterior of the subspace is a disjoint union of handlebodies, i.e., regular neighborhoods of embedded graphs, where a knot in a 3-subspace of S^3 is said to be transient if it can be moved by a homotopy within the subspace to the trivial knot in S^3. To show this, we discuss the relation between certain group-theoretic and homotopic properties of knots in a compact 3-manifold, which can be of independent interest. Further, using the notion of transient knots, we define an integer-valued invariant of knots in S^3 that we call the transient number. We then show that the union of the sets of knots of unknotting number one and tunnel number one is a proper subset of the set of knots of transient number one.

Introduction

In the list [Eilenberg 1949] of problems edited by Eilenberg, Fox proposed a program to distinguish 3-manifolds by the differences in their "knot theories". Following the program, Brody [1960] reobtained the topological classification of the 3-dimensional lens spaces using knot-theoretic invariants, which are the Alexander polynomials of knots suitably factored out so that it depends only on the homology classes of the knots. Bing's recognition theorem [1958] can be regarded as another example of works that follow Fox's program. The theorem asserts that a closed, connected 3-manifold M is homeomorphic to the 3-sphere if and only if every knot in M can be moved by an isotopy to lie within a 3-ball. We note here that if we replace *isotopy* in this statement by *homotopy*, the assertion implies the Poincaré conjecture, which was proved by Perelman [2002; 2003a; 2003b]. Bing's recognition theorem was generalized by Hass and Thompson [1989] and Kobayashi and Nishi [1994]

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proving that a closed, connected 3-manifold M admits a genus-g Heegaard splitting if and only if there exists a genus-g handlebody V embedded in M such that every knot in M can be moved by an isotopy to lie within V. We note that, as mentioned in [Nakamura 2015], the *homotopy* version of this statement holds when $g = 1$, again by the Poincaré conjecture, whereas the higher genus case fails in general. A result of Brin, Johannson, and Scott [Brin et al. 1985] can also be regarded as a work following Fox's program. This result asserts that if every knot in M can be moved by a homotopy to lie within a collar neighborhood of the boundary ∂M, then there exists a component F of ∂M such that the natural map $\pi_1(F) \to \pi_1(M)$ induced by the inclusion is surjective. In particular, for a compact, connected, orientable, irreducible, boundary-irreducible 3-manifold M, they proved that if every knot in M can be moved by a homotopy to lie within a collar neighborhood of ∂M, then M is homeomorphic to the 3-ball or the product $\Sigma \times [0, 1]$, where Σ is a closed, orientable surface of genus at least one. In the present paper, we will consider a relative version of Fox's program. Namely, we discuss "(extrinsic) knot theories" in 3-subspaces of the 3-sphere S^3 in order to characterize how the 3-subspaces are embedded in S^3.

Let M be a compact, connected, proper 3-submanifold of S^3. We say that M is *unknotted* if its exterior is a disjoint union of handlebodies. A famous theorem of Fox [1948] says that each M can be reembedded in S^3 so that its image is unknotted. A reembedding satisfying this property is called a *Fox reembedding*. Intuitively speaking, unknottedness of $M \subset S^3$ implies that M is embedded in S^3 in one of the "simplest" ways. We note that if M is a handlebody, an unknotted M in S^3 is actually unique up to isotopy [Waldhausen 1968]. The uniqueness up to isotopy and a reflection holds for each knot exterior by a celebrated result of Gordon and Luecke [1989]. However, in other cases M usually admits many mutually nonisotopic Fox reembeddings into S^3.

The unknottedness of a 3-submanifold, and so the existence of a Fox reembedding, can be considered for an arbitrary closed, connected 3-manifold. Scharlemann and Thompson [2005] generalized the above theorem of Fox by proving that any compact, connected, proper 3-submanifold of an irreducible non-Haken 3-manifold N admits a Fox reembedding into N or S^3. Another generalization is given by Nakamura [2015] who proved that a compact, connected, proper 3-submanifold M of a closed, connected 3-manifold N admits a Fox reembedding into N if every knot in N can be moved by an isotopy to lie within M. Here we remark that the property that *every knot in N can be moved by an isotopy to lie within M* does *not* imply that M itself is unknotted in N. This can be seen for example by considering the case where $N = S^3$ and M is not unknotted. In this paper, we will show that the property of a compact, connected, proper 3-submanifold M of S^3 that *every knot in M can be moved by a homotopy in M to be the trivial knot in S^3* implies that M is unknotted in S^3. Following [Letscher 2012], we say that a knot K in M is *transient in M* if

K can be deformed by a homotopy in M to be the trivial knot in S^3; K is said to be *persistent in* M otherwise. Using this terminology, we can state our main theorem:

Theorem 3.2. *Let M be a compact, connected, proper 3-submanifold of S^3. Then every knot in M is transient in M if and only if M is unknotted.*

Roughly speaking, the above theorem implies that a (homotopic) property of knots in M deduces an isotopic property of M inside S^3. We remark that the property that a given knot $K \subset M$ is transient is *extrinsic* with respect to the embedding $M \hookrightarrow S^3$, in the sense that it depends not only on the pair (M, K) but also on the way M is embedded in S^3. Indeed, we can find a persistent knot in a certain genus-two handlebody V embedded in S^3 in such a way that there exists another embedding of V into S^3 such that the reembedded knots in the reembedded V is transient. See Section 3. Now, we can say a little more precisely what is the relative version of Fox's program; we expect that extrinsic properties for knots in a compact, connected, proper 3-submanifold of S^3 distinguish the isotopy class of M inside S^3. Our main theorem is a first step for the program. To obtain the theorem, we discuss the relation between certain group-theoretic and homotopic properties of knots in a compact 3-manifold, which can be of independent interest. See Section 1.

Given a knot K in a compact, connected, proper 3-submanifold M of S^3, it is actually difficult in general to detect if K is persistent in M. One method provided by Letscher [2012] uses what he calls the *persistent Alexander polynomial*. In Section 4, we provide examples of persistent knots in a 3-subspace of S^3 whose persistence are shown by using the notion of *persistent lamination* and *accidental surface*.

In Section 5, we will introduce an integer-valued invariant, the *transient number* of knots in S^3, whose definition is related to Theorem 3.2 as follows. Given a knot K in S^3, we may consider a system of simple arcs in S^3 with their endpoints in K such that K is transient in a regular neighborhood of the union of K and the arcs. The transient number $\mathrm{tr}(K)$ is then defined to be the minimal number of simple arcs in such a system. By an easy observation, we see that the transient number is bounded from above by both the unknotting number and the tunnel number. Further, we will give a knot K that attains $\mathrm{tr}(K) = 1$ while $u(K) = t(K) = 2$, where $u(K)$ and $t(K)$ are the unknotting number and the tunnel number of K, respectively (see Proposition 5.2). In other words, the union of the sets of knots of unknotting number one and tunnel number one is actually a proper subset of the set of knots of transient number one. Section 6 contains some concluding remarks and open questions.

Throughout this paper, we will work in the piecewise linear category.

Notation. Let X be a subset of a given polyhedral space Y. We will denote the interior of X by $\mathrm{Int}\, X$. We will use $\mathrm{Nbd}(X; Y)$ to denote a closed regular neighborhood of X in Y. If the ambient space Y is clear from the context, we denote it briefly by $\mathrm{Nbd}(X)$. Let M be a 3-manifold. Let $L \subset M$ be a submanifold with or without boundary. When L is 1- or 2-dimensional, we write $E(L) = M \setminus \mathrm{Int}\,\mathrm{Nbd}(L)$. When L is 3-dimensional, we write $E(L) = M \setminus \mathrm{Int}\, L$. We shall often say "surfaces", "compression bodies", etc., in an ambient manifold to mean their isotopy classes.

1. Knots filling up a handlebody

Let F_g be a free group of rank g with a basis $X_g = \{x_1, x_2, \dots, x_g\}$. We set

$$X_g^{\pm} = X_g \cup \{{x_1}^{-1}, {x_2}^{-1}, \dots, {x_g}^{-1}\}.$$

A *word* on X_g is a finite sequence of letters of $X_g^{\pm}$. For an element x of a group G, we denote by $c_G(x)$ (or simply by $c(x)$) its conjugacy class in G.

