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# BRIDGE SPHERES FOR THE UNKNOT ARE TOPOLOGICALLY MINIMAL

JUNG HOON LEE

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# BRIDGE SPHERES FOR THE UNKNOT ARE TOPOLOGICALLY MINIMAL

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Topologically minimal surfaces were defined by Bachman as topological analogues of geometrically minimal surfaces, and one can associate a topological index to each topologically minimal surface. We show that an (n+1)-bridge sphere for the unknot is a topologically minimal surface of index at most n.

### 1. Introduction

Let S be a closed orientable separating surface embedded in a 3-manifold M. The structure of the set of compressing disks for S, such as how a pair of compressing disks on opposite sides of S intersects, reveals some topological properties of M. For example, if S is a minimal genus Heegaard surface of an irreducible manifold M and S has a pair of disjoint compressing disks on opposite sides, then M contains an incompressible surface [Casson and Gordon 1987].

The disk complex  $\mathcal{D}(S)$  of S is a simplicial complex defined as follows.

- Vertices of  $\mathcal{D}(S)$  are isotopy classes of compressing disks for S.
- A collection of k + 1 vertices forms a k-simplex if there are representatives for each that are pairwise disjoint.

The disk complex of an incompressible surface is empty. A surface *S* is *strongly irreducible* if *S* compresses to both sides and every compressing disk for *S* on one side intersects every compressing disk on the opposite side. So the disk complex of a strongly irreducible surface is disconnected. Extending these notions, Bachman [2010] defined topologically minimal surfaces, which can be regarded as topological analogues of (geometrically) minimal surfaces.

A surface S is *topologically minimal* if  $\mathcal{D}(S)$  is empty or  $\pi_i(\mathcal{D}(S))$  is nontrivial for some i. The *topological index* of S is 0 if  $\mathcal{D}(S)$  is empty, and the smallest n such that  $\pi_{n-1}(\mathcal{D}(S))$  is nontrivial, otherwise.

Topologically minimal surfaces share some useful properties. For example, if an irreducible manifold contains a topologically minimal surface and an incompressible

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surface, then the two surfaces can be isotoped so that any intersection loop is essential in both surfaces. There exist topologically minimal surfaces of arbitrarily high index [Bachman and Johnson 2010], and see also [Lee 2015] for possibly high index surfaces in (closed orientable surface)  $\times I$ . In this paper we consider bridge splittings of 3-manifolds, and show that the simplest bridge surfaces, bridge spheres for the unknot in  $S^3$ , are topologically minimal. The main idea is to construct a retraction from the disk complex of a bridge sphere to  $S^{n-1}$  as in [Bachman and Johnson 2010] and [Lee 2015].

**Theorem 1.1.** An (n + 1)-bridge sphere for the unknot is a topologically minimal surface of index at most n.

In particular, the topological index of a 3-bridge sphere for the unknot is two. We conjecture that the topological index of an (n+1)-bridge sphere for the unknot is n. There is another conjecture that the topological index of a genus n Heegaard surface of  $S^3$  is 2n-1. This correspondence may be due to the fact that a genus n Heegaard splitting of  $S^3$  can be obtained as a 2-fold covering of  $S^3$  branched along an unknot in (n+1)-bridge position.

### 2. Bridge splitting

For a closed 3-manifold M, a Heegaard splitting  $M = V^+ \cup_S V^-$  is a decomposition of M into two handlebodies  $V^+$  and  $V^-$  with  $\partial V^+ = \partial V^- = S$ . The surface S is called a Heegaard surface of the Heegaard splitting.

Let K be a knot in M such that  $V^{\pm} \cap K$  is a collection of n boundary-parallel arcs  $\{a_1^{\pm}, \ldots, a_n^{\pm}\}$  in  $V^{\pm}$ . Each  $a_i^{\pm}$  is called a *bridge*. The decomposition

$$(M, K) = (V^+, V^+ \cap K) \cup_S (V^-, V^- \cap K)$$

is called a *bridge splitting* of (M, K), and we say that K is in *n-bridge position* with respect to S. A bridge  $a_i^{\pm}$  cobounds a *bridge disk*  $\Delta_i^{\pm}$  with an arc in S. We can take the bridge disks  $\Delta_i^+$  ( $i=1,\ldots,n$ ) to be mutually disjoint, and similarly for  $\Delta_i^-$  ( $i=1,\ldots,n$ ). By a *bridge surface*, we mean S-K. The set of vertices of  $\mathcal{D}(S-K)$  consists of compressing disks for S-K in  $V^+-K$  and  $V^--K$ .

