Pacific Journal of Mathematics

CONJUGACY AND ELEMENT-CONJUGACY OF HOMOMORPHISMS OF COMPACT LIE GROUPS

YINGJUE FANG, GANG HAN AND BINYONG SUN

Volume 283 No. 1 July 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.283.75

CONJUGACY AND ELEMENT-CONJUGACY OF HOMOMORPHISMS OF COMPACT LIE GROUPS

YINGJUE FANG, GANG HAN AND BINYONG SUN

Let G be a connected compact Lie group. Among other things, we prove that the following are equivalent. (a) For all connected compact Lie groups H and all continuous homomorphisms $\phi, \phi' : H \to G$, if $\phi(h)$ and $\phi'(h)$ are conjugate in G for all $h \in H$, then ϕ and ϕ' are G-conjugate. (b) The Lie algebra of G contains no simple ideal of type D_n ($n \geq 4$), E_6, E_7, or E_8.

1. Introduction

Let G and H be two topological groups, and let $\phi, \phi' : H \to G$ be two continuous homomorphisms. We say that ϕ and ϕ' are conjugate if there is an element $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h) \quad \text{for all } h \in H.$$

We say that they are element-conjugate if for every $h \in H$, there is a $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h).$$

Clearly, conjugate homomorphisms are element-conjugate. Conversely, we are interested to know to what extent the following statement holds.

(1) If ϕ and ϕ' are element-conjugate, then they are conjugate.

This is closely related to the failure of multiplicity one for the cuspidal spectrum of reductive groups over number fields (see [Blasius 1994, Section 1.1; Lapid 1999, Section 3; Arthur 2002, page 471; Lafforgue 2014, Section 0.8]). Some counterexamples to (1) are used to construct nonisometric pairs of isospectral manifolds (see [Larsen 1996, Theorem 2.7]).

We say that G is H-acceptable if (1) holds for all continuous homomorphisms ϕ and ϕ'. M. Larsen [1994] defined G to be acceptable if it is H-acceptable whenever H is finite. In [Larsen 1996], he classified acceptable, connected, simply connected compact Lie groups. In this paper, we are more concerned with the

B. Sun was supported by NSFC Grants 11321101 and 11525105.
MSC2010: 22E46.
Keywords: element-conjugacy, conjugacy, compact Lie group.

statement (1) when both G and H are connected compact Lie groups. This is more relevant to the classification of reductive subalgebras of semisimple Lie algebras, as studied in [Dynkin 1952; Liebeck and Seitz 1996; Malcev 1944; Minchenko 2006].

It was essentially known to Dynkin how counterexamples to (1) can be constructed when G is a simple compact Lie group of type D_n, with $n \geq 4$ (see [Dynkin 1952, Theorem 1.4]; see also [Wang 2012]). M. Liebeck and G. Seitz [1996] found a counterexample when G is simple of type E_8, and H is simple of type A_2, in the setting of algebraic groups. For Lie algebra homomorphisms from a semisimple Lie algebra to a simple Lie algebra of type E_6, E_7 or E_8, all counterexamples of the Lie algebra analogue of (1) are listed in [Minchenko 2006, Table 9] (see Lemma 3.6).

Based on the main result of [Minchenko 2006], we prove the following theorem for connected compact Lie groups.

Theorem 1.1. *Let G be a connected compact Lie group. Then the following are equivalent.* (*a*) *The Lie algebra of G contains no simple ideal of type* D_n ($n \geq 4$), E_6, E_7 *or* E_8. (*b*) *The group G is H-acceptable for all connected compact Lie groups H.*

As a byproduct of the proof of Theorem 1.1, we get the following theorem for classical Lie groups.

Theorem 1.2. *Let G be a classical Lie group, that is, G is* $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{U}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Sp}(p,q)$ *or* $\mathrm{O}^*(2n)$, *where* $p, q, n \geq 0$. *Then G is H-acceptable for all compact Hausdorff topological groups H.*

By convention, when $n = 0$ or $p + q = 0$, the corresponding classical group of the above theorem is the trivial group, and the theorem is trivial in this case.

2. Classical Lie groups

In the rest of the paper, let H be a compact Hausdorff topological group. By abuse of notation, we do not distinguish a representation from its underlying vector space. All representations are assumed to be complex, finite-dimensional and continuous.

Classical complex groups. We begin with the following classical result.

