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COMMENSURATORS OF SOLVABLE S-ARITHMETIC GROUPS

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We show that the abstract commensurator of an S-arithmetic subgroup of a solvable algebraic group over $\mathbb{Q}$ is isomorphic to the $\mathbb{Q}$-points of an algebraic group, and compare this with examples of nonlinear abstract commensurators of S-arithmetic groups in positive characteristic. In particular, we include a description of the abstract commensurator of the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$.

1. Introduction

Overview. In this paper we show that the abstract commensurator of an S-arithmetic subgroup of a solvable $\mathbb{Q}$-group is isomorphic to the $\mathbb{Q}$-points of an algebraic group. We then include examples to show that the analogous result in positive characteristic does not hold. As part of these examples, we provide a description of the abstract commensurator of the lamplighter group.

Background. A $\mathbb{Q}$-*group* $\boldsymbol{G}$ is a linear algebraic group defined over $\mathbb{Q}$. For S any finite set of prime numbers, let $\boldsymbol{G}(S)$ denote the set of S-*integer points* of G, that is, those matrices in $\boldsymbol{G}(\mathbb{Q})$ whose entries have denominators with prime divisors belonging to S. A subgroup of $\boldsymbol{G}(\mathbb{Q})$ is S-*arithmetic* if it is commensurable with $\boldsymbol{G}(S)$. When $S = \varnothing$, an S-arithmetic group is called an *arithmetic* group.

Remark. Beware of our unconventional choice of notation for S, which by definition includes only *non-Archimedean* valuations on $\mathbb{Q}$.

The *abstract commensurator* of a group Γ, denoted $\mathrm{Comm}(\Gamma)$, is the group of equivalence classes of isomorphisms between finite-index subgroups of Γ, where two isomorphisms are equivalent if they agree on a finite-index subgroup of Γ.

The starting point for our work is the following result, immediate from the fact that S-arithmetic subgroups of $\mathbb{Q}$-groups are preserved by isomorphism of their ambient $\mathbb{Q}$-groups; see [Platonov and Rapinchuk 1994, Theorem 5.9, p. 269]. Let $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ denote the group of $\mathbb{Q}$-defined automorphisms of $\boldsymbol{G}$.

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Proposition 1.1. *Suppose $\boldsymbol{G}$ is any $\mathbb{Q}$-group. For any finite set of primes S, there is a natural map* $\Theta : \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \to \mathrm{Comm}(\boldsymbol{G}(S))$.

In the case that $\boldsymbol{G}$ is a higher-rank, connected, adjoint, semisimple linear algebraic group that is simple over $\mathbb{Q}$, rigidity theorems of Margulis [1991] imply that the map Θ of Proposition 1.1 is an isomorphism. Similarly, if $\boldsymbol{G}$ is unipotent then Θ is an isomorphism by Mal'cev rigidity; see Theorem 3.3. Moreover, in each of these cases the group $\mathrm{Aut}(\boldsymbol{G})$ has the structure of a $\mathbb{Q}$-group such that $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \cong \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q})$.

Main result. When $\boldsymbol{G}$ is solvable and not unipotent the group $\boldsymbol{G}(S)$ is not rigid in the above sense. One approach to remedying this lack of rigidity is taken in [Witte 1997], where solvable S-arithmetic groups are shown to satisfy a form of Archimedean superrigidity. For solvable arithmetic groups, another study of this failure of rigidity appears in [Grunewald and Platonov 1999]. Extending these methods, we prove the main theorem of this paper:

Theorem 1.2. *Let $\boldsymbol{G}$ be a solvable $\mathbb{Q}$-group and let S be a finite set of primes. Then there is a finite-index subgroup* $\mathrm{Comm}^0(\boldsymbol{G}(S)) \leq \mathrm{Comm}(\boldsymbol{G}(S))$ *and a $\mathbb{Q}$-group $\boldsymbol{D}$ such that*

$$\mathrm{Comm}^0(\boldsymbol{G}(S)) \cong \boldsymbol{D}(\mathbb{Q}).$$

The group $\boldsymbol{D}$ is constructed explicitly as a quotient of an iterated semidirect product of groups. See Section 3C for proof and details.

When $S = \varnothing$ the arithmetic group $\boldsymbol{G}(S) = \boldsymbol{G}(\mathbb{Z})$ is virtually polycyclic, and hence virtually a lattice in a connected, simply connected solvable Lie group. In [Studenmund 2015] it was shown that the abstract commensurator of a lattice in a connected, simply connected solvable Lie group is isomorphic to the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. Therefore the $S = \varnothing$ case of Theorem 1.2 is a consequence of [Studenmund 2015].

When $S \neq \varnothing$ the group $\boldsymbol{G}(S)$ is no longer necessarily polycyclic, so different methods are necessary. When $\boldsymbol{U}$ is a unipotent group, for any set of primes S we have

$$\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q}).$$

In particular the abstract commensurator is independent of S. For example, we have $\mathrm{Comm}(\mathbb{Z}[1/2]) \cong \mathrm{Comm}(\mathbb{Z}[1/3]) \cong \mathbb{Q}^*$. Note that for each nontrivial unipotent group this provides an infinite family of pairwise non-abstractly-commensurable groups with isomorphic abstract commensurator.

When $\boldsymbol{G}$ contains a torus, the abstract commensurator of an S-arithmetic subgroup may depend on S. For example, let $\boldsymbol{T}$ be the Zariski-closure of the cyclic subgroup generated by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Note that $\boldsymbol{T}$ is diagonalizable over $\mathbb{R}$ and over $\mathbb{Q}_{11}$ since 5 has an 11-adic square root, while $\boldsymbol{T}$ is not diagonalizable over either $\mathbb{Q}$

or $\mathbb{Q}_3$. It follows from Theorem 2.1 below that

$$\boldsymbol{T}(\varnothing) \doteq \mathbb{Z}, \quad \boldsymbol{T}(\{3\}) \doteq \mathbb{Z}, \quad \boldsymbol{T}(\{11\}) \doteq \mathbb{Z}^2, \quad \text{and} \quad \boldsymbol{T}(\{3, 11\}) \doteq \mathbb{Z}^2,$$

where we write $G \doteq H$ if G and H contain isomorphic subgroups of finite index. Then $\mathrm{Comm}(\boldsymbol{T}(\{11\}))$ and $\mathrm{Comm}(\boldsymbol{T}(\{3, 11\}))$ are each isomorphic to $\mathrm{GL}_2(\mathbb{Q})$, but neither is isomorphic to $\mathrm{Comm}(\boldsymbol{T}(\{3\})) \cong \mathbb{Q}^*$. This dependence on S appears even for groups whose maximal torus acts faithfully on the unipotent radical; see Theorem 1.3.

Explicit description of commensurator. A key case is when the action of any maximal torus of $\boldsymbol{G}$ on the unipotent radical of $\boldsymbol{G}$ is faithful. Such a solvable algebraic group is said to be *reduced*. When $\boldsymbol{G}$ is reduced, we have the following explicit statement whether or not $S = \varnothing$.

Theorem 1.3. *Let $\boldsymbol{G}$ be a connected and reduced solvable $\mathbb{Q}$-group, let S be a finite set of primes, and let Δ be an S-arithmetic subgroup of $\boldsymbol{G}$. Suppose $\boldsymbol{G}(S)$ is Zariski-dense in $\boldsymbol{G}$. There is an isomorphism of abstract groups*

$$\mathrm{Comm}(\Delta) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})) \rtimes \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}), \tag{1}$$

where N is the maximum rank of any torsion-free, free abelian subgroup of $\boldsymbol{T}(S)$ for any maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \leq \boldsymbol{G}$ and $\mathrm{Hom}_{\mathbb{Q}}$ *denotes the group of $\mathbb{Q}$-vector space homomorphisms under addition. There is a subgroup* $\mathrm{Comm}^0(\Delta) \leq \mathrm{Comm}(\Delta)$ *of finite index which has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group.*

Note that the semidirect product appearing in (1) is a semidirect product of abstract groups. However, there is a subgroup of finite index which has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. See Section 3 for details.

Remark. In the case $S = \varnothing$, Theorem 1.2 follows from Theorem 1.3 by the fact that any solvable arithmetic group Γ is abstractly commensurable with an arithmetic subgroup of a *reduced* solvable group. See [Grunewald and Platonov 1999, Theorem 3.4] for a proof of this fact. This is possible because arithmetic subgroups of tori are abstractly commensurable with arithmetic subgroups of abelian unipotent groups; both are virtually free abelian. The same method does not work when S is nonempty: S-arithmetic subgroups of tori are virtually free abelian while S-arithmetic subgroups of unipotent groups are not.

