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We study the Plancherel formula for a new class of homogeneous spaces for real reductive Lie groups; these spaces are fibered over non-Riemannian symmetric spaces, and they exhibit a phenomenon of uniform infinite multiplicities. The proof for this is new but rather elementary, and we give all details. As an application we use several results from the recent literature studying possible nontemperedness of homogeneous spaces; thus we provide examples of nontempered representations of the group appearing in the Plancherel formula for our homogeneous spaces. Several classes of examples are given, each building on different techniques and new results from the theory of symmetric spaces.

I. Introduction

Considerable efforts have been devoted to obtaining the Plancherel formula for homogeneous spaces of the form G/H with G a real reductive Lie group and H a symmetric subgroup, a program completed by T. Oshima, P. Delorme, E. van den Ban, and H. Schlichtkrull. This is a central theme in harmonic analysis, and there are a number of natural ways to extend such a program. One is to consider spherical spaces, i.e., where the homogeneous space admits an open orbit of a parabolic subgroup. In this paper we shall rather extend the interest to

(1) square-integrable sections of homogeneous line bundles over symmetric spaces, and

(2) spaces fibered over symmetric spaces.

Of course, these two questions are related, and we shall find several classes of spaces where rather explicit answers can be found. As an example consider $G = \mathrm{SL}(2, \mathbb{R})$ with H the connected diagonal subgroup; for each unitary character of H we may consider the space (1) and the corresponding Plancherel formula: This turns out to be independent of the character, and hence the space as in (2)

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above (in our case just the group G) has the same L^2-content as the symmetric space, only with infinite multiplicity. It is perhaps a little surprising, that one may thus find embeddings of, e.g., the discrete series of G in a uniform way in each of the spaces of sections (1).

To be more specific, our motivation in undertaking this work was to understand the disintegration of the representation of a semisimple Lie group G on the space $L^2(G/H_{ss})$ where H_{ss} is a semisimple subgroup which differs from a symmetric subgroup by a noncompact central real factor. In this paper we study this situation for the simplest nonsymmetric subgroups H_{ss} from the point of view of harmonic analysis and obtain a Plancherel theorem for space $L^2(G/H_{ss})$ in terms of the one for G/H.

Recently Y. Benoist and T. Kobayashi [2015] proved general criteria to determine if for a semisimple subgroup H the spectrum of $L^2(G/H)$ contains nontempered representations; this they use to determine in many examples if $L^2(G/H)$ is tempered. Here a representation is called tempered if it appears in the usual Plancherel formula for $L^2(G)$. However these authors do not obtain any results concerning the multiplicity of the representations in the Plancherel formula. By obtaining a Plancherel formula for $L^2(G/H_{ss})$ we are in a position to determine exactly in our examples which nontempered representations appear in the spectrum, and also to show that they appear with infinite multiplicities.

We consider a noncompact subgroup $H = H_{ss}Z_H$ where H is a subgroup of finite index in the fixpoints of an involution of G and $Z_H \simeq \mathbb{R}$ is a subgroup of finite index of the center of H. Under these assumptions we show the following.

Theorem. *As a left regular representation of G*

$$L^2(G/H_{ss}) \simeq L^2(G/H) \otimes L^2(Z_H).$$

It is instructive to compare with the situation where the central subgroup is compact, e.g., the case of G a simple noncompact Lie group and K a maximal compact subgroup with a one-dimensional center Z. Here G/K is a noncompact Riemannian symmetric space of Hermitian type, and $L^2(G/K)$ has a different Plancherel decomposition than $L^2(G/K, \chi)$, the square-integrable sections of the line bundle induced from a nontrivial unitary character χ of Z. In particular the first space contains no discrete series representations, whereas the second space typically does. Compare with Proposition III.3 for our situation of a noncompact center.

A related problem for spherical varieties over non-Archimedean fields is discussed in [Sakellaridis and Venkatesh 2014, Section 9.5].

