

*Pacific
Journal of
Mathematics*

**ENTIRE SIGN-CHANGING SOLUTIONS
WITH FINITE ENERGY TO THE
FRACTIONAL YAMABE EQUATION**

DANILO GARRIDO AND MONICA MUSSO

ENTIRE SIGN-CHANGING SOLUTIONS WITH FINITE ENERGY TO THE FRACTIONAL YAMABE EQUATION

DANILO GARRIDO AND MONICA MUSSO

We show the existence of infinitely many finite energy sign-changing solutions for the fractional Yamabe-type equation

$$(-\Delta)^s u = |u|^{\frac{4s}{n-2s}} u \quad \text{in } \mathbb{R}^n,$$

where $n \geq 3$ and $s \in (\frac{1}{2}, 1)$.

1. Introduction

We are interested in the existence of finite energy sign-changing solutions to the fractional Yamabe-type equation in \mathbb{R}^n ,

$$(1) \quad (-\Delta)^s u = \gamma |u|^{p-1} u \quad \text{in } \mathbb{R}^n,$$

where $n \geq 3$ and p is the fractional critical Sobolev exponent $p = (n + 2s)/(n - 2s)$. In (1), $\gamma > 0$ is a constant chosen for normalization purposes as

$$\gamma = \frac{\Gamma(\frac{n+2s}{2})}{\Gamma(\frac{n-2s}{2})}.$$

For any $s \in (0, 1)$, $(-\Delta)^s$ is the nonlocal operator defined as

$$(2) \quad \begin{aligned} (-\Delta)^s(x) &= c(n, s) \text{ P.V. } \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy \\ &= c(n, s) \lim_{\epsilon \rightarrow 0^+} \int_{\mathbb{R}^n \setminus B(x, \epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}} dy, \end{aligned}$$

where P.V. stands for *the principal value* and

$$c(n, s) = \pi^{-(2s + \frac{n}{2})} \frac{\Gamma(\frac{n}{2} + s)}{\Gamma(-s)}.$$

This nonlocal operator in \mathbb{R}^n can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators

MSC2010: 35J60, 35J61, 35J75, 35R11.

Keywords: fractional Laplacian, energy solutions, Yamabe-type equation.

defined on the upper halfspace $\mathbb{R}_+^{n+1} = \{(x, t) : x \in \mathbb{R}^n, t > 0\}$, as we learn from Caffarelli and Silvestre [2007]: given a solution $u = u(x)$ of $(-\Delta)^s u = f$ in \mathbb{R}^n , one can equivalently consider the dimensionally extended problem for $u = u(x, t)$ which solves

$$\begin{cases} \operatorname{div}(t^{1-2s} \nabla u) = 0, & \text{in } \mathbb{R}_+^{n+1}, \\ -\lim_{t \rightarrow 0} d_s t^{1-2s} \partial_t u(x, t) = f, & \text{on } \partial \mathbb{R}_+^{n+1}, \end{cases}$$

where d_s is the positive constant $d_s = 2^{2s-1} \Gamma(s) / \Gamma(1-s)$. By finite energy solutions of (1), we mean the following. Consider the Schwartz space \mathcal{S} of rapidly decaying C^∞ functions on \mathbb{R}^n , and for any $\tau \in \mathcal{S}$ we denote by

$$\mathcal{F}\tau(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot x} \tau(x) dx$$

the Fourier transformation of τ . We look for solutions u of (1) in the energy space

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) : \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)} < \infty\},$$

where $\|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$ is defined by $(\int_{\mathbb{R}^n} |\xi|^{2s} |\mathcal{F}u(\xi)|^2 d\xi)^{\frac{1}{2}}$, endowed with the norm $\|u\|_{\mathcal{D}^s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$. These solutions correspond to critical points of the functional

$$J(u) = \frac{1}{2} \int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}} u|^2 - \gamma \frac{n-2s}{2n} \int_{\mathbb{R}^n} |u|^{\frac{2n}{n+2s}}, \quad u \in \mathcal{D}^s(\mathbb{R}^n).$$

Following the work by Lieb [1983] — see also [Frank and Lieb 2010; 2012; Carlen and Loss 1990] for alternative proofs — positive solutions to (1) are given by the family of functions defined by

$$(3) \quad U(x) = \left(\frac{2}{1+|x|^2} \right)^{\frac{n-2s}{2}} \quad \text{and} \quad \mu^{-\frac{n-2s}{2}} U\left(\frac{x-\xi}{\mu} \right)$$

for any $\mu > 0$ and $\xi \in \mathbb{R}^n$. Indeed these functions realize the Hardy–Littlewood–Sobolev inequality, which states the existence of a positive number S such that for all $u \in C^\infty(\mathbb{R}^n)$,

$$S \|u\|_{L^{2^*}(\mathbb{R}^n)} \leq \|(-\Delta)^{\frac{s}{2}} u\|_{L^2(\mathbb{R}^n)}$$

where $2^* = p + 1 = 2n/(n-2s)$. Indeed, these functions are the only positive solutions to (1) under some decay conditions [Chen et al. 2006; Li 2004; Li and Zhu 1995]. In particular, this is true if $u \in L_{\text{loc}}^{2n/(n-2s)}(\mathbb{R}^n)$, as shown in [Chen et al. 2006].

On the other hand, (1) can be read on the sphere $S^n \subset \mathbb{R}^{n+1}$, after a stereographic projection. Indeed, the inverse of the stereographic projection $\pi : \mathbb{R}^n \rightarrow S^n \setminus \{S\}$,

where $S = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$, defined by

$$\pi(y) = \left(\frac{2y}{1 + |y|^2}, \frac{1 - |y|^2}{1 + |y|^2} \right)$$

is a conformal map and $\pi^* g_0 = U^{\frac{4s}{n-2s}}(y) dy$, where g_0 is the standard metric on S^n and U is defined in (3). In S^n , the fractional Laplacian $(-\Delta)^s$ reduces to an elliptic pseudodifferential operator $P_s^{g_0}$ of order $2s$ with principal symbol $\sigma_{2s}(P_s^{g_0}) = |\xi|_{g_0}^{2s}$. In [Chang and González 2011] a relation between this operator and a Dirichlet-to-Neumann operator of uniformly nondegenerate elliptic boundary value problems in the spirit of [Caffarelli and Silvestre 2007] is established. We have $\pi^*(L_s^{g_0} v) = U^{-(n+2s)/(n-2s)}(-\Delta)^s(U\pi^*v)$ for any v defined on S^n . Thus u is a solution to (1) if and only if w , defined by $u = U\pi^*w$, solves

$$(4) \quad \Delta_{g_0} w + \gamma(|w|^{\frac{4s}{n-2s}} w - w) = 0 \quad \text{in } S^n.$$

Positive solutions to (4) solve the so-called fractional Yamabe problem on the sphere S^n . We refer to [González and Qing 2013] for a general formulation of the fractional Yamabe problem and results concerning its solvability.

Finite energy sign-changing solutions to (1), or equivalently (4), are poorly understood.

The purpose of this paper is to give a first example of finite energy sign-changing solutions to (1), in all dimensions $n \geq 3$, and for $s \in (\frac{1}{2}, 1)$: we build a solution to (1) which looks like the solution U surrounded by k negative copies U properly scaled and distributed along the vertices of a regular polygon with radius 1. Our main result is the following theorem:

Theorem 1.1. *Let $n \geq 3$ and $s \in (\frac{1}{2}, 1)$. Write $\mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$ and let $\xi_j^k = (e^{2j\pi i/k}, 0)$, $j = 1, \dots, k$. Then for any sufficiently large k , there is a finite energy solution to Problem (1) of the form*

$$u_k(x) = U(x) - \sum_{j=1}^k \mu_k^{-\frac{n-2s}{2}} U(\mu_k^{-1}(x - \xi_j)) + o(1),$$

where

$$\mu_k = \left(k^2 2^{\frac{n-2s}{2}} \sum_{j=1}^{\infty} j^{2s-n} \right)^{-1} (1 + o(1))$$

Moreover,

$$(5) \quad J(u_k) = (k + 1)J(U) + O(1).$$

Here $O(1)$ remains bounded and $o(1) \rightarrow 0$ uniformly as $k \rightarrow +\infty$.

The proof of the result consists in defining a first approximation and then showing that a small perturbation of this approximation provides an actual solution to the

problem. This is done by linearizing the equation around the approximation and applying an invertibility theory for the linearized operator. In this step, we use the nondegeneracy property of U proved in [Dávila et al. 2013], which states that all bounded solutions of the linear problem

$$(-\Delta)^s \phi - \gamma \frac{n+2s}{n-2s} U^{\frac{-4s}{n-2s}} \phi = 0$$

are linear combinations of

$$\partial_{x_j} U(x), \quad \text{for } j = 1, \dots, n,$$

and

$$\frac{n-2s}{2} U(x) + x \cdot \nabla U(x).$$

Indeed, the above functions belong to the kernel of the linearized operator, due to the corresponding rigid motion under which (1) is invariant. These are the only nontrivial elements of the kernel according to [Dávila et al. 2013].

A second ingredient we take advantage of to produce an invertibility theory is the symmetry of the configuration. This reflects into the fact that our approximation, as well as our final solution, satisfy the symmetries

$$(6) \quad u(\bar{y}, y') = u(e^{\frac{2\pi j}{k} i} \bar{y}, y'), \quad j = 1, \dots, k-1,$$

$$(7) \quad u(y_1, y_2, \dots, y_j, \dots, y_n) = u(y_1, y_2, \dots, -y_j, \dots, y_n) \quad j = 2, \dots, n.$$

Furthermore, they are invariant under Kelvin transform, namely

$$u(y) = |y|^{2s-n} u\left(\frac{y}{|y|^2}\right).$$

The final step in the proof consists in adjusting properly the parameter μ_k . A detailed description of the scheme of the proof is given in Section 2.

Let us mention that a very similar construction for finite energy, sign-changing solutions to the classic Yamabe-type problem in \mathbb{R}^n :

$$\Delta u + |u|^{\frac{4}{n-2}} u = 0 \quad \text{in } \mathbb{R}^n,$$

namely when $s = 1$ in (1), has been done in [del Pino et al. 2011; 2013]. Indeed, our result extends to the case $s \in (\frac{1}{2}, 1)$, the construction done in [del Pino et al. 2011], from which we are inspired.