Let G be a group with a decomposition $G = G_1 * G_2$. Then G_1 and G_2 are called *free factors* of G. In particular, if $G_2 \neq 1$, then G_1 is called a *proper* free factor of G. Following [Lyon 1980], we say that an element x of G *binds* G if x is not contained in any proper free factor of G. Thus, for example, an element of $\mathbb{Z}$ binds $\mathbb{Z}$ if and only if it is nontrivial. We can also see that an element of a rank-2 free group $F_2 = \langle x_1, x_2 \rangle$ binds F_2 if and only if it is not a power of a primitive element, where an element of a free group is said to be *primitive* if it is a member of some free basis of the free group. For example $x_1x_2x_1x_2$ does not bind F_2, while $x_1x_2x_1{x_2}^3$ binds F. See, e.g., [Osborne and Zieschang 1981] and [Cho and Koda 2015]. Primitive elements of the rank-2 free group have been well understood by, e.g., Osborne and Zieschang [1981] and Cohen, Metzler, and Zimmermann [Cohen et al. 1981], whereas their classification in a free group of higher rank is known to be a hard problem. See [Puder and Wu 2014] (and also [Shpilrain 2005]) and [Puder and Parzanchevski 2015] for some of the deepest results on this problem. On the contrary, an algorithm to detect if a given element x of a free group F_g binds F_g is given by Stallings [1999] using the combinatorics of its Whitehead graph. See (2) in Section 6. It follows immediately from the definition that if x binds G, then any element of its conjugacy class $c(x)$ binds G. In fact, if x lies in G_1 for a decomposition $G = G_1 * G_2$, then $a^{-1}xa$ lies in $a^{-1}G_1a$ and $F = (a^{-1}G_1a) * (a^{-1}G_2a)$ is also a decomposition of G for any $a \in G$.

Let K be an oriented knot in a 3-manifold M. We denote by $c_{\pi_1(M)}(K)$ (or simply by $c(K)$) the conjugacy class in $\pi_1(M)$ defined by the homotopy class of K. Here we recall that two oriented knots K and K' in M are homotopic in M if and only if

$$c_{\pi_1(M)}(K) = c_{\pi_1(M)}(K').$$

We say that K *binds* $\pi_1(M)$ if an element (and so every element) of $c(K)$ binds $\pi_1(M)$. It is clear by definition that, if $\bar{K}$ is the knot K with the reversed orientation, K binds $\pi_1(M)$ if and only if $\bar{K}$ also does. For this reason, we can say whether or not a knot K binds $\pi_1(M)$, while ignoring the orientation of K.

Let M be a compact 3-manifold and F a subsurface of ∂M, or a surface properly embedded in M. Here we note that F is possibly disconnected. Recall that F is said to be *compressible* if

(1) there exists a component of F that bounds a 3-ball in M, or

(2) there exists an embedded disk D in M, called a *compression disk* for F, such that $D \cap F = \partial D$ and such that ∂D is an essential simple closed curve on F.

Otherwise, F is said to be *incompressible*. A 3-manifold is said to be *irreducible* if it contains no incompressible 2-spheres and *boundary-irreducible* if its boundary is incompressible. The following lemma is a generalization of [Lyon 1980, Corollary 1].

Lemma 1.1. *Let M be a compact, connected, orientable, irreducible 3-manifold with nonempty boundary. Let K be an oriented simple closed curve in the boundary of M. Then $\partial M \setminus K$ is incompressible in M if and only if K binds $\pi_1(M)$.*

Proof. We fix an orientation and a base point v of K.

Suppose first that K does not bind $\pi_1(M, v)$. Then there exists a decomposition $\pi_1(M, v) = G_1 * G_2$ with $G_2 \neq 1$ and $[K] \in G_1$. Let X_i be a $K(G_i, 1)$-space, and let p be a point not in $X_1 \cup X_2$. We define $\hat{X}_1$ and $\hat{X}_2$ to be the mapping cylinders of maps from p into X_1 and X_2, respectively. Let X denote the space obtained by identifying the copy of p in $\hat{X}_1$ with that of p in $\hat{X}_2$. By the construction, we have $\pi_1(X) = G_1 * G_2$ and $\pi_2(X_1) = \pi_2(X_2) = 0$. Thus there exists a continuous map $f : M \to X$ satisfying the following properties:

(1) $f(v) = p$,

(2) the induced map $f_* : \pi_1(M) \to \pi_1(X)$ is an isomorphism with $f_*(G_i) = \pi_1(X_i)$ for $i \in \{1, 2\}$, and

(3) $f^{-1}(p)$ consists of a finite number of compression disks for ∂M.

Here we use the assumption that M is irreducible. We may assume that $|f^{-1}(p) \cap K|$ is minimal among all continuous maps $M \to X$ satisfying (1)–(3). Suppose that $f^{-1}(p) \cap K$ is nonempty. Then $f(K)$ is a loop in X with base point p that can be decomposed as

$$f(K) = \alpha_1 * \alpha_2 * \cdots * \alpha_r,$$

where each α_i lies in $\hat{X}_1$ or $\hat{X}_2$, and α_i, α_{i+1} do not lie in one of $\hat{X}_1$ and $\hat{X}_2$ at the same time. We note that $r > 1$. Suppose that no $[\alpha_i]$ is trivial in G_1 or G_2. Then $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ is a *reduced sequence*, that is, $[\alpha_i]$ is in G_1 or G_2, and $[\alpha_i], [\alpha_{i+1}]$ do not lie in one of G_1 and G_2 at the same time. On the other hand, $[f(K)]$ lies in G_1 by the assumption. This contradicts the uniqueness of reduced sequences; see Theorem 4.1 of Magnus, Karrass, and Solitar's book [Magnus et al. 1976]. Thus at least one of $[\alpha_1], [\alpha_2], \ldots, [\alpha_r]$ is trivial. Consequently, there exists a subarc α of K such that

- $\alpha \cap f^{-1}(p) = \partial\alpha$,
- $f(\alpha) \subset X$ is a contractible loop, and
- α is essential in ∂M cut off by $\partial f^{-1}(p)$.

Then using a standard technique as in [Lyon 1980, Theorem 2], f can be deformed by a homotopy to be a continuous map $f' : M \to X$ satisfying the above (1)–(3) and $|f'^{-1}(p) \cap K| < |f^{-1}(p) \cap K|$. This contradicts the minimality of $|f^{-1}(p) \cap K|$. Thus we have $f^{-1}(p) \cap K = \varnothing$. This implies that $\partial M \setminus K$ is compressible in M.

Next suppose that there exists a compression disk D for $\partial M \setminus K$ in M. Suppose that D separates M into two components M_1 and M_2, where K lies in M_1. Then $\pi_1(M)$ can be decomposed as $\pi_1(M) = \pi_1(M_1) * \pi_1(M_2)$, where $[K] \in \pi_1(M_1)$. If $\pi_1(M_2) = 1$, then $M_2 \cong B^3$ by the Poincaré conjecture proved by Perelman [2002; 2003a; 2003b]. This is a contradiction. Hence $\pi_1(M_2) \neq 1$, which implies that K does not bind $\pi_1(M)$. Suppose that D does not separate M. Let M' be M cut off by D. Then we have $\pi_1(M) = \pi_1(M') * \mathbb{Z}$ and $[K] \in \pi_1(M')$. Hence, again, K does not bind $\pi_1(M)$. □

Let M be a compact, connected 3-manifold. Let K and K' be knots in M. We write $K \overset{M}{\sim} K'$ if K and K' are homotopic in M. Let K be a knot in the interior of M. We say that K *fills* up M if, for any knot K' in the interior of M such that $K \overset{M}{\sim} K'$, the exterior $E(K')$ is irreducible and boundary-irreducible.

Example. The knot K_1 shown on the left-hand side in Figure 1 does not fill up the handlebody V (because there exists a compression disk D for ∂V in $V \setminus K_1$ as shown), while the knot K_2 shown on the right-hand side fills up V (see Lemma 1.5).

By a *graph*, we mean the underlying space of a (possibly disconnected) finite 1-dimensional simplicial complex. A handlebody is a 3-manifold homeomorphic to a closed regular neighborhood of a connected graph embedded in the 3-sphere. The *genus* of a handlebody is defined to be the genus of its boundary surface. For a handlebody V, a *spine* is defined to be a graph Γ embedded in V so that V collapses onto Γ. By a 1-*vertex spine* we mean a spine with a single vertex. In other words,

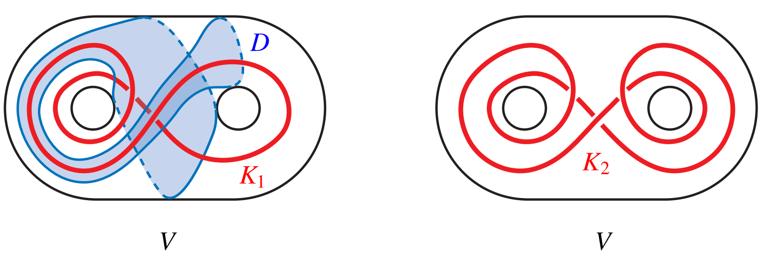


Figure 1. The knot K_1 does not fill up V, while K_2 fills up V.

a 1-vertex spine is a spine of a handlebody that is homeomorphic to a *rose*, i.e., a wedge of circles.

