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A NEW FAMILY OF SIMPLE $\mathfrak{gl}_{2n}(\mathbb{C})$-MODULES

JONATHAN NILSSON

We construct a new family of simple $\mathfrak{gl}_{2n}$-modules which depends on n^2 generic parameters. Each module in the family is isomorphic to the regular $U(\mathfrak{gl}_n)$-module when restricted the $\mathfrak{gl}_n$-subalgebra naturally embedded into the top-left corner.

1. Introduction

Classification of simple modules is one of the first natural questions which arises when studying the representation theory of some (Lie) algebra. Simple modules are, in some sense, "building blocks" for all other modules, and hence understanding simple modules is important. In some cases, for example for finite dimensional associative algebras, classification of simple modules is an easy problem. However, in most of the cases, the problem of classification of all simple modules is very difficult. Thus, if we consider simple, finite dimensional, complex Lie algebras, then the only algebra for which some kind of classification exists is the Lie algebra $\mathfrak{sl}_2$. This was obtained by R. Block [1981]; see also a detailed explanation in [Mazorchuk 2010, Chapter 6]. However, even in this case the "answer" only reduces the problem to classification of equivalence classes of irreducible elements in a certain noncommutative Euclidean ring.

At the moment, the problem of *classification* of simple modules over simple Lie algebras seems too hard. However, because of its importance, the problem of *construction* of new families of modules attracted a lot of attention over the years. The most studied case seem to be the one of the Virasoro Lie algebras, where many different multiparameter families of simple modules were constructed by various authors; see, for example, [Ondrus and Wiesner 2009; Lu et al. 2011; Lu and Zhao 2014 Liu et al. 2015; Mazorchuk and Zhao 2007; 2014; Mazorchuk and Wiesner 2014] and references therein.

In contrast to the Virasoro case, the "easier" case of simple, complex, finite dimensional Lie algebras does not yet have an equally large variety of families of simple modules. So, let $\mathfrak{g}$ be a complex, finite dimensional, simple Lie algebra. Some classes of simple $\mathfrak{g}$-modules are, of course, well understood. For example:

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- simple *finite dimensional* modules are classified already by Cartan [1913];
- simple *highest weight* modules related to a fixed triangular decomposition $\mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+$ of $\mathfrak{g}$ are classified by their highest weights and are extensively studied during last 50 years, see, for example, [Dixmier 1974; Humphreys 2008; Bernstein et al. 1976];
- simple Whittaker modules in the sense of [Kostant 1978] — see also [Arnal and Pinczon 1974; McDowell 1985; 1993];
- simple *Gelfand–Zeitlin* modules — see [Drozd et al. 1991; 1994; Mazorchuk 2001; Futorny et al. 2015];
- simple weight modules with *finite dimensional weight spaces* were classified in [Mathieu 2000] extending the previous work in [Fernando 1990; Futorny 1987];
- simple $\mathfrak{g}$-modules which are free of rank one over the universal enveloping algebra of the Cartan subalgebra were constructed and studied in [Nilsson 2015; 2016] (see also [Tan and Zhao 2013; 2015] for similar modules over infinite dimensional Lie algebras).

Some further classes of simple modules can be found in [Futorny et al. 2011]. We note that the largest known family of simple $\mathfrak{gl}_n$-modules is the one of Gelfand–Zeitlin-modules. It depends on $\frac{n(n+1)}{2}$ generic complex parameters, see [Drozd et al. 1991; 1994] for details.

Based on the above, it seems natural to look for new families of simple $\mathfrak{g}$-modules. The present paper contributes a new large family of simple $\mathfrak{gl}_{2n}$-modules. This family is parameterized by invertible $n \times n$ complex matrices. Let $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ be the four Lie subalgebras of $\mathfrak{gl}_{2n}$ of dimension n^2 as indicated in the following figure:

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}.$$

Then $\mathcal{B}$ is nilpotent (and even commutative), and the adjoint action of $\mathcal{B}$ on $\mathfrak{gl}_{2n}/\mathcal{B}$ is nilpotent, so $(\mathcal{B}, \mathfrak{gl}_{2n})$ is a *Whittaker pair* in the sense of [Batra and Mazorchuk 2011]. The original motivation for this paper was an attempt to describe generalized Whittaker modules (i.e., modules on which the action of $\mathcal{B}$ is locally finite) for this Whittaker pair. Our main result can be summarized as follows:

Theorem 1. *For each nondegenerate complex $n \times n$-matrix Q, there exists a simple $\mathfrak{gl}_{2n}$ module M with the following properties*:

- *M has Gelfand–Kirillov dimension n^2.*
- $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}} M$ *is isomorphic to the left regular $U(\mathcal{A})$-module.*

- $\operatorname{Res}_{\mathcal{B}}^{\mathfrak{gl}_{2n}} M$ *is locally finite. In other words*, M *is a generalized Whittaker module for the Whittaker pair* $(\mathcal{B}, \mathfrak{gl}_{2n})$.
- *With respect to a fixed PBW basis in* $U(\mathcal{A})$, *the action of each fixed element from* $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ *can be written explicitly as maps* $U(\mathcal{A}) \to U(\mathcal{A})$ *of degrees* $1, 0, 2, 1$, *respectively.*

Moreover, different matrices Q *give nonisomorphic modules.*

The paper is organized as follows. Section 2 introduces notation and lays down some motivation for the construction of our modules. In the same section, for each nondegenerate complex $n \times n$ matrix Q, we construct an $(\mathcal{A}+\mathcal{B})$-module having the first three properties listed in Theorem 1. We show that there must exist a simple quotient of the corresponding induced $\mathfrak{gl}_{2n}$ module that also has the fourth property. In Section 3 we explicitly construct such a module when Q is the identity matrix I and show that every other module in our family can be obtained by twisting this module by an explicit automorphism. Finally, we give explicit formulas for the $\mathfrak{gl}_{2n}$-action in all cases.

2. Motivation and existence

2.1. ***Setup.*** Let $\mathfrak{g} := \mathfrak{gl}_{2n}(\mathbb{C})$. Unless otherwise stated, all Lie algebras and vector spaces are over the complex numbers. $\mathbb{N}$ denotes the set of nonnegative integers.

First we observe that the subalgebras $\mathcal{A}$ and $\mathcal{D}$ defined above are both isomorphic to $\mathfrak{gl}_n$ while the subalgebras $\mathcal{B}$ and $\mathcal{C}$ are commutative. Let $e_{i,j}$ be the $2n \times 2n$-matrix with a single 1 in position (i, j) and zeros elsewhere. By convention, most indices i, j, etc. can be assumed to lie between 1 and n; in particular our canonical basis for $\mathfrak{gl}_{2n}$ will be written

$$\bigcup_{1 \le i,j \le n} \{e_{i,j}, e_{n+i,j}, e_{i,n+j}, e_{n+i,n+j}\}.$$

We denote the identity matrix by I, its size (n or $2n$) should be apparent by the context. The transpose of a matrix A is denoted A^T and if A is invertible we abbreviate $(A^{-1})^T$ by A^{-T}.

We also recall how to construct *twisted modules*. For every Lie algebra automorphism $\varphi \in \operatorname{Aut}(\mathfrak{g})$ we have a twisting functor $F_\varphi : \mathfrak{g}\text{-mod} \to \mathfrak{g}\text{-mod}$ which is an auto-equivalence. It maps a module M to ${}^{\varphi}M$ which is isomorphic to M as a vector space but has modified action: $x \bullet v := \varphi(x) \cdot v$ for all $x \in \mathfrak{g}$ and $v \in {}^{\varphi}M$.

2.2. ***Existence of simple generalized Whittaker modules for*** $\mathfrak{gl}_{2n}$. Following the idea in [Kostant 1978], we try to construct some modules on which the action of $\mathcal{B}$ is locally finite.

Fix Lie algebra homomorphisms $\lambda_A : \mathcal{A} \to \mathbb{C}$ and $\lambda_D : \mathcal{D} \to \mathbb{C}$. Let $\mathbb{C}_{\lambda_A\lambda_D}$ be the one dimensional $(\mathcal{A}+\mathcal{C}+\mathcal{D})$-module where $\mathcal{A}$ acts by λ_A, $\mathcal{D}$ acts by λ_D and $\mathcal{C}$ acts trivially. Now define a generalized Verma module

$$M_{\lambda_A\lambda_D} := U(\mathfrak{gl}_{2n}) \otimes_{U(\mathcal{A}+\mathcal{C}+\mathcal{D})} \mathbb{C}_{\lambda_A\lambda_A}.$$

Denote by $M^*_{\lambda_A\lambda_D}$ the full dual of $M_{\lambda_A\lambda_D}$. This is a $\mathfrak{gl}_{2n}$ module where the action is given by $(x \cdot f)(m) = -f(x \cdot m)$ as usual.

Proposition 2. *For every $\theta : \mathcal{B} \to \mathbb{C}$, there is a unique (up to multiple) eigenvector w in $M^*_{\lambda_A\lambda_D}$ with eigenvalue θ for $\mathcal{B}$.*

Proof. Note that $M_{\lambda_A\lambda_D} \simeq U(\mathcal{B})$ as a left and right $U(\mathcal{B})$-module. Let $\mathbb{C}(\theta)$ be the 1-dimensional $\mathcal{B}$-module where the action is given by θ. By the tensor-hom adjunction we have

$$\begin{aligned}
\mathrm{Hom}_{U(\mathcal{B})}(\mathbb{C}(\theta), M^*_{\lambda_A\lambda_D}) &= \mathrm{Hom}_{U(\mathcal{B})}\big(\mathbb{C}(\theta), \mathrm{Hom}_{\mathbb{C}}(M_{\lambda_A\lambda_D}, \mathbb{C})\big) \\
&\simeq \mathrm{Hom}_{U(\mathcal{B})}\big(\mathbb{C}(\theta), \mathrm{Hom}_{\mathbb{C}}(U(\mathcal{B}), \mathbb{C})\big) \\
&\simeq \mathrm{Hom}_{\mathbb{C}}\big(U(\mathcal{B}) \otimes_{U(\mathcal{B})} \mathbb{C}(\theta), \mathbb{C}\big) \\
&\simeq \mathrm{Hom}_{\mathbb{C}}(\mathbb{C}(\theta), \mathbb{C}) \simeq \mathbb{C}.
\end{aligned}$$

Thus there is a unique 1-dimensional subspace of $M^*_{\lambda_A\lambda_D}$ isomorphic to $\mathbb{C}(\theta)$ in $\mathcal{B}$-mod, which is equivalent to the statement of the proposition. □

The submodule generated by such an eigenvector must be simple (see [Batra and Mazorchuk 2011]), so we get the following result.

Corollary 3. *For the pair $(\mathcal{B}, \mathfrak{gl}_{2n})$, there exist simple generalized Whittaker modules and they can be realized as simple submodules in the dual of the generalized Verma module $M^*_{\lambda_A\lambda_D}$.*

The drawback with this approach in our case is that it is difficult to say anything more explicit about the resulting modules as $M^*_{\lambda_A\lambda_D}$ is very big and inconvenient to work in.

2.3. *An $(\mathcal{A}+\mathcal{B})$-module.*

2.3.1. *Construction and a formula for the action.* We now turn to a more explicit construction. Note that $\mathcal{B}$ is commutative. Let $Q = (q_{ij})$ be a nonsingular $n \times n$ matrix and define L_Q to be the 1-dimensional $U(\mathcal{B})$-module with generator v where the action of $\mathcal{B}$ is given by Q:

$$e_{i,n+j} \cdot v := q_{i,j} v \quad 1 \le i, j \le n.$$

Define an induced module

$$M_Q := \mathrm{Ind}_{\mathcal{B}}^{\mathcal{A}+\mathcal{B}} L_Q = U(\mathcal{A}+\mathcal{B}) \otimes_{U(\mathcal{B})} L_Q.$$

Then M_Q is clearly isomorphic to $U(\mathcal{A})$ as a left $\mathcal{A}$-module, and for $a \in U(\mathcal{A})$ we shall write just av or just a for $a \otimes v$. To explicitly see how $\mathcal{B}$ acts on M_Q, we introduce some more notation. Consider $U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$ as a tensor product in the category of unital associative algebras. This becomes an infinite dimensional Lie algebra under the commutator bracket. Note that $U(\mathcal{A}) \otimes \mathcal{A} \simeq \mathrm{Mat}_{n\times n}(U(\mathcal{A}))$ in a natural way and we shall even extend the trace function to $U(\mathcal{A}) \otimes \mathcal{A}$ by defining $\mathrm{tr}(a \otimes B) := a\ \mathrm{tr}(B)$. Note also that $\mathcal{A}$ embeds into $U(\mathcal{A}) \otimes \mathcal{A}$ (both as an associative algebra and as a Lie algebra) by the map $A \mapsto 1 \otimes A$, and we shall sometimes need to identify elements of $\mathcal{A}$ with their images under this map. To resolve some ambiguity in our notation, for $A, B \in \mathcal{A}$ we shall write AB for the product in $U(\mathcal{A})$ and $A.B$ for the product in the associative algebra $\mathcal{A}$ or $U(\mathcal{A}) \otimes \mathcal{A}$.

Let $\psi' : \mathcal{A} \to U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$ be the Lie algebra homomorphism defined by

$$\psi' : A \mapsto A \otimes I - 1 \otimes A^T.$$

This extends to an algebra homomorphism $\psi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes_{\mathbb{C}} \mathcal{A}$.

Lemma 4. *The action of $\mathcal{B}$ on M_Q is given by*

$$\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av = \mathrm{tr}(\psi(a).Q.B^T)v.$$

Proof. This follows by induction on the degree of a as follows. The lemma clearly holds for $a = 1$ by the definition of the action of $\mathcal{B}$ on L_Q: we have $\mathrm{tr}(Q.B^T) = \sum_{ij} q_{ij} b_{ij}$. Suppose the lemma holds for all monomials a of a fixed degree (with respect to any fixed PBW basis). We then have

$$\begin{aligned}
\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} (Aa)v &= A \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av + \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix} \right] av \\
&= A \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} av - \begin{pmatrix} 0 & A.B \\ 0 & 0 \end{pmatrix} av \\
&= A\ \mathrm{tr}(\psi(a).Q.B^T)v - \mathrm{tr}(\psi(a).Q.(A.B)^T)v \\
&= \mathrm{tr}((A \otimes I).\psi(a).Q.B^T)v - \mathrm{tr}(A^T.\psi(a).Q.B^T)v \\
&= \mathrm{tr}\big(((A \otimes I) - 1 \otimes A^T).\psi(a).Q.B^T\big)v \\
&= \mathrm{tr}(\psi(A).\psi(a).Q.B^T)v \\
&= \mathrm{tr}(\psi(Aa).Q.B^T)v.
\end{aligned}$$

This shows that the lemma holds for all monomials in $U(\mathcal{A})$ by induction. Since ψ is linear, it holds for all of $U(\mathcal{A})$. □

2.3.2. *Proof of simplicity.* We proceed to prove that M_Q is simple by first proving it for $Q = I$.

Lemma 5. *The following relations hold in $U(\mathcal{A}+\mathcal{B})$.*

$$[e_{j,k+n}, e_{i,j}^m] = \begin{cases} -m\, e_{i,j}^{m-1} e_{i,k+n} & \text{for } i \neq j, \\ ((e_{i,j}-1)^m - e_{i,j}^m)e_{i,k+n} & \text{for } i = j. \end{cases}$$

Proof. This follows easily by induction on m. □

Fix a PBW basis of $U(\mathcal{A})$ of form

$$\{e_{11}^{l_{11}} e_{12}^{l_{12}} \cdots e_{1n}^{l_{1n}} e_{21}^{l_{21}} \cdots\cdots e_{n1}^{l_{n1}} \cdots e_{nn}^{l_{nn}} \mid l_{ij} \in \mathbb{N}\},$$

Then $U(\mathcal{A}) \simeq M_I$ has a filtration:

$$M_I^{(0)} \subset M_I^{(1)} \subset M_I^{(2)} \subset \cdots$$

where $M_I^{(m)}$ is the span of all monomials f with $\deg f := \sum_{ij} l_{ij} \leq m$.

Lemma 6. *For each $1 \leq j, k \leq n$, the element $(e_{j,k+n} - \delta_{j,k}) \in U(\mathcal{B})$ has degree -1 with respect to the filtration of M_I. Moreover, the action on an arbitrary monomial in $M_I^{(d)}$ is given by*

$$(e_{j,k+n} - \delta_{j,k}) \cdot e_{11}^{l_{11}} \cdots e_{kj}^{l_{kj}} \cdots e_{nn}^{l_{nn}} = -l_{kj}\, e_{11}^{l_{11}} \cdots e_{kj}^{l_{kj}-1} \cdots e_{nn}^{l_{nn}} \mod M_I^{(d-2)}.$$

Proof. We have

$$(e_{j,k+n} - \delta_{j,k}) \cdot f = f(e_{j,k+n} - \delta_{j,k}) + [e_{j,k+n} - \delta_{j,k}, f] = [e_{j,k+n}, f],$$

so the fact that $(e_{j,k+n} - \delta_{j,k})$ has degree ≤ -1 follows from the previous lemma and the fact that $\mathrm{ad}_{e_{j,k+n}}$ is a derivation.

For the second, more precise, statement, let f be an arbitrary monomial of degree d. For each i let P_i, Q_i be the monomial factors of f such that $f = P_i e_{ij}^{l_{ij}} Q_i$ and $e_{ij} \not| P_i, Q_i$. We now calculate

$$\begin{aligned}(e_{j,k+n} - \delta_{j,k}) \cdot f = [e_{j,k+n}, f] &= \sum_i P_i [e_{j,k+n}, e_{ij}^{l_{ij}}] Q_i \\ &= P_j((e_{jj}-1)^{l_{jj}} - e_{jj}^{l_{jj}}) e_{j,k+n} \cdot Q_j + \sum_{i \neq j} -l_{ij}\, P_i e_{ij}^{l_{ij}-1} e_{i,k+n} \cdot Q_i.\end{aligned}$$

By writing

$$e_{i,k+n} = (e_{i,k+n} - \delta_{ik}) + \delta_{ik},$$

and using the fact that the first term has negative degree, we see that

$$\begin{aligned}(e_{j,k+n}-\delta_{j,k})\cdot f &= \delta_{j,k}P_j((e_{jj}-1)^{l_{jj}}-e_{jj}^{l_{jj}})Q_j \\ &\quad +\sum_{i\neq j}-\delta_{ik}l_{ij}\,P_ie_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -\delta_{j,k}l_{jj}\,P_je_{ij}^{l_{ij}-1}Q_j+\sum_{i\neq j}-\delta_{ik}l_{ij}\,P_ie_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -\sum_i \delta_{ik}l_{ij}\,P_ie_{ij}^{l_{ij}-1}Q_i \mod M_I^{(d-2)} \\ &= -l_{kj}\,P_ke_{kj}^{l_{kj}-1}Q_k \mod M_I^{(d-2)}.\end{aligned}$$

The lemma follows. □

Corollary 7. *For each $1\le i,j\le n$, the action of $(e_{i,n+j}-\delta_{i,j})$ on M_I is surjective. Its kernel is spanned by all monomials not divisible by $e_{j,i}$.*

Proposition 8. *The module M_I is simple in $U(\mathcal{A}+\mathcal{B})$-mod.*

Proof. It suffices to show that any $f\in M_I$ can be reduced to $1\in M_I^0$ via the $\mathcal{B}$-action. Fix $f\in M_I$ and let $p\in M_I^{(d)}$ be a nonzero monomial occurring in f with maximal degree d. If $p=\prod_{ij}e_{ij}^{l_{ij}}$ (in the PBW order), it is clear by the previous lemma that

$$B_p:=\prod_{ij}(e_{j,n+i}-\delta_{ij})^{l_{ij}}\in U(\mathcal{B})$$

maps p to a nonzero constant. By the maximality of d, B_p annihilates all other monomials occurring in f so in fact $B_p\cdot f\in M_I^{(0)}$ is a nonzero constant as desired. □

Corollary 9. *The module M_Q is simple if and only if Q is nonsingular.*

Proof. For each nonsingular $S\in\mathcal{A}$, define $\varphi_S:\mathcal{A}+\mathcal{B}\to\mathcal{A}+\mathcal{B}$ by

$$\varphi_S:\begin{pmatrix}A & B\\ 0 & 0\end{pmatrix}\mapsto\begin{pmatrix}A & B.S^{-1}\\ 0 & 0\end{pmatrix}.$$

It is easy to verify that φ_S is a Lie algebra automorphism and that $\varphi_S\circ\varphi_T=\varphi_{ST}$. It is also clear that the twisted module ${}^{\varphi_{Q^{-T}}}M_I$ is isomorphic to M_Q. Since M_I is simple by Proposition 8, and since twisting by automorphisms defines an auto-equivalence on $\mathfrak{gl}_{2n}$-Mod, M_Q is also simple for nonsingular Q.

Conversely, assume that Q is singular and let A be a nonzero matrix such that $Q^TA=0$. We shall show that $U(\mathcal{A})Av$ is a proper $(\mathcal{A}+\mathcal{B})$-submodule of M_Q. The subspace $U(\mathcal{A})Av$ is clearly $\mathcal{A}$-stable. For $a \in U(\mathcal{A})$ we compute

$$\begin{aligned}\begin{pmatrix}0 & B\\ 0 & 0\end{pmatrix}\cdot aAv &= \mathrm{tr}(\psi(aA).Q.B^T)v\\ &= \mathrm{tr}(\psi(a).\psi(A).Q.B^T)v = \mathrm{tr}(\psi(a).(A\otimes I - 1\otimes A^T).Q.B^T)v\\ &= \mathrm{tr}(Q.B^T.\psi(a).(A\otimes I))v - \mathrm{tr}(\psi(a).A^T.Q.B^T)v\\ &= \mathrm{tr}(Q.B^T.\psi(a))Av - \mathrm{tr}(\psi(a).(Q^T.A)^T.B^T)v\\ &= \mathrm{tr}(Q.B^T.\psi(a))Av.\end{aligned}$$

Thus $U(\mathcal{A})Av$ is also $\mathcal{B}$-stable, and is thus a proper submodule of M_Q. □

2.3.3. *Injectivity and an existence theorem.* Our next goal is to prove that for most Q's, the module M_Q is injective when restricted to $U(\mathcal{B})$. We begin by recalling a result about injective envelopes for the trivial module over polynomial rings. For a proof, see for example [Lam 1999, §3J].

Lemma 10. *Let k be a field, let $R = k[x_1,\dots x_n]$ and let L be the trivial R-module. Let E be the R-module $k[x_1^{-1},\dots x_n^{-1}]$, where x_i acts by*

$$x_i\cdot(x_1^{-k_1}\cdots x_n^{-k_n}) = \begin{cases} x_1^{-k_1}\cdots x_i^{-k_i+1}\cdots x_n^{-k_n} & \text{if } k_i>0,\\ 0 & \text{otherwise.}\end{cases}$$

Then $E = E(L)$ is the injective envelope of L.

By twisting E by automorphisms we obtain injective envelopes of all 1-dimensional R-modules as follows:

Corollary 11. *With notation as in the previous lemma, for scalars $q_i \in k$, let $L_{q_1,\dots,q_n}$ be the 1-dimensional R-module with action $x_i\cdot v = q_i v$. Then $E(L_{q_1,\dots,q_n}) \simeq {}^{\varphi}E(L)$ where φ is the R-automorphism mapping $x_i \mapsto x_i - q_i$.*

Proof. We have $L_{q_1,\dots,q_n} \simeq {}^{\varphi}L$ and since twisting by an automorphism is an autoequivalence on R-mod, the corollary follows. □

Proposition 12. *For nonsingular matrices Q, the module $\mathrm{Res}^{U(\mathcal{A}+\mathcal{B})}_{U(\mathcal{B})} M_Q$ is injective.*

Proof. Let $I(L_Q)$ be the injective envelope of L_Q. Applying the exact functor $\mathrm{Hom}_{\mathcal{B}}(-, I(L_Q))$ to the exact sequence

$$0 \to L_Q \to M_Q \to \mathrm{Coker} \to 0$$

we obtain the exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{B}}(\mathrm{Coker}, I(L_Q)) \to \mathrm{Hom}_{\mathcal{B}}(M_Q, I(L_Q)) \to \mathrm{Hom}_{\mathcal{B}}(L_Q, I(L_Q)) \to 0.$$

Hence the morphism $L_Q \to I(L_Q)$ mapping L_Q into its injective envelope is the image of some morphism $f : M_Q \to I(L_Q)$. Since f is nonzero on $\operatorname{span}(v) = \operatorname{soc}(M_Q)$, f is injective. Moreover, for all $k \in \mathbb{N}$ we have

$$\dim \operatorname{soc}_k(M_Q) = \binom{n^2+k-2}{k-1} = \dim \operatorname{soc}_k(I(L_Q)),$$

which shows that f is surjective. This shows that f is an isomorphism and in particular that M_Q is the injective envelope of L_Q. □

Remark 13. Indecomposable injectives over noetherian rings R correspond to $\operatorname{Spec}(R)$ via $\mathfrak{p} \mapsto$injective envelope of $(R/\mathfrak{p})$. Moreover $L_Q = U(\mathcal{B})/\mathfrak{m}$ where $\mathfrak{m}$ is the maximal ideal generated by $(e_{i,n+j} - q_{i,j})$, so if M_Q is injective, it must be the injective envelope of $U(\mathcal{B})/\mathfrak{m}$.

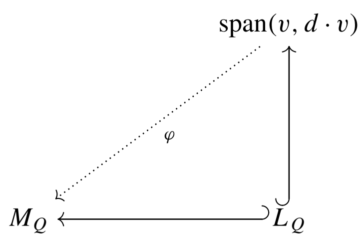
Theorem 14. *For each nonsingular matrix $n \times n$-matrix Q there exists a $\mathfrak{gl}_{2n}$-module M such that*:

- *M is generated by a single $\mathcal{B}$-eigenvector with eigenvalues corresponding to the entries of Q.*
- $\operatorname{Res}^{U(\mathfrak{gl}_{2n})}_{U(\mathcal{B})} M \simeq U(\mathcal{A}) \simeq U(\mathfrak{gl}_n)$.

Proof. As we've seen before, we take L_Q as the 1-dimensional $\mathcal{B}$-module corresponding to Q and we let $M_Q = U(\mathcal{A}+\mathcal{B}) \otimes_{U(\mathcal{B})} L_Q$. Then M_Q is injective in $\mathcal{B}$-mod. Next we define

$$W := U(\mathcal{A}+\mathcal{B}+\mathcal{D}) \otimes_{U(\mathcal{A}+\mathcal{B})} M_Q.$$

Fixing $d \in \mathcal{D}$ we note that $\operatorname{span}(v, d\cdot v)$ is a 2-dimensional $\mathcal{B}$-submodule of W, and moreover it is a nonsplit self-extension of L_Q with itself. Now by the injectivity of M_Q there exists a morphism φ such that the following diagram commutes in $\mathcal{B}$-mod:



Thus there exists $a_d \cdot v \in \operatorname{soc}_2(M_Q) = \mathcal{A}\cdot v$ such that $a_d \cdot v - d \cdot v$ spans a 1-dimensional B-submodule S_d of W. The module $W' := W/\sum_{d\in\mathcal{D}} U(\mathcal{A}+\mathcal{B}+\mathcal{D})S_d$ is then isomorphic to M_Q when restricted to $U(\mathcal{A}+\mathcal{B})$.

Next, let $W'' := U(\mathcal{A}+\mathcal{B}+\mathcal{C}+\mathcal{D}) \otimes_{U(\mathcal{A}+\mathcal{B}+\mathcal{D})} W'$. For a fixed $c \in \mathcal{C}$ we have a $\mathcal{B}$-submodule $\mathcal{B}^2(c\cdot v)$ with simple top and simple socle, both isomorphic to L_Q. By similar arguments, there exists $x \in \operatorname{soc}_3(M_Q) = \mathcal{A}^2 \cdot v$ such that $x - c\cdot v$ spans a $\mathcal{B}$-submodule of W''. Forming the quotient of all these submodules we get the module required by the theorem. □

In the next section we shall give explicit formulas for the elements a_d and x of the proof above in order to write down the action on the simple $\mathfrak{gl}_{2n}$-modules explicitly.

3. Explicit formulas for the $\mathfrak{gl}_{2n}$-modules

3.1. ***Preliminaries.*** The following formula will be particularly useful for $m = 2$.

Lemma 15. *Let* $F := (e_{j,i})_{i,j} = \sum_{i,j} e_{j,i} \otimes e_{i,j} \in U(\mathcal{A}) \otimes \mathcal{A}$. *For any* $A, B \in \mathfrak{gl}_n$ *and for all* $m \in \mathbb{N}$ *we have*

$$[A, \operatorname{tr}(B.F^m)] = \operatorname{tr}([A, B].F^m)$$

in $U(\mathfrak{gl}_n)$.

Proof. We proceed by induction on m. Since $\operatorname{tr}(X.F) = X$, the equality clearly holds for $m = 1$. The equation above is linear in both A and B so it suffices to verify it for $A = e_{ij}$, $B = e_{kl}$. Note that we explicitly have

$$\operatorname{tr}(e_{ij}.F^{m+1}) = \sum_{1\leq r_1,\ldots,r_m\leq n} e_{ir_1}e_{r_1r_2}\cdots e_{r_mj}.$$

Assume that the equality holds for some fixed m. We now compute

$$\begin{aligned}
[e_{ij}, \operatorname{tr}(e_{kl}.F^{m+1})] &= \left[e_{ij}, \sum_{r_1,\ldots,r_m} e_{kr_1}e_{r_1r_2}\cdots e_{r_ml}\right] \\
&= \sum_{r_1,\ldots,r_m} \big([e_{ij}, e_{kr_1}]e_{r_1r_2}\cdots e_{r_ml} + e_{kr_1}[e_{ij}, e_{r_1r_2}\cdots e_{r_ml}]\big) \\
&= \sum_{r_1,\ldots,r_m} (\delta_{jk}e_{ir_1} - \delta_{r_1i}e_{kj})e_{r_1r_2}\cdots e_{r_ml} \\
&\quad + \sum_{r_1} e_{kr_1}\left[e_{ij}, \sum_{r_2,\ldots,r_m} e_{r_1r_2}\cdots e_{r_ml}\right] \\
&= \delta_{jk}\operatorname{tr}(e_{il}.F^{m+1}) - e_{kj}\sum_{r_2,\ldots,r_m} e_{ir_2}\cdots e_{r_ml} \\
&\quad + \sum_{r_1} e_{kr_1}[e_{ij}, \operatorname{tr}(e_{r_1l}.F^m)] \\
&= \delta_{jk}\operatorname{tr}(e_{il}.F^{m+1}) - e_{kj}\operatorname{tr}(e_{il}.F^m) + \sum_{r_1} e_{kr_1}\operatorname{tr}([e_{ij}, e_{r_1l}].F^m) \\
&= \delta_{jk}\operatorname{tr}(e_{il}.F^{m+1}) - e_{kj}\operatorname{tr}(e_{il}.F^m) \\
&\quad + \sum_{r_1} e_{kr_1}\big(\delta_{jr_1}\operatorname{tr}(e_{il}.F^m) - \delta_{il}\operatorname{tr}(e_{r_1j}.F^m)\big)
\end{aligned}$$

$$
\begin{aligned}
&= \delta_{jk}\operatorname{tr}(e_{il}.F^{m+1}) - e_{kj}\operatorname{tr}(e_{il}.F^m) + e_{kj}\operatorname{tr}(e_{il}.F^m)\\
&\quad - \delta_{il}\sum_{r_1} e_{kr_1}\operatorname{tr}(e_{r_1 j}.F^m)\\
&= \delta_{jk}\operatorname{tr}(e_{il}.F^{m+1}) - \delta_{il}\operatorname{tr}(e_{kj}.F^{m+1})\\
&= \operatorname{tr}([e_{ij}, e_{kl}].F^{m+1}).
\end{aligned}
$$

By induction the lemma holds. □

Remark 16. Fixing B as the identity matrix above we obtain $[A, \operatorname{tr}(F^k)] = 0$ for all A in $\mathfrak{gl}_n$, which shows that $\operatorname{tr}(F^k)$ is central in $U(\mathfrak{gl}_n)$. In fact, $Z(\mathfrak{gl}_n) = \mathbb{C}[\operatorname{tr}(F), \operatorname{tr}(F^2), \dots, \operatorname{tr}(F^n)]$. The elements $\operatorname{tr}(F^k)$ are called Gelfand invariants.

