

*Pacific
Journal of
Mathematics*

**BRIDGE SPHERES FOR THE UNKNOT
ARE TOPOLOGICALLY MINIMAL**

JUNG HOON LEE

Volume 282 No. 2

June 2016

BRIDGE SPHERES FOR THE UNKNOT ARE TOPOLOGICALLY MINIMAL

JUNG HOON LEE

Topologically minimal surfaces were defined by Bachman as topological analogues of geometrically minimal surfaces, and one can associate a topological index to each topologically minimal surface. We show that an $(n + 1)$ -bridge sphere for the unknot is a topologically minimal surface of index at most n .

1. Introduction

Let S be a closed orientable separating surface embedded in a 3-manifold M . The structure of the set of compressing disks for S , such as how a pair of compressing disks on opposite sides of S intersects, reveals some topological properties of M . For example, if S is a minimal genus Heegaard surface of an irreducible manifold M and S has a pair of disjoint compressing disks on opposite sides, then M contains an incompressible surface [Casson and Gordon 1987].

The *disk complex* $\mathcal{D}(S)$ of S is a simplicial complex defined as follows.

- Vertices of $\mathcal{D}(S)$ are isotopy classes of compressing disks for S .
- A collection of $k + 1$ vertices forms a k -simplex if there are representatives for each that are pairwise disjoint.

The disk complex of an incompressible surface is empty. A surface S is *strongly irreducible* if S compresses to both sides and every compressing disk for S on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman [2010] defined topologically minimal surfaces, which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface S is *topologically minimal* if $\mathcal{D}(S)$ is empty or $\pi_i(\mathcal{D}(S))$ is nontrivial for some i . The *topological index* of S is 0 if $\mathcal{D}(S)$ is empty, and the smallest n such that $\pi_{n-1}(\mathcal{D}(S))$ is nontrivial, otherwise.

Topologically minimal surfaces share some useful properties. For example, if an irreducible manifold contains a topologically minimal surface and an incompressible

MSC2010: 57M50.

Keywords: disk complex, topologically minimal surface, bridge splitting, unknot.

surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces. There exist topologically minimal surfaces of arbitrarily high index [Bachman and Johnson 2010], and see also [Lee 2015] for possibly high index surfaces in (closed orientable surface) $\times I$. In this paper we consider bridge splittings of 3-manifolds, and show that the simplest bridge surfaces, bridge spheres for the unknot in S^3 , are topologically minimal. The main idea is to construct a retraction from the disk complex of a bridge sphere to S^{n-1} as in [Bachman and Johnson 2010] and [Lee 2015].

Theorem 1.1. *An $(n + 1)$ -bridge sphere for the unknot is a topologically minimal surface of index at most n .*

In particular, the topological index of a 3-bridge sphere for the unknot is two. We conjecture that the topological index of an $(n + 1)$ -bridge sphere for the unknot is n . There is another conjecture that the topological index of a genus n Heegaard surface of S^3 is $2n - 1$. This correspondence may be due to the fact that a genus n Heegaard splitting of S^3 can be obtained as a 2-fold covering of S^3 branched along an unknot in $(n + 1)$ -bridge position.

2. Bridge splitting

For a closed 3-manifold M , a *Heegaard splitting* $M = V^+ \cup_S V^-$ is a decomposition of M into two handlebodies V^+ and V^- with $\partial V^+ = \partial V^- = S$. The surface S is called a *Heegaard surface* of the Heegaard splitting.

Let K be a knot in M such that $V^\pm \cap K$ is a collection of n boundary-parallel arcs $\{a_1^\pm, \dots, a_n^\pm\}$ in V^\pm . Each a_i^\pm is called a *bridge*. The decomposition

$$(M, K) = (V^+, V^+ \cap K) \cup_S (V^-, V^- \cap K)$$

is called a *bridge splitting* of (M, K) , and we say that K is in *n -bridge position* with respect to S . A bridge a_i^\pm cobounds a *bridge disk* Δ_i^\pm with an arc in S . We can take the bridge disks Δ_i^+ ($i = 1, \dots, n$) to be mutually disjoint, and similarly for Δ_i^- ($i = 1, \dots, n$). By a *bridge surface*, we mean $S - K$. The set of vertices of $\mathcal{D}(S - K)$ consists of compressing disks for $S - K$ in $V^+ - K$ and $V^- - K$.

