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Topologically minimal surfaces were defined by Bachman as topological analogues of geometrically minimal surfaces, and one can associate a topological index to each topologically minimal surface. We show that an $(n+1)$-bridge sphere for the unknot is a topologically minimal surface of index at most n.

1. Introduction

Let S be a closed orientable separating surface embedded in a 3-manifold M. The structure of the set of compressing disks for S, such as how a pair of compressing disks on opposite sides of S intersects, reveals some topological properties of M. For example, if S is a minimal genus Heegaard surface of an irreducible manifold M and S has a pair of disjoint compressing disks on opposite sides, then M contains an incompressible surface [Casson and Gordon 1987].

The *disk complex* $\mathcal{D}(S)$ of S is a simplicial complex defined as follows.

- Vertices of $\mathcal{D}(S)$ are isotopy classes of compressing disks for S.
- A collection of $k+1$ vertices forms a k-simplex if there are representatives for each that are pairwise disjoint.

The disk complex of an incompressible surface is empty. A surface S is *strongly irreducible* if S compresses to both sides and every compressing disk for S on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman [2010] defined topologically minimal surfaces, which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface S is *topologically minimal* if $\mathcal{D}(S)$ is empty or $\pi_i(\mathcal{D}(S))$ is nontrivial for some i. The *topological index* of S is 0 if $\mathcal{D}(S)$ is empty, and the smallest n such that $\pi_{n-1}(\mathcal{D}(S))$ is nontrivial, otherwise.

Topologically minimal surfaces share some useful properties. For example, if an irreducible manifold contains a topologically minimal surface and an incompressible

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surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces. There exist topologically minimal surfaces of arbitrarily high index [Bachman and Johnson 2010], and see also [Lee 2015] for possibly high index surfaces in (closed orientable surface) $\times I$. In this paper we consider bridge splittings of 3-manifolds, and show that the simplest bridge surfaces, bridge spheres for the unknot in S^3, are topologically minimal. The main idea is to construct a retraction from the disk complex of a bridge sphere to S^{n-1} as in [Bachman and Johnson 2010] and [Lee 2015].

Theorem 1.1. *An $(n+1)$-bridge sphere for the unknot is a topologically minimal surface of index at most n.*

In particular, the topological index of a 3-bridge sphere for the unknot is two. We conjecture that the topological index of an $(n+1)$-bridge sphere for the unknot is n. There is another conjecture that the topological index of a genus n Heegaard surface of S^3 is $2n-1$. This correspondence may be due to the fact that a genus n Heegaard splitting of S^3 can be obtained as a 2-fold covering of S^3 branched along an unknot in $(n+1)$-bridge position.

2. Bridge splitting

For a closed 3-manifold M, a *Heegaard splitting* $M = V^+ \cup_S V^-$ is a decomposition of M into two handlebodies V^+ and V^- with $\partial V^+ = \partial V^- = S$. The surface S is called a *Heegaard surface* of the Heegaard splitting.

Let K be a knot in M such that $V^\pm \cap K$ is a collection of n boundary-parallel arcs $\{a_1^\pm, \ldots, a_n^\pm\}$ in $V^\pm$. Each $a_i^\pm$ is called a *bridge*. The decomposition

$$(M, K) = (V^+, V^+ \cap K) \cup_S (V^-, V^- \cap K)$$

is called a *bridge splitting* of (M, K), and we say that K is in *n-bridge position* with respect to S. A bridge $a_i^\pm$ cobounds a *bridge disk* $\Delta_i^\pm$ with an arc in S. We can take the bridge disks Δ_i^+ $(i = 1, \ldots, n)$ to be mutually disjoint, and similarly for Δ_i^- $(i = 1, \ldots, n)$. By a *bridge surface*, we mean $S - K$. The set of vertices of $\mathcal{D}(S-K)$ consists of compressing disks for $S - K$ in $V^+ - K$ and $V^- - K$.

Two bridge surfaces $S - K$ and $S' - K$ are equivalent if they are isotopic in $M - K$. An n-bridge position of the unknot in S^3 is unique for every n [Otal 1982], so for $n \geq 2$ it is *perturbed*, i.e., there exists a pair of bridge disks Δ_i^+ and Δ_j^- such that the arcs $\Delta_i^+ \cap S$ and $\Delta_j^- \cap S$ intersect at one endpoint. The uniqueness also holds for 2-bridge knots [Scharlemann and Tomova 2008] and torus knots [Ozawa 2011]. However, there are 3-bridge knots that admit multiple 3-bridge spheres [Birman 1976; Montesinos 1976].