Remark. Bogopolski [2012] has computed abstract commensurators of the solvable Baumslag–Solitar groups to be

$$\mathrm{Comm}(\mathrm{BS}(1, n)) \cong \mathbb{Q} \rtimes \mathbb{Q}^*.$$

Theorem 1.3 recovers Bogopolski's result in the case that n is a prime power, since $\mathrm{BS}(1, p^2)$ is isomorphic to the group $\boldsymbol{G}(S)$, where $S = \{p\}$ and $\boldsymbol{G} = \boldsymbol{B}_2/Z(\boldsymbol{B}_2)$ for

$$\boldsymbol{B}_2 = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \,\middle|\, xy = 1 \right\} \subseteq \mathrm{GL}_2(\mathbb{C}).$$

Note that $\mathrm{BS}(1, n^k)$ is a finite-index subgroup of $\mathrm{BS}(1, n)$; hence the two groups have isomorphic abstract commensurators.

When n is not a prime power, $\mathrm{BS}(1, n)$ is no longer commensurable with an S-arithmetic group. However, $\mathrm{BS}(1, n^2)$ embeds as a Zariski-dense subgroup of

$$(\boldsymbol{B}_2/Z(\boldsymbol{B}_2))(S),$$

where S consists of the prime factors of n. It may be possible to modify the proof of Theorem 1.3 to compute $\mathrm{Comm}(\mathrm{BS}(1, n))$ for any n from this embedding.

Number fields. Above we have defined S-arithmetic subgroups only of $\mathbb{Q}$-groups, but S-arithmetic groups may be defined over any global field. Our methods fail to prove any obvious analog of Theorem 1.2 for S-arithmetic groups over general number fields. In particular, if Γ is an S-arithmetic subgroup of a unipotent group $\boldsymbol{U}$ defined over K then $\mathrm{Comm}(\Gamma)$ may depend on S, in contrast with the case of $K = \mathbb{Q}$. This is explained in more detail in Section 4.

Despite this difference, the conclusion of Theorem 1.2 holds for unipotent groups $\boldsymbol{G}$ and may hold for general solvable $\boldsymbol{G}$. The difficulty in finding a proof lies in finding an alternative to the use of Proposition 1.1; see the remarks at the end of Section 4.

Function fields and the lamplighter group. In contrast to the case of S-arithmetic groups over number fields, Theorem 1.2 has no obvious analog for S-arithmetic groups over global fields of positive characteristic. Section 5 includes examples demonstrating this failure.

A well-known example of a solvable S-arithmetic group in characteristic 2 is the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. Section 6 describes the abstract commensurator of the lamplighter group, with the following main result.

Theorem 1.4. *Using the definitions in Equations* (6) *and* (7) *of Section 6, there is an isomorphism*

$$\mathrm{Comm}((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) \cong (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

Using this decomposition we show, for example, that the abstract commensurator of the lamplighter group contains every finite group as a subgroup.

2. Background and definitions

For any group Γ, a *partial automorphism* of Γ is an isomorphism between finite-index subgroups of Γ. Two partial automorphisms ϕ_1 and ϕ_2 are *equivalent* if there is some finite index $\Delta \le \Gamma$ such that $\phi_1\big|_\Delta = \phi_2\big|_\Delta$; an equivalence class of partial automorphisms is a *commensuration* of Γ. The *abstract commensurator* $\mathrm{Comm}(\Gamma)$ is the group of commensurations of Γ. If Γ_1 and Γ_2 are abstractly commensurable groups then $\mathrm{Comm}(\Gamma_1) \cong \mathrm{Comm}(\Gamma_2)$. We will implicitly use this fact often.

A subgroup $\Delta \le \Gamma$ is *commensuristic* if $\phi(\Delta \cap \Gamma_1)$ is commensurable with Δ for every partial automorphism $\phi : \Gamma_1 \to \Gamma_2$ of Γ. Say that Δ is *strongly commensuristic* if $\phi(\Delta \cap \Gamma_1) = \Delta \cap \Gamma_2$ for every such ϕ. If Δ is commensuristic, restriction induces a map $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Delta)$. If Δ is strongly commensuristic, then there is a natural map $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/\Delta)$.

A group Γ *virtually* has a property P if there is a subgroup $\Delta \le \Gamma$ of finite index with property P. For any Λ, a *virtual homomorphism* $\Gamma \to \Lambda$ is a homomorphism from a finite-index subgroup of Γ to Λ. Two such virtual homomorphisms are *equivalent* if they agree on a finite-index subgroup of Γ.

By a $\mathbb{Q}$-*defined linear algebraic group*, or $\mathbb{Q}$-*group*, we mean a subgroup $\boldsymbol{G} \le \mathrm{GL}_n(\mathbb{C})$ for some n that is closed in the Zariski topology and whose defining polynomials may be chosen to have coefficients in $\mathbb{Q}$. The $\mathbb{Q}$-*points* of $\boldsymbol{G}$ are $\boldsymbol{G}(\mathbb{Q}) = \boldsymbol{G} \cap \mathrm{GL}_n(\mathbb{Q})$. If S is a finite set of prime numbers, we define the group of S-*integer points* of $\boldsymbol{G}$, denoted $\boldsymbol{G}(S)$, to be the subgroup of elements of $\boldsymbol{G}(\mathbb{Q})$ with matrix coefficients having denominators divisible only by elements of S. A subgroup of $\boldsymbol{G}(\mathbb{Q})$ is S-*arithmetic* if it is commensurable with $\boldsymbol{G}(S)$. An abstract group Γ is S-*arithmetic* if it is abstractly commensurable with an S-arithmetic subgroup of some $\mathbb{Q}$-group $\boldsymbol{G}$.

Now let $\boldsymbol{G}$ be a solvable $\mathbb{Q}$-group, S be a finite set of primes, and $\Gamma = \boldsymbol{G}(S)$. Since $[\boldsymbol{G} : \boldsymbol{G}^0] < \infty$, we will assume $\boldsymbol{G}$ is connected. The subgroup $\boldsymbol{U} \le \boldsymbol{G}$ consisting of all unipotent elements of $\boldsymbol{G}$ is connected, is defined over $\mathbb{Q}$, and is called the *unipotent radical*. For any maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \le \boldsymbol{G}$, there is a semidirect product decomposition $\boldsymbol{G} = \boldsymbol{U} \rtimes \boldsymbol{T}$.

For any $\mathbb{Q}$-defined torus $\boldsymbol{T}$ and any field extension F of $\mathbb{Q}$, the F-*rank* of $\boldsymbol{T}$, denoted $\mathrm{rank}_F(\boldsymbol{T})$, is the dimension of any maximal subtorus of $\boldsymbol{T}$ diagonalizable over F. We will use the following special case of [Platonov and Rapinchuk 1994, Theorem 5.12, p. 276].

Theorem 2.1. *Let $\boldsymbol{T}$ be a torus defined over $\mathbb{Q}$ and S a finite set of prime numbers. Then $\boldsymbol{T}(S)$ is isomorphic to the product of a finite group and a free abelian group of rank*

$$N = \mathrm{rank}_{\mathbb{R}}(\boldsymbol{T}) - \mathrm{rank}_{\mathbb{Q}}(\boldsymbol{T}) + \sum_{p \in S} \mathrm{rank}_{\mathbb{Q}_p}(\boldsymbol{T}).$$

If $\boldsymbol{U}$ is a connected unipotent $\mathbb{Q}$-group, then $\mathrm{Aut}(\boldsymbol{U})$ may be identified with the automorphism group of the Lie algebra of $\boldsymbol{U}$ and thus has the structure of a $\mathbb{Q}$-group. This structure is such that $\mathrm{Aut}(\boldsymbol{U})(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{U})$, where $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{U})$ is the group of $\mathbb{Q}$-defined automorphisms of $\boldsymbol{U}$.

A solvable $\mathbb{Q}$-group $\boldsymbol{G}$ is said to be *reduced*, or to have *strong unipotent radical*, if the action of any maximal $\mathbb{Q}$-defined torus on the unipotent radical is faithful. If $\boldsymbol{G}$ is reduced then $\mathrm{Aut}(\boldsymbol{G})$ naturally has the structure of a $\mathbb{Q}$-group such that $\mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ (see [Grunewald and Platonov 1999, Section 4] or [Baues and Grunewald 2006, Section 3]) and the identity component $\mathrm{Aut}^0(\boldsymbol{G})$ is a finite-index subgroup of $\mathrm{Aut}(\boldsymbol{G})$ that acts trivially on the quotient of $\boldsymbol{G}$ by its unipotent radical.

3. Proof of main theorems

3A. *Setup.* In this section we begin the work necessary to prove Theorem 1.2, by way of Theorem 1.3. Let $\boldsymbol{G}$ be a connected solvable $\mathbb{Q}$-group, let S be a finite set of prime numbers, and let $\Gamma \le \boldsymbol{G}(\mathbb{Q})$ be an S-arithmetic subgroup. Replacing $\boldsymbol{G}$ by the Zariski-closure of Γ, we will assume going forward that Γ is Zariski-dense in $\boldsymbol{G}$.

Write $\boldsymbol{G} = \boldsymbol{U} \rtimes \boldsymbol{T}$ as above. We will assume without loss of generality that Γ decomposes as $\Gamma = \boldsymbol{U}(S) \rtimes \Gamma_{\boldsymbol{T}}$ for some finitely generated, torsion-free, free abelian S-arithmetic subgroup $\Gamma_{\boldsymbol{T}} \le \boldsymbol{T}(S)$; see [Platonov and Rapinchuk 1994, Lemma 5.9] and Theorem 2.1.