Two bridge surfaces S-K and S'-K are equivalent if they are isotopic in M-K. An n-bridge position of the unknot in  $S^3$  is unique for every n [Otal 1982], so for  $n \ge 2$  it is *perturbed*, i.e., there exists a pair of bridge disks  $\Delta_i^+$  and  $\Delta_j^-$  such that the arcs  $\Delta_i^+ \cap S$  and  $\Delta_j^- \cap S$  intersect at one endpoint. The uniqueness also holds for 2-bridge knots [Scharlemann and Tomova 2008] and torus knots [Ozawa 2011]. However, there are 3-bridge knots that admit multiple 3-bridge spheres [Birman 1976; Montesinos 1976].

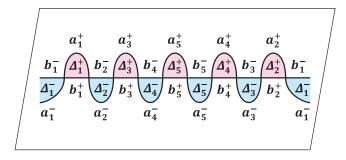


Figure 1. Bridges and bridge disks.

### 3. Proof of Theorem 1.1

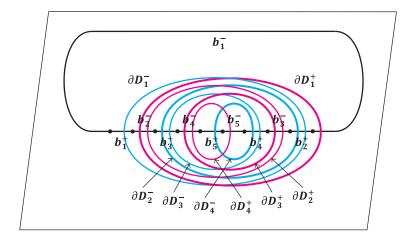
Let  $S^3$  be decomposed into two 3-balls  $B^+$  and  $B^-$  with common boundary S. Let K be an unknot in  $S^3$  which is in (n+1)-bridge position with respect to S. Then  $K \cap B^\pm$  is a collection of n+1 bridges  $a_i^\pm$  ( $i=1,\ldots,n+1$ ) in  $B^\pm$ . We assume that the bridges are arranged with  $a_1^\pm$  adjacent to  $a_1^\mp$  and  $a_2^\mp$ , with  $a_i^\pm$  adjacent to  $a_{i-1}^\mp$  and  $a_{i+1}^\mp$  for  $2 \le i \le n$ , and with  $a_{n+1}^\pm$  adjacent to  $a_n^\mp$  and  $a_{n+1}^\mp$ . Let  $\{\Delta_i^\pm\}$  be a collection of disjoint bridge disks  $\Delta_i^\pm$  for  $a_i^\pm$  with  $\Delta_i^\pm \cap S = b_i^\pm$ . We assume that int  $b_i^+ \cap \operatorname{int} b_i^- = \emptyset$  for any i and j. See Figure 1 for an example.

Let P be the (2n+2)-punctured sphere S-K. We define compressing disks  $D_i^{\pm}$   $(i=1,\ldots,n)$  for P in  $B^{\pm}-K$  as follows. Let  $D_1^+$  be a disk in  $B^+-K$  such that  $\partial D_1^+ = \partial N(b_1^+)$ , where  $N(b_1^+)$  is a neighborhood of  $b_1^+$  taken in S. Similarly, other disks are defined so as to satisfy the following.

$$\begin{split} \partial D_1^- &= \partial N(b_1^-), \\ \partial D_2^+ &= \partial N(b_1^+ \cup b_1^- \cup b_2^+), \\ \partial D_2^- &= \partial N(b_1^- \cup b_1^+ \cup b_2^-), \\ &\vdots \\ \partial D_i^+ &= \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{i-1}^+ \cup b_{i-1}^- \cup b_i^+), \\ \partial D_i^- &= \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{i-1}^- \cup b_{i-1}^+ \cup b_i^-), \\ &\vdots \\ \partial D_n^+ &= \partial N(b_1^+ \cup b_1^- \cup \cdots \cup b_{n-1}^+ \cup b_{n-1}^- \cup b_n^+), \\ \partial D_n^- &= \partial N(b_1^- \cup b_1^+ \cup \cdots \cup b_{n-1}^- \cup b_{n-1}^+ \cup b_n^-). \end{split}$$

The  $\partial D_i^{\pm}$ 's in *P* are depicted in Figure 2.

Now we define subsets  $C_i^{\pm}$  (i = 1, ..., n) of the set of vertices of  $\mathcal{D}(P)$  as



**Figure 2.**  $\partial D_i^{\pm}$  (i = 1, ..., n) in P.

follows. For odd i, let

$$\begin{split} C_i^+ &= \{D_i^+\}, \\ C_i^- &= \{\text{essential disks in } B^- - K \text{ that intersect } D_i^+ \\ &\quad \text{and are disjoint from } D_1^+, D_3^+, \dots, D_{i-2}^+\}. \end{split}$$

For even i, let

$$C_i^+ = \{\text{essential disks in } B^+ - K \text{ that intersect } D_i^-$$
 and are disjoint from  $D_2^-, D_4^-, \dots, D_{i-2}^-\},$   $C_i^- = \{D_i^-\}.$ 

Note that for all i,  $D_i^{\pm}$  belongs to  $C_i^{\pm}$ .