We learned recently of [Fang 2014], where the author constructs solutions to (1) similar to ours, covering the whole range $s \in (0, 1)$. Nevertheless, in that case, the concentration parameter μ_k is of order k^{-3} [Fang 2014, (2.4)], while our concentration parameter is $\mu_k \sim k^{-2}$, as $k \rightarrow \infty$. It is not clear to us how this choice of the parameter's rate provides a real solution to (1). Indeed, it is this choice of

the parameter's rate, in terms of k , that allows the author of [Fang 2014] to cover the whole range $s \in (0, 1)$.

Our restriction on s is consequence of two inequalities: we need a certain power of integrability q to be $q < n$ in order to have a good first approximation when estimated in proper norms, and at the same time we need $q > n/(2s)$ to guarantee enough regularity. These constraints restrict us to $s \in (\frac{1}{2}, 1)$. We believe that our construction should work in the whole range $s \in (0, 1)$, and in fact we think that $\mu_k \sim k^{-2}$, as $k \rightarrow \infty$, for the whole range $s \in (0, 1)$, but an invertibility theory on different weighted Sobolev spaces is needed. We will treat this problem in a forthcoming paper.

The rest of the paper will be devoted to the proof of [Theorem 1.1](#).

2. Ansatz for the solution and scheme of the proof

This section is devoted to define a first approximation for a solution to (1) and to describe the scheme of the proof of our result.

We start reminding that U defined in (3) is invariant under Kelvin transform, namely

$$U(y) := |y|^{2s-n} U(|y|^{-2}y).$$

Even more, it can be proved that also the family of solutions

$$\mu^{-\frac{n-2s}{2}} \left(\frac{y-\xi}{\mu} \right)$$

is invariant under Kelvin transform if and only if

$$|\xi|^2 + \mu^2 = 1.$$

Let k be a positive integer and define, for any $j = 1, \dots, k$, the k points

$$\xi_j = \sqrt{1 - \mu^2} (e^{2\pi i(j-1)/k}, 0, \dots, 0) \in \mathbb{R}^2 \times \mathbb{R}^{n-2},$$

where $\mu > 0$ is a positive number of the form

$$(8) \quad \mu = \frac{\delta}{k^2}, \quad \text{with } c < \delta < c^{-1}$$

for a certain constant $c > 0$, independent of k , as $k \rightarrow \infty$. Define

$$(9) \quad U_*(y) = U(y) - \sum_{j=1}^k U_j(y), \quad \text{where } U_j(y) = \mu^{-\frac{n-2s}{2}} U(\mu^{-1}(y - \xi_j)).$$

For large values of k , which at the same time make the scaling parameters μ very small, we shall show that U_* is a good approximate solution for (1). Observe that

the function U_* satisfies the symmetry properties (6) and (7). Furthermore, U_* is invariant under Kelvin transform

$$U_*(y) = |y|^{2-n} U_*\left(\frac{y}{|y|^2}\right).$$

This is consequence of a straightforward computation, using the fact that

$$\mu^2 + |\xi_j|^2 = 1 \quad \text{for any } j = 1, \dots, k.$$

We will show that (1) admits a solution of the form

$$u(y) = U_*(y) + \phi(y)$$

where ϕ is small when compared with U_* . It satisfies the symmetry conditions (6) and (7), and it is invariant under Kelvin transform. Then (1) can be rewritten in terms of ϕ as

$$(10) \quad (-\Delta)^s \phi - p\gamma |U_*|^{p-1} \phi - E - \gamma N(\phi) = 0,$$

where E is

$$(11) \quad \gamma^{-1} E = |U - \sum U_j|^{p-1} (U - \sum U_j) - (U^p - \sum U_j^p)$$

and

$$(12) \quad N(\phi) = |U_* + \phi|^{p-1} (U_* + \phi) - |U_*|^{p-1} - |U_*|^{p-1} U_* - p|U_*|^{p-1} \phi.$$

The size of the error term E defined in (11) turns out to be relatively small, as the number k tends to infinity, when estimated with proper norms. Let us fix a number $q > \frac{n}{2s}$; we define the weighted L^q norm

$$\|h\|_{**} := \|(1 + |y|)^{n+2s-2n/q} h\|_{L^q(\mathbb{R}^n)}$$

Let $\eta > 0$ be a small and fixed number, independent of k . The error can be estimated separately in the *exterior region* $\bigcap_j \{|y - \xi_j| > \frac{\eta}{k}\}$ and then in each of the *inner regions* $\{|y - \xi_j| < \frac{\eta}{k}\}$. Indeed, we shall prove that there exists a constant C such that, for all k large enough,

$$(13) \quad \|(1 + |y|)^{n+2s-2n/q} E\|_{L^q(\bigcap_j \{|y - \xi_j| > \frac{\eta}{k}\})} \leq Ck^{1-n/q}.$$

Observe that, in order to have a small (in k) size for the error in the exterior domain, we need $q < n$. On the other hand, for regularity issue we will discuss later, we assume that $q > \frac{n}{2s}$. The set of possible values for q , $\frac{n}{2s} < q < n$, is not empty since we are considering s in the range $s \in (\frac{1}{2}, 1)$.

If we change scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y)$, in $|y| < \eta/(\mu k)$, for any $j = 1, \dots, k$, we have the following estimate for the error in each *interior domain*:

$$(14) \quad \|(1 + |y|)^{n+2s-2n/q} \tilde{E}_j(y)\|_{L^q\{|y| < \frac{\eta}{k\mu}\}} \leq Ck^{-n/q}.$$

We shall prove the validity of estimates (13) and (14) at the end of this section.

In order to solve in ϕ the nonlinear Equation (10), we use a *gluing method*. Let ζ be a cutoff function defined as follows: $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We also defined $\zeta^-(t) = \zeta(2t)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta(k\eta^{-1}|y|^{-2}|y - \xi_j|) & \text{if } |y| > 1, \\ \zeta(k\eta^{-1}|y - \xi_j|) & \text{if } |y| \leq 1. \end{cases}$$

Observe that

$$\zeta_j(y) = \zeta_j(|y|^{-2}y)$$

A function ϕ of the form

$$(15) \quad \phi = \sum_{j=1}^k \tilde{\phi}_j + \psi.$$

is a solution of the problem (10), provided that we can solve the following coupled system of elliptic equation in $(\tilde{\phi}_1, \tilde{\phi}_2, \dots, \tilde{\phi}_k)$ and ψ :

$$(16) \quad \begin{aligned} (-\Delta)^s(\tilde{\phi}_j) - p\gamma|U_*|\zeta_j\tilde{\phi}_j \\ - \zeta_j\left(p\gamma|U_*|^{p-1}\psi + E + \gamma N\left(\tilde{\phi}_j + \sum_{i \neq j} \tilde{\phi}_i + \psi\right)\right) = 0, \end{aligned}$$

where $j = 1, 2, \dots, k$ and

$$(17) \quad \begin{aligned} (-\Delta)^s\psi - p\gamma U^{p-1}\psi \\ - \left(p\gamma(|U_*|^{p-1} - U^{p-1})\left(1 - \sum_{j=1}^k \zeta_j\right) + p\gamma U^{p-1} \sum_{j=1}^k \zeta_j\right)\psi \\ - p\gamma|U_*|^{p-1} \sum_j (1 - \zeta_j)\tilde{\phi}_j \\ - \left(1 - \sum_{j=1}^k \zeta_j\right)\left(E + \gamma N\left(\sum_{j=1}^k \tilde{\phi}_j + \psi\right)\right) = 0 \end{aligned}$$

To solve the above coupled system, we follow the following strategy. First we solve (17) in the unknown ψ , assuming that $\tilde{\phi}_j$ are fixed functions satisfying

$$(18) \quad \tilde{\phi}_j(\bar{y}, y') = \tilde{\phi}_1(e^{\frac{2\pi j}{k}i}\bar{y}, y'), \quad j = 1, 2, \dots, k-1,$$

$$(19) \quad \tilde{\phi}_1(y_1, y_2, \dots, y_j, \dots, y_n) = \tilde{\phi}_1(y_1, y_2, \dots, -y_j, \dots, y_n) \quad j = 2, \dots, n,$$

and the invariant condition under Kelvin's transform,

$$(20) \quad \tilde{\phi}_1 = |y|^{2s-n} \tilde{\phi}_1(|y|^{-2}y).$$

Furthermore, we assume that

$$(21) \quad \|\phi_1\|_* < \rho \quad \text{where } \phi_1 = \mu^{\frac{n-2s}{2}} \tilde{\phi}_1(\xi_1 + \mu y).$$

We have the validity of the following result:

Proposition 2.1. *There exist constants k_0, C, ρ_0 such that for all $k \geq k_0$, the following holds: Suppose that $\tilde{\phi}_j, j = 1, 2, \dots, k$, satisfy conditions (18)–(21) with $\rho < \rho_0$. Then there exists a unique solution $\psi = \Psi(\phi_1)$ to (17) that satisfies the symmetries*

$$\begin{aligned} \psi(\bar{y}, y') &= \psi(e^{\frac{2\pi j}{k}t} \bar{y}, y'), & j &= 1, 2, \dots, k-1, \\ \psi(\bar{y}, \dots, y_j, \dots, y_n) &= \psi(\bar{y}, \dots, -y_j, \dots, y_n), & j &= 3, \dots, n, \\ \psi &= |y|^{2s-n} \psi(|y|^{-2}y), \\ \|\psi\|_* &\leq \frac{C}{k^{n/q-1}} + C \|\phi_1\|_*^2. \end{aligned}$$

Moreover, the operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C \|\phi_1^1 - \phi_1^2\|_*.$$

Once we have the result of the above Proposition, under the assumption on $\tilde{\phi}_j$ we have that all equations (16) reduce to just one, say that for $\tilde{\phi}_1$. Then we will find a solution to our problem if we solve

$$(22) \quad (-\Delta)^s \tilde{\phi}_1 - p\gamma|U_1|^{p-1} \tilde{\phi}_1 - \zeta_1 E - \gamma \mathcal{N}(\phi_1) = 0 \quad \text{in } \mathbb{R}^n$$

where

$$\begin{aligned} \mathcal{N}(\phi_1) &= p(|U_*|^{p-1} \zeta_1 - |U_1|^{p-1}) \phi_1 \\ &\quad + \zeta_1 \left(p|U_*|^{p-1} \Psi(\phi_1) + N \left(\tilde{\phi}_1 + \sum_{i \neq 1} \tilde{\phi}_i + \Psi(\phi_1) \right) \right) \end{aligned}$$

Rather than solving (22) directly, we shall first solve the corresponding projected version of (22):

$$(23) \quad (-\Delta)^s \tilde{\phi}_1 - p\gamma|U_1|^{p-1} \tilde{\phi}_1 - \zeta_1 E + \gamma \mathcal{N}(\phi) = c_{n+1} U_1^{p-1} \tilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n$$

where

$$(24) \quad c_{n+1} = - \frac{\int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi)) \tilde{Z}_{n+1}}{\int_{\mathbb{R}^n} U_1^{p-1} \tilde{Z}_{n+1}^2}.$$

and

$$(25) \quad \tilde{Z}_{n+1}(y) = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y - \xi_1))$$

Proposition 2.2. *There exist constants k_0, C such that for all $k \geq k_0$, the following holds: Let $\Psi(\phi_1)$ the solution predicted by Proposition 2.1. Then there exists a unique solution $\psi_1 = \Phi(\delta)$, $c_{n+1} = c_{n+1}(\delta)$ to (23) and (24), which depends continuously on δ . Moreover,*

$$\|\Phi\|_* \leq Ck^{-\frac{n}{q}} \quad \text{and} \quad \|\mathcal{N}(\phi)\|_{**} \leq Ck^{-\frac{2n}{q}},$$

for some fixed positive constant C .