The paper is organized as follows: In Section II, we show that we can regard H as a subgroup of finite index in the Levi subgroup of a parabolic subgroup with abelian nilradical. In Section III we prove our main theorem above. In Section IV we discuss some examples. In particular we note that we find several examples of nontempered homogeneous spaces, some of them new; quite possibly our method could extend to other instances of Plancherel theorems, such as cases of vector bundles (as opposed to the cases of line bundles treated here).

II. Notation and preliminaries

We introduce the notation and prove some preliminary results.

Notation and assumptions. Let G be a real linear semisimple connected algebraic group with maximal compact subgroup K and complexification $G_{\mathbb{C}}$. We consider G and $K \subset G_{\mathbb{C}}$ as subgroups.

Proposition II.1. *Suppose that $P = LN$ is a maximal parabolic subgroup with an abelian nilradical N. Then L is the fixpoint set of an involution*

$$\tau : G \to G.$$

Proof (due to Dan Barbasch). We consider a maximal split Cartan subgroup and its corresponding complex Cartan subalgebra. A parabolic subalgebra is given by removing some simple roots from the diagram. The only way to get an abelian nilradical is to remove a single simple root which appears with coefficient at most one 1 if we write the roots as linear combinations of simple roots. The involution τ is then conjugation by $\exp(i\pi\varpi)$ where ϖ is the coroot of the simple root which was removed. □

Let H be a subgroup of the Levi subgroup L of P which contains the connected component L^0 of L. Then $H = H_{ss}Z_H$ where Z_H is a one dimensional connected subgroup in the center of H and H_{ss} is semisimple or discrete.

Example 1. $G = \mathrm{SL}(2, \mathbb{R})$, L diagonal matrices which are the fixed points under the conjugation by the diagonal matrix of order 2 and determinant -1. Alternatively we consider the adjoint representation. Then L is the stabilizer of a semisimple nontrivial element of order 2. It is also the fixed point set of the automorphism by the adjoint action of the matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(\pi i \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right).$$

We have to consider two subgroups H and H_{ss}:

(a) $H = L,\ H_{ss} = \mathbb{Z}_2,\ Z_H = \mathbb{R}_+ . G/H_{ss} = \mathrm{PSL}(2, \mathbb{R})$,

(b) $H = L^0$, $H_{ss} = I$ and $G/H_{ss} = \mathrm{SL}(2, \mathbb{R})$.

Proposition II.2. *Suppose that F is the fixed point set of an involution $\tau : G \to G$. Assume in addition that it is a product $F = F_{ss}Z_F$ where Z_F is a subgroup of the* *center of F isomorphic to $\mathbb{R}^+$ and F_{ss} is a semisimple group. Then F is contained in the Levi subgroup of a maximal parabolic subgroup P with abelian nilradical N.*

Proof. We choose maximally split Cartan subgroup $C \subset F$ with complexified Lie algebra $\mathfrak{h}_\mathbb{C}$. We choose the simple roots of $\mathfrak{h}_\mathbb{C}$, $\mathfrak{g}_\mathbb{C}$ so that they are simple roots in $\mathfrak{h}_\mathbb{C}$, $\mathfrak{f}_\mathbb{C}$. (In the lexicographical order we let $\mathfrak{f}$ come before $\mathfrak{g}$, $\mathfrak{f}$ the Lie algebra of F.) Then $\mathfrak{f}$ is the Levi subalgebra of a maximal parabolic subalgebra $\mathfrak{p}_\mathbb{C} = \mathfrak{f}_\mathbb{C} \oplus \mathfrak{n}_\mathbb{C}$ of $\mathfrak{g}_\mathbb{C}$.

It remains to show that N is abelian. Since τ leaves C invariant the induced homomorphism of $\tau : N \to N$ is equal to -1. Since τ induces a Lie algebra homomorphism and hence preserves the Lie bracket in $\mathfrak{n}_\mathbb{C}$, the results follows from the observation that $\tau(X) = -X$ and $\tau(Y) = -Y$ then $\tau([X, Y]) = [X, Y]$. □

Note that in the setting above, we have a direct product decomposition

$$G/H \times Z_H = G/H_{ss}.$$

This will be useful later in connection with integration over this space, and in considering the corresponding L^2-space.

