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A NEW FAMILY OF SIMPLE $\mathfrak{gl}_{2n}(\mathbb{C})$-MODULES

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We construct a new family of simple $\mathfrak{gl}_{2n}$-modules which depends on n^2 generic parameters. Each module in the family is isomorphic to the regular $U(\mathfrak{gl}_n)$-module when restricted the $\mathfrak{gl}_n$-subalgebra naturally embedded into the top-left corner.

1. Introduction

Classification of simple modules is one of the first natural questions which arises when studying the representation theory of some (Lie) algebra. Simple modules are, in some sense, "building blocks" for all other modules, and hence understanding simple modules is important. In some cases, for example for finite dimensional associative algebras, classification of simple modules is an easy problem. However, in most of the cases, the problem of classification of all simple modules is very difficult. Thus, if we consider simple, finite dimensional, complex Lie algebras, then the only algebra for which some kind of classification exists is the Lie algebra $\mathfrak{sl}_2$. This was obtained by R. Block [1981]; see also a detailed explanation in [Mazorchuk 2010, Chapter 6]. However, even in this case the "answer" only reduces the problem to classification of equivalence classes of irreducible elements in a certain noncommutative Euclidean ring.

At the moment, the problem of *classification* of simple modules over simple Lie algebras seems too hard. However, because of its importance, the problem of *construction* of new families of modules attracted a lot of attention over the years. The most studied case seem to be the one of the Virasoro Lie algebras, where many different multiparameter families of simple modules were constructed by various authors; see, for example, [Ondrus and Wiesner 2009; Lu et al. 2011; Lu and Zhao 2014 Liu et al. 2015; Mazorchuk and Zhao 2007; 2014; Mazorchuk and Wiesner 2014] and references therein.

In contrast to the Virasoro case, the "easier" case of simple, complex, finite dimensional Lie algebras does not yet have an equally large variety of families of simple modules. So, let $\mathfrak{g}$ be a complex, finite dimensional, simple Lie algebra. Some classes of simple $\mathfrak{g}$-modules are, of course, well understood. For example:

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- simple *finite dimensional* modules are classified already by Cartan [1913];
- simple *highest weight* modules related to a fixed triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$ are classified by their highest weights and are extensively studied during last 50 years, see, for example, [Dixmier 1974; Humphreys 2008; Bernstein et al. 1976];
- simple Whittaker modules in the sense of [Kostant 1978] — see also [Arnal and Pinczon 1974; McDowell 1985; 1993];
- simple *Gelfand–Zeitlin* modules — see [Drozd et al. 1991; 1994; Mazorchuk 2001; Futorny et al. 2015];
- simple weight modules with *finite dimensional weight spaces* were classified in [Mathieu 2000] extending the previous work in [Fernando 1990; Futorny 1987];
- simple $\mathfrak{g}$-modules which are free of rank one over the universal enveloping algebra of the Cartan subalgebra were constructed and studied in [Nilsson 2015; 2016] (see also [Tan and Zhao 2013; 2015] for similar modules over infinite dimensional Lie algebras).

Some further classes of simple modules can be found in [Futorny et al. 2011]. We note that the largest known family of simple $\mathfrak{gl}_n$-modules is the one of Gelfand–Zeitlin-modules. It depends on $\frac{n(n+1)}{2}$ generic complex parameters, see [Drozd et al. 1991; 1994] for details.

Based on the above, it seems natural to look for new families of simple $\mathfrak{g}$-modules. The present paper contributes a new large family of simple $\mathfrak{gl}_{2n}$-modules. This family is parameterized by invertible $n \times n$ complex matrices. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{D}$ be the four Lie subalgebras of $\mathfrak{gl}_{2n}$ of dimension n^2 as indicated in the following figure:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}.$$

Then $\mathcal{B}$ is nilpotent (and even commutative), and the adjoint action of $\mathcal{B}$ on $\mathfrak{gl}_{2n}/\mathcal{B}$ is nilpotent, so $(\mathcal{B}, \mathfrak{gl}_{2n})$ is a *Whittaker pair* in the sense of [Batra and Mazorchuk 2011]. The original motivation for this paper was an attempt to describe generalized Whittaker modules (i.e., modules on which the action of $\mathcal{B}$ is locally finite) for this Whittaker pair. Our main result can be summarized as follows:

Theorem 1. *For each nondegenerate complex $n \times n$-matrix Q, there exists a simple $\mathfrak{gl}_{2n}$ module M with the following properties*:

- *M has Gelfand–Kirillov dimension n^2.*
- $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}}$ *M is isomorphic to the left regular $U(\mathcal{A})$-module.*

- $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{B}} M$ *is locally finite. In other words*, M *is a generalized Whittaker module for the Whittaker pair* $(\mathcal{B}, \mathfrak{gl}_{2n})$.
- *With respect to a fixed PBW basis in* $U(\mathcal{A})$, *the action of each fixed element from* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *can be written explicitly as maps* $U(\mathcal{A}) \to U(\mathcal{A})$ *of degrees* $1, 0, 2, 1$, *respectively.*

Moreover, different matrices Q *give nonisomorphic modules.*

The paper is organized as follows. Section 2 introduces notation and lays down some motivation for the construction of our modules. In the same section, for each nondegenerate complex $n \times n$ matrix Q, we construct an $(\mathcal{A}+\mathcal{B})$-module having the first three properties listed in Theorem 1. We show that there must exist a simple quotient of the corresponding induced $\mathfrak{gl}_{2n}$ module that also has the fourth property. In Section 3 we explicitly construct such a module when Q is the identity matrix I and show that every other module in our family can be obtained by twisting this module by an explicit automorphism. Finally, we give explicit formulas for the $\mathfrak{gl}_{2n}$-action in all cases.

2. Motivation and existence

2.1. ***Setup.*** Let $\mathfrak{g} := \mathfrak{gl}_{2n}(\mathbb{C})$. Unless otherwise stated, all Lie algebras and vector spaces are over the complex numbers. $\mathbb{N}$ denotes the set of nonnegative integers.

First we observe that the subalgebras $\mathcal{A}$ and $\mathcal{D}$ defined above are both isomorphic to $\mathfrak{gl}_n$ while the subalgebras $\mathcal{B}$ and $\mathcal{C}$ are commutative. Let $e_{i,j}$ be the $2n \times 2n$-matrix with a single 1 in position (i, j) and zeros elsewhere. By convention, most indices i, j, etc. can be assumed to lie between 1 and n; in particular our canonical basis for $\mathfrak{gl}_{2n}$ will be written

$$\bigcup_{1\le i,j\le n} \{e_{i,j}, e_{n+i,j}, e_{i,n+j}, e_{n+i,n+j}\}.$$

We denote the identity matrix by I, its size (n or $2n$) should be apparent by the context. The transpose of a matrix A is denoted A^T and if A is invertible we abbreviate $(A^{-1})^T$ by A^{-T}.

We also recall how to construct *twisted modules*. For every Lie algebra automorphism $\varphi \in \mathrm{Aut}(\mathfrak{g})$ we have a twisting functor $F_\varphi : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ which is an auto-equivalence. It maps a module M to ${}^{\varphi}M$ which is isomorphic to M as a vector space but has modified action: $x \bullet v := \varphi(x) \cdot v$ for all $x \in \mathfrak{g}$ and $v \in {}^{\varphi}M$.