Two bridge surfaces $S - K$ and $S' - K$ are equivalent if they are isotopic in $M - K$. An n -bridge position of the unknot in S^3 is unique for every n [Otal 1982], so for $n \geq 2$ it is *perturbed*, i.e., there exists a pair of bridge disks Δ_i^+ and Δ_j^- such that the arcs $\Delta_i^+ \cap S$ and $\Delta_j^- \cap S$ intersect at one endpoint. The uniqueness also holds for 2-bridge knots [Scharlemann and Tomova 2008] and torus knots [Ozawa 2011]. However, there are 3-bridge knots that admit multiple 3-bridge spheres [Birman 1976; Montesinos 1976].

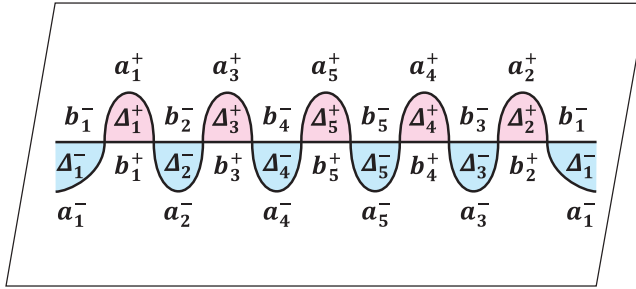


Figure 1. Bridges and bridge disks.

3. Proof of Theorem 1.1

Let S^3 be decomposed into two 3-balls B^+ and B^- with common boundary S . Let K be an unknot in S^3 which is in $(n + 1)$ -bridge position with respect to S . Then $K \cap B^\pm$ is a collection of $n + 1$ bridges a_i^\pm ($i = 1, \dots, n + 1$) in B^\pm . We assume that the bridges are arranged with a_1^\pm adjacent to a_1^\mp and a_2^\mp , with a_i^\pm adjacent to a_{i-1}^\mp and a_{i+1}^\mp for $2 \leq i \leq n$, and with a_{n+1}^\pm adjacent to a_n^\mp and a_{n+1}^\mp . Let $\{\Delta_i^\pm\}$ be a collection of disjoint bridge disks Δ_i^\pm for a_i^\pm with $\Delta_i^\pm \cap S = b_i^\pm$. We assume that $\text{int } b_i^+ \cap \text{int } b_j^- = \emptyset$ for any i and j . See Figure 1 for an example.

Let P be the $(2n + 2)$ -punctured sphere $S - K$. We define compressing disks D_i^\pm ($i = 1, \dots, n$) for P in $B^\pm - K$ as follows. Let D_1^+ be a disk in $B^+ - K$ such that $\partial D_1^+ = \partial N(b_1^+)$, where $N(b_1^+)$ is a neighborhood of b_1^+ taken in S . Similarly, other disks are defined so as to satisfy the following.

$$\begin{aligned} \partial D_1^- &= \partial N(b_1^-), \\ \partial D_2^+ &= \partial N(b_1^+ \cup b_1^- \cup b_2^+), \\ \partial D_2^- &= \partial N(b_1^- \cup b_1^+ \cup b_2^-), \\ &\vdots \\ \partial D_i^+ &= \partial N(b_1^+ \cup b_1^- \cup \dots \cup b_{i-1}^+ \cup b_{i-1}^- \cup b_i^+), \\ \partial D_i^- &= \partial N(b_1^- \cup b_1^+ \cup \dots \cup b_{i-1}^- \cup b_{i-1}^+ \cup b_i^-), \\ &\vdots \\ \partial D_n^+ &= \partial N(b_1^+ \cup b_1^- \cup \dots \cup b_{n-1}^+ \cup b_{n-1}^- \cup b_n^+), \\ \partial D_n^- &= \partial N(b_1^- \cup b_1^+ \cup \dots \cup b_{n-1}^- \cup b_{n-1}^+ \cup b_n^-). \end{aligned}$$

The ∂D_i^\pm 's in P are depicted in Figure 2.