About $L^2(G/H)$. Keep the assumptions on G, H, Z_H as above. We extend a unitary character $\chi \in \widehat{Z_H}$ to a character of H and consider the unitary induced representation $\mathrm{Ind}_H^G \chi$ on $L^2(G/H)_{\chi^{-1}}$. Normalize Plancherel measures on Z_H and its dual group in the usual way.

Proposition II.3. *As a representation of G*

$$L^2(G/H_{ss}) = \int_{\chi \in \widehat{Z_H}} L^2(G/H)_{\chi^{-1}} d\chi,$$

Proof. For $f \in L^2(G/H_{ss})$ and $\chi \in \widehat{Z_H}$ define

$$F(\chi, g) = \int_{Z_H} f(gz)\chi(z)^{-1}\, dz,$$

Then for $z_0 \in Z_H$

$$F(\chi, gz_0) = F(\chi, g)\chi^{-1}(z_0),$$

so $F(\chi) \in L^2(G/H)_{\chi^{-1}}$. By Fourier analysis on Z_H we have

$$\int_{\chi \in \widehat{Z_H}} |F(\chi, g)|^2 d\chi = \int_{z \in Z_H} |f(gz)|^2, dz.$$

So

$$\int_{G/H_{ss}} |f(g')|^2 dg' = \int_{G/H} \int_{Z_H} |f(g'z)|^2 dz dg'$$

completes the proof. □

III. Main results

In this section we relate the Plancherel formula for the left regular representation of G on $L^2(G/H_{ss})$ to the Plancherel formula for the left regular representation on $L^2(G/H)$. It turns out that these two spaces have the same content of unitary representations of G, only differing by their multiplicities.

Induction to the parabolic subgroup P.

Lemma III.1. *Let $\widehat{N}$ the dual group of N. There exist finitely many open H orbits $\mathbb{O}_i$ in $\widehat{N}$ so that $\widehat{N}$ is the closure of their union $\bigcup_i \mathbb{O}_i$.*

Proof. Here we refer to results by Wallach [2006]. Here he proves that our parabolic algebras are "very nice" since they have abelian nilradicals (see Corollary 6.4 of that reference). In particular there is only one open orbit of L on N.

Since our group H is a subgroup of finite index in L we will get a finite number of open orbits with dense union. Actually, the statement that "open orbit is generic" (i.e., "nice parabolic") would suffice for our purposes here. □

Let $\chi \in \widehat{Z_H}$. We consider again χ as a character of H and consider again the unitary induced representation $\mathrm{Ind}_H^P \chi$.

Proposition III.2. *Let χ and $\tilde{\chi}$ be unitary characters of Z_H considered as characters of H. Then we have (equivalence of representations)*

$$\mathrm{Ind}_H^P \chi = \mathrm{Ind}_H^P \tilde{\chi}.$$

Proof. We denote the induced representations acting on functions $F \in L^2(N)$ by

$$\rho_\chi(n_0)F(n) = F(n \cdot n_0),$$
$$\rho_\chi(h_0)F(n) = \chi(h_0)F(h_0^{-1} n h_0).$$

Using the Fourier transform we realize the representation $\mathrm{Ind}_H^P \chi$ on $L^2(\hat{N})$. It is a direct sum of irreducible representations on $L^2(\mathbb{O}_i)$ where

$$\hat{\rho}_\chi(n_0) \text{ is a multiplication operator,}$$
$$\hat{\rho}_\chi(h_0)\hat{F}(\xi) = \chi(h_0)J(h_0^t \xi)^{1/2}\hat{F}(h_0^t \xi).$$

The other representation on the orbit is obtained by multiplication of the right hand side of the second equation with a character $\chi_1 = \tilde{\chi}\chi^{-1}$ of H. In each orbit we fix an element ξ_i.

