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ON THE GEOMETRIC CONSTRUCTION OF COHOMOLOGY CLASSES FOR COCOMPACT DISCRETE SUBGROUPS OF $\mathrm{SL}_n(\mathbb{R})$ AND $\mathrm{SL}_n(\mathbb{C})$

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We construct nontrivial cohomology classes for certain cocompact discrete subgroups of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$ using a geometric method. The discrete subgroups are of arithmetic nature, i.e., they arise from arithmetic subgroups of suitably chosen algebraic groups. In certain cases, we show the nonvanishing of automorphic representations as a consequence.

1. Introduction

This paper contributes to the research on cohomology of arithmetic groups by providing a nonvanishing result for the cohomology of certain families of cocompact discrete subgroups of the real Lie groups $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$. The discrete subgroups are of arithmetic nature, i.e., they arise from arithmetic subgroups of suitably chosen algebraic groups. Our approach is via a geometric argument.

The main result. Let G be a semisimple real Lie group with finite center. Denote by K a maximal compact subgroup and by Γ a torsion-free discrete subgroup of G. The action of Γ on the symmetric space $X := K\backslash G$ is smooth, proper and free, and the quotient X/Γ is a $K(\Gamma, 1)$-space. In particular, one has $H^*(\Gamma, \mathbb{C}) = H^*(X/\Gamma; \mathbb{C})$, i.e., the group cohomology of Γ with respect to the trivial Γ-module $\mathbb{C}$ equals the singular cohomology of X/Γ with complex coefficients.

A particularly interesting case is the situation where the discrete subgroup Γ is cocompact, i.e., the locally symmetric space X/Γ is compact. General results by Borel [1963] and Borel and Harder [1978] imply that such cocompact subgroups can be constructed as arithmetic subgroups of suitable algebraic groups defined over some algebraic number field. One can then use geometric methods to study the cohomology of the compact locally symmetric space X/Γ. Assuming the space

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X/Γ is orientable, one approach is to construct certain oriented, totally geodesic submanifolds (so-called geometric cycles) and show that their fundamental homology classes contribute nontrivially to the cohomology of X/Γ via Poincaré duality. Such methods have been successfully applied to discrete subgroups of several classical and exceptional Lie groups including $\mathrm{SO}(p,q)$, $\mathrm{SU}(p,q)$, $\mathrm{SU}^*(2n)$ and G_2; see [Millson and Raghunathan 1981; Schwermer and Waldner 2011; Waldner 2010]. In this work, we deal with the special linear group over the real and the complex numbers. We obtain a result of the following form (see Theorems 5.6 and 6.5).

Theorem. *Let $n \in \mathbb{N}$, $n \geq 2$.*

(1) *Let $X := \mathrm{SO}(n)\backslash \mathrm{SL}_n(\mathbb{R})$ denote the symmetric space attached to the real Lie group $\mathrm{SL}_n(\mathbb{R})$. If n is even, there exists a discrete cocompact arithmetically defined subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{R})$ such that $H^k(X/\Gamma;\mathbb{C})$ contains nontrivial cohomology classes for all k of the form*

$$k = pq \quad \text{and} \quad k = \tfrac{1}{2}(p^2+q^2+n)-1,$$

where p and q are positive integers with $p+q=n$, and, if $n \neq 2$, for

$$k = \tfrac{1}{4}(n^2+2n) \quad \text{and} \quad k = \tfrac{1}{4}n^2-1.$$

(2) *Let $X := \mathrm{SU}(n)\backslash \mathrm{SL}_n(\mathbb{C})$ denote the symmetric space attached to the real Lie group $\mathrm{SL}_n(\mathbb{C})$. There exists a discrete cocompact arithmetically defined subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{C})$ such that $H^k(X/\Gamma;\mathbb{C})$ contains nontrivial cohomology classes for all k of the form*

$$k = 2pq \quad \text{and} \quad k = p^2+q^2-1,$$

where p and q are positive integers with $p+q=n$, and for

$$k = \tfrac{1}{2}(n^2-n) \quad \text{and} \quad k = \tfrac{1}{2}(n^2+n)-1.$$

Moreover, if n is even and $n \neq 2$, there are nontrivial cohomology classes in the degrees

$$k = \tfrac{1}{2}(n^2+n), \quad k = \tfrac{1}{2}(n^2-n)-1, \quad k = \tfrac{1}{2}n^2-1 \quad \text{and} \quad k = \tfrac{1}{2}n^2.$$

When $H^*(X/\Gamma,\mathbb{C})$ is interpreted as the cohomology of the de Rham complex $\Omega^*(X/\Gamma,\mathbb{C})$, the constructed classes are not represented by $\mathrm{SL}_n(\mathbb{R})$- or $\mathrm{SL}_n(\mathbb{C})$-invariant differential forms on X.

A geometric method. The geometric method we are using to obtain our result was developed by Millson and Raghunathan [1981] and is based on an earlier result of Millson [1976] about the nonvanishing of the first Betti number of certain compact hyperbolic manifolds.

Their approach applies to the situation where the Lie group G is the group of real points of a reductive algebraic $\mathbb{Q}$-group and Γ is a cocompact torsion-free arithmetic subgroup of this algebraic group. Under the assumption that the space X/Γ is orientable, they consider so-called geometric cycles, orientable totally geodesic submanifolds of X/Γ. Then the approach of Millson and Raghunathan is based on finding two such geometric cycles of complementary dimension in X/Γ that intersect transversally and with positive multiplicity in all points of intersection. Under this assumption, the fundamental classes of the two submanifolds have nontrivial intersection number, and hence they contribute nontrivially to the cohomology of X/Γ. In 1993, Rohlfs and Schwermer found a way to generalize the method in such a way that it also applies to nontransversal intersections, by using the theory of so-called excess bundles. Their work involves the investigation of deep orientability questions.

As the proof of our result is heavily based on the method of Rohlfs and Schwermer, we give an overview of the relevant notions and their main theorem in Section 3. Then, Section 4 is devoted to introducing the framework of algebraic groups in which the construction of geometric cycles and the associated cohomology classes is carried out: for our groups of interest, $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$, the algebraic group to start with is the special unitary group

$$\boldsymbol{G} := \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$$

defined over an algebraic number field F, where D is a division algebra with involution σ and h denotes a σ-hermitian form on D^m. Under certain assumptions, the associated real Lie group $\boldsymbol{G}_\infty$ is isomorphic to $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{C})$ up to compact factors and can be used for the construction of cocompact discrete subgroups. In this setting, the construction of geometric cycles and the application of the method of Rohlfs and Schwermer to obtain nontrivial cohomology classes is carried out in Sections 5 and 6, for the real and complex case, respectively. The main results are stated as Theorems 5.6 and 6.5.

Automorphic representations. Nonvanishing results for the cohomology of cocompact discrete subgroups can be applied to the theory of automorphic representations using a well-known result of Matsushima that allows one to interpret the cohomology of X/Γ in terms of the relative Lie algebra cohomology of irreducible unitary representations of G. Thus, we have devoted Section 7 to the study of representations with nontrivial $(\mathfrak{g}, K)$-cohomology occurring in Matsushima's formula for the group $G = \mathrm{SL}_n(\mathbb{C})$. Making explicit general results of Enright [1979] and Delorme [1979] for simply connected complex Lie groups for the case of $\mathrm{SL}_n(\mathbb{C})$, we obtain a complete classification of the equivalence classes of irreducible unitary representations with nontrivial $(\mathfrak{g}, K)$-cohomology. By comparing the occurring degrees in which X/Γ may possibly have nontrivial cohomology with those detected by special cycles, we can identify specific automorphic representations of G with respect to Γ for small values of n.

2. Notation

• For an algebraic number field k, we let $V = V(k)$ and $V_\infty = V_\infty(k)$ denote its set of places and archimedean places, respectively. For a place $v \in V$, we denote by k_v the completion of k at v.

• All algebraic groups are assumed to be linear, i.e., they can be considered as smooth affine algebraic group schemes. We denote algebraic groups by bold letters $(\boldsymbol{G}, \boldsymbol{H}, \dots)$. For an algebraic group $\boldsymbol{G}$ defined over a number field k, we set $\boldsymbol{G}_\infty := \prod_{v \in V_\infty} \boldsymbol{G}(k_v)$.

• Lie groups are denoted by standard Roman letters $(G, H, \dots)$. Whenever we speak of a semisimple Lie group, we assume that it has finite center and finitely many connected components.[1] We use the notion of a *reductive* Lie group as in [Knapp 1996, Section VII.2].

• For a semisimple Lie group G, we denote by $\widehat{G}$ the unitary dual of G, that is, the set of unitary equivalence classes of irreducible unitary representations.

• Lie algebras are denoted by small German letters $(\mathfrak{g}, \mathfrak{h}, \dots)$ and can be real or complex depending on context. If $\mathfrak{g}$ is a real Lie algebra, we will denote by $\mathfrak{g}_\mathbb{C}$ its complexification and if $\mathfrak{g}$ is complex we write $\mathfrak{g}_\mathbb{R}$ for the real Lie algebra underlying $\mathfrak{g}$. In general, we denote the Lie algebra of a Lie group G by $\mathfrak{g}$ and consider it as a real or complex Lie algebra depending on whether G is a real or a complex group.

• Let R be a ring and let $n \in \mathbb{N}$. We denote by I_n the $n \times n$ unity matrix in $M_n(R)$ and by $I_{p,q}$ the matrix $\operatorname{diag}(I_p, -I_q) \in M_n(R)$, for $p + q = n$. For even n, we set $J_n := \left(\begin{smallmatrix} 0 & -I_{n/2} \\ I_{n/2} & 0 \end{smallmatrix}\right)$.

3. A geometric method

This section gives a brief summary of the method of Rohlfs and Schwermer [1993] for the geometric construction of nontrivial cohomology classes.

3.1. ***Special cycles.*** Let $\boldsymbol{G}$ be a connected reductive algebraic group defined over $\mathbb{Q}$ and write G for its group of real points $\boldsymbol{G}(\mathbb{R})$. Then G is a real reductive Lie group with a maximal compact subgroup $K \subset G$ and we can form the associated symmetric space $X := K \backslash G$. Let now $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup. Then

[1] This is to ensure that the semisimple groups are also reductive in the sense of [Knapp 1996]. Lie groups arising as the groups of real or complex points of semisimple algebraic groups will always have this property.

Γ is a discrete subgroup of G and it acts on the symmetric space X by right translations. This action is smooth, proper and free, and the quotient X/Γ is a Riemannian locally symmetric space.

Let $\boldsymbol{\mu}$ be a $\mathbb{Q}$-rational automorphism of $\boldsymbol{G}$ and assume that K and Γ are invariant under $\boldsymbol{\mu}$. Then the fixed point group $\mathrm{Fix}(\boldsymbol{\mu}, \boldsymbol{G})$ is a reductive subgroup of $\boldsymbol{G}$ and we can associate with it the locally symmetric space $C(\mu, \Gamma) := \mathrm{Fix}(\boldsymbol{\mu}, K)\backslash \mathrm{Fix}(\boldsymbol{\mu}, \boldsymbol{G})(\mathbb{R})/\mathrm{Fix}(\boldsymbol{\mu}, \Gamma)$. This is a connected, totally geodesic submanifold of X/Γ and is called the *special cycle* associated with $\boldsymbol{\mu}$.

Let us now assume that Θ is the group generated by two commuting $\mathbb{Q}$-rational automorphisms $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ of $\boldsymbol{G}$ of finite order and that K and Γ are Θ-invariant.[2] In general, the locally symmetric space X/Γ and its special cycles need not be orientable. However, it was shown by Rohlfs and Schwermer [1993] that by passing to a suitable subgroup of finite index in Γ, one can always assume that the manifolds X/Γ, $C(\boldsymbol{\tau}_1, \Gamma)$, $C(\boldsymbol{\tau}_2, \Gamma)$ and the (finitely many) connected components of their intersection are orientable.

Let us denote by $[C(\boldsymbol{\tau}_i, \Gamma)]$ the fundamental homology class of $C(\boldsymbol{\tau}_i, \Gamma)$ in $H_*(C(\boldsymbol{\tau}_i, \Gamma))$ and for simplicity also its image in $H_*(X/\Gamma)$, for $i \in \{1, 2\}$. If we assume in addition that the two cycles are of complementary dimension in X/Γ, we can look at their intersection number $[C(\boldsymbol{\tau}_1, \Gamma)][C(\boldsymbol{\tau}_2, \Gamma)]$.

Since these submanifolds need not necessarily intersect transversally, the determination of their intersection number is a complicated issue. It involves the computation of Euler numbers of a certain *excess bundle*. Under certain assumptions connected to deep orientability questions for the involved manifolds, Rohlfs and Schwermer have come up with a nonvanishing result for the intersection number:

Theorem 3.2 [Rohlfs and Schwermer 1993, Theorem 4.11]. *Let $\boldsymbol{G}$ be a reductive algebraic $\mathbb{Q}$-group, let $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ be $\mathbb{Q}$-rational automorphisms of $\boldsymbol{G}$ of finite order, and let Γ be a torsion-free, $\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable, cocompact arithmetic subgroup of $\boldsymbol{G}$ such that X/Γ, $C(\boldsymbol{\tau}_1, \Gamma)$, $C(\boldsymbol{\tau}_2, \Gamma)$ and all connected components of their intersection are orientable. Suppose that the associated cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension. Assume that*

(i) *the real Lie groups $\boldsymbol{G}(\mathbb{R})$, $\mathrm{Fix}(\boldsymbol{\tau}_1, \boldsymbol{G})(\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \boldsymbol{G})(\mathbb{R})$ act orientation-preservingly on X, $X(\boldsymbol{\tau}_1)$ and $X(\boldsymbol{\tau}_2)$, respectively, and*

(ii) *the group $\mathrm{Fix}(\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle, \boldsymbol{G})(\mathbb{R})$ is compact.*

Then there exists a $\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable normal subgroup $\Gamma' \subset \Gamma$ of finite index such that

$$[C(\boldsymbol{\tau}_1, \Gamma')][C(\boldsymbol{\tau}_2, \Gamma')] \neq 0.$$

---

[2] Such a choice of K and Γ is always possible without loss of generality. For K, this follows from [Helgason 1978, Theorem 13.5]. For Γ, set $\Gamma' := \bigcap_{\theta\in\Theta}\theta(\Gamma)$. Then Γ' is of finite index in Γ and stable under Θ since the elements of Θ are automorphisms of finite order.