2.2. ***Existence of simple generalized Whittaker modules for*** $\mathfrak{gl}_{2n}$**.** Following the idea in [Kostant 1978], we try to construct some modules on which the action of $\mathcal{B}$ is locally finite.

Fix Lie algebra homomorphisms $\lambda_A : \mathcal{A} \to \mathbb{C}$ and $\lambda_D : \mathcal{D} \to \mathbb{C}$. Let $\mathbb{C}_{\lambda_A\lambda_D}$ be the one dimensional $(\mathcal{A}+\mathcal{C}+\mathcal{D})$-module where $\mathcal{A}$ acts by λ_A, $\mathcal{D}$ acts by λ_D and $\mathcal{C}$ acts trivially. Now define a generalized Verma module

$$M_{\lambda_A\lambda_D} := U(\mathfrak{gl}_{2n}) \otimes_{U(\mathcal{A}+\mathcal{C}+\mathcal{D})} \mathbb{C}_{\lambda_A\lambda_A}.$$

Denote by $M^*_{\lambda_A\lambda_D}$ the full dual of $M_{\lambda_A\lambda_D}$. This is a $\mathfrak{gl}_{2n}$ module where the action is given by $(x \cdot f)(m) = -f(x \cdot m)$ as usual.

Proposition 2. *For every $\theta : \mathcal{B} \to \mathbb{C}$, there is a unique (up to multiple) eigenvector w in $M^*_{\lambda_A\lambda_D}$ with eigenvalue θ for $\mathcal{B}$.*

Proof. Note that $M_{\lambda_A\lambda_D} \simeq U(\mathcal{B})$ as a left and right $U(\mathcal{B})$-module. Let $\mathbb{C}(\theta)$ be the 1-dimensional $\mathcal{B}$-module where the action is given by θ. By the tensor-hom adjunction we have

$$\begin{aligned}\mathrm{Hom}_{U(\mathcal{B})}(\mathbb{C}(\theta), M^*_{\lambda_A\lambda_D}) &= \mathrm{Hom}_{U(\mathcal{B})}\big(\mathbb{C}(\theta), \mathrm{Hom}_{\mathbb{C}}(M_{\lambda_A\lambda_D}, \mathbb{C})\big)\\ &\simeq \mathrm{Hom}_{U(\mathcal{B})}\big(\mathbb{C}(\theta), \mathrm{Hom}_{\mathbb{C}}(U(\mathcal{B}), \mathbb{C})\big)\\ &\simeq \mathrm{Hom}_{\mathbb{C}}\big(U(\mathcal{B}) \otimes_{U(\mathcal{B})} \mathbb{C}(\theta), \mathbb{C}\big)\\ &\simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(\theta), \mathbb{C}) \simeq \mathbb{C}.\end{aligned}$$

Thus there is a unique 1-dimensional subspace of $M^*_{\lambda_A\lambda_D}$ isomorphic to $\mathbb{C}(\theta)$ in $\mathcal{B}$-mod, which is equivalent to the statement of the proposition. $\square$

The submodule generated by such an eigenvector must be simple (see [Batra and Mazorchuk 2011]), so we get the following result.

Corollary 3. *For the pair $(\mathcal{B}, \mathfrak{gl}_{2n})$, there exist simple generalized Whittaker modules and they can be realized as simple submodules in the dual of the generalized Verma module $M^*_{\lambda_A\lambda_D}$.*

The drawback with this approach in our case is that it is difficult to say anything more explicit about the resulting modules as $M^*_{\lambda_A\lambda_D}$ is very big and inconvenient to work in.

2.3. *An $(\mathcal{A}+\mathcal{B})$-module.*

2.3.1. *Construction and a formula for the action.* We now turn to a more explicit construction. Note that $\mathcal{B}$ is commutative. Let $Q = (q_{ij})$ be a nonsingular $n \times n$ matrix and define L_Q to be the 1-dimensional $U(\mathcal{B})$-module with generator v where the action of $\mathcal{B}$ is given by Q:

$$e_{i,n+j} \cdot v := q_{i,j} v \quad 1 \le i, j \le n.$$

Define an induced module

$$M_Q := \mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}+\mathcal{B}} L_Q = U(\mathcal{A}+\mathcal{B}) \otimes_{U(\mathcal{B})} L_Q.$$

Then M_Q is clearly isomorphic to $U(\mathcal{A})$ as a left $\mathcal{A}$-module, and for $a \in U(\mathcal{A})$ we shall write just av or just a for $a \otimes v$. To explicitly see how $\mathcal{B}$ acts on M_Q, we introduce some more notation. Consider $U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$ as a tensor product in the category of unital associative algebras. This becomes an infinite dimensional Lie algebra under the commutator bracket. Note that $U(\mathcal{A}) \otimes \mathcal{A} \simeq \mathrm{Mat}_{n\times n}(U(\mathcal{A}))$ in a natural way and we shall even extend the trace function to $U(\mathcal{A}) \otimes \mathcal{A}$ by defining $\mathrm{tr}(a \otimes B) := a\ \mathrm{tr}(B)$. Note also that $\mathcal{A}$ embeds into $U(\mathcal{A}) \otimes \mathcal{A}$ (both as an associative algebra and as a Lie algebra) by the map $A \mapsto 1 \otimes A$, and we shall sometimes need to identify elements of $\mathcal{A}$ with their images under this map. To resolve some ambiguity in our notation, for $A, B \in \mathcal{A}$ we shall write AB for the product in $U(\mathcal{A})$ and $A.B$ for the product in the associative algebra $\mathcal{A}$ or $U(\mathcal{A}) \otimes \mathcal{A}$.

Let $\psi' : \mathcal{A} \to U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$ be the Lie algebra homomorphism defined by

$$\psi' : A \mapsto A \otimes I - 1 \otimes A^T.$$

This extends to an algebra homomorphism $\psi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$.

Lemma 4. *The action of $\mathcal{B}$ on M_Q is given by*

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av = \mathrm{tr}(\psi(a).Q.B^T)v.$$

Proof. This follows by induction on the degree of a as follows. The lemma clearly holds for $a = 1$ by the definition of the action of $\mathcal{B}$ on L_Q: we have $\mathrm{tr}(Q.B^T) = \sum_{ij} q_{ij} b_{ij}$. Suppose the lemma holds for all monomials a of a fixed degree (with respect to any fixed PBW basis). We then have

$$\begin{aligned}
\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} (Aa)v &= A \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av + \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right] av \\
&= A \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av - \begin{pmatrix} 0 & A.B \\ 0 & 0 \end{pmatrix} av \\
&= A\ \mathrm{tr}(\psi(a).Q.B^T)v - \mathrm{tr}(\psi(a).Q.(A.B)^T)v \\
&= \mathrm{tr}((A \otimes I).\psi(a).Q.B^T)v - \mathrm{tr}(A^T.\psi(a).Q.B^T)v \\
&= \mathrm{tr}\big(((A \otimes I) - 1 \otimes A^T).\psi(a).Q.B^T\big)v \\
&= \mathrm{tr}(\psi(A).\psi(a).Q.B^T)v \\
&= \mathrm{tr}(\psi(Aa).Q.B^T)v.
\end{aligned}$$

This shows that the lemma holds for all monomials in $U(\mathcal{A})$ by induction. Since ψ is linear, it holds for all of $U(\mathcal{A})$. □

2.3.2. *Proof of simplicity.* We proceed to prove that M_Q is simple by first proving it for $Q = I$.