We get a intertwining operator on each of the irreducible representations by

$$I(F)(\xi) = \chi_1(\xi)F(\xi).$$

Here $\xi = h\xi_i$ and $\chi_1(\xi) := \chi_1(h)$. □

Example 2. Consider the group $P = HN$ with

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \,\middle|\, a > 0 \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \,\middle|\, b \in \mathbb{R} \right\}.$$

We note that there are three orbits of H on

$$\hat{N} = \left\{ \xi_t \,\middle|\, \xi_t\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = e^{it\cdot b} \right\},$$

namely, $\mathbb{O}^+ = \{\xi_t \mid t > 0\}$, $\mathbb{O}^- = \{\xi_t \mid t < 0\}$ and $\mathbb{O}^1 = \{\xi_0\}$. The unitary representation ρ_1 of P induced from the trivial representation of H acts on $L^2(N)$ by

$$\rho_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) F(x) = a^{1/2} F(ax)$$

and

$$\rho_1\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) F(x) = F(x+b).$$

To analyze this representation we consider the Fourier transform of $L^2(N)$. The representation is a direct sum of two unitary representations of functions whose Fourier transform has support in $\xi \in \mathbb{O}^+$ and in $\xi \in \mathbb{O}^-$.

We consider $\chi_s : a \to a^{is}$ as a character of H. After applying the Fourier transform the representation $\widehat{\rho_s}$ induced from χ_s has the form

$$\widehat{\rho_s}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \hat{F}(\xi) = a^{-1/2} a^{is} \hat{F}(a^{-1}\xi)$$

and

$$\widehat{\rho_t}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \hat{F}(\xi) = e^{ib\xi} \hat{F}(\xi).$$

The equivalence of the representations ρ_s and ρ_1 follows from the intertwining operator

$$\mathcal{I}_s : \rho_0 \to \rho_s \quad \text{defined by } \mathcal{I}_s \hat{F}(\xi) = \xi^{is} \hat{F}(\xi).$$

Induction to G.

Proposition III.3. *Let χ and $\tilde{\chi}$ be characters of Z_H considered as characters of H. As representations of G we have (equivalence)*

$$\operatorname{Ind}_H^G \chi = \operatorname{Ind}_H^G \tilde{\chi}.$$

Proof. By induction by stages (Proposition III.2) we have

$$\operatorname{Ind}_H^G \chi = \operatorname{Ind}_P^G \operatorname{Ind}_H^P \chi = \operatorname{Ind}_P^G \operatorname{Ind}_H^P \tilde{\chi} = \operatorname{Ind}_H^G \tilde{\chi}.$$ □

Example 3. $G = \mathrm{SU}(1,1)$ and

$$H = A = \exp \mathbb{R}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We identify G/K with the complex unit disk $\mathfrak{D}$. We realize the discrete series representations D_n in the holomorphic functions on $\mathfrak{D}$. Then we have the H-invariant distribution vector, giving the embedding into $L^2(G/H)$,

$$v^* = (1+z^2)^{-n/2} \in D_n^{-\infty, H},$$

and similarly the distribution vector

$$v^* = (1+z^2)^{-n/2}\left(\frac{1-z}{1+z}\right)^{i\lambda} \in D_n^{-\infty, H, \chi_\lambda},$$

transforming by the character χ_λ of H. So indeed every discrete series representation occurs in every $L^2(G/H)_{\chi_\lambda}$.

Theorem III.4. *As a left regular representation of G*

$$L^2(G/H_{ss}) \simeq (\mathrm{Ind}_H^G 1) \otimes L^2(Z_H) \simeq L^2(G/H) \otimes L^2(Z_H).$$

Proof. This follows from Propositions II.3 and III.3. □

Corollary III.5. *All irreducible representations in the discrete spectrum of*

$$L^2(G/H_{ss})$$

have infinite multiplicity.

Definition. Following Benoist and Kobayashi we say that $L^2(G/H_{ss})$ is not tempered if the representations in the Plancherel formula for the right regular representation of G on $L^2(G/H_{ss})$ are not a subset of the representations of the Plancherel formula for G.