A group Γ is *uniquely p-radicable* if for every $\gamma \in \Gamma$ there is a unique element $\delta \in \Gamma$ such that $\delta^p = \gamma$.

Lemma 3.1. *Suppose Δ is any finite-index subgroup of Γ and $p \in S$. Then $\Delta \cap \boldsymbol{U}(S)$ is the unique maximal uniquely p-radicable subgroup of Δ.*

Proof. Since $\Gamma_{\boldsymbol{T}}$ is isomorphic to $\mathbb{Z}^N$ for some N, it suffices to show that $\boldsymbol{U}(S) \cap \Delta$ is uniquely p-radicable. Moreover, because the property of being uniquely p-radicable is inherited by subgroups of finite index, it suffices to check that $\boldsymbol{U}(S)$ is uniquely p-radicable. It is a standard fact that $\boldsymbol{U}$ is $\mathbb{Q}$-isomorphic to a subgroup of the group of $n \times n$ matrices with 1's on the diagonal, which we denote $\boldsymbol{U}_n$. Therefore $\boldsymbol{U}(S)$ is commensurable with a subgroup of $\boldsymbol{U}_n(S)$. The desired property is preserved by commensurability of torsion-free groups, so it suffices to show that $\boldsymbol{U}_n(S)$ is uniquely p-radicable. This may easily be done by induction on n. □

Corollary 3.2. *If $S \neq \varnothing$, then $\boldsymbol{U}(S)$ is strongly commensuristic in Γ.*

Remark. If $S = \varnothing$ then Corollary 3.2 is still true when $\boldsymbol{G}$ is reduced. This follows from the fact that $\Gamma \cap \boldsymbol{U}$ is the Fitting subgroup of Γ for any arithmetic subgroup $\Gamma \le \boldsymbol{G}(\mathbb{Q})$; see [Grunewald and Platonov 1999, Lemma 2.6] for a proof.

Theorem 3.3. *There is an isomorphism* $\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q})$.

Proof. Since $\boldsymbol{U}(S)$ has the property that for each $u \in \boldsymbol{U}(\mathbb{Q})$ there is some number k such that $u^k \in \boldsymbol{U}(\mathbb{Z})$, any partial automorphism ϕ of $\boldsymbol{U}(S)$ is determined by its values on $\boldsymbol{U}(\mathbb{Z})$. The resulting map $\phi\big|_{\boldsymbol{U}(\mathbb{Z})} : \boldsymbol{U}(\mathbb{Z}) \to \boldsymbol{U}(\mathbb{Q})$ uniquely extends to a $\mathbb{Q}$-defined homomorphism $\tilde{\phi} : \boldsymbol{U} \to \boldsymbol{U}$ by a theorem of Mal'cev (see, for example, the proof of [Raghunathan 1972, Theorem 2.11, p. 33].) Since the dimension of the Zariski-closure of $\phi(\boldsymbol{U}(\mathbb{Z}))$ is equal to the dimension of $\boldsymbol{U}$ by [Raghunathan 1972, Theorem 2.10, p. 32], the map $\hat{\phi}$ is an automorphism of $\boldsymbol{U}$.

The assignment $[\phi] \mapsto \tilde{\phi}$ gives a well-defined mapping $\xi : \mathrm{Comm}(\boldsymbol{U}(S)) \to \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q})$. We see that ξ is injective because $\boldsymbol{U}(S)$ is Zariski-dense in $\boldsymbol{U}$, and ξ is surjective because every $\mathbb{Q}$-defined automorphism of $\boldsymbol{U}$ induces a commensuration of $\boldsymbol{U}(S)$ by Proposition 1.1. □

3B. ***Reduced case.*** Now assume that $\boldsymbol{G}$ is reduced. We prove Theorem 1.3 using methods following those used to prove Theorems A and C of [Grunewald and Platonov 1999].

Proof of Theorem 1.3. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{G}$ and fix a maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \leq \boldsymbol{G}$. We assume without loss of generality that $\Delta = (\Delta \cap \boldsymbol{U}) \rtimes (\Delta \cap \boldsymbol{T})$.

Suppose $\phi : \Delta_1 \to \Delta_2$ is a partial automorphism of Δ. By Corollary 3.2 and Theorem 3.3, ϕ induces a $\mathbb{Q}$-defined automorphism $\Phi_{\boldsymbol{U}} \in \mathrm{Aut}(\boldsymbol{U})$. Define $\alpha : \boldsymbol{G} \to \mathrm{Aut}(\boldsymbol{U})$ to be the map induced by conjugation. Note that $\alpha\big|_{\boldsymbol{T}}$ is injective since $\boldsymbol{G}$ is reduced.

It is straightforward to check that for any $\delta \in \Delta_1$ we have

$$\Phi_{\boldsymbol{U}} \circ \alpha(\delta) \circ \Phi_{\boldsymbol{U}}^{-1} = \alpha(\phi(\delta)).$$

It follows that conjugation by $\Phi_{\boldsymbol{U}}$ preserves $\alpha(\boldsymbol{G})$ inside $\mathrm{Aut}(\boldsymbol{U})$. Conjugation by $\Phi_{\boldsymbol{U}}$ therefore induces an isomorphism between $\alpha(\boldsymbol{T})$ and $\alpha(\boldsymbol{T}')$ for some maximal $\mathbb{Q}$-defined torus $\boldsymbol{T}' \leq \boldsymbol{G}$, and hence an isomorphism $\Phi_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}'$. Note that $\Phi_{\boldsymbol{T}}$ is defined to satisfy the relation

$$\Phi_{\boldsymbol{U}} \circ \alpha(t) \circ \Phi_{\boldsymbol{U}}^{-1} = \alpha(\Phi_{\boldsymbol{T}}(t)) \tag{2}$$

for all $t \in \boldsymbol{T}$.

The maps $\Phi_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}}$ determine a self-map of $\boldsymbol{G}$: for each $g \in \boldsymbol{G}$, write $g = ut$ for $u \in \boldsymbol{U}$ and $t \in \boldsymbol{T}$ and set

$$\Phi_0(g) := \Phi_{\boldsymbol{U}}(u)\Phi_{\boldsymbol{T}}(t).$$

Equation (2) implies that Φ_0 is a $\mathbb{Q}$-defined automorphism of $\boldsymbol{G}$. However, the map $\mathrm{Comm}(\Delta) \to \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ defined by $[\phi] \mapsto \Phi_0$ is *not* necessarily a well-defined homomorphism of groups. We will show that Φ_0 can be modified in a unique way to produce an automorphism Φ so that $\Phi(\delta)\phi(\delta)^{-1} \in Z(\boldsymbol{G})$ for all $\delta \in \Delta_1$. This condition will guarantee the map $[\phi] \mapsto \Phi$ defines a homomorphism.

It is straightforward to check from our definitions that $\alpha(\Phi_0(\delta)\phi(\delta)^{-1})$ is trivial for all $\delta \in \Delta_1$. Therefore $v(\delta) := \Phi_0(\delta)\phi(\delta)^{-1}$ defines a function $v : \Delta_1 \to Z(\boldsymbol{U})(\mathbb{Q})$. One can check that

$$v(\delta_1\delta_2) = v(\delta_1)\phi(\delta_1)v(\delta_2)\phi(\delta_1)^{-1}.$$

That is, ϕ is a *derivation* when $Z(\boldsymbol{U})(\mathbb{Q})$ is given the structure of a left Δ_1-module by $\delta \cdot z = \phi(\delta)z\phi(\delta)^{-1}$ for $\delta \in \Delta_1$ and $z \in Z(\boldsymbol{U})(\mathbb{Q})$.

The derivation v is trivial on $\Delta_1 \cap \boldsymbol{U}$, and therefore descends to a derivation $\bar{v} : \Delta_1 \cap \boldsymbol{T} \to Z(\boldsymbol{U})(\mathbb{Q})$. Now decompose $Z(\boldsymbol{U})(\mathbb{Q})$ as a direct sum of weight spaces for the action of $\boldsymbol{T}$ and let V be the sum of all weight spaces with nontrivial weights. Let $v^{\perp}$ be the component of the derivation $\bar{v}$ in the submodule V. Since $C_V(\boldsymbol{T})$ is trivial, it follows from a standard cohomological fact (see [Segal 1983, Chapter 3, Theorem 2**, p. 44]) that $v^{\perp}$ is an inner derivation. That is, there is some $x \in V$ such that $v^{\perp}(\delta) = \phi(\delta)x\phi(\delta)^{-1}x^{-1}$ for all $\delta \in \Delta \cap \boldsymbol{T}$. It follows that

$$v(\delta)x\phi(\delta)x^{-1}\phi(\delta)^{-1} \in Z(\boldsymbol{G})(\mathbb{Q}).$$

When x is viewed as an element of $Z(\boldsymbol{U})(\mathbb{Q})$, the choice of x is unique up to $Z(\boldsymbol{G})(\mathbb{Q})$.