To conclude our argument, we shall show the existence of a number δ in the definition of μ in (8) so that the above constant c_{n+1} is equal to zero. In this way, we constructed a solution to (1) with the qualitative properties predicted by Theorem 1.1.

Scheme of the paper. In Section 3 we prove some basic results on linear problems in \mathbb{R}^n . These results will be applied to prove Propositions 2.1 and 2.2 in Section 4. Section 5 is dedicated to show the existence of $\delta > 0$ so that $c_{n+1} = 0$, concluding in this way the proof of our theorem.

We finish this section with the proof of estimates (13) and (14).

Proof of (13). This is in the region $\bigcap_j \{|y - \xi_j| > \frac{\eta}{k}\}$. For any y in this exterior region,

$$|E(y)| \leq C \left(\frac{1}{(1 + |y|^2)^{2s}} + \left| \sum_{j=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_j|^{n-2s}} \right|^{\frac{4s}{n-2s}} \right) \left(\sum_{j=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_j|^{n-2s}} \right),$$

for some positive constant $C > 0$. Since for any j fixed and $|y - \xi_j| = \frac{\eta}{k}$ we have

$$\sum_{i=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_i|^{n-2s}} = \frac{1}{k^{n-2s}} k^{n-2s} + \sum_{i \neq j}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_i|^{n-2s}} \leq 1 + \frac{k-1}{ck^{n-2s}},$$

then we conclude that

$$|E| \leq C \frac{\mu^{\frac{n-2s}{2}}}{(1 + |y|^2)^{2s}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{n-2s}}.$$

Thus a direct computation gives

$$\begin{aligned}
& \left\| (1 + |y|)^{n+2s-2n/q} E \right\|_{L^q(\text{Ext})} \\
& \leq C \mu^{\frac{n-2s}{2}} \left\| \frac{(1 + |y|)^{n+2s-2n/q}}{(1 + |y|^2)^{2s}} \sum_{j=1}^k \frac{1}{|y - \xi_j|^{n-2s}} \right\|_{L^q(\text{Ext})} \\
& \leq C \mu^{\frac{n-2s}{2}} \sum_{j=1}^k \left(\int_{|y - \xi_j| > \frac{\eta}{k}} \frac{(1 + |y|)^{(n+2s)q-2n}}{(1 + |y|^2)^{2sq}} \frac{1}{|y - \xi_j|^{(n-2s)q}} dy \right)^{\frac{1}{q}} \\
& \leq C \mu^{\frac{n-2s}{2}} k \left(\int_{\frac{\eta}{k}}^1 \frac{t^{n-1}}{t^{(n-2s)q}} dt \right)^{\frac{1}{q}} \\
& = C \mu^{\frac{n-2s}{2}} k (k^{(n-2s)q-n} - 1)^{\frac{1}{q}} \\
& \leq C \mu^{\frac{n-2s}{2}} k^{(n-2s)+1-\frac{n}{q}} \quad \square
\end{aligned}$$

Proof of (14). This is in the inner region $|y - \xi_j| < \frac{\eta}{k}$, for some j fixed. Observe that if y is close to ξ_j , then

$$U_j \sim O(\mu^{-(n-2s)/2}).$$

For any y in this region, there exists $t \in (0, 1)$ such that

$$E = p \left(-U_j + t \left(-\sum_{i \neq j} U_i + U \right) \right)^{p-1} \left(-\sum_{i \neq j} U_i + U \right) - U^p + \sum_{i \neq j} U_i.$$

We consider the change of scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y)$, $|y| < \frac{\eta}{\mu k}$. Therefore, we obtain that for some $t \in (0, 1)$

$$\begin{aligned}
\tilde{E}_j(y) &= p \left(-U(y) + t \left(\sum_{i \neq j} U(y - \mu^{-1}(\xi_i - \xi_j)) + \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y) \right) \right)^{p-1} \\
&\quad \times \left(-\sum_{i \neq j} U(y - \mu^{-1}(\xi_i - \xi_j)) + \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y) \right) \\
&\quad + \sum_{i \neq j} U^p(y - \mu^{-1}(\xi_i - \xi_j)) - \mu^{\frac{n+2s}{2}} U^p(\xi_j + \mu y).
\end{aligned}$$

Taking into account the configuration of the points ξ_j , we have

$$|\xi_i - \xi_j| \sim \frac{|i - j|}{k}.$$

Furthermore, for $i \neq j$ and $|y| < \frac{\eta}{k\mu}$,

$$\begin{aligned} U(y - \mu^{-1}(\xi_i - \xi_j)) &\leq C \frac{\mu^{n-2s}}{|\xi_j - \xi_i|^{n-2s}} \left(\frac{|\xi_j - \xi_i|^2}{\mu^2 + |\mu y - (\xi_j - \xi_i)|^2} \right)^{\frac{n-2s}{2}} \\ &\leq C \frac{\mu^{n-2s} k^{n-2s}}{|i-j|^{n-2s}}. \end{aligned}$$

Moreover,

$$\left| \sum_{i \neq j} U(y - \mu^{-1}(\xi_i - \xi_j)) \right| \leq C k^{n-2s} \mu^{n-2s} \quad \text{and} \quad \mu^{\frac{n-2s}{2}} U(\xi_j + \mu y) \leq C \mu^{\frac{n-2s}{2}}$$

for some constant $C > 0$. Thus we conclude that

$$|\tilde{E}_j(y)| \leq C \left(\frac{k^{n-2s} \mu^{n-2s}}{1 + |y|^{4s}} + \mu^{\frac{n+2s}{2}} \right),$$

and we have an estimate of the error in the inner region

$$\begin{aligned} \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j(y) \right\|_{L^q\{|y| < \frac{\eta}{k\mu}\}} \\ \leq C \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} \left(\frac{k^{n-2s} \mu^{n-2s}}{1 + |y|^{4s}} + \mu^{\frac{n+2s}{2}} \right) \right\|_{L^q\{|y| < \frac{\eta}{k\mu}\}}. \end{aligned}$$

Since

$$\begin{aligned} \left\| (1 + |y|)^{n-2s-\frac{2n}{q}} \right\|_{L^q\{|y| < \frac{\eta}{k\mu}\}}^q &\leq C \int_0^{\frac{\eta}{k\mu}} (1+r)^{(n-2s)q-n-1} dr \\ &\leq C \left(\frac{1}{k\mu} \right)^{(n-2s)q-n} \end{aligned}$$

and

$$\begin{aligned} \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} \right\|_{L^q\{|y| < \frac{\eta}{k\mu}\}}^q &\leq C \int_0^{\frac{\eta}{k\mu}} (1+r)^{(n+2s)q-n-1} dr \\ &\leq C \left(\frac{1}{k\mu} \right)^{(n+2s)q-n} \end{aligned}$$

it follows that

$$\left\| (1 + |y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j(y) \right\|_{L^q\{|y| < \frac{\eta}{k\mu}\}} \leq C k^{-\frac{n}{q}} (1 + k^{-4s}).$$

This gives the proof of (14). □

3. Some linear problems

Let L_0 be the linear operator defined by

$$L_0(\phi) := (-\Delta)^s(\phi) - p\gamma U^{p-1}\phi \quad \text{in } \mathbb{R}^n.$$

As we know from [Dávila et al. 2013], the set of bounded solutions of the homogeneous equation $L_0(\phi) = 0$ is spanned by the $n + 1$ functions defined by

$$Z_i = \partial_{x_i} U, \quad i = 1, \dots, n, \quad \text{and} \quad Z_{n+1} = \frac{1}{2}(n - 2s)U + x \cdot \nabla U.$$

We now establish a solvability result for the linear problem

$$L_0(\phi) = h \quad \text{in } \mathbb{R}^n,$$

under proper orthogonality conditions on h and ϕ . For this purpose, we introduce the norm

$$(26) \quad \|\phi\|_* := \|(1 + |y|^{n-2s})\phi\|_\infty.$$

Lemma 3.1. *Assume $q \in (\frac{n}{2s}, \frac{n}{s})$. Let h be such that $\|h\|_{**} < \infty$ and*

$$\int_{\mathbb{R}^n} U^{p-1} Z_l h \, dx = 0 \quad \text{for all } l = 1, 2, \dots, n + 1.$$

Then the equation

$$(27) \quad (-\Delta)^s \phi - pU^{p-1}\phi = h \quad \text{in } \mathbb{R}^n$$

has a unique solution ϕ with $\|\phi\|_ < +\infty$ such that*

$$\int_{\mathbb{R}^n} U^{p-1} Z_l \phi \, dx = 0 \quad \text{for all } l = 1, 2, \dots, n + 1.$$

Furthermore, there exists a constant $C > 0$, depending only on q, s , and n , such

$$(28) \quad \|\phi\|_* \leq C \|h\|_{**}.$$

Proof. Let H^s be the completion of $C_0^\infty(\mathbb{R}^n)$ equipped with the norm

$$\|\phi\|_{H^s} = \sqrt{\int_{\mathbb{R}^n} |\phi|^2 + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy},$$

and let $(H^s, \langle \cdot, \cdot \rangle_{H^s})$ be a Hilbert space with the product

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}^{2n}} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+2s}} \, dx \, dy.$$

Let us consider the subspace

$$H = \left\{ \phi \in H^s(\mathbb{R}^n) \text{ such that } \int U^{p-1} Z_l \phi \, dx = 0, \quad l = 1, 2, \dots, n + 1 \right\}.$$

We consider the problem of finding $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \tau \, dx - p\gamma \int_{\mathbb{R}^n} U^{p-1} \phi \tau + \int_{\mathbb{R}^n} h \tau = 0 \quad \text{for all } \tau \in H;$$

this variational formulation makes sense if we consider for instance $h \in L^{\frac{2n}{n+2s}}$, since $H^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ continuously; see, for instance, [Di Nezza et al. 2012].