Lemma 5. *The following relations hold in $U(\mathcal{A}+\mathcal{B})$.*

$$[e_{j,k+n}, e_{i,j}^m] = \begin{cases} -m\, e_{i,j}^{m-1} e_{i,k+n} & \text{for } i \neq j, \\ ((e_{i,j}-1)^m - e_{i,j}^m) e_{i,k+n} & \text{for } i = j. \end{cases}$$

Proof. This follows easily by induction on m. □

Fix a PBW basis of $U(\mathcal{A})$ of form

$$\{e_{11}^{l_{11}} e_{12}^{l_{12}} \cdots e_{1n}^{l_{1n}} e_{21}^{l_{21}} \cdots \cdots e_{n1}^{l_{n1}} \cdots e_{nn}^{l_{nn}} \mid l_{ij} \in \mathbb{N}\},$$

Then $U(\mathcal{A}) \simeq M_I$ has a filtration:

$$M_I^{(0)} \subset M_I^{(1)} \subset M_I^{(2)} \subset \cdots$$

where $M_I^{(m)}$ is the span of all monomials f with $\deg f := \sum_{ij} l_{ij} \leq m$.

Lemma 6. *For each $1 \leq j, k \leq n$, the element $(e_{j,k+n} - \delta_{j,k}) \in U(\mathcal{B})$ has degree -1 with respect to the filtration of M_I. Moreover, the action on an arbitrary monomial in $M_I^{(d)}$ is given by*

$$(e_{j,k+n} - \delta_{j,k}) \cdot e_{11}^{l_{11}} \cdots e_{kj}^{l_{kj}} \cdots e_{nn}^{l_{nn}} = -l_{kj}\, e_{11}^{l_{11}} \cdots e_{kj}^{l_{kj}-1} \cdots e_{nn}^{l_{nn}} \mod M_I^{(d-2)}.$$

Proof. We have

$$(e_{j,k+n} - \delta_{j,k}) \cdot f = f(e_{j,k+n} - \delta_{j,k}) + [e_{j,k+n} - \delta_{j,k}, f] = [e_{j,k+n}, f],$$

so the fact that $(e_{j,k+n} - \delta_{j,k})$ has degree ≤ -1 follows from the previous lemma and the fact that $\mathrm{ad}_{e_{j,k+n}}$ is a derivation.

For the second, more precise, statement, let f be an arbitrary monomial of degree d. For each i let P_i, Q_i be the monomial factors of f such that $f = P_i e_{ij}^{l_{ij}} Q_i$ and $e_{ij} \not| P_i, Q_i$. We now calculate

$$\begin{aligned}(e_{j,k+n} - \delta_{j,k}) \cdot f = [e_{j,k+n}, f] &= \sum_i P_i [e_{j,k+n}, e_{ij}^{l_{ij}}] Q_i \\ &= P_j((e_{jj}-1)^{l_{jj}} - e_{jj}^{l_{jj}}) e_{j,k+n} \cdot Q_j + \sum_{i \neq j} -l_{ij}\, P_i e_{ij}^{l_{ij}-1} e_{i,k+n} \cdot Q_i.\end{aligned}$$

By writing

$$e_{i,k+n} = (e_{i,k+n} - \delta_{ik}) + \delta_{ik},$$

and using the fact that the first term has negative degree, we see that

$$\begin{aligned}(e_{j,k+n}-\delta_{j,k})\cdot f &= \delta_{j,k}P_j((e_{jj}-1)^{l_{jj}}-e_{jj}^{l_{jj}})Q_j \\ &\qquad + \sum_{i\neq j} -\delta_{ik}l_{ij}\ P_i e_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -\delta_{j,k}l_{jj}\ P_j e_{ij}^{l_{ij}-1}Q_j + \sum_{i\neq j} -\delta_{ik}l_{ij}\ P_i e_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -\sum_i \delta_{ik}l_{ij}\ P_i e_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -l_{kj}\ P_k e_{kj}^{l_{kj}-1}Q_k \mod M_I^{(d-2)}.\end{aligned}$$

The lemma follows. □

Corollary 7. *For each $1 \le i, j \le n$, the action of $(e_{i,n+j}-\delta_{i,j})$ on M_I is surjective. Its kernel is spanned by all monomials not divisible by $e_{j,i}$.*

Proposition 8. *The module M_I is simple in $U(\mathcal{A}+\mathcal{B})$-mod.*

Proof. It suffices to show that any $f \in M_I$ can be reduced to $1 \in M_I^0$ via the $\mathcal{B}$-action. Fix $f \in M_I$ and let $p \in M_I^{(d)}$ be a nonzero monomial occurring in f with maximal degree d. If $p = \prod_{ij} e_{ij}^{l_{ij}}$ (in the PBW order), it is clear by the previous lemma that

$$B_p := \prod_{ij}(e_{j,n+i}-\delta_{ij})^{l_{ij}} \in U(\mathcal{B})$$

maps p to a nonzero constant. By the maximality of d, B_p annihilates all other monomials occurring in f so in fact $B_p \cdot f \in M_I^{(0)}$ is a nonzero constant as desired. □

Corollary 9. *The module M_Q is simple if and only if Q is nonsingular.*

Proof. For each nonsingular $S \in \mathcal{A}$, define $\varphi_S : \mathcal{A}+\mathcal{B} \to \mathcal{A}+\mathcal{B}$ by

$$\varphi_S : \begin{pmatrix} A & B \\ 0 & 0 \end{pmatrix} \mapsto \begin{pmatrix} A & B.S^{-1} \\ 0 & 0 \end{pmatrix}.$$

It is easy to verify that φ_S is a Lie algebra automorphism and that $\varphi_S \circ \varphi_T = \varphi_{ST}$. It is also clear that the twisted module ${}^{\varphi_{Q^{-T}}}M_I$ is isomorphic to M_Q. Since M_I is simple by Proposition 8, and since twisting by automorphisms defines an auto-equivalence on $\mathfrak{gl}_{2n}$-Mod, M_Q is also simple for nonsingular Q.

Conversely, assume that Q is singular and let A be a nonzero matrix such that $Q^T A = 0$. We shall show that $U(\mathcal{A})Av$ is a proper $(\mathcal{A}+\mathcal{B})$-submodule of M_Q. The subspace $U(\mathcal{A})Av$ is clearly $\mathcal{A}$-stable. For $a \in U(\mathcal{A})$ we compute

$$\begin{aligned}\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot aAv &= \operatorname{tr}(\psi(aA).Q.B^T)v \\ &= \operatorname{tr}(\psi(a).\psi(A).Q.B^T)v = \operatorname{tr}(\psi(a).(A \otimes I - 1 \otimes A^T).Q.B^T)v \\ &= \operatorname{tr}(Q.B^T.\psi(a).(A \otimes I))v - \operatorname{tr}(\psi(a).A^T.Q.B^T)v \\ &= \operatorname{tr}(Q.B^T.\psi(a))Av - \operatorname{tr}(\psi(a).(Q^T.A)^T.B^T)v \\ &= \operatorname{tr}(Q.B^T.\psi(a))Av.\end{aligned}$$

Thus $U(\mathcal{A})Av$ is also $\mathcal{B}$-stable, and is thus a proper submodule of M_Q. □

2.3.3. *Injectivity and an existence theorem.* Our next goal is to prove that for most Q's, the module M_Q is injective when restricted to $U(\mathcal{B})$. We begin by recalling a result about injective envelopes for the trivial module over polynomial rings. For a proof, see for example [Lam 1999, §3J].