3.2. ***The main result.*** We are now ready to state our main result. Define

$$\varphi' : \mathcal{A} \to U(\mathcal{A}) \otimes \mathcal{A}, \quad A \mapsto A \otimes I + 1 \otimes A.$$

This is a Lie algebra homomorphism and it extends to an algebra homomorphism $\varphi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes \mathcal{A}$. Also recall that we previously have defined

$$\psi : U(\mathcal{A}) \to U(\mathcal{A}) \otimes \mathcal{A}, \quad A \mapsto A \otimes I - 1 \otimes A^T,$$

for $A \in \mathcal{A}$. Using these two homomorphisms we now state our main theorem.

Theorem 17. *Define an action of $\mathfrak{gl}_{2n}$ on $M_I \simeq U(\mathcal{A})$ as follows: for any $a \in U(\mathcal{A})$, let*

$$\text{(1)} \quad \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot a = Aa - aD + \operatorname{tr}(\psi(a).B^T) - \operatorname{tr}(\varphi(a).F^2.C) - \operatorname{tr}(\varphi(a).C)\operatorname{tr}(F).$$

This is a $\mathfrak{gl}_{2n}$-module structure.

Proof. First, for all $X, Y \in \mathfrak{gl}_{2n}$, $A \in \mathcal{A}$ and $a \in U(\mathcal{A})$ we have

$$
\begin{aligned}
X \cdot Y \cdot Aa - Y \cdot X \cdot Aa &= A(X \cdot Y \cdot a) + [XY, A]a - A(Y \cdot X \cdot a) - [YX, A]a\\
&= A(X \cdot Y \cdot a - Y \cdot X \cdot a) + X \cdot [Y, A]a - [X, A] \cdot Ya\\
&\quad - Y \cdot [X, A]a - [Y, A] \cdot Xa\\
&= A \cdot [X, Y]a + [X, [Y, A]]a + [Y, [A, X]]a\\
&= A \cdot [X, Y]a - [A, [X, Y]]a\\
&= [X, Y] \cdot Aa.
\end{aligned}
$$

This shows that it suffices to check that

$$X \cdot Y \cdot 1 - Y \cdot X \cdot 1 = [X, Y] \cdot 1$$

for all $X, Y \in \mathfrak{gl}_{2n}$ in order to prove that (1) gives a module structure.

We first consider the case $Y := A_0 \in \mathcal{A}$. We compute

$$\begin{aligned}
&\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \cdot 1 - \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot 1 \\
&\qquad = AA_0 - A_0 D + \operatorname{tr}\big((A_0 \otimes I - 1 \otimes A_0^T).B^T\big) \\
&\qquad\quad - \operatorname{tr}\big((A_0 \otimes I + 1 \otimes A_0).F^2.C\big) - \operatorname{tr}\big((A_0 \otimes I + 1 \otimes A_0).C\big)\operatorname{tr}(F) \\
&\qquad\quad - \big(A_0 A - A_0 D + A_0 \operatorname{tr}(B^T) - A_0 \operatorname{tr}(F^2.C) - A_0 \operatorname{tr}(C)\operatorname{tr}(F)\big) \\
&\qquad = AA_0 - A_0 D + A_0 \operatorname{tr}(B^T) + \operatorname{tr}(A_0^T.B^T) - A_0 \operatorname{tr}(F^2.C) - \operatorname{tr}(A_0.F^2.C) \\
&\qquad\quad - A_0 \operatorname{tr}(C)\operatorname{tr}(F) - \operatorname{tr}(A_0.C)\operatorname{tr}(F) - A_0 A + A_0 D - A_0 \operatorname{tr}(B^T) \\
&\qquad\quad + A_0 \operatorname{tr}(F^2.C) + A_0 \operatorname{tr}(C)\operatorname{tr}(F) \\
&\qquad = [A, A_0] + \operatorname{tr}(A_0^T.B^T) - \operatorname{tr}(A_0.F^2.C) - \operatorname{tr}(A_0.C)\operatorname{tr}(F) \\
&\qquad = [A, A_0] + \operatorname{tr}((A_0.B)^T) - \operatorname{tr}(F^2.C.A_0) - \operatorname{tr}(C.A_0)\operatorname{tr}(F) \\
&\qquad = \begin{pmatrix} [A, A_0] & A_0.B \\ C.A_0 & 0 \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} A & B \\ C & D \end{pmatrix}, \begin{pmatrix} A_0 & 0 \\ 0 & 0 \end{pmatrix}\right] \cdot 1.
\end{aligned}$$

It remains to check that $X \cdot Y \cdot 1 - Y \cdot X \cdot 1 = [X, Y] \cdot v$ for $X, Y \in \mathcal{B}, \mathcal{C}, \mathcal{D}$. Moreover, since the right side of (1) is linear in A, B, C, and D it suffices to check it for the standard basis elements of $\mathfrak{gl}_{2n}$.

When $X, Y \in \mathcal{B}$ the calculation is easy:

$$\begin{aligned}
&\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot 1 = \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \operatorname{tr}(B') - \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix} \cdot \operatorname{tr}(B) \\
&\qquad = \operatorname{tr}(B)\operatorname{tr}(B') - \operatorname{tr}(B')\operatorname{tr}(B) = 0 = \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & B' \\ 0 & 0 \end{pmatrix}\right] \cdot 1.
\end{aligned}$$

Similarly, for $X, Y \in \mathcal{D}$ we have

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 = -\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot D' + \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix} \cdot D \\
&\qquad = D'D - DD' = [D', D] = \begin{pmatrix} 0 & 0 \\ 0 & [D, D'] \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D' \end{pmatrix}\right] \cdot 1.
\end{aligned}$$

For $X \in \mathcal{B}$, $Y \in \mathcal{D}$ we get

$$\begin{aligned}
&\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot 1 = -\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix} \cdot D - \operatorname{tr}(B^T)\begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 \\
&= -\operatorname{tr}((D \otimes I - 1 \otimes D^T).B^T) + \operatorname{tr}(B^T)D = -D\operatorname{tr}(B^T) + \operatorname{tr}(D^T.B^T) + \operatorname{tr}(B^T)D \\
&\qquad = \operatorname{tr}((D.B)^T) = \begin{pmatrix} 0 & D.B \\ 0 & 0 \end{pmatrix} \cdot 1 = \left[\begin{pmatrix} 0 & B \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix}\right] \cdot 1.
\end{aligned}$$

For $X \in \mathcal{C}$, $Y \in \mathcal{D}$ we apply Lemma 15 for $m = 1, 2$ to obtain

$$\begin{aligned}
&\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot 1 - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} v \cdot 1 \\
&\qquad = - \begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix} \cdot D + \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \cdot \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C)\operatorname{tr}(F)\big) \\
&\qquad = \operatorname{tr}\big((D \otimes I + 1 \otimes D).F^2.C\big) + \operatorname{tr}\big((D \otimes I + 1 \otimes D).C\big)\operatorname{tr}(F) \\
&\qquad\quad - \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C)\operatorname{tr}(F)\big)D \\
&\qquad = D\ \operatorname{tr}(F^2.C) + \operatorname{tr}(D.F^2.C) + D\ \operatorname{tr}(C)\operatorname{tr}(F) + \operatorname{tr}(D.C)\operatorname{tr}(F) \\
&\qquad\quad - \big(\operatorname{tr}(C.F^2) + \operatorname{tr}(C)\operatorname{tr}(F)\big)D \\
&\qquad = [D, \operatorname{tr}(C.F^2)] + \operatorname{tr}(C)[D, \operatorname{tr}(F)] + \operatorname{tr}(D.F^2.C) + \operatorname{tr}(D.C)\operatorname{tr}(F) \\
&\qquad = \operatorname{tr}([D, C].F^2) + \operatorname{tr}(C.D.F^2) + \operatorname{tr}(D.C)\operatorname{tr}(F) \\
&\qquad = \operatorname{tr}(F^2.D.C) + \operatorname{tr}(D.C)\operatorname{tr}(F) \\
&\qquad = \begin{pmatrix} 0 & 0 \\ -D.C & 0 \end{pmatrix} \cdot 1 \\
&\qquad = \left[\begin{pmatrix} 0 & 0 \\ C & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right] \cdot 1.
\end{aligned}$$

Next, for $X \in \mathcal{B}$, $Y \in \mathcal{C}$, take $X = e_{i,n+j}$ and $Y = e_{n+k,l}$. We then have

$$\begin{aligned}
&e_{i,n+j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{i,n+j} \cdot 1 \\
&\quad = -e_{i,n+j} \cdot \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl})\operatorname{tr}(F)\big) - e_{n+k,l} \cdot \operatorname{tr}(e_{ij}^T) \\
&\quad = -e_{i,n+j} \cdot \left(\sum_{r=1}^{n} e_{kr} e_{rl} + \delta_{kl}\operatorname{tr}(F) \right) + \delta_{ij}\big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl})\operatorname{tr}(F)\big) \\
&\quad = -\sum_{r=1}^{n} \operatorname{tr}(\psi(e_{kr}e_{rl}).e_{ji}) - \delta_{kl}\operatorname{tr}(\psi(\operatorname{tr}(F)).e_{ji}) + \delta_{ij}(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl})\operatorname{tr}(F)) \\
&\quad = -\sum_{r=1}^{n} \operatorname{tr}\big((e_{kr} \otimes I - 1 \otimes e_{rk}).(e_{rl} \otimes I - 1 \otimes e_{lr}).e_{ji}\big) \\
&\qquad - \delta_{kl}\operatorname{tr}\big(\big(\operatorname{tr}(F) \otimes I - 1 \otimes \operatorname{tr}(F)\big).e_{ji}\big) + \delta_{ij}\big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl})\operatorname{tr}(F)\big) \\
&\quad = \sum_{r=1}^{n} \big(-\operatorname{tr}(e_{ji}.e_{rk}.e_{lr}) + e_{kr}\operatorname{tr}(e_{ji}.e_{lr}) + e_{rl}\operatorname{tr}(e_{ji}.e_{rk}) - e_{kr}e_{rl}\operatorname{tr}(e_{ji})\big) \\
&\qquad + \delta_{kl}\big(\operatorname{tr}(e_{ji}\operatorname{tr}(F)) - \operatorname{tr}(F)\operatorname{tr}(e_{ji})\big) + \delta_{ij}\big(\operatorname{tr}(e_{kl}F^2) + \operatorname{tr}(e_{kl})\operatorname{tr}(F)\big) \\
&\quad = \big(-\delta_{kl}\operatorname{tr}(e_{ji}\operatorname{tr}(F)) + \delta_{li}e_{kj} + \delta_{jk}e_{il} - \delta_{ji}\operatorname{tr}(e_{kl}F^2)\big) \\
&\qquad + \delta_{kl}\delta_{ji} - \delta_{kl}\delta_{ji}\operatorname{tr}(F) + \delta_{ij}\operatorname{tr}(e_{kl}F^2) + \delta_{ij}\delta_{kl}\operatorname{tr}(F) \\
&\quad = -\delta_{kl}\delta_{ji} + \delta_{li}e_{kj} + \delta_{jk}e_{il} + \delta_{kl}\delta_{ji} \\
&\quad = \delta_{li}e_{kj} + \delta_{jk}e_{il} = \delta_{jk}e_{il} - \delta_{li}e_{n+k,n+j} = [e_{i,n+j}, e_{n+k,l}] \cdot 1.
\end{aligned}$$

It remains only to show that (1) holds for $X, Y \in \mathcal{C}$. Let $X = e_{n+i,j}$ and $Y = e_{n+k,l}$. In this case we have

$$
\begin{aligned}
&e_{n+i,j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{n+i,j} \cdot 1 \\
&\quad = -e_{n+i,j} \cdot \big(\operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\big) + e_{n+k,l} \cdot \big(\operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(e_{ij}) \operatorname{tr}(F)\big) \\
&\quad = -e_{n+i,j} \cdot \left(\sum_{r=1}^{n} e_{kr} e_{rl} + \delta_{kl} \operatorname{tr}(F)\right) + e_{n+k,l} \cdot \left(\sum_{r=1}^{n} e_{ir} e_{rj} + \delta_{ij} \operatorname{tr}(F)\right) \\
&\quad = \sum_{r=1}^{n} \Big(\operatorname{tr}(e_{ij}.e_{kr}.e_{rl}.F^2) + e_{kr} \operatorname{tr}(e_{ij}.e_{rl}.F^2) + e_{rl} \operatorname{tr}(e_{ij}.e_{kr}.F^2) + e_{kr} e_{rl} \operatorname{tr}(e_{ij}.F^2) \\
&\qquad\qquad + \big(\operatorname{tr}(e_{ij}.e_{kr}.e_{rl}) + e_{kr} \operatorname{tr}(e_{ij}.e_{rl}) + e_{rl} \operatorname{tr}(e_{ij}.e_{kr}) + e_{kr} e_{rl} \operatorname{tr}(e_{ij})\big) \operatorname{tr}(F)\Big) \\
&\qquad + \delta_{kl}\Big(\operatorname{tr}(e_{ij}. \operatorname{tr}(F).F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(e_{ij}. \operatorname{tr}(F)) \operatorname{tr}(F) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}) \operatorname{tr}(F)\Big) \\
&\qquad - \sum_{r=1}^{n} \Big(\operatorname{tr}(e_{kl}.e_{ir}.e_{rj}.F^2) + e_{ir} \operatorname{tr}(e_{kl}.e_{rj}.F^2) + e_{rj} \operatorname{tr}(e_{kl}.e_{ir}.F^2) + e_{ir} e_{rj} \operatorname{tr}(e_{kl}.F^2) \\
&\qquad\qquad + \big(\operatorname{tr}(e_{kl}.e_{ir}.e_{rj}) + e_{ir} \operatorname{tr}(e_{kl}.e_{rj}) + e_{rj} \operatorname{tr}(e_{kl}.e_{ir}) + e_{ir} e_{rj} \operatorname{tr}(e_{kl})\big) \operatorname{tr}(F)\Big) \\
&\qquad - \delta_{ij}\Big(\operatorname{tr}(e_{kl}. \operatorname{tr}(F).F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) + \operatorname{tr}(e_{kl}. \operatorname{tr}(F)) \operatorname{tr}(F) + \operatorname{tr}(F) \operatorname{tr}(e_{kl}) \operatorname{tr}(F)\Big) \\
&\quad = n \operatorname{tr}(e_{ij}.e_{kl}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) + \sum_r e_{rl} \operatorname{tr}(e_{ij}.e_{kr}.F^2) + \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(e_{ij}.F^2) \\
&\qquad + \left(n \operatorname{tr}(e_{ij}.e_{kl}) + e_{kj} \operatorname{tr}(e_{il}) + \sum_r e_{rl} \operatorname{tr}(e_{ij}.e_{kr}) + \delta_{ij} \operatorname{tr}(e_{kl}.F^2)\right) \operatorname{tr}(F) \\
&\qquad + \delta_{kl}\big(\operatorname{tr}(e_{ij}.F^2) + \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \delta_{ij} \operatorname{tr}(F) + \delta_{ij} \operatorname{tr}(F) \operatorname{tr}(F)\big) \\
&\qquad - n \operatorname{tr}(e_{kl}.e_{ij}.F^2) - e_{il} \operatorname{tr}(e_{kj}.F^2) - \sum_r e_{rj} \operatorname{tr}(e_{kl}.e_{ir}.F^2) - \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(e_{kl}.F^2) \\
&\qquad + \left(-n \operatorname{tr}(e_{kl}.e_{ij}) - e_{il} \operatorname{tr}(e_{kj}) - \sum_r e_{rj} \operatorname{tr}(e_{kl}.e_{ir}) - \delta_{kl} \operatorname{tr}(e_{ij}.F^2)\right) \operatorname{tr}(F) \\
&\qquad + \delta_{ij}\big(- \operatorname{tr}(e_{kl}.F^2) - \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) - \delta_{kl} \operatorname{tr}(F) - \delta_{kl} \operatorname{tr}(F) \operatorname{tr}(F)\big) \\
&\quad = n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) + \delta_{jk} \sum_r e_{rl} \operatorname{tr}(e_{ir}.F^2) + \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(e_{ij}.F^2) \\
&\qquad + n\delta_{jk}\delta_{il} \operatorname{tr}(F) + e_{kj}\delta_{il} \operatorname{tr}(F) + \delta_{jk}e_{il} \operatorname{tr}(F) + \delta_{ij} \operatorname{tr}(e_{kl}.F^2) \operatorname{tr}(F) \\
&\qquad + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) + \delta_{kl} \operatorname{tr}(F) \operatorname{tr}(e_{ij}.F^2) + \delta_{kl}\delta_{ij} \operatorname{tr}(F) + \delta_{kl}\delta_{ij} \operatorname{tr}(F)^2 \\
&\qquad - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) - e_{il} \operatorname{tr}(e_{kj}.F^2) - \delta_{li} \sum_r e_{rj} \operatorname{tr}(e_{kr}.F^2) - \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(e_{kl}.F^2) \\
&\qquad - n\delta_{li}\delta_{kj} \operatorname{tr}(F) - e_{il}\delta_{kj} \operatorname{tr}(F) - \delta_{li}e_{kj} \operatorname{tr}(F) - \delta_{kl} \operatorname{tr}(e_{ij}.F^2) \operatorname{tr}(F) \\
&\qquad - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - \delta_{ij} \operatorname{tr}(F) \operatorname{tr}(e_{kl}.F^2) - \delta_{ij}\delta_{kl} \operatorname{tr}(F) - \delta_{ij}\delta_{kl} \operatorname{tr}(F)^2 \\
&\quad = \delta_{jk} \sum_r e_{rl} \operatorname{tr}(e_{ir}.F^2) - \delta_{li} \sum_r e_{rj} \operatorname{tr}(e_{kr}.F^2) + e_{kj} \operatorname{tr}(e_{il}.F^2) \\
&\qquad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\qquad - e_{il} \operatorname{tr}(e_{kj}.F^2) + \big[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\big].
\end{aligned}
$$

We proceed to compute $\big[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\big]$ separately.

$$\begin{aligned}
\big[\operatorname{tr}(e_{kl}.F^2), \operatorname{tr}(e_{ij}.F^2)\big] &= \sum_r [e_{kr}e_{rl}, \operatorname{tr}(e_{ij}.F^2)] \\
&= \sum_r \big(e_{kr}[e_{rl}, \operatorname{tr}(e_{ij}.F^2)] + [e_{kr}, \operatorname{tr}(e_{ij}.F^2)]e_{rl}\big) \\
&= \sum_r \big(e_{kr} \operatorname{tr}([e_{rl}, e_{ij}].F^2) + \operatorname{tr}([e_{kr}, e_{ij}].F^2)e_{rl}\big) \\
&= \sum_r \big(\delta_{li} e_{kr} \operatorname{tr}(e_{rj}.F^2) - \delta_{jr} e_{kr} \operatorname{tr}(e_{il}.F^2) + \delta_{ri} \operatorname{tr}(e_{kj}.F^2)e_{rl} - \delta_{kj} \operatorname{tr}(e_{ir}.F^2)e_{rl}\big) \\
&= \delta_{li} \operatorname{tr}(e_{kj}.F^3) - e_{kj} \operatorname{tr}(e_{il}.F^2) + \operatorname{tr}(e_{kj}.F^2)e_{il} - \delta_{kj} \operatorname{tr}(e_{il}.F^3).
\end{aligned}$$

Inserting this into the previous expression gives

$$\begin{aligned}
&e_{n+i,j} \cdot e_{n+k,l} \cdot 1 - e_{n+k,l} \cdot e_{n+i,j} \cdot 1 \\
&= \delta_{jk} \sum_r e_{rl} \operatorname{tr}(e_{ir}.F^2) - \delta_{li} \sum_r e_{rj} \operatorname{tr}(e_{kr}.F^2) \\
&\quad + e_{kj} \operatorname{tr}(e_{il}.F^2) + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad - e_{il} \operatorname{tr}(e_{kj}.F^2) + \delta_{li} \operatorname{tr}(e_{kj}.F^3) - e_{kj} \operatorname{tr}(e_{il}.F^2) + \operatorname{tr}(e_{kj}.F^2)e_{il} - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk} \sum_r \big(\operatorname{tr}(e_{ir}.F^2)e_{rl} + [e_{rl}, \operatorname{tr}(e_{ir}.F^2)]\big) - \delta_{li} \sum_r \big(\operatorname{tr}(e_{kr}.F^2)e_{rj} + [e_{rj}, \operatorname{tr}(e_{kr}.F^2)]\big) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{li} \operatorname{tr}(e_{kj}.F^3) + [\operatorname{tr}(e_{kj}.F^2), e_{il}] - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk} \operatorname{tr}(e_{il}.F^3) + \delta_{jk} \sum_r \operatorname{tr}([e_{rl}, e_{ir}].F^2) - \delta_{li} \operatorname{tr}(e_{kj}.F^3) - \delta_{li} \sum_r \operatorname{tr}([e_{rj}, e_{kr}].F^2) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{li} \operatorname{tr}(e_{kj}.F^3) + \operatorname{tr}([e_{kj}, e_{il}].F^2) - \delta_{kj} \operatorname{tr}(e_{il}.F^3) \\
&= \delta_{jk}\big(\delta_{li} \operatorname{tr}(\operatorname{tr}(F).F^2) - n \operatorname{tr}(e_{il}.F^2)\big) - \delta_{li}\big(\delta_{jk} \operatorname{tr}(\operatorname{tr}(F).F^2) - n \operatorname{tr}(e_{kj}.F^2)\big) \\
&\quad + n\delta_{jk} \operatorname{tr}(e_{il}.F^2) + \delta_{kl} \operatorname{tr}(e_{ij}.F^2) - \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - n\delta_{li} \operatorname{tr}(e_{kj}.F^2) \\
&\quad + \delta_{ij} \operatorname{tr}(e_{kl}.F^2) - \delta_{lk} \operatorname{tr}(e_{ij}.F^2) \\
&= \delta_{jk}\delta_{li} \operatorname{tr}(F^2) - \delta_{li}\delta_{jk} \operatorname{tr}(F^2) = 0 = [e_{n+i,j}, e_{n+k,l}] \cdot 1.
\end{aligned}$$

□

Theorem 18. *Define an action of* $\mathfrak{gl}_{2n}$ *on* $M_Q \simeq U(\mathcal{A})$ *as follows: for any* $a \in U(\mathcal{A})$, *let*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot a = Aa - aD + \operatorname{tr}(\psi(a).Q.B^T) - \operatorname{tr}(\varphi(a).F^2.Q^{-T}.C) - \operatorname{tr}(\varphi(a).Q^{-T}.C) \operatorname{tr}(F).$$

This is a $\mathfrak{gl}_{2n}$*-module structure.*

Proof. For each nonsingular $S \in \mathrm{Mat}_{n\times n}$, define $\varphi_S : \mathfrak{gl}_{2n} \to \mathfrak{gl}_{2n}$ by

$$\varphi_S : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & B.S^{-1} \\ S.C & S.D.S^{-1} \end{pmatrix}.$$

It is easy to verify that φ_S is a Lie algebra automorphism and that $\varphi_S \circ \varphi_T = \varphi_{S.T}$, so the map $\Xi : \mathrm{Mat}_{n\times n}(\mathbb{C})^* \to \mathrm{Aut}(\mathfrak{gl}_{2n})$ with $S \mapsto \varphi_S$ is an injective algebra homomorphism. Let V be the $\mathfrak{gl}_{2n}$ module from Theorem 17. Now the action of $\mathfrak{gl}_{2n}$ on the twisted module $V_Q := {}^{\varphi_{Q^{-T}}}V$ is precisely as in the statement of this theorem. □

The modules V_Q now satisfy the conditions of Theorem 1 in the introduction:

Proof of Theorem 1. The module V_Q is simple since $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}+\mathcal{B}} V_Q \simeq M_Q$ is. That the GK-dimension is n^2 and that $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{A}} V_Q \simeq U(\mathcal{A})$ follows directly from the definition in Theorem 18. Since the linear maps $\mathrm{tr}(\psi(-).B^T) : U(\mathcal{A}) \to U(\mathcal{A})$ never increase the degree of a monomial, the module $\mathrm{Res}^{\mathfrak{gl}_{2n}}_{\mathcal{B}} V_Q$ is locally finite. The fourth point follows from similar arguments: the maps $\mathrm{tr}(\psi(-).F^2.C) : U(\mathcal{A}) \to U(\mathcal{A})$ have degree 2 and the maps $A(-)$ and $(-)D$ clearly have degree 1 (compare with Theorem 18). Finally, we note that any isomorphism $\varphi : V_Q \to V_{Q'}$ must map the generator of V_Q to a multiple of the generator of $V_{Q'}$. But then $q'_{ij}\varphi(1) = e_{i,n+j}\varphi(1) = \varphi(e_{i,n+j} \cdot 1) = q_{ij}\varphi(1)$, showing that $Q = Q'$ whenever such an isomorphism exists. □

3.3. *Alternative formula.* Since the automorphisms φ and ψ themselves are not very explicit, we present another formula for how elements of $\mathfrak{gl}_{2n}$ act on monomials of $U(\mathcal{A})$. We need some more conventions in notation for this formula.

In the argument of the trace functions, any product is by convention to be taken in $\mathrm{Mat}_{n\times n}(U(\mathfrak{gl}_n))$ (in particular we identify $\mathcal{A}$ with $\mathrm{Mat}_{n\times n}(\mathbb{C})$ here). Outside the trace function all products are in $U(\mathfrak{gl}_n)$. When $S \subset \mathbb{Z}$, the product $\prod_{i\in S} A_i$ means that the product is to be taken in the order inherited from $\mathbb{Z}$. For example, $\prod_{i\in\{3,2,5\}} A_i = A_2A_3A_5$. For $S \subset \{1,\dots,k\}$, we denote by S^* the complement $\{1,\dots,k\} \setminus S$ and by $|S|$ the cardinality of S.

Theorem 19. *Let $a = \prod_{i=1}^k A_i$ be a monomial in V_Q (see Theorem 18). The action of $\mathfrak{gl}_{2n}$ on the monomial a can be written explicitly as follows.*

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \cdot \prod_{i=1}^k A_i := A\prod_{i=1}^k A_i - \prod_{i=1}^k A_i(Q^{-T}.D.Q^T) + \sum_{S\subset\{1,\dots,k\}} \left(\prod_{i\in S^*} A_i\right) \times$$
$$\left((-1)^{|S|}\,\mathrm{tr}\left(B^T.\prod_{i\in S} A_i^T.Q\right) - \mathrm{tr}\left(Q^{-T}.C.\prod_{i\in S} A_i.F^2\right) - \mathrm{tr}\left(Q^{-T}.C.\prod_{i\in S} A_i\right)\mathrm{tr}(F)\right).$$

Proof. This follows by induction on k by comparing with the formula in Theorem 18. The verification is omitted here. □

Remark 20. When $n = 1$ the formula above simplifies significantly. In this case $Q = (q)$ is a nonzero scalar and we have $\mathcal{A} \simeq \mathbb{C}$. Letting $x := e_{11}$ we have $U(\mathcal{A}) = \mathbb{C}[x]$ where the $\mathfrak{gl}_2$-action is given by

$$\begin{aligned}
e_{11} \cdot f(x) &= xf(x), \\
e_{22} \cdot f(x) &= -xf(x), \\
e_{12} \cdot f(x) &= qf(x-1), \\
e_{21} \cdot f(x) &= -q^{-1}x(x+1)f(x+1).
\end{aligned}$$

When considered as an $\mathfrak{sl}_2$-module, this is a Whittaker module in Kostant's sense. Writing $\mathfrak{h}$ for the standard Cartan subalgebra of $\mathfrak{sl}_2$, we note that $U(\mathfrak{h})$ acts freely on these modules. The paper [Nilsson 2015] classifies $\mathfrak{sl}_n$-modules which are $U(\mathfrak{h})$-free of rank 1 and indeed, in the notation of [Nilsson 2015] the ($\mathfrak{sl}_2$-)module above would be written $\mathrm{F}_{(q,1)}(M_0')$.

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DERIVED CATEGORIES OF REPRESENTATIONS OF SMALL CATEGORIES OVER COMMUTATIVE NOETHERIAN RINGS

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We study the derived categories of small categories over commutative noetherian rings. Our main result is a parametrization of the localizing subcategories in terms of the spectrum of the ring and the localizing subcategories over residue fields. In the special case of representations of Dynkin quivers over a commutative noetherian ring, we give a complete description of the localizing subcategories of the derived category and a complete description of the thick subcategories of the perfect complexes. We also show that the telescope conjecture holds in this setting and we present some results concerning the telescope conjecture more generally.

1. Introduction

If T is a triangulated category with all coproducts, a localizing subcategory $L \subseteq T$ is a full triangulated subcategory closed under all coproducts in T. Localizing subcategories are so-named because in good cases (the Bousfield localizations) the Verdier quotient functor $T \to T/L$ possesses a right adjoint, i.e., they give rise to localization functors. Understanding the collection of localizing subcategories on a given triangulated category is a challenging and interesting problem which has been completely resolved in only a few classes of examples.

The history of such problems has roots in stable homotopy theory, where one would like to relate two localizations of the p-local stable homotopy category $\mathrm{SH}_{(p)}$: one which has excellent theoretical properties (localization with respect to the homology theory given by the Johnson–Wilson spectrum $\mathrm{E}(n)$) and one which is computable (the telescopic localization). The importance of such questions arose first in [Bousfield 1979] and [Ravenel 1984]. That these two localizations agree is the still-open telescope conjecture. Work on nilpotence closely related to the telescope

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conjecture by Devinatz, Hopkins, and Smith [Devinatz et al. 1988; Hopkins and Smith 1998] has led to the classification of all thick subcategories, i.e., triangulated subcategories closed under direct summands, of $\mathrm{SH}^{\mathrm{fin}}$, the homotopy category of finite spectra. Using similar ideas on the detection of nilpotent maps between objects in $\mathrm{D}(R)$, Neeman [1992] classified the localizing subcategories of $\mathrm{D}(R)$ and the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(R)$ when R is noetherian in terms of $\operatorname{Spec} R$.

Going beyond the example of $\mathrm{D}(R)$ where R is noetherian and commutative seems rather difficult. In terms of classification of thick subcategories of $\mathrm{D}^{\mathrm{perf}}(X)$, when X is a quasicompact and quasiseparated scheme, one has the result of [Thomason 1997], which says that the thick subcategories which are also tensor ideals correspond bijectively to unions of closed subsets of X with quasicompact complement. This kind of result has been taken up by other authors, such as Benson, Carlson, and Rickard [Benson et al. 1997] and Benson, Iyengar, and Krause [Benson et al. 2011], who study the tensor ideals of stable module categories of finite groups. This is part of a generalized framework of studying tensor ideals, pursued by Balmer [2005], Dell'Ambrogio and Stevenson [2013; 2014], and Stevenson [2013; 2014].

In contrast to all that is known about thick subcategories, very little is known about localizing subcategories outside of Neeman's theorem. For instance, one does not know all localizing subcategories of $\mathrm{D}_{\mathrm{qc}}(\mathbb{P}^1_{\mathbb{C}})$. We mention one more example, due to Brüning [2007], who classified the localizing subcategories of $\mathrm{D}(A)$ where A is a hereditary Artin algebra of finite representation type.

Let R be a noetherian commutative ring. We show that in many cases classification of the localizing subcategories of an R-linear triangulated category can be reduced to studying the localizing subcategories of the "fibers" over the residue fields of R.