Corollary III.6. *$L^2(G/H_{ss})$ is tempered if and only if $L^2(G/H)$ is tempered.*

Example 1 (continued). $G = \mathrm{SL}(2,\mathbb{R})$, H diagonal matrices, Then $X = G/H$ is a hyperboloid and

$$L^2(G/H) = \oplus_{\nu \in 2\mathbb{N}} D_\nu \oplus 2\int_0^\infty \pi_{it},$$

where D_ν are the discrete series representations with parameter ν and π_{it} are the tempered spherical principal series representations with parameter it. Here $H_{ss} = \mathbb{Z}_2$, then $L^2(G/H_{ss}) = L^2(\mathrm{PSL}(2,\mathbb{R}))$ and so the left regular representation contains the even discrete series representations with ∞ multiplicity.

If H is connected, then $L^2(G/H)$ contains all discrete series representations and so does the left regular representation of G on $L^2(G)$.

IV. More examples

We discuss in this section some interesting examples of groups G and H_{ss}, illustrating our results; one aspect is to find reductive spaces that are not tempered.

We use the Plancherel formula to determine if $L^2(G/H_{ss})$ is tempered. Some of our examples are also contained in [Benoist and Kobayashi 2015], where they are obtained with a different technique; others are new.

E. van den Ban and H. Schlichtkrull [2005] proved a Plancherel formula for $L^2(G/H)$ for a fixed point set H of an involution τ of G. They showed that only discrete series representations of $L^2(G/H)$ and principal series representations unitarily induced from a $\theta\tau$ invariant parabolic MAN, a discrete series representation π of $M/M \cap H$ and a unitary character of A contribute to the Plancherel formula. On the other hand the work of M. Flensted-Jensen and Oshima and Matsuki shows that the discrete spectrum of G/H is nontrivial if and only if

$$\operatorname{rank} G/H = \operatorname{rank} K/K \cap H.$$

A parametrization of the representations in the discrete spectrum was obtained by T. Matsuki and T. Oshima [1984]. See also [Schlichtkrull 1983]. We will make extensive use of these results in the proofs of our examples.

Remark 1. Induction by stages enlarges the set of pairs G, $\tilde{H}$ for which $L^2(G/\tilde{H})$ is tempered. (See [Fell 1962, Theorem 4.2]; here the point is that induction preserves weak containment, so if we have groups $H \subset \tilde{H} \subset G$ so that $L^2(G/H)$ is tempered and we know that $L^2(\tilde{H}/H)$ contains the trivial representation weakly, then also $L^2(G/\tilde{H})$ is tempered.)

Remark 2. The nontempered representations in the discrete spectrum of $L^2(G/H_{ss})$ are automorphic representations [Burger and Sarnak 1991]. Most of these automorphic representations are known and have been constructed using other techniques for example in [Kudla and Rallis 1990; Howe and Piatetski-Shapiro 1979; Schlichtkrull 1983; Mœglin and Waldspurger 1989].

Example 4. Let $G = \mathrm{SL}(2n, \mathbb{R})$, We take H as the connected component of $S\big(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})\big)$. Then $H_{ss} = \mathrm{SL}(p, \mathbb{R}) \times \mathrm{SL}(q, \mathbb{R})$ where $p + q = 2n$. and

$$\operatorname{rank} G/H = \operatorname{rank} K/K \cap H = \min(p, q).$$

The results of van den Ban and Schlichtkrull show that all the representations in the continuous spectrum are unitarily induced from $\theta\tau$-stable parabolic subgroups. It is easy to see that these parabolic subgroups are all cuspidal and thus the representations in the discrete spectrum determine whether $L^2(\mathrm{SL}(n, \mathbb{R})/H)$ is tempered.

