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## BRIDGE SPHERES FOR THE UNKNOT ARE TOPOLOGICALLY MINIMAL

Jung Hoon Lee

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#### Abstract

Topologically minimal surfaces were defined by Bachman as topological analogues of geometrically minimal surfaces, and one can associate a topological index to each topologically minimal surface. We show that an $(n+1)$ bridge sphere for the unknot is a topologically minimal surface of index at most $n$.


## 1. Introduction

Let $S$ be a closed orientable separating surface embedded in a 3-manifold $M$. The structure of the set of compressing disks for $S$, such as how a pair of compressing disks on opposite sides of $S$ intersects, reveals some topological properties of $M$. For example, if $S$ is a minimal genus Heegaard surface of an irreducible manifold $M$ and $S$ has a pair of disjoint compressing disks on opposite sides, then $M$ contains an incompressible surface [Casson and Gordon 1987].

The disk complex $\mathcal{D}(S)$ of $S$ is a simplicial complex defined as follows.

- Vertices of $\mathcal{D}(S)$ are isotopy classes of compressing disks for $S$.
- A collection of $k+1$ vertices forms a $k$-simplex if there are representatives for each that are pairwise disjoint.

The disk complex of an incompressible surface is empty. A surface $S$ is strongly irreducible if $S$ compresses to both sides and every compressing disk for $S$ on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman [2010] defined topologically minimal surfaces, which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface $S$ is topologically minimal if $\mathcal{D}(S)$ is empty or $\pi_{i}(\mathcal{D}(S))$ is nontrivial for some $i$. The topological index of $S$ is 0 if $\mathcal{D}(S)$ is empty, and the smallest $n$ such that $\pi_{n-1}(\mathcal{D}(S))$ is nontrivial, otherwise.

Topologically minimal surfaces share some useful properties. For example, if an irreducible manifold contains a topologically minimal surface and an incompressible

[^0]surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces. There exist topologically minimal surfaces of arbitrarily high index [Bachman and Johnson 2010], and see also [Lee 2015] for possibly high index surfaces in (closed orientable surface) $\times I$. In this paper we consider bridge splittings of 3-manifolds, and show that the simplest bridge surfaces, bridge spheres for the unknot in $S^{3}$, are topologically minimal. The main idea is to construct a retraction from the disk complex of a bridge sphere to $S^{n-1}$ as in [Bachman and Johnson 2010] and [Lee 2015].

Theorem 1.1. An $(n+1)$-bridge sphere for the unknot is a topologically minimal surface of index at most $n$.

In particular, the topological index of a 3-bridge sphere for the unknot is two. We conjecture that the topological index of an $(n+1)$-bridge sphere for the unknot is $n$. There is another conjecture that the topological index of a genus $n$ Heegaard surface of $S^{3}$ is $2 n-1$. This correspondence may be due to the fact that a genus $n$ Heegaard splitting of $S^{3}$ can be obtained as a 2 -fold covering of $S^{3}$ branched along an unknot in $(n+1)$-bridge position.

## 2. Bridge splitting

For a closed 3-manifold $M$, a Heegaard splitting $M=V^{+} \cup_{S} V^{-}$is a decomposition of $M$ into two handlebodies $V^{+}$and $V^{-}$with $\partial V^{+}=\partial V^{-}=S$. The surface $S$ is called a Heegaard surface of the Heegaard splitting.

Let $K$ be a knot in $M$ such that $V^{ \pm} \cap K$ is a collection of $n$ boundary-parallel $\operatorname{arcs}\left\{a_{1}^{ \pm}, \ldots, a_{n}^{ \pm}\right\}$in $V^{ \pm}$. Each $a_{i}^{ \pm}$is called a bridge. The decomposition

$$
(M, K)=\left(V^{+}, V^{+} \cap K\right) \cup_{S}\left(V^{-}, V^{-} \cap K\right)
$$

is called a bridge splitting of ( $M, K$ ), and we say that $K$ is in $n$-bridge position with respect to $S$. A bridge $a_{i}^{ \pm}$cobounds a bridge disk $\Delta_{i}^{ \pm}$with an arc in $S$. We can take the bridge disks $\Delta_{i}^{+}(i=1, \ldots, n)$ to be mutually disjoint, and similarly for $\Delta_{i}^{-}(i=1, \ldots, n)$. By a bridge surface, we mean $S-K$. The set of vertices of $\mathcal{D}(S-K)$ consists of compressing disks for $S-K$ in $V^{+}-K$ and $V^{-}-K$.