Given Φ_0 and x as above, the assignment $\mu(\phi) = c_x \circ \Phi_0$, where $c_x(g) = xgx^{-1}$ for all $g \in \boldsymbol{G}$, determines a well-defined map

$$\mu : \mathrm{Comm}(\Delta) \to \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}).$$

One can check using an obvious modification of [Grunewald and Platonov 1999, Lemma 2.9] that μ is a homomorphism. Because Γ is Zariski-dense in $\boldsymbol{G}$, the map

$$\Theta : \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \to \mathrm{Comm}(\boldsymbol{G}(S))$$

of Proposition 1.1 is injective. In fact Θ is a section of μ; to see this, note that if $\phi = \Theta(\Phi)$ then the associated maps $\Phi_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}}$ are $\Phi_{\boldsymbol{U}} = \Phi\big|_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}} = \Phi\big|_{\boldsymbol{T}}$, which clearly satisfy (2), and moreover the associated derivation ν is trivial. It follows that there is an isomorphism

$$\mathrm{Comm}(\Delta) \cong \ker(\mu) \rtimes \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}).$$

Now suppose that $[\phi] \in \ker(\mu)$. It follows from the above that ϕ is a virtual homomorphism $\Delta \to Z(\boldsymbol{G})(\mathbb{Q})$ trivial on $\Delta \cap \boldsymbol{U}$. We can view ϕ as a virtual homomorphism $\Delta \cap \boldsymbol{T} \to Z(\boldsymbol{G})(\mathbb{Q})$. Since $\Delta \cap \boldsymbol{T}$ is virtually $\mathbb{Z}^N$, the group of equivalence classes of such virtual homomorphisms is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$.

We therefore have a well-defined map

$$\xi : \ker(\mu) \to \mathrm{Hom}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})).$$

Clearly ξ is injective. On the other hand, suppose that $[\Delta \cap \boldsymbol{T} : \Lambda] < \infty$ and that $f : \Lambda \to Z(\boldsymbol{G})(\mathbb{Q})$ is a homomorphism. There is a finite-index subgroup $\widetilde{\Lambda} \leq \Lambda$ such that $f(\widetilde{\Lambda}) \leq Z(\boldsymbol{G})(S)$. The map

$$\phi : \boldsymbol{U}(S) \rtimes \widetilde{\Lambda} \to \boldsymbol{U}(S) \rtimes \widetilde{\Lambda}$$

defined by $\phi(u, \lambda) = (u \cdot f(\lambda), \lambda)$ induces a commensuration of Δ mapping to f under ξ; hence ξ is surjective. This completes the proof that $\mathrm{Comm}(\Delta)$ has the desired semidirect product decomposition.

Let

$$\mathrm{Comm}^0(\Delta) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})) \rtimes \mathrm{Aut}^0(\boldsymbol{G})(\mathbb{Q}).$$

Clearly $\mathrm{Comm}^0(\Delta)$ has finite index in $\mathrm{Comm}(\Delta)$. We will show that $\mathrm{Comm}^0(\Gamma)$ has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. We first understand the action of $\mathrm{Aut}(\boldsymbol{G})$ on $\mathrm{Hom}(\mathbb{Q}^N, Z(\boldsymbol{G}))$. Any $\Phi \in \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ induces a commensuration of Δ virtually preserving $\boldsymbol{U}(S)$, hence induces a commensuration of $\boldsymbol{T}(S)$. Let $\widetilde{\Phi}_{\boldsymbol{T}} \in \mathrm{GL}_N(\mathbb{Q})$ be the automorphism corresponding to the induced commensuration of $\boldsymbol{T}(S)$. Then the action is given by

$$(\Phi \cdot \alpha)(t) = \Phi_{\boldsymbol{U}}(\alpha(\widetilde{\Phi}_{\boldsymbol{T}}^{-1} t)).$$

Note that if $\Phi \in \mathrm{Aut}^0(\boldsymbol{G})$ then Φ acts trivially on the quotient $\boldsymbol{G}/\boldsymbol{U}$; hence the induced map $\widetilde{\Phi}_{\boldsymbol{T}}$ is trivial.

The group $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$ is isomorphic to the $\mathbb{Q}$-points of $(\boldsymbol{G}_a)^{Nd}$, a product of additive groups defined over $\mathbb{Q}$, where d is the dimension of $Z(\boldsymbol{G})$. Under this identification, the action of $\mathrm{Aut}(Z(\boldsymbol{G}))(\mathbb{Q})$ by postcomposition on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$ corresponds to the diagonal linear action of $\mathrm{Aut}(Z(\boldsymbol{G}))$ on $(\boldsymbol{G}_a)^{Nd}$. Since the restriction map $\mathrm{Aut}(\boldsymbol{G}) \to \mathrm{Aut}(Z(\boldsymbol{G}))$ is defined over $\mathbb{Q}$ by definition of the algebraic structure on $\mathrm{Aut}(\boldsymbol{G})$, the action map

$$\mathrm{Aut}^0(\boldsymbol{G}) \times (\boldsymbol{G}_a)^{Nd} \to (\boldsymbol{G}_a)^{Nd}$$

is defined over $\mathbb{Q}$. Hence the semidirect product $(\boldsymbol{G}_a)^{Nd} \rtimes \mathrm{Aut}^0(\boldsymbol{G})$ is an algebraic group whose $\mathbb{Q}$-points are identified with $\mathrm{Comm}^0(\Delta)$. □

3C. ***Nonreduced case.*** Now consider the case that $\boldsymbol{G}$ is a connected solvable group, not necessarily reduced. As above we will assume without loss of generality that Γ is Zariski-dense in $\boldsymbol{G}$ and decomposes as $\Gamma = \boldsymbol{U}(S) \rtimes \Gamma_{\boldsymbol{T}}$. Assume for the rest of this section that $S \neq \varnothing$. (The case that $S = \varnothing$ is addressed by the remarks following the statement of Theorem 1.2.) Our primary goal is to reduce to a situation where Theorem 1.3 can be applied. This reduction will occur over several steps.

Define $\boldsymbol{T}_0 \leq \boldsymbol{T}$ to be the centralizer of $\boldsymbol{U}$ in $\boldsymbol{T}$, a $\mathbb{Q}$-defined subgroup of $\boldsymbol{T}$. There is a $\mathbb{Q}$-defined subgroup $\boldsymbol{T}_1 \leq \boldsymbol{T}$ such that $\boldsymbol{T} = \boldsymbol{T}_0\boldsymbol{T}_1$ and $\boldsymbol{T}_0 \cap \boldsymbol{T}_1$ is finite. Without loss of generality we replace $\boldsymbol{G}$ by $\boldsymbol{G}/(\boldsymbol{T}_0 \cap \boldsymbol{T}_1)$ and henceforth assume that $\boldsymbol{T}_0 \cap \boldsymbol{T}_1 = \{1\}$. Note that now $\boldsymbol{U} \rtimes \boldsymbol{T}_1$ is a reduced solvable $\mathbb{Q}$-group. Moreover, without loss of generality we replace $\Gamma_{\boldsymbol{T}}$ with $\Gamma_0 \times \Gamma_1$, where $\Gamma_i \cong \mathbb{Z}^{N_i}$ is an S-arithmetic subgroup of $\boldsymbol{T}_i$ for each $i = 0, 1$. See Theorem 2.1 for the formula used to determine N_i.

From the semidirect product decomposition $\Gamma = (\boldsymbol{U}(S) \times \Gamma_0) \rtimes \Gamma_1$, let us denote elements of Γ by triples (u, γ_0, γ_1), where $u \in \boldsymbol{U}(S)$ and $\gamma_i \in \Gamma_i$ for $i = 0, 1$.

Define $Z_{\boldsymbol{U}}(\Gamma) = Z(\Gamma) \cap \boldsymbol{U}$. Clearly we have

$$Z(\Gamma) = Z_{\boldsymbol{U}}(\Gamma) \times \Gamma_0.$$

If Δ is any finite-index subgroup of Γ, then $Z(\Delta) = \Delta \cap Z(\boldsymbol{G})$ by the Zariski-density of Δ. It follows that $Z(\Gamma)$ is strongly commensuristic in Γ. Moreover, since $\boldsymbol{U}(S)$ is strongly commensuristic in Γ it follows that $Z_{\boldsymbol{U}}(\Gamma)$ is strongly commensuristic in Γ.

Any virtual homomorphism $\alpha : \Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$ determines a partial automorphism ψ_α of Γ defined on an appropriate subgroup of Γ by

$$\psi_\alpha(u, \gamma_0, \gamma_1) := (u + \alpha(\gamma_0, \gamma_1), \gamma_0, \gamma_1).$$

Let $\mathcal{W}$ denote the subgroup of $\mathrm{Comm}(\Gamma)$ arising in this way from equivalence classes of virtual homomorphisms $\Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$. There is an isomorphism

$$\mathcal{W} \cong \mathrm{Hom}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d),$$

where d is the dimension of $Z(\boldsymbol{G}) \cap \boldsymbol{U}$.

Let

$$\mathrm{Comm}_{\Gamma_0}(\Gamma) = \{[\phi : H \to K] \in \mathrm{Comm}(\Gamma) \mid \phi(H \cap \Gamma_0) = K \cap \Gamma_o\}.$$

Lemma 3.4. $\mathcal{W} \cdot \mathrm{Comm}_{\Gamma_0}(\Gamma) = \mathrm{Comm}(\Gamma)$.