In the remainder of the section we fix the following:

- A handlebody V of genus g at least 1 with a base point v_0.
- A 1-vertex spine Γ_0 of V having the vertex at v_0.
- A standard basis $X = \{x_1, x_2, \dots, x_g\}$ of $\pi_1(\Gamma_0, v_0) \cong \pi_1(V, v_0)$; that is, we can assign names $e_1^0, e_2^0, \dots, e_g^0$ and orientations to the edges of Γ_0 so that x_i corresponds to the oriented edge e_i^0 for each $i \in \{1, 2, \dots, g\}$.

In this setting, we identify $\pi_1(V) = \pi_1(V, v_0)$ with the free group F with basis X.

Let $\{y_1, y_2, \dots, y_g\}$ be a basis of F, where each y_i is a word on the standard basis X. We say that a 1-vertex spine Γ of V having the vertex at v_0 is *compatible with* the basis $\{y_1, y_2, \dots, y_g\}$ if we can assign names $e_1, e_2, \dots, e_g$ and orientations to the edges of Γ so that a word on X corresponding to the oriented edge e_i is y_i for each $i \in \{1, 2, \dots, g\}$.

Lemma 1.2. *For each basis $Y = \{y_1, y_2, \dots, y_g\}$ of F, there exists a* 1*-vertex spine of V with the vertex at v_0 that is compatible with Y.*

Proof. Let φ be the automorphism of F that maps x_i to y_i for each $i \in \{1, 2, \dots, g\}$. By [Nielsen 1924], the map φ can be factored into a composition $\varphi_n \circ \cdots \circ \varphi_2 \circ \varphi_1$, where each φ_j is an *elementary Nielsen transformation*. Here we recall that an elementary Nielsen transformation is one of the four automorphisms $\nu_1, \nu_2, \nu_3, \nu_4$ of F, where

- ν_1 switches x_1 and x_2,
- ν_2 cyclically permutes $x_1, x_2, \dots, x_g$ to $x_2, \dots, x_g, x_1$,
- ν_3 replaces x_1 with ${x_1}^{-1}$, and
- ν_4 replaces x_1 with $x_1 x_2$.

We refer the reader to [Magnus et al. 1976] for details. For each φ_i ($i \in \{1, 2, 3, 4\}$), it is easy to see that there exists a homeomorphism g_i of V such that g_i fixes v_0 and $g_i(\Gamma_0)$ is compatible with the basis $\{\nu_i(x_1), \nu_i(x_2), \dots, \nu_i(x_g)\}$. Let g_j be one of f_1, f_2, f_3, f_4 corresponding to φ_j. Then it is clear from the definition that $g_n \circ \cdots \circ g_2 \circ g_1(\Gamma_0)$ is a required 1-vertex spine of V. □

Let M be a compact, connected, orientable, irreducible 3-manifold with nonempty boundary and base point v. We say that M satisfies the *strong bounded Kneser conjecture* (*SBKC*) if, whenever we have subgroups G_1, G_2 of $\pi_1(M, v)$ with $G_1 \cap G_2 = 1$, $\pi_1(M, v) = G_1 * G_2$ and $G_i \ncong 1$ ($i = 1, 2$), there exists a properly embedded disk D in M containing v such that D separates M into two components M_1 and M_2 with $\iota_{i*}(\pi_1(M_i, v)) = G_i$ ($i = 1, 2$), where $\iota_i : M_i \hookrightarrow M$ is the natural embedding. As we will see in the remark after the proof of Lemma 1.4, there exists a 3-manifold that does not satisfy the SBKC. It follows directly from Lemma 1.2 that a genus-g handlebody V satisfies the SBKC. In fact, for each decomposition $\pi_1(V, v_0) = G_1 * G_2$, we have a 1-vertex spine Γ of V having the vertex at v_0 that is compatible with the basis $\{y_1, y_2, \dots, y_g\}$, where $\{y_1, y_2, \dots, y_{g_1}\}$ is a basis of G_1 and $\{y_{g_1+1}, y_{g_1+2}, \dots, y_g\}$ is a basis of G_2. Using the spine Γ, we have the required disk D. We note that a sufficient condition for a manifold to satisfy the SBKC was given by Jaco as follows.

Lemma 1.3 [Jaco 1969]. *Let M be a compact, connected, orientable, irreducible 3-manifold with nonempty, connected boundary. Suppose that $\pi_1(M)$ is freely reduced, that is, if we have a decomposition $G = G_1 * G_2$ then neither of G_1 and G_2 is a free group. Then M satisfies the SBKC.*

Lemma 1.4. *Let M be a compact, connected, orientable, irreducible 3-manifold with nonempty boundary. Let K be an oriented knot in the interior of M. If K binds $\pi_1(M)$, then K fills up M. Moreover, the converse is true when M satisfies the SBKC.*

Proof. Suppose that K does not fill up M. Then there exists an incompressible sphere or a compression disk D for ∂M in $M \setminus K'$, where K' is a knot with $K \overset{M}{\sim} K'$. By the same argument as in the second half of the proof of Lemma 1.1, using K' instead of K in the proof, we can show that K does not bind $\pi_1(M)$.

Next, suppose that M satisfies the SBKC and that K does not bind $\pi_1(M)$. We fix an orientation and a base point v of K. There exist subgroups G_1, G_2 of $\pi_1(M, v)$ with $G_1 \cap G_2 = 1$, $\pi_1(M, v) = G_1 * G_2$, $G_2 \ncong 1$, and $[K] \in G_1$. If $G_1 = 1$, then K is contractible and thus we are done. Suppose that $G_1 \ncong 1$. Then by the SBKC, there exists a properly embedded disk D in M containing v such that D separates M into two components M_1 and M_2 with $\iota_{i*}(\pi_1(M_i, v)) = G_i$ $(i \in \{1, 2\})$, where $\iota_i : M_i \hookrightarrow M$ is the natural embedding. We may assume that K is moved by a homotopy fixing v so that $|K \cap D|$ is minimal. If $|K \cap D| = 0$, we are done. Suppose that $|K \cap D| > 0$. Then $[K]$ can be decomposed into a product $x_1 x_2 \cdots x_r$, where x_i is in G_1 or G_2, and x_i, x_{i+1} do not lie in one of G_1 and G_2 at the same time. We note that $r > 1$. Since $[K] \subset G_1$, at least one, say x_{i_0}, of $x_1, x_2, \dots, x_r$ is trivial. Then moving a neighborhood of the subarc of K corresponding to x_{i_0} by a homotopy, we can reduce $|K \cap D|$. This contradicts the minimality of $|K \cap D|$. □

We remark that the converse of Lemma 1.4 is not true. This can be seen as follows. Let Σ be a closed orientable surface of genus at least one. Let M be a 3-manifold obtained by attaching a 1-handle H to $\Sigma \times [0, 1]$ so as to connect $D \times \{0\}$ and $D \times \{1\}$ and so that the resulting manifold M is orientable, where D is a disk in Σ. See Figure 2. Clearly, M is compact, connected, orientable and irreducible. Let $K \subset M$ be the knot obtained by extending the core of H

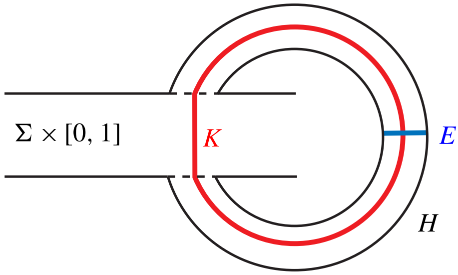


Figure 2. The manifold M.

along a vertical arc $\{*\} \times [0, 1]$ in $\Sigma \times [0, 1]$. We fix a base point v in K and an orientation of K. Then the fundamental group $\pi_1(M, v)$ can be naturally identified with $\pi_1(\Sigma) * \mathbb{Z}$, and under this identification $[K]$ is contained in the factor $\mathbb{Z}$. This implies that K does not bind $\pi_1(M)$. On the contrary, it is easy to see that the cocore E of the 1-handle H is the unique compression disk for ∂M up to isotopy. The algebraic intersection number of K and E is ± 1 after giving an orientation of E. This implies that after deforming K by any homotopy in M, K intersects E, whence K fills up M. We note that M does not satisfy the SBKC.