Remark. Condition (i) is quite restricting and we will see below natural choices for $\boldsymbol{G}$, $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ where it is not met. Note that the condition is satisfied if $\boldsymbol{G}(\mathbb{R})$, $\mathrm{Fix}(\boldsymbol{\tau}_1, \boldsymbol{G})(\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \boldsymbol{G})(\mathbb{R})$ are connected.

Clearly, the nonvanishing of the intersection number implies the nonvanishing of the homology classes $[C(\boldsymbol{\tau}_i, \Gamma')]$ in $H_*(X/\Gamma', \mathbb{C})$ and of the respective cohomology classes obtained via Poincaré duality, for $i \in \{1, 2\}$.

For a compact quotient X/Γ, it is well-known that there exists an injective homomorphism $\beta_\Gamma^* : H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma, \mathbb{C})$, where X_u denotes the compact dual symmetric space of X. When interpreting $H^*(X/\Gamma, \mathbb{C})$ in terms of de Rham cohomology, the classes in the image of this map can be identified with the G-invariant differential forms on X. It was shown by Millson and Raghunathan [1981] that under certain conditions the classes constructed with Theorem 3.2 are *new* in the sense that they do not lie in the image of β_Γ:

Theorem 3.3. *Let $\boldsymbol{G}$, $\boldsymbol{\tau}_1$, $\boldsymbol{\tau}_2$ and Γ satisfy the assumptions of Theorem 3.2 and suppose moreover that $\boldsymbol{\tau}_1$ and $\boldsymbol{\tau}_2$ are of order two. Then there exists a $\langle \boldsymbol{\tau}_1, \boldsymbol{\tau}_2 \rangle$-stable subgroup Γ'' of Γ' of finite index such that the nontrivial cohomology classes defined by $[C(\boldsymbol{\tau}_1, \Gamma'')]$ and $[C(\boldsymbol{\tau}_2, \Gamma'')]$ via Poincaré duality are not in the image of $\beta_{\Gamma''}^*$.*

Example 3.4. Consider the real Lie group $G = \mathrm{SO}(p, q)$ with maximal compact subgroup $K = S(O(p) \times O(q))$. The group K (and hence also G) is not connected but has two connected components that are distinguished by the determinant of the upper left $(p \times p)$-block. One can show that the action of G on the quotient $X := K \backslash G$ by left translations is orientation-preserving if and only if $n = p + q$ is even. Note that G is the fixed point group of the involution $x \mapsto I_{p,q}(x^t)^{-1} I_{p,q}$ in the connected real Lie group $\mathrm{SL}_n(\mathbb{R})$. Hence, for odd n, this is an example of a fixed point group that does not meet the orientability condition (i) in Theorem 3.2.

A similar argument also applies to the real Lie group $G = S(\mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R}))$ with maximal compact subgroup $K = S(O(p) \times O(q))$.

4. The setup: the construction of discrete cocompact subgroups of $\mathrm{SL}_n(\mathbb{R})$ and $\mathrm{SL}_n(\mathbb{C})$

In this section we will see how to construct cocompact discrete subgroups of $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{C})$ using an arithmetic method based on the compactness criterion by Borel and Harish-Chandra. The starting point is the special unitary group over a division algebra.

Let E be an algebraic number field and D a central division algebra over E of degree d endowed with an involution σ of the first or second kind. Recall that if σ is of the second kind, there exists a subfield F of E of index 2 such that $\sigma|_F = \mathrm{id}$ and $\sigma|_E = \iota$, where ι is the nontrivial Galois automorphism of E over F. For simplicity of notation, we set $F := E$ and $\iota := \mathrm{id}$ in case σ is an involution of the first kind.

Let m be a natural number and let h be a σ-hermitian (or σ-skew-hermitian) form on D^m. Then the *special unitary group of rank m over D* is defined as

$$\mathrm{SU}_m(h, D, \sigma) := \{x \in \mathrm{SL}_m(D) \mid h(xv, xw) = h(v, w) \text{ for all } v, w \in D^m\},$$

where $\mathrm{SL}_m(D)$ denotes the group of matrices in $M_m(D)$ with reduced norm 1.

It is well-known that there exists a simply connected semisimple algebraic group defined over F, whose F-rational points coincide with $\mathrm{SU}_m(h, D, \sigma)$. We denote this group by $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$. Indeed, on the algebraic F-group $\mathrm{Res}_{E/F}\,\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$ we can define an F-rational morphism $\boldsymbol{\psi}$ that is given on the F-rational points by $\psi(x) = H^{-1}\sigma(x^t)^{-1}H$, where H is the matrix of h with respect to a chosen basis, and we have $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma}) = \mathrm{Fix}(\boldsymbol{\psi}, \mathrm{Res}_{E/F}\,\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))$.

Being an F-rational algebraic group, we can look at the real Lie group of F_v-rational points of $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$ for any archimedean place $v \in V_\infty(F)$. The nature of this real Lie group depends on the properties of the place v and the splitting behavior of D at v. Recall that for a quadratic extension E/F a place $v \in V(F)$ is said to be *decomposed* in E if there are exactly two places $w \in V(E)$ such that $w \mid v$, and *nondecomposed* otherwise. We denote by ρ the involution on $M_m(D)$ given by $\rho(x) := H^{-1}\sigma(x)^t H$. The following result can be obtained as an application of results from the theory of algebras with involutions and some easy computations.

Proposition 4.1. (1) *Let σ be an involution of the first kind on D and assume that D splits at all real places of E. Let $w \in V_\infty(E)$ be an archimedean place of E. Then there are the following possibilities*:

- *If w is a complex place, we have*

$$\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(E_w) \cong \begin{cases} \mathrm{SO}(n, \mathbb{C}) & \text{if } \rho \text{ is of orthogonal type,} \\ \mathrm{Sp}(n, \mathbb{C}) & \text{if } \rho \text{ is of symplectic type.} \end{cases}$$

- *If w is a real place, we have*

$$\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(E_w) \cong \begin{cases} \mathrm{SO}(p, q) & \text{if } \rho \text{ is of orthogonal type,} \\ \mathrm{Sp}(n, \mathbb{R}) & \text{if } \rho \text{ is of symplectic type,} \end{cases}$$

  for suitable nonnegative integers p and q with $p + q = n$.

(2) *Let σ be an involution of the second kind on D and consider an archimedean place $v \in V_\infty(F)$. Then there are the following possibilities*:

- *If v is a complex place, we have* $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(F_v) \cong \mathrm{SL}_n(\mathbb{C})$.
- *If v is a nondecomposed real place, we have* $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(F_v) \cong \mathrm{SU}(p, q)$ *for nonnegative integers p, q with $p + q = n$.*

- *If v is a decomposed real place and $w_1 \mid v$ and $w_2 \mid v$ are the real places of E lying above v, we have*

$$\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(F_v) \cong \begin{cases} \mathrm{SL}_n(\mathbb{R}) & \text{if } D \text{ splits at } w_1 \text{ and } w_2, \\ \mathrm{SL}_{n/2}(\mathbb{H}) & \text{if } D \text{ ramifies at } w_1 \text{ and } w_2. \end{cases}$$

Using this result, we can now find certain conditions on the number fields E and F, the involution σ and the division algebra D such that arithmetic subgroups of $\mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$ give rise to cocompact discrete subgroups of either $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{SL}_n(\mathbb{C})$. An involution σ of the second kind on D is called *definite* if for every real nondecomposed place v of F we have an isomorphism $(D, \sigma)_v \cong (M_d(\mathbb{C}), {}^*)$, where $x \mapsto x^* = \bar{x}^t$ denotes the conjugate-transpose involution on $M_d(\mathbb{C})$. In general, an involution of the second kind need not be definite. However, if there exists an involution of the second kind on D, there is also a definite one (see [Scharlau 1985, Chapter 10, Remark 6.11]).

Theorem 4.2. *Let ℓ be an archimedean local field and let $n \in \mathbb{N}$ be fixed.*

(1) *If $\ell = \mathbb{R}$, let F be a totally real number field with $[F : \mathbb{Q}] \geq 2$, let E/F be a quadratic extension such that there is exactly one place $v \in V_\infty(F)$ that is decomposed in E, and let D be a division algebra of degree $d \mid n$ over E that splits at the places w_1 and w_2 of E lying above v. Moreover, assume that there is a definite involution σ of the second kind on D.*

(2) *If $\ell = \mathbb{C}$, let F be an algebraic number field with $[F : \mathbb{Q}] \geq 3$ that has exactly one complex place v, and let E/F be a quadratic extension such that all real places of F are nondecomposed in E. Let D be a division algebra of degree $d \mid n$ over E that admits a definite involution σ of the second kind.*

Let $m \in \mathbb{N}$ be such that $dm = n$. Then one can choose a σ-hermitian form h on D^m such that any arithmetic subgroup $\Gamma \subset \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})(F)$ gives rise to a discrete cocompact subgroup of $\mathrm{SL}_n(\ell)$.

Proof. Choose a hermitian form h in such a way that the matrix of h at each nondecomposed place v' of F has only positive eigenvalues (take the trivial hermitian form, for example). Set $\boldsymbol{G}' := \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$. Then, using Proposition 4.1 and the fact that σ is definite, it follows that

$$\boldsymbol{G}'(F_{v'}) \cong \mathrm{SU}(dm, 0) = \mathrm{SU}(n)$$

for all nondecomposed real places of F; in particular, these groups are compact. By our choice of E and F there is at least one such nondecomposed place and thus the group $\boldsymbol{G}'$ is anisotropic over F. On the other hand, at the decomposed place v we have $\boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\ell)$ by Proposition 4.1 and the fact that D is split at w_1 and

w_2 if $\ell = \mathbb{R}$. In particular, we have

$$\boldsymbol{G}'_\infty := \prod_{v \in V_\infty(F)} \boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\ell) \times \prod_{\substack{v' \in V_\infty(F) \\ \text{real, nondecomposed}}} \mathrm{SU}(n).$$

Let $\Gamma \subset \boldsymbol{G}'(F)$ be an arbitrary arithmetic subgroup of $\boldsymbol{G}'$. The image of Γ in $\boldsymbol{G}'_\infty$ under the diagonal embedding (still denoted by Γ) is a discrete subgroup. Since $\boldsymbol{G}'$ is semisimple and anisotropic over F, it follows from a well-know compactness criterion due to Borel and Harish-Chandra [1962] and Mostow and Tamagawa [1962] that the quotient $\boldsymbol{G}'_\infty/\Gamma$ is compact. Moreover, the image of Γ under the projection onto the noncompact factor of $\boldsymbol{G}'_\infty$ is a discrete cocompact subgroup of $\boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\ell)$. □

5. Geometric cycles for $\mathrm{SL}_n(\mathbb{R})$

Let F be a totally real number field of degree $r \geq 2$, and let E/F be a quadratic extension such that there is exactly one archimedean place $v \in V_\infty(F)$ that is decomposed in E and all other archimedean places of F are nondecomposed. Let us denote the nontrivial Galois automorphism of E/F by ι. Let D be a central division algebra of degree d over E with a definite involution σ of the second kind.

In this section we will restrict to the cases where D is either the field E itself (with $\sigma = \iota$) or a quaternion division algebra over E that splits at the places w_1 and w_2 of E lying above v and admits a definite involution σ of the second kind. In the latter case, we may assume by a theorem of Albert (see [Knus et al. 1998, Proposition 2.22]) that

$$D = Q(a, b \mid F) \otimes_F E = Q(a, b \mid E)$$

for some $a, b \in F^\times$ and that $\sigma = \tau_{c,0} \otimes \iota$, where $\tau_{c,0}$ denotes the conjugation on the quaternion algebra $Q(a, b \mid F)$.

Let $m \in \mathbb{N}$ be arbitrary and set $n := dm$. Then D, E, F, σ and m satisfy the conditions of Theorem 4.2(1) and we can find a hermitian form h such that any arithmetic subgroup of $\boldsymbol{G}' := \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$ gives rise to a discrete cocompact subgroup of $\mathrm{SL}_n(\mathbb{R})$. For technical reasons, we assume that the matrix H of h is a diagonal matrix in $M_m(F)$ that is positive definite under the embedding corresponding to the decomposed place v. If $D = E$ and if m is even, we assume in addition that H is a symplectic matrix, that is, it commutes with the matrix J_m.[3]

In order to construct special cycles for $\mathrm{SL}_n(\mathbb{R})$ we will now define suitable morphisms of finite order. To do this, we need a preparatory lemma. Recall that,

[3] Such a choice of H clearly exists: take the identity matrix, for example.

for each basis element $e \neq 1$ of a quaternion algebra, there exists an orthogonal involution τ_e that sends e to $-e$ and fixes all other basis elements.

Lemma 5.1. *Let E be a number field and $Q := Q(a, b \mid E)$ a quaternion algebra that splits at a real place w of E. Then there exist orthogonal involutions τ and $\tau_{(1,-1)}$ and an isomorphism $Q(a, b \mid E_w) \to M_2(\mathbb{R})$ such that*

$$(Q(a, b \mid E_w), \tau \otimes \mathrm{id}) \cong (M_2(\mathbb{R}),\ x \mapsto x^t),$$

and

$$(Q(a, b \mid E_w), \tau_{(1,-1)} \otimes \mathrm{id}) \cong \big(M_2(\mathbb{R}),\ x \mapsto \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) x^t \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)\big).$$

Proof. Let a_w and b_w denote the images of a and b under the embedding corresponding to w. Since D splits at w, exactly one of the elements a_w, b_w and $-a_w b_w$ is negative, i.e., there exists exactly one basis element $e_0 \in \{i, j, k\}$ such that e_0^2 is negative. Denote the remaining nontrivial basis elements by e_1 and e_2. We set $\tau := \tau_{e_0}$ and $\tau_{(1,-1)} := \tau_{e_1}$.