Proof. We first show that $\mathcal{W}$ is a normal subgroup of $\mathrm{Comm}(\Gamma)$ so that the product $\mathcal{W} \cdot \mathrm{Comm}_{\Gamma_0}(\Gamma)$ is well defined. To see this, take any $\phi \in \mathrm{Comm}(\Gamma)$. Since $\boldsymbol{U}(S)$ is commensuristic in Γ and is fixed by any $\psi_\alpha \in \mathcal{W}$ we see that $\phi \circ \psi_\alpha \circ \phi^{-1}$ is trivial on $\boldsymbol{U}(S)$. It follows by direct computation that

$$\phi \circ \psi_\alpha \circ \phi^{-1} = \psi_{\phi_{\boldsymbol{U}} \circ \alpha \circ \phi_{\boldsymbol{T}}^{-1}},$$

where $\phi_{\boldsymbol{U}}$ is the restriction of ϕ to $Z_{\boldsymbol{U}}(\Gamma)$ and $\phi_{\boldsymbol{T}}$ is the commensuration of $\Gamma_0 \times \Gamma_1$ induced by ϕ under the quotient map $\Gamma \to \Gamma/\boldsymbol{U}(S)$. The map $\phi_{\boldsymbol{U}} \circ \alpha \circ \phi_{\boldsymbol{T}}^{-1}$ is a virtual homomorphism from $\Gamma_o \times \Gamma_1$ to $Z_{\boldsymbol{U}}(\Gamma)$ because $Z_{\boldsymbol{U}}(\Gamma)$ is commensuristic in Γ. This shows that $\mathcal{W}$ is normal in $\mathrm{Comm}(\Gamma)$.

Suppose $\phi : H \to K$ is a partial automorphism of Γ. Since $\boldsymbol{U}(S)$ is strongly commensuristic, ϕ induces a commensuration $[\nu] \in \mathrm{Comm}(\Gamma_0 \times \Gamma_1)$. There is a function $\alpha : H \cap (\Gamma_0 \times \Gamma_1) \to K \cap Z_{\boldsymbol{U}}(\Gamma)$ such that

$$\phi(0, \gamma_0, \gamma_1) = (\alpha(\gamma_0), \nu(\gamma_0, \gamma_1))$$

for all $(\gamma_0, \gamma_1) \in H \cap (\Gamma_0 \times \Gamma_1)$. In fact the function α is a virtual homomorphism $\Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$.

Define a virtual homomorphism $\beta : \Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$ by $\beta = -\alpha \circ \nu^{-1}$. A straightforward computation shows that

$$(\psi_\beta \circ \phi)(0, \gamma_0, \gamma_1) = (0, \nu(\gamma_0, \gamma_1))$$

for all $(\gamma_0, \gamma_1) \in H \cap (\Gamma_0 \times \Gamma_1)$. Since $Z(\Gamma)$ is commensuristic in Γ, it follows that $(\psi_\beta \circ \phi)(0, \gamma_0, 0) = (0, \nu(\gamma_0), 0)$ for all $\gamma_0 \in H \cap \Gamma_0$. This means that $\psi_\beta \circ \phi \in \mathrm{Comm}_{\Gamma_0}(\Gamma)$, which completes the proof. $\square$

We now turn to the task of elucidating the structure of $\mathrm{Comm}_{\Gamma_0}(\Gamma)$. There is a natural map

$$\xi : \mathrm{Comm}_{\Gamma_0}(\Gamma) \to \mathrm{Comm}(\Gamma/\Gamma_0).$$

Define $\mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$ to be the kernel of ξ. Because Γ/Γ_0 is naturally identified with the subgroup $\boldsymbol{U}(S) \rtimes \Gamma_1 \leq \Gamma$, it is easy to see that ξ is surjective. Therefore there is a short exact sequence

$$1 \to \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{Comm}_{\Gamma_0}(\Gamma) \to \mathrm{Comm}(\Gamma/\Gamma_0) \to 1. \tag{3}$$

Because Γ decomposes as a direct product $\Gamma = (\boldsymbol{U}(S) \rtimes \Gamma_1) \times \Gamma_0$, the sequence (3) splits and we can identify $\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Comm}(\boldsymbol{U}(S) \rtimes \Gamma_1)$. By Theorem 1.3 there is an isomorphism

$$\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})) \rtimes \mathrm{Aut}(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}).$$

Note that $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1) = Z(\boldsymbol{G}) \cap \boldsymbol{U}$, so recalling that d is the dimension of $Z(\boldsymbol{G}) \cap \boldsymbol{U}$ we may write

$$\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d) \rtimes \mathrm{Aut}(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}).$$

Lemma 3.5. *Let $\Gamma_i \cong \mathbb{Z}^{N_i}$ for $i = 0, 1$ be as above. There is an isomorphism*

$$\mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0}) \rtimes \mathrm{GL}_{N_0}(\mathbb{Q}),$$

where the action is by postcomposition.

Proof. There is a homomorphism $\Psi : \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{GL}_{N_0}(\mathbb{Q})$ given by restriction to Γ_0. Because Γ_0 splits off as a direct product factor, Ψ is surjective and the following exact sequence splits:

$$1 \to \ker(\Psi) \to \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{GL}_{N_0}(\mathbb{Q}) \to 1.$$

The kernel of Ψ is given by equivalence classes of virtual homomorphisms $\boldsymbol{U}(S) \rtimes \Gamma_1 \to \Gamma_0$. There are no virtual homomorphisms $\boldsymbol{U}(S) \to \Gamma_0$ because Γ_0 is free abelian and every finite-index subgroup of $\boldsymbol{U}(S)$ is p-radicable for any $p \in S$. Therefore the kernel of Ψ may be identified with equivalence classes of virtual homomorphisms from Γ_1 to Γ_0, which form a group isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$. The fact that the action is by postcomposition is immediate. □

Define

$$\mathrm{Comm}^0_{\Gamma_0}(\Gamma) = \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \rtimes \mathrm{Comm}^0(\Gamma/\Gamma_0),$$

where $\mathrm{Comm}^0(\Gamma/\Gamma_0)$ is as defined in Theorem 1.3. This is a finite-index subgroup of $\mathrm{Comm}_{\Gamma_0}(\Gamma)$. Note that the subgroup $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d) \le \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts trivially on $\mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$, and the subgroup $\mathrm{GL}_{N_0}(\mathbb{Q}) \le \mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$ is centralized by the action of $\mathrm{Comm}^0(\Gamma/\Gamma_0)$. There is therefore a normal subgroup of $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ isomorphic to

$$\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0}) \times \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d),$$

which is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$. So we may write

$$\mathrm{Comm}^0_{\Gamma_0}(\Gamma) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d}) \rtimes (\mathrm{GL}_{N_0}(\mathbb{Q}) \times \mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})), \tag{4}$$

where the commuting actions of $\mathrm{GL}_{N_0}(\mathbb{Q})$ and $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ are each by postcomposition.

Lemma 3.6. *There is a $\mathbb{Q}$-group $\boldsymbol{C}$ such that* $\mathrm{Comm}^0_{\Gamma_0}(\Gamma) \cong \boldsymbol{C}(\mathbb{Q})$.

Proof. For each $i = 1, \dots, N_1$ and $j = 1, \dots, N_0 + d$, let $\boldsymbol{A}_{i,j}$ be a copy of the 1-dimensional additive $\mathbb{Q}$-group $\boldsymbol{G}_a$. Define

$$\boldsymbol{C}_{\boldsymbol{T}} = \prod_{i=1}^{N_1} \prod_{j=1}^{N_0+d} \boldsymbol{A}_{i,j}.$$

Fix bases $\{v_i\}_{i=1}^{N_1}$ for $\mathbb{Q}^{N_1}$, and $\{w_i\}_{i=1}^{N_0}$ for $\mathbb{Q}^{N_0}$, and $\{w_i\}_{i=N_0+1}^{N_0+d}$ for $\mathbb{Q}^d$, so that $\{w_i\}_{i=1}^{N_0+d}$ is a basis for $\mathbb{Q}^{N_0+d}$. Let $e_{i,j}$ be the element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ that sends v_i to w_j and each v_k to zero for $k \neq i$. Then the collection of $\{e_{i,j}\}$ are a basis for $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$. Fix an isomorphism $\boldsymbol{C}_{\boldsymbol{T}}(\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ that takes a generator of $\boldsymbol{A}_{i,j}$ to $e_{i,j}$ for each pair i, j.

The algebraic group GL_{N_0} acts on $\boldsymbol{C}_{\boldsymbol{T}}$ by acting in the standard way on each group $\prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$ for fixed i and trivially on each factor $\boldsymbol{A}_{i,j}$ for $j > N_o$. This action is defined over $\mathbb{Q}$. The restriction of this action to the group action of $\mathrm{GL}_{N_0}(\mathbb{Q})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$ inside $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ is the action in (4).