Lemma 10. *Let k be a field, let $R = k[x_1, \ldots x_n]$ and let L be the trivial R-module. Let E be the R-module $k[x_1^{-1}, \ldots x_n^{-1}]$, where x_i acts by*

$$x_i \cdot (x_1^{-k_1} \cdots x_n^{-k_n}) = \begin{cases} x_1^{-k_1} \cdots x_i^{-k_i+1} \cdots x_n^{-k_n} & \text{if } k_i > 0, \\ 0 & \text{otherwise.} \end{cases}$$

Then $E = E(L)$ is the injective envelope of L.

By twisting E by automorphisms we obtain injective envelopes of all 1-dimensional R-modules as follows:

Corollary 11. *With notation as in the previous lemma, for scalars $q_i \in k$, let $L_{q_1,\ldots,q_n}$ be the 1-dimensional R-module with action $x_i \cdot v = q_i v$. Then $E(L_{q_1,\ldots,q_n}) \simeq {}^{\varphi}E(L)$ where φ is the R-automorphism mapping $x_i \mapsto x_i - q_i$.*

Proof. We have $L_{q_1,\ldots,q_n} \simeq {}^{\varphi}L$ and since twisting by an automorphism is an auto-equivalence on R-mod, the corollary follows. □

Proposition 12. *For nonsingular matrices Q, the module $\operatorname{Res}^{U(\mathcal{A}+\mathcal{B})}_{U(\mathcal{B})} M_Q$ is injective.*

Proof. Let $I(L_Q)$ be the injective envelope of L_Q. Applying the exact functor $\operatorname{Hom}_{\mathcal{B}}(-, I(L_Q))$ to the exact sequence

$$0 \to L_Q \to M_Q \to \text{Coker} \to 0$$

we obtain the exact sequence

$$0 \to \operatorname{Hom}_{\mathcal{B}}(\text{Coker}, I(L_Q)) \to \operatorname{Hom}_{\mathcal{B}}(M_Q, I(L_Q)) \to \operatorname{Hom}_{\mathcal{B}}(L_Q, I(L_Q)) \to 0.$$

Hence the morphism $L_Q \to I(L_Q)$ mapping L_Q into its injective envelope is the image of some morphism $f : M_Q \to I(L_Q)$. Since f is nonzero on $\operatorname{span}(v) = \operatorname{soc}(M_Q)$, f is injective. Moreover, for all $k \in \mathbb{N}$ we have

$$\dim \operatorname{soc}_k(M_Q) = \binom{n^2+k-2}{k-1} = \dim \operatorname{soc}_k(I(L_Q)),$$

which shows that f is surjective. This shows that f is an isomorphism and in particular that M_Q is the injective envelope of L_Q. □

Remark 13. Indecomposable injectives over noetherian rings R correspond to $\operatorname{Spec}(R)$ via $\mathfrak{p} \mapsto$injective envelope of $(R/\mathfrak{p})$. Moreover $L_Q = U(\mathcal{B})/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal generated by $(e_{i,n+j} - q_{i,j})$, so if M_Q is injective, it must be the injective envelope of $U(\mathcal{B})/\mathfrak{m}$.

Theorem 14. *For each nonsingular matrix $n \times n$-matrix Q there exists a $\mathfrak{gl}_{2n}$-module M such that*:

- *M is generated by a single $\mathcal{B}$-eigenvector with eigenvalues corresponding to the entries of Q.*
- $\operatorname{Res}^{U(\mathfrak{gl}_{2n})}_{U(\mathcal{B})} M \simeq U(\mathcal{A}) \simeq U(\mathfrak{gl}_n)$.

Proof. As we've seen before, we take L_Q as the 1-dimensional $\mathcal{B}$-module corresponding to Q and we let $M_Q = U(\mathcal{A}+\mathcal{B}) \otimes_{U(\mathcal{B})} L_Q$. Then M_Q is injective in $\mathcal{B}$-mod. Next we define

$$W := U(\mathcal{A}+\mathcal{B}+\mathcal{D}) \otimes_{U(\mathcal{A}+\mathcal{B})} M_Q.$$

Fixing $d \in \mathcal{D}$ we note that $\operatorname{span}(v, d \cdot v)$ is a 2-dimensional $\mathcal{B}$-submodule of W, and moreover it is a nonsplit self-extension of L_Q with itself. Now by the injectivity of M_Q there exists a morphism φ such that the following diagram commutes in $\mathcal{B}$-mod:

$$\begin{tikzcd}
 & \operatorname{span}(v, d\cdot v) \arrow[dl, dotted, "\varphi"'] \\
M_Q & L_Q \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

Thus there exists $a_d \cdot v \in \operatorname{soc}_2(M_Q) = \mathcal{A} \cdot v$ such that $a_d \cdot v - d \cdot v$ spans a 1-dimensional B-submodule S_d of W. The module $W' := W / \sum_{d \in \mathcal{D}} U(\mathcal{A}+\mathcal{B}+\mathcal{D}) S_d$ is then isomorphic to M_Q when restricted to $U(\mathcal{A}+\mathcal{B})$.

Next, let $W'' := U(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}) \otimes_{U(\mathcal{A}+\mathcal{B}+\mathcal{D})} W'$. For a fixed $c \in \mathcal{C}$ we have a $\mathcal{B}$-submodule $\mathcal{B}^2(c \cdot v)$ with simple top and simple socle, both isomorphic to L_Q. By similar arguments, there exists $x \in \operatorname{soc}_3(M_Q) = \mathcal{A}^2 \cdot v$ such that $x - c \cdot v$ spans a $\mathcal{B}$-submodule of W''. Forming the quotient of all these submodules we get the module required by the theorem. □

In the next section we shall give explicit formulas for the elements a_d and x of the proof above in order to write down the action on the simple $\mathfrak{gl}_{2n}$-modules explicitly.

3. Explicit formulas for the $\mathfrak{gl}_{2n}$-modules

3.1. ***Preliminaries.*** The following formula will be particularly useful for $m = 2$.

Lemma 15. *Let $F := (e_{j,i})_{i,j} = \sum_{i,j} e_{j,i} \otimes e_{i,j} \in U(\mathcal{A}) \otimes \mathcal{A}$. For any $A, B \in \mathfrak{gl}_n$ and for all $m \in \mathbb{N}$ we have*

$$[A, \operatorname{tr}(B.F^m)] = \operatorname{tr}([A, B].F^m)$$

in $U(\mathfrak{gl}_n)$.

