Pacific Journal of Mathematics

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Volume 282 No. 2 June 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 282, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.282.479

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Blaschke's conjecture asserts that if a complete Riemannian manifold M satisfies $\mathrm{diam}(M) = \mathrm{Inj}(M) = \frac{\pi}{2}$, then M is isometric to $\mathbb{S}^n\left(\frac{1}{2}\right)$ or to the real, complex, quaternionic or octonionic projective plane with its canonical metric. We prove that the conjecture is true under the assumption that $\sec_M \geq 1$.

Introduction

The projective spaces $\mathbb{KP}^n$ (considered with their canonical metric, induced from the unit sphere) and the sphere $\mathbb{S}^n\left(\frac{1}{2}\right)$ are the only known examples of complete Riemannian manifolds M satisfying

$$\mathrm{diam}(M) = \mathrm{Inj}(M) = \tfrac{\pi}{2}. \tag{0-1}$$

Here $\mathrm{diam}(M)$ and $\mathrm{Inj}(M)$ are the diameter and injective radius of M, and $\mathbb{K}$ is one of the division algebras $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{C}\mathrm{a}$, with $n \leq 2$ if $\mathbb{K} = \mathbb{C}\mathrm{a}$. A longstanding conjecture, whose history is reviewed in [Besse 1978; Berger 2003; Bougas 2013], asserts that these are the only possibilities:

Blaschke's Conjecture. *If a complete Riemannian manifold M satisfies* (0-1), *then M is isometric to $\mathbb{S}^n\left(\frac{1}{2}\right)$ or a $\mathbb{KP}^n$ endowed with the canonical metric.*

(See (1-1) below for the reason why it is called Blaschke's conjecture.) Up to now, the conjecture is still almost open (there are only some partial answers to it) although (0-1) is an extremely strong condition. Note that the conjecture has no restriction on the curvature. The main purpose of the present paper is to give a positive answer to the conjecture under the additional assumption $\sec_M \geq 1$, which is stated as follows.

Main Theorem. *If a complete Riemannian manifold M satisfies* (0-1) *and* $\sec_M \geq 1$, *then M is isometric to $\mathbb{S}^n\left(\frac{1}{2}\right)$ or a $\mathbb{KP}^n$ endowed with the canonical metric.*

If the curvature has an upper bound, we have the following result of Rovenskii and Toponogov [1998] (see also [Shankar et al. 2005]).

Supported by NSFC 11471039. Yusheng Wang is the corresponding author.
MSC2010: 53C20.
Keywords: Blaschke's conjecture, Berger's rigidity theorem, Toponogov's comparison theorem.

Theorem 0.1. *If a complete, simply connected Riemannian manifold M satisfies* (0-1) *and* $\sec_M \le 4$, *then M is isometric to* $\mathbb{S}^n\left(\frac{1}{2}\right)$ *or a* $\mathbb{KP}^n$ $(\mathbb{K} \neq \mathbb{R})$ *endowed with the canonical metric.*

From our Main Theorem and Theorem 0.1, one can see how beautiful the following Berger's rigidity theorem [Cheeger and Ebin 1975] is.

Theorem 0.2. *Let M be a complete, simply connected Riemannian manifold with* $1 \le \sec_M \le 4$. *If* $\operatorname{diam}(M) = \frac{\pi}{2}$, *then M is isometric to* $\mathbb{S}^n\left(\frac{1}{2}\right)$ *or a* $\mathbb{KP}^n$ $(\mathbb{K} \neq \mathbb{R})$ *endowed with the canonical metric.*

In fact, "$1 \le \sec_M \le 4$" and "simply connected" imply that $\operatorname{Inj}(M) \ge \frac{\pi}{2}$ [Cheeger and Gromoll 1980], so "$\operatorname{diam}(M) = \frac{\pi}{2}$" implies that M (in Theorem 0.2) satisfies (0-1) (note that $\operatorname{Inj}(M) \le \operatorname{diam}(M)$). Hence, the Main Theorem implies Theorem 0.2 in the premise of (0-1) (as does Theorem 0.1). (Of course, "$\sec_M \ge 1$" implies that $\operatorname{diam}(M) \le \pi$, and the maximal diameter theorem asserts that *if* $\operatorname{diam}(M) = \pi$, *then M is isometric to* $\mathbb{S}^n(1)$, so Theorem 0.2 is also called the minimal diameter theorem. Moreover, inspired by Theorem 0.2, Grove and Shiohama, Gromoll and Grove, and Wilhelm supply some beautiful (but not purely isometric) classifications under the conditions "$\sec_M \ge 1$ and $\operatorname{diam}(M) \ge \frac{\pi}{2}$ or $\operatorname{Rad}(M) \ge \frac{\pi}{2}$" [Gromoll and Grove 1987; Wilhelm 1996].)