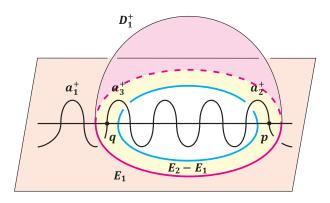
**Lemma 3.1.** The collection  $\{C_i^{\pm}\}$  (i = 1, ..., n) is a partition of the set of essential disks in  $B^{\pm} - K$ .

*Proof.* First we show that  $\{C_i^+\}$   $(i=1,\ldots,n)$  is a partition of the set of essential disks in  $B^+-K$ . We show that any essential disk in  $B^+-K$  belongs to one and only one  $C_i^+$ .

An essential disk in  $B^+ - K$  that intersects  $D_2^-$  belongs to  $C_2^+$  by definition. Let  $E_2 = N(b_1^- \cup b_1^+ \cup b_2^-)$  be the disk in S such that  $\partial E_2 = \partial D_2^-$ .

**Claim 1.** If an essential disk D in  $B^+ - K$  is disjoint from  $D_2^-$  and  $\partial D$  is in  $E_2$ , then D is isotopic to  $D_1^+ \in C_1^+$ .

*Proof of Claim 1*. We assume that D intersects  $D_1^+$  transversely and minimally, so  $D \cap D_1^+$  consists of arc components. Let  $E_1 = N(b_1^+)$  be the disk in S such that  $\partial E_1 = \partial D_1^+$ . See Figure 3. Suppose that  $D \cap D_1^+ \neq \emptyset$ . Consider an outermost



**Figure 3.**  $D_1^+$  in  $C_1^+$ .

disk  $\Delta$  of D cut off by an outermost arc of  $D \cap D_1^+$ . By the minimality of  $|D \cap D_1^+|$ ,  $\Delta$  cannot lie in the 3-ball B bounded by  $D_1^+ \cup E_1$  containing  $a_1^+$ . So  $\Delta$  lies outside of B. Let  $\overline{D}$  be one of the disks obtained from  $D_1^+$  by surgery along  $\Delta$  such that  $\partial \overline{D}$  bounds a disk  $\overline{E}$  in  $E_2 - E_1$ . Let p be the point  $a_2^+ \cap (E_2 - E_1)$  and q be the point  $a_3^+ \cap (E_2 - E_1)$ .

Suppose  $\overline{E}$  contains p. Then the sphere  $\overline{D} \cup \overline{E}$  intersects  $a_2^+ \cup b_2^+$  in a single point after a slight isotopy of int  $b_2^+$  into  $B^-$ , a contradiction. So  $\overline{E}$  does not contain p, and by similar reasoning  $\overline{E}$  does not contain q. Then  $\overline{E}$  is an inessential disk in  $E_2 - E_1 - K$ , so we can reduce  $|D \cap D_1^+|$ , a contradiction.

Hence  $D \cap D_1^+ = \emptyset$ . Let E be the disk in  $E_2$  such that  $\partial E = \partial D$ . If  $\partial E$  is in  $E_1$ , then D is isotopic to  $D_1^+$ . Suppose  $\partial E$  is in  $E_2 - E_1$ . Then E contains neither P nor P0, since otherwise P1 intersects P2 or P3 or P4 in a single point as above. So we get the conclusion that P3 is isotopic to P4.

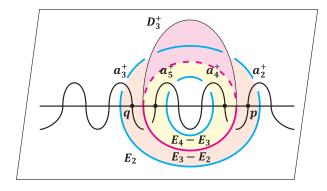
Therefore if an essential disk in  $B^+ - K$  is disjoint from  $D_2^-$  and its boundary is in  $S - E_2$ , then it belongs to one of  $C_3^+, \ldots, C_n^+$ .

An essential disk in  $B^+ - K$  that is disjoint from  $D_2^-$  and intersects  $D_4^-$  belongs to  $C_4^+$  by definition. Let  $E_4 = N(b_1^- \cup b_1^+ \cup \cdots \cup b_3^- \cup b_3^+ \cup b_4^-)$  be the disk in S such that  $\partial E_4 = \partial D_4^-$ . Let D be an essential disk in  $B^+ - K$  that is disjoint from  $D_2^-$  and  $D_4^-$  and such that  $\partial D \subset S - E_2$ .