Let $f \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$. By Riesz's theorem there exist a unique $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \tau \, dx + \int_{\mathbb{R}^n} f \tau \, dx = 0 \quad \text{for all } \tau \in H.$$

Thus $A(f) = \phi$ defines a linear operator between $L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$ and H . By the local compactness of Sobolev embedding [Di Nezza et al. 2012] and the decay at infinity of U^{p-1} , we have that the map $H \rightarrow L^{\frac{2n}{n+2s}}$, $\phi \mapsto U^{p-1}\phi$ is compact. Hence, Fredholm's alternative applies to the problem

$$(29) \quad \phi - A(p\gamma U^{p-1}\phi) = A(h).$$

For $h = 0$, we have $L_0(\phi) = 0$ and $\phi \in H$. Thus $(-\Delta)^s \phi = pU^{p-1}\phi$ in \mathbb{R}^n ; hence,

$$\phi(x) = \sigma_{n,s} p\gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi(y)}{|x-y|^{n-2s}},$$

for some explicit positive constant $\sigma_{n,s}$. We claim that ϕ is bounded. Indeed, let $\delta > 0$ be a fixed positive small number and write

$$(30) \quad \int_{\mathbb{R}^n} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} = \int_{|x-y| < \delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} + \int_{|x-y| > \delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} := I_1 + I_2.$$

We have

$$(31) \quad I_1 \leq C \|\phi\|_{\infty} \int_{|x-y| < \delta} \frac{1}{|x-y|^{n-2s}} \, dy \leq C\delta^{2s} \|\phi\|_{\infty}$$

and, using the Holder inequality repeatedly,

$$\begin{aligned} I_2 &\leq \left(\int_{|x-y| > \delta} \left(\frac{1}{|x-y|^{n-2s}} \right)^{\frac{2n}{n-2s}} \right)^{\frac{n-2s}{2n}} \left(\int_{|x-y| > \delta} \left(U^{p-1}\phi \right)^{\frac{2n}{n+2s}} \right)^{\frac{n+2s}{2n}} \\ &\leq C \left(\int_{|x-y| > \delta} \phi^{\frac{2n}{n-2s}} \right)^{\frac{n-2s}{2n}} \left(\int_{|x-y| > \delta} U^{(p-1)\frac{2n}{4s}} \right)^{\frac{4s}{2n}} \leq C \|\phi\|_{L^{2n/(n-2s)}} \end{aligned}$$

Choosing δ properly small, we obtain that ϕ is bounded. We can now apply the result in [Dávila et al. 2013] and conclude that ϕ is a linear combination of the functions Z_l , $l = 1, \dots, n+1$. Since $\phi \in H$ we have that $\phi \equiv 0$. Fredholm's alternative implies that, for any h satisfying the orthogonality condition, a function $\phi \in H$ solution to (29) exists.

Assume now that ϕ solves (27), we shall now show the a priori bound (28). We first show that ϕ is bounded. First we have

$$\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \leq \|\phi\|_{H^s(\mathbb{R}^n)} \leq \|h\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \leq \|(1+|y|)^{n+2s-2n/q} h\|_{L^q(\mathbb{R}^n)}.$$

Observe now that

$$\phi(x) = \sigma_{n,s} p \int_{\mathbb{R}^n} \frac{U^{p-1} \phi(y)}{|x-y|^{n-2s}} + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}}.$$

Fixing a small $\delta > 0$, we get

$$\int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}} dy = \int_{|x-y| < \delta} \frac{h(y)}{|x-y|^{n-2s}} dy + \int_{|x-y| < \delta} \frac{h(y)}{|x-y|^{n-2s}} dy = J_1 + J_2$$

with

$$J_1 \leq \int_{|x-y| < \delta} \left(\frac{1}{|x-y|^{(n-2s)q'}} \right)^{q'} \|h\|_{L^q(\mathbb{R}^n)} \leq C \|h\|_{L^q(\mathbb{R}^n)}$$

since $q > \frac{n}{2s}$, and

$$J_2 \leq \left(\int_{|x-y| > \delta} \frac{1}{|x-y|^{2n}} \right)^{\frac{n-2s}{2n}} \|h\|_{L^{2n/(n+2s)}} \leq C \|h\|_{L^{2n/(n+2s)}}.$$

Thus, thanks also to (30) and (31), for all $x \in \mathbb{R}^n$,

$$|\phi(x)| \leq C \delta^{2s} \|\phi\|_{\infty} + C (\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}}).$$

Choosing δ small, we conclude that ϕ is bounded since

$$(32) \quad \|\phi\|_{\infty} \leq C (\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}}).$$

Next we show the decay rate at infinity of ϕ . Consider

$$\tilde{\phi}(y) = |y|^{2s-n} \phi(|y|^{-2}y) \quad \text{and} \quad \tilde{h}(y) = |y|^{-n-2s} h(|y|^{-2}y).$$

A direct computation shows that

$$(-\Delta)^s \tilde{\phi} - p\gamma U^{p-1}(y) \tilde{\phi} = \tilde{h} \quad \text{on } \mathbb{R}^n \setminus \{0\},$$

and

$$\begin{aligned} \|\tilde{\phi}\|_{H^s(\mathbb{R}^n)} + \|\tilde{\phi}\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} &= \|\phi\|_{H^s(\mathbb{R}^n)} + \|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}, \\ \|\tilde{h}\|_{L^q(\mathbb{R}^n)} &= \|(1+|y|)^{n+2s-2n/q} h\|_{L^q(\mathbb{R}^n)} = \|h\|_{**}. \end{aligned}$$

Applying the estimate (32) to $\tilde{\phi}$, we get

$$\begin{aligned} \|\tilde{\phi}\|_{L^\infty(B(0,1))} &\leq \|\tilde{\phi}\|_{L^\infty(\mathbb{R}^n)} \leq C (\|\tilde{\phi}\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)} + \|\tilde{h}\|_{L^{2n/(n+2s)}}) \\ &\leq C (\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}}) \\ &\leq C (\|h\|_{**} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)}) = C \|h\|_{**}. \end{aligned}$$

Since $\||y|^{n-2s} \phi\|_{L^\infty\{|y|>1\}} = \|\tilde{\phi}\|_{L^\infty(B(0,1))}$, we conclude that $\|\phi\|_* \leq C \|h\|_{**}$. \square

Under further symmetry conditions on h and ϕ , (27) can be solved without the orthogonality conditions. For a general function ψ defined in \mathbb{R}^n , consider the symmetries

$$(33) \quad \psi(\bar{y}, y') = \psi(e^{\frac{2\pi j}{k}t} \bar{y}, y'), \quad j = 1, 2, \dots, k-1,$$

and

$$(34) \quad \psi(\bar{y}, \dots, y_j, \dots, y_n) = \psi(\bar{y}, \dots, -y_j, \dots, y_n), \quad j = 3, \dots, n,$$

together with invariance under the Kelvin transform

$$(35) \quad \psi(y) = |y|^{2s-n} \psi(|y|^{-2} y).$$

Lemma 3.2. *Assume that h satisfies (33), (34), and $\|h\|_{**} < \infty$. Furthermore, we assume that*

$$h(y) = |y|^{-n-2s} h(|y|^{-2} y).$$

Then (27) has a unique bounded solution $\phi = T(h)$ that satisfies symmetries (33), (34), and (35). Moreover, there exists C depending only on q , s , and n such that

$$\|\phi\|_* \leq \|h\|_{**}.$$

The proof of this result is very close to the proof of [del Pino et al. 2011, (4.19)]. We refer the interested reader to that reference.

For a later purpose, we need to establish a result like the one in Lemma 3.1 for a linear operator more general than L_0 .

Lemma 3.3. *Let $2s < v < n$. There exist numbers δ , C , depending on v , n such that the following holds: If g , a , and ϕ are functions such that $\|(1 + |y|^v)g\|_\infty < +\infty$, $\|(1 + |y|^{v-2s})\phi\|_\infty < +\infty$, and $\|(1 + |y|^{2s})a\|_\infty < \delta$, and*

$$(36) \quad L_0(\phi) + a(y)\phi = g(y) + \sum_{l=1}^{n+1} c_l U^{p-1} Z_l \quad \text{in } \mathbb{R}^n,$$

where

$$(37) \quad \int_{\mathbb{R}^n} U^{p-1} Z_l \phi = 0 \quad \text{for all } l = 1, \dots, n+1$$

and

$$(38) \quad c_l \int_{\mathbb{R}^n} U^{p-1} Z_l^2 = \int_{\mathbb{R}^n} (a(y)\phi - g(y)) Z_l \phi \quad \text{for all } l = 1, \dots, n+1,$$

then

$$(39) \quad \|(1 + |y|^{v-2s})\phi\|_\infty \leq C \|(1 + |y|^v)g\|_\infty.$$

Proof. By contradiction, let us assume the existence of functions ϕ_n , a_n , g_n and constants c_l^n such that (36)–(38) hold, and

$$(40) \quad \|(1 + |y|^\nu)g_n\|_\infty \rightarrow 0, \quad \|(1 + |y|^{\nu-2s})\phi_n\|_\infty = 1, \quad \|(1 + |y|^{2s})a_n\|_\infty \rightarrow 0.$$

Clearly, $\|(1 + |y|^\nu)a_n g_n\|_\infty \rightarrow 0$ and $c_l^n \rightarrow 0$, so without loss of generality we may assume that $a_n \equiv 0$ and $c_l^n = 0$. We claim first that

$$\|\phi_n\|_\infty \rightarrow 0.$$

Assume the opposite: there are numbers γ , $R > 0$ and points x_n such that

$$|\phi_n(x_n)| \geq \gamma, \quad |x_n| \leq R.$$

Passing to a subsequence, and arguing like in the proof of [Lemma 3.1](#), we find that ϕ_n converges in the energy space and locally uniformly over compact sets to a bounded function $\phi_0 \neq 0$ with

$$L_0(\phi_0) = 0, \quad \text{and} \quad \int_{\mathbb{R}^n} U^{p-1} \phi Z_l = 0 \quad \text{for all } l,$$

which gives $\phi_0 = 0$. This is a contradiction due to the result in [\[Dávila et al. 2013\]](#). Thus we have that $\|\phi_n\|_\infty \rightarrow 0$.