We recall the parametrization of the representations in the discrete spectrum. Using the decomposition $\mathfrak{h}\otimes\mathbb{C}\oplus\mathfrak{q}\otimes\mathbb{C}$ of $\mathfrak{gl}(2n,\mathbb{C})$ we conclude that the skew diagonal matrices in $\mathfrak{q}\otimes\mathbb{C}$ are a maximal abelian subspace of $\mathfrak{so}(2n,\mathbb{C})\cap\mathfrak{q}\otimes\mathbb{C}$. By [Ōshima and Matsuki 1984] their centralizer L is the Levi subgroup of a θ stable parabolic subgroup. The representations in the discrete spectrum are cohomologically induced from a character of the subgroup L. If the commutator subgroup L does not contain a noncompact semisimple subgroup then the representations are tempered. (For this, see [Knapp and Vogan 1995, Chapter XI] or [Vogan and Zuckerman 1984, Theorem 6.16]). Thus we conclude:

- If $p=q=n$ the subgroup $[L,L]$ is a product of n compact tori. Hence all representation $L^2(\mathrm{SL}(2n,\mathbb{R})/(\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R}))$ in the discrete spectrum are tempered and thus $L^2(\mathrm{SL}(2n,\mathbb{R})/H_{ss})$ is tempered.
- If $p-q\geq 2$ then L has a noncompact subgroup and hence the representations in the discrete spectrum of $L^2(G/H_{ss})$ are the Langlands subquotient of representations which is not unitarily induced. Hence
$$L^2(\mathrm{SL}(2n,\mathbb{R})/(\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R}))$$
is not tempered.
- Using Remark 2 we can construct a large number of additional semisimple subgroups H_{ss} so that $L^2(\mathrm{SL}(2n,\mathbb{R})/H_{ss})$ is tempered.

C. Mœglin and J. L. Waldspurger [1989] show that these representations are in the residual spectrum of a congruence subgroup of $\mathrm{GL}(n,\mathbb{R})$. Similar considerations for general linear groups can be found in [Venkatesh 2005].

Example 5. $G=\mathrm{SO}(p,q)$, $p+q=2n\geq 4$ with $p\geq q>2$ and
$$H=\mathrm{SO}(1,1)\times\mathrm{SO}(p-1,q-1)\quad\text{and}\quad H_{ss}=\mathrm{SO}(p-1,q-1).$$

Claim. $L^2\big(\mathrm{SO}(p,q)/\mathrm{SO}(p-1,q-1)\big)$ *is not tempered.*

We have
$$\operatorname{rank} G/H=\operatorname{rank} K/K\cap H=2.$$

We argue as in Example 2. The group $[L,L]$ has a factor isomorphic to
$$\mathrm{SO}(p-2,q-2),$$
and is hence is not compact. So there are nontempered representations in the discrete spectrum.

T. Kobayashi [1992] considered the case G/H_0 where $H=H_c\times H_0$. Here H_c is a compact orthogonal group and H_0 is a noncompact orthogonal group. He determined the parameter of the representations in the discrete spectrum of $L^2(G/H_0)$ and their multiplicities.

Example 6. $G = \mathrm{Sp}(n, \mathbb{R})$, $H = \mathrm{GL}(n, \mathbb{R})$, $H_{ss} = \mathrm{SL}(n, \mathbb{R})$.

Claim. $L^2(\mathrm{Sp}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R}))$ *is tempered.*

The proof proceeds as follows:

Step 1: All the representations in the discrete spectrum are tempered.

Step 2: Each conjugacy class of parabolic subgroups contains a $\theta\tau$-invariant parabolic subgroup MAN.

Step 3: All discrete series representations of $M/M \cap H$ are tempered.

For simplicity assume that the symplectic group is defined by the quadratic form defined by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the identity matrix. The subgroup $H = \mathrm{GL}(n, \mathbb{R})$ of G is the fixed point set of the automorphism τ defined by conjugation with

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The maximal compact subgroup K_H of H is $K \cap H = O(n)$. Furthermore

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

The one-dimensional torus T_0 in the center of K also defines a torus on $K/H \cap K$. Its Lie algebra $\mathfrak{t}_0$ is direct summand of the maximal abelian subalgebra $\mathfrak{a}_k$ of $\mathfrak{q}_k = \mathfrak{k} \cap \mathfrak{q}$. Since T_0 defines the complex structure on the symmetric space G/K the centralizer of $\mathfrak{a}_k$ in G is contained in K. Thus every representation in the discrete spectrum is tempered.