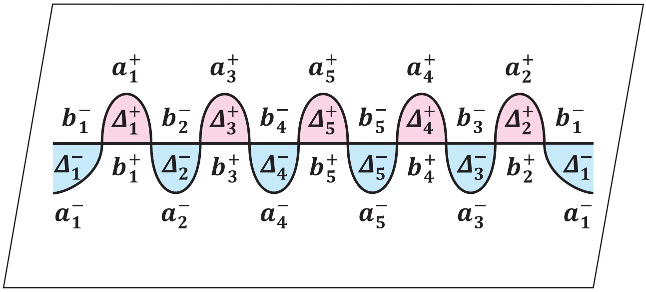

Figure 1. Bridges and bridge disks.

3. Proof of Theorem 1.1

Let S^3 be decomposed into two 3-balls B^+ and B^- with common boundary S. Let K be an unknot in S^3 which is in $(n+1)$-bridge position with respect to S. Then $K \cap B^{\pm}$ is a collection of $n+1$ bridges $a_i^{\pm}$ $(i = 1, \ldots, n+1)$ in $B^{\pm}$. We assume that the bridges are arranged with $a_1^{\pm}$ adjacent to $a_1^{\mp}$ and $a_2^{\mp}$, with $a_i^{\pm}$ adjacent to $a_{i-1}^{\mp}$ and $a_{i+1}^{\mp}$ for $2 \leq i \leq n$, and with $a_{n+1}^{\pm}$ adjacent to $a_n^{\mp}$ and $a_{n+1}^{\mp}$. Let $\{\Delta_i^{\pm}\}$ be a collection of disjoint bridge disks $\Delta_i^{\pm}$ for $a_i^{\pm}$ with $\Delta_i^{\pm} \cap S = b_i^{\pm}$. We assume that $\operatorname{int} b_i^+ \cap \operatorname{int} b_j^- = \varnothing$ for any i and j. See Figure 1 for an example.

Let P be the $(2n+2)$-punctured sphere $S - K$. We define compressing disks $D_i^{\pm}$ $(i = 1, \ldots, n)$ for P in $B^{\pm} - K$ as follows. Let D_1^+ be a disk in $B^+ - K$ such that $\partial D_1^+ = \partial N(b_1^+)$, where $N(b_1^+)$ is a neighborhood of b_1^+ taken in S. Similarly, other disks are defined so as to satisfy the following.

$$
\begin{aligned}
\partial D_1^- &= \partial N(b_1^-),\\
\partial D_2^+ &= \partial N(b_1^+ \cup b_1^- \cup b_2^+),\\
\partial D_2^- &= \partial N(b_1^- \cup b_1^+ \cup b_2^-),\\
&\ \ \vdots\\
\partial D_i^+ &= \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{i-1}^+ \cup b_{i-1}^- \cup b_i^+),\\
\partial D_i^- &= \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{i-1}^- \cup b_{i-1}^+ \cup b_i^-),\\
&\ \ \vdots\\
\partial D_n^+ &= \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{n-1}^+ \cup b_{n-1}^- \cup b_n^+),\\
\partial D_n^- &= \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{n-1}^- \cup b_{n-1}^+ \cup b_n^-).
\end{aligned}
$$

The $\partial D_i^{\pm}$'s in P are depicted in Figure 2.

Now we define subsets $C_i^{\pm}$ $(i = 1, \ldots, n)$ of the set of vertices of $\mathcal{D}(P)$ as

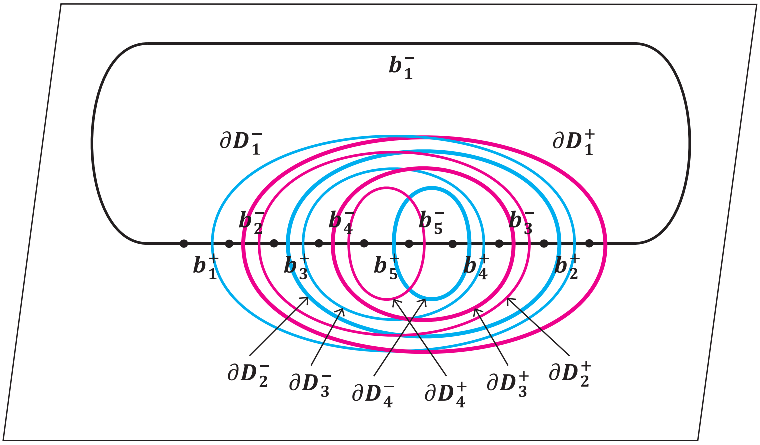

Figure 2. $\partial D_i^\pm$ $(i = 1, \dots, n)$ in P.

follows. For odd i, let

$$C_i^+ = \{D_i^+\},$$
$$C_i^- = \{\text{essential disks in } B^- - K \text{ that intersect } D_i^+ \text{ and are disjoint from } D_1^+, D_3^+, \dots, D_{i-2}^+\}.$$

For even i, let

$$C_i^+ = \{\text{essential disks in } B^+ - K \text{ that intersect } D_i^- \text{ and are disjoint from } D_2^-, D_4^-, \dots, D_{i-2}^-\},$$
$$C_i^- = \{D_i^-\}.$$

Note that for all i, $D_i^\pm$ belongs to $C_i^\pm$.