Two bridge surfaces $S-K$ and $S^{\prime}-K$ are equivalent if they are isotopic in $M-K$. An $n$-bridge position of the unknot in $S^{3}$ is unique for every $n$ [Otal 1982], so for $n \geq 2$ it is perturbed, i.e., there exists a pair of bridge disks $\Delta_{i}^{+}$and $\Delta_{j}^{-}$such that the arcs $\Delta_{i}^{+} \cap S$ and $\Delta_{j}^{-} \cap S$ intersect at one endpoint. The uniqueness also holds for 2-bridge knots [Scharlemann and Tomova 2008] and torus knots [Ozawa 2011]. However, there are 3-bridge knots that admit multiple 3-bridge spheres [Birman 1976; Montesinos 1976].


Figure 1. Bridges and bridge disks.

## 3. Proof of Theorem 1.1

Let $S^{3}$ be decomposed into two 3-balls $B^{+}$and $B^{-}$with common boundary $S$. Let $K$ be an unknot in $S^{3}$ which is in $(n+1)$-bridge position with respect to $S$. Then $K \cap B^{ \pm}$is a collection of $n+1$ bridges $a_{i}^{ \pm}(i=1, \ldots, n+1)$ in $B^{ \pm}$. We assume that the bridges are arranged with $a_{1}^{ \pm}$adjacent to $a_{1}^{\mp}$ and $a_{2}^{\mp}$, with $a_{i}^{ \pm}$ adjacent to $a_{i-1}^{\mp}$ and $a_{i+1}^{\mp}$ for $2 \leq i \leq n$, and with $a_{n+1}^{ \pm}$adjacent to $a_{n}^{\mp}$ and $a_{n+1}^{\mp}$. Let $\left\{\Delta_{i}^{ \pm}\right\}$be a collection of disjoint bridge disks $\Delta_{i}^{ \pm}$for $a_{i}^{ \pm}$with $\Delta_{i}^{ \pm} \cap S=b_{i}^{ \pm}$. We assume that int $b_{i}^{+} \cap \operatorname{int} b_{j}^{-}=\varnothing$ for any $i$ and $j$. See Figure 1 for an example.

Let $P$ be the $(2 n+2)$-punctured sphere $S-K$. We define compressing disks $D_{i}^{ \pm}$ $(i=1, \ldots, n)$ for $P$ in $B^{ \pm}-K$ as follows. Let $D_{1}^{+}$be a disk in $B^{+}-K$ such that $\partial D_{1}^{+}=\partial N\left(b_{1}^{+}\right)$, where $N\left(b_{1}^{+}\right)$is a neighborhood of $b_{1}^{+}$taken in $S$. Similarly, other disks are defined so as to satisfy the following.