Identify each group $\prod_{j=N_0+1}^{N_0+d} \boldsymbol{A}_{i,j}$ with $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$. This determines an action of the group $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$ on each group $\prod_{j=N_0+1}^{N_0+d} \boldsymbol{A}_{i,j}$ for fixed i, hence an action on all of $\boldsymbol{C_T}$. This action is defined over $\mathbb{Q}$, and its restriction to $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ agrees with the action in (4).

Using the actions defined above, the algebraic group

$$\boldsymbol{C} = (\boldsymbol{G}_a)^{N_1(N_0+d)} \rtimes (\mathrm{GL}_{N_0} \times \mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1))$$

is a $\mathbb{Q}$-group with $\boldsymbol{C}(\mathbb{Q}) = \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$. $\square$

The group $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts on $\mathcal{W}$ by conjugation. Under the identification $\mathcal{W} \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ and the decomposition of (4), this gives actions of each of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$, $\mathrm{GL}_{N_0}(\mathbb{Q})$, and $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$. We record here some facts about these actions that are straightforward to verify.

Lemma 3.7. *The action of* $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ *on* $\mathcal{W}$ *is given by the following*:

(1) *the action of* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ *on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *factors through the quotient* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$ *acting on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *by precomposition by the inverse*;

(2) *the action of* $\mathrm{GL}_{N_0}(\mathbb{Q})$ *is by precomposition by the inverse acting on* $\mathbb{Q}^{N_0} \leq \mathbb{Q}^{N_0+N_1}$;

(3) *the group* $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ *acts on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *by postcomposition, where* $\mathbb{Q}^d$ *is identified with* $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$.

We now complete the proof of the main theorem of this paper in the case $S \neq \varnothing$.

Proof of Theorem 1.2. Continue using the notation of Section 3C and Lemmas 3.4–3.7. We will define a $\mathbb{Q}$-group $\boldsymbol{D}$ so that $\boldsymbol{D}(\mathbb{Q}) \cong \mathcal{W} \cdot \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$. Because $\mathcal{W} \cdot \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ is a subgroup of finite index in $\mathrm{Comm}(\Gamma)$, this is the desired result.

Because $\mathcal{W}$ is normal in $\mathrm{Comm}(\Gamma)$, the group $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts on $\mathcal{W}$ by conjugation. This determines an action of $\boldsymbol{C}(\mathbb{Q})$ on $\mathcal{W}$. We will show there is an algebraic group $\boldsymbol{W}$ with $\boldsymbol{W}(\mathbb{Q}) \cong \mathcal{W}$ and an algebraic action of $\boldsymbol{C}$ on $\boldsymbol{W}$ such that the induced action of $\boldsymbol{C}(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ agrees with the action of $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ on $\mathcal{W}$ under our identifications.

Consider indexed copies of the additive group $\boldsymbol{G}_a^{i,j}$ for $i = 1, \dots, N_0 + N_1$ and $j = 1, \dots, d$. Let

$$\boldsymbol{W} = \prod_{i=1}^{N_0+N_1} \prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}.$$

Fix bases $\{x_i\}_{i=1}^{N_0}$ for $\mathbb{Q}^{N_0}$, and $\{x_i\}_{i=N_0+1}^{N_0+N_1}$ for $\mathbb{Q}^{N_1}$, and $\{y_i\}_{i=1}^{d}$ for $\mathbb{Q}^d$, so that $\{x_i\}_{i=1}^{N_0+N_1}$ is a basis for $\mathbb{Q}^{N_0+N_1}$. Let $f_{i,j}$ be the element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ that sends x_i to y_j and each x_k to zero for $k \neq i$. Then the collection of $\{f_{i,j}\}$ are a basis for $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$. Fix an isomorphism $\boldsymbol{W}(\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$

that takes a generator of $\boldsymbol{G}_a^{i,j}$ to $f_{i,j}$ for each pair i, j. This gives an isomorphism $\boldsymbol{W}(\mathbb{Q}) \cong \mathcal{W}$.

For each fixed i we may identify the group $\prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}$ with $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$. This identification determines an action of $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$ on each group $\prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}$, hence an action on all of $\boldsymbol{W}$ which is defined over $\mathbb{Q}$. This action restricts to an action of $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ which agrees under our identifications with the action of the subgroup $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}) \leq \mathrm{Comm}_{\Gamma_0}^0(\Gamma)$ on $\mathcal{W}$.

For each fixed j, the algebraic group GL_{N_0} acts on $\prod_{i=1}^{N_0} \boldsymbol{G}_a^{i,j}$ by the dual (inverse transpose) of the standard action. Letting GL_{N_0} act trivially on each $\boldsymbol{G}_a^{i,j}$ for $i > N_0$, this induces an action of GL_{N_0} on $\boldsymbol{W}$. The restriction of this action to $\mathrm{GL}_{N_0}(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ agrees with the action of the subgroup $\mathrm{GL}_{N_0}(\mathbb{Q}) \leq \mathrm{Comm}_{\Gamma_0}^0(\Gamma)$ on $\mathcal{W}$.

Finally, the group $\prod_{i=1}^{N_1} \prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$ embeds as a unipotent subgroup of $\mathrm{GL}_{N_0+N_1}$, and through this embedding acts by the inverse transpose on $\prod_{i=1}^{N_0+N_1} \boldsymbol{G}_a^{i,j}$ for each fixed j. There is a natural quotient map $\boldsymbol{C}_{\boldsymbol{T}} \to \prod_{i=1}^{N_1} \prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$, and through this map $\boldsymbol{C}_{\boldsymbol{T}}$ acts on $\boldsymbol{W}$ in such a way that the restriction to the $\mathbb{Q}$-points agrees with the action of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$.

In total these define an action of $\boldsymbol{C}$ on $\boldsymbol{W}$ which is defined over $\mathbb{Q}$. Therefore $\boldsymbol{W} \rtimes \boldsymbol{C}$ has the structure of a $\mathbb{Q}$-group.

The unipotent group $(\boldsymbol{G}_a)^{N_1 d}$ embeds in $\boldsymbol{W}$ and $\boldsymbol{C}_{\boldsymbol{T}}$, via maps $\alpha : (\boldsymbol{G}_a)^{N_1 d} \to \boldsymbol{W}$ and $\beta : (\boldsymbol{G}_a)^{N_1 d} \to \boldsymbol{C}_{\boldsymbol{T}}$, such that the image of $(\boldsymbol{G}_a)^{N_1 d}(\mathbb{Q})$ is identified with $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d)$ under each of α and β. Let $\Theta \leq \boldsymbol{W} \rtimes \boldsymbol{C}$ be the embedding of $(\boldsymbol{G}_a)^{N_1 d}$ under the product map $(-\alpha) \times \beta$. Note that Θ is a normal unipotent subgroup of $\boldsymbol{W} \rtimes \boldsymbol{C}$, so the quotient $\boldsymbol{D} = (\boldsymbol{W} \rtimes \boldsymbol{C})/\Theta$ is a $\mathbb{Q}$-group with $\boldsymbol{D}(\mathbb{Q}) = (\boldsymbol{W}(\mathbb{Q}) \rtimes \boldsymbol{C}(\mathbb{Q}))/\Theta(\mathbb{Q})$.

There are isomorphisms $\boldsymbol{W}(\mathbb{Q}) \to \mathcal{W}$ and $\boldsymbol{C}(\mathbb{Q}) \to \mathrm{Comm}_{\Gamma_0}^0(\Gamma)$ which induce a surjective map

$$\Phi : \boldsymbol{W}(\mathbb{Q}) \rtimes \boldsymbol{C}(\mathbb{Q}) \to \mathrm{Comm}^0(\Gamma)$$

because the action of $\boldsymbol{C}$ on $\boldsymbol{W}$ is compatible with the action of $\mathrm{Comm}_{\Gamma_0}^0(\Gamma)$ on $\mathcal{W}$. The kernel of Φ is precisely the subgroup $\Theta(\mathbb{Q})$, so Φ descends to an isomorphism $\boldsymbol{D}(\mathbb{Q}) \cong \mathrm{Comm}^0(\Gamma)$. □

4. Number fields

Linear algebraic groups can be defined over arbitrary fields. Let K be a global field and S a set of multiplicative valuations of K. The ring of *S-integral* elements of K, denoted $K(S)$, is the ring of $x \in K$ such that $v(x) \leq 1$ for each non-Archimedean valuation $v \notin S$. If $\boldsymbol{G}$ is a linear algebraic group defined over K, let $\boldsymbol{G}(K(S))$ denote the group of matrices in $\boldsymbol{G}$ with entries in $K(S)$. See [Margulis 1991, Chapter I] for details.

The following example shows that if $\boldsymbol{U}$ is a unipotent group defined over a number field K and S is a set of multiplicative valuations, then $\mathrm{Comm}(\boldsymbol{U}(K(S)))$ may depend on S. This stands in contrast with Theorem 3.3, which directly implies that $\mathrm{Comm}(\boldsymbol{U}(K(S)))$ is independent of S when $K = \mathbb{Q}$. The author is grateful to Dave Morris for suggesting this example.