Proposition 2.1. *The complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ ($n \geq 0$) *is H-acceptable.*

Proof. This is well known. Let $\phi, \phi' : H \to \mathrm{GL}_n(\mathbb{C})$ be two continuous homomorphisms, to be viewed as two n-dimensional representations of H. If they are element-conjugate, then they have the same character. By the classical character theory of representations of compact groups, these two representations are isomorphic. This is the same as saying that the homomorphisms ϕ and ϕ' are conjugate. □

We define an orthogonal representation of H to be a representation V of H together with an H-invariant orthogonal form $\langle\cdot,\cdot\rangle$ on it. Here by an orthogonal form, we mean a nondegenerate symmetric bilinear form. Two orthogonal representations $(V, \langle\cdot,\cdot\rangle)$ and $(V', \langle\cdot,\cdot\rangle')$ are called isomorphic if there is an H-intertwining linear isomorphism from V to V' which sends $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle'$.

Similarly, we define the notions of symplectic representations and isomorphisms of symplectic representations. The next result is well known (see [Malcev 1944]).

Proposition 2.2. *Let $(V, \langle\cdot,\cdot\rangle)$ and $(V', \langle\cdot,\cdot\rangle')$ be two orthogonal (symplectic) representations of H. If V and V' are isomorphic as representations of H, then $(V, \langle\cdot,\cdot\rangle)$ and $(V', \langle\cdot,\cdot\rangle')$ are isomorphic as orthogonal (symplectic) representations.*

It is clear that Propositions 2.1 and 2.2 imply the following result, which is stated and proved in [Larsen 1994, Propositions 2.3 and 2.4], in the setting that H is a finite group.

Proposition 2.3. *The complex orthogonal group $\mathrm{O}_n(\mathbb{C})$ and the complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ $(n \geq 0)$ are H-acceptable.*

Maximal compact subgroups. Let G be a Lie group with finitely many connected components. Let K be a maximal compact subgroup of G, which always exists and is unique up to conjugation [Borel 1998, Chapter VII, Theorem 1.2(i)]. Write $i_K : K \to G$ for the inclusion map.

Lemma 2.4. *Let $\phi_K, \phi'_K : H \to K$ be two continuous homomorphisms. Write $\phi := i_K \circ \phi_K$ and $\phi' := i_K \circ \phi'_K$. Then*

(a) *ϕ and ϕ' are conjugate if and only if ϕ_K and ϕ'_K are conjugate, and*

(b) *ϕ and ϕ' are element-conjugate if and only if ϕ_K and ϕ'_K are element-conjugate.*

Proof. We only prove (a), since (b) can be proved by the same method, and is also implied by (a). The "if" part of (a) is obvious. We prove the "only if" part below.

Write "Ad" for the conjugation action. By [Borel 1998, Chapter VII, Theorem 1.2(ii)], there is a closed analytic submanifold E of G such that

$$\mathrm{Ad}_k(E) = E \quad \text{for all } k \in K, \tag{2}$$

and every $g \in G$ is uniquely of the form

$$g = ke, \quad \text{with } k \in K,\ e \in E. \tag{3}$$

Assume that ϕ and ϕ' are conjugate, i.e., there is an element g in G such that

$$\phi' = \mathrm{Ad}_g \circ \phi.$$

Write $g = ke$ as in (3). Then

$$\mathrm{Ad}_{k^{-1}} \circ \phi' = \mathrm{Ad}_e \circ \phi.$$

Let $h \in H$ and put

$$k_1 := (\mathrm{Ad}_{k^{-1}} \circ \phi')(h), \quad k_2 := \phi(h).$$

Then $k_1, k_2 \in K$ and

$$k_1 = \mathrm{Ad}_e(k_2),$$

or the same,

$$k_1 e = k_2 \,\mathrm{Ad}_{k_2^{-1}}(e).$$

Now (2) and the uniqueness of the decomposition (3) imply that $k_1 = k_2$. This proves that $\mathrm{Ad}_{k^{-1}} \circ \phi' = \phi$, and thus ϕ_K and ϕ'_K are conjugate. □

The following result generalizes [Larsen 1994, Proposition 1.7].