Let C be a small category, and let $s : \mathcal{L} \to \operatorname{Spec} R$ denote the class constructed fiber by fiber over $\operatorname{Spec} R$, by letting $s^{-1}(p)$, for $p \in \operatorname{Spec} R$, be the class of localizing subcategories of $\mathrm{D}(k(p)C)$. Note that, a priori, the localizing subcategories of $\mathrm{D}(k(p)C)$ only form a proper class, which is the reason for the careful wording above. There is, however, no known example of a compactly generated triangulated category whose collection of localizing subcategories does not form a set. The following result is our first theorem.

Theorem (Corollary 4.3). *Let R be a noetherian commutative ring and C a small category. Then there is an isomorphism of lattices*

$$\{\textit{localizing subcategories } \mathrm{L} \textit{ of } \mathrm{D}(RC)\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{sections } l \textit{ of } \mathcal{L} \xrightarrow{s} \operatorname{Spec} R\},$$

where f takes a localizing subcategory L of $\mathrm{D}(RC)$ to the function $l : \operatorname{Spec} R \to \mathcal{L}$ such that $l(p) = \operatorname{add}(k(p) \otimes_R \mathrm{L})$, and where $g(l)$ is the localizing subcategory generated by all X such that $k(p) \otimes_R X \in l(p)$ for all $p \in \operatorname{Spec} R$.

In fact, our methods apply somewhat more generally, allowing one to replace $\mathrm{D}(RC)$ with derived categories of representations of R-flat R-linear categories.

Our second result is a classification of the telescopic localizations of $\mathrm{D}(RQ)$ and a classification of the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$ when Q is a Dynkin quiver.

Theorem (Corollaries 5.1, 5.10, and 5.11). *Let R be a noetherian commutative ring and Q a simply laced Dynkin quiver, and denote by RQ the R-linear path algebra of Q. There is an isomorphism of lattices*

$$\{\textit{localizing subcategories of } \mathrm{D}(RQ)\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{functions } \operatorname{Spec} R \to \mathrm{NC}(Q)\},$$

where $\mathrm{NC}(Q)$ denotes the lattice of noncrossing partitions associated to Q.

Moreover, the telescope conjecture holds for $\mathrm{D}(RQ)$, and the smashing subcategories, which by virtue of the telescope conjecture are in bijection with thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$, correspond to those $\sigma : \operatorname{Spec} R \to \mathrm{NC}(Q)$ such that whenever $p \subseteq q$ in $\operatorname{Spec} R$ we have $\sigma(p) \leq \sigma(q)$.

In terms of the localizing subcategories, this theorem basically combines Corollary 4.3 with the results of [Ingalls and Thomas 2009] on localizing subcategories of $\mathrm{D}(kQ)$ for fields k.

Initially, we had also hoped to prove the telescope conjecture for the telescopic localizations of $\mathrm{D}(RC)$ more generally, at least with some hopefully mild hypothesis. This turned out to be overly ambitious, but we present some partial results in Section 6.

2. Preliminaries on representations of small categories

Throughout we fix a commutative ring R. Let C be a small category.

Definition 2.1. The category of *right C-modules over R* is the functor category

$$\operatorname{Mod}_R C = \operatorname{Fun}(C^{\mathrm{op}}, \operatorname{Mod} R)$$

consisting of contravariant functors from C to the category of R-modules.

The following well-known lemma ensures that we can use the standard tools of homological algebra when dealing with C-modules.

Lemma 2.2. *The category $\operatorname{Mod}_R C$ of right C-modules over R is a Grothendieck category with enough projectives.*

Proof. Recall that a Grothendieck (abelian) category is an abelian category (1) satisfying axiom (AB5), on the existence and exactness of filtered colimits, and (2) possessing a generator. The lemma can be proved by showing that the direct sum of the set of representable objects is a generator, that filtered colimits are computed pointwise so that (AB5) follows from the satisfaction of that axiom for Mod_R itself, and finally that the projective objects of $\mathrm{Mod}_R\, C$ are summands of direct sums of representables. Details are left to the reader. □

We can also approach C-modules via R-linear functors.

Definition 2.3. The *R-linearization* of C, which we will denote by RC, is the category with the same objects as C and whose hom-objects are free R-modules on the hom-sets of C

$$RC(c, c') = \bigoplus_{f \in C(c,c')} Rf,$$

with the obvious composition rule. In other words, RC is the free R-linear category on C.

Definition 2.4. An R-linear category D is a small category enriched in R-modules. It is *flat* if $D(c, c')$ is a flat R-module for all pairs of objects c, c' in D.

Definition 2.5. If D is an R-linear category, then the category of right D-modules over R is defined to be the functor category

$$\mathrm{Mod}_R\, D = \mathrm{Fun}_R(D, \mathrm{Mod}\, R)$$

of R-linear functors.

Evidently, RC is a flat R-linear category for any small category C, since the hom-objects are free. The reason for looking at these more general categories is to capture the representation theory of R-algebras "with many objects", whereas the representations of RC are representations of monoids with many objects. In the case where C has one object with monoid of endomorphisms M, the category of representations of C in R-modules is equivalent to the category of right $R[M]$-modules, where $R[M]$ is the monoid algebra of M. On the other hand, if D is an R-linear category with one object having endomorphism algebra S, then S is an R-algebra, and the category of R-linear representations of D is equivalent to the category of right S-modules. Of course, not every R-algebra is a monoid algebra, so the R-linear categories capture more examples.

Of course, we should now check that $\mathrm{Mod}_R\, C$ and $\mathrm{Mod}_R\, RC$ are equivalent. We do this in a moment, but we first want to introduce extra structure that will be preserved. Tensoring an RC-module objectwise with an R-module defines a bifunctor

$$\mathrm{Mod}\, R \times \mathrm{Mod}_R\, RC \xrightarrow{\otimes_R} \mathrm{Mod}_R\, RC$$

which is explicitly given by $(M \otimes_R F)(c) = M \otimes_R F(c)$ for an R-module M, an RC-module F, and $c \in C$. This gives an action of the category of R-modules on the category of RC-modules. We note that this action is nothing other than the existence of copowers for the R-linear category $\mathrm{Mod}_R\, RC$. There is, of course, a similar action on $\mathrm{Mod}_R\, D$ when D is an R-linear category.

Remark 2.6. Here and in the sequel we will work with categories of the form RC since our main examples are of this form. However, our results are equally valid for flat R-linear categories; the only changes which need to be made are cosmetic.

Lemma 2.7. *The natural map* $\mathrm{Mod}_R\, RC \to \mathrm{Mod}_R\, C$ *is an equivalence for any small category* C*. This equivalence is compatible with the actions described above.*

Proof. This follows from the standard 2-adjunction relating categories and R-linear categories; see for instance [Kelly 1982, Chapter 2.5]. □

Lemma 2.8. *Given a morphism of commutative rings* $R \xrightarrow{\phi} S$, *the natural base change functor*

$$\phi^* : \mathrm{Mod}_R\, RC \to \mathrm{Mod}_S\, SC$$

has a right adjoint ϕ_*.

Proof. The functor ϕ^* is given by applying $S \otimes_R -$ objectwise, and ϕ_* is induced by restriction of scalars. This is again induced by a standard 2-adjunction between R-linear and S-linear categories corresponding to ϕ. □

3. Generalities on derived categories of small categories over a commutative ring

Again R is a fixed commutative ring which we now also assume is noetherian, and C is a small category with R-linearization RC. The (unbounded) derived category $\mathrm{D}(RC)$ of RC is the category of complexes of right RC-modules where quasi-isomorphisms have been inverted. We note that this is a compactly generated triangulated category and the compact objects are, up to quasi-isomorphism, precisely the bounded complexes of projective RC-modules.

Recall that a localizing subcategory of $\mathrm{D}(RC)$ is a full triangulated subcategory of $\mathrm{D}(RC)$ closed under coproducts (any such subcategory is automatically closed under direct summands). We want to consider to what extent the localizing subcategories of $\mathrm{D}(RC)$ are determined by the localizing subcategories of $\mathrm{D}(k(p)C)$ as p ranges over the prime ideals of R. This is inspired by work of Neeman [1992] who showed that in the case where C is the terminal category, i.e., $RC = R$, the localizing subcategories of $\mathrm{D}(R)$ are determined by those of the $\mathrm{D}(k(p))$. We restrict to noetherian rings as, even in the case $RC = R$, it is known that $\mathrm{Spec}\, R$ does not determine the localizing subcategories of $\mathrm{D}(R)$ in general.

Let us begin by observing that the action of $\mathrm{Mod}\, R$ on $\mathrm{Mod}_R\, C$ can be derived:

Lemma 3.1. *The bifunctor* $\mathrm{Mod}\, R \times \mathrm{Mod}_R\, C \to \mathrm{Mod}_R\, C$ *is left-balanced, with respect to flat* R*-modules and objectwise* R*-flat* RC*-modules, i.e., it is exact when either the first variable is flat or the second variable is objectwise flat. It admits a left-derived functor, independent up to isomorphism of which variable it is derived in, which gives a left action* $\mathrm{D}(R) \times \mathrm{D}(RC) \to \mathrm{D}(RC)$ *in the sense of* [Stevenson 2013].

Proof. Given $F \in \mathrm{Mod}_R\, C$ such that F is objectwise R-flat, it is clear that $- \otimes_R F$ is exact. As $\mathrm{Mod}_R\, C$ has enough projectives, and the projective RC-modules are componentwise projective, we see that $\mathrm{Mod}_R\, C$ has enough objectwise R-flat modules. It is thus clear that the functor can be left-derived, using resolutions either in $\mathrm{Mod}\, R$ or $\mathrm{Mod}_R\, C$, and that it does not matter, up to quasi-isomorphism, on which side the resolution is taken (i.e., $- \otimes_R -$ is balanced as claimed). It is straightforward to check that this gives an associative and unital action of $\mathrm{D}(R)$ on $\mathrm{D}(RC)$. □

Remark 3.2. Given $E \in \mathrm{D}(R)$ and $F \in \mathrm{D}(RC)$, we will simply denote $E \otimes_R^{\mathrm{L}} F$ by $E \otimes_R F$ or even $E \otimes F$; no confusion should result as we will almost exclusively work with derived functors (frequently with R fixed or clear from the context).

This allows us to utilize the machinery of tensor actions to analyze localizing subcategories of $\mathrm{D}(RC)$. After giving a convenient lemma and some notation, we will recall the main result that we will need from this theory.

Lemma 3.3. *Any localizing subcategory* $\mathrm{L} \subseteq \mathrm{D}(RC)$ *is closed under tensoring with complexes of R-modules. Explicitly, for any* $M \in \mathrm{D}(R)$ *and* $X \in \mathrm{L}$, *we have* $M \otimes_R X \in \mathrm{L}$.

Proof. Evidently, if $X \in \mathrm{L}$, then $R \otimes_R X \simeq X \in \mathrm{L}$. Since $- \otimes_R X$ preserves coproducts, it follows that the subcategory of $\mathrm{D}(R)$ consisting of complexes of R-modules M such that $M \otimes_R X \in \mathrm{L}$ is localizing and contains R. Since R is a compact generator of $\mathrm{D}(R)$, the lemma follows. □

Let f be an element of R. We denote by $K_\infty(f)$ the *stable Koszul complex* $R \to R_f$ *of* f, where the map is the canonical one. Given a prime ideal p of R, we set

$$K_\infty(p) = K_\infty(f_1) \otimes_R \cdots \otimes_R K_\infty(f_n),$$

where $f_1, \ldots, f_n$ is a choice of generators for p. The resulting complex is independent of the choice of generators up to quasi-isomorphism (independence is usually left as an exercise but a proof can be found, for instance, in [Greenlees 1993, Lemma 2.3]).

Given $p \in \mathrm{Spec}\, R$, we define the object $\Gamma_p R$ to be $K_\infty(p) \otimes_R R_p$. We recall from [Stevenson 2013] that $\Gamma_p R \otimes_R \Gamma_p R \simeq \Gamma_p R$ and for $p \neq q$ in $\mathrm{Spec}\, R$ we have $\Gamma_p R \otimes_R \Gamma_q R = 0$.

Remark 3.4. In more familiar language, the object $K_\infty(p)$ corresponds to taking local cohomology with support in $V(p)$ in the sense that the local cohomology functor is isomorphic to $K_\infty(p) \otimes (-)$. Thus $\Gamma_p R$ can be thought of as corresponding to "p-localized local cohomology on $V(p)$". In general it differs from the residue field $k(p)$, which is rarely tensor idempotent. In certain situations, for instance if $R = \mathbb{Z}$, one can express $\Gamma_p R$ as a desuspension of a flat resolution of $E(k(p))$, the injective envelope of the residue field at p; for instance, given a prime $p \in \mathbb{Z}$, one has $\Gamma_{(p)}\mathbb{Z} \cong \Sigma^{-1}E(\mathbb{Z}/p\mathbb{Z})$. However, in general the precise relationship between $\Gamma_p R$, $k(p)$, and $E(k(p))$ seems to be more subtle.

As a final point of notation, we will use $\langle S\rangle$ to denote the smallest localizing subcategory of a triangulated category generated by some collection of objects S.

Theorem 3.5 [Stevenson 2013, Theorem 6.9]. *Given an object X of $\mathrm{D}(RC)$, there is an equality of localizing subcategories*

$$\langle X\rangle = \langle \Gamma_p R \otimes_R X \mid p \in \operatorname{Spec} R\rangle.$$

It follows that $\Gamma_p R \otimes_R X \simeq 0$ for all prime ideals p if and only if $X \simeq 0$.

Corollary 3.6. *If $X \in \mathrm{D}(RC)$ is nonzero, then there is some prime ideal p of R such that $k(p) \otimes_R X$ is not zero.*

Proof. By the theorem there is a p such that $\Gamma_p R \otimes_R X$ is nonzero. The result now follows as $\langle \Gamma_p R\rangle = \langle k(p)\rangle$ in $\mathrm{D}(R)$ by [Neeman 1992, Section 2], which implies $k(p) \otimes_R X \simeq 0$ if and only if $\Gamma_p R \otimes_R X \simeq 0$. □

We now turn to analyzing the localizing subcategories of $\mathrm{D}(RC)$ in terms of the "fibers" $\mathrm{D}(k(p)C)$ for $p \in \operatorname{Spec} R$. Let $\mathcal{L}$ be the class defined in the following way. It comes equipped with a surjective map $\mathcal{L} \xrightarrow{s} \operatorname{Spec} R$, and the fiber over $p \in \operatorname{Spec} R$ is the class of localizing subcategories of $\mathrm{D}(k(p)C)$. We will define a pair of maps

$$\{\text{localizing subcategories } \mathrm{L} \text{ of } \mathrm{D}(RC)\} \underset{g}{\overset{f}{\rightleftarrows}} \{\text{sections } l \text{ of } \mathcal{L} \xrightarrow{s} \operatorname{Spec} R\}.$$

In order to define the maps in the most convenient manner, we require a little preparation.

Lemma 3.7. *If X is in the image of the forgetful functor $\mathrm{D}(k(p)C) \to \mathrm{D}(RC)$, then $k(p) \otimes_R X$ is a direct sum of suspensions of X. In particular, the base change functor $\mathrm{D}(RC) \to \mathrm{D}(k(p)C)$ is essentially surjective up to summands.*

Proof. Let X be as in the statement, i.e., X is a complex of $k(p)C$-modules regarded as a complex of RC-modules. Then

$$k(p) \otimes_R X \simeq (k(p) \otimes_R k(p)) \otimes_{k(p)} X$$

is a coproduct of suspensions of X since $k(p) \otimes_R k(p)$ is a coproduct of suspensions of $k(p)$. As the base change functor $\mathrm{D}(RC) \to \mathrm{D}(k(p)C)$ is just $k(p) \otimes_R -$, the final statement of the lemma is an immediate consequence. □

Lemma 3.8. *Let L be a localizing subcategory of $\mathrm{D}(RC)$. Then $\operatorname{add}(k(p) \otimes_R \mathrm{L})$, the closure of $k(p) \otimes_R \mathrm{L}$ under summands and isomorphisms in $\mathrm{D}(k(p)C)$, is a localizing subcategory of $\mathrm{D}(k(p)C)$.*

Proof. It is evident that $\mathrm{add}(k(p) \otimes_R \mathrm{L})$ is closed under suspensions and coproducts in $\mathrm{D}(k(p)C)$ as derived base change is exact and coproduct-preserving. Thus it is sufficient to show that $\mathrm{add}(k(p) \otimes_R \mathrm{L})$ is closed under triangles. Suppose $X \to Y \to Z \to$ is a triangle with $X, Y \in \mathrm{add}(k(p) \otimes_R \mathrm{L})$. Without loss of generality we may assume $X, Y \in k(p) \otimes_R \mathrm{L}$. By Lemma 3.3 the restrictions of X and Y lie in L, so we deduce that the restriction of Z lies in L. Hence $k(p) \otimes_R Z$ is in $k(p) \otimes_R \mathrm{L}$ and using Lemma 3.7 we see that Z is in $\mathrm{add}(k(p) \otimes_R \mathrm{L})$, proving the lemma. □

The function f is defined as follows: we set $f(\mathrm{L})(p) = \mathrm{add}(k(p) \otimes_R \mathrm{L})$ which is localizing by Lemma 3.8. Given a section l of s, define $g(l)$ as the localizing subcategory

$$\{X \in \mathrm{D}(RC) \mid k(p) \otimes_R X \in l(p) \text{ for all primes } p \in \operatorname{Spec} R\}.$$

There is another natural function

$$\{\text{localizing subcategories L of } \mathrm{D}(RC)\} \xleftarrow[g']{} \{\text{sections } l \text{ of } \mathcal{L} \xrightarrow{s} \operatorname{Spec} R\}$$

defined as follows: let g' be the function that takes l to the localizing subcategory generated by the objects X of $l(p)$ for all p, viewed as RC-modules in the natural way, i.e.,

$$g'(l) = \langle l(p) \mid p \in \operatorname{Spec} R \rangle.$$

Lemma 3.9. *If* L *is a localizing subcategory of* $\mathrm{D}(RC)$ *then* $g'(f(\mathrm{L})) \subseteq \mathrm{L} \subseteq g(f(\mathrm{L}))$.

Proof. The inclusion $\mathrm{L} \subseteq g(f(\mathrm{L}))$ is clear:

$$g(f(\mathrm{L})) = \{X \in \mathrm{D}(RC) \mid k(p) \otimes_R X \in \mathrm{add}(k(p) \otimes_R \mathrm{L}) \text{ for all } p \in \operatorname{Spec} R\} \supseteq \mathrm{L}.$$

To show the other inclusion, note that $g'(f(\mathrm{L}))$ is generated by $k(p) \otimes_R X$, as X ranges over the objects of L and p ranges over the primes of R. But, by Lemma 3.3, these are all in L. □

Lemma 3.10. *Suppose* l *is a section of* s*. Then* $f(g'(l)) = l = f(g(l))$*. In particular,* f *is surjective.*

Proof. The value of $f(g'(l))$ at a prime p consists of the localizing subcategory of $\mathrm{D}(k(p)C)$ generated by the complexes $k(p) \otimes_R X$ for $X \in l(p)$. By Lemma 3.7 $k(p) \otimes_R X$ is a direct sum of suspensions of X and thus $f(g'(l)) = l$. Similarly $l = f(g(l))$, proving the lemma. □

Our goal is to show that $g'(f(\mathrm{L})) = \mathrm{L} = g(f(\mathrm{L}))$. This will prove that g and f are inverse bijections and so gives a description of the lattice of localizing subcategories of $\mathrm{D}(RC)$ in terms of the corresponding derived categories over the residue fields of $\operatorname{Spec} R$.

4. Proof of the main theorem

This section is dedicated to proving $g'(f(L)) = L = g(f(L))$.

Write $\Gamma_p\mathrm{D}(RC)$ for the localizing subcategory consisting of objects X supported at $p \in \operatorname{Spec} R$, i.e., those X satisfying $k(q) \otimes_R X \simeq 0$ for $q \neq p$. Equivalently, one can describe $\Gamma_p\mathrm{D}(RC)$ as the essential image of $\Gamma_p R \otimes_R -$ in $\mathrm{D}(RC)$. We can restrict f to the class of localizing subcategories of $\Gamma_p\mathrm{D}(RC)$.

Proposition 4.1. *The following are equivalent*:

(1) *the functions f and g are inverse bijections*;

(2) *the restrictions*

$$\left\{\begin{matrix}\textit{localizing subcategories}\\ \textit{of } \Gamma_p\mathrm{D}(RC)\end{matrix}\right\} \underset{g_p}{\overset{f_p}{\rightleftarrows}} \left\{\begin{matrix}\textit{localizing subcategories}\\ \textit{of } \mathrm{D}(k(p)C)\end{matrix}\right\}$$

are inverse bijections for all primes p;

(3) *for every prime ideal p in* Spec R *and for every object X of $\Gamma_p\mathrm{D}(RC)$, the localizing subcategories $\langle k(p) \otimes_R X\rangle$ and $\langle X\rangle$ are the same.*

Proof. Clearly (1) implies (2). That (2) implies (3) follows from the fact that the localizing subcategories $\langle X\rangle$ and $\langle k(p) \otimes_R X\rangle$ have the same image under f_p. Since f is surjective, to prove that (3) implies (1), it suffices to prove that (3) implies f is injective. Assuming this for a moment, Lemma 3.10 says that both g and g' are inverses for f, which must then coincide.

Assume now that L is a localizing subcategory of $\mathrm{D}(RC)$ and that $X \in \mathrm{L}$. It suffices to show that $X \in g'(f(\mathrm{L}))$ since we have the other containment by Lemma 3.9. Under the assumption (3), $\Gamma_p R \otimes_R X \in g'(f(\mathrm{L}))$ for every prime ideal p in $\operatorname{Spec} R$ because $k(p) \otimes_R \Gamma_p R \otimes_R X \cong k(p) \otimes_R X$. Hence there is a containment of localizing subcategories

$$\langle \Gamma_p R \otimes_R X \mid p \in \operatorname{Spec} R\rangle \subseteq g'(f(\mathrm{L})).$$

By Theorem 3.5, $X \in \langle \Gamma_p R \otimes_R X \mid p \in \operatorname{Spec} R\rangle$, and so $X \in g'(f(\mathrm{L}))$, completing the proof. □

The following observation is our main 'theorem'.

Theorem 4.2. *Let p be a prime ideal of R and X an object of $\Gamma_p\mathrm{D}(RC)$. Then $X \in \langle k(p) \otimes_R X\rangle$ and hence*

$$\langle k(p) \otimes_R X\rangle = \langle X\rangle.$$

Proof. Let X be as in the lemma and consider the full subcategory

$$\mathrm{M} = \{E \in \mathrm{D}(R) \mid E \otimes_R X \in \langle k(p) \otimes_R X\rangle\}$$

of $\mathrm{D}(R)$. As $\langle k(p) \otimes_R X\rangle$ is a localizing subcategory, it follows that M is also localizing (this is relatively straightforward but a proof can be found in [Stevenson 2013, Lemma 3.8]). It is immediate from the definition that $k(p) \in \mathrm{M}$ and so $\langle k(p)\rangle \subseteq \mathrm{M}$. By Neeman's classification result [1992] we have $\Gamma_p R \in \langle k(p)\rangle$, and hence $\Gamma_p R$ also lies in M. Thus $\Gamma_p R \otimes_R X \in \langle k(p) \otimes_R X\rangle$ and it only remains to observe that $X \in \Gamma_p \mathrm{D}(RC)$ implies $\Gamma_p R \otimes_R X \simeq X$. □

Corollary 4.3. *Let R be a commutative noetherian ring and C a small category. Then the assignments*

$$\{\textit{localizing subcategories } \mathrm{L} \textit{ of } \mathrm{D}(RC)\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{sections } l \textit{ of } \mathcal{L} \xrightarrow{s} \operatorname{Spec} R\}$$

are inverse to one another.

Proof. It is sufficient to verify condition (3) of Proposition 4.1, i.e., that for every $X \in \Gamma_p \mathrm{D}(RC)$ we have $X \in \langle k(p) \otimes_R X\rangle$. This is precisely the content of the theorem and so we see that f and g are inverse. □

Remark 4.4. As noted in Remark 2.6, our results are also valid in the case where D is a flat R-linear category and we consider $\mathrm{D}(\mathrm{Mod}_R\, D)$. One just needs to replace $k(p)C$ by $k(p) \otimes_R D$, the base change of D to $k(p)$; the arguments don't change.

5. Dynkin quivers

In this section we give a concrete application of the formalism above by considering the case where C is the path category of a simply laced Dynkin quiver. Let Q be a quiver whose underlying graph is a simply laced Dynkin diagram. We can naturally view Q as a poset, i.e., a small category, and apply our result to the study of the derived category, $\mathrm{D}(RQ)$, of representations of Q over R. This yields the following extension of work of Ingalls and Thomas [2009], where we refer the reader for information about noncrossing partitions.

Corollary 5.1. *Let R be a commutative noetherian ring and Q a simply laced Dynkin quiver, and denote by RQ the R-linear path algebra of Q. There is an isomorphism of lattices*

$$\{\textit{localizing subcategories of } \mathrm{D}(RQ)\} \underset{g}{\overset{f}{\rightleftarrows}} \{\textit{functions } \operatorname{Spec} R \to \mathrm{NC}(Q)\},$$

where $\mathrm{NC}(Q)$ denotes the lattice of noncrossing partitions associated to Q.

Proof. Corollary 4.3 applies so it just remains to demonstrate that there is a bijection

$$\{\text{sections of } \mathcal{L} \xrightarrow{s} \operatorname{Spec} R\} \simeq \operatorname{Hom}(\operatorname{Spec} R, \mathrm{NC}(Q)).$$

This follows from [Krause 2012, Theorem 6.10] which shows, without restriction on the field k, that there is a bijection between the lattice of thick subcategories of $\mathrm{D}^b(kQ)$ and $\mathrm{NC}(Q)$. As kQ is hereditary and of finite representation type, $\mathrm{D}(kQ)$ is pure-semisimple, i.e., every object is a direct sum of compact objects, and so we deduce a bijection between the lattice of localizing subcategories of $\mathrm{D}(kQ)$ and $\mathrm{NC}(Q)$. Thus sections of $\mathcal{L} \to \operatorname{Spec} R$ are nothing but functions from $\operatorname{Spec} R$ to $\mathrm{NC}(Q)$. □

Remark 5.2. One can also use Lemma 3.10 and Krause's extension [2012, Theorem 6.10] of a result by Igusa and Schiffler to get partial information on the lattice of localizing subcategories of $\mathrm{D}(RQ)$ for an arbitrary quiver Q.

In this situation we can obtain a classification of the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$, the category of perfect complexes of RQ-modules. Recall that $\mathrm{D}^{\mathrm{perf}}(RQ)$ is the full subcategory of $\mathrm{D}(RQ)$ consisting of those objects quasi-isomorphic to a bounded complex of finitely generated projective modules; it is a thick subcategory and is the subcategory of compact objects in $\mathrm{D}(RQ)$. As in the case of $\mathrm{D}^{\mathrm{perf}}(R)$, the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$ are given by a sublattice of the lattice of localizing subcategories defined by a certain specialization closure condition.

Definition 5.3. We call a function $\sigma : \operatorname{Spec} R \to \mathrm{NC}(Q)$ *specialization closed* if whenever $p \subseteq q$ we have $\sigma(p) \leq \sigma(q)$ in $\mathrm{NC}(Q)$.

Remark 5.4. This recovers the usual notion of specialization closure of subsets of $\operatorname{Spec} R$ when $Q = A_1$ and so $\mathrm{NC}(Q) = \{0, 1\}$. Moreover, returning to the general simply laced case, if L is a localizing subcategory with $f(\mathrm{L})$ specialization closed then for $p \subseteq q$ we have

$$k(p) \otimes \mathrm{L} \neq 0 \quad \Rightarrow \quad k(q) \otimes \mathrm{L} \neq 0.$$

We will show that specialization closed functions $\operatorname{Spec} R \to \mathrm{NC}(Q)$ classify smashing subcategories of $\mathrm{D}(RQ)$ and that the telescope conjecture holds. Combining these two results gives the claimed classification result for thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$. We begin by recalling a useful fact and then present the easiest part of the argument.

Lemma 5.5. *Let p be a prime ideal of R and let M be an indecomposable $k(p)Q$-module with dimension vector α. Then there is a rigid lattice $\widetilde{M}$ over RQ, i.e., $\widetilde{M}$ is R-free and* $\operatorname{Ext}^1_{RQ}(\widetilde{M}, \widetilde{M}) = 0$, *with rank vector α. Moreover, for any $q \in \operatorname{Spec} R$ the module $k(q) \otimes \widetilde{M}$ is the unique indecomposable $k(q)Q$-module with dimension vector α. In particular,*

$$k(p) \otimes \widetilde{M} \cong M.$$

Proof. This is a (very) special case of [Crawley-Boevey 1996, Theorem 1]. □

Lemma 5.6. *Let $\sigma : \operatorname{Spec} R \to \mathrm{NC}(Q)$ be specialization closed. Then the localizing subcategory* $\mathrm{L} = g(\sigma)$ *is generated by objects of* $\mathrm{D}^{\mathrm{perf}}(RQ)$.

Proof. We prove this by just writing down a (rather redundant) generating set for L. For each prime ideal p such that $k(p)\otimes\mathrm{L}\neq 0$, let $M(p)$ be a compact generator for the localizing subcategory of $\mathrm{D}(k(p)Q)$ generated by $k(p)\otimes\mathrm{L}$. Since $M(p)$ is a finite sum of (suspensions of) indecomposable modules in $\mathrm{D}(k(p)Q)$, we can lift it to a lattice $\widetilde{M(p)}$ in $\mathrm{D}(RQ)$ using Lemma 5.5. In particular, it is easily seen that $\widetilde{M(p)}$ is compact in $\mathrm{D}(RQ)$. Set

$$G=\left\{K(p)\otimes\widetilde{M(p)}\mid p\in\operatorname{Spec}R \text{ with } k(p)\otimes\mathrm{L}\neq 0\right\}\quad\text{and}\quad \mathrm{L}'=\langle G\rangle,$$

where $K(p)$ denotes the Koszul complex for p defined by

$$K(p)=\bigotimes_{i=1}^{r}\operatorname{cone}(R\xrightarrow{f_i}R),$$

where p is generated by $f_1,\ldots,f_r$. (Recall that this implicitly means the derived tensor product over R.) Since $K(p)\in\mathrm{D}^{\mathrm{perf}}(R)$ and $\widetilde{M(p)}\in\mathrm{D}^{\mathrm{perf}}(RQ)$, the set G consists of compact objects by [Stevenson 2013, Lemma 4.6].

For primes $p\subseteq q\in\operatorname{Spec}R$ the object $k(q)\otimes\big(K(p)\otimes\widetilde{M(p)}\big)$ is a finite sum of suspensions of copies of the $k(q)Q$-module $k(q)\otimes\widetilde{M(p)}$. This latter module can be described as follows: each indecomposable summand of $M(p)$ corresponds to an indecomposable $k(q)Q$-module, namely the indecomposable $k(q)Q$-module with the same dimension vector, and $k(q)\otimes\widetilde{M(p)}$ is the corresponding sum of these indecomposable $k(q)Q$-modules. In particular, $M(p)$ and $k(q)\otimes\widetilde{M(p)}$ correspond to the same element of $\mathrm{NC}(Q)$. If, on the other hand, $p\nsubseteq q$ then $k(q)\otimes\big(K(p)\otimes\widetilde{M(p)}\big)=0$.