Proof. We proceed by induction on m. Since $\operatorname{tr}(X.F) = X$, the equality clearly holds for $m = 1$. The equation above is linear in both A and B so it suffices to verify it for $A = e_{ij}$, $B = e_{kl}$. Note that we explicitly have

$$\operatorname{tr}(e_{ij}.F^{m+1}) = \sum_{1 \le r_1, \dots, r_m \le n} e_{ir_1} e_{r_1 r_2} \cdots e_{r_m j}.$$

Assume that the equality holds for some fixed m. We now compute

$$\begin{aligned}
[e_{ij}, \operatorname{tr}(e_{kl}.F^{m+1})] &= \left[e_{ij}, \sum_{r_1, \dots, r_m} e_{kr_1} e_{r_1 r_2} \cdots e_{r_m l} \right] \\
&= \sum_{r_1, \dots, r_m} \big([e_{ij}, e_{kr_1}] e_{r_1 r_2} \cdots e_{r_m l} + e_{kr_1} [e_{ij}, e_{r_1 r_2} \cdots e_{r_m l}] \big) \\
&= \sum_{r_1, \dots, r_m} (\delta_{jk} e_{ir_1} - \delta_{r_1 i} e_{kj}) e_{r_1 r_2} \cdots e_{r_m l} \\
&\quad + \sum_{r_1} e_{kr_1} \left[e_{ij}, \sum_{r_2, \dots, r_m} e_{r_1 r_2} \cdots e_{r_m l} \right] \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^{m+1}) - e_{kj} \sum_{r_2, \dots, r_m} e_{ir_2} \cdots e_{r_m l} \\
&\quad + \sum_{r_1} e_{kr_1} [e_{ij}, \operatorname{tr}(e_{r_1 l}.F^m)] \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^{m+1}) - e_{kj} \operatorname{tr}(e_{il}.F^m) + \sum_{r_1} e_{kr_1} \operatorname{tr}([e_{ij}, e_{r_1 l}].F^m) \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^{m+1}) - e_{kj} \operatorname{tr}(e_{il}.F^m) \\
&\quad + \sum_{r_1} e_{kr_1} \big(\delta_{jr_1} \operatorname{tr}(e_{il}.F^m) - \delta_{il} \operatorname{tr}(e_{r_1 j}.F^m) \big)
\end{aligned}$$

$$
\begin{aligned}
&= \delta_{jk} \operatorname{tr}(e_{il}.F^{m+1}) - e_{kj} \operatorname{tr}(e_{il}.F^m) + e_{kj} \operatorname{tr}(e_{il}.F^m) \\
&\quad - \delta_{il} \sum_{r_1} e_{kr_1} \operatorname{tr}(e_{r_1 j}.F^m) \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^{m+1}) - \delta_{il} \operatorname{tr}(e_{kj}.F^{m+1}) \\
&= \operatorname{tr}([e_{ij}, e_{kl}].F^{m+1}).
\end{aligned}
$$

By induction the lemma holds. □

Remark 16. Fixing B as the identity matrix above we obtain $[A, \operatorname{tr}(F^k)] = 0$ for all A in $\mathfrak{gl}_n$, which shows that $\operatorname{tr}(F^k)$ is central in $U(\mathfrak{gl}_n)$. In fact, $Z(\mathfrak{gl}_n) = \mathbb{C}[\operatorname{tr}(F), \operatorname{tr}(F^2), \ldots, \operatorname{tr}(F^n)]$. The elements $\operatorname{tr}(F^k)$ are called Gelfand invariants.

3.2. ***The main result.*** We are now ready to state our main result. Define

$$\varphi' : \mathcal{A} \to U(\mathcal{A}) \otimes \mathcal{A}, \quad A \mapsto A \otimes I + 1 \otimes A.$$

This is a Lie algebra homomorphism and it extends to an algebra homomorphism $\varphi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes \mathcal{A}$. Also recall that we previously have defined

$$\psi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes \mathcal{A}, \quad A \mapsto A \otimes I - 1 \otimes A^T,$$

for $A \in \mathcal{A}$. Using these two homomorphisms we now state our main theorem.

Theorem 17. *Define an action of $\mathfrak{gl}_{2n}$ on $M_I \simeq U(\mathcal{A})$ as follows: for any $a \in U(\mathcal{A})$, let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot a = Aa - aD + \operatorname{tr}(\psi(a).B^T) - \operatorname{tr}(\varphi(a).F^2.C) - \operatorname{tr}(\varphi(a).C) \operatorname{tr}(F). \tag{1}$$

This is a $\mathfrak{gl}_{2n}$-module structure.

Proof. First, for all $X, Y \in \mathfrak{gl}_{2n}$, $A \in \mathcal{A}$ and $a \in U(\mathcal{A})$ we have

$$
\begin{aligned}
X \cdot Y \cdot Aa - Y \cdot X \cdot Aa &= A(X \cdot Y \cdot a) + [XY, A]a - A(Y \cdot X \cdot a) - [YX, A]a \\
&= A(X \cdot Y \cdot a - Y \cdot X \cdot a) + X \cdot [Y, A]a - [X, A] \cdot Ya \\
&\quad - Y \cdot [X, A]a - [Y, A] \cdot Xa \\
&= A \cdot [X, Y]a + [X, [Y, A]]a + [Y, [A, X]]a \\
&= A \cdot [X, Y]a - [A, [X, Y]]a \\
&= [X, Y] \cdot Aa.
\end{aligned}
$$

This shows that it suffices to check that

$$X \cdot Y \cdot 1 - Y \cdot X \cdot 1 = [X, Y] \cdot 1$$

for all $X, Y \in \mathfrak{gl}_{2n}$ in order to prove that (1) gives a module structure.

We first consider the case $Y := A_0 \in \mathcal{A}$. We compute

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \cdot 1 - \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot 1$$
$$\begin{aligned}
&= AA_0 - A_0D + \operatorname{tr}\big((A_0 \otimes I - 1 \otimes A_0^T).B^T\big) \\
&\quad - \operatorname{tr}\big((A_0 \otimes I + 1 \otimes A_0).F^2.C\big) - \operatorname{tr}\big((A_0 \otimes I + 1 \otimes A_0).C\big)\operatorname{tr}(F) \\
&\quad - \big(A_0A - A_0D + A_0 \operatorname{tr}(B^T) - A_0 \operatorname{tr}(F^2.C) - A_0 \operatorname{tr}(C)\operatorname{tr}(F)\big) \\
&= AA_0 - A_0D + A_0 \operatorname{tr}(B^T) + \operatorname{tr}(A_0^T.B^T) - A_0 \operatorname{tr}(F^2.C) - \operatorname{tr}(A_0.F^2.C) \\
&\quad - A_0 \operatorname{tr}(C)\operatorname{tr}(F) - \operatorname{tr}(A_0.C)\operatorname{tr}(F) - A_0A + A_0D - A_0 \operatorname{tr}(B^T) \\
&\quad + A_0 \operatorname{tr}(F^2.C) + A_0 \operatorname{tr}(C)\operatorname{tr}(F) \\
&= [A, A_0] + \operatorname{tr}(A_0^T.B^T) - \operatorname{tr}(A_0.F^2.C) - \operatorname{tr}(A_0.C)\operatorname{tr}(F) \\
&= [A, A_0] + \operatorname{tr}((A_0.B)^T) - \operatorname{tr}(F^2.C.A_0) - \operatorname{tr}(C.A_0)\operatorname{tr}(F) \\
&= \begin{pmatrix} [A, A_0] & A_0.B \\ C.A_0 & 0 \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\right] \cdot 1.
\end{aligned}$$

It remains to check that $X \cdot Y \cdot 1 - Y \cdot X \cdot 1 = [X, Y] \cdot v$ for $X, Y \in \mathcal{B}, \mathcal{C}, \mathcal{D}$. Moreover, since the right side of (1) is linear in A, B, C, and D it suffices to check it for the standard basis elements of $\mathfrak{gl}_{2n}$.