Example 4.1. Take $\boldsymbol{U}$ to be the additive group $\boldsymbol{G}_a$ defined over $K = \mathbb{Q}(i)$. On the one hand, we have $\boldsymbol{U}(K(\varnothing)) = \mathbb{Z}[i]$ and so

$$\mathrm{Comm}(\boldsymbol{U}(K(\varnothing))) \cong \mathrm{GL}_2(\mathbb{Q}).$$

On the other hand, let $p = 5$ and write $p = ab$ for $a = 2+i$ and $b = 2-i$. Let v_a and v_b be the valuations corresponding to the distinct prime ideals (a) and (b) of $\mathbb{Z}[i]$, respectively. Set $S = \{v_a\}$ and $\Gamma = \boldsymbol{U}(K(S))$. Note that $\Gamma = \mathbb{Z}[i, 1/a]$. We will show that $\mathrm{Comm}(\Gamma)$ is much smaller than $\mathrm{GL}_2(\mathbb{Q})$.

Let K_b be the Cauchy completion of K with respect to the valuation v_b, and let $\mathcal{O}_b$ be the ring of integers of K_b. Note that K_b is a finite extension of $\mathbb{Q}_5$, and that Γ is a dense subgroup of $\mathcal{O}_b$. Any commensuration $[\phi] \in \mathrm{Comm}(\Gamma)$ induces a map $\Phi : K_b \to K_b$ that is continuous and $\mathbb{Q}$-linear, hence K_b-linear. Therefore Φ is multiplication by some nonzero $x \in K_b$. In fact it follows that $x \in K$ since Γ is virtually preserved and Zariski-dense in K. Every element of $K^\times$ induces a nontrivial commensuration, so we have

$$\mathrm{Comm}(\Gamma) \cong \mathbb{Q}(i)^\times.$$

In this example, $\mathrm{Comm}(\Gamma)$ has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. Hence the conclusion of Theorem 1.2 holds even though the method of proof does not.

Dave Morris has pointed out that the arguments of Example 4.1 extend to prove the following:

Proposition 4.2. *Let $\boldsymbol{U}$ be a unipotent group defined over a number field K. For every finite set S of valuations of K, there is a subfield $L \leq K$ such that*

$$\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(R_{K/L}\boldsymbol{U})(L),$$

where $R_{K/L}$ is the restriction of scalars operator.

With this, much of the proof of Theorem 1.2 still applies. For example, Theorem 2.1 generalizes to tori $\boldsymbol{T}$ defined over number fields K to show that $\boldsymbol{T}(K(S))$ is virtually a finitely generated, free abelian group for any finite S. However, there is an obstruction to extending the proof of Theorem 1.2: Proposition 1.1 no longer applies on passage to the restriction of scalars over L.

5. Function fields

In this section we provide examples of S-arithmetic groups over a global field of positive characteristic for which no obvious analog of Theorem 1.2 holds.

In what follows we use the global field $K = \mathbb{F}_q(t)$, the field of rational functions in one variable over the finite field with q elements. Choose $S = \{v_t, v_\infty\}$, where the valuations v_∞ and v_t are defined as follows. Given any $r \in \mathbb{F}_q(t)$, write $r(t) = t^k(f(t)/g(t))$, where f and g are polynomials with nontrivial constant term and $k \in \mathbb{Z}$. Then define

$$v_t(r) = q^{-k} \quad \text{and} \quad v_\infty(r) = q^{\deg(f)+k-\deg(g)}.$$

In this case, $K(S)$ is the ring of Laurent polynomials over $\mathbb{F}_q$, denoted $\mathbb{F}_q[t, t^{-1}]$.

Example 5.1. Consider the 1-dimensional additive algebraic group

$$\boldsymbol{G}_a = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_2 .$$

Then $\boldsymbol{G}_a(K(S)) \cong K(S)$ is an S-arithmetic group. There is an isomorphism of abstract groups

$$K(S) \cong \bigoplus_{k=-\infty}^{\infty} \mathbb{F}_q .$$

Proposition 5.2. *For any field F and any linear algebraic group $\boldsymbol{G}$ over F, there is no embedding* $\mathrm{Comm}(K(S)) \to \boldsymbol{G}(F)$.

Proof. It suffices to treat the case that $\boldsymbol{G} = \mathrm{GL}_d$ for some d. We will show that $\mathrm{Comm}(K(S))$ contains $\mathrm{GL}_n(\mathbb{F}_q)$ for every n, which implies that $\mathrm{Comm}(K(S))$ contains every finite group. This completes the proof, since $\mathrm{GL}_d(F)$ does not contain every finite group. (See, for example, [Serre 2007, Theorem 5].)

For each $n \in \mathbb{N}$, embed $\mathrm{GL}_n(\mathbb{F}_q)$ into $\mathrm{Comm}(K(S))$ "diagonally" as follows: Let $V = \bigoplus_{k=-\infty}^{\infty} \mathbb{F}_q$, and for each $\ell \in \mathbb{Z}$ define a subgroup $V_\ell \leq V$ by $V_\ell = \bigoplus_{k=n\ell}^{n(\ell+1)-1} \mathbb{F}_q$. Given any automorphism $\phi \in \mathrm{GL}_n(\mathbb{F}_q)$, define an automorphism $\Phi \in \mathrm{Aut}(V)$ piecewise by $\Phi\big|_{V_\ell} = \phi$. In this way every nontrivial element of $\mathrm{GL}_n(\mathbb{F}_q)$ determines a nontrivial commensuration of $V \cong K(S)$. $\square$

In particular, Proposition 5.2 implies that Theorem 1.2 does not hold when $\mathbb{Q}$ is replaced by a global field of positive characteristic.

Example 5.3 (lamplighter group). Consider the algebraic group

$$\boldsymbol{B}_2 = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \,\middle|\, xy = 1 \right\} \subseteq \mathrm{GL}_2 .$$

Set $q = 2$. The S-arithmetic group $\boldsymbol{B}_2(\mathbb{F}_2[t, t^{-1}])$ is isomorphic to the (restricted) wreath product $\mathbb{F}_2^2 \wr \mathbb{Z}$, which is an index-2 subgroup of the *lamplighter group* $\mathbb{F}_2 \wr \mathbb{Z}$. The lamplighter group is isomorphic to the semidirect product

$$\Bigl(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\Bigr) \rtimes \mathbb{Z},$$

where the $\mathbb{Z}$ acts by permutation of the $\mathbb{Z}/2\mathbb{Z}$ factors through the usual left action on the index set.

The abstract commensurator of $\mathbb{F}_2 \wr \mathbb{Z}$ is fairly complicated, and has not been well studied. See Section 6 for a more detailed discussion of $\mathrm{Comm}(\mathbb{F}_2 \wr \mathbb{Z})$. For now we use the fact that $\mathrm{Comm}(\mathbb{F}_2 \wr \mathbb{Z})$ contains the direct limit

$$\varinjlim_{n \in \mathbb{N}} \mathrm{Aut}(\mathbb{F}_2^n),$$

where the maps are the diagonal inclusions of $\mathrm{Aut}(\mathbb{F}_2^n)$ into $\mathrm{Aut}(\mathbb{F}_2^m)$ whenever $n \mid m$. It follows now as in Proposition 5.2 that $\mathrm{Comm}(\boldsymbol{B}_2(\mathbb{F}_2[t, t^{-1}]))$ is not a linear group over any field. This shows that Theorem 1.2 does not apply in positive characteristic even in the presence of a nontrivial action by a torus.

6. Commensurations of the lamplighter group

Define K to be the direct product

$$K := \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

The group of integers $\mathbb{Z}$ acts on itself by left-translation, inducing an action on K by permutation of indices. The *lamplighter group*, which we will denote by Γ throughout this section, is the semidirect product $\Gamma = K \rtimes \mathbb{Z}$. The goal of this section is to show that $\mathrm{Comm}(\Gamma)$ admits the following decomposition.

Theorem 1.4. *Using the definitions in* (6) *and* (7) *below*, *there is an isomorphism*

$$\mathrm{Comm}(\Gamma) \cong (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}). \tag{5}$$

See [Houghton 1962] for an analogous description of automorphism groups of unrestricted wreath products.

Let $e_i \in \Gamma$ be the element of the direct sum subgroup which is nontrivial only at the i-th index and let $t \in \Gamma$ be a generator for $\mathbb{Z}$. By definition we have the relation $t^m e_i t^{-m} = e_{i+m}$. Then Γ is generated by the set $\{e_0, t\}$ and has the presentation

$$\Gamma = \langle e_0, t \mid e_0^2 = 1 \text{ and } [t^k e_0 t^{-k}, t^\ell e_0 t^{-\ell}] = 1 \text{ for all } k, \ell \in \mathbb{Z} \rangle.$$

Lemma 6.1. *The quotient map $\Gamma \to \Gamma/K$ induces a surjective homomorphism $\Theta : \mathrm{Comm}(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$.*

Proof. The subgroup $K \leq \Gamma$ is equal to the set of torsion elements of Γ, and is therefore strongly commensuristic. It follows that there is a homomorphism $\Theta : \mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/K) \cong \mathrm{Comm}(\mathbb{Z})$. The nontrivial automorphism of $\mathbb{Z}$ induces an automorphism, hence a commensuration, of Γ by $t \mapsto t^{-1}$ and $e_i \mapsto e_{-i}$ for each $i \in \mathbb{Z}$. It remains to show that the image of Θ is in $\mathrm{Aut}(\mathbb{Z}) \leq \mathrm{Comm}(\mathbb{Z})$.