$$
\begin{aligned}
\partial D_{1}^{-} & =\partial N\left(b_{1}^{-}\right) \\
\partial D_{2}^{+} & =\partial N\left(b_{1}^{+} \cup b_{1}^{-} \cup b_{2}^{+}\right) \\
\partial D_{2}^{-} & =\partial N\left(b_{1}^{-} \cup b_{1}^{+} \cup b_{2}^{-}\right) \\
& \vdots \\
\partial D_{i}^{+} & =\partial N\left(b_{1}^{+} \cup b_{1}^{-} \cup \cdots \cup b_{i-1}^{+} \cup b_{i-1}^{-} \cup b_{i}^{+}\right) \\
\partial D_{i}^{-} & =\partial N\left(b_{1}^{-} \cup b_{1}^{+} \cup \cdots \cup b_{i-1}^{-} \cup b_{i-1}^{+} \cup b_{i}^{-}\right) \\
& \vdots \\
\partial D_{n}^{+} & =\partial N\left(b_{1}^{+} \cup b_{1}^{-} \cup \cdots \cup b_{n-1}^{+} \cup b_{n-1}^{-} \cup b_{n}^{+}\right) \\
\partial D_{n}^{-} & =\partial N\left(b_{1}^{-} \cup b_{1}^{+} \cup \cdots \cup b_{n-1}^{-} \cup b_{n-1}^{+} \cup b_{n}^{-}\right)
\end{aligned}
$$

The $\partial D_{i}^{ \pm}$'s in $P$ are depicted in Figure 2.
Now we define subsets $C_{i}^{ \pm}(i=1, \ldots, n)$ of the set of vertices of $\mathcal{D}(P)$ as


Figure 2. $\partial D_{i}^{ \pm}(i=1, \ldots, n)$ in $P$.
follows. For odd $i$, let

$$
\begin{aligned}
C_{i}^{+} & =\left\{D_{i}^{+}\right\}, \\
C_{i}^{-} & =\left\{\text {essential disks in } B^{-}-K \text { that intersect } D_{i}^{+}\right. \\
& \text {and are disjoint from } \left.D_{1}^{+}, D_{3}^{+}, \ldots, D_{i-2}^{+}\right\} .
\end{aligned}
$$

For even $i$, let

$$
\begin{aligned}
C_{i}^{+} & =\left\{\text {essential disks in } B^{+}-K \text { that intersect } D_{i}^{-}\right. \\
& \text {and are disjoint from } \left.D_{2}^{-}, D_{4}^{-}, \ldots, D_{i-2}^{-}\right\}, \\
C_{i}^{-} & =\left\{D_{i}^{-}\right\} .
\end{aligned}
$$

Note that for all $i, D_{i}^{ \pm}$belongs to $C_{i}^{ \pm}$.
Lemma 3.1. The collection $\left\{C_{i}^{ \pm}\right\}(i=1, \ldots, n)$ is a partition of the set of essential disks in $B^{ \pm}-K$.

Proof. First we show that $\left\{C_{i}^{+}\right\}(i=1, \ldots, n)$ is a partition of the set of essential disks in $B^{+}-K$. We show that any essential disk in $B^{+}-K$ belongs to one and only one $C_{i}^{+}$.

An essential disk in $B^{+}-K$ that intersects $D_{2}^{-}$belongs to $C_{2}^{+}$by definition. Let $E_{2}=N\left(b_{1}^{-} \cup b_{1}^{+} \cup b_{2}^{-}\right)$be the disk in $S$ such that $\partial E_{2}=\partial D_{2}^{-}$.
Claim 1. If an essential disk $D$ in $B^{+}-K$ is disjoint from $D_{2}^{-}$and $\partial D$ is in $E_{2}$, then $D$ is isotopic to $D_{1}^{+} \in C_{1}^{+}$.

Proof of Claim 1. We assume that $D$ intersects $D_{1}^{+}$transversely and minimally, so $D \cap D_{1}^{+}$consists of arc components. Let $E_{1}=N\left(b_{1}^{+}\right)$be the disk in $S$ such that $\partial E_{1}=\partial D_{1}^{+}$. See Figure 3. Suppose that $D \cap D_{1}^{+} \neq \varnothing$. Consider an outermost


Figure 3. $D_{1}^{+}$in $C_{1}^{+}$.
disk $\Delta$ of $D$ cut off by an outermost arc of $D \cap D_{1}^{+}$. By the minimality of $\left|D \cap D_{1}^{+}\right|$, $\Delta$ cannot lie in the 3 -ball $B$ bounded by $D_{1}^{+} \cup E_{1}$ containing $a_{1}^{+}$. So $\Delta$ lies outside of $B$. Let $\bar{D}$ be one of the disks obtained from $D_{1}^{+}$by surgery along $\Delta$ such that $\partial \bar{D}$ bounds a disk $\bar{E}$ in $E_{2}-E_{1}$. Let $p$ be the point $a_{2}^{+} \cap\left(E_{2}-E_{1}\right)$ and $q$ be the point $a_{3}^{+} \cap\left(E_{2}-E_{1}\right)$.