When $X, Y \in \mathcal{B}$ the calculation is easy:

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot 1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \operatorname{tr}(B') - \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot \operatorname{tr}(B)$$
$$= \operatorname{tr}(B)\operatorname{tr}(B') - \operatorname{tr}(B')\operatorname{tr}(B) = 0 = \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix}\right] \cdot 1.$$

Similarly, for $X, Y \in \mathcal{D}$ we have

$$\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 = - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot D' + \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot D$$
$$= D'D - DD' = [D', D] = \begin{pmatrix} 0 & 0 \\ 0 & [D, D'] \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix}\right] \cdot 1.$$

For $X \in \mathcal{B}$, $Y \in \mathcal{D}$ we get

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot 1 = - \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot D - \operatorname{tr}(B^T) \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1$$
$$= -\operatorname{tr}((D \otimes I - 1 \otimes D^T).B^T) + \operatorname{tr}(B^T)D = -D\operatorname{tr}(B^T) + \operatorname{tr}(D^T.B^T) + \operatorname{tr}(B^T)D$$
$$= \operatorname{tr}((D.B)^T) = \begin{pmatrix} 0 & D.B \\ 0 & 0 \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}\right] \cdot 1.$$

For $X \in \mathcal{C}$, $Y \in \mathcal{D}$ we apply Lemma 15 for $m = 1, 2$ to obtain

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} v \cdot 1 \\
&\qquad = - \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \cdot D + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C) \operatorname{tr}(F)\big) \\
&\qquad = \operatorname{tr}\big((D \otimes I + 1 \otimes D).F^2.C\big) + \operatorname{tr}\big((D \otimes I + 1 \otimes D).C\big) \operatorname{tr}(F) \\
&\qquad\quad - \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C) \operatorname{tr}(F)\big) D \\
&\qquad = D \operatorname{tr}(F^2.C) + \operatorname{tr}(D.F^2.C) + D \operatorname{tr}(C) \operatorname{tr}(F) + \operatorname{tr}(D.C) \operatorname{tr}(F) \\
&\qquad\quad - \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C) \operatorname{tr}(F)\big) D \\
&\qquad = [D, \operatorname{tr}(C.F^2)] + \operatorname{tr}(C)[D, \operatorname{tr}(F)] + \operatorname{tr}(D.F^2.C) + \operatorname{tr}(D.C) \operatorname{tr}(F) \\
&\qquad = \operatorname{tr}([D, C].F^2) + \operatorname{tr}(C.D.F^2) + \operatorname{tr}(D.C) \operatorname{tr}(F) \\
&\qquad = \operatorname{tr}(F^2.D.C) + \operatorname{tr}(D.C) \operatorname{tr}(F) \\
&\qquad = \begin{pmatrix} 0 & 0 \\ -D.C & 0 \end{pmatrix} \cdot 1 \\
&\qquad = \left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right] \cdot 1.
\end{aligned}$$

Next, for $X \in \mathcal{B}$, $Y \in \mathcal{C}$, take $X = e_{i,n+j}$ and $Y = e_{n+k,l}$. We then have

$$\begin{aligned}
&e_{i,n+j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{i,n+j} \cdot 1 \\
&\quad = -e_{i,n+j} \cdot \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) - e_{n+k,l} \cdot \operatorname{tr}(e_{ij}^T) \\
&\quad = -e_{i,n+j} \cdot \left(\sum_{r=1}^{n} e_{kr} e_{rl} + \delta_{kl} \operatorname{tr}(F) \right) + \delta_{ij} \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) \\
&\quad = - \sum_{r=1}^{n} \operatorname{tr}(\psi(e_{kr} e_{rl}).e_{ji}) - \delta_{kl} \operatorname{tr}(\psi(\operatorname{tr}(F)).e_{ji}) + \delta_{ij} (\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)) \\
&\quad = - \sum_{r=1}^{n} \operatorname{tr}\big((e_{kr} \otimes I - 1 \otimes e_{rk}).(e_{rl} \otimes I - 1 \otimes e_{lr}).e_{ji}\big) \\
&\qquad - \delta_{kl} \operatorname{tr}\big(\big(\operatorname{tr}(F) \otimes I - 1 \otimes \operatorname{tr}(F)\big).e_{ji}\big) + \delta_{ij} \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) \\
&\quad = \sum_{r=1}^{n} \big(- \operatorname{tr}(e_{ji}.e_{rk}.e_{lr}) + e_{kr} \operatorname{tr}(e_{ji}.e_{lr}) + e_{rl} \operatorname{tr}(e_{ji}.e_{rk}) - e_{kr} e_{rl} \operatorname{tr}(e_{ji}) \big) \\
&\qquad + \delta_{kl} \big(\operatorname{tr}(e_{ji} \operatorname{tr}(F)) - \operatorname{tr}(F) \operatorname{tr}(e_{ji})\big) + \delta_{ij} \big(\operatorname{tr}(e_{kl} F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) \\
&\quad = \big(-\delta_{kl} \operatorname{tr}(e_{ji} \operatorname{tr}(F)) + \delta_{li} e_{kj} + \delta_{jk} e_{il} - \delta_{ji} \operatorname{tr}(e_{kl} F^2) \big) \\
&\qquad + \delta_{kl} \delta_{ji} - \delta_{kl} \delta_{ji} \operatorname{tr}(F) + \delta_{ij} \operatorname{tr}(e_{kl} F^2) + \delta_{ij} \delta_{kl} \operatorname{tr}(F) \\
&\quad = -\delta_{kl} \delta_{ji} + \delta_{li} e_{kj} + \delta_{jk} e_{il} + \delta_{kl} \delta_{ji} \\
&\quad = \delta_{li} e_{kj} + \delta_{jk} e_{il} = \delta_{jk} e_{il} - \delta_{li} e_{n+k,n+j} = [e_{i,n+j}, e_{n+k,l}] \cdot 1.
\end{aligned}$$

It remains only to show that (1) holds for $X, Y \in \mathcal{C}$. Let $X = e_{n+i,j}$ and $Y = e_{n+k,l}$. In this case we have