Now we define subsets C_i^\pm ($i = 1, \dots, n$) of the set of vertices of $\mathcal{D}(P)$ as

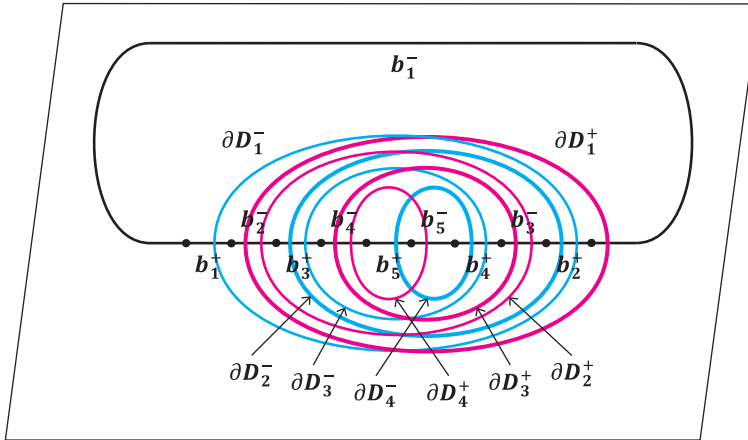


Figure 2. ∂D_i^\pm ($i = 1, \dots, n$) in P .

follows. For odd i , let

$$C_i^+ = \{D_i^+\},$$

$$C_i^- = \{\text{essential disks in } B^- - K \text{ that intersect } D_i^+ \text{ and are disjoint from } D_1^+, D_3^+, \dots, D_{i-2}^+\}.$$

For even i , let

$$C_i^+ = \{\text{essential disks in } B^+ - K \text{ that intersect } D_i^- \text{ and are disjoint from } D_2^-, D_4^-, \dots, D_{i-2}^-\},$$

$$C_i^- = \{D_i^-\}.$$

Note that for all i , D_i^\pm belongs to C_i^\pm .

Lemma 3.1. *The collection $\{C_i^\pm\}$ ($i = 1, \dots, n$) is a partition of the set of essential disks in $B^\pm - K$.*

Proof. First we show that $\{C_i^+\}$ ($i = 1, \dots, n$) is a partition of the set of essential disks in $B^+ - K$. We show that any essential disk in $B^+ - K$ belongs to one and only one C_i^+ .

An essential disk in $B^+ - K$ that intersects D_2^- belongs to C_2^+ by definition. Let $E_2 = N(b_1^- \cup b_1^+ \cup b_2^-)$ be the disk in S such that $\partial E_2 = \partial D_2^-$.

Claim 1. *If an essential disk D in $B^+ - K$ is disjoint from D_2^- and ∂D is in E_2 , then D is isotopic to $D_1^+ \in C_1^+$.*

Proof of Claim 1. We assume that D intersects D_1^+ transversely and minimally, so $D \cap D_1^+$ consists of arc components. Let $E_1 = N(b_1^+)$ be the disk in S such that $\partial E_1 = \partial D_1^+$. See Figure 3. Suppose that $D \cap D_1^+ \neq \emptyset$. Consider an outermost

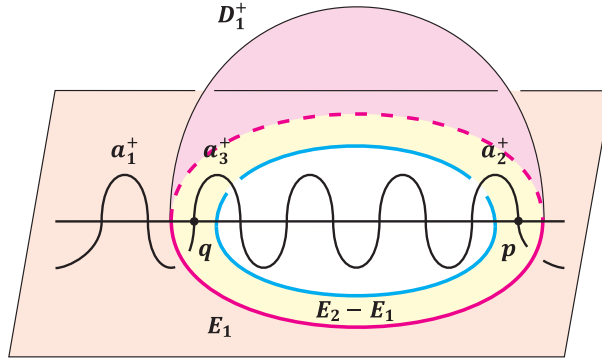


Figure 3. D_1^+ in C_1^+ .