Next we shall show that $\|(1 + |y|^{\nu-2s})\phi_n\|_\infty \rightarrow 0$, thus getting to a contradiction with (40), and the proof of the Lemma. Using the equation, we have that

$$(41) \quad \phi_n(x) = \sigma_{n,s} p \gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi_n(y)}{|x-y|^{n-2s}} dy + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{g_n(y)}{|x-y|^{n-2s}} dy$$

for some explicit positive constant $\sigma_{n,s}$. Since $2s < \nu < n$, and taking into account that $\|(1 + |y|^\nu)g_n\|_\infty \rightarrow 0$, as well as the behavior of U^{p-1} at infinity, there exists a positive constant C , independent of n , such that

$$|\phi_n(x)| \leq C \left(\frac{\|\phi_n\|_\infty}{(1 + |x|^{2s})} + \frac{o(1)}{(1 + |x|^{\nu-2s})} \right)$$

for some $o(1) \rightarrow 0$, as $n \rightarrow \infty$. Replacing the above estimate in (41) and repeating the same procedure a finite number of times, we get that

$$|\phi_n(x)| \leq C \frac{\|\phi_n\|_\infty + o(1)}{(1 + |x|^{\nu-2s})}. \quad \square$$

4. Proof of Propositions 2.1 and 2.2

Proof of Proposition 2.1. Let us fix functions $\tilde{\phi}_j$ and we assume that they satisfy the symmetry assumptions (6), (7) and the invariance under Kelvin transform

$$\tilde{\phi}_1 = |y|^{2s-n} \tilde{\phi}_1(|y|^{-2}y).$$

Finally, we assume

$$(42) \quad \|\phi_1\|_* < \rho, \quad \text{where } \phi_1 = \mu^{\frac{n-2s}{2}} \tilde{\phi}_1(\xi_1 + \mu y).$$

for a small, fixed $\rho > 0$.

We next solve (17). To do so, we write it in the form

$$(-\Delta)^s(\psi) - p\gamma U^{p-1}(y)\psi - \gamma V(y)\psi - \underbrace{p\gamma|U_*|^{p-1} \sum_{j=1}^k (1 - \zeta_j) \tilde{\phi}_j}_{:=h} - M(\psi) = 0,$$

where

$$V(y) := \underbrace{p(|U_*|^{p-1} - U^{p-1}) \left(1 - \sum_{j=1}^k \zeta_j\right)}_{:=V_1} + \underbrace{pU^{p-1} \sum_{j=1}^k \zeta_j}_{:=V_2} := V_1 + V_2$$

and

$$M(\psi) := \left(1 - \sum_{j=1}^k \zeta_j\right) \left(E + \gamma N \left(\sum_{j=1}^k \tilde{\phi}_j + \psi\right)\right)$$

A basic observation is that the function h as defined above satisfies the conditions (33), (34), and $\|h\|_{**} < \infty$. Furthermore, we have that

$$h(y) = |y|^{-n-2s} h(|y|^{-2}y).$$

Hence, we can define the linear operator T in the Lemma 3.2 and we can write our problem (17) in fixed point as

$$(43) \quad \psi = -T \left(V\psi + p\gamma|U_*|^{p-1} \sum_j (1 - \zeta) \tilde{\phi}_j + M(\psi) \right) =: \mathcal{M}(\psi)$$

We notice that \mathcal{M} is well defined in space X of continuous functions ψ with $\|\psi\|_* \leq \infty$, and satisfying

$$\begin{aligned} \psi(\bar{y}, y') &= \psi(e^{\frac{2\pi j}{k}t} \bar{y}, y'), & j &= 1, 2, \dots, k-1, \\ \psi(\bar{y}, \dots, y_j, \dots, y_n) &= \psi(\bar{y}, \dots, -y_j, \dots, y_n), & j &= 3, \dots, n, \\ \psi &= |y|^{2s-n} \psi(|y|^{-2}y). \end{aligned}$$

We claim that

$$(44) \quad \|V\psi(y)\|_{**} \leq Ck^{1-\frac{n}{q}} \|\psi\|_*$$

and

$$(45) \quad \left\| p\gamma|U_*|^{p-1} \sum_{j=1}^k (1 - \zeta_j) \tilde{\phi}_j \right\|_{**} \leq Ck^{1-\frac{n}{q}} \|\psi\|_*.$$

We claim that if

$$\|\psi\|_* + \|\phi_1\|_* \leq 2\rho,$$

then

$$(46) \quad \|M(\psi)\|_{**} \leq C(k^{1-\frac{n}{q}} + k^{1-\frac{n}{q}}\|\phi_1\|_*^2 + \|\psi\|_*^2).$$

Furthermore, for ψ_1, ψ_2 in X ,

$$\|M(\psi_1) - M(\psi_2)\|_{**} \leq C\rho\|\psi_1 - \psi_2\|_*$$

We can thus conclude that, for ρ small enough, the operator \mathcal{M} defines a contraction map in the set of functions $\psi \in X$ with

$$(47) \quad \|\psi\|_* \leq C(\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}).$$

From the estimate (47), we get the Lipschitz dependence

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C\|\phi_1^1 - \phi_1^2\|_*.$$

We shall next show the validity of (44), (45), and (46).

Proof of (44). Consider

$$f(t) = \left| U - t \sum_{j=1}^k U_j \right|^{p-1}.$$

By the mean value theorem,

$$|V_1| \leq p(p-1) \left| U - s \sum_{j=1}^k U_j \right|^{p-2} \left(\sum_{j=1}^k U_j \right) \leq C U^{p-2} \sum_{j=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_j|^{n-2s}}.$$

Thus, if for all j , $|y - \xi_j| > \frac{n}{k}$, then

$$|V_1 \psi(y)| \leq C \|\psi\|_* U^{p-1}(y) \sum_{j=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|y - \xi_j|^{n-2s}}.$$

Since $\zeta_j \equiv 1$ on $|y - \xi_j| < \frac{n}{k}$,

$$\begin{aligned} \|(1 + |y|)^{n+2s-\frac{2n}{q}} V_1 \psi\|_{L^q(\mathbb{R}^n)} &= \|(1 + |y|)^{n+2s-\frac{2n}{q}} V_1 \psi\|_{L^q(\mathbb{R}^n \setminus \bigcup_j B(\xi_j, \frac{n}{k}))} \\ &\leq Ck \left(\int_{B(\xi_1, \frac{n}{k})^c \cap B(0,2)} \frac{\mu^{\frac{(n-2s)q}{2}}}{|y - \xi_j|^{(n-2s)q}} dy \right)^{\frac{1}{q}} \|\psi\|_* \\ &\leq Ck \mu^{\frac{n-2s}{2}} k^{(n-2s)-\frac{n}{q}} \|\psi\|_*, \end{aligned}$$

for some positive constant C . Thus $\|V_1 \psi(y)\|_{**} \leq Ck^{1-\frac{n}{q}} \|\psi\|_*$. On the other hand,

$$\begin{aligned} \|V_2 \psi\|_{**} &= \|(1+|y|)^{n+2s-\frac{2n}{q}} pU^{p-1} \sum_{j=1}^k \zeta_j \psi\|_{L^q(\mathbb{R}^n)} \\ &\leq C \left(\int_{B(0,1)} \left((1+|y|)^{n+2s-\frac{2n}{q}} U^{p-1} \sum_{j=1}^k \zeta_j \psi \right)^q dy \right)^{\frac{1}{q}} \end{aligned}$$

with

$$\begin{aligned} &\left(\int_{B(0,1)} \left((1+|y|)^{n+2s-\frac{2n}{q}} U^{p-1} \sum_{j=1}^k \zeta_j \psi \right)^q dy \right)^{\frac{1}{q}} \\ &\leq C \sum_j \left(\int_{B(\xi_j, \frac{2n}{k})} \frac{U^{(p-1)q} (1+|y|)^{(n+2s)q-2n}}{(1+|y|)^{(n-2s)q}} dy \right)^{\frac{1}{q}} \|\psi\|_* \\ &\leq Ck^{1-n} \|\psi\|_* \quad \square \end{aligned}$$

Proof of (45). Estimate (45) can be obtained arguing as in the proof of estimate (44), after noticing that

$$|\tilde{\phi}_j(y)| \leq CU(y) \|\phi_1\|_* \frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}. \quad \square$$

Proof of (46). For the moment we shall assume that

$$\|\psi\|_* + \|\phi_1\|_* \leq 2\rho$$

for a ρ sufficiently small. Let us assume that $|y-\xi_j| > \frac{\eta}{k}$ for all j . First we recall that

$$\left\| (1+|y|)^{n+2s-\frac{2n}{q}} \left(1 - \sum_{j=1}^k \zeta_j \right) E \right\|_{L^q(\mathbb{R}^n)} = \|(1+|y|)^{n+2s-\frac{2n}{q}} E\|_{L^q(\text{Ext})} \leq Ck^{1-\frac{n}{q}}$$

Then we find in this region

$$\left| N \left(\sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right| \leq CU^{p-2} \left(\left| \sum_{j=1}^k \tilde{\phi}_j \right|^2 + |\psi|^2 \right).$$

But

$$U^{p-2} \left| \sum_{j=1}^k \tilde{\phi}_j \right|^2 \leq C \|\phi_1\|_*^2 U^p \sum_{j=1}^k \frac{\mu^{n-2s}}{|y-\xi_j|^{2(n-2s)}}, \quad U^{p-2} |\psi|^2 \leq U^p \|\psi\|_*^2$$

Thus, we have

$$\begin{aligned}
& \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} \left(1 - \sum_{j=1}^k \zeta_j \right) \left(\gamma N \left(\sum_{j=1}^k \tilde{\phi}_j + \psi \right) \right) \right\|_{L^q(\mathbb{R}^n)} \\
&= \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} (\gamma N(\phi)) \right\|_{L^q(\text{Ext})} \\
&\leq C \|\phi_1\|_*^2 \left\| (1 + |y|)^{n+2s-\frac{2n}{q}} U^p \left(\sum_{j=1}^k \frac{\mu^{n-2s}}{|y - \xi_j|^{2(n-2s)}} + \psi \right) \right\|_{L^q(\text{Ext})} \\
&\leq \frac{C \mu^{n-2s}}{k^{-2(n-2s)+\frac{n}{q}-1}} \|\phi_1\|_*^2 + C \|\psi\|_*^2
\end{aligned}$$

Using the above inequalities, we get

$$\|M(\psi)\|_{**} \leq C k^{1-\frac{n}{q}} + k^{1-\frac{n}{q}} \|\phi_1\|_*^2 + C \|\psi\|_*^2, \quad \square$$

This concludes the proof of [Proposition 2.1](#).