Now let φ be the $\mathbb{R}$-linear map given on the basis of $Q(a, b \mid E_w)$ by $1 \mapsto I_2$ and

$$e_0 \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{-e_0^2} \\ -\sqrt{-e_0^2} & 0 \end{pmatrix}, \quad e_1 \otimes 1 \mapsto \begin{pmatrix} 0 & \sqrt{e_1^2} \\ \sqrt{e_1^2} & 0 \end{pmatrix}, \quad e_2 \otimes 1 \mapsto \begin{pmatrix} \sqrt{e_2^2} & 0 \\ 0 & -\sqrt{e_2^2} \end{pmatrix}.$$

Then $\varphi : Q(a, b \mid E_w) \to M_2(\mathbb{R})$ is a well-defined isomorphism under which $\tau_{e_0} \otimes \mathrm{id}$ goes over to $x \mapsto x^t$ and $\tau_{e_1} \otimes \mathrm{id}$ goes over to $x \mapsto \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) x^t \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. □

Remark. Note that the two orthogonal involutions τ and $\tau_{(1,-1)}$ commute and that we have $\tau \circ \tau_{(1,-1)} = \tau_{(1,-1)} \circ \tau = \mathrm{Int}(e_2)$. Moreover, τ commutes with the conjugation τ_c of Q and we have $\tau \circ \tau_c = \mathrm{Int}(e_0)$.

Let us return to our specific choice of a division algebra $D = Q(a, b \mid F) \otimes_F E$ as described above. Applying Lemma 5.1 to D, we get the existence of orthogonal involutions τ and $\tau_{(1,-1)}$ that are mapped to $x \mapsto x^t$ and $x \mapsto \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right) x^t \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$ under a suitable splitting isomorphism at the real place w_1.

Using these involutions, we can now define the following automorphisms of order two on $\mathrm{SL}_m(D)$:

$$\theta : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D),$$
$$\theta(x) = \begin{cases} H^{-1}(x^t)^{-1}H & \text{if } D = E, \\ H^{-1}(\tau(x)^t)^{-1}H & \text{if } D = Q(a, b \mid E), \end{cases}$$

$$\mu : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D),$$
$$\mu(x) = \begin{cases} H^{-1}J_m(x^t)^{-1}J_m^{-1}H & \text{if } D = E \text{ and } m > 2 \text{ even}, \\ H^{-1}(\tau_c(x)^t)^{-1}H & \text{if } D = Q(a, b \mid E) \text{ and } m > 1. \end{cases}$$

Moreover, for certain positive integers p and q such that $p+q=n$, we define a family of automorphisms $\nu_{p,q} : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D)$ by

$$\nu_{p,q}(x) = \begin{cases} H^{-1} I_{p,q}((x)^t)^{-1} I_{p,q} H & \text{if } D = E, \\ H^{-1} I_{p/2,q/2}(\tau(x)^t)^{-1} I_{p/2,q/2} H & \text{if } D = Q(a, b \mid E) \text{ and } p, q \text{ even}, \\ H^{-1}(\tau_{(1,-1)}(x)^t)^{-1} H & \text{if } D = Q(a, b \mid E) \\ & \qquad \text{and } p = q = n/2 \text{ is odd.} \end{cases}$$

Note that θ commutes with any of the other automorphisms.

5.2. To avoid case distinctions, we put in place the following general assumptions. Whenever we deal with the maps $\nu_{p,q}$ it should be understood that the parameters p and q are nonzero natural numbers satisfying $p+q=n$. Moreover, if D is a quaternion algebra, we assume that both p and q are even or that $p=q=n/2$. Furthermore, statements involving the map μ are only applicable when n is even and $n>2$.

The maps θ, $\nu_{p,q}$ and μ are basically built out of E-linear maps (involutions of the first kind on $\mathrm{SL}_m(D)$) and the group inversion, so they define E-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$ on the algebraic E-group $\mathbf{SL_m}(\boldsymbol{D})$ and F-rational automorphisms $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$, $\mathrm{Res}_{E/F}\,\boldsymbol{\nu_{p,q}}$ and $\mathrm{Res}_{E/F}\,\boldsymbol{\mu}$ on $\mathrm{Res}_{E/F}\,\mathbf{SL_m}(\boldsymbol{D})$ by restriction of scalars. A straightforward computation shows that these maps commute with the morphism $\boldsymbol{\psi}$ whose fixed points in $\mathrm{Res}_{E/F}\,\mathbf{SL_m}(\boldsymbol{D})$ define the group $\boldsymbol{G}'$ and can thus be restricted to $\boldsymbol{G}'$.

The fixed points of these morphisms define algebraic subgroups of $\boldsymbol{G}'$ whose F_v-rational points are certain Lie subgroups of $\boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\mathbb{R})$. We will now determine these subgroups. Recall the definition $\mathrm{GL}_r^{(1)}(\mathbb{C}) := \{g \in \mathrm{GL}_r(\mathbb{C}),\ |\det(g)| = 1\}$ for a natural number r.

Proposition 5.3. *Let F be a totally real number field. For $\boldsymbol{G}'$ defined as above, we have $\boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\mathbb{R})$. The fixed points of the morphisms $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$, $\mathrm{Res}_{E/F}\,\boldsymbol{\nu_{p,q}}$, $\mathrm{Res}_{E/F}(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta})$, $\mathrm{Res}_{E/F}\,\boldsymbol{\mu}$ and $\mathrm{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ define the following subgroups of $\mathrm{SL}_n(\mathbb{R})$:*

$$\begin{aligned} \mathrm{Fix}(\mathrm{Res}_{E/F}\,\boldsymbol{\theta}, \boldsymbol{G}')(F_v) &\cong \mathrm{SO}(n), \\ \mathrm{Fix}(\mathrm{Res}_{E/F}\,\boldsymbol{\nu_{p,q}}, \boldsymbol{G}')(F_v) &\cong \mathrm{SO}(p,q), \\ \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}), \boldsymbol{G}')(F_v) &\cong S(\mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R})), \\ \mathrm{Fix}(\mathrm{Res}_{E/F}\,\boldsymbol{\mu}, \boldsymbol{G}')(F_v) &\cong \mathrm{Sp}(n,\mathbb{R}), \\ \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta}), \boldsymbol{G}')(F_v) &\cong \mathrm{GL}_{n/2}^{(1)}(\mathbb{C}). \end{aligned}$$

In particular, $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$ induces a Cartan involution on $\mathrm{SL}_n(\mathbb{R})$.

Proof. We start with a general observation. Let $\boldsymbol{\varphi}$ denote an E-rational morphism of $\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$ such that $\operatorname{Res}_{E/F}\boldsymbol{\varphi}$ commutes with $\boldsymbol{\psi}$. Then $\boldsymbol{\varphi}$ can be restricted to $\boldsymbol{G}'$ and we have

$$\operatorname{Fix}(\operatorname{Res}_{E/F}\boldsymbol{\varphi},\boldsymbol{G}')=\operatorname{Fix}(\operatorname{Res}_{E/F}\boldsymbol{\varphi},\operatorname{Res}_{E/F}\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))\cap\boldsymbol{G}'$$

as a subgroup of $\operatorname{Res}_{E/F}\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$. At the F_v-rational points, there exists an isomorphism

$$\begin{aligned}\operatorname{Res}_{E/F}(\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(F_v)&\cong\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})(E\otimes_F F_v)\\&\cong\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})(E_{w_1}\oplus E_{w_2})\cong\mathrm{SL}_n(\mathbb{R})\times\mathrm{SL}_n(\mathbb{R}).\end{aligned}$$

Moreover,

$$\begin{aligned}\operatorname{Fix}(\operatorname{Res}_{E/F}\boldsymbol{\varphi},\operatorname{Res}_{E/F}\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(F_v)&\\&=\operatorname{Res}_{E/F}\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(F_v)\\&=\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})\times\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_2})\\&\subset\mathrm{SL}_n(\mathbb{R})\times\mathrm{SL}_n(\mathbb{R}).\end{aligned}$$

On the other hand, the defining condition of $\boldsymbol{G}'$ identifies the two copies of $\mathrm{SL}_n(\mathbb{R})$ (see Proposition 4.1). Thus we can restrict to one of the components (we choose the first one without loss of generality) and get

$$\begin{aligned}\operatorname{Fix}(\operatorname{Res}_{E/F}\boldsymbol{\varphi},\boldsymbol{G}')(F_v)&=\operatorname{Fix}(\operatorname{Res}_{E/F}\boldsymbol{\varphi},\operatorname{Res}_{E/F}\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(F_v)\cap\boldsymbol{G}'(F_v)\\&\cong\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})\subset\mathrm{SL}_n(\mathbb{R}).\end{aligned}$$

Let us now specify $\boldsymbol{\varphi}$ to be one of the above morphisms. For $\boldsymbol{\varphi}=\boldsymbol{\theta}$ or $\boldsymbol{\varphi}=\boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}}$, the group $\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))$ is a special unitary group with respect to a hermitian form and an orthogonal involution. Therefore, by Proposition 4.1, we have $\operatorname{Fix}(\boldsymbol{\varphi},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})\cong\mathrm{SO}(p',q')$ for suitable $p',q'\in\mathbb{N}$. Since H is positive definite at the place v, it does not influence the signature (p',q'). When $D=E$ it is clear from the definitions of θ and $\nu_{p,q}$ that the signature comes from the matrices $I_{p,q}$. When $D=Q(a,b\mid E)$, we can conclude from Lemma 5.1 that, under a suitable splitting at the place w_1, the involution τ^t is mapped to $x\mapsto x^t$ on $M_n(\mathbb{R})$ and $\tau^t_{(1,-1)}$ is mapped to $x\mapsto\operatorname{Int}(I_{n/2,n/2})(x^t)$. Moreover, the matrices $I_{p/2,q/2}$ are mapped to $I_{p,q}$ at the place w_1. Therefore, for both choices of D, we get

$$\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\theta}),\boldsymbol{G}')(F_v)=\operatorname{Fix}(\boldsymbol{\theta},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})\cong\mathrm{SO}(n)$$

and

$$\operatorname{Fix}(\operatorname{Res}_{E/F}(\boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}}),\boldsymbol{G}')(F_v)=\operatorname{Fix}(\boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}},\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})\cong\mathrm{SO}(p,q).$$

In particular, $\mathrm{Res}_{E/F}\,\boldsymbol{\theta}$ induces a Cartan involution on $\mathrm{SL}_n(\mathbb{R})$, as the group of F_v-rational points of its fixed point group is isomorphic to $\mathrm{SO}(n)$, a maximal compact subgroup of $\mathrm{SL}_n(\mathbb{R})$.

For $\boldsymbol{\varphi} = \boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}$ one can easily see from the definition of $\nu_{p,q}$ that

$$\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta} = \begin{cases} \mathrm{Int}(I_{p,q}) & \text{if } D = E, \\ \mathrm{Int}(I_{p/2,q/2}) & \text{if } D = Q(a, b \mid E) \text{ and } p, q \text{ even}, \\ \mathrm{Int}(\mathrm{diag}(e_2, \ldots, e_2)) & \text{if } D = Q(a, b \mid E) \text{ and } p = q = n/2. \end{cases}$$

Here, we use the notation of Lemma 5.1 and its remark for the statement in the last line. Under a suitable splitting isomorphism at the place w_1 of E, these morphisms go over to $\mathrm{Int}(I_{p,q})$ on $\mathrm{SL}_n(\mathbb{R})$ (to see this in the third case, use the isomorphism given in Lemma 5.1). Therefore, we have

$$\begin{aligned} \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}), \boldsymbol{G}')(F_v) &= \mathrm{Fix}(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1}) \\ &\cong \{x \in \mathrm{SL}_n(\mathbb{R}) \mid I_{p,q} x I_{p,q} = x\} \\ &\cong S(\mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R})). \end{aligned}$$

Let now $\boldsymbol{\varphi} = \boldsymbol{\mu}$. The group $\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))$ is either a special unitary group with respect to a skew-hermitian form and an orthogonal involution (when $D = E$, the matrix $H^{-1} J_m$ occurring in the definition of μ is skew-symmetric and hence it describes a skew-hermitian form over E) or a special unitary group with respect to a hermitian form and a symplectic involution (when D is a quaternion algebra, H is a diagonal matrix with entries in F and thus τ_c-invariant). In both cases, Proposition 4.1 implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\mu}), \boldsymbol{G}')(F_v) = \mathrm{Fix}(\boldsymbol{\mu}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1}) \cong \mathrm{Sp}(n, \mathbb{R}).$$

Finally, we consider $\boldsymbol{\varphi} = \boldsymbol{\mu} \circ \boldsymbol{\theta}$. If $D = E$, we have $(\boldsymbol{\mu} \circ \boldsymbol{\theta})(E) = \mathrm{Int}\, J_m$. In the case $D = Q(a, b \mid E)$, we note that $(\boldsymbol{\mu} \circ \boldsymbol{\theta})(E) = \tau_c \circ \tau = \mathrm{Int}\,\mathrm{diag}(e_0, \ldots, e_0)$ on $M_m(D)$, by the remark following Lemma 5.1. Under a suitable splitting isomorphism, $\mathrm{diag}(e_0, \ldots, e_0)$ is mapped to J_n at the place w_1 of D (see Lemma 5.1). Therefore, for both choices of D, we have

$$\begin{aligned} \mathrm{Fix}(\mathrm{Res}_{E/F}(\boldsymbol{\mu} \circ \boldsymbol{\theta}), \boldsymbol{G}')(F_v) &= \mathrm{Fix}(\boldsymbol{\mu} \circ \boldsymbol{\theta}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1}) \\ &\cong \{x \in \mathrm{SL}_n(\mathbb{R}) \mid J_n x J_n^{-1} = x\} \\ &\cong \mathrm{GL}^{(1)}_{n/2}(\mathbb{C}). \end{aligned}$$ □