Lemma 3.1. *The collection $\{C_i^\pm\}$ $(i = 1, \dots, n)$ is a partition of the set of essential disks in $B^\pm - K$.*

Proof. First we show that $\{C_i^+\}$ $(i = 1, \dots, n)$ is a partition of the set of essential disks in $B^+ - K$. We show that any essential disk in $B^+ - K$ belongs to one and only one C_i^+.

An essential disk in $B^+ - K$ that intersects D_2^- belongs to C_2^+ by definition. Let $E_2 = N(b_1^- \cup b_1^+ \cup b_2^-)$ be the disk in S such that $\partial E_2 = \partial D_2^-$.

Claim 1. *If an essential disk D in $B^+ - K$ is disjoint from D_2^- and ∂D is in E_2, then D is isotopic to $D_1^+ \in C_1^+$.*

Proof of Claim 1. We assume that D intersects D_1^+ transversely and minimally, so $D \cap D_1^+$ consists of arc components. Let $E_1 = N(b_1^+)$ be the disk in S such that $\partial E_1 = \partial D_1^+$. See Figure 3. Suppose that $D \cap D_1^+ \neq \varnothing$. Consider an outermost

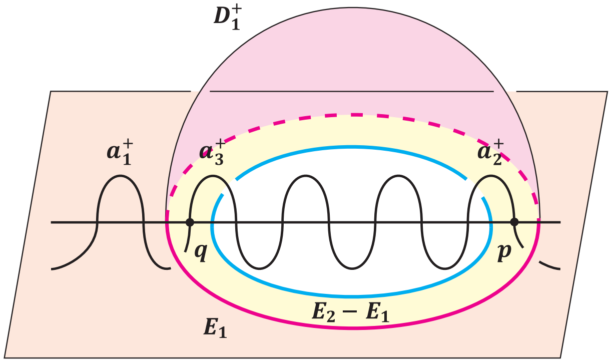

Figure 3. D_1^+ in C_1^+.

disk Δ of D cut off by an outermost arc of $D \cap D_1^+$. By the minimality of $|D \cap D_1^+|$, Δ cannot lie in the 3-ball B bounded by $D_1^+ \cup E_1$ containing a_1^+. So Δ lies outside of B. Let $\overline{D}$ be one of the disks obtained from D_1^+ by surgery along Δ such that $\partial \overline{D}$ bounds a disk $\overline{E}$ in $E_2 - E_1$. Let p be the point $a_2^+ \cap (E_2 - E_1)$ and q be the point $a_3^+ \cap (E_2 - E_1)$.

Suppose $\overline{E}$ contains p. Then the sphere $\overline{D} \cup \overline{E}$ intersects $a_2^+ \cup b_2^+$ in a single point after a slight isotopy of $\operatorname{int} b_2^+$ into B^-, a contradiction. So $\overline{E}$ does not contain p, and by similar reasoning $\overline{E}$ does not contain q. Then $\overline{E}$ is an inessential disk in $E_2 - E_1 - K$, so we can reduce $|D \cap D_1^+|$, a contradiction.

Hence $D \cap D_1^+ = \varnothing$. Let E be the disk in E_2 such that $\partial E = \partial D$. If ∂E is in E_1, then D is isotopic to D_1^+. Suppose ∂E is in $E_2 - E_1$. Then E contains neither p nor q, since otherwise $D \cup E$ intersects $a_2^+ \cup b_2^+$ or $a_3^+ \cup b_3^+$ in a single point as above. So we get the conclusion that D is isotopic to D_1^+. □

Therefore if an essential disk in $B^+ - K$ is disjoint from D_2^- and its boundary is in $S - E_2$, then it belongs to one of $C_3^+, \ldots, C_n^+$.

An essential disk in $B^+ - K$ that is disjoint from D_2^- and intersects D_4^- belongs to C_4^+ by definition. Let $E_4 = N(b_1^- \cup b_1^+ \cup \cdots \cup b_3^- \cup b_3^+ \cup b_4^-)$ be the disk in S such that $\partial E_4 = \partial D_4^-$. Let D be an essential disk in $B^+ - K$ that is disjoint from D_2^- and D_4^- and such that $\partial D \subset S - E_2$.

Claim 2. *If ∂D is in E_4 (hence in $E_4 - E_2$), then D is isotopic to $D_3^+ \in C_3^+$.*

Proof of Claim 2. We assume that $|D \cap D_3^+|$ is minimal up to isotopy, so $D \cap D_3^+$ consists of arc components. Let $E_3 = N(b_1^+ \cup b_1^- \cup b_2^+ \cup b_2^- \cup b_3^+)$ be the disk in S such that $\partial E_3 = \partial D_3^+$. See Figure 4.