Lemma 1.5. *Let V be a handlebody. Then there exists a knot in the interior of V that fills up V.*

Proof. Let K be a simple closed curve in ∂V such that $\partial V \setminus K$ is incompressible in V. Such a simple closed curve does exist. In fact, the simple closed curve shown in Figure 3 satisfies this condition (see for instance [Wu 1996, Section 1]). Then by Lemma 1.1 K binds $\pi_1(V)$. It follows from Lemma 1.4 that a knot obtained by moving K by an isotopy to lie in the interior of V fills up V. □

2. Knots filling up a 3-subspace of the 3-sphere

Let V be a handlebody. A (possibly disconnected) subgraph of a spine of V is called a *subspine* if it does not contain a contractible component. A *compression body* W is the complement of an open regular neighborhood of a (possibly empty) subspine Γ

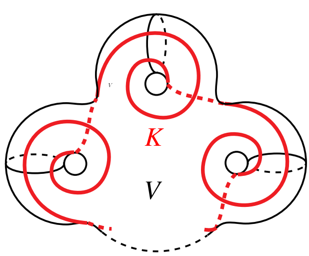
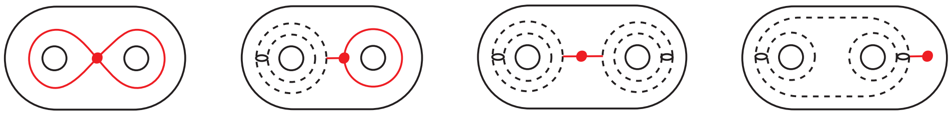


Figure 3. The surface $\partial V \setminus K$ is incompressible in V.



(i) (ii) (iii) (iv)

Figure 4

of a handlebody V. The component $\partial_+ W = \partial V$ is called the *exterior boundary* of W, and $\partial_- W = \partial W \setminus \partial_+ W = \partial \operatorname{Nbd}(\Gamma)$ is called the *interior boundary* of W. We remark that the interior boundary is incompressible in W; see [Bonahon 1983].

For a compression body W, a *spine* is defined to be a graph Γ embedded in W such that

(1) $\Gamma \cap \partial W = \Gamma \cap \partial_- W$ consists only of vertices of valence one, and

(2) W collapses onto $\Gamma \cup \partial_- W$.

We note that this is a generalization of a spine of a handlebody. We also note that if V is a handlebody and Γ is a subspine of $\hat{\Gamma}$ of V such that $W \cong V \setminus \operatorname{Int}\operatorname{Nbd}(\Gamma;\, V)$, then $\hat{\Gamma} \setminus \operatorname{Int}\operatorname{Nbd}(\Gamma;\, V)$ is a spine of W. As a generalization of the case of handlebodies, a 1-*vertex spine* of a compression body W is defined to be a (possibly empty) connected spine Γ such that

(1) Γ is homeomorphic to the empty set, an interval, a circle, or a graph with a single vertex of valence at least 3,

(2) Γ intersects each component of $\partial_- W$ in a single univalent vertex, and

(3) Γ has no univalent vertices in the interior of W.

If Γ is an interval or a circle, we regard it as a graph containing a unique vertex of valence 2. The spines shown in Figure 4(i)–(iii) are 1-vertex spines while the one shown in Figure 4(iv) is not so because it has a univalent vertex in the interior of the illustrated compression body. We call a vertex of valence at least 2 the *interior vertex*. We note that every 1-vertex spine has a unique interior vertex. This is the reason why it is named so.

Let W be a compression body. Suppose that $\partial_- W$ consists of n closed surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$. A (possibly empty) set $\mathcal{D} = \{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ of pairwise disjoint compression disks for $\partial_+ W$ is called a *cut-system* for W if

(1) each E_{Σ_i} separates from W a component that is homeomorphic to $\Sigma_i \times [0, 1]$ and contains Σ_i,

(2) W cut off by $E_{\Sigma_1} \cup E_{\Sigma_2} \cup \cdots \cup E_{\Sigma_n}$ has at most one handlebody component V, and

(3) $D_1 \cup D_2 \cup \cdots \cup D_m$ cuts off V into a single 3-ball.

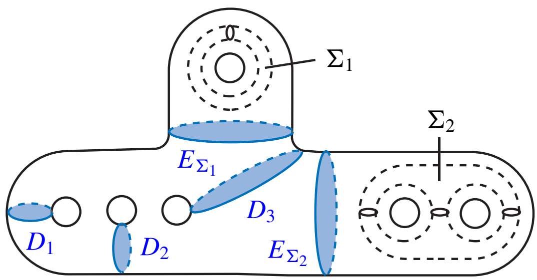


Figure 5. A cut system.

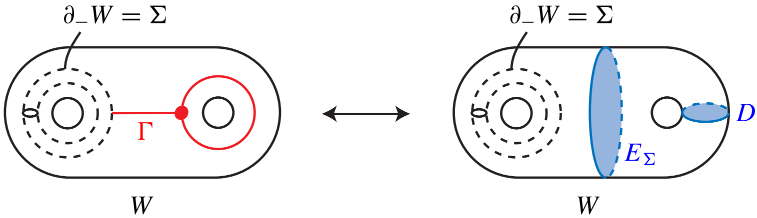


Figure 6. Poincaré–Lefschez duality.

See Figure 5. We note that if $W = \Sigma \times [0, 1]$, where Σ is a closed orientable surface, then $m = n = 0$. If W is a handlebody, then n is 0 and m is its genus.

By virtue of Poincaré–Lefschez duality, we have a one-to-one correspondence between the 1-vertex spines and cut-systems of a compression body W modulo isotopy (see Figure 6). The correspondence can be described as follows. The 1-vertex spine Γ *dual to* a given cut-system $\mathcal{D}$ for a compression body W is obtained by regarding a regular neighborhood of each disk D in $\mathcal{D}$ as a 1-handle with D as the cocore, and then extending the core arcs of the 1-handles in each component W_0 of the exterior of the union of the disks in $\mathcal{D}$ in such a way that

(1) if W_0 is a 3-ball, then the extension is given by radial arcs, and

(2) if W_0 is the product of a closed surface with an interval, then the extension is given by a vertical arc.

By conversing the construction, we get the cut-system *dual to* a 1-vertex spine of W.

Let V be a handlebody of genus g and Γ a subspine of V. Assume that each component of Γ is a rose. A *cut-system* for the pair (V, Γ) is a cut-system for V dual to a spine $\hat{\Gamma}$, where $\hat{\Gamma}$ is obtained by contracting a maximal subtree of a spine Γ' of V that contains Γ as a subgraph. See Figure 7.

Lemma 2.1. *Let W be a compression body. Let D be a compression disk for $\partial_+ W$. Then there exists a cut-system for W disjoint from D.*

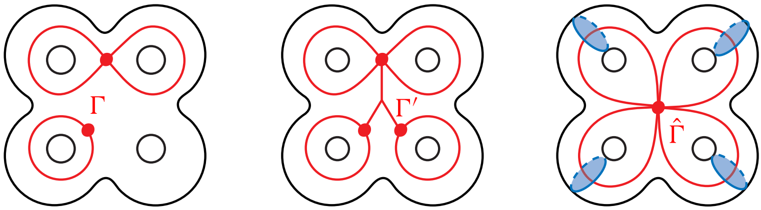


Figure 7. A cut-system for (V, Γ) is a cut-system for V dual to a spine $\hat{\Gamma}$.

Proof. We may identify W with a genus-g handlebody V with an open regular neighborhood of a subspine Γ removed. Further, we may assume that each component of Γ is a rose. Let $\Gamma_1, \Gamma_2, \ldots, \Gamma_n$ be the components of Γ. Choose a cut-system $\{D_1, D_2, \ldots, D_g\}$ for the pair (V, Γ) so that $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$ is minimal among all cut-systems for (V, Γ). We note here that each component of the intersection $D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)$ is an arc, for simple closed curves of the intersection can be eliminated by a standard argument.

Suppose for a contradiction that $D \cap (D_1 \cup D_2 \cup \cdots \cup D_g) \neq \varnothing$. Choose an outermost subdisk δ of D cut off by $D_1 \cup D_2 \cup \cdots \cup D_g$. We may assume that $\delta \cap D_1 \neq \varnothing$. Let D_1' and D_1'' be the disks obtained from D_1 by surgery along δ. Then exactly one of $\{D_1', D_2, \ldots, D_g\}$ and $\{D_1'', D_2, \ldots, D_g\}$, say $\{D_1', D_2, \ldots, D_g\}$, is a cut-system for the handlebody V. We note that D_1'' separates the handlebody V cut off by $D_2 \cup D_3 \cup \cdots \cup D_g$. Recall that D_1 intersects Γ in at most one point. If D_1 does not intersect Γ, then it follows that $\{D_1', D_2, \ldots, D_g\}$ is a cut-system for the pair (V, Γ) with $|D \cap (D_1' \cup D_2 \cup \cdots \cup D_g)| < |D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. This contradicts the minimality of $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. Suppose that D_1 intersects Γ. If D_1'' intersects Γ, then D_1'' cannot separate the handlebody V cut off by $D_2 \cup D_3 \cup \cdots \cup D_g$. This is a contradiction. Thus D_1' intersects Γ. This implies that $\{D_1', D_2, \ldots, D_g\}$ is a cut-system for the pair (V, Γ). This contradicts, again, the minimality of $|D \cap (D_1 \cup D_2 \cup \cdots \cup D_g)|$. Thus, we have $D \cap (D_1 \cup D_2 \cup \cdots \cup D_g) = \varnothing$ and $D \cap \Gamma = \varnothing$.