5.4. Now that we have defined certain F-rational morphisms on $\boldsymbol{G}'$ and studied their fixed point groups, we are ready to define the corresponding special cycles. To do this, we pass to the algebraic $\mathbb{Q}$-group $\boldsymbol{G} := \operatorname{Res}_{F/\mathbb{Q}} \boldsymbol{G}'$. We have $\boldsymbol{G}(\mathbb{Q}) \cong \boldsymbol{G}'(F)$ and

$$\boldsymbol{G}(\mathbb{R}) \cong \boldsymbol{G}'(\mathbb{R} \otimes_{\mathbb{Q}} F) = \prod_{v' \in V_\infty(F)} \boldsymbol{G}'(F_{v'}) = \mathrm{SL}_n(\mathbb{R}) \times \prod_{\substack{v' \in V_\infty \\ v' \neq v}} \mathrm{SU}(n).$$

Moreover, there exist $\mathbb{Q}$-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$ of order two on $\boldsymbol{G}$ that are induced from the corresponding morphisms of $\boldsymbol{G}'$. Let $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ be an arithmetic subgroup of $\boldsymbol{G}$. The image of Γ under the isomorphism $\boldsymbol{G}(\mathbb{Q}) \cong \boldsymbol{G}'(F)$ is an arithmetic subgroup of $\boldsymbol{G}'$ and thus it gives rise to a discrete cocompact subgroup of $\mathrm{SL}_n(\mathbb{R})$ that we will still denote by Γ for simplicity of notation.[4] Let K' denote a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$ and $X := K' \backslash \boldsymbol{G}(\mathbb{R})$ the symmetric space attached to $\boldsymbol{G}(\mathbb{R})$. Since $\boldsymbol{G}(\mathbb{R})$ is a product of $\mathrm{SL}_n(\mathbb{R})$ and compact factors, X is isomorphic to $\mathrm{SO}(n) \backslash \mathrm{SL}_n(\mathbb{R})$ and Γ acts on X by right translations. Note that X is a symmetric space of dimension

$$\dim X = \dim(\mathrm{SL}_n(\mathbb{R})) - \dim(\mathrm{SO}(n)) = n^2 - 1 - \tfrac{1}{2} n(n-1) = \tfrac{1}{2} n(n+1) - 1.$$

Let now $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup of $\boldsymbol{G}$ and assume that Γ and K' are invariant under the morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\mu}$.[5] Then these morphisms induce certain special geometric cycles in X/Γ, as explained in Section 3.1.

Theorem 5.5. *The morphisms $\boldsymbol{\nu_{p,q}}$ and $\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}$ of $\boldsymbol{G}$ induce a family of pairs of special geometric cycles $C(\boldsymbol{\nu_{p,q}})$, $C(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta})$ in X/Γ, for positive integers p and q with $p+q=n$ if $\boldsymbol{G}$ comes from a special unitary group over an algebraic number field, and for positive integers p and q with $p+q=n$ and p and q even or $p=q=n/2$ if $\boldsymbol{G}$ comes from a special unitary group over a quaternion algebra. If n is even and $n>2$, the morphisms $\boldsymbol{\mu}$ and $\boldsymbol{\mu} \circ \boldsymbol{\theta}$ induce a pair of geometric cycles $C(\boldsymbol{\mu})$, $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ in X/Γ. Some properties of these cycles are summarized in Table 1.*

Proof. The existence of the cycles is clear from Section 3.1. The isomorphisms in the second column of Table 1 follow from Proposition 5.3 and the fact that $\boldsymbol{G}(\mathbb{R}) \cong \boldsymbol{G}'(F_v)$ up to compact factors. The dimensions of the cycles can be computed as the dimensions of the associated symmetric spaces, using the dimensions of the occurring real Lie groups and their maximal compact subgroups (for a list of dimensions of classical Lie groups, see, e.g., [Helgason 1978, Table IV, p. 516]). Note that both $\mathrm{SO}(p,q)$ and $S(\mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R}))$ have maximal compact

[4] To be precise, the arithmetic subgroup $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ can be regarded as a subgroup of $\boldsymbol{G}(\mathbb{R}) = \mathrm{SL}_n(\mathbb{R}) \times$ compact factors, and the discrete cocompact subgroup is in fact the projection of Γ to the noncompact factor of $\boldsymbol{G}(\mathbb{R})$.

[5] Such a choice of K' is possible by [Helgason 1978, Theorem 13.5]

$C = C(\boldsymbol{\varphi})$	$\mathrm{Fix}(\boldsymbol{\varphi}, \boldsymbol{G})(\mathbb{R}) \cong$	$\dim C$
$C(\boldsymbol{\nu_{p,q}})$	$\mathrm{SO}(p,q)$	pq
$C(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta})$	$S(\mathrm{GL}_p(\mathbb{R}) \times \mathrm{GL}_q(\mathbb{R}))$	$\frac{1}{2}(p^2+q^2+n)-1$
$C(\boldsymbol{\mu})$	$\mathrm{Sp}(n,\mathbb{R})$	$\frac{1}{4}(n^2+2n)$
$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$\mathrm{GL}^{(1)}_{n/2}(\mathbb{C})$	$\frac{1}{4}n^2-1$

Table 1. Geometric cycles in $\mathrm{SO}(n)\backslash \mathrm{SL}_n(\mathbb{R})/\Gamma$: the isomorphism in the second column is up to compact factors and the lower half of the table is only applicable if n is even and $n > 2$.

subgroup $S(O(p) \times O(q))$ and that both $\mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{GL}^{(1)}_{n/2}(\mathbb{C})$ have maximal compact subgroup isomorphic to $U(n/2)$.[6] □

Finally, we can apply Theorem 3.2 to the constructed cycles to obtain a nonvanishing result for the cohomology of X/Γ. As before, we denote by $X_u \cong \mathrm{SO}(n)\backslash \mathrm{SU}(n)$ the compact dual symmetric space of X.

Theorem 5.6. *Let $n \in \mathbb{N}$ be even.*

(1) *There exists a cocompact discrete subgroup Γ_1 of $\mathrm{SL}_n(\mathbb{R})$ that arises from an arithmetic subgroup of a special unitary group over an algebraic number field, such that $H^k(X/\Gamma_1, \mathbb{C})$ contains nontrivial cohomology classes for*
$$k = pq \quad \textit{and} \quad k = \tfrac{1}{2}(p^2+q^2+n)-1,$$
where p and q are positive integers with $p+q=n$, and, if $n \neq 2$, for
$$k = \tfrac{1}{4}(n^2+2n) \quad \textit{and} \quad k = \tfrac{1}{4}n^2-1.$$

(2) *There exists a cocompact discrete subgroup Γ_2 of $\mathrm{SL}_n(\mathbb{R})$ that arises from an arithmetic subgroup of a special unitary group over a quaternion algebra, such that $H^k(X/\Gamma_2, \mathbb{C})$ contains nontrivial cohomology classes for*
$$k = pq \quad \textit{and} \quad k = \tfrac{1}{2}(p^2+q^2+n)-1,$$
where p and q are positive, even integers with $p+q=n$ or $p=q=n/2$, and, if $n \neq 2$, for
$$k = \tfrac{1}{4}(n^2+2n) \quad \textit{and} \quad k = \tfrac{1}{4}n^2-1.$$

In both cases, these classes are not in the image of the respective injective map
$$\beta^*_{\Gamma_i} : H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma_i, \mathbb{C}),$$
i.e., they are not represented by $\mathrm{SL}_n(\mathbb{R})$-invariant forms on X.

[6] Here $U(n/2)$ is considered as a subgroup of $\mathrm{Sp}(n)$ via the embedding $\phi : \mathrm{GL}_{n/2}(\mathbb{C}) \to \mathrm{GL}_n(\mathbb{R})$, $X = A + iB \mapsto \left(\begin{smallmatrix} A & -B \\ B & A \end{smallmatrix}\right)$, for $A, B \in \mathrm{GL}_{n/2}(\mathbb{R})$.

Proof. We give a detailed proof of (1), then (2) follows analogously.

Let $\boldsymbol{G}$ denote the algebraic $\mathbb{Q}$-group whose real points are isomorphic to $\mathrm{SL}_n(\mathbb{R})$ up to compact factors and which is defined via a special unitary group over an algebraic number field. Set $\Psi := \{\boldsymbol{\nu}_{p,q} \mid p+q=n,\ p\neq 0\neq q\}\cup\{\boldsymbol{\mu}\}$ and let Γ be a torsion-free arithmetic subgroup of $\boldsymbol{G}$ that is stable under the group generated by $\Psi\cup\{\boldsymbol{\theta}\}$. If we choose $\boldsymbol{\tau}_1\in\Psi$ and set $\boldsymbol{\tau}_2 := \boldsymbol{\tau}_1\circ\boldsymbol{\theta}$, the pair $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ is a pair of commuting morphisms of order two and it defines a pair of geometric cycles on X/Γ, whose properties are given in Theorem 5.5. As discussed in Section 3 we may assume that the cycles and the connected components of their intersection are orientable. Moreover, it follows from Table 1 and the fact that $\boldsymbol{\tau}_1\circ\boldsymbol{\tau}_2=\boldsymbol{\theta}$ induces a Cartan involution on $\mathrm{SL}_n(\mathbb{R})$ that the cycles $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are of complementary dimension and satisfy condition (ii) of Theorem 3.2.

To apply Theorem 3.2, it remains to check condition (i). It suffices to look at the action of the noncompact factor of $\mathrm{Fix}(\boldsymbol{\tau}_1, \boldsymbol{G})(\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \boldsymbol{G})(\mathbb{R})$ on the respective symmetric space.

For $\boldsymbol{\tau}_1=\boldsymbol{\mu}$, we have $\mathrm{Fix}(\boldsymbol{\tau}_1, \boldsymbol{G})(\mathbb{R})\cong \mathrm{Sp}(n,\mathbb{R})$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \boldsymbol{G})(\mathbb{R})\cong \mathrm{GL}^{(1)}_{n/2}(\mathbb{C})$ up to compact factors (see Table 1). These are connected Lie subgroups of $\mathrm{SL}_n(\mathbb{R})$ and hence they act orientation-preservingly on the respective symmetric spaces. For $\boldsymbol{\tau}_1=\boldsymbol{\nu}_{p,q}$, we have $\mathrm{Fix}(\boldsymbol{\tau}_1, \boldsymbol{G})(\mathbb{R})\cong \mathrm{SO}(p,q)$ and $\mathrm{Fix}(\boldsymbol{\tau}_2, \boldsymbol{G})(\mathbb{R})\cong S(\mathrm{GL}_p(\mathbb{R})\times \mathrm{GL}_q(\mathbb{R}))$ up to compact factors, where $p+q=n$. Since n is even, it follows from Example 3.4 that these groups act orientation-preservingly on their associated symmetric spaces.

We conclude that, for any choice of $\boldsymbol{\tau}_1\in\Psi$, the pair $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ meets all assumptions of Theorem 3.2. Therefore, for each such $\boldsymbol{\tau}_1$, we can find a normal, $\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable subgroup $\Gamma_{\tau_1}\subset\Gamma$ of finite index such that $H^k(X/\Gamma_{\tau_1}, \mathbb{C})\neq 0$ for $k\in\{\dim C(\boldsymbol{\tau}_1), \dim C(\boldsymbol{\tau}_2)\}$. Moreover, Γ_{τ_1} can be chosen such that the nontrivial cohomology classes detected by $C(\boldsymbol{\tau}_1)$ and $C(\boldsymbol{\tau}_2)$ are not represented by $\mathrm{SL}_n(\mathbb{R})$-invariant differential forms on X, as follows from Theorem 3.3. Set

$$\Gamma' := \bigcap_{\boldsymbol{\tau}_1\in\Psi}\Gamma_{\tau_1} \quad\text{and}\quad \Gamma_1 := \bigcap_{\boldsymbol{\tau}_1\in\Psi}\boldsymbol{\tau}_1(\Gamma')\cap\boldsymbol{\tau}_2(\Gamma').$$

Then Γ_1 is a cocompact discrete subgroup of $\mathrm{SL}_n(\mathbb{R})$ that is of finite index in each Γ_{τ_1} and $\langle\boldsymbol{\tau}_1, \boldsymbol{\tau}_2\rangle$-stable for each $\boldsymbol{\tau}_1\in\Psi$. The group Γ_1 admits nontrivial cohomology classes in all degrees $k\in\{\dim C(\boldsymbol{\tau}_1), \dim C(\boldsymbol{\tau}_2)\}$ for possible pairs $(\boldsymbol{\tau}_1, \boldsymbol{\tau}_2)$ with $\boldsymbol{\tau}_1\in\Psi$, and these classes are not represented by $\mathrm{SL}_n(\mathbb{R})$-invariant differential forms on X.[7] The exact dimensions can be read off from Table 1. □

Remark. (1) We do not get any result in the case where n is odd. The morphism $\boldsymbol{\mu}$ is not defined in this case, so we are left with the cycles $C(\boldsymbol{\nu}_{p,q})$ and $C(\boldsymbol{\nu}_{p,q}\circ\boldsymbol{\theta})$.

[7] Here we use the fact that the results of Theorems 3.2 and 3.3 carry over to finite index subgroups of Γ.