Proof of Proposition 2.2. In order to prove [Proposition 2.2](#), we need to consider the linear problem

$$(48) \quad (-\Delta)^s \tilde{\phi}_1 - p\gamma U_1^{p-1} \tilde{\phi} - \tilde{h}(y) = c_{n+1} U_1^{p-1} \tilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n$$

for a general function \tilde{h} , where

$$\tilde{Z}_{n+1}(y) = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y - \xi_1)) \quad \text{and} \quad c_{n+1} = \frac{\int_{\mathbb{R}^n} \tilde{h} \tilde{Z}_{n+1}}{\int_{\mathbb{R}^n} U_1^{p-1} \tilde{Z}_{n+1}^2}.$$

Lemma 4.1. *Assume that \tilde{h} is even with respect to each variable y_2, \dots, y_n and it satisfies the invariance*

$$\tilde{h}(y) = |y|^{-n-2s} h(|y|^{-2}y)$$

Assume in addition that

$$h(y) = \mu^{\frac{n+2s}{2}} \tilde{h}(\xi_1 + \mu y)$$

satisfies $\|h\|_{**} \leq \infty$. Then (48) has a unique solution $\tilde{\phi} := \tilde{T}(\tilde{h})$ that is even with respect to each of the variables y_2, \dots, y_n , invariant under Kelvin's transformations

$$\tilde{\phi}(y) = |y|^{2s-n} \tilde{\phi}(|y|^{-2}y),$$

where $\phi(y) = \mu^{\frac{n-2s}{2}} \tilde{\phi}(\xi_1 + \mu y)$ and satisfies

$$\int_{\mathbb{R}^n} \phi U^{p-1} Z_{n+1} = 0.$$

Moreover, there exists C such that

$$\|\phi\|_* \leq C \|h\|_{**}.$$

Proof. We consider ϕ and h such that

$$(-\Delta)^s \phi - p\gamma|U|^{p-1}\phi = h(y) \quad \text{in } \mathbb{R}^n, \quad \text{and} \quad \int_{\mathbb{R}^n} \tilde{h} \tilde{Z}_{n+1} = 0.$$

The evenness of h in the last $(n - 1)$ coordinates guarantees that

$$\int_{\mathbb{R}^n} h Z_l = 0, \quad l = 2, \dots, n, n + 1.$$

We have that to prove that $\int_{\mathbb{R}^n} h Z_1 = 0$. Let

$$I(t) = \int_{\mathbb{R}^n} w_\mu(y - t\xi_1) \tilde{h}(y) dy.$$

We notice that

$$(49) \quad (\xi_1)_1 \int_{\mathbb{R}^n} h Z_1 = \partial_t I(t) \Big|_{t=0} = -(\xi_1)_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(y - \xi_1) h(y) dy;$$

after a change of variable,

$$I(t) = \int_{\mathbb{R}^n} w_\mu(|y|^{-2}y - t\xi_1) \tilde{h}(|y|^{-2}y) |y|^{-2n} = \int_{\mathbb{R}^n} w_{\mu(t)}(y - a(t)\xi_1) \tilde{h}(y) dy$$

where

$$\mu(t) = \frac{\mu t}{\mu^2 + |\xi_1|^2 t^2} \quad \text{and} \quad s(t) = \frac{t}{\mu^2 + |\xi_1|^2 t^2}.$$

Hence,

$$(50) \quad \begin{aligned} \partial_t I(t) \Big|_{t=1} &= \mu'(1) \int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1) \Big|_{\mu=1} \tilde{h}(y) dy \\ &\quad - s'(1) \xi_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(t)(y - \xi_1) h(y) dy = 0. \end{aligned}$$

We can check that

$$\int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1) \Big|_{\mu=1} \tilde{h}(y) dy = \int_{\mathbb{R}^n} Z_{n+1}(y) h(y) dy = 0$$

and $s'(1) = 1 - 2|\xi_1|^2$. Hence, using (49) and (50), we obtain $\int_{\mathbb{R}^n} h Z_1 = 0$. It follows from Lemma 3.1 that there exists a unique solution ϕ_1 for (48) with

$$\int_{\mathbb{R}^n} h Z_l = 0, \quad l = 1, \dots, n + 1 \quad \text{and} \quad \|\phi\|_* \leq C \|h\|_{**}.$$

Arguing by uniqueness, as in proof of Lemma 3.2, we find that $\tilde{\phi}$ satisfies the corresponding symmetries. \square

We use the above lemma to solve (23) and (24). We consider the operator \tilde{T} defined in the lemma. We are going to prove the existence of a solution to (23) by a fixed point argument

$$(51) \quad \tilde{\phi}_1 = \tilde{T}(\zeta_1 + \gamma \mathcal{N}(\phi_1)) =: \mathcal{M}(\phi_1).$$

For any f we set $\bar{f}(y) = \mu^{\frac{n+2s}{2}} f(\xi + \mu y)$. Let

$$f_1(y) = p\zeta_1(|U_*|^{p-1} - |U_1|^{p-1})\tilde{\phi}_1.$$

For $|y| < \frac{\eta}{k\mu}$,

$$|\bar{f}_1(y)| \leq C \left(\mu^{n-2s} k^{n-2s} \sum_{j=1}^{k-1} \frac{1}{j^{n-2s}} + \mu^{\frac{n-2s}{2}} \right) U^{p-1} \|\phi_1\|_*$$

and so

$$\|\bar{f}_1(y)\|_{**} \leq C \left(\mu^{n-2s} k^{n-2s} + \mu^{\frac{n-2s}{2}} \right) (\mu k)^{-n+2s+\frac{n}{q}} \|\phi_1\|_* = C \mu^{\frac{2n}{q}} \|\phi_1\|_*.$$

Analogously for $f_2 = (\zeta_1 - 1)U_1^{p-1}\tilde{\phi}_1$ in the region $|y| < \frac{\eta}{\mu k}$,

$$|\bar{f}_2(y)| \leq U^p \|\phi_1\|_*;$$

hence $\|\bar{f}_2\|_{**} \leq C k^{-\frac{n}{q}} \|\phi_1\|_*$. Now we consider $f_3 = \zeta_1 p |U_*|^{p-1} \Psi(\phi_1)$ on $|y| < \frac{\eta}{\mu k}$,

$$|\bar{f}_3| \leq C U^{p-1} \mu^{\frac{n-2s}{2}} \|\Psi(\phi_1)\|_\infty \leq C U^{p-1} \mu^{\frac{n-2s}{2}} (\|\phi_1\|_* + k^{1-\frac{n}{q}});$$

thus,

$$\|\bar{f}_3(y)\|_{**} \leq C \mu^{\frac{n}{2q}} (\|\phi_1\|_* + k^{1-\frac{n}{q}}).$$

Now, for

$$f_4 = \zeta_1 N \left(\tilde{\phi}_1 + \sum_{i=2} \tilde{\phi}_i \right) \Psi(\phi_1)$$

we notice that

$$\bar{N}(\phi) = (V_* + \hat{\phi})^p - V_*^p - p V_*^{p-1} \hat{\phi}$$

where $\hat{\phi}(y) := \mu^{\frac{n-2s}{2}} \phi(\xi_1 + \mu y)$ and

$$V_*(y) = U(y) + \sum_{i=2}^k U(y + \mu^{-1}(\xi_1 - \xi_j)) - \mu^{\frac{n-2s}{2}} U(\xi_1 + \mu y)$$

with

$$\phi = \tilde{\phi}_1 + \sum_{i=2}^k \tilde{\phi}_i + \Psi(\phi_1).$$

Therefore

$$|\bar{f}_4| \leq C \left(U^{p-1} \mu^{\frac{n-2s}{2}} \|\phi_1\|_* + U^{p-1} \mu^{\frac{n-2s}{2}} (\|\phi_1\|_* + k^{1-\frac{n}{q}}) \right),$$

and hence,

$$\|\bar{f}_4\|_{**} \leq C \left(\mu^{\frac{n}{2q}} \|\phi_1\|_* + \mu^{\frac{n}{2q}} (\|\phi_1\|_* + k^{1-\frac{n}{q}})^2 \right).$$

Concerning $f_5 = \zeta_1 E$, we recall that

$$\|\bar{f}_5\|_{**} \leq C \mu^{\frac{n}{2q}}.$$

The above estimates suggest that it is possible to apply a fixed point argument of contraction type in the set of all continuous functions $\phi_1 = \Phi(\delta)$ such that $\|\phi_1\|_* \leq C \mu^{\frac{n}{2q}}$. This gives the existence and the estimate for ϕ_1 , satisfying

$$\|\Phi\|_* \leq C k^{-\frac{n}{q}},$$

and

$$\|\mathcal{N}(\phi)\|_{**} \leq C k^{-\frac{2n}{q}}.$$

Straightforward computations shows also the continuous dependence of $\phi_1 = \Phi(\delta)$ and c_{n+1} on the parameter δ . This concludes the proof of [Proposition 2.2](#).