From now on, we assume that each of $D_1, D_2, \ldots, D_m$ does not intersect Γ, while each of $D_{m+1}, D_{m+2}, \ldots, D_g$ does so. Let B be the 3-ball obtained by cutting V along $D_1 \cup D_2 \cup \cdots \cup D_g$. Then $B \cap \Gamma_i$ is a cone on an even number of points. We note that D is a separating disk in B disjoint from the cones $B \cap \Gamma$. For each $i \in \{1, 2, \ldots, m\}$ let D_i^+ and D_i^- be the disks on the boundary of B coming from D_i. Then there exists a set $\{E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ of mutually disjoint disks properly embedded in B such that

(1) $E_{\Sigma_1} \cup E_{\Sigma_2} \cup \cdots \cup E_{\Sigma_n}$ is disjoint from $\Gamma \cup D \cup D_1^{\pm} \cup D_2^{\pm} \cup \cdots \cup D_g^{\pm}$, and

(2) E_{Σ_i} separates from B a 3-ball B_i such that $B_i \cap (D_1^{\pm} \cup D_2^{\pm} \cup \cdots \cup D_m^{\pm}) = \varnothing$ and $B_i \cap \Gamma = B \cap \Gamma_i$.

Now $\{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ is a required cut-system for W. □

Let M be a compact, connected, orientable, irreducible 3-manifold with connected boundary. Following [Bonahon 1983], a *characteristic compression body* W of M is defined to be a compression body embedded in M such that

(1) $\partial_+ W = \partial M$, and

(2) the closure of $M \setminus W$ is boundary-irreducible.

We remark that, for a given characteristic compression body W of M, by the irreducibility of M, every compression disk for ∂M can be moved by an isotopy to lie in W.

Theorem 2.2 [Bonahon 1983]. *A compact, connected, orientable, irreducible 3-manifold with connected boundary has a unique (up to isotopy) characteristic compression body.*

Lemma 2.3. *Let M be a compact, connected, orientable 3-manifold with connected boundary. Let W be a compression body in M such that $\partial M = \partial_+ W$. Let K be a knot in the interior of W. If K fills up M, then K fills up W. Further, when M is irreducible and W is the characteristic compression body, then K fills up M if and only if K fills up W.*

Proof. Since any knot K' in the interior of W with $K \overset{W}{\sim} K'$ satisfies $K \overset{M}{\sim} K'$, it follows immediately from the definition that if K fills up M, then K fills up W.

Suppose M is irreducible, W is the characteristic compression body, and K is a knot in W that fills up W. We will show that K fills up M. If M is a handlebody, then we have $M = W$ and there is nothing to prove. Suppose that M is not a handlebody. Then M can be decomposed as $M = W \cup X$, where $W \cap X = \partial_- W = \partial X$ and X is the union of boundary-irreducible 3-manifolds. The interior boundary $\partial_- W$ consists of a finite number of closed surfaces $\Sigma_1, \Sigma_2, \ldots, \Sigma_n$ of genus at least 1. Let g_i be the genus of Σ_i ($i \in \{1, 2, \ldots, n\}$). We recall that each Σ_i is incompressible in M. Suppose for a contradiction that there exists a knot K' in the interior of M with $K \overset{M}{\sim} K'$ such that ∂M is compressible in $M \setminus K'$. Let D be a compression disk for ∂M in $M \setminus K'$. We may assume that D is contained in W.

Suppose first that D does not separate W. By Lemma 2.1, there exists a cut-system for W disjoint from D. By replacing a suitable disk in the system with D, we obtain a cut-system $\mathcal{D} = \{D_1, D_2, \ldots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \ldots, E_{\Sigma_n}\}$ where $D = D_1$. Let Γ be the 1-vertex spine of W dual to $\mathcal{D}$. Fix a presentation of the fundamental group of each surface Σ_i as $\pi_1(\Sigma_i) = \langle a_{i,j}, b_{i,j}\ (j \in \{1, 2, \ldots, g_i\}) \mid \prod_{j=1}^{g_i} [a_{i,j}, b_{i,j}] \rangle$, where we take the base point at $\Gamma \cap \Sigma_i$.

Let v_0 be the interior vertex of Γ. Let V be the unique component of W cut off by the union of disks in $\mathcal{D}$ that is homeomorphic to a handlebody. We fix a generating set $\{x_1, x_2, \ldots, x_m\}$ of $\pi_1(V, v_0)$ so that an element x_i is defined by the loop in Γ dual to D_i. Then by the Seifert–van Kampen theorem, $\pi_1(W, v_0)$ is generated by the x_i, $a_{i,j}$ and $b_{i,j}$. Set

$$G=\{{x_i}^{\pm 1} \mid i\in\{1,2,\dots,m\}\}\cup\{{a_{i,j}}^{\pm 1}, {b_{i,j}}^{\pm 1}\ (j\in\{1,2,\dots,g_i\}) \mid i\in\{1,2,\dots,n\}\}.$$

Let $H_1, H_2, \dots, H_l$ be 1-handles in X attached to $\partial_- W$ so that the closure of $M \setminus (W \cup H_1 \cup H_2 \cup \cdots \cup H_l)$ is the union of handlebodies. Let $h_1, h_2, \dots, h_l$ be the element of $\pi_1(M, v_0)$ corresponding to the core of the 1-handles $H_1, H_2, \dots, H_l$, respectively. We set

$$\hat{G} = G \cup \{{h_i}^{\pm 1} \mid i \in \{1, 2, \dots, l\}\}.$$

We note that the elements of $\hat{G}$ generate the group $\pi_1(M, v_0)$. In other words, any element of $\pi_1(M, v_0)$ can be represented by a word on $\hat{G}$.

Since each Σ_i is incompressible in M, $\pi_1(W, v_0)$ is a subgroup of $\pi_1(M, v_0)$. Consider the conjugation class $c_{\pi_1(W,v_0)}(K)$. Since K fills up W, every word w on G representing an element of $c_{\pi_1(W,v_0)}(K)$ contains $x_1^{\pm 1}$.

By the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. Let u be a word on $\hat{G}$ such that $u^{-1}wu$ represents the same element as w' in $\pi_1(M, v_0)$. Let $\varphi : \pi_1(M, v_0) \to \pi_1(W, v_0)$ be the epimorphism obtained by adding the relations $h_i = 1$ for each $i \in \{1, 2, \dots, l\}$. For a word v, we denote by $\varphi(v)$ the word on G obtained from v by replacing each ${h_i}^{\pm}$ in the word with $\varnothing$. Then $\varphi(u^{-1}wu) = \varphi(u)^{-1}w\varphi(u)$ represents an element contained in $c_{\pi_1(W,v_0)}(K)$. It follows that $\varphi(w')$ is a word on $G \setminus \{{x_1}^{\pm}\}$ representing an element of $c_{\pi_1(W,v_0)}(K)$. This is a contradiction.

Next, suppose D separates W into two components W_1 and W_2. By Lemma 2.1, there exists a cut-system $\mathcal{D} = \{D_1, D_2, \dots, D_m, E_{\Sigma_1}, E_{\Sigma_2}, \dots, E_{\Sigma_n}\}$ for W disjoint from D. Without loss of generality, we can assume that the set of disks of $\mathcal{D}$ contained in W_1 is $\{D_1, D_2, \dots, D_{m_1}, E_{\Sigma_1}, E_{\Sigma_2}, \dots, E_{\Sigma_{n_1}}\}$, where $m_1 \in \{1, 2, \dots, m\}$ and $n_1 \in \{0, 1, \dots, n\}$. Here we set $n_1 = 0$ if none of $\{E_{\Sigma_1}, E_{\Sigma_2}, \dots, E_{\Sigma_n}\}$ is contained in W_1.

Let Γ be the 1-vertex spine of W dual to $\mathcal{D}$. Using the spine Γ, fix generating sets

$$G=\{{x_i}^{\pm 1} \mid i\in\{1,2,\dots,m\}\}\cup\{{a_{i,j}}^{\pm 1}, {b_{i,j}}^{\pm 1} \mid i\in\{1,2,\dots,n\},\ j\in\{1,2,\dots,g_i\}\}$$

of $\pi_1(W, v_0)$ and

$$\hat{G} = G \cup \{{h_i}^{\pm 1} \mid i \in \{1, 2, \dots, l\}\}.$$

of $\pi_1(M, v_0)$ and an epimorphism $\varphi : \pi_1(M, v_0) \to \pi_1(W, v_0)$ as above.