Putting everything together we see that

$$\begin{aligned}\langle k(q)\otimes\mathrm{L}'\rangle&=\langle k(q)\otimes K(p)\otimes\widetilde{M(p)}\mid p\in\operatorname{Spec}R\text{ with }k(p)\otimes\mathrm{L}\neq 0\rangle\\&=\langle k(q)\otimes\widetilde{M(q)}\rangle=\langle M(q)\rangle=\langle k(q)\otimes\mathrm{L}\rangle,\end{aligned}$$

where the second equality follows from the computation in the preceding paragraph together with specialization closure of σ, and the third and fourth equalities are by definition of $M(q)$ and $\widetilde{M(q)}$. This shows that $f(\mathrm{L})=f(\mathrm{L}')$ and thus, by the classification of localizing subcategories, $\mathrm{L}=\mathrm{L}'$. We have thus exhibited a set of generators $G\subseteq\mathrm{D}^{\mathrm{perf}}(RQ)$ for L. □

We now continue with proving that the specialization closed functions $\operatorname{Spec}R\to\mathrm{NC}(Q)$ classify smashing subcategories of $\mathrm{D}(RQ)$. Combined with the above lemma, this proves the telescope conjecture and classifies the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$.

Fix a smashing subcategory S of $\mathrm{D}(RQ)$, i.e., consider a localization sequence

$$\mathrm{S}\underset{i^!}{\overset{i_*}{\rightleftarrows}}\mathrm{D}(RQ)\underset{j_*}{\overset{j^*}{\rightleftarrows}}\mathrm{S}^{\perp},$$

where $i^!$ and j_* are the right adjoints of the inclusion functors i_* and the localization functor j^*, respectively, and all of these functors preserve coproducts. In particular, $\mathrm{S}^{\perp}$ is also a localizing subcategory of $\mathrm{D}(RQ)$. In order to prove the result indicated above we start with two elementary lemmas.

Lemma 5.7. *Let* S *be as above. Then for any* $Y \in \mathrm{D}(R)$ *and* $X \in \mathrm{D}(RQ)$ *we have canonical isomorphisms*

$$i_* i^!(Y \otimes X) \cong Y \otimes i_* i^! X \quad \textit{and} \quad j_* j^*(Y \otimes X) \cong Y \otimes j_* j^* X.$$

Proof. Consider the localization triangle for X

$$i_* i^! X \to X \to j_* j^* X \to \Sigma i_* i^! X.$$

Acting on this triangle with Y gives a new triangle

$$Y \otimes i_* i^! X \to Y \otimes X \to Y \otimes j_* j^* X \to \Sigma(Y \otimes i_* i^! X).$$

By Lemma 3.3 both S and $\mathrm{S}^{\perp}$ are closed under the $\mathrm{D}(R)$ action and so we have $Y \otimes i_* i^! X \in \mathrm{S}$ and $Y \otimes j_* j^* X \in \mathrm{S}^{\perp}$. The claimed isomorphisms follow immediately from the uniqueness of localization triangles. □

Lemma 5.8. *Let* $p' \in \operatorname{Spec} R$. *Let* M, N *be indecomposable* $k(p')Q$*-modules with*

$$\operatorname{Hom}_{k(p')Q}(M, N) \neq 0$$

and denote choices of their respective rigid lattice lifts by $\widetilde{M}$ *and* $\widetilde{N}$. *Then, given* $p \subseteq q \in \operatorname{Spec} R$, *we have*

$$\operatorname{Hom}_{RQ}\big(E(k(p)) \otimes \widetilde{M}, E(k(q)) \otimes \widetilde{N}\big) \neq 0,$$

where $E(k(p))$, $E(k(q))$ *denote the injective envelopes of the residue fields* $k(p)$, $k(q)$.

Proof. We know there are rigid lattice lifts of M and N by Lemma 5.5. We can choose, using the classification of indecomposable modules over Q, a nonzero $\phi : M \to N$ given on each component by matrices involving only zero and identity maps. It is then clear that we can lift it to a nonzero $\tilde{\phi} : \widetilde{M} \to \widetilde{N}$ such that $\tilde{\phi}$, like ϕ, is given componentwise by matrices whose only entries are zero and identity maps. On the other hand, since $p \subseteq q$, there is a nonzero map $\psi : E(k(p)) \to E(k(q))$. It is thus evident by our choice of $\tilde{\phi}$ that either of the equal composites in the commutative square

$$\begin{array}{ccc} E(k(q)) \otimes \widetilde{M} & \xrightarrow{1\otimes\tilde{\phi}} & E(k(q)) \otimes \widetilde{N} \\ {\scriptstyle \psi\otimes 1}\uparrow & & \uparrow{\scriptstyle \psi\otimes 1} \\ E(k(p)) \otimes \widetilde{M} & \xrightarrow[1\otimes\tilde{\phi}]{} & E(k(p)) \otimes \widetilde{N} \end{array}$$

gives the desired nonzero morphism. □

Using this series of easy observations we can now dispose of the proof of the theorem in short order.

Theorem 5.9. *Let* S *be a smashing subcategory of* $\mathrm{D}(RQ)$ *with notation as introduced above. Then* $f(\mathrm{S}) : \operatorname{Spec} R \to \mathrm{NC}(Q)$ *is specialization closed.*

Proof. Fix $p \subseteq q \in \operatorname{Spec} R$ and an indecomposable module $M \in k(p) \otimes \mathrm{S} \subseteq \mathrm{D}(k(p)Q)$ with dimension vector α. By Lemma 5.5 there is a lattice $\widetilde{M} \in \mathrm{D}^{\mathrm{perf}}(RQ)$ with $k(p) \otimes \widetilde{M} \cong M$ and $k(q) \otimes \widetilde{M}$ the unique indecomposable $k(q)Q$-module with dimension vector α. We have to show that $k(q) \otimes \widetilde{M}$ is in $k(q) \otimes \mathrm{S}$. To this end consider the localization triangle

$$i_* i^! \widetilde{M} \to \widetilde{M} \to j_* j^* \widetilde{M} \to \Sigma i_* i^! \widetilde{M}.$$

Pick an indecomposable summand N of $k(q) \otimes j_* j^* \widetilde{M}$ and note that, by Lemma 5.7, $N \in \mathrm{S}^{\perp}$. We assume N is nonzero since if $k(q) \otimes j_* j^* \widetilde{M}$ is zero then $k(q) \otimes \widetilde{M}$ is in S and we are done. Let $\widetilde{N}$ be a lattice lift of N. As we have assumed $k(q) \otimes j_* j^* \widetilde{M}$ is nonzero, the morphism

$$\phi = k(q) \otimes \widetilde{M} \to k(q) \otimes j_* j^* \widetilde{M} \to N \cong k(q) \otimes \widetilde{N}$$

must also be nonzero. Thus we can apply Lemma 5.8 to produce a nonzero morphism

$$\gamma : E(k(p)) \otimes \widetilde{M} \to E(k(q)) \otimes \widetilde{N}$$

in $\mathrm{D}(RQ)$.

On the other hand, by assumption $k(p) \otimes \widetilde{M} \in \mathrm{S}$ and $k(q) \otimes \widetilde{N} \in \mathrm{S}^{\perp}$. Since both S and $\mathrm{S}^{\perp}$ are localizing, and since for any prime ideal p' we have $E(k(p')) \in \langle k(p') \rangle$, we see (as in the proof of Theorem 4.2) that

$$E(k(p)) \otimes \widetilde{M} \in \mathrm{S} \quad \text{and} \quad E(k(q)) \otimes \widetilde{N} \in \mathrm{S}^{\perp}.$$

But this contradicts the existence of the nonzero morphism γ. Hence N must have been zero, showing that $k(q) \otimes j_* j^* \widetilde{M} \cong 0$, which in turn implies (via Lemma 5.7) that $k(q) \otimes \widetilde{M} \in \mathrm{S}$ as desired. □

This theorem has the following, more palatable, consequences.

Corollary 5.10. *Let* R *be a commutative noetherian ring and* Q *a simply laced Dynkin quiver. Then* $\mathrm{D}(RQ)$ *satisfies the telescope conjecture: every smashing subcategory is generated by objects of* $\mathrm{D}^{\mathrm{perf}}(RQ)$.

Proof. Suppose S is a smashing subcategory. Then by the classification given in Corollary 5.1 we know $\mathrm{S} = gf(\mathrm{S})$. By Theorem 5.9 the function $f(\mathrm{S})$ is specialization closed and so by Lemma 5.6 we see that $\mathrm{S} = gf(\mathrm{S})$ is generated by objects of $\mathrm{D}^{\mathrm{perf}}(RQ)$ as claimed. □

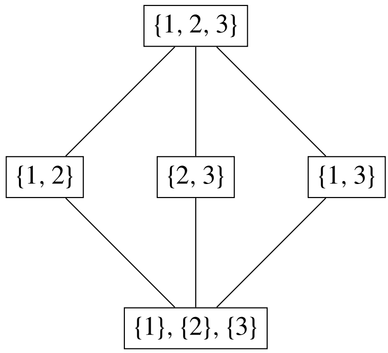

Figure 1. The lattice of noncrossing partitions of $\{1, 2, 3\}$. The coarser partitions are decreed to be bigger in the lattice structure.

Corollary 5.11. *Let R be a commutative noetherian ring and Q a simply laced Dynkin quiver. There is an isomorphism of lattices*

$$\left\{\begin{array}{c}\textit{thick subcategories}\\ \textit{of } \mathrm{D}^{\mathrm{perf}}(RQ)\end{array}\right\} \underset{g}{\overset{f}{\rightleftarrows}} \left\{\begin{array}{c}\textit{specialization closed functions}\\ \operatorname{Spec} R \to \mathrm{NC}(Q)\end{array}\right\},$$

where $\mathrm{NC}(Q)$ *denotes the lattice of noncrossing partitions associated to Q.*

Proof. Considering the classification of Corollary 5.1 and combining Theorem 5.9 and Lemma 5.6 gives a classification of the smashing subcategories of $\mathrm{D}(RQ)$ in terms of the specialization closed functions $\operatorname{Spec} R \to \mathrm{NC}(Q)$. By the previous corollary this is also the classification of the localizing subcategories of $\mathrm{D}(RQ)$ generated by objects of $\mathrm{D}^{\mathrm{perf}}(RQ)$. One obtains the isomorphism we have asserted in the statement in the standard way: by Thomason's localization theorem (see, for example, [Neeman 1996, Theorem 2.1]) the thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RQ)$ are in order-preserving bijection with the localizing subcategories of $\mathrm{D}(RQ)$ which are generated by perfect complexes. □

Example 5.12. Let R be a local 1-dimensional domain. Then $\operatorname{Spec} R$ consists of two points: a generic point η and a closed point x. We will consider the case of $Q = A_2$ in Corollary 5.11. The lattice $\mathrm{NC}(A_2)$ consists of the noncrossing partitions of the set $\{1, 2, 3\}$. A noncrossing partition of a cyclically ordered set S determined by an equivalence relation $\sim$ is one where $x < y < z < w$, $x \sim z$, and $y \sim w$ together imply $x \sim y \sim z \sim w$.

In Figure 1 we display each partition as determined by its largest equivalence classes. The class of all localizing subcategories of $\mathrm{D}(RA_2)$ in this case is simply two copies of this lattice, indexed on η and x. Figure 2 shows the lattice of specialization closed functions $\operatorname{Spec} R \to \mathrm{NC}(A_2)$, which by the results above is the lattice of thick subcategories of $\mathrm{D}^{\mathrm{perf}}(RA_2)$.

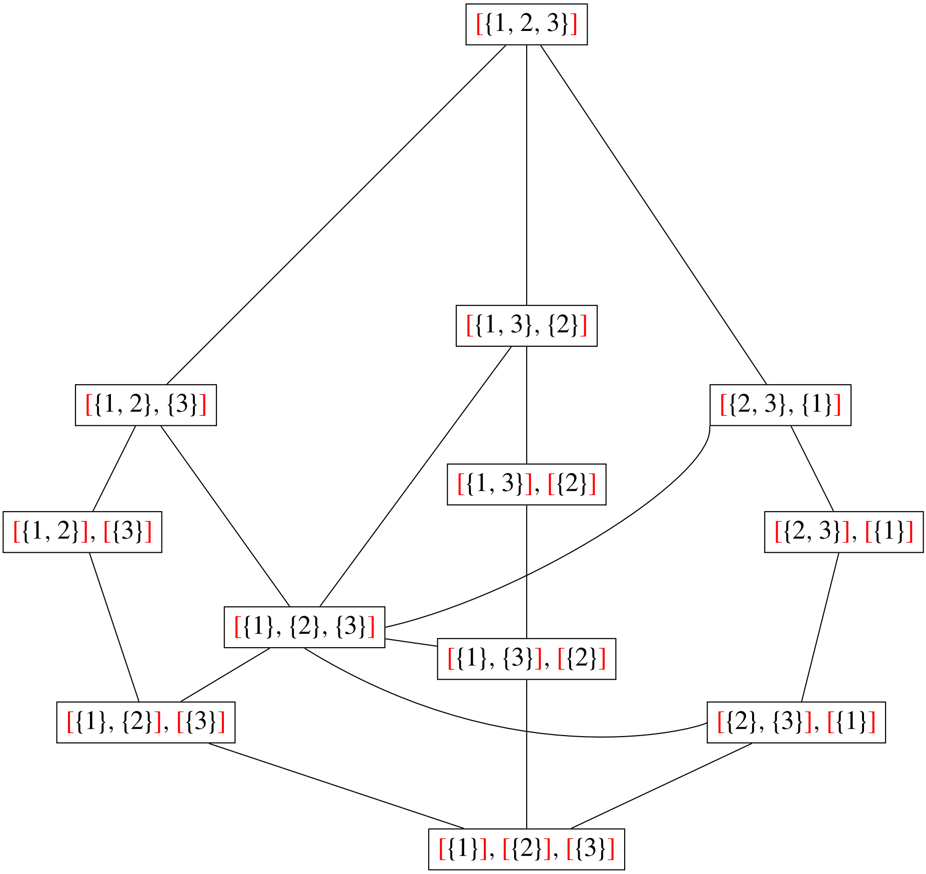

Figure 2. The lattice of specialization closed functions Spec $R \to \mathrm{NC}(A_2)$ for R a 1-dimensional local domain. The partition given by the black parentheses is the noncrossing partition corresponding to the generic point η, while the partition determined by the red parentheses is the partition corresponding to the closed point x.

6. Towards telescopy

We have seen in Corollary 5.10 that the telescope conjecture holds for $\mathrm{D}(RQ)$ when Q is an ADE quiver and R is any commutative noetherian ring. Unfortunately we were not able to prove such a general statement for even arbitrary quivers, let alone arbitrary small categories. However, we do have some partial results and remarks that we present in this section which revolve around the following question.

Question 6.1. Let R be a noetherian commutative ring. Does the telescope conjecture hold for $\mathrm{D}(RC)$ when C is an ordinary (not R-linear) category if it holds for $\mathrm{D}(k(p)C)$ for all $p \in \operatorname{Spec} R$?

We begin to answer this question by showing that the bijection of Proposition 4.1(2) restricts to a bijection between the collections of smashing subcategories. Given a localizing subcategory L of some triangulated category, we will denote the associated acyclization and localization functors by Γ_{L} and L_{L} respectively.

Remark 6.2. Throughout we will prove that some localizing subcategory S is smashing by exhibiting that the right orthogonal $\mathrm{S}^{\perp}$ is also localizing. In order for this condition to be equivalent to S being smashing, one needs to know that the inclusion of S admits a right adjoint. In all of the cases we consider S will clearly be generated by a set of objects; for instance, it will be the localizing subcategory generated by the image of some other smashing subcategory under an exact functor, and so the existence of the adjoint follows from Brown representability. Indeed, in this case one has a generating set, as any smashing subcategory of a compactly generated triangulated category has a set of generators by [Krause 2010, Theorem 7.4.1], and so one can apply Brown representability for well-generated categories as in [Neeman 2001] (or see [Krause 2010, Theorem 5.1.1]). Thus we will suppress this part of the arguments throughout.

For the moment, fix some $p \in \operatorname{Spec} R$ and denote by i^* the functor $k(p) \otimes (-) : \Gamma_p \mathrm{D}(RC) \to \mathrm{D}(k(p)C)$ and by i_* its right adjoint.

Lemma 6.3. *Suppose* S *is a smashing subcategory of* $\Gamma_p\mathrm{D}(RC)$ *and set*

$$\mathrm{T} = f(\mathrm{S}) = \operatorname{add}(k(p) \otimes \mathrm{S}) \quad \textit{and} \quad \mathrm{T}' = f(\mathrm{S}^{\perp}) = \operatorname{add}(k(p) \otimes \mathrm{S}^{\perp}).$$

Then T' *is the right orthogonal of* T, *and hence* T *is a smashing subcategory of* $\mathrm{D}(k(p)C)$.

Proof. If $X \in \mathrm{T}'$ then there is, by definition, some $\bar{X} \in \mathrm{S}^{\perp}$ such that X is a summand of $i^*\bar{X}$. Given $Y \in \mathrm{T}$, which we can assume to be of the form $i^*\bar{Y}$ with $\bar{Y} \in \mathrm{S}$, we have

$$\operatorname{Hom}(i^*\bar{Y}, i^*\bar{X}) \cong \operatorname{Hom}(\bar{Y}, i_*i^*\bar{X}).$$

This latter hom-set is zero, as $\bar{Y} \in \mathrm{S}$ and $i_*i^*\bar{X} \in \mathrm{S}^{\perp}$ by the closure of localizing subcategories under the $\mathrm{D}(R)$ action. Thus $\mathrm{T}' \subseteq \mathrm{T}^{\perp}$.

On the other hand, if $\operatorname{Hom}(i^*\mathrm{S}, Z) = 0$ for some $Z \in \mathrm{D}(k(p)C)$, then by adjunction $i_*Z \in \mathrm{S}^{\perp}$. Hence $i^*i_*Z \in \mathrm{T}'$ and we know, by Lemma 3.7, that Z is a summand of i^*i_*Z. So Z is in T', proving that $\mathrm{T}^{\perp} \subset \mathrm{T}'$ and completing the argument. □

Now we fix a smashing subcategory T of $\mathrm{D}(k(p)C)$ and set

$$\mathrm{S} = g(\mathrm{T}) = \langle i_*\mathrm{T} \rangle \quad \text{and} \quad \mathrm{S}' = g(\mathrm{T}^{\perp}) = \langle i_*\mathrm{T}^{\perp} \rangle.$$

We wish to show that S is smashing with right orthogonal S'. We prove this in the following four statements.

Lemma 6.4. *The subcategories* S *and* S' *generate* $\Gamma_p\mathrm{D}(RC)$, *i.e., we have*

$$\langle \mathrm{S} \cup \mathrm{S}' \rangle = \Gamma_p\mathrm{D}(RC).$$

Proof. Let X be an object of $\Gamma_p\mathrm{D}(RC)$. By Theorem 4.2 we know X is in the localizing subcategory $\langle i_* i^* X\rangle$. We have a localization triangle in $\mathrm{D}(k(p)C)$

$$\Gamma_{\mathrm{T}} i^* X \to i^* X \to L_{\mathrm{T}} i^* X \to \Sigma \Gamma_{\mathrm{T}} i^* X,$$

where $\Gamma_{\mathrm{T}} i^* X \in \mathrm{T}$ and $L_{\mathrm{T}} i^* X \in \mathrm{T}^{\perp}$. Applying i_* gives a triangle in $\mathrm{D}(RC)$

$$i_*\Gamma_{\mathrm{T}} i^* X \to i_* i^* X \to i_* L_{\mathrm{T}} i^* X \to \Sigma i_* \Gamma_T i^* X$$

with $i_*\Gamma_{\mathrm{T}} i^* X \in \mathrm{S}$ and $i_* L_{\mathrm{T}} i^* X \in \mathrm{S}'$ by definition. Thus $X \in \langle i_* i^* X \rangle \subseteq \langle \mathrm{S} \cup \mathrm{S}' \rangle$, as claimed. □

Lemma 6.5. *There is a containment of triangulated subcategories* $\mathrm{S}' \subseteq \mathrm{S}^{\perp}$.

Proof. It is enough to check that for every $t \in \mathrm{T}$ and $t' \in \mathrm{T}^{\perp}$ we have

$$\mathrm{Hom}(i_* t, i_* t') = 0.$$

The required vanishing follows from the isomorphisms

$$\mathrm{Hom}(i_* t, i_* t') \cong \mathrm{Hom}(i^* i_* t, t') \cong \mathrm{Hom}\Big(\coprod_{\lambda} \Sigma^{n_\lambda} t, t'\Big) \cong \prod_{\lambda} \mathrm{Hom}(\Sigma^{n_\lambda} t, t') = 0,$$

where the first isomorphism is by adjunction, the second is by Lemma 3.7, and the final hom-set vanishes by assumption. □

Lemma 6.6. *There is an equality*

$$\Gamma_p\mathrm{D}(RC) = \{X \in \Gamma_p\mathrm{D}(RC) \mid \textit{there exists a triangle } X' \to X \to X'' \to \Sigma X' \textit{ with } X' \in \mathrm{S} \textit{ and } X'' \in \mathrm{S}'\}.$$

Proof. It is routine to verify that the full subcategory defined on the right-hand side above is localizing, and it contains S and S' by definition. The equality then follows from Lemma 6.4. □

Proposition 6.7. S *is smashing in* $\Gamma_p\mathrm{D}(RC)$ *with right orthogonal* S'.

Proof. We already know by Lemma 6.5 that $\mathrm{S}' \subseteq \mathrm{S}^{\perp}$. Let X be an object of $\mathrm{S}^{\perp}$. By the last lemma we know there is a triangle

$$X' \to X \to X'' \to \Sigma X'$$

with $X' \in \mathrm{S}$ and $X'' \in \mathrm{S}'$. But, since $X \in \mathrm{S}^{\perp}$, the first map must vanish, implying $X'' \cong X \oplus \Sigma X'$. This in turn implies $X' \cong 0$ since $\mathrm{S} \cap \mathrm{S}' = 0$. We thus conclude that $X \cong X''$, i.e., $X \in \mathrm{S}'$, proving $\mathrm{S}^{\perp} = \mathrm{S}'$. In particular, S is smashing. □

We now have enough to prove that we can describe the smashing subcategories of $\Gamma_p \mathrm{D}(RC)$ in terms of the smashing subcategories of $\mathrm{D}(k(p)C)$.

Theorem 6.8. *There is an order-preserving bijection*

$$\left\{\begin{matrix}\textit{smashing subcategories}\\ \textit{of } \Gamma_p \mathrm{D}(RC)\end{matrix}\right\} \underset{g_p}{\overset{f_p}{\rightleftarrows}} \left\{\begin{matrix}\textit{smashing subcategories}\\ \textit{of } \mathrm{D}(k(p)C)\end{matrix}\right\}.$$

Proof. We know from Proposition 4.1(2) that there is a bijection between the sets of localizing subcategories of $\Gamma_p \mathrm{D}(RC)$ and $\mathrm{D}(k(p)C)$ given by f_p and g_p. By Lemma 6.3 and Proposition 6.7 both f_p and g_p send smashing subcategories to smashing subcategories and so the bijection restricts as claimed. □

Obtaining the corresponding result for localizing subcategories generated by compact objects of $\Gamma_p \mathrm{D}(RC)$ and $\mathrm{D}(k(p)C)$ seems more subtle. However, if R is sufficiently nice at the prime ideal p this is possible. In order to state the result we need a simple preparatory lemma.

Lemma 6.9. *Let p be a prime ideal of* Spec R. *The category $\Gamma_p \mathrm{D}(RC)$ is a compactly generated triangulated category.*

Proof. Recall that $\Gamma_p \mathrm{D}(RC)$ is the essential image of acting by

$$\Gamma_p R = K_\infty(p) \otimes_R R_p.$$

It is clear that $\mathrm{D}(R_p C)$, the essential image of acting by R_p, is a compactly generated triangulated category. By [Stevenson 2013, Corollary 4.11] the essential image of $K_\infty(p)_p \otimes_{R_p} (-)$ acting on $\mathrm{D}(R_p C)$, namely $\Gamma_p \mathrm{D}(RC)$, is also compactly generated (even by objects of $\mathrm{D}^{\mathrm{perf}}(R_p C)$). □

In the statement and proof of the following proposition, $(\Gamma_p \mathrm{D}(RC))^c$ denotes the full subcategory of compact objects of $\Gamma_p \mathrm{D}(RC)$.

Proposition 6.10. *Let p be a prime ideal of R such that R_p is regular. Then the assignments f_p and g_p of Proposition 4.1*(2) *induce an order-preserving bijection between localizing subcategories of* $\Gamma_p \mathrm{D}(RC)$ *generated by objects of* $(\Gamma_p \mathrm{D}(RC))^c$ *and localizing subcategories of* $\mathrm{D}(k(p)C)$ *generated by objects of* $\mathrm{D}^{\mathrm{perf}}(k(p)C)$.

Proof. The base change functor $\Gamma_p \mathrm{D}(RC) \to \mathrm{D}(k(p)C)$ has a coproduct-preserving right adjoint and so sends compacts to compacts by [Neeman 1996, Theorem 5.1]. Thus it is clear that f_p sends any localizing subcategory of $\Gamma_p \mathrm{D}(RC)$ generated by objects of $(\Gamma_p \mathrm{D}(RC))^c$ to a localizing subcategory generated by objects of $\mathrm{D}^{\mathrm{perf}}(k(p)C)$. The argument for g_p is similar, using the fact that, as R_p is regular, the residue field $k(p)$ is compact, and so the right adjoint of the restriction functor $\mathrm{Hom}_R(k(p), -)$ is also coproduct-preserving. □

As an immediate consequence of the theorem and the proposition we deduce the following corollary.

Corollary 6.11. *Suppose R_p is regular. Then $\Gamma_p\mathrm{D}(RC)$ satisfies the telescope conjecture if and only if $\mathrm{D}(k(p)C)$ satisfies the telescope conjecture.*

Proof. Suppose $\mathrm{D}(k(p)C)$ satisfies the telescope conjecture and let S be a smashing subcategory of $\Gamma_p\mathrm{D}(RC)$. Then $f_p(S)$ is smashing in $\mathrm{D}(k(p)C)$ by Theorem 6.8 and we have $g_p f_p(\mathsf{S}) = \mathsf{S}$. Since we have assumed the telescope conjecture for $\mathrm{D}(k(p)C)$, we know $f_p(S)$ is generated by objects of $\mathrm{D}^{\mathrm{perf}}(k(p)C)$. Applying Proposition 6.10 we deduce that $\mathsf{S} = g_p f_p(\mathsf{S})$ is generated by objects which are compact in $\Gamma_p\mathrm{D}(RC)$. Thus the telescope conjecture holds for $\Gamma_p\mathrm{D}(RC)$. The other implication is clear since i^* preserves compact objects. □

This corollary already buys us something in a concrete setting, although it is not clear how to extend it to all of $\mathrm{D}(RC)$.

Corollary 6.12. *Let Q be a quiver and let R be a commutative noetherian ring. For each $p \in \operatorname{Spec} R$ such that R_p is regular, the telescope conjecture holds for $\Gamma_p\mathrm{D}(RC)$.*

Proof. By the previous corollary it is sufficient to verify the telescope conjecture for $\mathrm{D}(k(p)Q)$. This has been done by Krause and Šťovíček [2010, Theorem 7.1]. □

We give one additional lemma that could prove useful in resolving Question 6.1.

Lemma 6.13. *If S is a smashing subcategory of $\mathrm{D}(RC)$ then for every $p \in \operatorname{Spec} R$ the localizing subcategory $\Gamma_p\mathsf{S}$ is smashing in $\Gamma_p\mathrm{D}(RC)$.*

Proof. It is not hard to check that both $\Gamma_p\mathsf{S}$ and $\Gamma_p(\mathsf{S}^\perp)$ are localizing subcategories of $\Gamma_p\mathrm{D}(RC)$. Moreover,

$$\Gamma_p\mathsf{S} \subseteq \mathsf{S} \quad \text{and} \quad \Gamma_p(\mathsf{S}^\perp) \subseteq \mathsf{S}^\perp$$

by Lemma 3.3. In particular, $\Gamma_p(\mathsf{S}^\perp) \subseteq (\Gamma_p\mathsf{S})^\perp$. Applying $\Gamma_p R \otimes_R (-)$ to localization triangles for S shows that every object X of $\Gamma_p\mathrm{D}(RC)$ fits into a triangle

$$X' \to X \to X'' \to \Sigma X'$$

with $X' \in \Gamma_p\mathsf{S}$ and $X'' \in \Gamma_p(\mathsf{S}^\perp)$. One can conclude the proof by arguing as in the proof of Proposition 6.7. □

In summary, we understand what happens at "points" and we can pass from a smashing subcategory of $\mathrm{D}(RC)$ to a smashing subcategory at each prime. What is not clear is how to use this pointwise information to deduce something about the original smashing subcategory. The naive idea, based on the existing proofs of the telescope conjecture in various instances, would be to prove some sort of specialization closure condition for the section corresponding to a smashing subcategory as in Theorem 5.9. One could then hope to combine such a condition with the fiberwise results above. However, the following example shows that one cannot always expect specialization closure.

Example 6.14. Consider the projection $\operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x]$. We then view $\operatorname{Mod} k[x, y]$ as a $k[x]$-linear category. This gives rise to an action of $\mathrm{D}(k[x])$ on $\mathrm{D}(k[x, y])$. Let S be the smashing subcategory of $\mathrm{D}(k[x, y])$ determined by the closed curve $xy = 1$. Then the support of S with respect to the action of $\mathrm{D}(k[x])$ is open in $\operatorname{Spec} k[x]$. Of course, in this case the telescope conjecture does hold.

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VECTOR BUNDLES OVER A REAL ELLIPTIC CURVE

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Given a geometrically irreducible smooth projective curve of genus 1 defined over the field of real numbers, and a pair of integers *r* and *d*, we determine the isomorphism class of the moduli space of semistable vector bundles of rank *r* and degree *d* on the curve. When *r* and *d* are coprime, we describe the topology of the real locus and give a modular interpretation of its points. We also study, for arbitrary rank and degree, the moduli space of indecomposable vector bundles of rank *r* and degree *d*, and determine its isomorphism class as a real algebraic variety.

1. Introduction

1A. ***Notation.*** In this paper, a *real elliptic curve* will be a triple (X, x_0, σ) where (X, x_0) is a complex elliptic curve (i.e., a compact connected Riemann surface of genus 1 with a marked point x_0) and $\sigma : X \to X$ is an antiholomorphic involution (also called a real structure). We do not assume that x_0 is fixed under σ. In particular, $X^\sigma := \mathrm{Fix}(\sigma)$ is allowed to be empty.

The gcd of two integers r and d will be denoted by $r \wedge d$.

In the introduction, we omit the definitions of stability and semistability of vector bundles, as well as that of real and quaternionic structures; all these definitions will be recalled in Section 2.