	$\dim X/\Gamma$	Cycle	Subgroup of $SL_n(\mathbb{R})$	Contributing to degree	Occurs for $\Gamma = \Gamma_1$	Occurs for $\Gamma = \Gamma_2$
$n=2$	2	$C(\boldsymbol{\nu}_{1,1})$	$SO(1, 1)$	1	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,1} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_1)$	1	$\times$	$\times$
$n=4$	9	$C(\boldsymbol{\nu}_{1,3} \circ \boldsymbol{\theta})$	$S(GL_1 \times GL_3)$	3	$\times$	
		$C(\boldsymbol{\mu})$	$Sp(4, \mathbb{R})$	3	$\times$	$\times$
		$C(\boldsymbol{\nu}_{2,2} \circ \boldsymbol{\theta})$	$S(GL_2 \times GL_2)$	4	$\times$	$\times$
		$C(\boldsymbol{\nu}_{2,2})$	$SO(2, 2)$	5	$\times$	$\times$
		$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$GL_2^{(1)}(\mathbb{C})$	6	$\times$	$\times$
		$C(\boldsymbol{\nu}_{1,3})$	$SO(1, 3)$	6	$\times$	

Table 2. Real case: degrees in $H^*(X/\Gamma)$ in which we have nontrivial cohomology classes coming from special cycles.

These are indeed of complementary dimension and $\mathrm{Fix}(\langle \boldsymbol{\nu}_{p,q}, \boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta} \rangle, \boldsymbol{G})(\mathbb{R})$ is compact. However, the cycles do not satisfy condition (i) in Theorem 3.2 (see Example 3.4). Therefore, the result of Rohlfs and Schwermer is not applicable and we cannot deduce any statement about the intersection number of $C(\boldsymbol{\nu}_{p,q})$ and $C(\boldsymbol{\nu}_{p,q} \circ \boldsymbol{\theta})$. It is still an open question whether or not this number is nontrivial.

(2) Ash and Ginzburg [1994] show a part of our result in Theorem 5.6(1) to use it in the proof of their Lemma 5.4.2. More precisely, in the case where n is even and $\boldsymbol{G}$ is the algebraic group associated with a special unitary group over a number field, they construct the pair of special cycles $C(\boldsymbol{\nu}_{n/2,n/2})$, $C(\boldsymbol{\nu}_{n/2,n/2} \circ \boldsymbol{\theta})$ (in our notation). Then, using the result of Rohlfs and Schwermer, they show that the intersection number of these cycles is nonzero and deduce the existence of a nonvanishing homology class.

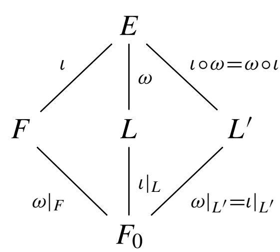
Example 5.7. Let Γ be a cocompact discrete subgroup of $SL_n(\mathbb{R})$ chosen as in Theorem 5.6(1) or (2). In Table 2 we give an overview of the occurring cycles, the associated subgroups of $SL_n(\mathbb{R})$ and the degrees in the cohomology of X/Γ to which these cycles contribute,[8] for some choices of n. The last two columns indicate if the respective cycle exists for the choice of Γ as in Theorem 5.6(1) or (2).

Remark. Using the method of crosswise intersection (see [Waldner 2010]), one can show that for $n = 2$ the two cohomology classes contributing to degree 1 are linearly independent. Unfortunately, for $n > 2$ we do not get a result on the linear independence of the constructed cohomology classes using this technique.

[8]Note that these degrees are not the dimensions of the cycles but their complements, since we are looking at the cohomology classes obtained via Poincaré duality.

6. Geometric cycles for $\mathrm{SL}_n(\mathbb{C})$

We will work in the following general setting. Let F_0 be a totally real number field and let E/F_0 be a totally complex biquadratic extension.[9] Assume that there is a quadratic extension F/F_0 such that F is a subfield of E with exactly one complex place v and denote by v_0 the real place of F_0 with $v \mid v_0$. Moreover, let L and L' denote the other two intermediate fields of the extension E/F_0. Then L is a quadratic extension of F_0 that has two real places v_1, v_2 lying above v_0 and only complex archimedean places otherwise, and L' is a totally complex quadratic extension of F_0. This is to say, we have $F = F_0(\sqrt{D_1})$, $L = F_0(\sqrt{D_2})$ and $L' = F_0(\sqrt{D_1 D_2})$ for $D_1, D_2 \in F_0$ such that none of D_1, D_2 and $D_1 D_2$ is a square in F_0 and such that $(D_1)_{v_0} < 0$, $(D_2)_{v_0} > 0$ and $(D_1)_{v'} > 0 > (D_2)_{v'}$ for $v' \in V_\infty(F_0)$ and $v' \neq v_0$. We write ι and ω for the nontrivial Galois automorphisms of E/F and E/L, respectively. Then ι and ω generate the Galois group of E/F_0 and the third nontrivial element $\iota \circ \omega = \omega \circ \iota$ is the nontrivial Galois automorphism of E over L'. Moreover, we note that $\iota|_L$ is the nontrivial Galois automorphism of the quadratic extension L/F_0, $\omega|_F$ is the one of F/F_0 and $\iota|_{L'} = \omega|_{L'}$ is the one of L'/F_0. The field extension E/F_0, its intermediate subfields and the nontrivial Galois automorphisms corresponding to each extension are illustrated in the following diagram:



Now we let D be a division algebra of degree d over E with an involution σ of the second kind. Again we will restrict to the cases where D is either the field E itself and $\sigma = \iota$ or D is a quaternion division algebra over E constructed in the following way. Let D' over L' be a quaternion division algebra that does not split over E and that admits an involution γ of the second kind (with respect to the subfield $F_0 \subset L'$). Without loss of generality, we may assume that γ is definite. Moreover, by Albert's theorem, we find a quaternion division algebra $D_0 = Q(a, b \mid F_0)$ over F_0 with $a, b \in F_0^\times$ such that $(D', \gamma) \cong (D_0 \otimes_{F_0} L', \tau_{c,0} \otimes \omega|_{L'})$, where $\tau_{c,0}$ denotes the conjugation on D_0. Now set $D := D_0 \otimes_{F_0} E = D' \otimes_{L'} E$. By our choice of D', this is a quaternion division algebra over E that admits the two involutions $\sigma := \tau_{c,0} \otimes \iota$ and $\sigma' := \tau_{c,0} \otimes \omega$, both of the second kind. Note that σ is trivial on the subfield

[9] A *biquadratic extension* of a number field F_0 is an extension of degree 4 with Galois group $\mathrm{Gal}(E/F_0) \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.

F of E, and σ' is trivial on the subfield L of E. Moreover, it follows from the choice of γ that both involutions are definite. For simplicity of notation, we will set $D_0 := F_0$, $D' = L'$, $\gamma := \omega|_{L'} = \iota|_{L'}$ and $\sigma' := \omega$ when $D = E$.

Let $m \in \mathbb{N}$ be arbitrary and set $n := 2m$. Then D, E, F, σ and m satisfy the conditions of Theorem 4.2(2), and we can choose a hermitian form h on D^m such that each arithmetic subgroup of $\boldsymbol{G}' := \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}, \boldsymbol{D}, \boldsymbol{\sigma})$ gives rise to a cocompact discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$. As above, h can be chosen such that its diagonal representation H is an element of $M_m(F_0)$ and we will restrict to this situation for technical reasons. Moreover, we suppose that H is positive definite under the embedding corresponding to the real place v_0 of F_0.

We can now define some automorphisms of order two on $\mathrm{SL}_m(D)$:

$$\theta : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D),\qquad \theta(x) = H^{-1}(\sigma'(x)^t)^{-1}H,$$

$$\eta : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D),\qquad \eta(x) = \begin{cases} H^{-1}(x^t)^{-1}H & \text{if } D = E, \\ H^{-1}(\tau_k(x)^t)^{-1}H & \text{if } D = Q(a, b \mid E), \end{cases}$$

$$\mu : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D),\qquad \mu(x) = \begin{cases} H^{-1}J_m(x^t)^{-1}J_m^{-1}H & \text{if } D = E \text{ and } m > 2 \text{ even}, \\ H^{-1}(\tau_c(x)^t)^{-1}H & \text{if } D = Q(a, b \mid E) \text{ and } m > 1. \end{cases}$$

Moreover, for certain positive integers p and q such that $p + q = n$, we define a family of automorphisms $\nu_{p,q} : \mathrm{SL}_m(D) \to \mathrm{SL}_m(D)$ by

$$\nu_{p,q}(x) = \begin{cases} H^{-1}I_{p,q}(\sigma'(x)^t)^{-1}I_{p,q}H & \text{if } D = E, \\ H^{-1}I_{p/2,q/2}(\sigma'(x)^t)^{-1}I_{p/2,q/2}H & \text{if } D = Q(a, b \mid E) \text{ and } p, q \text{ even}. \end{cases}$$

Again, θ commutes with each of the other automorphisms.

6.1. As in Section 5.2, we will assume from now on that p and q are positive integers such that $p + q = n$ and that p and q are both even, whenever we deal with the case where D is a quaternion algebra. Statements involving the map μ will again only be applicable if n is even and $n > 2$.

The maps η and μ are built out of E-linear maps (involutions of the first kind on $\mathrm{SL}_m(D)$) and the group inversion, and hence they define E-rational morphisms $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ on the algebraic E-group $\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$, as expected. However, the maps θ and $\nu_{p,q}$ involve involutions of the second kind with respect to the subfield L of E, and therefore they only define L-rational morphisms $\boldsymbol{\theta}$ and $\boldsymbol{\nu}_{\boldsymbol{p},\boldsymbol{q}}$ of the algebraic L-group $\mathrm{Res}_{E/L}\,\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$. On the other hand, the morphism $\boldsymbol{\psi}$ defining the algebraic group $\boldsymbol{G}'$ is an F-rational morphism of the group $\mathrm{Res}_{E/F}\,\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$. To work with all of these morphisms simultaneously, we need to pass to an algebraic group over the common subfield F_0 of F, L and E.[10] Using restriction of scalars with respect to the field F_0, the morphisms $\boldsymbol{\eta}$, $\boldsymbol{\mu}$, $\boldsymbol{\theta}$ and $\boldsymbol{\nu_{p,q}}$ give rise to F_0-rational morphisms on $\operatorname{Res}_{E/F_0}\mathbf{SL_m}(\boldsymbol{D})$. An easy computation shows that these morphisms commute with $\operatorname{Res}_{F/F_0}\boldsymbol{\psi}$ and can thus be restricted to the group $\boldsymbol{G}'' := \operatorname{Res}_{F/F_0}\boldsymbol{G}' = \operatorname{Fix}(\operatorname{Res}_{F/F_0}\boldsymbol{\psi}, \operatorname{Res}_{E/F_0}\mathbf{SL_m}(\boldsymbol{D}))$.

Their fixed points define algebraic subgroups of $\boldsymbol{G}''$ whose F_{0,v_0}-rational points are certain Lie subgroups of $\mathrm{SL}_n(\mathbb{C})$. We now determine these subgroups.

Proposition 6.2. *The algebraic F_0-group $\boldsymbol{G}''$ satisfies $\boldsymbol{G}''(F_{0,v_0}) \cong \mathrm{SL}_n(\mathbb{C})$. The fixed points of (certain compositions of) the above-defined F_0-rational morphisms define the following subgroups of* $\mathrm{SL}_n(\mathbb{C})$:

$$\begin{aligned}
\operatorname{Fix}(\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SU}(n),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}\boldsymbol{\eta}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SO}(n,\mathbb{C}),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}\boldsymbol{\eta}\circ\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SL}(n,\mathbb{R}),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}\boldsymbol{\mu}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{Sp}(n,\mathbb{C}),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}\boldsymbol{\mu}\circ\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SU}^*(n),\\
\operatorname{Fix}(\operatorname{Res}_{L/F_0}\boldsymbol{\nu_{p,q}}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SU}(p,q),\\
\operatorname{Fix}(\operatorname{Res}_{L/F_0}(\boldsymbol{\nu_{p,q}}\circ\boldsymbol{\theta}), \boldsymbol{G}'')(F_{0,v_0}) &\cong S(\mathrm{GL}_p(\mathbb{C})\times\mathrm{GL}_q(\mathbb{C})),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}(\boldsymbol{\eta}\circ\boldsymbol{\mu}), \boldsymbol{G}'')(F_{0,v_0}) &\cong S(\mathrm{GL}_{n/2}(\mathbb{C})\times\mathrm{GL}_{n/2}(\mathbb{C})),\\
\operatorname{Fix}(\operatorname{Res}_{E/F_0}(\boldsymbol{\eta}\circ\boldsymbol{\mu})\circ\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) &\cong \mathrm{SU}(n/2,n/2).
\end{aligned}$$

In particular, $\operatorname{Res}_{L/F_0}\boldsymbol{\theta}$ *induces a Cartan involution on* $\mathrm{SL}_n(\mathbb{C})$.