Moreover, from the proof in [Cheeger and Ebin 1975] for Theorem 0.2, it is not hard to see the following.

Theorem 0.3. *Let M be a complete Riemannian manifold satisfying* (0-1) *and* $1 \le \sec_M \le 4$. *Then M is isometric to* $\mathbb{S}^n\left(\frac{1}{2}\right)$ *or a* $\mathbb{KP}^n$ *endowed with the canonical metric.*

We end this section with the idea of our proof of the Main Theorem. We first prove that for any $p \in M$, denoting by $|pq|$ the distance between p and q,

$$\{p\}^{=\pi/2} \triangleq \left\{q \in M \mid |pq| = \tfrac{\pi}{2}\right\}$$

is a complete totally geodesic submanifold in M. Then using Theorem 1.3 below and Toponogov's comparison theorem, we derive by induction that $1 \le \sec_M \le 4$, and thus the proof is done by Theorem 0.3. (We would like to point out that, in the premise of Theorem 1.3, we can use the method in [Gromoll and Grove 1987; 1988; Wilhelm 1996] to give the proof (which involves many significant classification results). By comparison, however, our proof is much more direct.)

1. Blaschke manifolds

A closed Riemannian manifold M is called a Blaschke manifold if it is *Blaschke* at each point $p \in M$, i.e., $\Uparrow_q^p$ *is a great sphere in* $\Sigma_q M$ *for any q in the cut locus of p*

[Besse 1978], where

$$\Sigma_q M \triangleq \{v \in T_q M \mid |v| = 1\},$$
$$\Uparrow_q^p \triangleq \{\text{the unit tangent vector at } q \text{ of a minimal geodesic from } q \text{ to } p\}.$$

On a Blaschke manifold, one can get the following not so obvious fact (p. 137 in [Besse 1978]).

Proposition 1.1. *For a Blaschke manifold M, we have that* $\operatorname{diam}(M) = \operatorname{Inj}(M)$.

A much more difficult observation is the following (p. 138 in [Besse 1978]).

Proposition 1.2. *Given a closed Riemannian manifold M and a point $p \in M$, if $|pq|$ is a constant for all q in the cut locus of p, then M is Blaschke at p.*

Obviously, it follows from Propositions 1.1 and 1.2 that

(1-1) a closed Riemannian manifold M is Blaschke $\Leftrightarrow$ $\operatorname{diam}(M) = \operatorname{Inj}(M)$.

Up to now, Blaschke's conjecture has been solved only for spheres.

Theorem 1.3 [Besse 1978; Berger 2003]. *If a Blaschke manifold is homeomorphic to a sphere, then it is isometric to the unit sphere (up to a rescaling).*

2. Proof of the Main Theorem

We first give our main tool of the paper: Toponogov's comparison theorem.

Theorem 2.1 [Petersen 1998; Grove and Markvorsen 1995]. *Let M be a complete Riemannian manifold with $\sec_M \geq \kappa$, and let $\mathbb{S}^2_\kappa$ be the complete, simply connected 2-manifold of curvature κ.*

(i) *To any $p \in M$ and minimal geodesic $[qr] \subset M$, we associate $\tilde{p}$ and a minimal geodesic $[\tilde{q}\tilde{r}]$ in $\mathbb{S}^2_\kappa$ with $|\tilde{p}\tilde{q}| = |pq|$, $|\tilde{p}\tilde{r}| = |pr|$ and $|\tilde{r}\tilde{q}| = |rq|$. Then for any $s \in [qr]$ and $\tilde{s} \in [\tilde{q}\tilde{r}]$ with $|qs| = |\tilde{q}\tilde{s}|$, we have that $|ps| \geq |\tilde{p}\tilde{s}|$.*

(ii) *To any minimal geodesics $[qp]$ and $[qr]$ in M, we associate minimal geodesics $[\tilde{q}\tilde{p}]$ and $[\tilde{q}\tilde{r}]$ in $\mathbb{S}^2_\kappa$ with $|\tilde{q}\tilde{p}| = |qp|$, $|\tilde{q}\tilde{r}| = |qr|$ and $\angle \tilde{p}\tilde{q}\tilde{r} = \angle pqr$. Then we have that $|\tilde{p}\tilde{r}| \geq |pr|$.*

(iii) *If equality in (ii) (or in (i) for some s in the interior part of $[qr]$) holds, then there exists a minimal geodesic $[pr]$ such that the triangle formed by $[qp]$, $[qr]$ and $[pr]$ bounds a surface which is convex[1] and can be isometrically embedded into $\mathbb{S}^2_\kappa$.*

[1]We say that a subset A is convex (resp. totally convex) in M if, between any $x \in A$ and $y \in A$, some minimal geodesic $[xy]$ (resp. all minimal geodesics) belongs to A.