If $m_1 \neq m$, then, by the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ or $\hat{G} \setminus \{{x_m}^{\pm 1}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. By the same argument as in the case where D is nonseparating, this is a contradiction. If $m_1 = m$, then $n_1 \neq n$. Hence, by the existence of K', there exists a word w' on $\hat{G} \setminus \{{x_1}^{\pm 1}\}$ or

$\hat{G} \setminus \{a_{n,j}{}^{\pm 1}, b_{n,j}{}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$ representing an element of $c_{\pi_1(M,v_0)}(K)$. It follows that $\varphi(w')$ is a word on $G \setminus \{x_1{}^{\pm 1}\}$ or $G \setminus \{a_{n,j}{}^{\pm 1}, b_{n,j}{}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$ representing an element of $c_{\pi_1(W,v_0)}(K)$. However, this is again a contradiction because the fact that K fills up W implies that every word on G representing an element of $c_{\pi_1(W,v_0)}(K)$ contains both one of $\{a_{n,j}{}^{\pm 1}, b_{n,j}{}^{\pm 1} \mid j \in \{1, 2, \ldots, g_n\}\}$ and one of $x_1^{\pm 1}$. This completes the proof. □

Theorem 2.4. *Let M be a compact, connected, orientable, irreducible 3-manifold with connected boundary. Then there exists a knot K in the interior of M that fills up M. Moreover, such a knot K can be taken to lie in* $\mathrm{Nbd}(\partial M; M)$.

Proof. If M is a handlebody, the assertion follows from Lemma 1.5. Suppose that M is not a handlebody. Let W be the characteristic compression body of M. We may identify W with the complement of an open regular neighborhood of a subspine Γ of a handlebody V. Let K be a knot in the interior of V that fills up V. Since K can be taken not to intersect a spine of V containing Γ as a subgraph, we may assume that K lies in a collar neighborhood of $\partial_+ W = \partial M$. By Lemma 2.3, K fills up W. Thus, again by Lemma 2.3, K fills up M. □

3. Transient knots in a subspace of the 3-sphere

Let M be a compact, connected, proper 3-submanifold of S^3. A knot K in $M \subset S^3$ is said to be *transient in M* if K can be deformed by a homotopy in M to be the trivial knot in S^3. Otherwise, K is said to be *persistent in M*.

Example. The knot K_1 described on the left-hand side in Figure 8 is transient in the handlebody V_1 in S^3, while the knot K_2 described on the right-hand side is persistent in V_2.

The next lemma follows straightforwardly from the definition.

Lemma 3.1. *Let M be a compact, connected, proper 3-submanifold of S^3 and let N be a compact, connected 3-submanifold of M. If a knot K in N is persistent in M, then it is also persistent in N.*

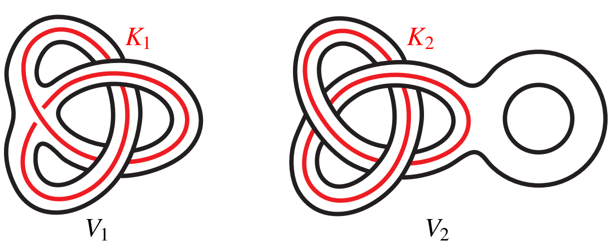


Figure 8. The knot K_1 is transient in V_1, while K_2 is persistent in V_2.

A compact, connected, proper 3-submanifold M of S^3 is said to be *unknotted* if the exterior $E(M)$ is a disjoint union of handlebodies. Otherwise M is said to be *knotted*. We recall that a theorem of Fox [1948] says that any compact, connected, proper 3-submanifold of S^3 can be reembedded in S^3 in such a way that its image is unknotted. See [Scharlemann and Thompson 2005] and [Ozawa and Shimokawa 2015] for certain generalizations and refinements of Fox's theorem.

Remark. As mentioned in the introduction, M usually admits many nonisotopic embeddings into S^3 with the unknotted image. The uniqueness holds for a handlebody by [Waldhausen 1968]. Here the uniqueness is up to isotopy for subsets of S^3, where we recall that two subsets M_1 and M_2 of S^3 are isotopic if and only if there exists an orientation-preserving homeomorphism f of S^3 carrying M_1 onto M_2. If we consider isotopies not between the embedded subsets but between embeddings, it is far from being unique even for a handlebody. This can be explained under a general setting as follows. Let M be a compact, connected 3-submanifold M that can be embedded in S^3. Then its mapping class group $\mathcal{MCG}_+(M)$ is defined to be the group of isotopy classes of orientation-preserving homeomorphisms of M. We fix an embedding $\iota_0 : M \to S^3$. Let $\mathcal{G}_{\iota_0(M)} = \mathcal{MCG}_+(S^3, \iota_0(M))$ be the mapping class group of the pair $(S^3, \iota_0(M))$, that is, the group of isotopy classes of orientation-preserving homeomorphisms of S^3 that preserve $\iota_0(M)$. See [Koda 2015] for details of this group when M is a knotted handlebody. We can define an injective homomorphism $\iota_0^* : \mathcal{G}_{\iota_0(M)} \hookrightarrow \mathcal{MCG}_+(M)$ by assigning to each homeomorphism $\varphi \in \mathcal{G}_{\iota_0(M)}$ a unique element f of $\mathcal{MCG}_+(M)$ satisfying $\varphi \circ \iota_0 = \iota_0 \circ f$. Then the set of embeddings of M into S^3 with the same image up to isotopy can be identified with the right cosets $\iota_0^*(\mathcal{G}_{\iota_0(M)}) \backslash \mathcal{MCG}_+(M)$, where the identification is given by assigning to $f \in \mathcal{MCG}_+(M)$ the embedding $\iota_0 \circ f : M \to S^3$. When M is a handlebody of genus at least two, it is clear that this is an infinite set. We note that, when $\iota_0(M)$ is an unknotted handlebody of genus two, the group $\mathcal{G}_{\iota_0(M)}$ is called the genus-two Goeritz group of S^3 and studied in [Goeritz 1933; Scharlemann 2004; Akbas 2008; Cho 2008].

Let K be a knot in M. Let f be contained in the coset $\iota_0^*(\mathcal{G}_{\iota_0(M)})\,\mathrm{id}_M$. By the observation above and the definition of the persistence of knots in $M \subset S^3$, it follows immediately that $\iota_0 \circ f(K)$ is persistent in M if and only if K is. We note that if f is not contained in the coset $\iota_0^*(\mathcal{G}_{\iota_0(M)})\,\mathrm{id}_M$, then the knot $\iota_0 \circ f(K)$ is not necessarily persistent in M even if K is persistent in M. See Figure 9. Be that as it may, we discuss in this paper extrinsic properties of knots embedded in submanifolds of S^3, not intrinsic ones.

Theorem 3.2. *Let M be a compact, connected, proper 3-submanifold of S^3. Then every knot in M is transient if and only if M is unknotted.*

Proof. Suppose first that M is unknotted, i.e., $M = S^3 \setminus \operatorname{Int} \operatorname{Nbd}(\Gamma)$, where Γ is a graph embedded in M. Let K be a knot in M. Considering a diagram of the spatial

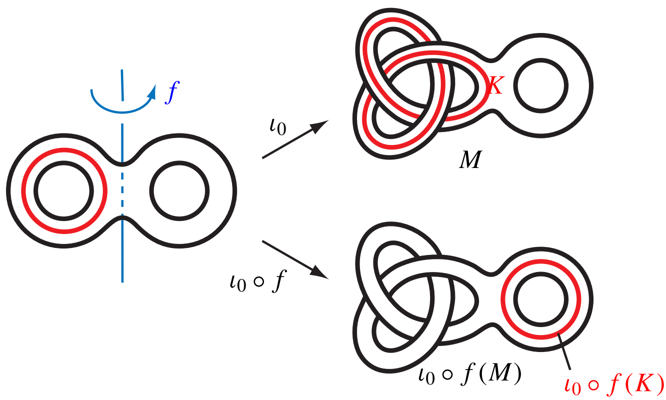


Figure 9. Persistence is an extrinsic property.

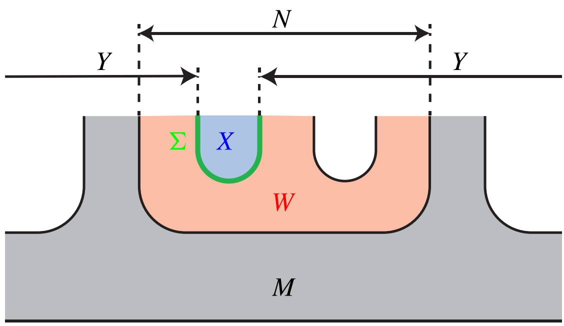


Figure 10. The configurations of M, N, W, Σ, X and Y.

graph $K \cup \Gamma$, we easily see that K can be converted into the trivial knot in S^3 by a finite number of crossing changes of K itself. This implies that K is transient in M.