5. Conclusion

In this section we show the existence of $\delta > 0$ such that $c_{n+1}(\delta) = 0$ in [\(23\)](#). Indeed this fact guarantees that the function

$$U_* + \phi,$$

where $U_* = U - \sum U_j$ is defined in [\(9\)](#) and $\phi = \sum_{j=1}^k \tilde{\phi}_j + \psi$ is defined in [\(15\)](#), is a solution for the original problem [\(1\)](#). Let

$$\tilde{Z}_{n+1} = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y - \xi_1)).$$

We recall that

$$Z_{n+1}(y) = y \cdot \nabla U + \frac{n-2s}{2} U.$$

We need the existence of a δ such that

$$(52) \quad c_{n+1} = \int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi_1)) \tilde{Z}_{n+1} = 0.$$

Since we are assuming that $s > \frac{1}{2}$, we claim that

$$(53) \quad \int_{\mathbb{R}^n} \zeta_1 E \tilde{Z}_{n+1} = A \delta k^{2s-n} \left(-2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \right) \delta + 1 \right) + k^{1-n} \Theta_k(\delta)$$

and

$$(54) \quad \int_{\mathbb{R}^n} \gamma \mathcal{N}(\phi_1) \tilde{Z}_{n+1} = k^{-(n-2s)} k^{1-\frac{n}{q}} \Theta_k(\delta),$$

where $\Theta_k(\delta)$ denotes a continuous function of δ , which is uniformly bounded, as $k \rightarrow \infty$. Since $n - 2s > 1$ for any $s \in (\frac{1}{2}, 1)$, from (53) and (54) we obtain the existence of a unique δ solution to (52) with

$$\delta = \left(2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \right) \right)^{-1} (1 + O(k^{1-2s})).$$

What is left of this section is devoted to the proof of (53) and (54).

Proof of (53). We write

$$\int_{\mathbb{R}^n} \zeta_1 E \tilde{Z}_{n+1} = \int_{\mathbb{R}^n} E \tilde{Z}_{n+1} + \int_{\mathbb{R}^n} (\zeta_1 - 1) E \tilde{Z}_{n+1}.$$

Expanding the first term, we get

$$\int_{\mathbb{R}^n} E \tilde{Z}_{n+1} = \int_{B_1} E \tilde{Z}_{n+1} + \int_{\mathbb{R}^n \setminus \cup B_j} E \tilde{Z}_{n+1} + \sum_{j \neq 1} \int_{B_j} E \tilde{Z}_{n+1} := I_1 + I_2 + I_3,$$

where $B_j = B(\xi_j, \frac{\eta}{k})$. With the scaling $x = \mu y + \xi_1$ and writing

$$\tilde{E}(y) = \mu^{\frac{n+2s}{2}} E(\xi_1 + \mu y),$$

we get

$$\int_{B_1} E \tilde{Z}_{n+1} = \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1(y) Z_{n+1}(y) dy.$$

Thus

$$\begin{aligned} I_1 &= \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1 Z_{n+1}(y) dy \\ &= -\gamma p \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\ &\quad + \gamma p \mu^{\frac{n-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} dy \\ &\quad + \gamma p \int_{B(0, \frac{\eta}{\mu k})} ((U(y) + sV(y))^{p-1} - U^{p-1}) V(y) Z_{n+1} dy \\ &\quad + \gamma \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\ &\quad - \mu^{\frac{n+2s}{2}} \gamma \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) Z_{n+1} dy, \end{aligned}$$

where

$$V(y) = \left(- \sum_{j \neq 1} U(y - \mu^{-1}(\xi_j - \xi_1)) + \mu^{\frac{n-2s}{2}} U(\xi_1 + \mu y) \right).$$

For $j \neq 1$, and by Taylor expansion,

$$U(y + \mu^{-1}(\xi_1 - \xi_j)) = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} (1 + O(\mu^2 k^2)),$$

where $\hat{\xi}_1 = (1, 0, \dots, 0)$ and

$$\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}} \hat{\xi}_1;$$

thus

$$\begin{aligned} & \int_{B(0, \frac{n}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\ &= \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \int_{B(0, \frac{n}{\mu k})} U^{p-1} (1 + O(\mu^2 k^2)) Z_{n+1} \\ &= \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \left(\int_{\mathbb{R}^n} U^{p-1} Z_{n+1} - \int_{\mathbb{R}^n \setminus B(0, \frac{n}{\mu k})} U^{p-1} Z_{n+1} \right. \\ & \quad \left. + O(\mu^2 k^2) \int_{B(0, \frac{n}{\mu k})} U^{p-1} Z_{n+1} \right) \\ &= \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} (C_1 + O(\mu^{2s} k^{2s}) + O(\mu^2 k^2)) \\ &= \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} C_1 (1 + O(\mu^{2s} k^{2s})), \end{aligned}$$

where

$$C_1 = \int_{\mathbb{R}^n} U^{p-1} Z_{n+1}.$$

For the second term,

$$\mu^{\frac{n-2s}{2}} \int_{B(0, \frac{n}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} dy = \mu^{\frac{n-2s}{2}} C_1 (1 + O(\mu^{2s} k^{2s})).$$

Furthermore,

$$\begin{aligned}
& \left| \int_{B(0, \frac{\eta}{\mu k})} ((U(y) + sV(y))^{p-1} - U^{p-1}) V(y) Z_{n+1} dy \right| \\
& \leq \left| \sum_{i \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \right| \\
& \leq C \sum_{i \neq 1} \frac{\mu^{n+2s}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2s}} \int_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{n-2s}} \\
& \leq C(\mu k)^{-2s} \sum_{i \neq 1} \frac{\mu^{n+2s}}{|\hat{\xi}_1 - \hat{\xi}_i|^{n+2s}}
\end{aligned}$$

and

$$\begin{aligned}
& \left| \mu^{\frac{n+2s}{2}} \gamma \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) Z_{n+1} dy \right| \\
& \leq C \mu^{\frac{n+2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} \frac{1}{(1 + |y|)^{n-2s}} dy \leq C \mu^{\frac{n-2s}{2}} k^{-2s}.
\end{aligned}$$

Therefore, we conclude that

$$I_1 = A \delta k^{-(n-2s)} \left(-2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \right) \delta + 1 \right) + k^{-n} \Theta_k(\delta),$$

where $\Theta_k(\delta)$ is a smooth function of δ , which is uniformly bounded as $k \rightarrow \infty$.

Now we are going to estimate I_2 . The Holder inequality gives

$$\begin{aligned}
& \left| \int_{\mathbb{R}^n \setminus \cup B_j} E \tilde{Z}_{n+1} \right| \\
& \leq C \|(1 + |y|)^{n+2s - \frac{2n}{q}} E\|_{L^q(\mathbb{R}^n \setminus \cup B_j)} \\
& \quad \times \|(1 + |y|)^{-n-2s + \frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y + \mu^{-1}(\xi_j - \xi_1))\|_{L^{q/(q-1)}(\mathbb{R}^n \setminus \cup B_j)}.
\end{aligned}$$

A direct computation gives that

$$\|(1 + |y|)^{-n-2s + \frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y + \mu^{-1}(\xi_j - \xi_1))\|_{L^{q/(q-1)}(\mathbb{R}^n \setminus \cup B_j)} \leq C k^{-n \frac{q-1}{q}}$$

for some constant $C > 0$. Thus we conclude that

$$|I_2| \leq C k^{1-n}$$

since we have already proved — see (13) — that

$$\|(1 + |y|)^{n+2s - \frac{2n}{q}} E\|_{L^q(\mathbb{R}^n \setminus \cup B_j)} \leq C k^{1 - \frac{n}{q}}.$$

Let $j \neq 1$ be fixed and $\tilde{E}_j(y) = \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y)$. After the change of variable $x = \mu y + \xi_j$, we obtain

$$\begin{aligned} \left| \int_{B_j} E \tilde{Z}_{n+1} \right| &= \left| \mu^{\frac{n-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_j \tilde{Z}_{n+1}(\mu y + \xi_j) \right| \\ &\leq C \mu^{\frac{n-2s}{2}} \|(1 + |y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j\|_{L^q(B(0, \frac{\eta}{\mu k}))} \\ &\quad \times \|(1 + |y|)^{-n-2s+\frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y + \mu^{-1}(\xi_j - \xi_1))\|_{L^{q/(q-1)}(B(0, \frac{\eta}{\mu k}))}. \end{aligned}$$

We have

$$\begin{aligned} \|(1 + |y|)^{-n-2s+\frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y + \mu^{-1}(\xi_j - \xi_1))\|_{L^{q/(q-1)}(B(0, \frac{\eta}{\mu k}))} \\ \leq C \frac{\mu^{\frac{n-2s}{2}}}{|\xi_j - \xi_1|^{n-2s}} \left(\int_1^{\frac{\eta}{\mu k}} \frac{t^{n-1}}{t^{(n+2s-\frac{2n}{q})\frac{q}{q-1}}} dt \right)^{\frac{q-1}{q}} \\ \leq C \frac{\mu^{\frac{n-2s}{2}}}{|\xi_j - \xi_1|^{n-2s}} (\mu k)^{2s-\frac{n}{q}} \end{aligned}$$

and

$$\|(1 + |y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j\|_{L^q(B(0, \frac{\eta}{\mu k}))} \leq (\mu k)^{\frac{n}{q}} (1 + k^{-(n+2s)} \mu^{-\frac{n-2s}{2}}).$$

Hence,

$$\begin{aligned} |I_3| &= \left| \sum_{j \neq 1} \int_{B_j} E \tilde{Z}_{n+1} \right| \\ &\leq \mu^{\frac{n-2s}{2}} (\mu k)^{\frac{n}{q}} (1 + k^{-(n+2s)} \mu^{-\frac{n-2s}{2}}) \sum_{j=1}^k \frac{\mu^{\frac{n-2s}{2}}}{|\xi_j - \xi_1|^{n-2s}} (\mu k)^{2s-\frac{n}{q}} \\ &\leq C \mu^{\frac{n-2s}{2}} k^{-2s} \end{aligned}$$

Finally, we conclude that

$$(55) \quad \int_{\mathbb{R}^n} E \tilde{Z}_{n+1} = A \delta k^{-(n-2s)} \left(-2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \right) \delta + 1 \right) + k^{1-n} \Theta_k(\delta),$$

where $\Theta_k(\delta)$ is a smooth function of δ , which is uniformly bounded as $k \rightarrow \infty$.