**Claim 2.** If  $\partial D$  is in  $E_4$  (hence in  $E_4 - E_2$ ), then D is isotopic to  $D_3^+ \in C_3^+$ .

*Proof of Claim 2.* We assume that  $|D \cap D_3^+|$  is minimal up to isotopy, so  $D \cap D_3^+$  consists of arc components. Let  $E_3 = N(b_1^+ \cup b_1^- \cup b_2^+ \cup b_2^- \cup b_3^+)$  be the disk in S such that  $\partial E_3 = \partial D_3^+$ . See Figure 4.

Suppose that  $D \cap D_3^+ \neq \emptyset$ . Consider an outermost disk  $\Delta$  of D cut off by an outermost arc of  $D \cap D_3^+$ . Without loss of generality, we assume that  $\partial \Delta \cap S$  lies in  $E_3 - E_2$ . Let  $\overline{D}$  be one of the disks obtained from  $D_3^+$  by surgery along  $\Delta$  such



**Figure 4.**  $D_3^+$  in  $C_3^+$ .

that  $\partial \overline{D}$  bounds a disk  $\overline{E}$  in  $E_3 - E_2$ . Let p be the point  $a_2^+ \cap (E_3 - E_2)$  and q be the point  $a_3^+ \cap (E_3 - E_2)$ .

Suppose  $\overline{E}$  contains p. Then the sphere  $\overline{D} \cup \overline{E}$  intersects  $a_2^+ \cup b_2^+$  in a single point after a slight isotopy, a contradiction. So  $\overline{E}$  does not contain p, and similarly  $\overline{E}$  does not contain q. Then  $\overline{E}$  is an inessential disk in  $E_3 - E_2 - K$ , so we can reduce  $|D \cap D_3^+|$ , a contradiction. Hence  $D \cap D_3^+ = \emptyset$ . Then, reasoning as we did for Claim 1, we see that D is isotopic to  $D_3^+$ .

Therefore if an essential disk in  $B^+ - K$  is disjoint from  $D_2^-$  and  $D_4^-$  and its boundary is in  $S - E_4$ , then it belongs to one of  $C_5^+, \ldots, C_n^+$ .

In general, let  $E_{2i} = N(b_1^- \cup b_1^+ \cup \cdots \cup b_{2i-1}^- \cup b_{2i-1}^+ \cup b_{2i}^-)$  be the disk in S such that  $\partial E_{2i} = \partial D_{2i}^-$ . Let D be an essential disk in  $B^+ - K$  that is disjoint from  $D_2^-, D_4^-, \ldots, D_{2i-2}^-$  and such that  $\partial D \subset S - E_{2i-2}$ .

- If  $\partial D \subset E_{2i} E_{2i-2}$ , then D is isotopic to  $D_{2i-1}^+ \in C_{2i-1}^+$ .
- If D intersects  $D_{2i}^-$ , then D belongs to  $C_{2i}^+$  by definition.
- If  $\partial D \subset S E_{2i}$ , then D belongs to one of  $C_{2i+1}^+, \ldots, C_n^+$ .

An inductive argument in this way leads to the conclusion that any essential disk in  $B^+ - K$  belongs to one and only one  $C_i^+$ . A similar argument shows that  $\{C_i^-\}$  (i = 1, ..., n) is a partition of the set of essential disks in  $B^- - K$ .

The collection of disks  $\{D_1^+, D_1^-, \ldots, D_n^+, D_n^-\}$  spans an (n-1)-sphere  $S^{n-1}$  in  $\mathcal{D}(P)$ . There is no edge in  $\mathcal{D}(P)$  connecting  $C_i^+$  and  $C_i^-$  by definition. There exists an edge in  $\mathcal{D}(P)$  connecting  $C_i^\pm$  and  $C_j^\pm$  for  $i \neq j$ , e.g., an edge between  $D_i^\pm$  and  $D_j^\pm$ , and there exists an edge in  $\mathcal{D}(P)$  connecting  $C_i^+$  and  $C_j^-$  for  $i \neq j$ , e.g., an edge between  $D_i^+$  and  $D_j^-$ . Hence if we define a map  $\bar{r}$  from the set of vertices of  $\mathcal{D}(P)$  to the set of vertices of  $S^{n-1}$  by

$$\bar{r}(v) = D_i^{\pm}$$
 if  $v \in C_i^{\pm}$ ,

then  $\bar{r}$  extends to a continuous map from the 1-skeleton of  $\mathcal{D}(P)$  to the 1-skeleton of  $S^{n-1}$ . Since higher-dimensional simplices of  $\mathcal{D}(P)$  are determined by 1-simplices, the map  $\bar{r}$  can be extended to a retraction  $r:\mathcal{D}(P)\to S^{n-1}$ . Hence  $\pi_{n-1}(\mathcal{D}(P))\neq 1$ , and the topological index of P is at most n.

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