In the rest of this paper, M always denotes the manifold in the Main Theorem, and N denotes $\{p\}^{=\pi/2} \triangleq \{q \in M \mid |pq| = \frac{\pi}{2}\}$ for an arbitrary fixed point $p \in M$. We first give an easy observation following from (0-1) (i.e., $\mathrm{Inj}(M) = \mathrm{diam}(M) = \frac{\pi}{2}$), namely that

(2-1) for any $x \in M$,

there is a minimal geodesic $[pq]$ with $q \in N$ such that $x \in [pq]$.

Lemma 2.2. *N is a complete totally geodesic submanifold in M; if* $\dim(N) = 0$, *then N consists of a single point.*

Remark 2.3. Since $\sec_M \geq 1$, it follows from (i) of Theorem 2.1 that

$$\{p\}^{\geq \pi/2} \triangleq \{q \in M \mid |pq| \geq \tfrac{\pi}{2}\}$$

is totally convex in M. Note that $N = \{p\}^{\geq \pi/2}$ because $\mathrm{diam}(M) = \frac{\pi}{2}$, and that N is closed in M. On the other hand, since M is a Blaschke manifold, we know that N is a submanifold in M [Besse 1978]. It then follows that N is a totally geodesic submanifold in M. This proof is short because we apply the proposition that N is a submanifold in M, which is a significant property of a Blaschke manifold [Besse 1978]. Here, in order to show the importance of "$\sec_M \geq 1$", we will supply a proof only based on the definition of a Blaschke manifold.

Proof of Lemma 2.2. From Remark 2.3, we know that N is totally convex in M, which implies that N consists of a single point if $\dim(N) = 0$. Hence, we can assume that $\dim(N) > 0$; for any geodesic $\gamma(t)|_{t\in[0,\ell]} \subset N$, we need only to show that its prolonged geodesic $\gamma(t)|_{t\in[0,\ell+\varepsilon]}$ in M also belongs to N for some small $\varepsilon > 0$. Note that, without loss of generality, we can assume that there is a unique minimal geodesic between $\gamma(0)$ and $\gamma(\ell+\varepsilon)$. Due to (2-1), we can select $q \in N$ such that $\gamma(\ell+\varepsilon) \in [pq]$. Observe that $q \neq \gamma(0)$ (otherwise, $\gamma(\ell) \in [pq]$ must hold, contradicting $\gamma(\ell) \in N$). Let $[q\gamma(0)]$ be a minimal geodesic in N (note that N is convex in M). By the first variation formula, it is easy to see that

$$|\uparrow_q^{\gamma(0)} \xi| \geq \tfrac{\pi}{2} \quad \text{in } \Sigma_q M, \quad \text{for any } \xi \in \Uparrow_q^p.$$

On the other hand, $\Uparrow_q^p$ is a great sphere in $\Sigma_q M$ because M is Blaschke at p (see Proposition 1.2). It follows that in fact

$$|\uparrow_q^{\gamma(0)} \xi| = \tfrac{\pi}{2} \quad \text{for any } \xi \in \Uparrow_q^p.$$

Then by (iii) of Theorem 2.1, there is a minimal geodesic $[p\gamma(0)]$ such that the triangle formed by $[q\gamma(0)]$, $[pq]$ and $[p\gamma(0)]$ bounds a surface (containing $[\gamma(0)\gamma(\ell+\varepsilon)]$) which is convex and can be isometrically embedded into $\mathbb{S}^2(1)$. It then has to hold that $[\gamma(0)\gamma(\ell+\varepsilon)] = [\gamma(0)q]$ because $[\gamma(0)\gamma(\ell)]$ belongs to N, and so $[\gamma(0)\gamma(\ell+\varepsilon)] \subset N$. □

Since N is a complete totally geodesic submanifold in M, for any $q \in N$, any minimal geodesic $[pq]$ is perpendicular to N at q, i.e.,

$$\Uparrow_q^p \subseteq (\Sigma_q N)^{=\pi/2} \quad \text{in } \Sigma_q M. \tag{2-2}$$

Then from the proof of Lemma 2.2, we have the following corollary.

Corollary 2.4. *For any minimal geodesics $[pq]$ and $[qq'] \subset N$, there is a minimal geodesic $[pq']$ such that the triangle formed by $[pq]$, $[qq']$ and $[pq']$ bounds a surface which is convex and can be isometrically embedded into $\mathbb{S}^2(1)$.*

Moreover, the "$\subseteq$" in (2-2) can in fact be changed to "$=$".