Suppose $\bar{E}$ contains $p$. Then the sphere $\bar{D} \cup \bar{E}$ intersects $a_{2}^{+} \cup b_{2}^{+}$in a single point after a slight isotopy of int $b_{2}^{+}$into $B^{-}$, a contradiction. So $\bar{E}$ does not contain $p$, and by similar reasoning $\bar{E}$ does not contain $q$. Then $\bar{E}$ is an inessential disk in $E_{2}-E_{1}-K$, so we can reduce $\left|D \cap D_{1}^{+}\right|$, a contradiction.

Hence $D \cap D_{1}^{+}=\varnothing$. Let $E$ be the disk in $E_{2}$ such that $\partial E=\partial D$. If $\partial E$ is in $E_{1}$, then $D$ is isotopic to $D_{1}^{+}$. Suppose $\partial E$ is in $E_{2}-E_{1}$. Then $E$ contains neither $p$ nor $q$, since otherwise $D \cup E$ intersects $a_{2}^{+} \cup b_{2}^{+}$or $a_{3}^{+} \cup b_{3}^{+}$in a single point as above. So we get the conclusion that $D$ is isotopic to $D_{1}^{+}$.

Therefore if an essential disk in $B^{+}-K$ is disjoint from $D_{2}^{-}$and its boundary is in $S-E_{2}$, then it belongs to one of $C_{3}^{+}, \ldots, C_{n}^{+}$.

An essential disk in $B^{+}-K$ that is disjoint from $D_{2}^{-}$and intersects $D_{4}^{-}$belongs to $C_{4}^{+}$by definition. Let $E_{4}=N\left(b_{1}^{-} \cup b_{1}^{+} \cup \cdots \cup b_{3}^{-} \cup b_{3}^{+} \cup b_{4}^{-}\right)$be the disk in $S$ such that $\partial E_{4}=\partial D_{4}^{-}$. Let $D$ be an essential disk in $B^{+}-K$ that is disjoint from $D_{2}^{-}$ and $D_{4}^{-}$and such that $\partial D \subset S-E_{2}$.
Claim 2. If $\partial D$ is in $E_{4}$ (hence in $E_{4}-E_{2}$ ), then $D$ is isotopic to $D_{3}^{+} \in C_{3}^{+}$.
Proof of Claim 2. We assume that $\left|D \cap D_{3}^{+}\right|$is minimal up to isotopy, so $D \cap D_{3}^{+}$ consists of arc components. Let $E_{3}=N\left(b_{1}^{+} \cup b_{1}^{-} \cup b_{2}^{+} \cup b_{2}^{-} \cup b_{3}^{+}\right)$be the disk in $S$ such that $\partial E_{3}=\partial D_{3}^{+}$. See Figure 4.

Suppose that $D \cap D_{3}^{+} \neq \varnothing$. Consider an outermost disk $\Delta$ of $D$ cut off by an outermost arc of $D \cap D_{3}^{+}$. Without loss of generality, we assume that $\partial \Delta \cap S$ lies in $E_{3}-E_{2}$. Let $\bar{D}$ be one of the disks obtained from $D_{3}^{+}$by surgery along $\Delta$ such


Figure 4. $D_{3}^{+}$in $C_{3}^{+}$.
that $\partial \bar{D}$ bounds a disk $\bar{E}$ in $E_{3}-E_{2}$. Let $p$ be the point $a_{2}^{+} \cap\left(E_{3}-E_{2}\right)$ and $q$ be the point $a_{3}^{+} \cap\left(E_{3}-E_{2}\right)$.