The $\theta\tau$-stable parabolic subalgebras are determined by maximal abelian subspaces $\mathfrak{i}$ in $\mathfrak{p} \cap \mathfrak{q}$. Now

$$\tilde{\mathfrak{h}} := \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q} = \mathfrak{gl}(n, \mathbb{R})$$

is the fixed point set of the involution $\theta\tau$ since the fixed point set of $\theta\tau$ is conjugate in $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{R})$. This implies that there is a n-dimensional abelian split subalgebra $\tilde{\mathfrak{a}}_H$ in $p \cap q$ consisting of the matrices

$$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix},$$

where D is a real diagonal matrix. Hence every conjugacy classes of parabolic subgroups contains a $\theta\tau$-stable parabolic $P_s = M_s A_s N_s$ whose Levi subgroup is a centralizer of $\tilde{\mathfrak{a}}_H$.

Next we have to determine $M_s \cap H$. Note that M_s is a product of general linear groups and a symplectic group. The factors isomorphic to general linear groups are subgroups of $\tilde{H}$. Since $\tilde{H} \cap H = K \cap H$ is an orthogonal group, the intersection of the general linear subgroups of M_s with H are orthogonal groups and hence the corresponding symmetric space has no discrete spectrum. Thus we may assume that $M_s = \mathrm{Sp}(m, \mathbb{R})$ with $m < n$. In this case $\theta\tau$ is an involution of M_s with fixed points $\tilde{H} \cap M_s = \mathrm{GL}(m, \mathbb{R})$. Furthermore since θ and τ commute their restriction to M_s also defines an automorphisms of M_s. So the fixed point set of $\theta\tau_{|M_s}$ is conjugate to the fixpoint set of $\tau_{|M_s}$ in $\mathrm{GL}(2n, \mathbb{R})$. Hence we conclude that $M_s \cap H$ is isomorphic to $\mathrm{GL}(m, \mathbb{R})$. By Step 1 the representations in the discrete spectrum of $\mathrm{Sp}(m, \mathbb{R})/\mathrm{GL}(m, \mathbb{R})$ are tempered and thus by [van den Ban and Schlichtkrull 2005] all the representations in the continuous spectrum of $\mathrm{Sp}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R})$ are tempered.

Example 7. Cayley-type spaces are considered in [Ólafsson and Ørsted 1999; Faraut and Korányi 1994]. These are

(a) $G = \mathrm{Sp}(n, \mathbb{R})$, $H = \mathrm{GL}(n, R)$ and $H_{ss} = \mathrm{SL}(n, \mathbb{R})$, $n > 1$;

(b) $G = \mathrm{SO}(2, n)$, $H = \mathrm{SO}(1, 1)\,\mathrm{SO}(1, n-1)$ and $H_{ss} = \mathrm{SO}(1, n)$, $n > 2$;

(c) $G = \mathrm{SU}(n, n)$, $H = \mathrm{SL}(n, C)\mathbb{R}^+$ and $H_{ss} = \mathrm{SL}(n, \mathbb{C})$;

(d) $G = O^*(2n)$, $H = \mathbb{R}^+\,\mathrm{SU}^*(2n)$ and $H_{ss} = \mathrm{SU}^*(2n)$;

(e) $G = E_{7(-25)}$, $H = E_{6(-26)}\mathbb{R}^+$ and $H_{ss} = E_{6(-26)}$.

Claims.

- *In Example 7*(b)–(d) *with n large enough $L^2(G/H_{ss})$ is not tempered.*
- *In Example 7*(a) *$L^2(G/H_{ss})$ is tempered.*
- *We expect that in Example 7(e) $L^2(G/H_{ss})$ is tempered.*

Proof. The proof is based on case by case considerations of the spectrum of $L^2(G/H)$. Ólafsson and Ørsted [1999] proved that all these spaces are of equal rank and hence $L^2(G/H)$ has a discrete spectrum.