Proposition 2.5. *The group G is H-acceptable if and only if so is K.*

Proof. Let us prove the "if" part first. Assume that K is H-acceptable, and let $\phi, \phi' : H \to G$ be two continuous homomorphisms which are element-conjugate. We need to prove that ϕ and ϕ' are conjugate. Since every compact subgroup of G is conjugate to a subgroup of K, we assume without loss of generality that the images of ϕ and ϕ' are both contained in K. Let ϕ_K and ϕ'_K be as in Lemma 2.4 so that $\phi := i_K \circ \phi_K$ and $\phi' := i_K \circ \phi'_K$. Then Lemma 2.4 implies that ϕ_K and ϕ'_K are element-conjugate, and they are conjugate since K is H-acceptable. This implies that ϕ and ϕ' are conjugate.

To prove the "only if" part, we assume that G is H-acceptable. Write ϕ, ϕ', ϕ_K and ϕ'_K as before. Assume that ϕ_K and ϕ'_K are element-conjugate. Then ϕ and ϕ' are element-conjugate, and therefore conjugate since G is H-acceptable. Now Lemma 2.4 implies that ϕ_K and ϕ'_K are conjugate. This proves that K is H-acceptable. □

Corollary 2.6. *The compact groups* $\mathrm{U}(n)$, $\mathrm{O}(n)$ *and* $\mathrm{Sp}(n)$, *where* $n \geq 0$, *are H-acceptable.*

Proof. Note that $\mathrm{U}(n)$, $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$ is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$, respectively. Therefore the corollary is a consequence of Propositions 2.1, 2.3 and 2.5. □

The following lemma is obvious.

Lemma 2.7. *Let G_1 and G_2 be two topological groups. Then $G_1 \times G_2$ is H-acceptable if and only if so are both G_1 and G_2.*

Now we come to the proof of Theorem 1.2. When G is $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ or $\mathrm{O}^*(2n)$, its maximal compact subgroup is $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$, $\mathrm{O}(p)\times\mathrm{O}(q)$, $\mathrm{O}(n)$, $\mathrm{U}(p)\times\mathrm{U}(q)$, $\mathrm{Sp}(p)\times\mathrm{Sp}(q)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ or $\mathrm{U}(n)$, respectively. By Corollary 2.6 and Lemma 2.7, all these compact groups are H-acceptable. Therefore G is H-acceptable by Proposition 2.5. This proves Theorem 1.2.

We record the following two results for later use.

Corollary 2.8. *The compact groups* $\mathrm{SU}(n)$ *and* $\mathrm{SO}(2n+1)$, *where* $n \geq 0$, *are* H*-acceptable.*

Proof. By Corollary 2.6, the groups $\mathrm{U}(n)$ and $\mathrm{O}(2n+1)$ are H-acceptable. Then the corollary follows by noting that every inner automorphism of $\mathrm{U}(n)$ or $\mathrm{O}(2n+1)$ restricts to an inner automorphism of $\mathrm{SU}(n)$ or $\mathrm{SO}(2n+1)$, respectively. $\square$

The following lemma is obvious.

Lemma 2.9. *If* G *is commutative*, *then it is* H*-acceptable.*

3. A proof of Theorem 1.1

In this section, we concentrate on connected compact Lie groups.

Lemma 3.1. *Let* $\phi, \phi' : H \to G$ *be two continuous homomorphisms of connected compact Lie groups. Then they are element-conjugate if and only if* $\phi|_S$ *and* $\phi'|_S$ *are conjugate*, *where* S *is a maximal torus in* H.

Proof. This is because S is topologically cyclic, and every element of H is H-conjugate to an element of S. $\square$

Lemma 3.2. *Let* $\rho : \widetilde{G} \to G$ *be a surjective continuous homomorphism with finite kernel of connected compact Lie groups. Let* $\tilde{\phi}, \tilde{\phi}' : H \to \widetilde{G}$ *be two continuous homomorphisms of connected compact Lie groups. Then* $\tilde{\phi}$ *and* $\tilde{\phi}'$ *are conjugate if and only if* $\rho \circ \tilde{\phi}$ *and* $\rho \circ \tilde{\phi}'$ *are conjugate*, *and* $\tilde{\phi}$ *and* $\tilde{\phi}'$ *are element-conjugate if and only if* $\rho \circ \tilde{\phi}$ *and* $\rho \circ \tilde{\phi}'$ *are element-conjugate.*

Proof. The first assertion easily follows from the observation that

$$\tilde{\phi} = \tilde{\phi}' \qquad \text{if and only if} \qquad \rho \circ \tilde{\phi} = \rho \circ \tilde{\phi}'.$$

We leave the details to the reader. The second assertion is a consequence of the first one and Lemma 3.1. $\square$

Lemma 3.3. *Let* G *and* G' *be two connected compact Lie groups with isomorphic Lie algebras. If* G *is* H*-acceptable for all connected compact Lie groups* H, *then so is* G'.