Suppose $\bar{E}$ contains $p$. Then the sphere $\bar{D} \cup \bar{E}$ intersects $a_{2}^{+} \cup b_{2}^{+}$in a single point after a slight isotopy, a contradiction. So $\bar{E}$ does not contain $p$, and similarly $\bar{E}$ does not contain $q$. Then $\bar{E}$ is an inessential disk in $E_{3}-E_{2}-K$, so we can reduce $\left|D \cap D_{3}^{+}\right|$, a contradiction. Hence $D \cap D_{3}^{+}=\varnothing$. Then, reasoning as we did for Claim 1, we see that $D$ is isotopic to $D_{3}^{+}$.

Therefore if an essential disk in $B^{+}-K$ is disjoint from $D_{2}^{-}$and $D_{4}^{-}$and its boundary is in $S-E_{4}$, then it belongs to one of $C_{5}^{+}, \ldots, C_{n}^{+}$.

In general, let $E_{2 i}=N\left(b_{1}^{-} \cup b_{1}^{+} \cup \cdots \cup b_{2 i-1}^{-} \cup b_{2 i-1}^{+} \cup b_{2 i}^{-}\right)$be the disk in $S$ such that $\partial E_{2 i}=\partial D_{2 i}^{-}$. Let $D$ be an essential disk in $B^{+}-K$ that is disjoint from $D_{2}^{-}, D_{4}^{-}, \ldots, D_{2 i-2}^{-}$and such that $\partial D \subset S-E_{2 i-2}$.

- If $\partial D \subset E_{2 i}-E_{2 i-2}$, then $D$ is isotopic to $D_{2 i-1}^{+} \in C_{2 i-1}^{+}$.
- If $D$ intersects $D_{2 i}^{-}$, then $D$ belongs to $C_{2 i}^{+}$by definition.
- If $\partial D \subset S-E_{2 i}$, then $D$ belongs to one of $C_{2 i+1}^{+}, \ldots, C_{n}^{+}$.

An inductive argument in this way leads to the conclusion that any essential disk in $B^{+}-K$ belongs to one and only one $C_{i}^{+}$. A similar argument shows that $\left\{C_{i}^{-}\right\}$ $(i=1, \ldots, n)$ is a partition of the set of essential disks in $B^{-}-K$.

The collection of disks $\left\{D_{1}^{+}, D_{1}^{-}, \ldots, D_{n}^{+}, D_{n}^{-}\right\}$spans an $(n-1)$-sphere $S^{n-1}$ in $\mathcal{D}(P)$. There is no edge in $\mathcal{D}(P)$ connecting $C_{i}^{+}$and $C_{i}^{-}$by definition. There exists an edge in $\mathcal{D}(P)$ connecting $C_{i}^{ \pm}$and $C_{j}^{ \pm}$for $i \neq j$, e.g., an edge between $D_{i}^{ \pm}$ and $D_{j}^{ \pm}$, and there exists an edge in $\mathcal{D}(P)$ connecting $C_{i}^{+}$and $C_{j}^{-}$for $i \neq j$, e.g., an edge between $D_{i}^{+}$and $D_{j}^{-}$. Hence if we define a map $\bar{r}$ from the set of vertices of $\mathcal{D}(P)$ to the set of vertices of $S^{n-1}$ by

$$
\bar{r}(v)=D_{i}^{ \pm} \quad \text { if } v \in C_{i}^{ \pm},
$$

then $\bar{r}$ extends to a continuous map from the 1 -skeleton of $\mathcal{D}(P)$ to the 1 -skeleton of $S^{n-1}$. Since higher-dimensional simplices of $\mathcal{D}(P)$ are determined by 1 -simplices, the map $\bar{r}$ can be extended to a retraction $r: \mathcal{D}(P) \rightarrow S^{n-1}$. Hence $\pi_{n-1}(\mathcal{D}(P)) \neq 1$, and the topological index of $P$ is at most $n$.

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