1B. ***The case of genus zero.*** Vector bundles over a real Riemann surface of genus $g \geq 2$ have been studied from various points of view in the past few years: moduli spaces of real and quaternionic vector bundles were introduced through gauge-theoretic techniques in [Biswas et al. 2010], then related to the real points of the usual moduli variety in [Schaffhauser 2012]. In genus 0, there are, up to isomorphism, only two possible real Riemann surfaces: the only compact Riemann surface of genus 0 is the Riemann sphere $\mathbb{CP}^1$ and it can be endowed either with the real structure $[z_1 : z_2] \mapsto [\overline{z}_1 : \overline{z}_2]$ or with the real structure $[z_1 : z_2] \mapsto [-\overline{z}_2 : \overline{z}_1]$. The real locus of the first real structure is $\mathbb{RP}^1$ while the real locus of the second one is empty. Now, over $\mathbb{CP}^1$, two holomorphic line bundles are isomorphic if and

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only if they have the same degree and, by a theorem due to Grothendieck [1957], any holomorphic vector bundle over the Riemann sphere is isomorphic to a direct sum of line bundles. So, over $\mathbb{CP}^1$, the only stable vector bundles are the line bundles, a semistable vector bundle is necessarily polystable, and any vector bundle is isomorphic to a direct sum of semistable vector bundles, distinguished by their respective slopes and ranks. In particular, if $\mathcal{E}$ is semistable of rank r and degree d, then r divides d and

$$\mathcal{E} \simeq \mathcal{O}(d/r) \oplus \cdots \oplus \mathcal{O}(d/r),$$

where $\mathcal{O}(1)$ is the positive degree generator of the Picard group of $\mathbb{CP}^1$ and $\mathcal{O}(k)$ is its k-th tensor power. This means that the moduli space of semistable vector bundles of rank r and degree d over $\mathbb{CP}^1$ is

$$\mathcal{M}_{\mathbb{CP}^1}(r,d) = \begin{cases} \{\text{pt}\} & \text{if } r \mid d, \\ \varnothing & \text{if } r \nmid d. \end{cases}$$

Assume now that a real structure σ has been given on $\mathbb{CP}^1$. Then, if $\mathcal{L}$ is a holomorphic line bundle of degree d over $\mathbb{CP}^1$, it is isomorphic to its Galois conjugate $\overline{\sigma^*\mathcal{L}}$, since they have the same degree. This implies that $\mathcal{L}$ is either real or quaternionic. Moreover, this real or quaternionic structure is unique up to real or quaternionic isomorphism, respectively; see Proposition 2.10. If the real structure σ has real points, then quaternionic bundles must have even rank. Thus, when $\mathrm{Fix}(\sigma) \neq \varnothing$ in $\mathbb{CP}^1$, any line bundle (more generally, any direct sum of holomorphic line bundles) admits a canonical real structure. Of course, given a real vector bundle of the form $(\mathcal{L} \oplus \mathcal{L}, \tau \oplus \tau)$, where τ is a real structure on the line bundle $\mathcal{L}$, one can also construct the quaternionic structure

$$\begin{pmatrix} 0 & -\tau \\ \tau & 0 \end{pmatrix}$$

on $\mathcal{L} \oplus \mathcal{L}$. Note that the real vector bundle

$$\left(\mathcal{L} \oplus \mathcal{L}, \begin{pmatrix} 0 & \tau \\ \tau & 0 \end{pmatrix}\right)$$

is isomorphic to $(\mathcal{L} \oplus \mathcal{L}, \tau \oplus \tau)$. When $\mathbb{CP}^1$ is equipped with its real structure with no real points, a given line bundle $\mathcal{L}$ of degree k is again necessarily self-conjugate, so it has to be either real or quaternionic, but now real line bundles must have even degree and quaternionic line bundles must have odd degree [Biswas et al. 2010], so $\mathcal{L}$ admits a canonical real structure if k is even and a canonical quaternionic structure if k is odd. Consequently, when $\mathrm{Fix}(\sigma) = \varnothing$ in $\mathbb{CP}^1$, semistable holomorphic vector bundles of rank r and degree $d = rk$ over $\mathbb{CP}^1$ admit a canonical real structure if k is even and a canonical quaternionic structure if k is odd.

1C. *Description of the results.* The goal of the present paper is to analyze that same situation in the case of real Riemann surfaces of genus one. In particular, we completely identify the moduli space of semistable holomorphic vector bundles of rank r and degree d as a real algebraic variety (Theorem 1.1 below). Our main references are [Atiyah 1957b; Tu 1993]. In what follows, we denote by (X, x_0) a complex elliptic curve and by $\mathcal{M}_X(r, d)$ the moduli space of semistable vector bundles of rank r and degree d over X, i.e., the set of S-equivalence classes of semistable holomorphic vector bundles of rank r and degree d over X [Seshadri 1967]. Since (X, x_0) is an elliptic curve, the results of Atiyah show that any holomorphic vector bundle on X is again (as in genus 0) a direct sum of semistable vector bundles (see Theorem 3.3) but now there can be semistable vector bundles which are not polystable (see (3-2)) and also there can be stable vector bundles of rank higher than 1. Moreover, the moduli space $\mathcal{M}_X(r, d)$ is a nonsingular complex algebraic variety of dimension $h := r \wedge d$. As a matter of fact, $\mathcal{M}_X(r, d)$ is isomorphic, as a complex algebraic variety, to the $(r \wedge d)$-fold symmetric product $\mathrm{Sym}^{r\wedge d}(X)$ of the complex elliptic curve X, and it contains stable bundles if and only if $r \wedge d = 1$, in which case all semistable bundles are in fact stable. Let now $\sigma : X \to X$ be a real structure on X (recall that the marked point x_0 is not assumed to be fixed under σ). Then the map $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$ induces a real structure, again denoted by σ, on $\mathcal{M}_X(r, d)$, since it preserves the rank, the degree, and the S-equivalence class of semistable vector bundles [Schaffhauser 2012]. Our main result is then the following, to be proved in Section 2C.

Theorem 1.1. *Let $h := r \wedge d$. Then, as a real algebraic variety,*

$$\mathcal{M}_X(r, d) \simeq_{\mathbb{R}} \begin{cases} \mathrm{Sym}^h(X) & \text{if } X^\sigma \neq \varnothing, \\ \mathrm{Sym}^h(X) & \text{if } X^\sigma = \varnothing \text{ and } d/h \text{ is odd}, \\ \mathrm{Sym}^h(\mathrm{Pic}^0_X) & \text{if } X^\sigma = \varnothing \text{ and } d/h \text{ is even}. \end{cases}$$

We recall that Pic^0_X is isomorphic to X over $\mathbb{C}$ (via the choice of x_0) but not over $\mathbb{R}$ when $X^\sigma = \varnothing$ because Pic^0_X has the real point corresponding to the trivial line bundle. In contrast, Pic^1_X is always isomorphic to X over $\mathbb{R}$, as we shall recall in Section 2A. For any $d \in \mathbb{Z}$, the real structure of Pic^d_X is induced by the map $\mathcal{L} \mapsto \overline{\sigma^*\mathcal{L}}$, while the real structure of the h-fold symmetric product $\mathrm{Sym}^h(Y)$ of a real variety (Y, σ) is induced by that of Y in the following way: $[y_1, \dots, y_h] \mapsto [\sigma(y_1), \dots, \sigma(y_h)]$. Note that, if $r \wedge d = 1$, then by Theorem 1.1 we have $\mathcal{M}_X(r, d) \simeq_{\mathbb{R}} X$ if $X^\sigma \neq \varnothing$ or d is odd, and $\mathcal{M}_X(r, d) \simeq_{\mathbb{R}} \mathrm{Pic}^0_X$ if $X^\sigma = \varnothing$ and d is even. This will eventually imply the following results on the topology and modular interpretation of the set of real points of $\mathcal{M}_X(r, d)$, analogous to those of [Schaffhauser 2012] for real curves of genus $g \geq 2$ (see Section 2D for a proof of Theorem 1.2; we point out that it will only be valid under the assumption that $r \wedge d = 1$, in which case all semistable bundles are in fact stable; in particular a real point of $\mathcal{M}_X(r,d)$ is given by either a real bundle or a quaternionic bundle, in an essentially unique way; see Proposition 2.10).

Theorem 1.2. *Assume that* $r \wedge d = 1$.

(1) *If* $X^\sigma \neq \varnothing$, *then* $\mathcal{M}_X(r,d)^\sigma \simeq X^\sigma$ *has either* 1 *or* 2 *connected components. Points in either component correspond to real isomorphism classes of real vector bundles of rank r and degree d over* (X,σ), *and two such bundles* $(\mathcal{E}_1,\tau_1)$ *and* $(\mathcal{E}_2,\tau_2)$ *lie in the same connected component if and only if* $w_1(\mathcal{E}^{\tau_1}) = w_1(\mathcal{E}_2^{\tau_2})$.

(2) *If* $X^\sigma = \varnothing$ *and* $d = 2e+1$, *then* $\mathcal{M}_X(r,2e+1)^\sigma \simeq X^\sigma$ *is empty.*

(3) *If* $X^\sigma = \varnothing$ *and* $d = 2e$, *then* $\mathcal{M}_X(r,2e)^\sigma \simeq (\mathrm{Pic}^0_X)^\sigma$ *has two connected components, one consisting of real isomorphism classes of real bundles, the other consisting of quaternionic isomorphism classes of quaternionic bundles. These two components become diffeomorphic under the operation of tensoring a given bundle by a quaternionic line bundle of degree* 0.

In cases (1) *and* (3), *each connected component of* $\mathcal{M}_X(r,d)^\sigma$ *is diffeomorphic to* S^1.

In particular, the formulae of Liu and Schaffhauser [2013] (see also [Baird 2014]), giving the mod 2 Betti numbers of the connected components of $\mathcal{M}_X(r,d)^\sigma$ when $r \wedge d = 1$ are still valid for $g = 1$. In contrast, when $r \wedge d \neq 1$, the formulae do not seem to be interpretable in any way since, over an elliptic curve, the dimension of $\mathcal{M}_X(r,d)$ is $r \wedge d$, not $r^2(g-1)+1$.

In the third and final section of the paper, we investigate the properties of indecomposable vector bundles over real elliptic curves. Recall that a holomorphic vector bundle $\mathcal{E}$ over a complex curve X is said to be indecomposable if it is not isomorphic to a direct sum of nontrivial holomorphic bundles. When X is of genus 1, there exists a moduli variety $\mathcal{I}_X(r,d)$ whose points are isomorphism classes of indecomposable vector bundles of rank r and degree d: it was constructed by Atiyah [1957b] and revisited by Tu [1993], as will be recalled in Theorems 3.2 and 3.4. We will then see in Section 3C that we can extend their approach to the case of real elliptic curves and obtain the following characterization of $\mathcal{I}_X(r,d)$ as a real algebraic variety.

Theorem 1.3. *Let* (X, x_0, σ) *be a real elliptic curve. Let* $\mathcal{I}_X(r,d)$ *be the set of isomorphism classes of indecomposable vector bundles of rank r and degree d and set* $h := r \wedge d$, $r' := r/h$, $d' := d/h$. *Then*

$$\mathcal{I}_X(r,d) \simeq_{\mathbb{R}} \mathcal{M}_X(r',d') \simeq_{\mathbb{R}} \begin{cases} X & \text{if } X^\sigma \neq \varnothing, \\ X & \text{if } X^\sigma = \varnothing \text{ and } d' \text{ is odd}, \\ \mathrm{Pic}^0_X & \text{if } X^\sigma = \varnothing \text{ and } d' \text{ is even}. \end{cases}$$

By combining Theorems 1.2 and 1.3, we obtain the following topological description of the set of real points of $\mathcal{I}_X(r,d)$, valid even when $r \wedge d \neq 1$.

Theorem 1.4. *Denote by $\mathcal{I}_X(r,d)^\sigma$ the fixed points of the real structure $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$ in $\mathcal{I}_X(r,d)$.*

(1) *If $X^\sigma \neq \varnothing$, then $\mathcal{I}_X(r,d)^\sigma \simeq X^\sigma$ consists of real isomorphism classes of real and indecomposable vector bundles of rank r and degree d. It has either one or two connected components, according to whether X^σ has one or two connected components, and these are distinguished by the Stiefel–Whitney classes of the real parts of the real bundles that they contain.*

(2) *If $X^\sigma = \varnothing$ and $d/(r \wedge d) = 2e+1$, then $\mathcal{I}_X(r,d)^\sigma \simeq X^\sigma$ is empty.*

(3) *If $X^\sigma = \varnothing$ and $d/(r \wedge d) = 2e$, then $\mathcal{I}_X(r,d)^\sigma \simeq (\mathrm{Pic}^0_X)^\sigma$ has two connected components, one consisting of real isomorphism classes of vector bundles which are both real and indecomposable and one consisting of quaternionic isomorphism classes of vector bundles which are both quaternionic and indecomposable. These two components become diffeomorphic under the operation of tensoring a given bundle by a quaternionic line bundle of degree* 0*.*

In cases (1) *and* (3), *each connected component of the set of real points of $\mathcal{I}_X(r,d)$ is diffeomorphic to S^1.*

2. Moduli spaces of semistable vector bundles over an elliptic curve

2A. ***Real elliptic curves and their Picard varieties.*** The real points of Picard varieties of real algebraic curves were studied, for instance, by Gross and Harris [1981]. We summarize here some of their results, specializing to the case of genus 1 curves.

Let X be a compact connected Riemann surface of genus 1. To each point $x \in X$, there is associated a holomorphic line bundle $\mathcal{L}(x)$, of degree 1, whose holomorphic sections have a zero of order 1 at x and no other zeros or poles. Since X is compact, the map $X \to \mathrm{Pic}^1_X$ thus defined, called the Abel–Jacobi map, is injective. And since X has genus 1, it is also surjective. The choice of a point $x_0 \in X$ defines an isomorphism $\mathrm{Pic}^0_X \xrightarrow{\simeq} \mathrm{Pic}^1_X$, obtained by tensoring by $\mathcal{L}(x_0)$. In particular, $X \simeq \mathrm{Pic}^1_X$ is isomorphic to Pic^0_X as a complex analytic manifold and inherits, moreover, a structure of Abelian group with x_0 as the neutral element.

If $\sigma : X \to X$ is a real structure on X, the Picard variety Pic^d_X, whose points represent isomorphism classes of holomorphic line bundles of degree d, has a canonical real structure, defined by $\mathcal{L} \mapsto \overline{\sigma^*\mathcal{L}}$ (observe that this antiholomorphic involution, which we will still denote by σ, indeed preserves the degree). Since $\mathcal{L}(\sigma(x)) \simeq \overline{\sigma^*(\mathcal{L}(x))}$, the Abel–Jacobi map $X \to \mathrm{Pic}^1_X$ is defined over $\mathbb{R}$, meaning that it commutes with the real structures of X and Pic^1_X. We also call such a map a real map. If $X^\sigma \neq \varnothing$, we can choose $x_0 \in X^\sigma$ and then $\mathcal{L}(x_0)$ will satisfy

$\overline{\sigma^*\mathcal{L}(x_0)} \simeq \mathcal{L}(x_0)$ so the isomorphism $\mathrm{Pic}^0_X \xrightarrow{\sim} \mathrm{Pic}^1_X$ obtained by tensoring by $\mathcal{L}(x_0)$ will also be defined over $\mathbb{R}$. More generally, by tensoring by a suitable power of $\mathcal{L}(x_0)$, we obtain real isomorphisms $\mathrm{Pic}^d_X \simeq \mathrm{Pic}^1_X$ for any $d \in \mathbb{Z}$. If now $X^\sigma = \varnothing$, then we actually cannot choose x_0 in such a way that $\mathcal{L}(\sigma(x_0)) \simeq \mathcal{L}(x_0)$ — see [Gross and Harris 1981] or Theorem 2.8 below; the reason is that such a line bundle would be either real or quaternionic but, over a real curve of genus 1 with no real points, real and quaternionic line bundles must have even degree — but we may consider the holomorphic line bundle of degree 2 defined by the divisor $x_0 + \sigma(x_0)$, call it $\mathcal{L}$, say. Then $\overline{\sigma^*\mathcal{L}} \simeq \mathcal{L}$ and, by tensoring by an appropriate tensor power of it, we have the real isomorphisms

$$\mathrm{Pic}^d_X \simeq_{\mathbb{R}} \begin{cases} \mathrm{Pic}^1_X & \text{if } d = 2e+1, \\ \mathrm{Pic}^0_X & \text{if } d = 2e. \end{cases}$$

So, when the genus of X is 1, we have the following result:

Theorem 2.1. *Let (X, x_0, σ) be a real elliptic curve.*

(1) *If $X^\sigma \neq \varnothing$, then for all $d \in \mathbb{Z}$,*

$$\mathrm{Pic}^d_X \simeq_{\mathbb{R}} X.$$

(2) *If $X^\sigma = \varnothing$, then*

$$\mathrm{Pic}^d_X \simeq_{\mathbb{R}} \begin{cases} X & \text{if } d = 2e+1, \\ \mathrm{Pic}^0_X & \text{if } d = 2e. \end{cases}$$

2B. ***Semistable vector bundles.*** Let X be a compact connected Riemann surface of genus g and recall that the slope of a nonzero holomorphic vector bundle $\mathcal{E}$ on X is by definition the ratio $\mu(\mathcal{E}) = \deg(\mathcal{E})/\operatorname{rk}(\mathcal{E})$ of its degree by its rank. The vector bundle $\mathcal{E}$ is called *stable* if for any nonzero proper subbundle $\mathcal{F} \subset \mathcal{E}$, one has $\mu(\mathcal{F}) < \mu(\mathcal{E})$. Analogously, $\mathcal{E}$ is called *semistable* if $\mu(\mathcal{F}) \leq \mu(\mathcal{E})$. By a theorem of Seshadri [1967], any semistable vector bundle $\mathcal{E}$ of rank r and degree d admits a filtration whose successive quotients are stable bundles of the same slope, necessarily equal to d/r. Such a filtration, called a Jordan–Hölder filtration, is not unique but the graded objects associated to any two such filtrations are isomorphic. The isomorphism class thus defined is denoted by $\mathrm{gr}(\mathcal{E})$ and holomorphic vector bundles which are isomorphic to direct sums of stable vector bundles of equal slope are called *polystable* vector bundles. Moreover, two semistable vector bundles $\mathcal{E}_1$ and $\mathcal{E}_2$ are called S-equivalent if $\mathrm{gr}(\mathcal{E}_1) = \mathrm{gr}(\mathcal{E}_2)$ and Seshadri proved in [Seshadri 1967] that, when $g \geq 2$, the set of S-equivalence classes of semistable vector bundles of rank r and degree d admits a structure of complex projective variety of dimension $r^2(g-1)+1$ and is nonsingular when $r \wedge d = 1$ but usually singular when $r \wedge d \neq 1$ (unless, in fact, $g = 2$, $r = 2$, and $d = 0$). Finally, when $g \geq 2$, there are always stable bundles of rank r and degree d over X [Narasimhan and Seshadri 1965].

These come from irreducible rank r unitary representations of a certain central extension of $\pi_1(X)$ by $\mathbb{Z}$, determined by d up to isomorphism). If now $g = 1$, then the results of Atiyah [1957b] and Tu [1993] show that the set of S-equivalence classes of semistable vector bundles of rank r and degree d admits a structure of nonsingular complex projective variety of dimension $r \wedge d$ (which is consistent with the formula for $g \geq 2$ only when r and d are coprime). But now stable vector bundles of rank r and degree d can only exist if $r \wedge d = 1$, as Tu showed [1993, Theorem A] following Atiyah's results. In particular, the structure of polystable vector bundles over a complex elliptic curve is rather special, as recalled next.

Proposition 2.2 (Atiyah–Tu). *Let $\mathcal{E}$ be a polystable holomorphic vector bundle of rank r and degree d over a compact connected Riemann surface X of genus* 1. *Set $h := r \wedge d$, $r' := r/h$, and $d' := d/h$. Then $\mathcal{E} \simeq \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h$ where each $\mathcal{E}_i$ is a stable holomorphic vector bundle of rank r' and degree d'.*

Proof. By definition, a polystable bundle of rank r and degree d is isomorphic to a direct sum $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ of stable bundles of slope $d/r = d'/r'$. Since $d' \wedge r' = 1$ and each $\mathcal{E}_i$ is stable of slope d'/r', each $\mathcal{E}_i$ must have rank r' and degree d' (because stable bundles over elliptic curves must have coprime rank and degree). Since $\mathrm{rk}(\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k) = kr' = \mathrm{rk}(\mathcal{E}) = r$, we have indeed $k = h$. □

To understand the moduli space $\mathcal{M}_X(r, d)$ of semistable holomorphic vector bundles of rank r and degree d over a complex elliptic curve X, one then has the next two theorems.

Theorem 2.3 (Atiyah–Tu). *Let X be a compact connected Riemann surface of genus* 1 *and assume that $r \wedge d = 1$. Then the determinant map* $\det : \mathcal{M}_X(r, d) \to \mathrm{Pic}_X^d$ *is an isomorphism of complex analytic manifolds of dimension* 1.

Note that, when $r \wedge d = 1$, any semistable vector bundle of rank r and degree d is in fact stable (over a curve of arbitrary genus) and that, to prove Theorem 2.3, it is in particular necessary to show that a stable vector bundle $\mathcal{E}$ of rank r and degree d over a complex elliptic curve X satisfies $\mathcal{E} \otimes \mathcal{L} \simeq \mathcal{E}$ if and only if $\mathcal{L}$ is an r-torsion point in Pic_X^0 (i.e., $\mathcal{L}^{\otimes r} \simeq \mathcal{O}_X$), a phenomenon which only occurs in genus 1.

If now $h := r \wedge d \geq 2$, then we know, by Proposition 2.2, that a semistable vector bundle of rank r and degree d is isomorphic to the direct sum of h stable vector bundles of rank $r' = r/h$ and degree $d' = d/h$. Combining this with Theorem 2.3, one obtains:

Theorem 2.4 [Tu 1993, Theorem 1]. *Let X be a compact connected Riemann surface of genus* 1 *and denote by $h := r \wedge d$. Then there is an isomorphism of complex analytic manifolds*:

$$\mathcal{M}_X(r, d) \xrightarrow{\sim} \mathrm{Sym}^h(\mathrm{Pic}_X^{d/h}), \quad \mathcal{E} \simeq \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h \mapsto [\det \mathcal{E}_i]_{1 \leq i \leq h}.$$

In particular, $\mathcal{M}_X(r,d)$ has dimension $h = r \wedge d$. Since the choice of a point $x_0 \in X$ provides an isomorphism $\mathrm{Pic}_X^d \simeq_{\mathbb{C}} X$, we have indeed $\mathcal{M}_X(r,d) \simeq_{\mathbb{C}} \mathrm{Sym}^h(X)$. In the next section, we will analyze the corresponding situation over $\mathbb{R}$. But first we recall the basics about real and quaternionic vector bundles.

Let (X, σ) be a real Riemann surface, i.e., a Riemann surface X endowed with a real structure σ. A real holomorphic vector bundle over (X, σ) is a pair $(\mathcal{E}, \tau)$ such that $\mathcal{E} \to X$ is a holomorphic vector bundle over X and $\tau : \mathcal{E} \to \mathcal{E}$ is an antiholomorphic map such that

(1) the diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\tau} & \mathcal{E}_2 \\ \downarrow & & \downarrow \\ X & \xrightarrow{\sigma} & X \end{array}$$

is commutative;

(2) the map τ is fiberwise $\mathbb{C}$-antilinear: $\forall v \in \mathcal{E}, \forall \lambda \in \mathbb{C}, \tau(\lambda v) = \bar{\lambda}(v)$;

(3) $\tau^2 = \mathrm{Id}_{\mathcal{E}}$.

A quaternionic holomorphic vector bundle over (X, σ) is a pair $(\mathcal{E}, \tau)$ satisfying conditions (1) and (2) above, as well as a modified third condition:

(3′) $\tau^2 = -\mathrm{Id}_{\mathcal{E}}$.

A homomorphism $\varphi : (\mathcal{E}_1, \tau_1) \to (\mathcal{E}_2, \tau_2)$ between two real or quaternionic vector bundles is a holomorphic map $\varphi : \mathcal{E}_1 \to \mathcal{E}_2$ such that

(1) the diagram

$$\begin{array}{ccc} \mathcal{E}_1 & \xrightarrow{\varphi} & \mathcal{E}_2 \\ \downarrow & & \downarrow \\ X & = & X \end{array}$$

is commutative;

(2) $\varphi \circ \tau_1 = \tau_2 \circ \varphi$.

A real or quaternionic holomorphic vector bundle is called *stable* if for any τ-invariant subbundle $\mathcal{F} \subset \mathcal{E}$, one has $\mu(\mathcal{F}) < \mu(\mathcal{E})$. It is called *semistable* if for any such $\mathcal{F}$, one has $\mu(\mathcal{F}) \leq \mu(\mathcal{F})$. As shown in [Schaffhauser 2012], $(\mathcal{E}, \tau)$ is semistable as a real vector bundle if and only $\mathcal{E}$ is semistable as a holomorphic vector bundle but $(\mathcal{E}, \tau)$ may be stable as a real vector bundle while being only polystable as a holomorphic vector bundle (when $\mathcal{E}$ is in fact stable, we will say that $(\mathcal{E}, \tau)$ is geometrically stable). However, any semistable real vector bundle admits real Jordan–Hölder filtrations (where the successive quotients can sometimes be stable in the real sense only) and there is a corresponding notion of polystable real vector bundle, which turns out to be equivalent to being polystable and real. Analogous results hold for quaternionic vector bundles. Real and quaternionic vector bundles over a compact connected real Riemann surface (X, σ) were topologically classified in [Biswas et al. 2010]. If $X^\sigma \neq \varnothing$, a real vector bundle $(\mathcal{E}, \tau)$ over (X, σ) defines in particular a real vector bundle in the ordinary sense $\mathcal{E}^\tau \to X^\sigma$; hence an associated first Stiefel–Whitney class $w_1(\mathcal{E}^\tau) \in H^1(X^\sigma; \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^n$, where $n \in \{0, \dots, g+1\}$ is the number of connected components of X^σ. The topological classification of real and quaternionic vector bundles then goes as follows.

Theorem 2.5 [Biswas et al. 2010]. *Let (X, σ) be a compact connected real Riemann surface.*

(1) *If $X^\sigma \neq \varnothing$, real vector bundles over (X, σ) are classified up to smooth isomorphism by the numbers $r = \mathrm{rk}(\mathcal{E})$, $d = \deg(\mathcal{E})$, and $(s_1, \dots, s_n) = w_1(\mathcal{E}^\tau)$, subject to the condition $s_1 + \cdots + s_n \equiv d \pmod 2$. Quaternionic vector bundles must have even rank and degree in this case and are classified up to smooth isomorphism by the pair $(2r, 2d)$.*

(2) *If $X^\sigma = \varnothing$, real vector bundles over (X, σ) must have even degree and are classified up to smooth isomorphism by the pair $(r, 2d)$. Quaternionic vector bundles are classified up to smooth isomorphism by the pair (r, d), subject to the condition $d + r(g-1) \equiv 0 \pmod 2$. In particular, if $g = 1$, real and quaternionic vector bundles alike must have even degree.*

Theorem 2.5 will be useful in Section 2D, for the proof of Theorem 1.2.

2C. ***The real structure of the moduli space.*** Let first (X, σ) be a real Riemann surface of arbitrary genus g. Then the involution $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$ preserves the rank and the degree of a holomorphic vector bundle and the bundle $\overline{\sigma^*\mathcal{E}}$ is stable if and only if $\mathcal{E}$ is. The analogous statement holds for semistable bundles. Moreover, if $\mathcal{E}$ is semistable, a Jordan–Hölder filtration of $\mathcal{E}$ is mapped to a Jordan–Hölder filtration of $\overline{\sigma^*\mathcal{E}}$, so, for any g, the moduli space $\mathcal{M}_X(r, d)$ of semistable holomorphic vector bundles of rank r and degree d on X has an induced real structure. Assume now that $g = 1$ and let us prove Theorem 1.1.

Proof of Theorem 1.1. Since, for any vector bundle $\mathcal{E}$ one has $\det(\overline{\sigma^*\mathcal{E}}) = \overline{\sigma^*(\det \mathcal{E})}$, the map

$$\mathcal{M}_X(r, d) \xrightarrow{\sim} \mathrm{Sym}^h(\mathrm{Pic}_X^{d/h}), \quad \mathcal{E} \simeq \mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_h \mapsto [\det \mathcal{E}_i]_{1 \le i \le h}$$

of Theorem 2.4 is a real map. If $X^\sigma \neq \varnothing$, then by Theorem 2.1 we have

$$\mathrm{Pic}_X^d \simeq_{\mathbb{R}} \mathrm{Pic}_X^0 \simeq_{\mathbb{R}} X,$$

so $\mathcal{M}_X(r, d) \simeq_{\mathbb{R}} \mathrm{Sym}^h(X)$ in this case. And if $X^\sigma = \varnothing$, we distinguish between the cases $d = 2e + 1$ and $d = 2e$ to obtain, again by Theorem 2.1, that

$$\mathcal{M}_X(r, d) \simeq_{\mathbb{R}} \begin{cases} \mathrm{Sym}^h(X) & \text{if } d/h \text{ is odd,} \\ \mathrm{Sym}^h(\mathrm{Pic}^0_X) & \text{if } d/h \text{ is even,} \end{cases}$$

which finishes the proof of Theorem 1.1. □

Let us now focus on the case $d = 0$, where there is a nice alternate description of the moduli variety in terms of representations of the fundamental group of the elliptic curve (X, x_0). Since $\pi_1(X, x_0) \simeq \mathbb{Z}^2$ is a free Abelian group on two generators, a rank r unitary representation of it is entirely determined by the data of two commuting unitary matrices u_1, u_2 in $\boldsymbol{U}(r)$ (in particular, such a representation is never irreducible unless $r = 1$), and we may assume that these two matrices lie in the maximal torus $\boldsymbol{T}_r \subset \boldsymbol{U}(r)$ consisting of diagonal unitary matrices. The Weyl group of $\boldsymbol{T}_r$ is $\boldsymbol{W}_r \simeq \mathfrak{S}_r$, the symmetric group on r letters, and one has

$$\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{U}(r))/\boldsymbol{U}(r) \simeq \mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{T}_r)/\boldsymbol{W}_r. \tag{2-1}$$

Note that since $\pi_1(X, x_0)$ is Abelian, there is a well-defined action of σ on it even if $x_0 \notin X^\sigma$: a loop γ at x_0 is sent to the loop $\sigma \circ \gamma$ at $\sigma(x_0)$ then brought back to x_0 by conjugation by an arbitrary path between x_0 and $\sigma(x_0)$. Combining this with the involution $u \mapsto \bar{u}$ of $\boldsymbol{U}(r)$, we obtain an action of σ on $\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{U}(r))$, defined by sending a representation ρ to the representation $\sigma\rho\sigma$. This action preserves the subset $\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{T}_r)$ and is compatible with the conjugacy action of $\boldsymbol{U}(r)$ in the sense that $\sigma(\mathrm{Ad}_u\, \rho)\sigma = \mathrm{Ad}_{\sigma(u)}(\sigma\rho\sigma^{-1})$, so it induces an involution on the representation varieties $\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{U}(r))/\boldsymbol{U}(r)$ and $\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{T}_r)/\boldsymbol{W}_r$ and the bijection (2-1) is equivariant for the actions just described. By the results of Friedman, Morgan and Witten [Friedman et al. 1998] and Laszlo [1998], this representation variety is in fact isomorphic to the moduli space $\mathcal{M}_X(r, 0)$. Moreover, the involution $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$ on bundles corresponds to the involution $\rho \mapsto \sigma\rho\sigma$ on unitary representations. Moreover,

$$\boldsymbol{T}_r \simeq \underbrace{\boldsymbol{U}(1) \times \cdots \times \boldsymbol{U}(1)}_{r \text{ times}} \simeq \boldsymbol{U}(1) \otimes_{\mathbb{Z}} \mathbb{Z}^r$$

as Abelian Lie groups, where $\mathbb{Z}^r$ can be interpreted as $\pi_1(\boldsymbol{T}_r)$. In particular, the Galois action induced on $\mathbb{Z}^r$ by the complex conjugation on $\boldsymbol{T}_r$ is simply $(n_1, \ldots, n_r) \mapsto (-n_1, \ldots, -n_r)$ and the isomorphism $\boldsymbol{T}_r \simeq \boldsymbol{U}(1) \otimes \mathbb{Z}^r$ is equivariant with respect to these natural real structures. Finally, the bijection

$$\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{T}_r) \simeq \mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{U}(1)) \otimes \mathbb{Z}^r$$

is also equivariant, and the representation variety $\mathrm{Hom}(\pi_1(X, x_0); \boldsymbol{U}(1))$ is isomorphic to Pic^0_X as a real variety. We have thus proved the following result, which is an analogue over $\mathbb{R}$ of one of the results in [Friedman et al. 1998; Laszlo 1998].

Theorem 2.6. *Let (X, x_0, σ) be a real elliptic curve. Then the map*

$$(\mathrm{Pic}^0_X \otimes_{\mathbb{Z}} \mathbb{Z}^r) \simeq (\mathrm{Pic}^0_X)^r \to \mathcal{M}(r, 0), \quad (\mathcal{L}_1, \dots, \mathcal{L}_r) \mapsto \mathcal{L}_1 \oplus \cdots \oplus \mathcal{L}_r$$

induces an isomorphism

$$\mathcal{M}_X(r, 0) \simeq_{\mathbb{R}} (\mathrm{Pic}^0_X \otimes \mathbb{Z}^r)/\mathfrak{S}_r,$$

where the symmetric group $\mathfrak{S}_r$ acts on $\mathbb{Z}^r$ by permutation.