Proof. Note that G and G' have a common finite fold covering group, that is, there is a connected compact Lie group $\widetilde{G}$, and surjective continuous homomorphisms $\rho : \widetilde{G} \to G$ and $\rho' : \widetilde{G} \to G'$ with finite kernels.

Assume that G is H-acceptable for all connected compact Lie groups H. Let $\tilde{\phi}_1, \tilde{\phi}_2 : H \to \widetilde{G}$ be two element-conjugate continuous homomorphisms. Then $\rho \circ \tilde{\phi}_1$ and $\rho \circ \tilde{\phi}_2$ are element-conjugate. Therefore by the assumption on G, $\rho \circ \tilde{\phi}_1$ and $\rho \circ \tilde{\phi}_2$ are conjugate. Therefore by Lemma 3.2, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are conjugate. This shows that $\widetilde{G}$ is H-acceptable for all connected compact Lie groups H.

To prove that G' is H-acceptable for all connected compact Lie groups H, we let $\phi'_1, \phi'_2 : H \to G'$ be two element-conjugate continuous homomorphisms. Then there is a connected compact Lie group $\widetilde{H}$, with a surjective continuous homomorphism $\rho_H : \widetilde{H} \to H$ with finite kernel, and two continuous homomorphisms $\tilde{\phi}'_1, \tilde{\phi}'_2 : \widetilde{H} \to \widetilde{G}$ such that the diagram

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{\tilde{\phi}'_i} & \widetilde{G} \\ {\scriptstyle \rho_H}\big\downarrow & & \big\downarrow{\scriptstyle \rho'} \\ H & \xrightarrow{\phi'_i} & G' \end{array}$$

commutes ($i = 1, 2$).

Since ϕ'_1 and ϕ'_2 are element-conjugate, we know that $\phi'_1 \circ \rho_H$ and $\phi'_2 \circ \rho_H$ are element-conjugate, or equivalently, $\rho' \circ \tilde{\phi}'_1$ and $\rho' \circ \tilde{\phi}'_2$ are element-conjugate. Then by Lemma 3.2, $\tilde{\phi}'_1$ and $\tilde{\phi}'_2$ are element-conjugate. Therefore $\tilde{\phi}'_1$ and $\tilde{\phi}'_2$ are conjugate since $\widetilde{G}$ is $\widetilde{H}$-acceptable. Thus $\rho' \circ \tilde{\phi}'_1$ and $\rho' \circ \tilde{\phi}'_2$ are conjugate, or equivalently, $\phi'_1 \circ \rho_H$ and $\phi'_2 \circ \rho_H$ are conjugate. This implies that ϕ'_1 and ϕ'_2 are conjugate. Thus G' is H-acceptable. □

We say that two homomorphisms $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ between two finite-dimensional complex Lie algebras are conjugate if there is an inner automorphism φ of $\mathfrak{g}$ such that $\varphi \circ \phi = \phi'$.

Lemma 3.4 [Minchenko 2006, Theorem 3; Dynkin 1952, Theorem 1.1]. *Let* $\mathfrak{g}$ *be a simple complex Lie algebra of type* G_2 *or* F_4. *Let* $\mathfrak{h}$ *be a reductive complex Lie algebra, and let* $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ *be two injective Lie algebra homomorphisms whose images are reductive Lie subalgebras of* $\mathfrak{g}$. *If* $\phi|_{\mathfrak{s}}$ *and* $\phi'|_{\mathfrak{s}}$ *are conjugate, then* ϕ *and* ϕ' *are conjugate, where* $\mathfrak{s}$ *is a Cartan subalgebra of* $\mathfrak{h}$.

Recall that a Lie subalgebra of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is said to be reductive if its adjoint representation on $\mathfrak{g}$ is completely reducible.

Lemma 3.4 has the following consequence.