In order to complete the proof of (53), we first estimate:

$$\left| \int_{\mathbb{R}^n} (\xi_1 - 1) E \tilde{Z}_{n+1} \right| \leq C \left| \int_{|y-\xi_1| > \frac{\eta}{k}} E \tilde{Z}_{n+1} \right|.$$

Then we split the domain of integration:

$$\int_{|y-\xi_1|>\frac{\eta}{k}} E\tilde{Z}_{n+1} = \int_{\bigcap_j |y-\xi_j|>\frac{\eta}{k}} E\tilde{Z}_{n+1} + \sum_{j=2}^k \int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1}$$

In the exterior region, we already proved that

$$\int_{\bigcap_j |y-\xi_j|>\frac{\eta}{k}} E\tilde{Z}_{n+1} = k^{1-n} \Theta_k(\delta),$$

for some smooth function Θ_k of δ , which is uniformly bounded as $k \rightarrow \infty$. On the another hand, to estimate

$$\sum_{j=2}^k \int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1}$$

we can argue like in the estimate of the term I_3 above, thus concluding that

$$\left| \sum_{j=2}^k \int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1} \right| \leq Ck^{-n}$$

for some constant $C > 0$. □

Proof of (54). It is convenient to decompose

$$\mathcal{N}(\phi_1) = \tilde{\mathcal{N}}(\phi_1) + N(\tilde{\phi}_1)$$

where

$$\begin{aligned} \tilde{\mathcal{N}}(\phi_1) = p(|U_*|^{p-1}\zeta_1 - U_1^{p-1})\tilde{\phi}_1 + p\zeta_1|U_*|^{p-1}\Psi(\phi_1) \\ + N\left(\tilde{\phi}_1 + \sum_{j \neq 1} \tilde{\phi}_j + \Psi(\phi_1)\right) - N(\tilde{\phi}_1) \end{aligned}$$

and

$$N(\tilde{\phi}_1) = |U_* + \tilde{\phi}_1|^{p-1}(U_* + \tilde{\phi}_1) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\tilde{\phi}_1$$

We have that

$$I := \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)\tilde{Z}_{n+1} = \mu^{\frac{n+2}{2}} \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)(\xi_1 + \mu x)Z_{n+1}(x) dx$$

so that, from the estimates found, we readily check

$$(56) \quad |I| \leq Ck^{2s-n}k^{1-\frac{n}{q}} \int_{R^n} U^{p-1}|Z_{n+1}|.$$

On the other hand, if we let

$$II := \int_{R^n} N(\tilde{\phi}_1)\tilde{Z}_{n+1},$$

we find that

$$|II| \leq \|\phi_1\|_* \int_{R^n} U^{p-1} |\phi_1| |Z_{n+1}|.$$

Now, we notice that from (23), we can write

$$L_0(\phi_1) + a\phi_1 = g + \sum_l c_l U^{p-1} Z_l, \quad \text{where } a = \mu^{\frac{n+2s}{2}} \gamma N(\tilde{\phi}_1)(\xi_1 + \mu y)$$

so that

$$|a| \leq C U^{p-1} \|\phi_1\|_* \quad \text{and} \quad |g| \leq C \mu^{\frac{n-2s}{2}} (1 + |y|)^{-4s}.$$

Thus, applying Lemma 3.3 with $v = 4s$, we find

$$|\phi_1| \leq C \mu^{\frac{n-2s}{2}} (1 + |y|)^{-2s}$$

and we conclude that

$$|II| \leq C \|\phi_1\|_* \mu^{\frac{n-2s}{2}} \leq C k^{2s-n-\frac{n}{q}}.$$

Combining this with (56), we find

$$\left| \int_{R^n} \mathcal{N}(\phi_1) \tilde{Z}_{n+1} \right| \leq C k^{2s-n} k^{1-\frac{n}{q}}. \quad \square$$

Acknowledgments

Musso is partially supported by Fondecyt Grant 1120151 and Millennium Nucleus Center for Analysis of PDE, NC130017. Garrido is partially supported by Mecesup Grant 0711 and VRI Scholarship.

References

- [Caffarelli and Silvestre 2007] L. Caffarelli and L. Silvestre, “An extension problem related to the fractional Laplacian”, *Comm. Partial Differential Equations* **32** (2007), 1245–1260. MR 2009k:35096 Zbl 1143.26002
- [Carlen and Loss 1990] E. A. Carlen and M. Loss, “Extremals of functionals with competing symmetries”, *J. Funct. Anal.* **88**:2 (1990), 437–456. MR 91f:42021 Zbl 0705.46016
- [Chang and González 2011] S.-Y. A. Chang and M. d. M. González, “Fractional Laplacian in conformal geometry”, *Adv. Math.* **226**:2 (2011), 1410–1432. MR 2012k:58057 Zbl 1214.26005
- [Chen et al. 2006] W. Chen, C. Li, and B. Ou, “Classification of solutions for an integral equation”, *Comm. Pure Appl. Math.* **59**:3 (2006), 330–343. MR 2006m:45007a Zbl 1093.45001
- [Dávila et al. 2013] J. Dávila, M. del Pino, and Y. Sire, “Nondegeneracy of the bubble in the critical case for nonlocal equations”, *Proc. Amer. Math. Soc.* **141**:11 (2013), 3865–3870. MR 3091775 Zbl 1275.35028
- [Di Nezza et al. 2012] E. Di Nezza, G. Palatucci, and E. Valdinoci, “Hitchhiker’s guide to the fractional Sobolev spaces”, *Bull. Sci. Math.* **136**:5 (2012), 521–573. MR 2944369 Zbl 1252.46023
- [Fang 2014] F. Fang, “Infinitely many non-radial sign-changing solutions for a fractional Laplacian equation with critical nonlinearity”, preprint, 2014. arXiv 1408.3187

- [Frank and Lieb 2010] R. L. Frank and E. H. Lieb, “Inversion positivity and the sharp Hardy–Littlewood–Sobolev inequality”, *Calc. Var. Partial Differential Equations* **39**:1-2 (2010), 85–99. MR 2012a:26027 Zbl 1204.39024
- [Frank and Lieb 2012] R. L. Frank and E. H. Lieb, “A new, rearrangement-free proof of the sharp Hardy–Littlewood–Sobolev inequality”, pp. 55–67 in *Spectral theory, function spaces and inequalities*, edited by B. M. Brown et al., Operator Theory: Advances and Applications **219**, Birkhäuser, Basel, 2012. MR 2848628 Zbl 1297.39023
- [González and Qing 2013] M. d. M. González and J. Qing, “Fractional conformal Laplacians and fractional Yamabe problems”, *Anal. PDE* **6**:7 (2013), 1535–1576. MR 3148060 Zbl 1287.35039
- [Li 2004] Y. Y. Li, “Remark on some conformally invariant integral equations: the method of moving spheres”, *J. Eur. Math. Soc.* **6**:2 (2004), 153–180. MR 2005e:45007 Zbl 1075.45006
- [Li and Zhu 1995] Y. Y. Li and M. Zhu, “Uniqueness theorems through the method of moving spheres”, *Duke Math. J.* **80**:2 (1995), 383–417. MR 96k:35061 Zbl 0846.35050
- [Lieb 1983] E. H. Lieb, “Sharp constants in the Hardy–Littlewood–Sobolev and related inequalities”, *Ann. of Math. (2)* **118**:2 (1983), 349–374. MR 86i:42010 Zbl 0527.42011
- [del Pino et al. 2011] M. del Pino, M. Musso, F. Pacard, and A. Pistoia, “Large energy entire solutions for the Yamabe equation”, *J. Differential Equations* **251**:9 (2011), 2568–2597. MR 2012k:35176 Zbl 1233.35008
- [del Pino et al. 2013] M. del Pino, M. Musso, F. Pacard, and A. Pistoia, “Torus action on S^n and sign-changing solutions for conformally invariant equations”, *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* **12**:1 (2013), 209–237. MR 3088442 Zbl 1267.53040

Received January 5, 2015.

DANILO GARRIDO
DEPARTAMENTO DE MATEMÁTICA
PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
AVDA. VICUÑA MACKENNA 4860
X MACUL
CHILE
dggarrido@mat.puc.cl

MONICA MUSSO
DEPARTAMENTO DE MATEMÁTICA
PONTIFICIA UNIVERSIDAD CATOLICA DE CHILE
AVDA. VICUÑA MACKENNA
4860 MACUL
CHILE
mmusso@mat.puc.cl

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.

See inside back cover or msp.org/pjm for submission instructions.


The subscription price for 2016 is US \$/year for the electronic version, and \$/year for print and electronic.

Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by [Mathematical Reviews](#), [Zentralblatt MATH](#), [PASCAL CNRS Index](#), [Referativnyi Zhurnal](#), [Current Mathematical Publications](#) and [Web of Knowledge \(Science Citation Index\)](#).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW[®] from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2016 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 283 No. 1 July 2016

A New family of simple $\mathfrak{gl}_{2n}(\mathbb{C})$ -modules JONATHAN NILSSON	1
Derived categories of representations of small categories over commutative noetherian rings BENJAMIN ANTIEAU and GREG STEVENSON	21
Vector bundles over a real elliptic curve INDRANIL BISWAS and FLORENT SCHAFFHAUSER	43
$\mathbb{Q}(\mathbb{N})$ -graded Lie superalgebras arising from fermionic-bosonic representations JIN CHENG	63
Conjugacy and element-conjugacy of homomorphisms of compact Lie groups YINGJUE FANG, GANG HAN and BINYONG SUN	75
Entire sign-changing solutions with finite energy to the fractional Yamabe equation DANILO GARRIDO and MONICA MUSSO	85
Calculation of local formal Mellin transforms ADAM GRAHAM-SQUIRE	115
The untwisting number of a knot KENAN INCE	139
A Plancherel formula for $L^2(G/H)$ for almost symmetric subgroups BENT ØRSTED and BIRGIT SPEH	157
Multiplicative reduction and the cyclotomic main conjecture for GL_2 CHRISTOPHER SKINNER	171
Commensurators of solvable S -arithmetic groups DANIEL STUDENMUND	201
Gerstenhaber brackets on Hochschild cohomology of quantum symmetric algebras and their group extensions SARAH WITHERSPOON and GUODONG ZHOU	223