$$
\begin{aligned}
&e_{n+i,j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{n+i,j} \cdot 1 \\
&\quad = -e_{n+i,j} \cdot \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) + e_{n+k,l} \cdot \big(\operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(e_{ij}) \operatorname{tr}(F)\big) \\
&\quad = -e_{n+i,j} \cdot \left(\sum_{r=1}^{n} e_{kr} e_{rl} + \delta_{kl} \operatorname{tr}(F)\right) + e_{n+k,l} \cdot \left(\sum_{r=1}^{n} e_{ir} e_{rj} + \delta_{ij} \operatorname{tr}(F)\right) \\
&\quad = \sum_{r=1}^{n} \big(\operatorname{tr}(e_{ij}.e_{kr}.e_{rl}.F^2) + e_{kr} \operatorname{tr}(e_{ij}.e_{rl}.F^2) + e_{rl} \operatorname{tr}(e_{ij}.e_{kr}.F^2) + e_{kr} e_{rl} \operatorname{tr}(e_{ij}.F^2) \\
&\qquad\qquad + \big(\operatorname{tr}(e_{ij}.e_{kr}.e_{rl}) + e_{kr} \operatorname{tr}(e_{ij}.e_{rl}) + e_{rl} \operatorname{tr}(e_{ij}.e_{kr}) + e_{kr} e_{rl} \operatorname{tr}(e_{ij})\big) \operatorname{tr}(F)\big) \\
&\qquad + \delta_{kl} \big(\operatorname{tr}(e_{ij}. \operatorname{tr}(F).F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(e_{ij}. \operatorname{tr}(F)) \operatorname{tr}(F) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}) \operatorname{tr}(F)\big) \\
&\qquad - \sum_{r=1}^{n} \big(\operatorname{tr}(e_{kl}.e_{ir}.e_{rj}.F^2) + e_{ir} \operatorname{tr}(e_{kl}.e_{rj}.F^2) + e_{rj} \operatorname{tr}(e_{kl}.e_{ir}.F^2) + e_{ir} e_{rj} \operatorname{tr}(e_{kl}.F^2) \\
&\qquad\qquad + \big(\operatorname{tr}(e_{kl}.e_{ir}.e_{rj}) + e_{ir} \operatorname{tr}(e_{kl}.e_{rj}) + e_{rj} \operatorname{tr}(e_{kl}.e_{ir}) + e_{ir} e_{rj} \operatorname{tr}(e_{kl})\big) \operatorname{tr}(F)\big) \\
&\qquad - \delta_{ij} \big(\operatorname{tr}(e_{kl}. \operatorname{tr}(F).F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}. \operatorname{tr}(F)) \operatorname{tr}(F) + \operatorname{tr}(F) \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) \\
&\quad = n \operatorname{tr}(e_{ij}.e_{kl}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) + \sum_{r} e_{rl} \operatorname{tr}(e_{ij}.e_{kr}.F^2) + \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(e_{ij}.F^2) \\
&\qquad + \left(n \operatorname{tr}(e_{ij}.e_{kl}) + e_{kj} \operatorname{tr}(e_{il}) + \sum_{r} e_{rl} \operatorname{tr}(e_{ij}.e_{kr}) + \delta_{ij} \operatorname{tr}(e_{kl}.F^2)\right) \operatorname{tr}(F) \\
&\qquad + \delta_{kl} \big(\operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \delta_{ij} \operatorname{tr}(F) + \delta_{ij} \operatorname{tr}(F) \operatorname{tr}(F)\big) \\
&\qquad - n \operatorname{tr}(e_{kl}.e_{ij}.F^2) - e_{il} \operatorname{tr}(e_{kj}.F^2) - \sum_{r} e_{rj} \operatorname{tr}(e_{kl}.e_{ir}.F^2) - \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(e_{kl}.F^2) \\
&\qquad + \left(-n \operatorname{tr}(e_{kl}.e_{ij}) - e_{il} \operatorname{tr}(e_{kj}) - \sum_{r} e_{rj} \operatorname{tr}(e_{kl}.e_{ir}) - \delta_{kl} \operatorname{tr}(e_{ij}.F^2)\right) \operatorname{tr}(F) \\
&\qquad + \delta_{ij} \big(- \operatorname{tr}(e_{kl}.F^2) - \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) - \delta_{kl} \operatorname{tr}(F) - \delta_{kl} \operatorname{tr}(F) \operatorname{tr}(F)\big) \\
&\quad = n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) + \delta_{jk} \sum_{r} e_{rl} \operatorname{tr}(e_{ir}.F^2) + \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(e_{ij}.F^2) \\
&\qquad + n\delta_{jk}\delta_{il} \operatorname{tr}(F) + e_{kj}\delta_{il} \operatorname{tr}(F) + \delta_{jk} e_{il} \operatorname{tr}(F) + \delta_{ij} \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(F) \\
&\qquad + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) + \delta_{kl} \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \delta_{kl}\delta_{ij} \operatorname{tr}(F) + \delta_{kl}\delta_{ij} \operatorname{tr}(F)^2 \\
&\qquad - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) - e_{il} \operatorname{tr}(e_{kj}.F^2) - \delta_{li} \sum_{r} e_{rj} \operatorname{tr}(e_{kr}.F^2) - \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(e_{kl}.F^2) \\
&\qquad - n\delta_{li}\delta_{kj} \operatorname{tr}(F) - e_{il}\delta_{kj} \operatorname{tr}(F) - \delta_{li} e_{kj} \operatorname{tr}(F) - \delta_{kl} \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(F) \\
&\qquad - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - \delta_{ij} \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) - \delta_{ij}\delta_{kl} \operatorname{tr}(F) - \delta_{ij}\delta_{kl} \operatorname{tr}(F)^2 \\
&\quad = \delta_{jk} \sum_{r} e_{rl} \operatorname{tr}(e_{ir}.F^2) - \delta_{li} \sum_{r} e_{rj} \operatorname{tr}(e_{kr}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) \\
&\qquad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\qquad - e_{il} \operatorname{tr}(e_{kj}.F^2) + \big[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\big].
\end{aligned}
$$

We proceed to compute $\left[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\right]$ separately.

$$\begin{aligned}
\left[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\right] &= \sum_r [e_{kr}e_{rl}, \operatorname{tr}(e_{ij}.F^2)] \\
&= \sum_r \big(e_{kr}[e_{rl}, \operatorname{tr}(e_{ij}.F^2)] + [e_{kr}, \operatorname{tr}(e_{ij}.F^2)]e_{rl}\big) \\
&= \sum_r \big(e_{kr} \operatorname{tr}([e_{rl}, e_{ij}].F^2) + \operatorname{tr}([e_{kr}, e_{ij}].F^2)e_{rl}\big) \\
&= \sum_r \big(\delta_{li} e_{kr} \operatorname{tr}(e_{rj}.F^2) - \delta_{jr} e_{kr} \operatorname{tr}(e_{il}.F^2) + \delta_{ri} \operatorname{tr}(e_{kj}.F^2)e_{rl} - \delta_{kj} \operatorname{tr}(e_{ir}.F^2)e_{rl}\big) \\
&= \delta_{li} \operatorname{tr}(e_{kj}.F^3) - e_{kj} \operatorname{tr}(e_{il}.F^2) + \operatorname{tr}(e_{kj}.F^2)e_{il} - \delta_{kj} \operatorname{tr}(e_{il}.F^3).
\end{aligned}$$