disk Δ of D cut off by an outermost arc of $D \cap D_1^+$. By the minimality of $|D \cap D_1^+|$, Δ cannot lie in the 3-ball B bounded by $D_1^+ \cup E_1$ containing a_1^+ . So Δ lies outside of B . Let \bar{D} be one of the disks obtained from D_1^+ by surgery along Δ such that $\partial \bar{D}$ bounds a disk \bar{E} in $E_2 - E_1$. Let p be the point $a_2^+ \cap (E_2 - E_1)$ and q be the point $a_3^+ \cap (E_2 - E_1)$.

Suppose \bar{E} contains p . Then the sphere $\bar{D} \cup \bar{E}$ intersects $a_2^+ \cup b_2^+$ in a single point after a slight isotopy of $\text{int } b_2^+$ into B^- , a contradiction. So \bar{E} does not contain p , and by similar reasoning \bar{E} does not contain q . Then \bar{E} is an inessential disk in $E_2 - E_1 - K$, so we can reduce $|D \cap D_1^+|$, a contradiction.

Hence $D \cap D_1^+ = \emptyset$. Let E be the disk in E_2 such that $\partial E = \partial D$. If ∂E is in E_1 , then D is isotopic to D_1^+ . Suppose ∂E is in $E_2 - E_1$. Then E contains neither p nor q , since otherwise $D \cup E$ intersects $a_2^+ \cup b_2^+$ or $a_3^+ \cup b_3^+$ in a single point as above. So we get the conclusion that D is isotopic to D_1^+ . \square

Therefore if an essential disk in $B^+ - K$ is disjoint from D_2^- and its boundary is in $S - E_2$, then it belongs to one of C_3^+, \dots, C_n^+ .

An essential disk in $B^+ - K$ that is disjoint from D_2^- and intersects D_4^- belongs to C_4^+ by definition. Let $E_4 = N(b_1^- \cup b_1^+ \cup \dots \cup b_3^- \cup b_3^+ \cup b_4^-)$ be the disk in S such that $\partial E_4 = \partial D_4^-$. Let D be an essential disk in $B^+ - K$ that is disjoint from D_2^- and D_4^- and such that $\partial D \subset S - E_2$.

Claim 2. *If ∂D is in E_4 (hence in $E_4 - E_2$), then D is isotopic to $D_3^+ \in C_3^+$.*

Proof of Claim 2. We assume that $|D \cap D_3^+|$ is minimal up to isotopy, so $D \cap D_3^+$ consists of arc components. Let $E_3 = N(b_1^+ \cup b_1^- \cup b_2^+ \cup b_2^- \cup b_3^+)$ be the disk in S such that $\partial E_3 = \partial D_3^+$. See Figure 4.

Suppose that $D \cap D_3^+ \neq \emptyset$. Consider an outermost disk Δ of D cut off by an outermost arc of $D \cap D_3^+$. Without loss of generality, we assume that $\partial \Delta \cap S$ lies in $E_3 - E_2$. Let \bar{D} be one of the disks obtained from D_3^+ by surgery along Δ such

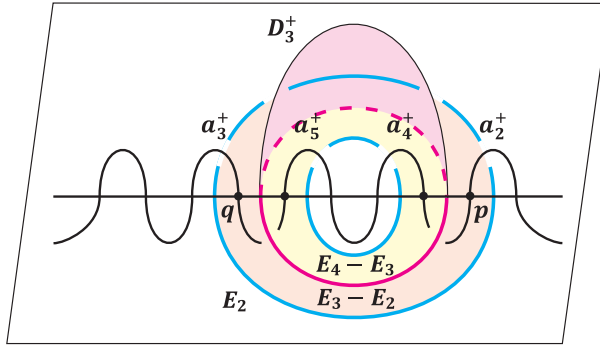


Figure 4. D_3^+ in C_3^+ .

that $\partial \bar{D}$ bounds a disk \bar{E} in $E_3 - E_2$. Let p be the point $a_2^+ \cap (E_3 - E_2)$ and q be the point $a_3^+ \cap (E_3 - E_2)$.