Proposition 3.5. *Let G be a connected compact Lie group whose complexified Lie algebra is simple of type* G_2 *or* F_4. *Then G is H-acceptable for all connected compact Lie groups H.*

Proof. Let $\phi, \phi' : H \to G$ be two element-conjugate homomorphisms. We want to show that they are conjugate. Note that ϕ and ϕ' have the same kernel. Replacing H by its quotient by the kernel, we assume without loss of generality that both ϕ and ϕ' are injective. Let S be a maximal torus in H. Write $c_G : G \to G_{\mathbb{C}}$ for the universal complexification of G, which is injective (see [Hochschild 1966]). Write $\mathfrak{s}$, $\mathfrak{h}$ and $\mathfrak{g}$ for the complexified Lie algebras of S, H and G, respectively.

By Lemma 3.1, $\phi|_S$ and $\phi'|_S$ are conjugate. Therefore their complexified differentials

$$\mathrm{d}(\phi|_S) : \mathfrak{s} \to \mathfrak{g} \quad \text{and} \quad \mathrm{d}(\phi'|_S) : \mathfrak{s} \to \mathfrak{g}$$

are conjugate. Then Lemma 3.4 implies that the complexified differentials

$$\mathrm{d}(\phi) : \mathfrak{h} \to \mathfrak{g} \quad \text{and} \quad \mathrm{d}(\phi') : \mathfrak{h} \to \mathfrak{g}$$

are conjugate. This implies that the homomorphisms

$$c_G \circ \phi : H \to G_{\mathbb{C}} \quad \text{and} \quad c_G \circ \phi' : H \to G_{\mathbb{C}}$$

are conjugate. Since G is a maximal compact subgroup of $G_{\mathbb{C}}$, Lemma 2.4 implies that ϕ and ϕ' are conjugate. This proves the proposition. □

Lemma 3.6. *Let $\mathfrak{g}$ be a simple complex Lie algebra of type D_n ($n \geq 4$), E_6, E_7 or E_8. Then there are a semisimple complex Lie algebra $\mathfrak{h}$ and two nonconjugate injective Lie algebra homomorphisms $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ such that $\phi|_{\mathfrak{s}}$ and $\phi'|_{\mathfrak{s}}$ are conjugate. Here $\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{h}$.*

Proof. The lemma is a consequence of [Dynkin 1952, Theorem 1.4] when $\mathfrak{g}$ has type D_n ($n \geq 4$) and a consequence of [Minchenko 2006, Theorem 7] when $\mathfrak{g}$ has type E_6, E_7 or E_8. □

Lemma 3.6 has the following consequence.

Proposition 3.7. *Let G be a connected compact Lie group whose complexified Lie algebra is simple of type D_n ($n \geq 4$), E_6, E_7 or E_8. Then G is not H-acceptable for some connected compact Lie group H.*

Proof. Write $\mathfrak{g}$ for the complexified Lie algebra of G. Let $\mathfrak{h}$, $\mathfrak{s}$, and $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ be as in Lemma 3.6. As in the proof of Proposition 3.5, write $c_G : G \to G_{\mathbb{C}}$ for the universal complexification of G. Let $H_{\mathbb{C}}$ be a simply connected, connected complex Lie group whose Lie algebra is identified with $\mathfrak{h}$. Then ϕ, ϕ' integrate to holomorphic homomorphisms

$$\psi_{\mathbb{C}}, \psi'_{\mathbb{C}} : H_{\mathbb{C}} \to G_{\mathbb{C}}. \tag{4}$$

Take a maximal compact subgroup H of $H_{\mathbb{C}}$, and a maximal torus S in H such that the complexified Lie algebra of S equals $\mathfrak{s}$. Replacing ϕ and ϕ' by their conjugations by appropriate elements of $G_{\mathbb{C}}$, we assume without loss of generality that $\psi_{\mathbb{C}}(H) \subset G$ and $\psi'_{\mathbb{C}}(H) \subset G$. Then the homomorphisms in (4) restrict to two homomorphisms

$$\psi, \psi' : H \to G.$$

Since ϕ and ϕ' are nonconjugate, we know that $\psi_{\mathbb{C}}$ and $\psi'_{\mathbb{C}}$ are nonconjugate, which implies that ψ and ψ' are nonconjugate. On the other hand, since $\phi|_{\mathfrak{s}}$ and $\phi'|_{\mathfrak{s}}$ are conjugate, we know that $\psi_{\mathbb{C}}|_S$ and $\psi'_{\mathbb{C}}|_S$ are conjugate. Then Lemma 2.4 implies that $\psi|_S$ and $\psi'|_S$ are conjugate. This implies that ψ and ψ' are element-conjugate by Lemma 3.1. This proves the proposition. □

Finally, in view of Lemmas 3.3 and 2.7, Theorem 1.1 is a consequence of Lemma 2.9, Corollaries 2.6 and 2.8, and Propositions 3.5 and 3.7.