Proof. We have $\boldsymbol{G}''(F_{0,v_0}) = \boldsymbol{G}'(F_v) \cong \mathrm{SL}_n(\mathbb{C})$ by construction of the algebraic group $\boldsymbol{G}'$. To determine the fixed points of the morphisms, we need to study each map separately. We start with the morphism $\boldsymbol{\theta}$. The F_0-group $\operatorname{Fix}(\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')$ is defined by the equations $\theta(x) = x = \psi(x)$ on $\mathrm{SL}_m(D) = \operatorname{Res}_{E/F_0}\mathbf{SL_m}(\boldsymbol{D})(F_0)$. We have

$$\begin{aligned}
&\operatorname{Fix}(\operatorname{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_0)\\
&\qquad= \{x \in \mathrm{SL}_m(D) \mid \theta(x) = x = \psi(x)\}\\
&\qquad= \{x \in \mathrm{SL}_m(D) \mid H^{-1}(\sigma'(x)^t)^{-1}H = x = H^{-1}(\sigma(x)^t)^{-1}H\}\\
&\qquad= \{x \in \mathrm{SL}_m(D) \mid (\sigma'\circ\sigma)(x) = x \text{ and } x = H^{-1}(\sigma(x)^t)^{-1}H\}\\
&\qquad= \{x \in \mathrm{SL}_m(D') \mid x = H^{-1}(\gamma(x)^t)^{-1}H\},
\end{aligned}$$

[10] The reason for this additional complication is that we want the map $\boldsymbol{\theta}$ to define a Cartan involution of $\mathrm{SL}_n(\mathbb{C})$. Unlike in the real case, this involves complex conjugation and can hence not be defined by an E-rational morphism.

which yields $\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'') = \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}|_{\boldsymbol{D}'}, \boldsymbol{D}', \boldsymbol{\gamma})$. Now Proposition 4.1 implies

$$\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) \cong \mathrm{SU}(n),$$

since the real place v_0 of F_0 is nondecomposed in L', the map γ is a definite involution on D' and the matrix $H \in M_m(F_0)$ is chosen positive definite at the place v_0. In particular, this shows that $\mathrm{Res}_{L/F_0}\boldsymbol{\theta}$ induces a Cartan involution on $\mathrm{SL}_n(\mathbb{C})$.

Next, we consider the maps $\mathrm{Res}_{L/F_0}\boldsymbol{\nu_{p,q}}$ and $\mathrm{Res}_{L/F_0}(\boldsymbol{\nu_{p,q}}\circ\boldsymbol{\theta})$. On $\mathrm{SL}_m(D)$ we have $\nu_{p,q} = \mathrm{Int}(I_{p,q})\circ\theta$ and $\nu_{p,q}\circ\theta = \mathrm{Int}(I_{p,q})$ if $D = E$, and $\nu_{p,q} = \mathrm{Int}(I_{p/2,q/2})\circ\theta$ and $\nu_{p,q}\circ\theta = \mathrm{Int}(I_{p/2,q/2})$ if $D = Q(a,b\mid E)$.[11] However, in the latter case, the matrices $I_{p/2,q/2}$ are mapped to $I_{p,q}$ under a suitable splitting of $M_m(D)\otimes\mathbb{C}\to M_n(\mathbb{C})$, and therefore these maps induce the groups

$$\mathrm{Fix}(\mathrm{Res}_{L/F_0}\boldsymbol{\nu_{p,q}}, \boldsymbol{G}'')(F_{0,v_0}) \cong \{x\in\mathrm{SL}_n(\mathbb{C}) \mid I_{p,q}(x^*)^{-1}I_{p,q} = x\} \cong \mathrm{SU}(p,q)$$

and

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{L/F_0}(\boldsymbol{\nu_{p,q}}\circ\boldsymbol{\theta}), \boldsymbol{G}'')(F_{0,v_0}) &\cong \{x\in\mathrm{SL}_n(\mathbb{C}) \mid I_{p,q}xI_{p,q} = x\}\\ &\cong S(\mathrm{GL}_p(\mathbb{C})\times\mathrm{GL}_q(\mathbb{C}))\end{aligned}$$

for both choices of D.

To deal with the maps $\mathrm{Res}_{E/F_0}\boldsymbol{\eta}$, $\mathrm{Res}_{E/F_0}\boldsymbol{\mu}$ and $\mathrm{Res}_{E/F_0}(\boldsymbol{\eta}\circ\boldsymbol{\mu})$ we proceed as in the proof of Proposition 5.3. In fact, for any E-rational morphism $\boldsymbol{\varphi}$ on $\mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D})$ such that $\mathrm{Res}_{E/F}\boldsymbol{\varphi}$ commutes with $\boldsymbol{\psi}$, we have

$$\begin{aligned}\mathrm{Fix}(\mathrm{Res}_{E/F_0}\boldsymbol{\varphi}, \boldsymbol{G}'')(F_{0,v_0}) &= \mathrm{Fix}(\mathrm{Res}_{E/F}\boldsymbol{\varphi}, \boldsymbol{G}')(F_v)\\ &\cong \mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1}) \subset \mathrm{SL}_n(\mathbb{C}).\end{aligned}$$

Here, the isomorphism is chosen as in the proof of Proposition 5.3. However, since v is now a complex place of F, we obtain a subgroup of $\mathrm{SL}_n(\mathbb{C})$ instead of $\mathrm{SL}_n(\mathbb{R})$. The result then follows from the determination of $\mathrm{Fix}(\boldsymbol{\varphi}, \mathbf{SL}_{\boldsymbol{m}}(\boldsymbol{D}))(E_{w_1})$ for $\boldsymbol{\varphi}\in\{\boldsymbol{\eta}, \boldsymbol{\mu}, \boldsymbol{\eta}\circ\boldsymbol{\mu}\}$, where we use in the third case the fact that $\eta\circ\mu$ is mapped to $\mathrm{Int}(I_{n/2,n/2})$ under a suitable splitting isomorphism $M_m(D)\otimes\mathbb{C}\to M_n(\mathbb{C})$.

For the remaining morphisms $\mathrm{Res}_{E/F_0}\boldsymbol{\eta}\circ\mathrm{Res}_{L/F_0}\boldsymbol{\theta}$, $\mathrm{Res}_{E/F_0}\boldsymbol{\mu}\circ\mathrm{Res}_{L/F_0}\boldsymbol{\theta}$ and $\mathrm{Res}_{E/F_0}(\boldsymbol{\eta}\circ\boldsymbol{\mu})\circ\mathrm{Res}_{L/F_0}\boldsymbol{\theta}$, the result follows from straightforward calculations if $D = E$. Thus, we only deal with the more complicated case where $D = Q(a,b\mid E)$.

Recall that $D = D_0\otimes_{F_0}E$. One can show that D_0 ramifies at all archimedean places of F_0 since the involution $\gamma = \tau_{c,0}\otimes\omega|_{L'}$ on $D_0\otimes_{F_0}L'$ is definite and all real places of F_0 are nondecomposed in L'. In particular, we have $a_{v_0} < 0$ and $b_{v_0} < 0$. Moreover, recall that $F = F_0(\sqrt{D_1})$ for some square-free element $D_1\in F_0$ such that D_1 is negative under the embedding corresponding to the place v_0 of F_0. Therefore, the quaternion algebra $Q_0 := Q(D_1a, D_1b\mid F_0)$ is a division algebra

[11] Recall that in the case of a quaternion algebra the maps $\nu_{p,q}$ are only defined for even p and q.

that splits at the place v_0 of F_0. Note that $x \in (Q_0 \otimes_{F_0} L)$ if and only if $x \in D$ and $(\tau_k \circ \tau_c)(x) = (\mathrm{id} \otimes \omega)(x)$. With the help of these observations, we can describe the fixed points of $\eta \circ \theta$ in $\boldsymbol{G}''(F_0)$ with the equation

$$\begin{aligned}
&\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \boldsymbol{G}'')(F_0) \\
&\quad = \{x \in \mathrm{SL}_m(D) \mid (\eta \circ \theta)(x) = x = \psi(x)\} \\
&\quad = \{x \in \mathrm{SL}_m(D) \mid (\tau_r \circ (\tau_{c,0} \otimes \omega))(x) = x = H^{-1}(\sigma(x)^t)^{-1} H\} \\
&\quad = \{x \in \mathrm{SL}_m(D) \mid (\tau_r \circ \tau_c)(x) = (\mathrm{id} \otimes \omega)(x) \text{ and } x = H^{-1}((\tau_{c,0} \otimes \iota)(x)^t)^{-1} H\} \\
&\quad = \{x \in \mathrm{SL}_m(Q_0 \otimes_{F_0} L) \mid x = H^{-1}((\tau_{c,Q_0} \otimes \iota|_L)(x)^t)^{-1} H\},
\end{aligned}$$

where τ_{c,Q_0} denotes the canonical symplectic involution of Q_0. This implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \boldsymbol{G}'') = \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}|_{\boldsymbol{Q_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}}, \boldsymbol{Q_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}, \boldsymbol{\tau_{c,Q_0}} \otimes \boldsymbol{\iota}|_{\boldsymbol{L}}),$$

and hence by Proposition 4.1

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\eta} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) \cong \mathrm{SL}_n(\mathbb{R}),$$

since the real place v_0 of F_0 is decomposed in L and Q_0 splits at v_0.

A similar calculation for the other two morphisms leads to

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \boldsymbol{G}'') = \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}|_{\boldsymbol{D_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}}, \boldsymbol{D_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}, \boldsymbol{\tau_{c,0}} \otimes \boldsymbol{\iota}|_{\boldsymbol{L}})$$

and

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta}, \boldsymbol{G}'') = \mathbf{SU}_{\boldsymbol{m}}(\boldsymbol{h}|_{\boldsymbol{Q_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}'}, \boldsymbol{Q_0} \otimes_{\boldsymbol{F_0}} \boldsymbol{L}', \boldsymbol{\tau_{c,Q_0}} \otimes \boldsymbol{\iota}|_{\boldsymbol{L}'}).$$

For the first group, Proposition 4.1 implies

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0} \boldsymbol{\mu} \circ \mathrm{Res}_{F/F_0} \boldsymbol{\theta}, \boldsymbol{G}'')(F_{0,v_0}) \cong \mathrm{SL}_{n/2}(\mathbb{H}) \cong \mathrm{SU}^*(n),$$

since the real place v_0 of F_0 is decomposed in L and D_0 ramifies at v_0. For the second group, we note that the involution $\tau_{c,Q_0} \otimes \iota|_{L'}$ of the second kind cannot be definite on $Q_0 \otimes_{F_0} L'$ because Q_0 does not ramify at the real place v_0 of F_0 that is nondecomposed in L'. This means we get a signature of $(n/2, n/2)$ when passing to the F_{0,v_0}-rational points:

$$\mathrm{Fix}(\mathrm{Res}_{E/F_0}(\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \mathrm{Res}_{L/F_0} \boldsymbol{\theta})(F_{0,v_0}) \cong \mathrm{SU}(n/2, n/2). \qquad \square$$

6.3. In this section, we study the geometric cycles defined by the various morphisms on $\boldsymbol{G}''$. To do this, we pass to the algebraic $\mathbb{Q}$-group $\boldsymbol{G} := \mathrm{Res}_{F_0/\mathbb{Q}} \boldsymbol{G}''$. This is an algebraic group over $\mathbb{Q}$ with $\boldsymbol{G}(\mathbb{Q}) \cong \boldsymbol{G}''(F_0)$ and

$$\boldsymbol{G}(\mathbb{R}) \cong \boldsymbol{G}''(\mathbb{R} \otimes_{\mathbb{Q}} F_0) = \boldsymbol{G}'(\mathbb{R} \otimes_{\mathbb{Q}} F) = \prod_{v' \in V_\infty(F)} \boldsymbol{G}'(F_{v'}) = \mathrm{SL}_n(\mathbb{C}) \times \prod_{\substack{v' \in V_\infty \\ v' \neq v}} \mathrm{SU}(n).$$

Moreover, we have $\mathbb{Q}$-rational morphisms $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$, $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$ of order two on $\boldsymbol{G}$ that are induced from the corresponding morphisms of $\boldsymbol{G}''$.

Let $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ be an arithmetic subgroup of $\boldsymbol{G}$. In analogy to the real case, Γ gives rise to a discrete cocompact subgroup of $\mathrm{SL}_n(\mathbb{C})$ that we still denote by Γ for simplicity of notation. Let K' denote a maximal compact subgroup of $\boldsymbol{G}(\mathbb{R})$ and $X := K'\backslash \boldsymbol{G}(\mathbb{R})$ the symmetric space attached to $\boldsymbol{G}(\mathbb{R})$. Since $\boldsymbol{G}(\mathbb{R})$ is a product of $\mathrm{SL}_n(\mathbb{C})$ and compact factors, X is isomorphic to $\mathrm{SU}(n)\backslash \mathrm{SL}_n(\mathbb{C})$ and Γ acts on X by right translations. Note that X is a symmetric space of real dimension

$$\dim X = \dim(\mathrm{SL}_n(\mathbb{C})) - \dim(\mathrm{SU}(n)) = 2n^2 - 2 - (n^2 - 1) = n^2 - 1.$$

Let now $\Gamma \subset \boldsymbol{G}(\mathbb{Q})$ be a torsion-free arithmetic subgroup of $\boldsymbol{G}$ and assume that Γ and K' are invariant under $\boldsymbol{\theta}$, $\boldsymbol{\nu_{p,q}}$, $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$. Then these morphisms induce certain special geometric cycles in X/Γ, as explained in Section 3:

Theorem 6.4. *The pair of morphisms* $(\boldsymbol{\eta}, \boldsymbol{\eta} \circ \boldsymbol{\theta})$ *and, if n is even and $n > 2$, the pairs* $(\boldsymbol{\mu}, \boldsymbol{\mu} \circ \boldsymbol{\theta})$ *and* $(\boldsymbol{\eta} \circ \boldsymbol{\mu}, (\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$ *induce pairs of special geometric cycles* $C(\boldsymbol{\eta})$, $C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$, $C(\boldsymbol{\mu})$, $C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$ *and* $C(\boldsymbol{\eta} \circ \boldsymbol{\mu})$, $C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$ *in* X/Γ. *Moreover, the morphisms* $\boldsymbol{\nu_{p,q}}$ *and* $\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta}$ *induce a family of pairs of special geometric cycles* $C(\boldsymbol{\nu_{p,q}})$, $C(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta})$ *in* X/Γ, *for positive integers p and q with $p + q = n$ if* $\boldsymbol{G}$ *is induced from a special unitary group over an algebraic number field, and for positive, even integers p and q with $p+q = n$ if* $\boldsymbol{G}$ *is induced from a special unitary group over a quaternion algebra. The properties of these cycles are summarized in Table 3.*

Proof. This is proved completely analogously to Theorem 5.5. □

$C = C(\varphi)$	$\mathrm{Fix}(\boldsymbol{\varphi}, \boldsymbol{G})(\mathbb{R}) \cong$	$\dim C$
$C(\boldsymbol{\nu_{p,q}})$	$\mathrm{SU}(p,q)$	$2pq$
$C(\boldsymbol{\nu_{p,q}} \circ \boldsymbol{\theta})$	$S(\mathrm{GL}_p(\mathbb{C}) \times \mathrm{GL}_q(\mathbb{C}))$	$p^2 + q^2 - 1$
$C(\boldsymbol{\eta})$	$\mathrm{SO}(n, \mathbb{C})$	$\frac{1}{2}(n^2 - n)$
$C(\boldsymbol{\eta} \circ \boldsymbol{\theta})$	$\mathrm{SL}_n(\mathbb{R})$	$\frac{1}{2}(n^2 + n) - 1$
$C(\boldsymbol{\mu})$	$\mathrm{Sp}(n, \mathbb{C})$	$\frac{1}{2}(n^2 + n)$
$C(\boldsymbol{\mu} \circ \boldsymbol{\theta})$	$\mathrm{SU}^*(n)$	$\frac{1}{2}(n^2 - n) - 1$
$C(\boldsymbol{\eta} \circ \boldsymbol{\mu})$	$S(\mathrm{GL}_{n/2}(\mathbb{C}) \times \mathrm{GL}_{n/2}(\mathbb{C}))$	$\frac{1}{2}n^2 - 1$
$C((\boldsymbol{\eta} \circ \boldsymbol{\mu}) \circ \boldsymbol{\theta})$	$\mathrm{SU}(n/2, n/2)$	$\frac{1}{2}n^2$

Table 3. Geometric cycles in $\mathrm{SU}(n)\backslash \mathrm{SL}_n(\mathbb{C})/\Gamma$: the isomorphism in the second column is up to compact factors and the bottom half of the table is only applicable if n is even and $n > 2$.