Suppose \bar{E} contains p . Then the sphere $\bar{D} \cup \bar{E}$ intersects $a_2^+ \cup b_2^+$ in a single point after a slight isotopy, a contradiction. So \bar{E} does not contain p , and similarly \bar{E} does not contain q . Then \bar{E} is an inessential disk in $E_3 - E_2 - K$, so we can reduce $|D \cap D_3^+|$, a contradiction. Hence $D \cap D_3^+ = \emptyset$. Then, reasoning as we did for Claim 1, we see that D is isotopic to D_3^+ . \square

Therefore if an essential disk in $B^+ - K$ is disjoint from D_2^- and D_4^- and its boundary is in $S - E_4$, then it belongs to one of C_5^+, \dots, C_n^+ .

In general, let $E_{2i} = N(b_1^- \cup b_1^+ \cup \dots \cup b_{2i-1}^- \cup b_{2i-1}^+ \cup b_{2i}^-)$ be the disk in S such that $\partial E_{2i} = \partial D_{2i}^-$. Let D be an essential disk in $B^+ - K$ that is disjoint from $D_2^-, D_4^-, \dots, D_{2i-2}^-$ and such that $\partial D \subset S - E_{2i-2}$.

- If $\partial D \subset E_{2i} - E_{2i-2}$, then D is isotopic to $D_{2i-1}^+ \in C_{2i-1}^+$.
- If D intersects D_{2i}^- , then D belongs to C_{2i}^+ by definition.
- If $\partial D \subset S - E_{2i}$, then D belongs to one of C_{2i+1}^+, \dots, C_n^+ .

An inductive argument in this way leads to the conclusion that any essential disk in $B^+ - K$ belongs to one and only one C_i^+ . A similar argument shows that $\{C_i^-\}$ ($i = 1, \dots, n$) is a partition of the set of essential disks in $B^- - K$. \square

The collection of disks $\{D_1^+, D_1^-, \dots, D_n^+, D_n^-\}$ spans an $(n - 1)$ -sphere S^{n-1} in $\mathcal{D}(P)$. There is no edge in $\mathcal{D}(P)$ connecting C_i^+ and C_i^- by definition. There exists an edge in $\mathcal{D}(P)$ connecting C_i^\pm and C_j^\pm for $i \neq j$, e.g., an edge between D_i^\pm and D_j^\pm , and there exists an edge in $\mathcal{D}(P)$ connecting C_i^+ and C_j^- for $i \neq j$, e.g., an edge between D_i^+ and D_j^- . Hence if we define a map \bar{r} from the set of vertices of $\mathcal{D}(P)$ to the set of vertices of S^{n-1} by

$$\bar{r}(v) = D_i^\pm \quad \text{if } v \in C_i^\pm,$$

then \bar{r} extends to a continuous map from the 1-skeleton of $\mathcal{D}(P)$ to the 1-skeleton of S^{n-1} . Since higher-dimensional simplices of $\mathcal{D}(P)$ are determined by 1-simplices, the map \bar{r} can be extended to a retraction $r : \mathcal{D}(P) \rightarrow S^{n-1}$. Hence $\pi_{n-1}(\mathcal{D}(P)) \neq 1$, and the topological index of P is at most n .

Acknowledgement

The author was supported by the Basic Science Research Program through the National Research Foundation of Korea (NRF), funded by the Ministry of Education (2015R1D1A1A01056953).