Suppose $\phi : \Delta_1 \to \Delta_2$ is a partial automorphism of Γ. In what follows, let $i = 1, 2$. Let $K_i = K \cap \Delta_i$. Choose $g_i \in \Delta_i$ so that its equivalence class $[g_i]$ generates the image of the quotient map $\Delta_i \to \Delta_i/K_i$. Let $G_i = \langle g_i \rangle$. Note that Δ_i admits a product decomposition $\Delta_i = K_i G_i$.

Let m_i be the integer such that $g_i = at^{m_i}$ for some $a \in K_i$. Replacing g_i with its inverse if necessary, assume that $m_i > 0$. Each group G_i naturally acts on K/K_i. Since K/K_i is finite, after replacing g_i with a power if necessary we assume that the action of G_i on K/K_i is trivial for $i = 1, 2$. Our goal is to prove $m_1 = m_2$.

One can check that ϕ induces an isomorphism $[K_1, G_1] \cong [K_2, G_2]$, where $[K_i, G_i]$ is the group generated by commutators of the form $[a, g] := aga^{-1}g^{-1}$ for $a \in K_i$ and $g \in G_i$. (In fact, in this case we know $[K_i, G_i]$ is equal to the *set* of elements of the form $[a, g_i]$, which is equal to $[a, t^{m_i}]$, for some $a \in K_i$. This is helpful in understanding the proof of the claim below.) Since ϕ induces an isomorphism

$$K_1/[K_1, G_1] \cong K_2/[K_2, G_2],$$

the desired result is apparent from the following claim.

Claim. *There are isomorphisms $K_i/[K_i, G_i] \cong (\mathbb{Z}/2\mathbb{Z})^{m_i}$ for $i = 1, 2$.*

Proof of claim. Let $H_{m_i} \leq K$ be the subgroup generated by the set $\{e_0, e_1, \ldots, e_{m_i-1}\}$. Clearly H_{m_i} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{m_i}$. Let $P_i = K_i \cap H_{m_i}$, and let $Q_i \leq H_{m_i}$ be a complement to P_i so that $H_{m_i} = P_i \oplus Q_i$. Consider the subset $S_i \subseteq K_i$ defined by

$$S_i = \{g \in K \mid g = p[q, g_i] \text{ for some } p \in P_i \text{ and } q \in Q_i\}.$$

The condition that G_i act trivially on K/K_i ensures that $[a, g_i] \in K_i$ for any $a \in K$, and so $S_i \subseteq K_i$. By construction S_i is in bijection with H_{m_i}, hence has cardinality 2^{m_i}. Consider the map of sets $\rho_i : S_i \to K_i/[K_i, G_i]$ sending an element to its equivalence class. Since $[K_i, G_i]$ consists of elements of the form $[a, g_i]$ for some $a \in K_i$, it is not hard to see from the construction of S_i that ρ_i is injective. We leave as an exercise to check that ρ_i is surjective. □

Let Θ be the surjection of Lemma 6.1. The short exact sequence

$$1 \to \ker(\Theta) \to \mathrm{Comm}(\Gamma) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

splits, so that $\mathrm{Comm}(\Gamma) \cong \ker(\Theta) \rtimes (\mathbb{Z}/2\mathbb{Z})$. Since K is strongly commensuristic, there is a natural map $\Phi : \ker(\Theta) \to \mathrm{Comm}(K)$. We describe first the kernel of Φ and then the image of Φ.

If G is a group and A is a left G-module, then $\tau : G \to A$ is a *derivation* if $\tau(g_1 g_2) = \tau(g_1) + g_1 \cdot \tau(g_2)$ for all $g_1, g_2 \in G$. The set of derivations from G to A forms an abelian group, denoted $\mathrm{Der}(G, A)$. A *virtual derivation* from G to A is a derivation from a finite-index subgroup of G to A. Two virtual derivations are *equivalent* if they agree on a finite-index subgroup of G. The set of equivalence classes of virtual derivations forms a group

$$\mathrm{VDer}(G, A) := \varinjlim_{[G:H]<\infty} \mathrm{Der}(H, A). \tag{6}$$

Lemma 6.2. *There is an isomorphism* $\ker(\Phi) \cong \mathrm{VDer}(\mathbb{Z}, K)$.

Proof. Given any $[\phi] \in \ker(\Phi)$, find $m \in \mathbb{Z}$ so that $\phi(t^m)$ is defined. Then define a map $\tau : m\mathbb{Z} \to K$ by $\tau(t^k) = \phi(t^k)t^{-k}$ for any $k \in m\mathbb{Z}$. It is easy to check that τ is a derivation from $m\mathbb{Z}$ to K, and that the assignment $[\phi] \mapsto \tau$ gives a homomorphism $\mathrm{Comm}(\Gamma) \to \mathrm{VDer}(\mathbb{Z}, K)$. This assignment is clearly injective. On the other hand, if $\tau \in \mathrm{Der}(m\mathbb{Z}, K)$ then setting $\phi(xt^\ell) = x\tau(t^\ell)t^\ell$ for $x \in K$ defines an automorphism ϕ of $\Gamma_m \leq \Gamma$. $\square$

Let $\mathrm{Comm}(K)^{m\mathbb{Z}}$ denote the group of $m\mathbb{Z}$-equivariant commensurations of K. There are natural inclusions $\mathrm{Comm}(K)^{m\mathbb{Z}} \to \mathrm{Comm}(K)^{n\mathbb{Z}}$ whenever $m \mid n$. Define

$$\mathrm{Comm}_\infty(K) := \varinjlim_{m} \mathrm{Comm}(K)^{m\mathbb{Z}}. \tag{7}$$

Lemma 6.3. *There is an isomorphism* $\Phi(\ker(\Theta)) \cong \mathrm{Comm}_\infty(K)$.

Proof. Suppose $\alpha = \Phi([\phi])$ for some partial automorphism ϕ of Γ. Find $m \in \mathbb{Z}$ so that t^m is in the domain of ϕ. Define $x_0 = \phi(t^m)t^{-m} \in K$. Then given any $x \in K$, we have

$$\phi(t^m x t^{-m}) = x_0 t^m \phi(x) t^{-m} x_0^{-1} = t^m \phi(x) t^{-m}.$$

From this we see that any $\alpha \in \Phi(\ker(\Theta))$ is $m\mathbb{Z}$-equivariant for some m.

On the other hand, suppose $\beta : H_1 \to H_2$ is any partial automorphism of K that is $m\mathbb{Z}$-equivariant. Define $\Gamma_m = K \rtimes \langle t^m \rangle$, an index-$m$ subgroup of Γ. The formula $\phi(xt^\ell) = \alpha(x)t^\ell$ defines an automorphism $\phi \in \mathrm{Aut}(\Gamma_m)$. Hence $[\phi]$ is a commensuration of Γ which evidently satisfies $\Phi([\phi]) = \beta$. $\square$

Proof of Theorem 1.4. It is clear from the proof of Lemma 6.3 that the short exact sequence

$$1 \to \mathrm{VDer}(\mathbb{Z}, K) \to \ker(\Theta) \to \mathrm{Comm}_\infty(K) \to 1$$

splits. Putting together Lemmas 6.1, 6.2, and 6.3, we have the semidirect product description of (5):

$$\mathrm{Comm}(\Gamma) = (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

The action of $\mathrm{Comm}_\infty(K)$ on $\mathrm{VDer}(\mathbb{Z}, K)$ is the action by postcomposition. The factor of $\mathbb{Z}/2\mathbb{Z}$ preserves $\mathrm{VDer}(\mathbb{Z}, K)$ and $\mathrm{Comm}_\infty(K)$, and acts on $\mathrm{VDer}(\mathbb{Z}, K)$ by precomposition. □

It is not clear whether a more explicit description of $\mathrm{Comm}_\infty(K)$ exists, but we can describe some subgroups. For example, the "diagonal embedding" construction of Proposition 5.2 shows that $\mathrm{Comm}_\infty(K)$ contains the direct limit

$$\varinjlim_m \mathrm{GL}_m(\mathbb{F}_2),$$

where $\mathrm{GL}_m(\mathbb{F}_2)$ includes into $\mathrm{GL}_n(\mathbb{F}_2)$ diagonally whenever $m \mid n$. So $\mathrm{Comm}_\infty(K)$ contains every finite group.

Note that $\mathrm{VDer}(\mathbb{Z}, K)$ contains every commensuration induced by conjugation by some $a \in K$. However, some elements of $\mathrm{VDer}(\mathbb{Z}, K)$ do not arise in this way. For example, any virtual derivation $\tau : m\mathbb{Z} \to K$ such that $\tau(t^m)$ is nontrivial in an odd number of coordinates cannot arise from conjugation.

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