Theorem 6.5. *Let $n \in \mathbb{N}$ be arbitrary.*

(1) *There exists a cocompact discrete subgroup Γ_1 of $\mathrm{SL}_n(\mathbb{C})$ that arises from an arithmetic subgroup of a special unitary group over an algebraic number field, such that $H^k(X/\Gamma_1, \mathbb{C})$ contains nontrivial cohomology classes for*

$$k = 2pq \quad \text{and} \quad k = p^2 + q^2 - 1,$$

where p and q are positive integers with $p + q = n$, and for

$$k = \tfrac{1}{2}(n^2 - n) \quad \text{and} \quad k = \tfrac{1}{2}(n^2 + n) - 1.$$

Moreover, if n is even and $n \neq 2$, there are nontrivial cohomology classes in the degrees

$$k = \tfrac{1}{2}(n^2 + n), \quad k = \tfrac{1}{2}(n^2 - n) - 1, \quad k = \tfrac{1}{2}n^2 - 1 \quad \text{and} \quad k = \tfrac{1}{2}n^2.$$

(2) *If n is even, there exists a discrete, cocompact subgroup Γ_2 of $\mathrm{SL}_n(\mathbb{C})$ that arises from an arithmetic subgroup of a special unitary group over a quaternion algebra, such that $H^k(X/\Gamma_2, \mathbb{C})$ contains nontrivial cohomology classes for*

$$k = 2pq \quad \text{and} \quad k = p^2 + q^2 - 1,$$

where p and q are positive, even integers with $p + q = n$, and for

$$k = \tfrac{1}{2}(n^2 - n) \quad \text{and} \quad k = \tfrac{1}{2}(n^2 + n) - 1.$$

Moreover, if $n \neq 2$, there exist nontrivial cohomology classes in the degrees

$$k = \tfrac{1}{2}(n^2 + n), \quad k = \tfrac{1}{2}(n^2 - n) - 1, \quad k = \tfrac{1}{2}n^2 - 1 \quad \text{and} \quad k = \tfrac{1}{2}n^2.$$

In both cases these classes are not in the image of the respective injective map

$$\beta^*_{\Gamma_i} : H^*(X_u, \mathbb{C}) \to H^*(X/\Gamma_i, \mathbb{C}),$$

i.e., they are not represented by $\mathrm{SL}_n(\mathbb{C})$-invariant forms on X.

Proof. The proof is completely analogous to the proof of Theorem 5.6; details are left to the reader. In contrast to the real case, orientability questions are not an issue here, as all occurring fixed point groups are connected Lie subgroups of $\mathrm{SL}_n(\mathbb{C})$. □

Example 6.6. Table 4 summarizes the occurring cycles and the degrees in which they contribute to the cohomology for small values of n. The group Γ denotes a cocompact discrete subgroup of $\mathrm{SL}_n(\mathbb{C})$ chosen as in Theorem 6.5(1) or (2).

Remark. (1) Looking at these examples, the question arises of whether the degrees in which we have constructed nontrivial cohomology classes exhaust all degrees in the cohomology of X/Γ in which there is cohomology that is not coming from the compact dual symmetric space. In general, this is not the case, as we will see in Section 7 using methods from representation theory. For certain choices of n, this

	$\dim X/\Gamma$	Cycle	Subgroup of $SL_n(\mathbb{C})$	Contributing to degree	Occurs for $\Gamma=\Gamma_1$	$\Gamma=\Gamma_2$
$n=2$	3	$C(\boldsymbol{\nu}_{1,1})$	$SU(1,1)$	1	×	
		$C(\boldsymbol{\eta}\circ\boldsymbol{\theta})$	$SL_2(\mathbb{R})$	1	×	×
		$C(\boldsymbol{\nu}_{1,1}\circ\boldsymbol{\theta})$	$S(GL_1\times GL_1)$	2	×	
		$C(\boldsymbol{\eta})$	$SO(2,\mathbb{C})$	2	×	×
$n=3$	8	$C(\boldsymbol{\eta}\circ\boldsymbol{\theta})$	$SL_3(\mathbb{R})$	3	×	
		$C(\boldsymbol{\nu}_{1,2})$	$SU(1,2)$	4	×	
		$C(\boldsymbol{\nu}_{1,2}\circ\boldsymbol{\theta})$	$S(GL_1\times GL_2)$	4	×	
		$C(\boldsymbol{\eta})$	$SO(3,\mathbb{C})$	5	×	

Table 4. Complex case: degrees in $H^*(X/\Gamma)$ in which we have nontrivial cohomology classes coming from special cycles.

can also be seen using the Euler characteristic: it is a consequence of the Gauss–Bonnet formula that for compact quotients X/Γ, where $X=SU(n)\backslash SL_n(\mathbb{C})$ and $n\geq 2$, the Euler characteristic of X/Γ is always 0. This implies that the sum over the Betti numbers in even degrees equals the sum over the Betti numbers in odd degrees.

Now for certain choices of n (in fact, whenever $n\geq 2$ and $n\equiv 1 \pmod 4$) the cohomology classes constructed in Theorem 6.5 all contribute to even degrees in the cohomology of X/Γ. Therefore, the vanishing of the Euler characteristic implies the existence of at least one nontrivial cohomology class in an odd degree that does not lie in the image of the cohomology of the compact dual symmetric space.

The smallest n to which our argumentation applies is $n=5$. For this case, one can easily read off from Theorem 6.5 that the constructed cycles do indeed only contribute to even degrees.

(2) Again, by using the technique of intersecting crosswise, one can show that when $n=2$ and $\Gamma=\Gamma_1$ the two cohomology classes in each of the degrees 1 and 2 are linearly independent. However, for the case $n=3$ it remains an open question whether or not the two classes in degree 4 are linearly independent.

7. Representation theory and Matsushima's formula

Let G be a connected semisimple Lie group (with finite center), K a maximal compact subgroup, $X := K\backslash G$ the associated symmetric space and $\Gamma\subset G$ a discrete, cocompact subgroup. By a well-known result of Matsushima [1962], the cohomology of X/Γ decomposes as a finite algebraic sum over the set of equivalence classes of irreducible unitary representations of G,

$$H^*(X/\Gamma, \mathbb{C}) = \bigoplus_{\pi \in \widehat{G}} m(\pi, \Gamma) H^*(\mathfrak{g}, K; H^{\infty}_{\pi,K}),$$

where the $m(\pi, \Gamma)$ are nonnegative integers and we denote by $H^{\infty}_{\pi,K}$ the Harish-Chandra module of K-finite, smooth vectors associated with an element $\pi \in \widehat{G}$. Moreover, $m(\mathbb{C}, \Gamma) = 1$, i.e., there is an injection of the $(\mathfrak{g}, K)$-cohomology of the trivial representation into $H^*(X/\Gamma, \mathbb{C})$.

The *unitary representations with nonvanishing* $(\mathfrak{g}, K)$*-cohomology* contributing to the right-hand side of Matsushima's formula are classified by the work of Enright [1979] (for complex groups) and Vogan and Zuckerman [1984] (for real groups). Note that, by a well-known result of Wigner (see [Borel and Wallach 2000, Theorem 5.3(ii)]), the representations π with $H^*(\mathfrak{g}, K; \mathbb{C} \otimes H^{\infty}_{\pi,K}) \neq 0$ are only those with trivial infinitesimal character. Representations occurring with a nontrivial multiplicity are called *automorphic representations of G with respect to* Γ. In general, given an irreducible unitary representation π of G, it is still an open question whether the corresponding multiplicity $m(\pi, \Gamma)$ is nontrivial or not. For groups admitting discrete series representations, there are nonvanishing results by DeGeorge and Wallach [1978], Wallach [1990], Langlands [1966], and others (see [Schwermer 1990]). We point out that for our cases of interest (i.e., $G = \mathrm{SL}_n(\mathbb{R})$ or $G = \mathrm{SL}_n(\mathbb{C})$), there is no discrete series except for the case $G = \mathrm{SL}_2(\mathbb{R})$.

Against this background, the result from Section 6 can be interpreted as a result in the theory of automorphic representations. To make a precise statement and possibly identify one (or several) automorphic representations explicitly, we will devote this section to the classification of all irreducible unitary representations with nonvanishing $(\mathfrak{g}, K)$-cohomology of the group $\mathrm{SL}_n(\mathbb{C})$ and the determination of their cohomology.

7.1. First we need to fix some notation. Let G be a complex simply connected semisimple Lie group with Lie algebra $\mathfrak{g}$. Considered as a real Lie algebra, $\mathfrak{g}$ has a Cartan involution θ and a corresponding Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $\mathfrak{h}$ denote a θ-stable Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h}$ admits the structure of a complex Lie algebra and we denote by $\Phi(\mathfrak{g}, \mathfrak{h})$ and $\Phi^+(\mathfrak{g}, \mathfrak{h})$ the set of roots and a system of positive roots of the pair $(\mathfrak{g}, \mathfrak{h})$, respectively. We denote by $\mathfrak{q}_0$ the minimal parabolic subalgebra of $\mathfrak{g}$ associated with the system of positive roots Φ^+ and by $\mathfrak{q} \supset \mathfrak{q}_0$ a standard parabolic subalgebra of $\mathfrak{g}$. Let $\mathfrak{l}$ denote the Levi factor of $\mathfrak{q}$ and $\mathfrak{s} = [\mathfrak{l}, \mathfrak{l}]$ its derived Lie algebra. Then $\mathfrak{h} \cap \mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{s}$ and we can identify the root system $\Phi_{\mathfrak{s}}$ of $\mathfrak{s}$ with respect to $\mathfrak{h} \cap \mathfrak{s}$ with the set of roots in Φ that are trivial on the center $Z_{\mathfrak{l}}$ of $\mathfrak{l}$. Using this identification, we can set $\Phi^+_{\mathfrak{s}} := \Phi_{\mathfrak{s}} \cap \Phi^+$ and this is a system of positive roots for $\Phi_{\mathfrak{s}}$.

On the other hand, we may consider $\mathfrak{q}$ as a real parabolic subalgebra of $\mathfrak{g}$ and as such it has a Langlands decomposition of the form $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$. We denote by Q_0, Q, L, S, M, A and N the connected Lie subgroups of G with Lie algebras $\mathfrak{q}_0$, $\mathfrak{q}$, $\mathfrak{l}$, $\mathfrak{s}$, $\mathfrak{m}$, $\mathfrak{a}$ and $\mathfrak{n}$, respectively.

The irreducible unitary representations of G with trivial infinitesimal character have been completely classified by the work of Delorme and Enright. They have shown that one can associate with each standard parabolic subgroup $Q \supset Q_0$ a principal series representation π_Q that has the desired properties and that these representations exhaust the set of irreducible unitary representations of G with trivial infinitesimal character up to unitary equivalence. Being principal series representations, the $(\mathfrak{g}, K)$-cohomology of the π_Q can be computed with the help of a well-known theorem [Borel and Wallach 2000]. This leads to the following general result.