References

- [Bachman 2010] D. Bachman, “Topological index theory for surfaces in 3-manifolds”, *Geom. Topol.* **14**:1 (2010), 585–609. MR 2011f:57042 Zbl 1206.57020
- [Bachman and Johnson 2010] D. Bachman and J. Johnson, “On the existence of high index topologically minimal surfaces”, *Math. Res. Lett.* **17**:3 (2010), 389–394. MR 2011e:57025 Zbl 1257.57026
- [Birman 1976] J. S. Birman, “On the stable equivalence of plat representations of knots and links”, *Canad. J. Math.* **28**:2 (1976), 264–290. MR 53 #6529 Zbl 0339.55005
- [Casson and Gordon 1987] A. J. Casson and C. M. Gordon, “Reducing Heegaard splittings”, *Topology Appl.* **27**:3 (1987), 275–283. MR 89c:57020 Zbl 0632.57010
- [Lee 2015] J. H. Lee, “On topologically minimal surfaces of high genus”, *Proc. Amer. Math. Soc.* **143**:6 (2015), 2725–2730. MR 3326050 Zbl 1315.57023
- [Montesinos 1976] J. M. Montesinos, “Minimal plat representations of prime knots and links are not unique”, *Canad. J. Math.* **28**:1 (1976), 161–167. MR 53 #14470 Zbl 0314.57011
- [Otal 1982] J.-P. Otal, “Présentations en ponts du nœud trivial”, *C. R. Acad. Sci. Paris Sér. I Math.* **294**:16 (1982), 553–556. MR 84a:57006 Zbl 0498.57001
- [Ozawa 2011] M. Ozawa, “Nonminimal bridge positions of torus knots are stabilized”, *Math. Proc. Cambridge Philos. Soc.* **151**:2 (2011), 307–317. MR 2012m:57015 Zbl 1226.57012
- [Scharlemann and Tomova 2008] M. Scharlemann and M. Tomova, “Uniqueness of bridge surfaces for 2-bridge knots”, *Math. Proc. Cambridge Philos. Soc.* **144**:3 (2008), 639–650. MR 2009c:57020 Zbl 1152.57006

Received December 22, 2014. Revised May 29, 2015.

JUNG HOON LEE
 DEPARTMENT OF MATHEMATICS AND INSTITUTE OF PURE AND APPLIED MATHEMATICS
 CHONBUK NATIONAL UNIVERSITY
 JEONJU 54896
 SOUTH KOREA
 junghoon@jbnu.ac.kr

PACIFIC JOURNAL OF MATHEMATICS

msp.org/pjm

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US \$440/year for the electronic version, and \$600/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 282 No. 2 June 2016

Exhausting curve complexes by finite rigid sets	257
JAVIER ARAMAYONA and CHRISTOPHER J. LEININGER	
A variational characterization of flat spaces in dimension three	285
GIOVANNI CATINO, PAOLO MASTROLIA and DARIO D. MONTICELLI	
Estimates of the gaps between consecutive eigenvalues of Laplacian	293
DAGUANG CHEN, TAO ZHENG and HONGCANG YANG	
Liouville type theorems for the p -harmonic functions on certain manifolds	313
JINGYI CHEN and YUE WANG	
Cartan–Fubini type rigidity of double covering morphisms of quadratic manifolds	329
HOSUNG KIM	
On the uniform squeezing property of bounded convex domains in \mathbb{C}^n	341
KANG-TAE KIM and LIYOU ZHANG	
Lefschetz pencils and finitely presented groups	359
RYOMA KOBAYASHI and NAOYUKI MONDEN	
Knot homotopy in subspaces of the 3-sphere	389
YUYA KODA and MAKOTO OZAWA	
On the relationship of continuity and boundary regularity in prescribed mean curvature Dirichlet problems	415
KIRK E. LANCASTER and JARON MELIN	
Bridge spheres for the unknot are topologically minimal	437
JUNG HOON LEE	
On the geometric construction of cohomology classes for cocompact discrete subgroups of $SL_n(\mathbb{R})$ and $SL_n(\mathbb{C})$	445
SUSANNE SCHIMPF	
On Blaschke’s conjecture	479
XIAOLE SU, HONGWEI SUN and YUSHENG WANG	
The role of the Jacobi identity in solving the Maurer–Cartan structure equation	487
ORI YUDILEVICH	



0030-8730(2016)282:2;1-3