When $X^\sigma \neq \varnothing$, one can further identify Pic^0_X *with X over $\mathbb{R}$ and obtain the isomorphism*

$$\mathcal{M}_X(r, 0) \simeq_{\mathbb{R}} (X \otimes \mathbb{Z}^r)/\mathfrak{S}_r.$$

The results of Section 3 will actually give an alternate proof of Theorem 2.6, by using the theory of indecomposable vector bundles over elliptic curves. We point out that algebraic varieties of the form $(X \otimes \pi_1(\boldsymbol{T}))/\boldsymbol{W_T}$ for X a complex elliptic curve have been studied for instance by Looijenga [1976], who identified them with certain weighted projective spaces determined by the root system of $\boldsymbol{T}$, when the ambient group $G \supset \boldsymbol{T}$ is semisimple. Theorem 2.6 shows that, over $\mathbb{R}$, it may sometimes be necessary to replace X by Pic^0_X.

To conclude on the case where $d = 0$, we recall that, on $\mathcal{M}_X(r, 0)$, there exists another real structure, obtained from the real structure $\mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}$ by composing it with the holomorphic involution $\mathcal{E} \mapsto \mathcal{E}^*$, which in general sends a vector bundle of degree d to a vector bundle of degree $-d$, so preserves only the moduli spaces $\mathcal{M}_X(r, 0)$. Denote by

$$\eta_r : \mathcal{M}_X(r, 0) \to \mathcal{M}_X(r, 0), \quad \mathcal{E} \mapsto \overline{\sigma^*\mathcal{E}}^{\,*}$$

this new real structure on the moduli space $\mathcal{M}_X(r, 0)$. In particular, we have

$$\eta_1 : \mathrm{Pic}^0_X \to \mathrm{Pic}^0_X, \quad \mathcal{L} \mapsto \overline{\sigma^*\mathcal{L}}^{\,*}$$

and we note that η_1 has real points because it fixes the trivial line bundle. The real elliptic curve $(\mathrm{Pic}^0_X, \eta_1)$ is, in general, not isomorphic to (X, σ), even when σ has fixed points. We can nonetheless characterize the new real structure of the moduli spaces $\mathcal{M}_X(r, 0)$ in the following way:

Proposition 2.7. *The real variety $(\mathcal{M}_X(r, 0), \eta_r)$ is isomorphic to the r-fold symmetric product of the real elliptic curve* $(\mathrm{Pic}^0_X, \eta_1)$.

Proof. The proposition is proved in the same way as Theorem 2.6, changing only the real structures under consideration. □

2D. ***Topology of the set of real points in the coprime case.*** In rank 1, the topology of the set of real points of Pic^d_X is well understood and so is the modular interpretation of its elements.

Theorem 2.8 ([Gross and Harris 1981], case $g = 1$). *Let (X, σ) be a compact real Riemann surface of genus 1 and let $d \in \mathbb{Z}$.*

(1) *If $X^\sigma \neq \varnothing$, then $(\mathrm{Pic}^d_X)^\sigma \simeq X^\sigma$ has 1 or 2 connected components. Elements of $(\mathrm{Pic}^d_X)^\sigma$ correspond to real isomorphism classes of real holomorphic line bundles over (X, σ) and two such real line bundles $(\mathcal{L}_1, \tau_1)$ and $(\mathcal{L}_2, \tau_2)$ lie in the same connected component of $(\mathrm{Pic}^d_X)^\sigma$ if and only if $w_1(\mathcal{L}_1^{\tau_1}) = w_1(\mathcal{L}_2^{\tau_2})$.*

(2) *If $X^\sigma = \varnothing$ and $d = 2e + 1$, then $(\mathrm{Pic}^d_X)^\sigma \simeq X^\sigma$ is empty.*

(3) *If $X^\sigma = \varnothing$ and $d = 2e$, then $(\mathrm{Pic}^d_X)^\sigma \simeq (\mathrm{Pic}^0_X)^\sigma$ has 2 connected components, corresponding to isomorphism classes of either real or quaternionic line bundles of degree d, depending on the connected component of $(\mathrm{Pic}^d_X)^\sigma$ in which they lie.*

In cases (1) and (3), any given connected component of $(\mathrm{Pic}^d_X)^\sigma$ is diffeomorphic to S^1.

For real Riemann surfaces of genus $g \geq 2$, the topology of $(\mathrm{Pic}^d_X)^\sigma$, in particular the number of connected components, is a bit more involved but also covered in [Gross and Harris 1981], the point being that these components are indexed by the possible topological types of real and quaternionic line bundles over (X, σ). For vector bundles of rank $r \geq 2$ on real Riemann surfaces of genus $g \geq 2$, a generalization of the results of Gross and Harris was obtained in [Schaffhauser 2012]: we recall here the result for coprime rank and degree (in general, a similar but more complicated result holds provided one restricts one's attention to the stable locus in $\mathcal{M}_X(r, d)$). The coprime case is the case that we will actually generalize to genus 1 curves (where stable bundles can only exist in coprime rank and degree).

Theorem 2.9 [Schaffhauser 2012]. *Let (X, σ) be a compact real Riemann surface of genus $g \geq 2$ and assume that $r \wedge d = 1$. The number of connected component of $\mathcal{M}_X(r, d)^\sigma$ is equal to:*

(1) *2^{n-1} if X^σ has $n > 0$ connected components. In this case, elements of $\mathcal{M}_X(r, d)^\sigma$ correspond to real isomorphism classes of real holomorphic vector bundles of rank r and degree d and two such bundles $(\mathcal{E}_1, \tau_1)$ and $(\mathcal{E}_2, \tau_2)$ lie in the same connected component if and only if $w_1(\mathcal{E}^{\tau_1}) = w_1(\mathcal{E}_2^{\tau_2})$.*

(2) *0 if $X^\sigma = \varnothing$, d is odd and $r(g - 1)$ is even.*

(3) *1 if $X^\sigma = \varnothing$, d is odd and $r(g - 1)$ is odd, in which case the elements of $\mathcal{M}_X(r, d)^\sigma$ correspond to quaternionic isomorphism classes of quaternionic vector bundles of rank r and degree d.*

(4) *1 if $X^\sigma = \varnothing$, d is even and $r(g - 1)$ is odd, in which case the elements of $\mathcal{M}_X(r, d)^\sigma$ correspond to real isomorphism classes of real vector bundles of rank r and degree d.*

(5) 2 *if* $X^\sigma = \varnothing$, d *is even and* $r(g-1)$ *is even, in which case there is one component consisting of real isomorphism classes of real vector bundles of rank* r *and degree* d *while the other consists of quaternionic isomorphism classes of quaternionic vector bundles of rank* r *and degree* d.

Now, using Theorem 1.1, we can extend Theorem 2.9 to the case $g = 1$. Indeed, to prove Theorem 1.2, we only need to combine Theorem 2.8 and the coprime case of Theorem 1.1 (i.e., $h = 1$), with the following result, for a proof of which we refer to either [Biswas et al. 2010] or [Schaffhauser 2012].

Proposition 2.10. *Let* (X, σ) *be a compact connected real Riemann surface and let* $\mathcal{E}$ *be a stable holomorphic vector bundle over* X *satisfying* $\overline{\sigma^*\mathcal{E}} \simeq \mathcal{E}$. *Then* $\mathcal{E}$ *is either real or quaternionic and cannot be both. Moreover, two different real or quaternionic structures on* $\mathcal{E}$ *are conjugate by a holomorphic automorphism of* $\mathcal{E}$.

Note that it is easy to show that two real or two quaternionic structures on $\mathcal{E}$ *differ* by a holomorphic automorphism $e^{i\theta} \in S^1 \subset \mathbb{C}^* \simeq \mathrm{Aut}(\mathcal{E})$ but, in order to prove that these two structures τ_1 and τ_2, say, *are conjugate*, we need to observe that $\tau_2(\,\cdot\,) = e^{i\theta}\tau_1(\,\cdot\,) = e^{i\theta/2}\tau_1(e^{-i\theta/2}\,\cdot\,)$. Then, to finish the proof of Theorem 1.2, we proceed as follows:

Proof of Theorem 1.2. Recall that X here has genus 1. If $X^\sigma \neq \varnothing$, quaternionic vector bundles must have even rank and degree by Theorem 2.5, so, by Proposition 2.10, points of $\mathcal{M}_X(r,d)^\sigma$ correspond in this case to real isomorphism classes of geometrically stable real vector bundles of rank r and degree d. By Theorem 1.1, one indeed has $\mathcal{M}_X(r,d)^\sigma \simeq (\mathrm{Pic}^d_X)^\sigma \simeq X^\sigma$ in this case. Moreover, since the diffeomorphism $\mathcal{M}_X(r,d)^\sigma \simeq (\mathrm{Pic}^d_X)^\sigma$ is provided by the determinant map, the connected components of $\mathcal{M}_X(r,d)^\sigma$, or equivalently of $(\mathrm{Pic}^d_X)^\sigma$ are indeed distinguished by the first Stiefel–Whitney class of the real part of the real bundles that they parametrize, as in Theorem 2.8. If now $X^\sigma = \varnothing$, then by Theorem 2.5, real and quaternionic vector bundles must have even degree and we can again use Theorem 2.8 to conclude: note that since the diffeomorphism $\mathcal{M}_X(r,d)^\sigma \simeq (\mathrm{Pic}^d_X)^\sigma$ is provided by the determinant map, when d is even r must be odd (because r is assumed to be coprime to d), so the determinant indeed takes real vector bundles to real line bundles and quaternionic vector bundles to quaternionic line bundles. □

Had we not assumed $r \wedge d = 1$, then the situation would have been more complicated to analyze, because the determinant of a quaternionic vector bundle of even rank is a real line bundle and also because, when $h = r \wedge d$ is even, the real space $\mathrm{Sym}^h(X)$ may have real points even if X does not (points of the form $[x_i, \sigma(x_i)]_{1 \le i \le h/2}$ for $x_i \in X$).

2E. ***Real vector bundles of fixed determinant.*** Let us now consider spaces of vector bundles of fixed determinant. By [Tu 1993, Theorem 3], one has, for any

$\mathcal{L} \in \mathrm{Pic}_X^d$, $\mathcal{M}_X(r, \mathcal{L}) := \det^{-1}(\{\mathcal{L}\}) \simeq_{\mathbb{C}} \mathbb{P}_{\mathbb{C}}^{h-1}$ where $d = \deg(\mathcal{L})$ and $h = r \wedge d$. This is proved in the following way: under the identification $\mathcal{M}_X(r, d) \simeq_{\mathbb{C}} \mathrm{Sym}^h(X)$, there is a commutative diagram

$$\begin{array}{ccc} \mathcal{M}_X(r, d) & \xrightarrow{\simeq} & \mathrm{Sym}^h(X) \\ \downarrow \det & & \downarrow \mathrm{AJ} \\ \mathrm{Pic}_X^d & \xrightarrow{\simeq} & \mathrm{Pic}_X^h \end{array}$$

where

$$\mathrm{AJ} : \mathrm{Sym}^h(X) \to \mathrm{Pic}_X^h, \quad (x_1, \dots, x_h) \mapsto \mathcal{L}(x_1 + \cdots + x_h)$$

is the Abel–Jacobi map (taking a finite family of points $(x_1, \dots, x_h)$ to the line bundle associated to the divisor $x_1 + \cdots + x_h$) and the fiber of the Abel–Jacobi map above a holomorphic line bundle $\mathcal{L}$ of degree h is the projective space $\mathbb{P}(H^0(X, \mathcal{L}))$ which, since $\deg(\mathcal{L}) = h \geq 1$ and X has genus 1, is isomorphic to $\mathbb{P}_{\mathbb{C}}^{h-1}$. Evidently, the same proof will work over $\mathbb{R}$ whenever we can identify Pic_X^d and Pic_X^h as real varieties, which happens in particular when $X^\sigma \neq \varnothing$.

Theorem 2.11. *Let (X, x_0, σ) be a real elliptic curve satisfying $X^\sigma \neq \varnothing$ and let $\mathcal{L}$ be a real line bundle of degree d on X. Then, for all $r \geq 1$,*

$$\mathcal{M}_X(r, \mathcal{L}) \simeq_{\mathbb{R}} \mathbb{P}_{\mathbb{R}}^{h-1},$$

where $h = r \wedge d$.

Proof. When $X^\sigma \neq \varnothing$, we can choose $x_0 \in X^\sigma$ and use Theorem 2.1 to identify all Picard varieties Pic_X^k over $\mathbb{R}$, then reproduce Tu's proof recalled above. □

3. Indecomposable vector bundles

3A. ***Indecomposable vector bundles over a complex elliptic curve.*** As recalled in the introduction, a theorem of Grothendieck [1957] shows that any holomorphic vector bundle on $\mathbb{CP}^1$ is isomorphic to a direct sum of holomorphic line bundles, and this can be easily recast in modern perspective by using the notions of stability and semistability of vector bundles over curves, introduced by Mumford [1963] and first studied by himself and Seshadri [1967]: the moduli variety of semistable vector bundles of rank r and degree d over $\mathbb{CP}^1$ is a single point if r divides d and is empty otherwise. As for vector bundles over a complex elliptic curve, the study was initiated by Atiyah [1957b], thus at a time when the notion of stability was not yet available. Rather, Atiyah's starting point is the notion of an indecomposable vector bundle: a holomorphic vector bundle $\mathcal{E}$ over a complex curve X is said to be *indecomposable* if it is not isomorphic to a direct sum of nontrivial holomorphic bundles. In the present paper, we shall denote by $\mathcal{I}_X(r, d)$

the set of isomorphism classes of indecomposable vector bundles of rank r and degree d. It is immediate from the definition that a holomorphic vector bundle is a direct sum of indecomposable ones. Moreover, one has the following result, which is a consequence of the categorical Krull–Schmidt theorem, also due to Atiyah, showing that the decomposition of a bundle into indecomposable ones is essentially unique.

Proposition 3.1 [Atiyah 1956, Theorem 3]. *Let X be a compact connected complex analytic manifold. Any holomorphic vector bundle $\mathcal{E}$ over X is isomorphic to a direct sum $\mathcal{E}_1 \oplus \cdots \oplus \mathcal{E}_k$ of indecomposable vector bundles. If one also has $\mathcal{E} \simeq \mathcal{E}'_1 \oplus \cdots \oplus \mathcal{E}'_l$, then $l = k$ and there exists a permutation σ of the indices such that $\mathcal{E}'_{\sigma(i)} \simeq \mathcal{E}_i$.*

Going back to the case of a compact, connected Riemann surface X of genus 1, Atiyah completely describes all indecomposable vector bundles on X. He first shows [op. cit., Theorem 5] the existence, for any $h \geq 1$, of a unique (isomorphism class of) indecomposable vector bundle F_h of rank h and degree 0 such that

$$\dim H^0(X; F_h) = 1. \tag{3-1}$$

As a matter of fact, this is the only vector bundle of rank h and degree 0 over X with a nonzero space of sections. Let us call F_h the *Atiyah bundle* of rank h and degree 0. The construction of F_h is by induction, starting from $F_1 = \mathcal{O}_X$, the trivial line bundle over X, and showing the existence and uniqueness of an extension of the form

$$0 \to \mathcal{O}_X \to F_h \to F_{h-1} \to 0. \tag{3-2}$$

In particular, $\det(F_h) = \mathcal{O}_X$. Moreover, Since F_h is the unique indecomposable vector bundle with nonzero space of sections, one has [op. cit., Corollary 1]:

$$F_h^* \simeq F_h. \tag{3-3}$$

Note that F_h is an extension of semistable bundles so it is semistable. The associated polystable bundle is the trivial bundle of rank h, which is not isomorphic to F_h (in particular, F_h is not itself polystable).

Atiyah then shows that any indecomposable vector bundle $\mathcal{E}$ of rank h and degree 0 is isomorphic to $F_h \otimes \mathcal{L}$ for a line bundle $\mathcal{L}$ of degree 0 which is unique up to isomorphism [op. cit., Theorem 5(ii)]. Since it follows from the construction of F_h recalled in (3-2) that $\det(F_h) = \mathcal{O}_X$, one has $\det \mathcal{E} = \mathcal{L}^h$. This sets up a bijection

$$\mathrm{Pic}^0_X \to \mathcal{I}_X(h, 0), \quad \mathcal{L} \mapsto F_h \otimes \mathcal{L}. \tag{3-4}$$

Note that the map (3-4) is just the identity map if $h = 1$. Atiyah then uses a marked point $x_0 \in X$ to further identify Pic^0_X with X. In particular, the set $\mathcal{I}_X(h, 0)$ inherits a natural structure of complex analytic manifold of dimension 1.

The next step in Atiyah's characterization of indecomposable vector bundles is to consider the case of vector bundles of nonvanishing degree. He shows [op. cit., Theorem 6] that, associated to the choice of a marked point $x_0 \in X$, there is, for all r and d, a unique bijection (subject to certain conditions)

$$\alpha_{r,d}^{x_0} : \mathcal{I}_X(r \wedge d, 0) \to \mathcal{I}_X(r, d) \tag{3-5}$$

between the set of isomorphism classes of indecomposable vector bundles of rank $h := r \wedge d$ and degree 0 and the set of isomorphism classes of indecomposable vector bundles of rank r and degree d. As a consequence, Atiyah can define a canonical indecomposable vector bundle of rank r and degree d, namely

$$F_{x_0}(r, d) := \alpha_{r,d}^{x_0}(F_{r \wedge d}),$$

where $F_{r \wedge d}$ is the indecomposable vector bundle of rank $r \wedge d$ and degree 0 whose construction was recalled in (3-2). We will call the bundle $F_{x_0}(r, d)$ the Atiyah bundle of rank r and degree d (in particular $F_{x_0}(r, 0) = F_r$). Atiyah then obtains the following description of indecomposable vector bundles:

Theorem 3.2 [Atiyah 1957b, Theorem 10]. *Set $h = r \wedge d$, $r' = r/h$, and $d' = d/h$. Then every indecomposable vector bundle of rank r and degree d is isomorphic to a bundle of the form $F_{x_0}(r, d) \otimes \mathcal{L}$ where $\mathcal{L}$ is a line bundle of degree 0. Moreover, $F_{x_0}(r, d) \otimes \mathcal{L} \simeq F_{x_0}(r, d) \otimes \mathcal{L}'$ if and only if $(\mathcal{L}' \otimes \mathcal{L}^{-1})^{r'} \simeq \mathcal{O}_X$.*

Thus, as a generalization to (3-4), Theorem 3.2 shows that there is a surjective map $\mathrm{Pic}_X^0 \to \mathcal{I}_X(r, d)$, whose fiber is isomorphic to the group $T_{r'}$ of r'-torsion elements in Pic_X^0. This in particular induces a bijection between the Riemann surface $\mathrm{Pic}_X^0 / T_{r'} \simeq \mathrm{Pic}_X^0 \simeq X$ and the set $\mathcal{I}_X(r, d)$ for all r and d, and the set $\mathcal{I}_X(r, d)$ inherits in this way a natural structure of complex analytic manifold of dimension 1.

3B. ***Relation to semistable and stable bundles.*** It is immediate to prove that stable bundles (over a curve of arbitrary genus) are indecomposable. Moreover, over an elliptic curve, we have the following result, proved in Tu's paper:

Theorem 3.3 [Tu 1993, Appendix A]. *Every indecomposable vector bundle of rank r and degree d over a complex elliptic curve is semistable. It is stable if and only if $r \wedge d = 1$.*

In particular, the Atiyah bundles $F_{x_0}(r, d)$ are semistable (and stable if and only if $r \wedge d = 1$) and, by Proposition 3.1, every holomorphic vector bundle over a complex elliptic curve is isomorphic to a direct sum of semistable bundles. Next, there is a very important relation between indecomposable vector bundles and stable vector bundles, which will be useful in the next section.

Theorem 3.4 (Atiyah–Tu). *Set $h = r \wedge d$, $r' = r/h$, and $d' = d/h$. Then the map*

$$\mathcal{M}_X(r', d') \to \mathcal{I}_X(r, d), \quad \mathcal{E}' \mapsto \mathcal{E}' \otimes F_h$$

is a bijection: any indecomposable vector bundle of rank r and degree d is isomorphic to a bundle of the form $\mathcal{E}' \otimes F_h$ where $\mathcal{E}'$ is a stable vector bundle of rank r' and degree d', unique up to isomorphism, and F_h is the Atiyah bundle of rank h and degree 0.

In particular, $\mathcal{I}_X(r, d)$ inherits in this way a structure of complex analytic manifold of dimension $r' \wedge d' = 1$. This result, which generalizes (3-4) in a different direction than Theorem 3.2, can be deduced from Atiyah and Tu's papers but we give a proof below for the sake of completeness. It is based on the following lemma.

Lemma 3.5 [Atiyah 1957b, Lemma 24]. *The Atiyah bundles $F_{x_0}(r, d)$ and $F_{x_0}(r', d')$ are related in the following way:*

$$F_{x_0}(r, d) \simeq F_{x_0}(r', d') \otimes F_h.$$

Proof of Theorem 3.4. Let $\mathcal{E} \in \mathcal{I}_X(r, d)$. By Theorem 3.2, there exists a line bundle $\mathcal{L}$ of degree 0 such that $\mathcal{E} \simeq F_{x_0}(r, d) \otimes \mathcal{L}$. By Lemma 3.5, $F_{x_0}(r, d) \simeq F_{x_0}(r', d') \otimes F_h$. Since $r' \wedge d' = 1$, Theorem 3.3 shows that $F_{x_0}(r', d')$, hence also $\mathcal{E}' := F_{x_0}(r', d') \otimes \mathcal{L}$, are stable bundles of rank r' and degree d'. And one has indeed $\mathcal{E} \simeq \mathcal{E}' \otimes F_h$. Let now $\mathcal{E}'$ and $\mathcal{E}''$ be two stable bundles of rank r' and degree d' such that $\mathcal{E}' \otimes F_h \simeq \mathcal{E}'' \otimes F_h$. Since stable bundles are indecomposable, Theorem 3.2 shows the existence of two line bundles $\mathcal{L}'$ and $\mathcal{L}''$ of degree 0 such that $\mathcal{E}' \simeq F_{x_0}(r', d') \otimes \mathcal{L}'$ and $\mathcal{E}'' \simeq F_{x_0}(r', d') \otimes \mathcal{L}''$. Tensoring by F_h and applying Lemma 3.5, we obtain that $F_{x_0}(r, d) \otimes \mathcal{L}' \simeq F_{x_0}(r, d) \otimes \mathcal{L}''$, which, again by Theorem 3.2, implies that $\mathcal{L}'$ and $\mathcal{L}''$ differ by an r'-torsion point of Pic^0_X. But then a final application of Theorem 3.2 shows that $F_{x_0}(r', d') \otimes \mathcal{L}' \simeq F_{x_0}(r', d') \otimes \mathcal{L}''$, i.e., $\mathcal{E}' \simeq \mathcal{E}''$. ☐

Thus, the complex variety $\mathcal{I}_X(r, d) \simeq \mathcal{M}_X(r', d') \simeq X$ is a 1-dimensional subvariety of the h-dimensional moduli variety $\mathcal{M}_X(r, d) \simeq \mathrm{Sym}^h(X)$, and these two nonsingular varieties coincide exactly when r and d are coprime. More explicitly, under the identifications $\mathcal{I}_X(r, d) \simeq X$ and $\mathcal{M}_X(r, d) \simeq \mathrm{Sym}^h(X)$, the inclusion map $\mathcal{I}_X(r, d) \hookrightarrow \mathcal{M}_X(r, d)$, implicit in Theorem 3.3, is simply the diagonal map

$$X \to \mathrm{Sym}^h(X), \quad x \mapsto [x, \dots, x]$$

and it commutes to the determinant map, the latter being, on $\mathrm{Sym}^h(X)$, just the Abel–Jacobi map $[x_1, \dots, x_h] \mapsto x_1 + \cdots + x_h$; see [Tu 1993, Theorem 2].

3C. ***Indecomposable vector bundles over a real elliptic curve.*** Over a real elliptic curve, the description of indecomposable vector bundles is a bit more complicated than in the complex case, because the Atiyah map $\alpha^{x_0}_{r,d}$ defined in (3-5) is not a real map unless the point x_0 is a real point, which excludes the case where $X^\sigma = \varnothing$. Of course the case $X^\sigma \neq \varnothing$ is already very interesting and if we follow Atiyah's paper in that case, then the Atiyah map $\alpha^{x_0}_{r,d}$ is a real map and the Atiyah bundles $F_{x_0}(r, d)$ are all real bundles. In particular, the description given by Atiyah of the ring structure of the set of isomorphism class of all vector bundles (namely the way to decompose the tensor product of two Atiyah bundles into a direct sum of Atiyah bundles, see for instance [Tu 1993, Appendix A] for a concise exposition) directly applies to the subring formed by isomorphism classes real bundles. (Note that, in contrast, isomorphism classes of quaternionic bundles do not form a ring, as the tensor product of two quaternionic bundles is a real bundle.) To obtain a description of indecomposable bundles over a real elliptic curve which holds without assuming that the curve has real points, we need to replace the Atiyah isomorphism

$$\alpha^{x_0}_{r,d} : \mathcal{I}_X(r \wedge d, 0) \to \mathcal{I}_X(r, d)$$

(which cannot be a real map when $X^\sigma = \varnothing$) by the isomorphism $\mathcal{I}_X(r, d) \simeq \mathcal{M}_X(r', d')$ of Theorem 3.4 and show that the latter is always a real map. The first step is the following result, about the Atiyah bundle of rank h and degree 0, whose definition was recalled in (3-2).

Proposition 3.6. *Let (X, σ) be a real Riemann surface of genus 1. For any $h \geq 1$, the indecomposable vector bundle F_h has a canonical real structure.*

Proof. We proceed by induction. Since X is assumed to be real, $\mathcal{O}_X$ has a canonical real structure. So, if $h = 1$, then F_h is canonically real. Assume now that $h > 1$ and that F_{h-1} has a fixed real structure. Following again Atiyah [1957a], extensions of the form (3-2) are parametrized by the sheaf cohomology group $H^1(X; \mathrm{Hom}_{\mathcal{O}_X}(F_{h-1}; \mathcal{O}_X)) = H^1(X; F^*_{h-1})$. The uniqueness part of the statement in Atiyah's construction above says that this cohomology group is a complex vector space of dimension 1, which, in any case, can be checked by Riemann–Roch using properties (3-1) and (3-3). Indeed, since $\deg(F^*_{h-1}) = 0$ and X is of genus $g = 1$, one has

$$h^0(F^*_{h-1}) - h^1(F^*_{h-1}) = \deg(F^*_{h-1}) + (\mathrm{rk}\, F^*_{h-1})(1 - g) = 0,$$

where $h^i(\,\cdot\,) = \dim H^i(X;\,\cdot\,)$, so $h^1(F^*_{h-1}) = h^0(F^*_{h-1}) = 1$. Now, since X and F_{h-1} have real structures, so does $H^1(X; F^*_{h-1})$ and the fixed point space of that real structure corresponds to isomorphism classes of real extensions of F_{h-1} by $\mathcal{O}_X$. Since the fixed point space of the real structure of $H^1(X; F^*_{h-1})$ is a 1-dimensional real vector space, the real structure of F_h is unique up to isomorphism. □

Thus, in contrast to Atiyah bundles of nonvanishing degree, F_h is always canonically a real bundle. In particular, $\overline{\sigma^* F_h} \simeq F_h$. It is then clear that the isomorphism

$$\mathcal{M}_X(r', d') \to \mathcal{I}_X(r, d), \quad \mathcal{E}' \mapsto \mathcal{E}' \otimes F_h$$

is a real map: $\overline{\sigma^*\mathcal{E}'} \otimes F_h \simeq \overline{\sigma^*\mathcal{E}'} \otimes \overline{\sigma^* F_h} \simeq \overline{\sigma^*(\mathcal{E}' \otimes F_h)}$, which readily implies Theorem 1.3. Moreover, one can make the following observation:

Proposition 3.7. *Let $\mathcal{E}$ be an indecomposable vector bundle of rank r and degree d over the real elliptic curve (X, x_0, σ) and assume that $\overline{\sigma^*\mathcal{E}} \simeq \mathcal{E}$. Then $\mathcal{E}$ admits either a real or a quaternionic structure.*

Proof. By Theorem 3.4, we can write $\mathcal{E} \simeq \mathcal{E}' \otimes F_h$, with $\mathcal{E}'$ stable. Therefore,

$$\overline{\sigma^*\mathcal{E}} \simeq \overline{\sigma^*(\mathcal{E}' \otimes F_h)} \simeq \overline{\sigma^*\mathcal{E}'} \otimes \overline{\sigma^* F_h} \simeq \overline{\sigma^*\mathcal{E}'} \otimes F_h.$$

The assumption $\overline{\sigma^*\mathcal{E}} \simeq \mathcal{E}$ then translates to $\overline{\sigma^*\mathcal{E}'} \otimes F_h \simeq \mathcal{E}' \otimes F_h$ which, since the map from Theorem 3.4 is a bijection, shows that $\overline{\sigma^*\mathcal{E}'} \simeq \mathcal{E}'$. As $\mathcal{E}'$ is stable, the fact that $\mathcal{E}'$ admits a real or quaternionic structure τ' follows from Proposition 2.10. If τ_h denotes the real structure of F_h, we then have that $\tau' \otimes \tau_h$ is a real or quaternionic structure on $\mathcal{E}$, depending on whether τ' is real or quaternionic. □

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Q(N)-GRADED LIE SUPERALGEBRAS ARISING FROM FERMIONIC-BOSONIC REPRESENTATIONS

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We use fermionic-bosonic representations to obtain a class of $Q(N)$-graded Lie superalgebras coordinatized by quantum tori.

1. Introduction

Root graded Lie algebras were first introduced by Berman and Moody [1992] to understand the generalized intersection matrix algebras of Slodowy. Berman and Moody [1992] classified Lie algebras graded by the root systems of type A_l, D_l, and E_6, E_7, E_8 up to central isogeny. Benkart and Zelmanov [1996] classified Lie algebras graded by the root systems of type B_n, C_n, F_4, G_2 up to central isogeny. Allison et al. [2000] completed the classifications of the above root graded Lie algebras by figuring out explicitly the centers of the universal coverings of those root graded Lie algebras. It turns out that the classification of those root graded Lie algebras played a crucial role in classifying the newly developed extended affine Lie algebras (see [Berman et al. 1996]), which is a generalization of many important Lie algebras, such as affine and toroidal Lie algebras.

Root graded Lie superalgebras are a "super" analog of root graded Lie algebras. Lie superalgebras graded by the root systems of type $A(m,n)$, $B(m,n)$, $C(n)$, $D(m,n)$, and $D(2,1;\alpha)$, $F(4)$, $G(3)$ were classified by G. Benkart and A. Elduque. Lie superalgebras graded by the root systems of type $P(N)$, $Q(N)$ were introduced and classified by C. Martínez and E. I. Zelmanov [2003].

Fermionic representations for the affine Kac–Moody Lie algebras were first developed by Frenkel [1980] and Kac and Peterson [1981] independently. Feingold and Frenkel [1985] constructed representations for all classical affine Lie algebras by using Clifford or Weyl algebras with infinitely many generators. Gao [2002] gave bosonic and fermionic representations for the extended affine Lie algebra $\widetilde{\mathrm{gl}}_N(\mathbb{C}_q)$, where $\mathbb{C}_q$ is the quantum torus in two variables. Chen and Gao [2007] constructed fermionic modules for some BC_N-graded Lie algebras, Chen et al. [2006] constructed modules for some $B(0,N)$-graded Lie superalgebras.