Lemma 2.5. *For any $q \in N$, we have that $\Uparrow_q^p = (\Sigma_q N)^{=\pi/2}$ in $\Sigma_q M$.*

Proof. According to (2-2), it suffices to show that for any $\zeta \in (\Sigma_q N)^{=\pi/2}$ there is a minimal geodesic $[qp]$ such that $\uparrow_q^p = \zeta$. Note that there is a minimal geodesic $[qx]$ ($x \in M$) such that $\uparrow_q^x = \zeta$, and we can assume that there is a unique geodesic between q and x. It follows from (2-1) that there is a minimal geodesic $[pq_x]$ with $q_x \in N$ such that $x \in [pq_x]$. Hence, we need only to show that $q_x = q$. If this is not true, then by Corollary 2.4 there are minimal geodesics $[pq]$ and $[qq_x] \subset N$ such that the triangle formed by $[pq]$, $[pq_x]$ and $[qq_x]$ bounds a surface D which is convex and can be isometrically embedded into $\mathbb{S}^2(1)$. Note that $[qx]$ belongs to D. This is impossible because both $[qp]$ (see (2-2)) and $[qx]$ are perpendicular to $[qq_x]$ at q (in D). □

Now we give the proof of our Main Theorem.

Proof of the Main Theorem. Note that, according to Theorem 0.3, we need only to show that

$$1 \le \sec_M \le 4. \tag{2-3}$$

We will apply induction on $\dim(N)$.

- $\dim(N) = 0$: By Lemma 2.2, N consists of a point, so M is homeomorphic to a sphere (because M consists of minimal geodesics between p and N). It follows from Theorem 1.3 that M is isometric to $\mathbb{S}^n\left(\frac{1}{2}\right)$ (which implies (2-3)).

- $\dim(N) = 1$: Note that N is a closed geodesic of length π. Let q_1 and q_2 be two antipodal points of N (i.e., $|q_1q_2| = \frac{\pi}{2}$). It follows that there are only two minimal geodesics between q_1 and q_2 (note that N is totally convex in M). Similarly, we consider $L \triangleq \{q_2\}^{=\pi/2}$ containing p and q_1, which is a totally geodesic submanifold in M of dimension > 0 by Lemma 2.2. Then similar to Lemma 2.5, we have that

$$\Uparrow_p^{q_2} = (\Sigma_p L)^{=\pi/2} = (\Sigma_{q_1} L)^{=\pi/2} = \Uparrow_{q_1}^{q_2}.$$

This implies that there are only two minimal geodesics between p and any $q \in N$ (by Lemma 2.5). It is then easy to see that $\sec_M \equiv 1$ by Corollary 2.4 (in fact, M is isometric to $\mathbb{RP}^2$ with the canonical metric).

• $\dim(N) > 1$: Since N is a complete totally geodesic submanifold in M (see Lemma 2.2), (0-1) implies that

$$\operatorname{diam}(N) = \operatorname{Inj}(N) = \tfrac{\pi}{2}. \tag{2-4}$$

By the inductive assumption on N, we have that

$$1 \le \sec_N \le 4. \tag{2-5}$$

On the other hand, we claim:

Claim. *For any $q \in N$,*

$$S(p, q) \triangleq \{\textit{the point on a minimal geodesic between } p \textit{ and } q\}$$

is totally geodesic in M and is isometric to $\mathbb{S}^m\left(\tfrac{1}{2}\right)$, where $m = \dim(M) - \dim(N)$.

Note that (2-3) is implied by the claim, (2-5), Lemma 2.5, Corollary 2.4 and Lemma 2.2. Hence, in the rest of the proof, we need only to verify the claim.

By (2-4), we can select $r \in N$ such that $|qr| = \frac{\pi}{2}$. Similarly, we consider $K \triangleq \{r\}^{=\pi/2}$ containing p and q, which is a complete totally geodesic submanifold in M with $\dim(K) > 0$; moreover, we have that

$$\Uparrow_p^r = (\Sigma_p K)^{=\pi/2},$$

and $\Uparrow_p^r$ is isometric to a unit sphere by Lemma 2.5. On the other hand, note that $\Uparrow_r^p$ is isometric to $\mathbb{S}^{m-1}(1)$ by Lemma 2.5, and that $\Uparrow_r^p$ is isometric to $\Uparrow_p^r$. Therefore, it is easy to see (again from Lemma 2.5 on K) that

$$\dim(K) = \dim(N).$$

Hence, by the inductive assumption on K (similar to on N), K is isometric to $\mathbb{S}^l\left(\tfrac{1}{2}\right)$ or a $\mathbb{KP}^l$ endowed with the canonical metric, which implies the claim above. □

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Received June 11, 2015.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 282 No. 2 June 2016