Inserting this into the previous expression gives

$$\begin{aligned}
&e_{n+i,j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{n+i,j} \cdot 1 \\
&= \delta_{jk} \sum_r e_{rl} \operatorname{tr}(e_{ir}.F^2) - \delta_{li} \sum_r e_{rj} \operatorname{tr}(e_{kr}.F^2) \\
&\quad + e_{kj} \operatorname{tr}(e_{il}.F^2) + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad - e_{il} \operatorname{tr}(e_{kj}.F^2) + \delta_{li} \operatorname{tr}(e_{kj}.F^3) - e_{kj} \operatorname{tr}(e_{il}.F^2) + \operatorname{tr}(e_{kj}.F^2)e_{il} - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk} \sum_r \big(\operatorname{tr}(e_{ir}.F^2)e_{rl} + [e_{rl}, \operatorname{tr}(e_{ir}.F^2)]\big) - \delta_{li} \sum_r \big(\operatorname{tr}(e_{kr}.F^2)e_{rj} + [e_{rj}, \operatorname{tr}(e_{kr}.F^2)]\big) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{li} \operatorname{tr}(e_{kj}.F^3) + [\operatorname{tr}(e_{kj}.F^2), e_{il}] - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^3) + \delta_{jk} \sum_r \operatorname{tr}([e_{rl}, e_{ir}].F^2) - \delta_{li} \operatorname{tr}(e_{kj}.F^3) - \delta_{li} \sum_r \operatorname{tr}([e_{rj}, e_{kr}].F^2) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{li} \operatorname{tr}(e_{kj}.F^3) + \operatorname{tr}([e_{kj}, e_{il}].F^2) - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk}\big(\delta_{li} \operatorname{tr}(\operatorname{tr}(F).F^2) - n\ \operatorname{tr}(e_{il}.F^2)\big) - \delta_{li}\big(\delta_{jk} \operatorname{tr}(\operatorname{tr}(F).F^2) - n\ \operatorname{tr}(e_{kj}.F^2)\big) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - \delta_{lk} \operatorname{tr}(e_{ij}.F^2) \\
&= \delta_{jk}\delta_{li} \operatorname{tr}(F^2) - \delta_{li}\delta_{jk} \operatorname{tr}(F^2) = 0 = [e_{n+i,j}, e_{n+k,l}] \cdot 1. \qquad \square
\end{aligned}$$

Theorem 18. *Define an action of $\mathfrak{gl}_{2n}$ on $M_Q \simeq U(\mathcal{A})$ as follows: for any $a \in U(\mathcal{A})$, let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot a = Aa - aD + \operatorname{tr}(\psi(a).Q.B^T) - \operatorname{tr}(\varphi(a).F^2.Q^{-T}.C) - \operatorname{tr}(\varphi(a).Q^{-T}.C) \operatorname{tr}(F).$$

This is a $\mathfrak{gl}_{2n}$-module structure.

Proof. For each nonsingular $S \in \mathrm{Mat}_{n\times n}$, define $\varphi_S : \mathfrak{gl}_{2n} \to \mathfrak{gl}_{2n}$ by

$$\varphi_S : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & B.S^{-1} \\ S.C & S.D.S^{-1} \end{pmatrix}.$$

It is easy to verify that φ_S is a Lie algebra automorphism and that $\varphi_S \circ \varphi_T = \varphi_{S.T}$, so the map $\Xi : \mathrm{Mat}_{n\times n}(\mathbb{C})^* \to \mathrm{Aut}(\mathfrak{gl}_{2n})$ with $S \mapsto \varphi_S$ is an injective algebra homomorphism. Let V be the $\mathfrak{gl}_{2n}$ module from Theorem 17. Now the action of $\mathfrak{gl}_{2n}$ on the twisted module $V_Q := {}^{\varphi_{Q^{-T}}}V$ is precisely as in the statement of this theorem. □

The modules V_Q now satisfy the conditions of Theorem 1 in the introduction:

Proof of Theorem 1. The module V_Q is simple since $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}+\mathcal{B}} V_Q \simeq M_Q$ is. That the GK-dimension is n^2 and that $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}} V_Q \simeq U(\mathcal{A})$ follows directly from the definition in Theorem 18. Since the linear maps $\mathrm{tr}(\psi(-).B^T) : U(\mathcal{A}) \to U(\mathcal{A})$ never increase the degree of a monomial, the module $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{B}} V_Q$ is locally finite. The fourth point follows from similar arguments: the maps $\mathrm{tr}(\psi(-).F^2.C) : U(\mathcal{A}) \to U(\mathcal{A})$ have degree 2 and the maps $A(-)$ and $(-)D$ clearly have degree 1 (compare with Theorem 18). Finally, we note that any isomorphism $\varphi : V_Q \to V_{Q'}$ must map the generator of V_Q to a multiple of the generator of $V_{Q'}$. But then $q'_{ij}\varphi(1) = e_{i,n+j}\varphi(1) = \varphi(e_{i,n+j} \cdot 1) = q_{ij}\varphi(1)$, showing that $Q = Q'$ whenever such an isomorphism exists. □

3.3. *Alternative formula.* Since the automorphisms φ and ψ themselves are not very explicit, we present another formula for how elements of $\mathfrak{gl}_{2n}$ act on monomials of $U(\mathcal{A})$. We need some more conventions in notation for this formula.

In the argument of the trace functions, any product is by convention to be taken in $\mathrm{Mat}_{n\times n}(U(\mathfrak{gl}_n))$ (in particular we identify $\mathcal{A}$ with $\mathrm{Mat}_{n\times n}(\mathbb{C})$ here). Outside the trace function all products are in $U(\mathfrak{gl}_n)$. When $S \subset \mathbb{Z}$, the product $\prod_{i\in S} A_i$ means that the product is to be taken in the order inherited from $\mathbb{Z}$. For example, $\prod_{i\in\{3,2,5\}} A_i = A_2A_3A_5$. For $S \subset \{1,\dots,k\}$, we denote by S^* the complement $\{1,\dots,k\} \setminus S$ and by $|S|$ the cardinality of S.

Theorem 19. *Let $a = \prod_{i=1}^k A_i$ be a monomial in V_Q (see Theorem 18). The action of $\mathfrak{gl}_{2n}$ on the monomial a can be written explicitly as follows.*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \prod_{i=1}^k A_i := A\prod_{i=1}^k A_i - \prod_{i=1}^k A_i(Q^{-T}.D.Q^T) + \sum_{S\subset\{1,\dots,k\}} \Big(\prod_{i\in S^*} A_i\Big) \times$$
$$\Big((-1)^{|S|}\,\mathrm{tr}\Big(B^T.\prod_{i\in S} A_i^T.Q\Big) - \mathrm{tr}\Big(Q^{-T}.C.\prod_{i\in S} A_i.F^2\Big) - \mathrm{tr}\Big(Q^{-T}.C.\prod_{i\in S} A_i\Big)\,\mathrm{tr}(F)\Big).$$

Proof. This follows by induction on k by comparing with the formula in Theorem 18. The verification is omitted here. □

Remark 20. When $n = 1$ the formula above simplifies significantly. In this case $Q = (q)$ is a nonzero scalar and we have $\mathcal{A} \simeq \mathbb{C}$. Letting $x := e_{11}$ we have $U(\mathcal{A}) = \mathbb{C}[x]$ where the $\mathfrak{gl}_2$-action is given by

$$\begin{aligned} e_{11} \cdot f(x) &= xf(x), \\ e_{22} \cdot f(x) &= -xf(x), \\ e_{12} \cdot f(x) &= qf(x-1), \\ e_{21} \cdot f(x) &= -q^{-1}x(x+1)f(x+1). \end{aligned}$$

When considered as an $\mathfrak{sl}_2$-module, this is a Whittaker module in Kostant's sense. Writing $\mathfrak{h}$ for the standard Cartan subalgebra of $\mathfrak{sl}_2$, we note that $U(\mathfrak{h})$ acts freely on these modules. The paper [Nilsson 2015] classifies $\mathfrak{sl}_n$-modules which are $U(\mathfrak{h})$-free of rank 1 and indeed, in the notation of [Nilsson 2015] the ($\mathfrak{sl}_2$-)module above would be written $\mathrm{F}_{(q,1)}(M'_0)$.

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