Case (b). The arguments in Example 5 show that the representations in the discrete spectrum of $L^2(\mathrm{SO}(n, 2)/\mathrm{SO}(n-1, 1))$ are tempered if and only if $n \leq 2$. So we can conclude that $L^2(\mathrm{SO}(n, 2)/\mathrm{SO}(n-1, 1))$ is not tempered if $3 \leq n$.

Case (c). It was proved in [Ólafsson and Ørsted 1988] that the discrete spectrum for $\mathrm{SU}(n, n)/H$ contains some nontempered highest weight representations. Hence $L^2(\mathrm{SU}(n, n)/\mathrm{SL}(n, \mathbb{C}))$ is not tempered.

Case (a). This was proved in Example 6.

Case (d). We have $\mathrm{rank}(G/H) = n$. The Levi of the θ-stable parabolic subgroup also contains a subgroup of type A_{2n-1}. Since it is not the maximal compact subgroup, L has a noncompact subgroup. This implies that the discrete spectrum of $L^2(O^*(2n)/\operatorname{SU}^*(2n))$ is not tempered.

Case (e). We only prove that the discrete spectrum is tempered. The arguments are the same as in Example 4. Recall that

(1) the rank of G/H is 3;

(2) the maximal compact subgroup K of G is $E_6\operatorname{SO}(2)$;

(3) the maximal compact subgroup K_H of H is F_4;

(4) $K/H\cap K$ has a one-dimensional compact torus T_0 as factor.

The centralizer of this torus T_0 is K. Its Lie algebra is a direct summand of the maximal abelian subalgebra $\mathfrak{a}_k$ of $\mathfrak{q}_k = \mathfrak{k}\cap\mathfrak{q}$. Since T_0 defines the complex structure on the symmetric space G/K the centralizer of $\mathfrak{a}_k$ in G is contained in K. Thus every representation in the discrete spectrum is tempered.

As in Example 6 we conclude that the fixed point set of $\theta\tau$ is a subgroup isomorphic to $H = E_{6(-26)}\mathbb{R}^+$, which has real rank 3. Hence every conjugacy classes of parabolic subgroups contains a $\theta\tau$ -stable parabolic $P_s = M_s A_s N_s$ whose Levi subgroup is a centralizer of $\tilde{\mathfrak{a}}_H$. □

Example 8. $G = \operatorname{SL}(2n,\mathbb{C})$ and H_{ss} has a covering $T^1\operatorname{SL}(p,\mathbb{C})\times\operatorname{SL}(q,\mathbb{C})$, p+q = 2n for a one dimensional torus T^1. Then

$$L^2(\operatorname{SL}(n,\mathbb{C})/\operatorname{SL}(p,\mathbb{C})\times\operatorname{SL}(q,\mathbb{C})) = \oplus_{\delta\in\hat{T}} L^2(\operatorname{SL}(n,\mathbb{C})/H_{ss},\delta),$$

where $L^2(\operatorname{SL}(n,\mathbb{C})/H_{ss},\delta)$ are the L^2-sections of the line bundle defined by the character δ of H_{ss}. As in Example 2 we are in the equal rank case.

The same arguments as in Example 4 show:

- If $p = q = n$ the subgroup $[L, L]$ is compact. Hence all representations in the discrete spectrum of $L^2(\operatorname{SL}(2n,\mathbb{C})/H)$ are tempered, which implies that $L^2(\operatorname{SL}(2n,\mathbb{C})/H_{ss})$ is tempered.
- If $p-q\geq 2$ then $[L, L]$ is not compact and hence the representations in the discrete spectrum of $L^2(G/H_{ss})$ are the Langlands subquotients of representations which are not unitarily induced. Hence $L^2(\operatorname{SL}(2n,\mathbb{C})/\operatorname{SL}(n,\mathbb{C})\times\operatorname{SL}(n,\mathbb{C}))$ is not tempered.

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