References

[Arthur 2002] J. Arthur, "A note on the automorphic Langlands group", *Canad. Math. Bull.* **45**:4 (2002), 466–482. MR 1941222 Zbl 1031.11066

[Blasius 1994] D. Blasius, "On multiplicities for SL(n)", *Israel J. Math.* **88**:1-3 (1994), 237–251. MR 1303497 Zbl 0826.11023

[Borel 1998] A. Borel, *Semisimple groups and Riemannian symmetric spaces*, Texts and Readings in Mathematics **16**, Hindustan Book Agency, New Delhi, 1998. MR 1661166 Zbl 0954.53002

[Dynkin 1952] E. B. Dynkin, "Полупростые подалгебры полупростых алгебр Ли", *Mat. Sbornik* (*N.S.*) **30**:72 (1952), 349–462. Translated as "Semisimple subalgebras of semisimple Lie algebras" in *Amer. Math. Soc. Transl.* (2) **6** (1957), 111–243. MR 0047629 Zbl 0077.03404

[Hochschild 1966] G. Hochschild, "Complexification of real analytic groups", *Trans. Amer. Math. Soc.* **125** (1966), 406–413. MR 0206141 Zbl 0149.27603

[Lafforgue 2014] V. Lafforgue, "Introduction to chtoucas for reductive groups and to the global Langlands parameterization", preprint, 2014. arXiv 1404.6416

[Lapid 1999] E. M. Lapid, "Some results on multiplicities for SL(n)", *Israel J. Math.* **112** (1999), 157–186. MR 1714998 Zbl 0937.22002

[Larsen 1994] M. Larsen, "On the conjugacy of element-conjugate homomorphisms", *Israel J. Math.* **88**:1-3 (1994), 253–277. MR 1303498 Zbl 0898.20025

[Larsen 1996] M. Larsen, "On the conjugacy of element-conjugate homomorphisms II", *Quart. J. Math. Oxford Ser.* (2) **47**:185 (1996), 73–85. MR 1380951 Zbl 0898.20026

[Liebeck and Seitz 1996] M. W. Liebeck and G. M. Seitz, *Reductive subgroups of exceptional algebraic groups*, Memoirs of the American Mathematical Society **121:**580, American Mathematical Society, Providence, RI, 1996. MR 1329942 Zbl 0851.20045

[Malcev 1944] A. I. Malcev, "О полупростых подгруппах групп Ли", *Izvestia Akad. Nauk SSSR* **8**:4 (1944), 143–174. Translated as "On semi-simple subgroups of Lie groups" in *Amer. Math. Soc. Transl.* **33** (1950). MR 0011303 Zbl 0061.04701

[Minchenko 2006] A. N. Minchenko, "Полупростые подалгебры особых алгебр Ли", *Tr. Mosk. Mat. Obs.* **67** (2006), 256–293. Translated as "Semisimple subalgebras of exceptional Lie algebras" in *Trans. Mosc. Math. Soc.* **2006** (2006), 225–259. MR 2301595 Zbl 1152.17003

[Wang 2012] S. Wang, "On dimension data and local vs. global conjugacy", pp. 365–382 in *Fifth International Congress of Chinese Mathematicians* (Beijing, 2010), edited by L. Ji et al., AMS/IP

Studies in Advanced Mathematics **51**, part 1, American Mathematical Society, Providence, RI, 2012. MR 2908081 Zbl 1246.22028

Received September 29, 2015. Revised January 13, 2016.

YINGJUE FANG
COLLEGE OF MATHEMATICS AND STATISTICS
SHENZHEN UNIVERSITY
SHENZHEN, 518060
CHINA
joyfang@szu.edu.cn

GANG HAN
DEPARTMENT OF MATHEMATICS
ZHEJIANG UNIVERSITY
P.O. BOX 1511
HANG ZHOU, 310027
CHINA
mathhgg@zju.edu.cn

BINYONG SUN
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCES
BEIJING, 100190
CHINA
sun@math.ac.cn

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.
STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA
UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US \$/year for the electronic version, and \$/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

© 2016 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 283 No. 1 July 2016

0030-8730(201607)283:1;1-X