Next suppose that M is knotted. Then there exists a component N of the exterior of M that is not a handlebody. Let W be the characteristic compression body of N. We note that if N is boundary-irreducible, then W is a collar neighborhood of ∂N in N. Since W is not a handlebody, we can take a nonempty component Σ of $\partial_- W$. Then Σ separates S^3 into two components X and Y so that X is boundary-irreducible and Y contains $M \cup W$. See Figure 10.

By Theorem 2.4, there exists a knot K lying in $\mathrm{Nbd}(\partial Y; Y)$ that fills up Y. In particular K lies in W. Thus by an isotopy we can move K to lie within M. Let $K' \subset M$ be an arbitrary knot with $K \overset{M}{\sim} K'$. Since K fills up Y, Σ is incompressible in $Y \setminus K'$. Thus Σ is incompressible in $S^3 \setminus K'$. This implies that K' is not the trivial knot in S^3. Therefore K is persistent in M. □

Remark. Let M be a compact, connected, knotted, proper 3-submanifold of S^3. In the proof of Theorem 3.2, we explained how to obtain a knot in M that is persistent

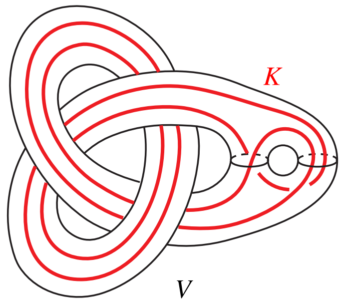


Figure 11. The knot K fills up V, whereas K is transient in V.

in M. In the process, some readers may have guessed that if a knot $K \subset M$ filled up M, then K would already be persistent. If so, the process to consider the characteristic compression body of a nonhandlebody component of the exterior in the proof would not be necessary. However, the guess is not true in fact. Let K be the knot in the genus-two knotted handlebody $V \subset S^3$ as shown in Figure 11. Then we see that K fills up V by the same reason as in the proof of Lemma 1.5 (see also (2) in Section 6, whereas K is apparently transient in V.

4. Construction of persistent knots

Persistent laminations and persistent knots. Let M be a compact, connected, proper 3-submanifold of S^3 whose exterior consists of boundary-irreducible 3-manifolds. It is easy to see that every knot filling up M is persistent in M. Indeed, if a knot K in M fills up M, then each component of ∂M will be an incompressible surface in the exterior of any knot K' homotopic to K in V, hence K' is not the trivial knot in S^3. However, the converse is false in general as we see now:

Proposition 4.1. *There exists a genus-two handlebody V embedded in S^3 with the boundary-irreducible exterior such that there exists a knot $K \subset V$ which is persistent in V, and which does not fill up V.*

Proof. Let V be the genus-two handlebody in S^3 and K the knot in V as shown in Figure 12. We note that the handlebody V is the exterior of Brittenham's branched surface [1999] constructed from a disk spanning the trivial knot in S^3. In particular, the exterior of V is boundary-irreducible. We note that K does not fill up V since there exists a compression disk D for ∂V in $V \setminus K$ as shown in the figure.

We will show that K is persistent in V. As illustrated in the figure, there are meridian disks D_1, D_2 of V each of which intersects K once and transversely. Let K' be any knot homotopic to K in V. Then K' intersects each of D_1 and D_2 at least once. By [Hirasawa and Kobayashi 2001] or [Lee and Oh 2002], which generalizes the result of [Brittenham 1999], in the exterior of V there exists a

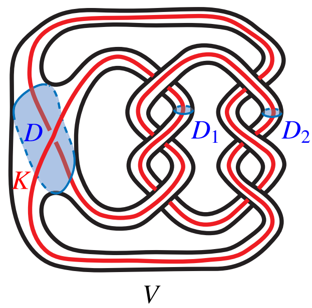


Figure 12. A handlebody V in S^3 with the boundary-irreducible exterior such that there exists a knot $K \subset V$ which is persistent in V, and which does not fill up V.

persistent lamination, that is, an essential lamination that remains essential after performing any nontrivial Dehn surgeries along K'. This implies that K' is not the trivial knot. Thus K is persistent in V. □

Accidental surfaces and persistent knots. A closed essential surface Σ in the exterior of a knot K in the 3-sphere is called an *accidental surface* if there exists an annulus A, called an *accidental annulus*, embedded in the exterior $E(K)$ such that

- the interior of A does not intersect $\Sigma \cup \partial E(K)$,
- $A \cap \Sigma \neq \varnothing$ and $A \cap \partial E(K) \neq \varnothing$, and
- $A \cap \Sigma$ and $A \cap \partial E(K)$ are essential simple closed curves in Σ and $\partial E(K)$, respectively.

In [Ichihara and Ozawa 2000] it is shown that, for each accidental surface in the exterior of a knot in S^3, the boundary curves of accidental annuli determine a unique slope on the boundary of a regular neighborhood of the knot. This slope is called an *accidental slope* for Σ. By the work of Culler, Gordon, Luecke, and Shalen [Culler et al. 1987], an accidental slope is either meridional or integral.

Proposition 4.2. *Let M be a compact, connected, proper 3-submanifold of S^3 with connected boundary such that the exterior of M is boundary-irreducible. Let K be a knot in M such that ∂M is incompressible in $M \setminus K$. If ∂M is an accidental surface with integral accidental slope in the exterior of K, then K is persistent in the submanifold M of S^3 bounded by Σ and containing K.*

Proof. Let $A \subset M$ be an accidental annulus connecting K and a simple closed curve in ∂M. Using this annulus, we move K to a knot K^* lying in ∂M by an isotopy. Since ∂M is incompressible in $E(K)$, $\partial M \setminus K^*$ is incompressible in M. Thus by Lemma 1.1 K^* binds $\pi_1(M)$, and so does K. By Lemma 1.4, K fills up M. Let

$K' \subset M$ be an arbitrary knot lying in the interior of M with $K \overset{M}{\sim} K'$. Since K fills up M, ∂M is incompressible in $M \setminus K'$. Thus ∂M is incompressible in $S^3 \setminus K'$. This implies that K' is not the trivial knot in S^3. Therefore, K is persistent in M. □

5. Transient number of knots

Let K be a knot in S^3. A *crossing move* on a knot K is the operation of passing one strand of K through another. The *unknotting number* $u(K)$ of K, which was first defined by Wendt [1937], is then the minimal number of crossing moves required to convert the knot into the trivial knot. We note that to each crossing move we can associate a simple arc α in S^3 such that $\alpha \cap K = \partial\alpha$ and such that the crossing move is performed in $\mathrm{Nbd}(\alpha)$.

An *unknotting tunnel system* for K is a set $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ of mutually disjoint simple arcs in S^3 such that $\gamma_i \cap K = \partial\gamma_i$ for each $i \in \{1, 2, \ldots, n\}$ and such that the exterior of the union $K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n$ is a handlebody. The *tunnel number* $t(K)$ of K, first defined in [Clark 1980], is the minimal number of arcs in any of the unknotting tunnel systems for K.

We introduce a new invariant for a knot in the 3-sphere that is strongly related to the above two classical invariants. We define a *transient system* for K to be a set $\{\tau_1, \tau_2, \ldots, \tau_n\}$ of mutually disjoint simple arcs in S^3 such that $\tau_i \cap K = \partial\tau_i$ for each $i \in \{1, 2, \ldots, n\}$ and such that K is transient in $\mathrm{Nbd}(K \cup \tau_1 \cup \tau_2 \cup \cdots \cup \tau_n)$. The *transient number* $\mathrm{tr}(K)$ of K is defined to be the minimal number of arcs in any of the transient systems for K.

Proposition 5.1. *Let K be a knot in S^3. Then* $\mathrm{tr}(K) \leqslant u(K)$ *and* $\mathrm{tr}(K) \leqslant t(K)$.

Proof. Suppose that $u(K) = m$. Let $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ be a set of mutually disjoint simple arcs associated to m crossing moves that convert K into the trivial knot. Then K is transient in the handlebody $\mathrm{Nbd}(K \cup \alpha_1 \cup \alpha_2 \cup \cdots \cup \alpha_m)$. In other words, $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$ is a transient tunnel system for K. This implies that $\mathrm{tr}(K) \leqslant m$.

Suppose that $t(K) = n$. Let $\{\gamma_1, \gamma_2, \ldots, \gamma_n\}$ be an unknotting tunnel system for K. Since the handlebody $\mathrm{Nbd}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ is unknotted, K is transient in $\mathrm{Nbd}(K \cup \gamma_1 \cup \gamma_2 \cup \cdots \cup \gamma_n)$ by Theorem 3.2. This implies that $\mathrm{tr}(K) \leqslant n$. □

Proposition 5.2. *There exists a knot K in S^3 with* $\mathrm{tr}(K) = 1$ *and* $u(K) = t(K) = 2$.