Theorem 7.2. *Let G be a connected, simply connected complex Lie group. The correspondence $Q \leftrightarrow \pi_Q$ is a bijective correspondence between the standard parabolic subgroups $Q \supset Q_0$ of G and the set of equivalence classes of irreducible unitary representations of G with trivial infinitesimal character.*

The relative Lie algebra cohomology of the representations π_Q is given by

$$H^{k+d_Q}(\mathfrak{g}, K; H^{\infty}_{\pi_Q,K}) = \bigoplus_{r+s=k} \bigl(H^r(\mathfrak{m}, K_Q; \mathbb{C}) \otimes \textstyle\bigwedge^s \mathfrak{a}_{\mathbb{C}}\bigr), \tag{1}$$

where $K_Q := K \cap Q$ and $d_Q := |\Phi^+(\mathfrak{g}, \mathfrak{h})| - |\Phi^+_{\mathfrak{s}}|$.[12]

7.3. Let us apply the above result to the case $G = \mathrm{SL}_n(\mathbb{C})$. On $\mathfrak{sl}_n(\mathbb{C})$ considered as a real Lie algebra, we have a Cartan involution $\theta : X \mapsto -\overline{X}^t$. The subalgebra $\mathfrak{h} := \{X \in \mathfrak{sl}_n(\mathbb{C}) \mid X = \mathrm{diag}(x_1, \dots, x_n)\}$ of diagonal matrices is a θ-stable Cartan subalgebra of $\mathfrak{sl}_n(\mathbb{C})$ and, with the usual choice of positive roots, the algebra $\mathfrak{q}_0$ of upper triangular matrices is a Borel subalgebra in $\mathfrak{sl}_n(\mathbb{C})$. Then the standard parabolic subalgebras of $\mathfrak{g}$ are in bijective correspondence with the set of compositions of n; with a composition $n = \ell_1 + \cdots + \ell_m$ we associate the parabolic subalgebra

$$\mathfrak{q} = \mathfrak{q}_{\ell_1,\dots,\ell_m} = \left\{ X \in \mathfrak{g} \,\middle|\, X = \begin{pmatrix} X_1 & * & * \\ & \ddots & * \\ 0 & & X_m \end{pmatrix}, \text{ where } X_j \in \mathrm{GL}_{\ell_j}(\mathbb{C}),\ 1 \le j \le m \right\}.$$

The Levi component of $\mathfrak{q}$ is given by the subalgebra of block diagonal matrices $\mathfrak{l} = \{X \in \mathfrak{q} \mid X = \mathrm{diag}(X_1, \dots, X_m)\}$ and it decomposes into its semisimple part

[12]Note that $K_Q = K \cap Q \subset M$ and that K_Q is a maximal compact subgroup of M by [Borel and Wallach 2000, Section 0.1.6], so taking the relative Lie algebra cohomology of $\mathfrak{m}$ with respect to K_Q is defined.

and its center, given respectively by $\mathfrak{s} = \{X \in \mathfrak{l} \mid \operatorname{tr}(X_j) = 0 \text{ for all } 1 \le j \le m\}$ and $Z_{\mathfrak{l}} = \{X \in \mathfrak{l} \mid X_j = x_j I_{\ell_j} \text{ for some } x_j \in \mathbb{C},\ 1 \le j \le m\}$.

Considering $\mathfrak{q}$ as a real Lie algebra, it also has a Langlands decomposition of the form $\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}$, where

$$\mathfrak{m} = \{X \in \mathfrak{q} \mid X = \operatorname{diag}(X_1, \dots, X_m),\ \operatorname{tr}(X_j) \in i \cdot \mathbb{R} \text{ for all } 1 \le j \le m\},$$
$$\mathfrak{a} = \{X \in \mathfrak{q} \mid X = \operatorname{diag}(X_1, \dots, X_m),\ X_j = x_j I_{\ell_j} \text{ for some } x_j \in \mathbb{R},\ 1 \le j \le m\}$$

and

$$\mathfrak{n} = \left\{ X \in \mathfrak{q} \;\middle|\; X = \begin{pmatrix} 0 & * & * \\ & \ddots & * \\ 0 & & 0 \end{pmatrix}, \text{ where the } j\text{-th diagonal 0-block is of size } \ell_j \times \ell_j \right\}.$$

We use capital letters to denote the connected Lie subgroups of $\mathrm{SL}_n(\mathbb{C})$ corresponding to these Lie algebras.

Theorem 7.4. *The equivalence classes of irreducible unitary representations of* $\mathrm{SL}_n(\mathbb{C})$ *with trivial infinitesimal character are in one-to-one correspondence with the standard parabolic subgroups* $Q \supset Q_0$ *of* $\mathrm{SL}_n(\mathbb{C})$*; a standard parabolic subgroup corresponds to the induced representation* π_Q*. The* $(\mathfrak{g}, K)$*-cohomology of the representation* π_Q *is given by the Poincaré polynomial*

$$P_{H^*(\mathfrak{g},K;H^\infty_{\pi_Q,K})}(t) = t^d \cdot \left(\sum_{k=0}^{m-1} \binom{m-1}{k} t^k \right) \cdot \prod_{\substack{j=1 \\ \ell_j \neq 1}}^{m} \prod_{k=2}^{\ell_j} (1 + t^{2k-1}),$$

where m *denotes the number of blocks of* Q, ℓ_j *denotes the length of the* j*-th block and*

$$d := \tfrac{1}{2} n(n-1) - \sum_{j=1}^{m} \tfrac{1}{2} \ell_j (\ell_j - 1).$$

Proof. The first part of the theorem is a direct application of Theorem 7.2 to the connected simply connected complex Lie group $\mathrm{SL}_n(\mathbb{C})$.

To compute the cohomology of the representations π_Q, we use the formula from Theorem 7.2. Note that, in terms of Poincaré polynomials, this formula says

$$P_{H^*(\mathfrak{g},K;H^\infty_{\pi_Q,K})}(t) = t^{d_Q} \cdot P_{H^*(\mathfrak{m},K_Q;\mathbb{C})}(t) \cdot P_{\bigwedge(\mathfrak{a}_{\mathbb{C}})}(t). \tag{2}$$

Therefore, it suffices to determine the number d_Q and the Poincaré polynomials of $H^*(\mathfrak{m}, K_Q; \mathbb{C})$ and $\bigwedge(\mathfrak{a}_{\mathbb{C}})$.

(1) *The Poincaré polynomial of* $\bigwedge(\mathfrak{a}_{\mathbb{C}})$: From the structure of $\mathfrak{a}$ given above we conclude that $\mathfrak{a}_{\mathbb{C}}$ has complex dimension $m-1$, and thus

$$P_{\bigwedge(\mathfrak{a}_{\mathbb{C}})}(t) = \sum_{k=0}^{m-1} \binom{m-1}{k} t^k$$

by the general formula for the Poincaré polynomial of the exterior algebra of a complex vector space.

(2) *The Poincaré polynomial of* $H^*(\mathfrak{m}, K_Q; \mathbb{C})$: First, we consider $Q = Q_0$. The group Q_0 has the Langlands decomposition $Q_0 = M_0 A_0 N_0$, where M_0 is compact. This implies $K_{Q_0} = K \cap M_0 = M_0$, so in fact we consider the relative Lie algebra cohomology $H^*(\mathfrak{m}_0, M_0; \mathbb{C})$. By the definition of relative Lie algebra cohomology, this is one-dimensional in degree 0 and trivial in all higher degrees. In particular,

$$P_{H^*(\mathfrak{m}_0, K_{Q_0}; \mathbb{C})}(t) = 1.$$

Now let $Q \neq Q_0$. The Lie algebra $\mathfrak{m}$ is reductive, has semisimple part $\mathfrak{s}$ and center $Z_{\mathfrak{m}} \subset \mathfrak{k}$. Using the Künneth rule (see [Borel and Wallach 2000, Section I.1.3]) and the fact that K_Q and $K_Q \cap S$ are connected, we obtain $H^*(\mathfrak{m}, K_Q; \mathbb{C}) = H^*(\mathfrak{s}, K_Q \cap S; \mathbb{C})$, so we can restrict to the semisimple part. From the structure of $\mathfrak{s}$ given above we deduce that $S \cong \prod_{j=1}^{m} \mathrm{SL}_{\ell_j}(\mathbb{C})$, which is clearly the group of real points of a reductive algebraic $\mathbb{R}$-group. Therefore, by [Vogan 1997, Theorem 2.10], $H^*(\mathfrak{s}, K_Q \cap S; \mathbb{C})$ equals the cohomology of the compact symmetric space $\prod_{j=1}^{m} \mathrm{SU}(\ell_j)$, the compact dual symmetric space of S. For $\ell_j \geq 2$, the Poincaré polynomial of $H^*(\mathrm{SU}(\ell_j); \mathbb{C})$ is given by

$$P_{H^*(\mathrm{SU}(\ell_j); \mathbb{C})}(t) = \prod_{k=2}^{\ell_j} (1 + t^{2k-1})$$

(see [Greub et al. 1976, Theorem VI.X]). For $\ell_j = 1$, we have $\mathrm{SU}(1) = S^1$, so the Poincaré polynomial is given by $P_{H^*(\mathrm{SU}(1), \mathbb{C})}(t) = 1$. Putting everything together, we obtain the formula

$$P_{H^*(\mathfrak{m}, K_Q; \mathbb{C})}(t) = \prod_{j=1}^{m} P_{H^*(\mathrm{SU}(\ell_j), \mathbb{C})}(t) = \prod_{\substack{j=1 \\ \ell_j \neq 1}}^{m} \prod_{k=2}^{\ell_j} (1 + t^{2k-1}),$$

where we have used the Künneth rule for singular cohomology in the first step.

(3) *Determination of* d_Q: Recall from Theorem 7.2 that $d_Q = |\Phi^+| - |\Phi^+_{\mathfrak{s}}|$. From the structure of the set of positive roots $\Phi^+(\mathfrak{g}, \mathfrak{h})$ and the definition of $\Phi_{\mathfrak{s}}$ as given above, we conclude that

$$d_Q = \tfrac{1}{2} n(n-1) - \sum_{j=1}^{m} \tfrac{1}{2} \ell_j (\ell_j - 1). \qquad \square$$

$(\ell_1, \dots, \ell_m)$	$P_{H^*(\mathfrak{g},K;\pi_{Q_{\ell_1,\dots,\ell_m}})}(t)$
(1, 1, 1)	$t^3+2t^4+t^5$
(1, 2)	$t^2+t^3+t^5+t^6$
(2, 1)	$t^2+t^3+t^5+t^6$
(3)	$1+t^3+t^5+t^8$

Table 5. Poincaré polynomials of the irreducible unitary representations of $\mathrm{SL}_3(\mathbb{C})$ with nontrivial $(\mathfrak{g}, K)$-cohomology.

Example 7.5. Let's look at some examples for small values of n.

In the case $n = 2$, $\mathrm{SL}_2(\mathbb{C})$ only has two standard parabolic subgroups, corresponding to the compositions $2 = 1 + 1$ and $2 = 2$ of 2. These are the minimal parabolic subgroup $Q = Q_0$ and the whole group $Q = G$, with associated representations π_{Q_0} and π_G (the latter being the trivial representation). An application of Theorem 7.4 gives us the Poincaré polynomials of the $(\mathfrak{g}, K)$-cohomology of these representations:

$$P_{H^*(\mathfrak{g},K;\pi_{Q_0})}(t) = t + t^2, \quad P_{H^*(\mathfrak{g},K;\pi_G)}(t) = 1 + t^3.$$

For $n = 3$, the situation is more complicated and we will give the results in Table 5. We denote a composition $n = \ell_1 + \cdots + \ell_m$ by the m-tuple $(\ell_1, \dots, \ell_m)$ and the associated parabolic subgroup by $Q_{\ell_1,\dots,\ell_m}$.

7.6. Let us relate our findings to the results of Theorem 6.5. Assume we are given a cocompact discrete subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{C})$. The irreducible unitary representations with trivial infinitesimal character that we have classified in the previous sections are exactly the representations that can possibly contribute to the cohomology of X/Γ via Matsushima's formula. In general, the question of whether or not a given representation $\pi \in \widehat{G}$ does actually contribute to the cohomology, i.e., $m(\Gamma, \pi) \neq 0$, is still open. However, the nonvanishing results for the cohomology in Theorem 6.5 imply the existence of (at least) one nontrivial automorphic representation for $\mathrm{SL}_n(\mathbb{C})$ with respect to Γ. For small values of n, we can even identify explicit representations with nonvanishing multiplicity. If for one of the degrees for which we have constructed nontrivial cohomology classes in Theorem 6.5 there is exactly one representation π with nontrivial $(\mathfrak{g}, K)$-cohomology that contributes in that degree, we can deduce that the corresponding multiplicity $m(\pi, \Gamma)$ is not zero.

To be able to compare the degrees in which we have cohomology coming from geometric cycles and the degrees to which our representations can possibly contribute, we summarize this information in Tables 6 and 7 for $n = 2, 3$. As above, we denote a representation π_Q by the associated tuple $(\ell_1, \dots, \ell_m)$.

	0	1	2	3
Cycles		×	×	
Trivial representation	×			×
(1, 1)		×	×	

Table 6. Complex case: contribution to the cohomology of X/Γ, $n = 2$.

	0	1	2	3	4	5	6	7	8
Cycles				×	×	×			
Trivial representation	×			×		×			×
(1, 1, 1)				×	×	×			
(2, 1)			×	×		×	×		
(1, 2)			×	×		×	×		

Table 7. Complex case: contribution to the cohomology of X/Γ, $n = 3$.

Corollary 7.7. *Let $n \in \{2, 3\}$ and let Q_0 denote the minimal parabolic subgroup of upper triangular matrices of $\mathrm{SL}_n(\mathbb{C})$. Then there exists a cocompact discrete subgroup $\Gamma \subset \mathrm{SL}_n(\mathbb{C})$ such that the multiplicity $m(\Gamma, \pi_{Q_0})$ is not* 0.

Proof. We choose Γ as in Theorem 6.5 for $n = 2$ or $n = 3$. Then the result can be read off from Tables 6 and 7: when $n = 2$, we have cycles contributing to the cohomology in degrees 1 and 2, and π_{Q_0} is the only unitary representation that has cohomology in these degrees. Therefore, we conclude $m(\Gamma, \pi_{Q_0}) \neq 0$. Similarly, for $n = 3$, we have cycles contributing to degree 4, and π_{Q_0} is the only unitary representation of $\mathrm{SL}_3(\mathbb{C})$ that has cohomology in degree 4. □

Remark. Unfortunately, for bigger n the situation is more complicated and this reasoning is not successful anymore. Already in the case $n = 4$ one can easily see (by looking at a similar table) that there is no degree in the cohomology of X/Γ in which we have a nontrivial class coming from a cycle and to which only one representation can contribute.

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