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In this paper, we use fermions and bosons to obtain a class of $Q(N)$-graded Lie superalgebras coordinatized by quantum tori.

The structure of this paper is as follows. In Section 2, we review the definition of $Q(N)$-graded Lie superalgebras and give examples of $Q(N)$-graded Lie superalgebras which coordinatized by quantum tori. In Section 3, we use a tensor product of a fermionic module and a bosonic module to construct the representations for those examples of $Q(N)$-graded Lie superalgebras.

Throughout this paper, we denote the field of complex numbers and the ring of integers by $\mathbb{C}$ and $\mathbb{Z}$ respectively. Let $\mathbb{F}$ be a field of characteristic zero.

2. Lie superalgebras graded by $Q(N)$

In this section, we first recall the definition of $Q(N)$-graded Lie superalgebras. Then we construct examples of $Q(N)$-graded Lie superalgebras coordinatized by quantum tori.

Following the notations in [Kac 1977], the finite-dimensional split simple Lie superalgebra $Q(N-1)$ over $\mathbb{F}$ equals $\widetilde{Q}(N-1)/\mathbb{F}I_{2N}$, where $\widetilde{Q}(N-1)$ consists of the matrices of the form $\left(\begin{smallmatrix} a & b \\ b & a \end{smallmatrix}\right)$, where $a, b \in M_N(\mathbb{F})$, and $\operatorname{tr}(b) = 0$. Let

$$\mathcal{H} = \left\{ \sum_{i=1}^{N} a_i (e_{ii} + e_{N+i,N+i}) \,\middle|\, a_i \in \mathbb{C}, \sum_{i=1}^{N} a_i = 0 \right\},$$

then $\mathcal{H}$ is a Cartan subalgebra of $Q(N-1)_{\bar{0}}$.

Define $\varepsilon_i \in \mathcal{H}^*$, $i = 1, \ldots, N$, by

$$\varepsilon_i \left(\sum_{j=1}^{N} a_j (e_{jj} + e_{N+j,N+j}) \right) = a_i$$

for $i = 1, \cdots, N$. Set

$$Q(N-1)_\alpha = \{x \in Q(N-1) \mid [h, x] = \alpha(h)x \text{ for all } h \in \mathcal{H}\}$$

as usual. Then

$$Q(N-1) = \mathcal{H} + \sum_{\alpha \in \Delta_{\bar{0}}} Q(N-1)_{\bar{0}\alpha} + \sum_{\beta \in \Delta_{\bar{1}}} Q(N-1)_{\bar{1}\beta}$$

is the root space decomposition of $Q(N-1)$ with respect to the action of $\mathcal{H}$, $\Delta_{Q(N-1)} = \Delta_{\bar{0}} \cup \Delta_{\bar{1}}$, where

$$\Delta_{\bar{0}} = \Delta_{\bar{1}} = \{\varepsilon_i - \varepsilon_j \mid 1 \le i \ne j \le N\}.$$

Definition 2.1 [Martínez and Zelmanov 2003]. A Lie superalgebra L over $\mathbb{F}$ is graded by $Q(N-1)$ if (*ı*) L contains a subsuperalgebra

$$Q(N-1)=\mathcal{H}+\sum_{\alpha\in\Delta_{Q(N-1)}} Q(N-1)_\alpha;$$

(*ıı*) $L=\sum_{\alpha\in\Delta_{Q(N-1)}\cup\{0\}} L_\alpha$;

(*ııı*) $L_0=\sum_{\alpha\in\Delta_{Q(N-1)}}[L_{-\alpha}, L_\alpha]$.

Let $0\neq q\in\mathbb{C}$. A quantum torus associated to q is the unital associative $\mathbb{C}$-algebra $\mathbb{C}_q[x^{\pm1}, y^{\pm1}]$ (or simply $\mathbb{C}_q$) with generators $x^\pm$,$y^\pm$ and relations

$$xx^{-1}=x^{-1}x=yy^{-1}=y^{-1}y=1 \quad\text{and}\quad yx=qxy.$$

Let $\mathrm{Matr}_{m,n}(\mathbb{C}_q)$ denote the associative algebra consisting of $m\times n$ matrices with entries in $\mathbb{C}_q$.

For two arbitrary positive integers M and N we have an associative superalgebra $\mathrm{Matr}(M, N)(\mathbb{C}_q)$ consisting of (M, N)-block matrices with entries in $\mathbb{C}_q$, whose $\mathbb{Z}_2$-grading is given as follows:

$$\mathrm{Matr}(M, N)(\mathbb{C}_q)_{\bar 0}=\left\{\begin{pmatrix}A & 0\\ 0 & B\end{pmatrix} \,\middle|\, A\in \mathrm{Matr}_{M,M}(\mathbb{C}_q),\, B\in\mathrm{Matr}_{N,N}(\mathbb{C}_q)\right\},$$
$$\mathrm{Matr}(M, N)(\mathbb{C}_q)_{\bar 1}=\left\{\begin{pmatrix}0 & C\\ D & 0\end{pmatrix} \,\middle|\, C\in \mathrm{Matr}_{M,N}(\mathbb{C}_q),\, D\in\mathrm{Matr}_{N,M}(\mathbb{C}_q)\right\}.$$

$\mathrm{Matr}(M, N)(\mathbb{C}_q)$ forms a Lie superalgebra under the supercommutator product $[x, y]:=xy-(-1)^{|x||y|}yx$ for homogeneous $x, y\in\mathrm{Matr}(M, N)(\mathbb{C}_q)$. We denote this Lie superalgebra by $\mathrm{gl}(M, N)(\mathbb{C}_q)$.

Set $\Lambda(q)=\{n\in\mathbb{Z}\mid q^n=1\}$.

We form a central extension of the Lie superalgebra $\mathrm{gl}(M, N)(\mathbb{C}_q)$ as was done in [Gao 2002] and [Chen and Gao 2007]:

$$\widehat{\mathrm{gl}}(M, N)(\mathbb{C}_q)=\mathrm{gl}(M, N)(\mathbb{C}_q)\oplus\Bigl(\bigoplus_{n\in\Lambda(q)}\mathbb{C}c(n)\Bigr)\oplus\mathbb{C}c_y$$

with Lie superbracket

$$\begin{aligned}(2\text{-}1)\quad &[A(x^my^n), B(x^py^s)]\\ &=A(x^my^n)B(x^py^s)-(-1)^{\deg A\deg B}B(x^py^s)A(x^my^n)\\ &\quad+mq^{np}\,\mathrm{str}(AB)\delta_{m+p,0}\delta_{\overline{n+s},\bar 0}c(n+s)+nq^{np}\,\mathrm{str}(AB)\delta_{m+p,0}\delta_{n+s,0}c_y\end{aligned}$$

for $m, p, n, s\in\mathbb{Z}$, $A, B\in\mathrm{gl}(M, N)_\alpha$, $\alpha=\bar 0$ or $\bar 1$, where str is the supertrace of the Lie superalgebra $\mathrm{gl}(M, N)$, $c(u)$ for $u\in\Lambda(q)$ and c_y are central elements of $\widehat{\mathrm{gl}}(M, N)(\mathbb{C}_q)$, and $\bar t\in\mathbb{Z}/\Lambda(q)$ for $t\in\mathbb{Z}$.

Let $G=\sqrt{-1}\left(\begin{smallmatrix}0 & I_n\\ -I_n & 0\end{smallmatrix}\right)$. Using the matrix G, we define a $\mathbb{Z}_2$-graded subspace $\widetilde{\mathcal{Q}}$ with

$$\widetilde{\mathcal{Q}}_{\bar{0}}=\{X\in \mathrm{gl}(N,N)(\mathbb{C}_q)_{\bar{0}} \mid XG-GX=0\},$$
$$\widetilde{\mathcal{Q}}_{\bar{1}}=\{X\in \mathrm{gl}(N,N)(\mathbb{C}_q)_{\bar{1}} \mid XG+GX=0\}.$$

Proposition 2.2. *The general form of a matrix in $\widetilde{\mathcal{Q}}$ is*

$$\begin{pmatrix} A & B\\ B & A\end{pmatrix},$$

where A, B are $N\times N$ submatrices.

As in [Allison et al. 1997], we know that, for the Lie superalgebra $\mathcal{Q}=[\widetilde{\mathcal{Q}},\widetilde{\mathcal{Q}}]$, we have

$$\mathcal{Q}=\left\{\begin{pmatrix} A & B\\ B & A\end{pmatrix}\in\widetilde{\mathcal{Q}} \;\middle|\; \mathrm{tr}(B)\equiv 0 \bmod [\mathbb{C}_q,\mathbb{C}_q]\right\}.$$

Let

$$\tilde{g}_{ij}(m,n)=x^my^ne_{ij}+x^my^ne_{N+i,N+j},$$
$$\tilde{h}_{ij}(m,n)=x^my^ne_{i,N+j}+x^my^ne_{N+i,j}.$$

Then we have the root space decomposition

$$\mathcal{Q}=\mathcal{Q}_0\oplus\bigoplus_{1\le i\ne j\le N}\mathcal{Q}_{\bar{0}(\varepsilon_i-\varepsilon_j)}\oplus\bigoplus_{1\le i\ne j\le N}\mathcal{Q}_{\bar{1}(\varepsilon_i-\varepsilon_j)},$$

where

$$\mathcal{Q}_{\bar{0}(\varepsilon_i-\varepsilon_j)}=\mathrm{span}_{\mathbb{C}}\{\tilde{g}_{ij}(m,n)\mid m,n\in\mathbb{Z}\},$$
$$\mathcal{Q}_{\bar{1}(\varepsilon_i-\varepsilon_j)}=\mathrm{span}_{\mathbb{C}}\{\tilde{h}_{ij}(m,n)\mid m,n\in\mathbb{Z}\},$$

and

$$\begin{aligned}\mathcal{Q}_0=\;&\mathrm{span}_{\mathbb{C}}\{\tilde{g}_{ii}(m,n)\mid 1\le i\le N, m,n\in\mathbb{Z}\}\\ &\oplus\mathrm{span}_{\mathbb{C}}\{\tilde{h}_{ii}(m,n)-\tilde{h}_{NN}(m,n)\mid 1\le i\le N-1,\; m,n\in\mathbb{Z}\}\\ &\oplus\mathrm{span}_{\mathbb{C}}\big\{\tilde{h}_{NN}(m,n)\mid m,n\in(\mathbb{Z}\times\mathbb{Z})\backslash(\Lambda(q)\times\Lambda(q))\big\}.\end{aligned}$$

As in [Chen and Gao 2007], one easily sees that $\mathcal{Q}$ is a Lie superalgebra graded by $Q(N-1)$. By a direct calculation, we get the central extension of $\mathcal{Q}$ with superbracket as in (2-1) is trivial, and we have:

Proposition 2.3.

(2-2) $[\tilde{g}_{ij}(m,n),\tilde{g}_{kl}(p,t)]_+=\delta_{jk}q^{np}\tilde{g}_{il}(m+p,n+t)-\delta_{il}q^{tm}\tilde{g}_{kj}(m+p,n+t),$

(2-3) $[\tilde{h}_{ij}(m,n),\tilde{h}_{kl}(p,t)]_+=\delta_{jk}q^{np}\tilde{g}_{il}(m+p,n+t)+\delta_{il}q^{tm}\tilde{g}_{kj}(m+p,n+t),$

(2-4) $[\tilde{g}_{ij}(m,n),\tilde{h}_{kl}(p,t)]_-=\delta_{jk}q^{np}\tilde{h}_{il}(m+p,n+t)-\delta_{il}q^{tm}\tilde{h}_{kj}(m+p,n+t),$

for all $m,p,n,t\in\mathbb{Z}$ and $1\le i,j,k,l\le N$.

3. Module construction

Let $\mathcal{R}$ be an arbitrary associative algebra, $\rho = \pm 1$. We define a ρ-bracket on $\mathcal{R}$ by

$$\{a, b\}_\rho = ab + \rho ba, \quad a, b \in \mathcal{R}.$$

Let $\mathfrak{a}$ be the unital associative algebra with $2N$ generators $a_i, a_i^*,\ 1 \le i \le N$, subject to relations

$$\{a_i, a_j\}_\rho = \{a_i^*, a_j^*\}_\rho = 0$$

and

$$\{a_i, a_j^*\}_\rho = \delta_{ij}. \tag{3-1}$$

Let the associative algebra $\alpha(N, \rho)$ be generated by

$$\left\{ u(m) \,\middle|\, u \in \bigoplus_{i=1}^{N} (\mathbb{C}a_i \oplus \mathbb{C}a_i^*),\ m \in \mathbb{Z} \right\}$$

subject to relations

$$\{u(m), v(n)\}_\rho = \{u, v\}_\rho \delta_{m+n,0}.$$

Then we define the normal ordering as in [Feingold and Frenkel 1985]:

$$
\begin{aligned}
{:}u(m)v(n){:} &= \begin{cases} u(m)v(n) & \text{if } n > m, \\ \frac{1}{2}\big(u(m)v(n) - \rho v(n)u(m)\big) & \text{if } m = n, \\ -\rho v(n)u(m) & \text{if } m > n, \end{cases} \\
&= -\rho\, {:}v(n)u(m){:}
\end{aligned}
$$

for $n, m \in \mathbb{Z},\ u, v \in \mathfrak{a}$. Set

$$\theta(n) = \begin{cases} 1 & \text{for } n > 0, \\ \frac{1}{2} & \text{for } n = 0, \\ 0 & \text{for } n < 0, \end{cases} \qquad \text{then } 1 - \theta(n) = \theta(-n). \tag{3-2}$$

Then we have

$$
\begin{aligned}
{:}a_i(m)a_j(n){:} &= a_i(m)a_j(n) = -\rho a_j(n)a_i(m), \\
{:}a_i^*(m)a_j^*(n){:} &= a_i^*(m)a_j^*(n) = -\rho a_j^*(n)a_i^*(m),
\end{aligned}
$$

and

$$
\begin{aligned}
a_i(m)a_j^*(n) &= {:}a_i(m)a_j^*(n){:} + \delta_{ij}\delta_{m+n,0}\theta(m-n), \\
a_j^*(n)a_i(m) &= {:}a_i(m)a_j^*(n){:} - \delta_{ij}\delta_{m+n,0}\theta(n-m).
\end{aligned}
\tag{3-3}
$$

Proposition 3.1. *In the Clifford algebra $\alpha(N, +1)$ case, the subspaces of quadratic operators are closed under the Lie bracket $[\,\cdot\,,\cdot\,]_-$. We have the commutator* *relations*

$$
\begin{aligned}
[a_i(m)a_j(n), a_k(p)a_l(t)]_- &= 0,\\
[a_i(m)a_j(n), a_k(p)a_l^*(t)]_- &= -\delta_{il}\delta_{m,-t}a_k(p)a_j(n) + \delta_{jl}\delta_{n,-t}a_k(p)a_i(m),\\
[a_i(m)a_j^*(n), a_k(p)a_l^*(t)]_- &= -\delta_{il}\delta_{m,-t}a_k(p)a_j^*(n) + \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t),\\
[a_i(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- &= -\delta_{il}\delta_{m,-t}a_k^*(p)a_j^*(n) - \delta_{ik}\delta_{m,-p}a_j^*(n)a_l^*(t),\\
[a_i^*(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- &= 0,\\
[a_i(m)a_j(n), a_k^*(p)a_l^*(t)]_- &= -\delta_{il}\delta_{m,-t}a_k^*(p)a_j(n) + \delta_{ik}\delta_{m,-p}a_l^*(t)a_j(n)\\
&\qquad + \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t) - \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\
&= \delta_{il}\delta_{m,-t}a_j(n)a_k^*(p) - \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\
&\qquad + \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t) - \delta_{ik}\delta_{m,-p}a_j(n)a_l^*(t)\\
&\qquad + \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t} - \delta_{il}\delta_{jk}\delta_{m,-t}\delta_{n,-p}.
\end{aligned}
$$

Proposition 3.2. *In the Weyl algebra $\alpha(N,-1)$ case, the subspaces of quadratic operators are closed under the Lie bracket $[\,\cdot\,,\cdot\,]_-$. We have the commutator relations*

$$
\begin{aligned}
[a_i(m)a_j(n), a_k(p)a_l(t)]_- &= 0,\\
[a_i(m)a_j(n), a_k(p)a_l^*(t)]_- &= \delta_{il}\delta_{m,-t}a_k(p)a_j(n) + \delta_{jl}\delta_{n,-t}a_k(p)a_i(m),\\
[a_i(m)a_j^*(n), a_k(p)a_l^*(t)]_- &= \delta_{il}\delta_{m,-t}a_k(p)a_j^*(n) - \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t),\\
[a_i(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- &= \delta_{il}\delta_{m,-t}a_k^*(p)a_j^*(n) + \delta_{ik}\delta_{m,-p}a_j^*(n)a_l^*(t),\\
[a_i^*(m)a_j^*(n), a_k^*(p)a_l^*(t)]_- &= 0,\\
[a_i(m)a_j(n), a_k^*(p)a_l^*(t)]_- &= \delta_{il}\delta_{m,-t}a_j(n)a_k^*(p) + \delta_{ik}\delta_{m,-p}a_l^*(t)a_j(n)\\
&\qquad + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p) + \delta_{jk}\delta_{n,-p}a_l^*(t)a_i(m)\\
&= \delta_{il}\delta_{m,-t}a_j(n)a_k^*(p) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\
&\qquad + \delta_{jk}\delta_{n,-p}a_i(m)a_l^*(t) + \delta_{ik}\delta_{m,-p}a_j(n)a_l^*(t)\\
&\qquad - \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t} - \delta_{il}\delta_{jk}\delta_{m,-t}\delta_{n,-p}.
\end{aligned}
$$

Remark. The subspaces of fermionic or bosonic quadratic operators are not closed under $[\,\cdot\,,\cdot\,]_-$, then we see that the fermionic or bosonic quadratic operators can only correspond to even root vectors.

In the tensor product algebra $\alpha(N,+1)\otimes\alpha(N,-1)$ case, we will identify $u(m)\bigotimes v(n) = u(m)v(n)$. Then we have

Proposition 3.3. *If we express the generators of $\alpha(N,+1)$ and $\alpha(N,-1)$ by $a_i(m)$, $a_j^*(n)$ and $e_i(m)$, $e_j^*(n)$ respectively, we get, for the quadric operators $a_i(m)\otimes e_j(n)$,*

$a_i(m)\otimes e_j^*(n)$, $a_i^*(m)\otimes e_j(n)$ *and* $a_i^*(m)\otimes e_j^*(n)$, *the anticommutation relations*

$$[a_i(m)e_j(n), a_k(p)e_l(t)]_+ = 0,$$

$$[a_i(m)e_j(n), a_k^*(p)e_l(t)]_+ = \delta_{ik}\delta_{m,-p}e_j(n)e_l(t), \tag{3-4}$$

$$[a_i(m)e_j(n), a_k(p)e_l^*(t)]_+ = \delta_{jl}\delta_{n,-t}a_i(m)a_k(p),$$

$$\begin{aligned}[a_i(m)e_j(n), a_k^*(p)e_l^*(t)]_+ &= \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\ &= \delta_{ik}\delta_{m,-p}e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\ &\qquad - \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t},\end{aligned} \tag{3-5}$$

$$\begin{aligned}
&[a_i(m)e_j^*(n), a_k(p)e_l^*(t)]_+ = 0,\\
&[a_i(m)e_j^*(n), a_k^*(p)e_l(t)]_+ = \delta_{ik}\delta_{m,-p}e_l(t)e_j^*(n) - \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p),\\
&[a_i(m)e_j^*(n), a_k^*(p)e_l^*(t)]_+ = \delta_{ik}\delta_{m,-p}e_j^*(n)e_l^*(t),\\
&[a_i^*(m)e_j(n), a_k^*(p)e_l(t)]_+ = 0,\\
&[a_i^*(m)e_j(n), a_k^*(p)e_l^*(t)]_+ = \delta_{jl}\delta_{n,-t}a_i^*(m)a_k^*(p),\\
&[a_i^*(m)e_j^*(n), a_k^*(p)e_l^*(t)]_+ = 0.
\end{aligned}$$

Proof. We only check (3-4) and (3-5):

$$\begin{aligned}
[a_i(m)e_j(n), a_k^*(p)e_l(t)]_+ &= a_i(m)e_j(n)a_k^*(p)e_l(t) + a_k^*(p)e_l(t)a_i(m)e_j(n)\\
&= a_i(m)e_j(n)a_k^*(p)e_l(t) + \delta_{ik}\delta_{m,-p}e_j(n)e_l(t)\\
&\qquad - a_i(m)a_k^*(p)e_l(t)e_j(n)\\
&= \delta_{ik}\delta_{m,-p}e_j(n)e_l(t);\\
[a_i(m)e_j(n), a_k^*(p)e_l^*(t)]_+ &= a_i(m)e_j(n)a_k^*(p)e_l^*(t) + a_k^*(p)e_l^*(t)a_i(m)e_j(n)\\
&= a_i(m)e_j(n)a_k^*(p)e_l(t) + \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n)\\
&\qquad - a_i(m)a_k^*(p)e_l^*(t)e_j(n)\\
&= a_i(m)e_j(n)a_k^*(p)e_l^*(t) + \delta_{ik}\delta_{m,-p}e_l^*(t)e_j(n)\\
&\qquad - a_i(m)a_k^*(p)e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\
&= \delta_{ik}\delta_{m,-p}e_j(n)e_l^*(t) + \delta_{jl}\delta_{n,-t}a_i(m)a_k^*(p)\\
&\qquad - \delta_{ik}\delta_{jl}\delta_{m,-p}\delta_{n,-t}.
\end{aligned}$$

The proofs of the others are similar. □

As in [Feingold and Frenkel 1985; Gao 2002], let $\alpha(N,\rho)^+$ be the subalgebra generated by $a_i(n)$, $a_j^*(m)$, $a_k^*(0)$ for $n, m > 0$ and $1 \le i, j, k \le N$. Let $\alpha(N,\rho)^-$ be the subalgebra generated by $a_i(n)$, $a_j^*(m)$, $a_k(0)$ for $n, m < 0$ and $1 \le i, j, k \le N$. Those generators in $\alpha(N,\rho)^+$ are called annihilation operators while those in $\alpha(N,\rho)^-$ are called creation operators. Let $V(N,\rho)$ be a simple $\alpha(N,\rho)$-module containing an element v_0^ρ, called a“vacuum vector” and satisfying

$$\alpha(N,\rho)^+ v_0^\rho = 0.$$

So all annihilation operators kill v_0^ρ and

$$V(N,\rho) = \alpha(N,\rho)^- v_0^\rho.$$

The normal orderings of the mixed quadratic elements are given as follows:

$$:a_i(m)e_j(n): = a_i(m)e_j(n), \qquad :a_i(m)e_j^*(n): = a_i(m)e_j^*(n),$$
$$:a_i^*(m)e_j(n): = a_i^*(m)e_j(n), \quad :a_i^*(m)e_j^*(m): = a_i^*(m)e_j^*(m).$$

We see that the $\alpha(N,+1)\otimes\alpha(N,-1)$-module

$$V(N) := V(N,+1)\otimes V(N,-1) = \alpha(N,+1)\otimes\alpha(N,-1)v_0^+\otimes v_0^-$$

is simple.

Motivated by Propositions 2.3, 3.1, 3.2, and 3.3, we let

$$h_{ij}(m,n) = \sum_{s\in Z} q^{-ns} :a_i(m-s)e_j(s): + \sum_{s\in Z} q^{-ns} :a_j^*(s)e_i^*(m-s): .$$

Lemma 3.4.

$$\begin{aligned}
&[h_{ij}(m,n), h_{kl}(p,t)]_+ \\
&\quad = \delta_{il} q^{tm} \sum_{s\in Z} q^{-(n+t)s}\{:a_k(m+p-s)a_j^*(s): + :e_j(s)e_k^*(m+p-s):\} \\
&\qquad\qquad + \delta_{jk} q^{np} \sum_{s\in Z} q^{-(n+t)s}\{:a_i(m+p-s)a_l^*(s): + :e_l(s)e_i^*(m+p-s):\}.
\end{aligned}$$

Proof. First we have

$$\begin{aligned}
&[h_{ij}(m,n), h_{kl}(p,t)]_+ \\
&\quad = \delta_{il} \sum_{s_1,s_2\in Z} q^{-ns_1-ts_2}\{\delta_{m-s_1,-s_2} e_k^*(p-s_2)e_j(s_1) + \delta_{m-s_1,-s_2} a_k(p-s_2)a_j^*(s_1)\} \\
&\qquad + \delta_{jk} \sum_{s_1,s_2\in Z} q^{-ns_1-ts_2}\{\delta_{s_1,s_2-p} e_i^*(m-s_1)e_l(s_2) + \delta_{s_1,s_2-p} a_i(m-s_1)a_l^*(s_2)\}.
\end{aligned}$$

Secondly notice that

$$e_k^*(p-s_2)e_j(s_1) = e_j(s_1)e_k^*(p-s_2) - \delta_{jk}\delta_{s_1,s_2-p},$$

and by the property (3-3) of the normal ordering we have

$$a_k(p-s_2)a_j^*(s_1) = :a_k(p-s_2)a_j^*(s_1): + \delta_{jk}\delta_{s_1,s_2-p}\theta(p-s_2-s_1),$$
$$e_j(s_1)e_k^*(p-s_2) = :e_j(s_1)e_k^*(p-s_2): + \delta_{jk}\delta_{s_1,s_2-p}\theta(s_1+s_2-p).$$

Then

$$\begin{aligned}e_k^*(p-s_2)e_j(s_1)+a_k(p-s_2)a_j^*(s_1) &= {:}a_k(p-s_2)a_j^*(s_1){:}+{:}e_j(s_1)e_k^*(p-s_2){:}\\ &\quad +\delta_{jk}\delta_{s_1,s_2-p}\theta(s_1+s_2-p)\\ &\quad +\delta_{jk}\delta_{s_1,s_2-p}\theta(p-s_1-s_2)-\delta_{jk}\delta_{s_1,s_2-p}\\ &= {:}a_k(p-s_2)a_j^*(s_1){:}+{:}e_j(s_1)e_k^*(p-s_2){:}\end{aligned}$$

since $\theta(s_1+s_2-p)+\theta(p-s_1-s_2)=1$. We get

$$\begin{aligned}&[h_{ij}(m,n),h_{kl}(p,t)]_+\\ &\quad=\delta_{il}\sum_{s_1,s_2\in Z}q^{-ns_1-ts_2}\delta_{m-s_1,-s_2}\{{:}a_k(p-s_2)a_j^*(s_1){:}+{:}e_j(s_1)e_k^*(p-s_2){:}\}\\ &\qquad+\delta_{jk}\sum_{s_1,s_2\in Z}q^{-ns_1-ts_2}\delta_{s_1,s_2-p}\{{:}a_i(m-s_1)a_l^*(s_2){:}+{:}e_i^*(m-s_1)e_l(s_2){:}\}\\ &\quad=\delta_{il}q^{tm}\sum_{s\in Z}q^{-(n+t)s}\{{:}a_k(m+p-s)a_j^*(s){:}+{:}e_j(s)e_k^*(m+p-s){:}\}\\ &\qquad+\delta_{jk}q^{np}\sum_{s\in Z}q^{-(n+t)s}\{{:}a_i(m+p-s)a_l^*(s){:}+{:}e_l(s)e_i^*(m+p-s){:}\}.\end{aligned}$$ □

Comparing with Proposition 2.3, let

$$g_{ij}(m,n)=\sum_{s\in Z}q^{-ns}{:}a_i(m-s)a_j^*(s){:}+\sum_{s\in Z}q^{-ns}{:}e_j(s)e_i^*(m-s){:}.$$

Then we only need to check the remaining Lie brackets (2-2) and (2-4).

Lemma 3.5.

$$[g_{ij}(m,n),h_{kl}(p,t)]_-=\delta_{jk}q^{np}h_{il}(m+p,n+t)-\delta_{il}q^{tm}h_{kj}(m+p,n+t).$$

Proof. Notice that removing the normal ordering has no effect on Lie bracket; then we have

$$\begin{aligned}[g_{ij}(m,n),h_{kl}(p,t)]_-=\sum_{s_1,s_2\in Z}q^{-ns_1-ts_2}\big[&a_i(m-s_1)a_j^*(s_1)+e_j(s_1)e_i^*(m-s_1),\\ &a_k(p-s_2)e_l(s_2)+a_l^*(s_2)e_k^*(p-s_2)\big]_-.\end{aligned}$$

Secondly, for $[a_i(m-s_1)a_j^*(s_1),a_k(p-s_2)e_l(s_2)]_-$ we have

$$[a_i(m-s_1)a_j^*(s_1),a_k(p-s_2)e_l(s_2)]_-=\delta_{jk}\delta_{s_1,s_2-p}a_i(m-s_1)e_l(s_2).$$

Similarly, we have

$$\begin{aligned}[a_i(m-s_1)a_j^*(s_1),a_l^*(s_2)e_k^*(p-s_2)]_-&=-\delta_{il}\delta_{m-s_1,-s_2}a_j^*(s_1)e_k^*(p-s_2),\\ [e_j(s_1)e_i^*(m-s_1),a_k(p-s_2)e_l(s_2)]_-&=-\delta_{il}\delta_{m-s_1,-s_2}e_j(s_1)a_k(p-s_2),\\ [e_j(s_1)e_i^*(m-s_1),a_l^*(s_2)e_k^*(p-s_2)]_-&=\delta_{jk}\delta_{s_1,s_2-p}a_l^*(s_2)e_i^*(m-s_1).\end{aligned}$$

Then we replace s_1 or s_2 in the above four terms by s:

$$\begin{aligned}[g_{ij}(m,n), h_{kl}(p,t)]_{-}i &= \delta_{jk}q^{np}\sum_{s\in Z} q^{-(n+t)s}(a_i(m+p-s)e_l(s)+a_l^*(s)e_i^*(m+p-s)) \\ &\quad -\delta_{il}q^{tm}\sum_{s\in Z} q^{-(n+t)s}(a_k(m+p-s)e_j(s)+a_j^*(s)e_k^*(m+p-s)) \\ &= \delta_{jk}q^{np}h_{il}(m+p,n+t)-\delta_{il}q^{tm}h_{kj}(m+p,n+t). \qquad \square\end{aligned}$$

Lemma 3.6.

$$[g_{ij}(m,n), g_{kl}(p,t)]_{-} = \delta_{jk}q^{np}g_{il}(m+p,n+t)-\delta_{il}q^{tm}g_{kj}(m+p,n+t).$$

Proof.

$$\begin{aligned}[g_{ij}(m,n), g_{kl}(p,t)]_{-} = \sum_{s_1,s_2\in Z} q^{-ns_1-ts_2}\big[&a_i(m-s_1)a_j^*(s_1)+e_j(s_1)e_i^*(m-s_1),\\ &a_k(p-s_2)a_l^*(s_2)+e_l(s_2)e_k^*(p-s_2)\big]_{-}.\end{aligned}$$

Then, for $[a_i(m-s_1)a_j^*(s_1), a_k(p-s_2)a_l^*(s_2)]_{-}$, by using Proposition 3.1 we have

$$\begin{aligned}&[a_i(m-s_1)a_j^*(s_1), a_k(p-s_2)a_l^*(s_2)]_{-}\\ &\qquad = -\delta_{il}\delta_{m-s_1,-s_2}a_k(p-s_2)a_j^*(s_1)+\delta_{jk}\delta_{s_1,s_2-p}a_i(m-s_1)a_l^*(s_2).\end{aligned}$$

Using Proposition 3.2,

$$\begin{aligned}&[e_j(s_1)e_i^*(m-s_1), e_l(s_2)e_k^*(p-s_2)]_{-}\\ &\qquad = -\delta_{il}\delta_{m-s_1,-s_2}e_j(s_1)e_k^*(p-s_2)+\delta_{jk}\delta_{s_1,s_2-p}e_l(s_2)e_i^*(m-s_1).\end{aligned}$$

Clearly,

$$[a_i(m-s_1)a_j^*(s_1), e_l(s_2)e_k^*(p-s_2)]_{-} = [e_j(s_1)e_i^*(m-s_1), a_k(p-s_2)a_l^*(s_2)]_{-} = 0.$$

From (3-3) and (3-2), we have

$$\begin{aligned}a_k(p-s_2)a_j^*(s_1) &= :a_k(p-s_2)a_j^*(s_1): + \delta_{jk}\delta_{s_1,s_2-p}\theta(p-s_1-s_2),\\ e_j(s_1)e_k^*(p-s_2) &= :e_j(s_1)e_k^*(p-s_2): + \delta_{jk}\delta_{s_1,s_2-p}\theta(s_1+s_2-p),\\ a_i(m-s_1)a_l^*(s_2) &= :a_i(m-s_1)a_l^*(s_2): + \delta_{il}\delta_{m-s_1,-s_2}\theta(m-s_1-s_2),\\ e_l(s_2)e_i^*(m-s_1) &= :e_l(s_2)e_i^*(m-s_1): + \delta_{il}\delta_{m-s_1,-s_2}\theta(s_1+s_2-m),\end{aligned}$$

$$\theta(p-s_1-s_2)+\theta(s_1+s_2-p) = \theta(m-s_1-s_2)+\theta(s_1+s_2-m) = 1.$$

So

$$
\begin{aligned}
-\delta_{il}\delta_{m-s_1,-s_2}a_k(p-s_2)a_j^*(s_1)+\delta_{jk}\delta_{s_1,s_2-p}a_i(m-s_1)a_l^*(s_2)\\
-\delta_{il}\delta_{m-s_1,-s_2}e_j(s_1)e_k^*(p-s_2)+\delta_{jk}\delta_{s_1,s_2-p}e_l(s_2)e_i^*(m-s_1)\\
=\delta_{jk}\delta_{s_1,s_2-p}\big(:a_i(m-s_1)a_l^*(s_2):+:e_l(s_2)e_i^*(m-s_1):\big)\\
-\delta_{il}\delta_{m-s_1,-s_2}\big(:a_k(p-s_2)a_j^*(s_1):+:e_j(s_1)e_k^*(p-s_2):\big).
\end{aligned}
$$

Then we get

$$
\begin{aligned}
&[g_{ij}(m,n),g_{kl}(p,t)]_-\\
&\quad=\sum_{s_1,s_2\in Z}q^{-ns_1-ts_2}\big\{\delta_{jk}\delta_{s_1,s_2-p}\big(:a_i(m-s_1)a_l^*(s_2):+:e_l(s_2)e_i^*(m-s_1):\big)\\
&\qquad\qquad-\delta_{il}\delta_{m-s_1,-s_2}\big(:a_k(p-s_2)a_j^*(s_1):+:e_j(s_1)e_k^*(p-s_2):\big)\big\}.
\end{aligned}
$$

Now we replace s_1 or s_2 in the above terms by s; we get

$$
\begin{aligned}
&[g_{ij}(m,n),g_{kl}(p,t)]_-\\
&\quad=\delta_{jk}q^{np}\sum_{s\in Z}q^{-(n+t)s}(a_i(m+p-s)a_l^*(s)+e_l(s)e_i^*(m+p-s))\\
&\qquad-\delta_{il}q^{tm}\sum_{s\in Z}q^{-(n+t)s}(a_k(m+p-s)a_j^*(s)+e_j(s)e_k^*(m+p-s))\\
&\quad=\delta_{jk}q^{np}g_{il}(m+p,n+t)-\delta_{il}q^{tm}g_{kj}(m+p,n+t).
\end{aligned}
$$

□

Although $g_{ij}(m,n)$ and $h_{ij}(m,n)$ are infinite sums, they are well defined as operators on $V(N)$ since at most finitely many terms can have a nontrivial action on any $v\in V(N)=\alpha(N,+1)\otimes\alpha(N,-1)v_0^+\otimes v_0^-$.