Proof. Let K be the satellite knot of the figure-eight knot shown in Figure 13. Clearly, the genus of K is one. The transient number of K is one because K admits a transient tunnel as shown in the figure. In [Kobayashi 1989] and [Scharlemann and Thompson 1989], it is proved that the only knots of genus one and unknotting number one are the doubled knots. It follows that the unknotting number of K is at least two. It is then straightforward to see that the unknotting number is exactly two.

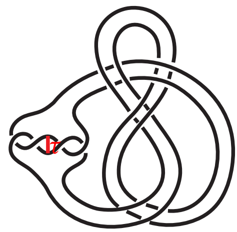


Figure 13. A knot K with $\mathrm{tr}(K) = 1$ and $u(K) = t(K) = 2$.

It is proved in [Morimoto and Sakuma 1991] that the only nonsimple knots having unknotting tunnels are certain satellites of torus knots. It follows that the tunnel number of K is at least two. It is then straightforward to see that the tunnel number is exactly two. □

6. Concluding remarks

(1) Let M be a compact, connected, proper 3-submanifold of S^3. Let K be a knot in the interior of M. In the earlier sections, we have introduced various homotopic properties of knots in M. We summarize their relations. We say that K is *accidental* in M if K can be moved to a knot K' in ∂M by a homotopy in M so that $\partial M \setminus K'$ is incompressible in M. Then we have the following:

(a) If K is accidental, then K binds $\pi_1(M)$ (see Lemma 1.1).

(b) If K binds $\pi_1(M)$, then K fills up M (see Lemma 1.4).

(c) By (a) and (b), if K is accidental, then K fills up M.

The converse of each of these is false. To see this, suppose that M is the exterior of a nontrivial knot in S^3. We note that $\pi_1(M)$ is freely indecomposable by the Kneser conjecture. Let K be a knot in M that cannot be moved by any homotopy in M to lie in ∂M. Such a knot K always exists by, for instance, the work of Brin, Johannson, and Scott [Brin et al. 1985]. This implies that K binds $\pi_1(M)$, whereas K is not accidental in M. A somewhat more subtle example is shown on the left in Figure 14. In the figure, the knot K lies in a genus-two handlebody V, and thus K can be moved by homotopy to lie within a collar neighborhood of ∂V. If K is accidental, then by attaching a 2-handle to V we obtain a 3-manifold M with toroidal boundary whose fundamental group has the presentation $\langle x, y \mid xyx^{-2}y^{-1} \rangle$. This group is called the Baumslag–Solitar group, BS(1), and is known not to be a 3-manifold group; see the work of Aschenbrenner, Friedl, and Wilton [Aschenbrenner et al. 2015]. This implies that K is not accidental in V. On the other hand, it follows straightforwardly

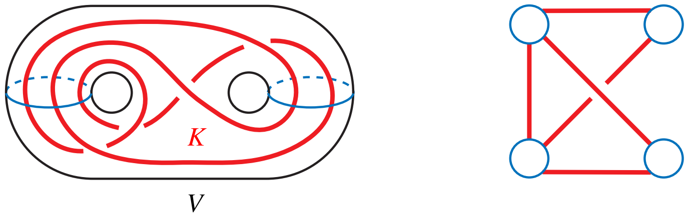


Figure 14. The knot K binds V and is not accidental in V.

from Theorem 6.1 that K binds V since the corresponding Whitehead graph, shown on the right in Figure 14, is connected and contains no cut vertex.

The remark after the proof of Lemma 1.4 shows that the converse of Lemma 1.4 is false. However, the 3-manifold M introduced in the example is not embeddable in S^3. To have a counterexample of the converse of (b), let Σ be a closed orientable surface of genus at least one. Let M be an orientable 3-manifold obtained by attaching a 1-handle to each component of $\partial(\Sigma \times [0, 1])$. We note that M can be embedded in S^3. Let D_0 and D_1 be the cocore of the 1-handles. Then we can easily show as in the remark that there exists a knot K in M, intersecting each of D_0 and D_1 once and transversely, that fills up M, whereas K does not bind $\pi_1(M)$. The relations of these three intrinsic properties are shown on the left-hand side in Figure 15. It is worth noting that, to show that a given knot K in $M \subset S^3$ is persistent, we have used an intrinsic property of K in a subset of S^3 containing M. See Theorem 3.2 and Propositions 4.1 and 4.2.

(2) Let F_g be a rank-g free group. As mentioned in Section 1, an algorithm to detect whether a given element x of a free group F_g binds F_g is described by Stallings using the combinatorics of its Whitehead graph. In fact, the following is proved:

Theorem 6.1 [Stallings 1999]. *Let x be a cyclically reduced word on the set $X_g = \{x_1, x_2, \dots, x_g\}$. If the Whitehead graph of x is connected and contains no cut vertex, then x binds F_g.*

For a simple closed curve in the boundary of a handlebody, this can be seen clearly as follows. Let x be an element of the rank-g free group F_g. We identify

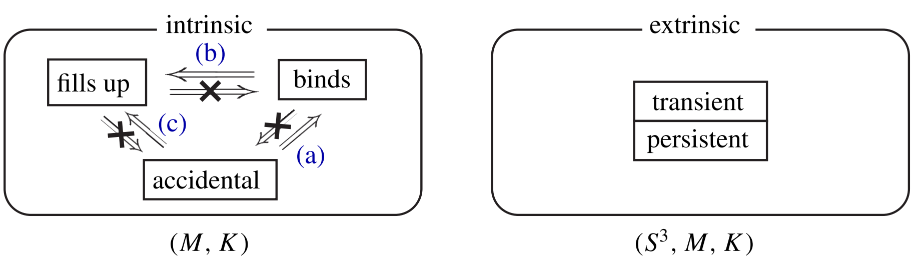


Figure 15. Correlation diagrams of extrinsic and intrinsic properties.

F_g with the fundamental group of a genus-g handlebody. In the case of $M = V_g$ in Lemma 1.1, which is actually [Lyon 1980, Corollary 1], we have seen that if x can be represented by an oriented simple closed curve K in ∂V_g, then x binds F_g if and only if $\partial V_g \setminus K$ is incompressible. On the other hand, Starr [1992] (see also [Wu 1996, Theorem 1.2]) showed that $\partial V_g \setminus K$ is incompressible if and only if there is a complete meridian disk system $D_1, D_2, \dots, D_g$ of V_g such that the planar graph with “fat” vertices obtained by cutting ∂V_g along $\bigcup_{i=1}^{g} D_i$ is connected and contains no cut vertex. This graph is actually nothing else but the Whitehead graph of x. (As explained in [Stallings 1999], we can obtain a geometric interpretation of this for an arbitrary element of F_g if we consider the connected sum of g copies of $S^2 \times S^1$ instead of V_g.)

(3) Let M be a compact, connected, proper 3-submanifold of S^3. In the proofs of Theorem 3.2 and Propositions 4.1 and 4.2, we provided a way to show that a given knot $K \subset M$ is persistent in M. The key idea is to find an essential surface (or lamination) in the exterior of M that is also essential in the exterior of any knot K' homotopic to K in M. As mentioned in the introduction, another way to show persistence was provided by Letscher [2012] and uses what he calls the *persistent Alexander polynomial*.

Problem 1. Provide more methods for detecting whether a knot $K \subset M$ is persistent.

(4) As we have summarized in Figure 15, the only extrinsic property of knots in a 3-subspace of S^3 we have considered in the present paper is transience (or persistence). Using this property, we have actually gotten an “if and only if” condition for a 3-subspace of S^3 being unknotted in Theorem 3.2. This is a first step for a relative version of Fox’s program and further progress will be expected.

Problem 2. Consider other extrinsic properties of knots in $M \subset S^3$ in order to characterize how M is embedded in S^3.

We note that the case where M is a handlebody is already a very interesting problem. See, e.g., [Ishii 2008; Koda 2015; Koda and Ozawa 2015].

(5) As mentioned in the introduction, the unknottedness of a 3-submanifold can be considered for an arbitrary closed, connected 3-manifold. Thus it is natural to ask:

Question 1. Can Theorem 3.2 be generalized for M in an arbitrary 3-manifold N?

(6) Finally, in Section 5, we defined an integer-valued invariant $\mathrm{tr}(K)$, the transient number, for a knot K in S^3. This invariant is nice in the sense that it shows the knots of unknotting number 1 and those of tunnel number 1 from the same perspective as we have seen in Proposition 5.1. However, it remains unknown whether there exists a knot whose transient number is more than 1.

Question 2. Can the transient number $\mathrm{tr}(K)$ be arbitrarily large?

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