Suppose that $D \cap D_3^+ \neq \varnothing$. Consider an outermost disk Δ of D cut off by an outermost arc of $D \cap D_3^+$. Without loss of generality, we assume that $\partial \Delta \cap S$ lies in $E_3 - E_2$. Let $\overline{D}$ be one of the disks obtained from D_3^+ by surgery along Δ such

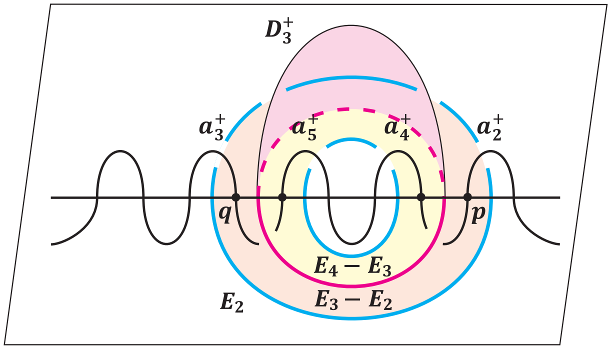

Figure 4. D_3^+ in C_3^+.

that $\partial \bar{D}$ bounds a disk $\bar{E}$ in $E_3 - E_2$. Let p be the point $a_2^+ \cap (E_3 - E_2)$ and q be the point $a_3^+ \cap (E_3 - E_2)$.

Suppose $\bar{E}$ contains p. Then the sphere $\bar{D} \cup \bar{E}$ intersects $a_2^+ \cup b_2^+$ in a single point after a slight isotopy, a contradiction. So $\bar{E}$ does not contain p, and similarly $\bar{E}$ does not contain q. Then $\bar{E}$ is an inessential disk in $E_3 - E_2 - K$, so we can reduce $|D \cap D_3^+|$, a contradiction. Hence $D \cap D_3^+ = \varnothing$. Then, reasoning as we did for Claim 1, we see that D is isotopic to D_3^+. □

Therefore if an essential disk in $B^+ - K$ is disjoint from D_2^- and D_4^- and its boundary is in $S - E_4$, then it belongs to one of $C_5^+, \dots, C_n^+$.

In general, let $E_{2i} = N(b_1^- \cup b_1^+ \cup \cdots \cup b_{2i-1}^- \cup b_{2i-1}^+ \cup b_{2i}^-)$ be the disk in S such that $\partial E_{2i} = \partial D_{2i}^-$. Let D be an essential disk in $B^+ - K$ that is disjoint from $D_2^-, D_4^-, \dots, D_{2i-2}^-$ and such that $\partial D \subset S - E_{2i-2}$.

- If $\partial D \subset E_{2i} - E_{2i-2}$, then D is isotopic to $D_{2i-1}^+ \in C_{2i-1}^+$.
- If D intersects D_{2i}^-, then D belongs to C_{2i}^+ by definition.
- If $\partial D \subset S - E_{2i}$, then D belongs to one of $C_{2i+1}^+, \dots, C_n^+$.

An inductive argument in this way leads to the conclusion that any essential disk in $B^+ - K$ belongs to one and only one C_i^+. A similar argument shows that $\{C_i^-\}$ $(i = 1, \dots, n)$ is a partition of the set of essential disks in $B^- - K$. □

The collection of disks $\{D_1^+, D_1^-, \dots, D_n^+, D_n^-\}$ spans an $(n-1)$-sphere S^{n-1} in $\mathcal{D}(P)$. There is no edge in $\mathcal{D}(P)$ connecting C_i^+ and C_i^- by definition. There exists an edge in $\mathcal{D}(P)$ connecting $C_i^\pm$ and $C_j^\pm$ for $i \neq j$, e.g., an edge between $D_i^\pm$ and $D_j^\pm$, and there exists an edge in $\mathcal{D}(P)$ connecting C_i^+ and C_j^- for $i \neq j$, e.g., an edge between D_i^+ and D_j^-. Hence if we define a map $\bar{r}$ from the set of vertices of $\mathcal{D}(P)$ to the set of vertices of S^{n-1} by

$$\bar{r}(v) = D_i^\pm \quad \text{if } v \in C_i^\pm,$$

then $\bar{r}$ extends to a continuous map from the 1-skeleton of $\mathcal{D}(P)$ to the 1-skeleton of S^{n-1}. Since higher-dimensional simplices of $\mathcal{D}(P)$ are determined by 1-simplices, the map $\bar{r}$ can be extended to a retraction $r : \mathcal{D}(P) \to S^{n-1}$. Hence $\pi_{n-1}(\mathcal{D}(P)) \neq 1$, and the topological index of P is at most n.

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