Then from Lemmas 3.4, 3.5 and 3.6 we have:

Theorem 3.7. *$V(N)$ is a module for the $Q(N-1)$-graded Lie superalgebra $\mathcal{Q}$ under the action given by*

$$
\pi(\tilde g_{ij}(m,n))=g_{ij}(m,n),
$$
$$
\pi(\tilde h_{ij}(m,n))=h_{ij}(m,n),
$$

for all $m,n\in\mathbb{Z}$ and $1\le i,j\le N$.

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CONJUGACY AND ELEMENT-CONJUGACY OF HOMOMORPHISMS OF COMPACT LIE GROUPS

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Let G be a connected compact Lie group. Among other things, we prove that the following are equivalent. (a) For all connected compact Lie groups H and all continuous homomorphisms $\phi, \phi' : H \to G$, if $\phi(h)$ and $\phi'(h)$ are conjugate in G for all $h \in H$, then ϕ and ϕ' are G-conjugate. (b) The Lie algebra of G contains no simple ideal of type D_n ($n \geq 4$), E_6, E_7, or E_8.

1. Introduction

Let G and H be two topological groups, and let $\phi, \phi' : H \to G$ be two continuous homomorphisms. We say that ϕ and ϕ' are conjugate if there is an element $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h) \quad \text{for all } h \in H.$$

We say that they are element-conjugate if for every $h \in H$, there is a $g \in G$ such that

$$g\phi(h)g^{-1} = \phi'(h).$$

Clearly, conjugate homomorphisms are element-conjugate. Conversely, we are interested to know to what extent the following statement holds.

(1) If ϕ and ϕ' are element-conjugate, then they are conjugate.

This is closely related to the failure of multiplicity one for the cuspidal spectrum of reductive groups over number fields (see [Blasius 1994, Section 1.1; Lapid 1999, Section 3; Arthur 2002, page 471; Lafforgue 2014, Section 0.8]). Some counterexamples to (1) are used to construct nonisometric pairs of isospectral manifolds (see [Larsen 1996, Theorem 2.7]).

We say that G is H-acceptable if (1) holds for all continuous homomorphisms ϕ and ϕ'. M. Larsen [1994] defined G to be acceptable if it is H-acceptable whenever H is finite. In [Larsen 1996], he classified acceptable, connected, simply connected compact Lie groups. In this paper, we are more concerned with the

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statement (1) when both G and H are connected compact Lie groups. This is more relevant to the classification of reductive subalgebras of semisimple Lie algebras, as studied in [Dynkin 1952; Liebeck and Seitz 1996; Malcev 1944; Minchenko 2006].

It was essentially known to Dynkin how counterexamples to (1) can be constructed when G is a simple compact Lie group of type D_n, with $n \geq 4$ (see [Dynkin 1952, Theorem 1.4]; see also [Wang 2012]). M. Liebeck and G. Seitz [1996] found a counterexample when G is simple of type E_8, and H is simple of type A_2, in the setting of algebraic groups. For Lie algebra homomorphisms from a semisimple Lie algebra to a simple Lie algebra of type E_6, E_7 or E_8, all counterexamples of the Lie algebra analogue of (1) are listed in [Minchenko 2006, Table 9] (see Lemma 3.6).

Based on the main result of [Minchenko 2006], we prove the following theorem for connected compact Lie groups.

Theorem 1.1. *Let G be a connected compact Lie group. Then the following are equivalent.* (*a*) *The Lie algebra of G contains no simple ideal of type* D_n $(n \geq 4)$, E_6, E_7 *or* E_8. (*b*) *The group G is H-acceptable for all connected compact Lie groups H.*

As a byproduct of the proof of Theorem 1.1, we get the following theorem for classical Lie groups.

Theorem 1.2. *Let G be a classical Lie group, that is, G is* $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{U}(p,q)$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$, $\mathrm{Sp}(p,q)$ *or* $\mathrm{O}^*(2n)$, *where* $p, q, n \geq 0$. *Then G is H-acceptable for all compact Hausdorff topological groups H.*

By convention, when $n = 0$ or $p + q = 0$, the corresponding classical group of the above theorem is the trivial group, and the theorem is trivial in this case.

2. Classical Lie groups

In the rest of the paper, let H be a compact Hausdorff topological group. By abuse of notation, we do not distinguish a representation from its underlying vector space. All representations are assumed to be complex, finite-dimensional and continuous.

Classical complex groups. We begin with the following classical result.

Proposition 2.1. *The complex general linear group* $\mathrm{GL}_n(\mathbb{C})$ $(n \geq 0)$ *is H-acceptable.*

Proof. This is well known. Let $\phi, \phi' : H \to \mathrm{GL}_n(\mathbb{C})$ be two continuous homomorphisms, to be viewed as two n-dimensional representations of H. If they are element-conjugate, then they have the same character. By the classical character theory of representations of compact groups, these two representations are isomorphic. This is the same as saying that the homomorphisms ϕ and ϕ' are conjugate. □

We define an orthogonal representation of H to be a representation V of H together with an H-invariant orthogonal form $\langle\cdot,\cdot\rangle$ on it. Here by an orthogonal form, we mean a nondegenerate symmetric bilinear form. Two orthogonal representations $(V,\langle\cdot,\cdot\rangle)$ and $(V',\langle\cdot,\cdot\rangle')$ are called isomorphic if there is an H-intertwining linear isomorphism from V to V' which sends $\langle\cdot,\cdot\rangle$ to $\langle\cdot,\cdot\rangle'$.

Similarly, we define the notions of symplectic representations and isomorphisms of symplectic representations. The next result is well known (see [Malcev 1944]).

Proposition 2.2. *Let $(V,\langle\cdot,\cdot\rangle)$ and $(V',\langle\cdot,\cdot\rangle')$ be two orthogonal (symplectic) representations of H. If V and V' are isomorphic as representations of H, then $(V,\langle\cdot,\cdot\rangle)$ and $(V',\langle\cdot,\cdot\rangle')$ are isomorphic as orthogonal (symplectic) representations.*

It is clear that Propositions 2.1 and 2.2 imply the following result, which is stated and proved in [Larsen 1994, Propositions 2.3 and 2.4], in the setting that H is a finite group.

Proposition 2.3. *The complex orthogonal group $\mathrm{O}_n(\mathbb{C})$ and the complex symplectic group $\mathrm{Sp}_{2n}(\mathbb{C})$ $(n\geq 0)$ are H-acceptable.*

Maximal compact subgroups. Let G be a Lie group with finitely many connected components. Let K be a maximal compact subgroup of G, which always exists and is unique up to conjugation [Borel 1998, Chapter VII, Theorem 1.2(i)]. Write $i_K : K \to G$ for the inclusion map.

Lemma 2.4. *Let $\phi_K,\phi'_K : H\to K$ be two continuous homomorphisms. Write $\phi := i_K\circ\phi_K$ and $\phi' := i_K\circ\phi'_K$. Then*

(a) *ϕ and ϕ' are conjugate if and only if ϕ_K and ϕ'_K are conjugate, and*

(b) *ϕ and ϕ' are element-conjugate if and only if ϕ_K and ϕ'_K are element-conjugate.*

Proof. We only prove (a), since (b) can be proved by the same method, and is also implied by (a). The "if" part of (a) is obvious. We prove the "only if" part below.

Write "Ad" for the conjugation action. By [Borel 1998, Chapter VII, Theorem 1.2(ii)], there is a closed analytic submanifold E of G such that

$$\mathrm{Ad}_k(E)=E \quad \text{for all } k\in K, \tag{2}$$

and every $g\in G$ is uniquely of the form

$$g=ke, \quad \text{with } k\in K,\ e\in E. \tag{3}$$

Assume that ϕ and ϕ' are conjugate, i.e., there is an element g in G such that

$$\phi'=\mathrm{Ad}_g\circ\phi.$$

Write $g = ke$ as in (3). Then

$$\mathrm{Ad}_{k^{-1}} \circ \phi' = \mathrm{Ad}_e \circ \phi.$$

Let $h \in H$ and put

$$k_1 := (\mathrm{Ad}_{k^{-1}} \circ \phi')(h), \quad k_2 := \phi(h).$$

Then $k_1, k_2 \in K$ and

$$k_1 = \mathrm{Ad}_e(k_2),$$

or the same,

$$k_1 e = k_2 \,\mathrm{Ad}_{k_2^{-1}}(e).$$

Now (2) and the uniqueness of the decomposition (3) imply that $k_1 = k_2$. This proves that $\mathrm{Ad}_{k^{-1}} \circ \phi' = \phi$, and thus ϕ_K and ϕ'_K are conjugate. □

The following result generalizes [Larsen 1994, Proposition 1.7].

Proposition 2.5. *The group G is H-acceptable if and only if so is K.*

Proof. Let us prove the "if" part first. Assume that K is H-acceptable, and let $\phi, \phi' : H \to G$ be two continuous homomorphisms which are element-conjugate. We need to prove that ϕ and ϕ' are conjugate. Since every compact subgroup of G is conjugate to a subgroup of K, we assume without loss of generality that the images of ϕ and ϕ' are both contained in K. Let ϕ_K and ϕ'_K be as in Lemma 2.4 so that $\phi := i_K \circ \phi_K$ and $\phi' := i_K \circ \phi'_K$. Then Lemma 2.4 implies that ϕ_K and ϕ'_K are element-conjugate, and they are conjugate since K is H-acceptable. This implies that ϕ and ϕ' are conjugate.

To prove the "only if" part, we assume that G is H-acceptable. Write ϕ, ϕ', ϕ_K and ϕ'_K as before. Assume that ϕ_K and ϕ'_K are element-conjugate. Then ϕ and ϕ' are element-conjugate, and therefore conjugate since G is H-acceptable. Now Lemma 2.4 implies that ϕ_K and ϕ'_K are conjugate. This proves that K is H-acceptable. □

Corollary 2.6. *The compact groups* $\mathrm{U}(n)$, $\mathrm{O}(n)$ *and* $\mathrm{Sp}(n)$, *where* $n \geq 0$, *are H-acceptable.*

Proof. Note that $\mathrm{U}(n)$, $\mathrm{O}(n)$ or $\mathrm{Sp}(n)$ is a maximal compact subgroup of $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{O}_n(\mathbb{C})$ or $\mathrm{Sp}_{2n}(\mathbb{C})$, respectively. Therefore the corollary is a consequence of Propositions 2.1, 2.3 and 2.5. □

The following lemma is obvious.

Lemma 2.7. *Let G_1 and G_2 be two topological groups. Then $G_1 \times G_2$ is H-acceptable if and only if so are both G_1 and G_2.*

Now we come to the proof of Theorem 1.2. When G is $\mathrm{GL}_n(\mathbb{R})$, $\mathrm{GL}_n(\mathbb{C})$, $\mathrm{GL}_n(\mathbb{H})$, $\mathrm{O}(p,q)$, $\mathrm{O}_n(\mathbb{C})$, $\mathrm{U}(p,q)$, $\mathrm{Sp}(p,q)$, $\mathrm{Sp}_{2n}(\mathbb{R})$, $\mathrm{Sp}_{2n}(\mathbb{C})$ or $\mathrm{O}^*(2n)$, its maximal compact subgroup is $\mathrm{O}(n)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$, $\mathrm{O}(p)\times\mathrm{O}(q)$, $\mathrm{O}(n)$, $\mathrm{U}(p)\times\mathrm{U}(q)$, $\mathrm{Sp}(p)\times\mathrm{Sp}(q)$, $\mathrm{U}(n)$, $\mathrm{Sp}(n)$ or $\mathrm{U}(n)$, respectively. By Corollary 2.6 and Lemma 2.7, all these compact groups are H-acceptable. Therefore G is H-acceptable by Proposition 2.5. This proves Theorem 1.2.

We record the following two results for later use.

Corollary 2.8. *The compact groups* $\mathrm{SU}(n)$ *and* $\mathrm{SO}(2n+1)$, *where* $n \geq 0$, *are H-acceptable.*

Proof. By Corollary 2.6, the groups $\mathrm{U}(n)$ and $\mathrm{O}(2n+1)$ are H-acceptable. Then the corollary follows by noting that every inner automorphism of $\mathrm{U}(n)$ or $\mathrm{O}(2n+1)$ restricts to an inner automorphism of $\mathrm{SU}(n)$ or $\mathrm{SO}(2n+1)$, respectively. □

The following lemma is obvious.

Lemma 2.9. *If G is commutative, then it is H-acceptable.*

3. A proof of Theorem 1.1

In this section, we concentrate on connected compact Lie groups.

Lemma 3.1. *Let $\phi, \phi' : H \to G$ be two continuous homomorphisms of connected compact Lie groups. Then they are element-conjugate if and only if $\phi|_S$ and $\phi'|_S$ are conjugate, where S is a maximal torus in H.*

Proof. This is because S is topologically cyclic, and every element of H is H-conjugate to an element of S. □

Lemma 3.2. *Let $\rho : \widetilde{G} \to G$ be a surjective continuous homomorphism with finite kernel of connected compact Lie groups. Let $\tilde{\phi}, \tilde{\phi}' : H \to \widetilde{G}$ be two continuous homomorphisms of connected compact Lie groups. Then $\tilde{\phi}$ and $\tilde{\phi}'$ are conjugate if and only if $\rho\circ\tilde{\phi}$ and $\rho\circ\tilde{\phi}'$ are conjugate, and $\tilde{\phi}$ and $\tilde{\phi}'$ are element-conjugate if and only if $\rho\circ\tilde{\phi}$ and $\rho\circ\tilde{\phi}'$ are element-conjugate.*

Proof. The first assertion easily follows from the observation that

$$\tilde{\phi} = \tilde{\phi}' \qquad \text{if and only if} \qquad \rho\circ\tilde{\phi} = \rho\circ\tilde{\phi}'.$$

We leave the details to the reader. The second assertion is a consequence of the first one and Lemma 3.1. □

Lemma 3.3. *Let G and G' be two connected compact Lie groups with isomorphic Lie algebras. If G is H-acceptable for all connected compact Lie groups H, then so is G'.*

Proof. Note that G and G' have a common finite fold covering group, that is, there is a connected compact Lie group $\widetilde{G}$, and surjective continuous homomorphisms $\rho : \widetilde{G} \to G$ and $\rho' : \widetilde{G} \to G'$ with finite kernels.

Assume that G is H-acceptable for all connected compact Lie groups H. Let $\tilde{\phi}_1, \tilde{\phi}_2 : H \to \widetilde{G}$ be two element-conjugate continuous homomorphisms. Then $\rho \circ \tilde{\phi}_1$ and $\rho \circ \tilde{\phi}_2$ are element-conjugate. Therefore by the assumption on G, $\rho \circ \tilde{\phi}_1$ and $\rho \circ \tilde{\phi}_2$ are conjugate. Therefore by Lemma 3.2, $\tilde{\phi}_1$ and $\tilde{\phi}_2$ are conjugate. This shows that $\widetilde{G}$ is H-acceptable for all connected compact Lie groups H.

To prove that G' is H-acceptable for all connected compact Lie groups H, we let $\phi'_1, \phi'_2 : H \to G'$ be two element-conjugate continuous homomorphisms. Then there is a connected compact Lie group $\widetilde{H}$, with a surjective continuous homomorphism $\rho_H : \widetilde{H} \to H$ with finite kernel, and two continuous homomorphisms $\tilde{\phi}'_1, \tilde{\phi}'_2 : \widetilde{H} \to \widetilde{G}$ such that the diagram

$$\begin{array}{ccc} \widetilde{H} & \xrightarrow{\tilde{\phi}'_i} & \widetilde{G} \\ {\scriptstyle \rho_H}\downarrow & & \downarrow{\scriptstyle \rho'} \\ H & \xrightarrow{\phi'_i} & G' \end{array}$$

commutes ($i = 1, 2$).

Since ϕ'_1 and ϕ'_2 are element-conjugate, we know that $\phi'_1 \circ \rho_H$ and $\phi'_2 \circ \rho_H$ are element-conjugate, or equivalently, $\rho' \circ \tilde{\phi}'_1$ and $\rho' \circ \tilde{\phi}'_2$ are element-conjugate. Then by Lemma 3.2, $\tilde{\phi}'_1$ and $\tilde{\phi}'_2$ are element-conjugate. Therefore $\tilde{\phi}'_1$ and $\tilde{\phi}'_2$ are conjugate since $\widetilde{G}$ is $\widetilde{H}$-acceptable. Thus $\rho' \circ \tilde{\phi}'_1$ and $\rho' \circ \tilde{\phi}'_2$ are conjugate, or equivalently, $\phi'_1 \circ \rho_H$ and $\phi'_2 \circ \rho_H$ are conjugate. This implies that ϕ'_1 and ϕ'_2 are conjugate. Thus G' is H-acceptable. □

We say that two homomorphisms $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ between two finite-dimensional complex Lie algebras are conjugate if there is an inner automorphism φ of $\mathfrak{g}$ such that $\varphi \circ \phi = \phi'$.

Lemma 3.4 [Minchenko 2006, Theorem 3; Dynkin 1952, Theorem 1.1]. *Let $\mathfrak{g}$ be a simple complex Lie algebra of type* G_2 *or* F_4. *Let $\mathfrak{h}$ be a reductive complex Lie algebra, and let $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ be two injective Lie algebra homomorphisms whose images are reductive Lie subalgebras of $\mathfrak{g}$. If $\phi|_{\mathfrak{s}}$ and $\phi'|_{\mathfrak{s}}$ are conjugate, then ϕ and ϕ' are conjugate, where $\mathfrak{s}$ is a Cartan subalgebra of $\mathfrak{h}$.*

Recall that a Lie subalgebra of a finite-dimensional complex Lie algebra $\mathfrak{g}$ is said to be reductive if its adjoint representation on $\mathfrak{g}$ is completely reducible.

Lemma 3.4 has the following consequence.

Proposition 3.5. *Let G be a connected compact Lie group whose complexified Lie algebra is simple of type* G_2 *or* F_4. *Then G is H-acceptable for all connected compact Lie groups H.*

Proof. Let $\phi, \phi' : H \to G$ be two element-conjugate homomorphisms. We want to show that they are conjugate. Note that ϕ and ϕ' have the same kernel. Replacing H by its quotient by the kernel, we assume without loss of generality that both ϕ and ϕ' are injective. Let S be a maximal torus in H. Write $c_G : G \to G_{\mathbb{C}}$ for the universal complexification of G, which is injective (see [Hochschild 1966]). Write $\mathfrak{s}$, $\mathfrak{h}$ and $\mathfrak{g}$ for the complexified Lie algebras of S, H and G, respectively.

By Lemma 3.1, $\phi|_S$ and $\phi'|_S$ are conjugate. Therefore their complexified differentials

$$\mathrm{d}(\phi|_S) : \mathfrak{s} \to \mathfrak{g} \quad \text{and} \quad \mathrm{d}(\phi'|_S) : \mathfrak{s} \to \mathfrak{g}$$

are conjugate. Then Lemma 3.4 implies that the complexified differentials

$$\mathrm{d}(\phi) : \mathfrak{h} \to \mathfrak{g} \quad \text{and} \quad \mathrm{d}(\phi') : \mathfrak{h} \to \mathfrak{g}$$

are conjugate. This implies that the homomorphisms

$$c_G \circ \phi : H \to G_{\mathbb{C}} \quad \text{and} \quad c_G \circ \phi' : H \to G_{\mathbb{C}}$$

are conjugate. Since G is a maximal compact subgroup of $G_{\mathbb{C}}$, Lemma 2.4 implies that ϕ and ϕ' are conjugate. This proves the proposition. □

Lemma 3.6. *Let* $\mathfrak{g}$ *be a simple complex Lie algebra of type* D_n $(n \geq 4)$, E_6, E_7 *or* E_8. *Then there are a semisimple complex Lie algebra* $\mathfrak{h}$ *and two nonconjugate injective Lie algebra homomorphisms* $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ *such that* $\phi|_{\mathfrak{s}}$ *and* $\phi'|_{\mathfrak{s}}$ *are conjugate. Here* $\mathfrak{s}$ *is a Cartan subalgebra of* $\mathfrak{h}$.

Proof. The lemma is a consequence of [Dynkin 1952, Theorem 1.4] when $\mathfrak{g}$ has type D_n $(n \geq 4)$ and a consequence of [Minchenko 2006, Theorem 7] when $\mathfrak{g}$ has type E_6, E_7 or E_8. □

Lemma 3.6 has the following consequence.

Proposition 3.7. *Let* G *be a connected compact Lie group whose complexified Lie algebra is simple of type* D_n $(n \geq 4)$, E_6, E_7 *or* E_8. *Then* G *is not* H*-acceptable for some connected compact Lie group* H.

Proof. Write $\mathfrak{g}$ for the complexified Lie algebra of G. Let $\mathfrak{h}$, $\mathfrak{s}$, and $\phi, \phi' : \mathfrak{h} \to \mathfrak{g}$ be as in Lemma 3.6. As in the proof of Proposition 3.5, write $c_G : G \to G_{\mathbb{C}}$ for the universal complexification of G. Let $H_{\mathbb{C}}$ be a simply connected, connected complex Lie group whose Lie algebra is identified with $\mathfrak{h}$. Then ϕ, ϕ' integrate to holomorphic homomorphisms

$$\psi_{\mathbb{C}}, \psi'_{\mathbb{C}} : H_{\mathbb{C}} \to G_{\mathbb{C}}. \tag{4}$$

Take a maximal compact subgroup H of $H_{\mathbb{C}}$, and a maximal torus S in H such that the complexified Lie algebra of S equals $\mathfrak{s}$. Replacing ϕ and ϕ' by their conjugations by appropriate elements of $G_{\mathbb{C}}$, we assume without loss of generality that $\psi_{\mathbb{C}}(H) \subset G$ and $\psi'_{\mathbb{C}}(H) \subset G$. Then the homomorphisms in (4) restrict to two homomorphisms

$$\psi, \psi' : H \to G.$$

Since ϕ and ϕ' are nonconjugate, we know that $\psi_{\mathbb{C}}$ and $\psi'_{\mathbb{C}}$ are nonconjugate, which implies that ψ and ψ' are nonconjugate. On the other hand, since $\phi|_{\mathfrak{s}}$ and $\phi'|_{\mathfrak{s}}$ are conjugate, we know that $\psi_{\mathbb{C}}|_S$ and $\psi'_{\mathbb{C}}|_S$ are conjugate. Then Lemma 2.4 implies that $\psi|_S$ and $\psi'|_S$ are conjugate. This implies that ψ and ψ' are element-conjugate by Lemma 3.1. This proves the proposition. □

Finally, in view of Lemmas 3.3 and 2.7, Theorem 1.1 is a consequence of Lemma 2.9, Corollaries 2.6 and 2.8, and Propositions 3.5 and 3.7.

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ENTIRE SIGN-CHANGING SOLUTIONS WITH FINITE ENERGY TO THE FRACTIONAL YAMABE EQUATION

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We show the existence of infinitely many finite energy sign-changing solutions for the fractional Yamabe-type equation

$$(-\Delta)^s u = |u|^{\frac{4s}{n-2s}} u \quad \text{in } \mathbb{R}^n,$$

where $n \geq 3$ and $s \in \left(\frac{1}{2}, 1\right)$.

1. Introduction

We are interested in the existence of finite energy sign-changing solutions to the fractional Yamabe-type equation in $\mathbb{R}^n$,

$$(-\Delta)^s u = \gamma\, |u|^{p-1} u \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $n \geq 3$ and p is the fractional critical Sobolev exponent $p = (n+2s)/(n-2s)$. In (1), $\gamma > 0$ is a constant chosen for normalization purposes as

$$\gamma = \frac{\Gamma\left(\frac{n+2s}{2}\right)}{\Gamma\left(\frac{n-2s}{2}\right)}.$$

For any $s \in (0, 1)$, $(-\Delta)^s$ is the nonlocal operator defined as

$$\begin{aligned}(-\Delta)^s(x) &= c(n, s) \text{ P.V.} \int_{\mathbb{R}^n} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy \\ &= c(n, s) \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^n \setminus B(x,\epsilon)} \frac{u(x) - u(y)}{|x - y|^{n+2s}}\, dy,\end{aligned} \tag{2}$$

where P.V. stands for *the principal value* and

$$c(n, s) = \pi^{-(2s+\frac{n}{2})} \frac{\Gamma\left(\frac{n}{2} + s\right)}{\Gamma(-s)}.$$

This nonlocal operator in $\mathbb{R}^n$ can be expressed as a generalized Dirichlet-to-Neumann map for a certain elliptic boundary value problem with local differential operators

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defined on the upper halfspace $\mathbb{R}^{n+1}_+ = \{(x,t) : x \in \mathbb{R}^n,\ t > 0\}$, as we learn from Caffarelli and Silvestre [2007]: given a solution $u = u(x)$ of $(-\Delta)^s u = f$ in $\mathbb{R}^n$, one can equivalently consider the dimensionally extended problem for $u = u(x,t)$ which solves

$$\begin{cases} \operatorname{div}(t^{1-2s}\nabla u) = 0, & \text{in } \mathbb{R}^{n+1}_+, \\ -\lim_{t\to 0} d_s t^{1-2s}\partial_t u(x,t) = f, & \text{on } \partial\mathbb{R}^{n+1}_+, \end{cases}$$

where d_s is the positive constant $d_s = 2^{2s-1}\Gamma(s)/\Gamma(1-s)$. By finite energy solutions of (1), we mean the following. Consider the Schwartz space $\mathcal{S}$ of rapidly decaying C^∞ functions on $\mathbb{R}^n$, and for any $\tau \in \mathcal{S}$ we denote by

$$\mathcal{F}\tau(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\tau(x)\,dx$$

the Fourier transformation of τ. We look for solutions u of (1) in the energy space

$$\mathcal{D}^s(\mathbb{R}^n) = \{u \in L^{\frac{2n}{n-2s}}(\mathbb{R}^n) : \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)} < \infty\},$$

where $\|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)}$ is defined by $\left(\int_{\mathbb{R}^n}|\xi|^{2s}|\mathcal{F}u(\xi)|^2\,d\xi\right)^{\frac{1}{2}}$, endowed with the norm $\|u\|_{\mathcal{D}^s(\mathbb{R}^n)} = \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)}$. These solutions correspond to critical points of the functional

$$J(u) = \frac{1}{2}\int_{\mathbb{R}^n} |(-\Delta)^{\frac{s}{2}}u|^2 - \gamma\frac{n-2s}{2n}\int_{\mathbb{R}^n}|u|^{\frac{2n}{n+2s}}, \quad u \in \mathcal{D}^s(\mathbb{R}^n).$$

Following the work by Lieb [1983] — see also [Frank and Lieb 2010; 2012; Carlen and Loss 1990] for alternative proofs — positive solutions to (1) are given by the family of functions defined by

$$U(x) = \left(\frac{2}{1+|x|^2}\right)^{\frac{n-2s}{2}} \quad \text{and} \quad \mu^{-\frac{n-2s}{2}}\,U\!\left(\frac{x-\xi}{\mu}\right) \tag{3}$$

for any $\mu > 0$ and $\xi \in \mathbb{R}^n$. Indeed these functions realize the Hardy–Littlewood–Sobolev inequality, which states the existence of a positive number S such that for all $u \in C^\infty(\mathbb{R}^n)$,

$$S\,\|u\|_{L^{2^*}(\mathbb{R}^n)} \le \|(-\Delta)^{\frac{s}{2}}u\|_{L^2(\mathbb{R}^n)}$$

where $2^* = p+1 = 2n/(n-2s)$. Indeed, these functions are the only positive solutions to (1) under some decay conditions [Chen et al. 2006; Li 2004; Li and Zhu 1995]. In particular, this is true if $u \in L^{2n/(n-2s)}_{\text{loc}}(\mathbb{R}^n)$, as shown in [Chen et al. 2006].

On the other hand, (1) can be read on the sphere $S^n \subset \mathbb{R}^{n+1}$, after a stereographic projection. Indeed, the inverse of the stereographic projection $\pi : \mathbb{R}^n \to S^n \setminus \{S\}$, where $S = (0, \dots, 0, -1) \in \mathbb{R}^{n+1}$, defined by

$$\pi(y) = \left(\frac{2y}{1+|y|^2}, \frac{1-|y|^2}{1+|y|^2} \right)$$

is a conformal map and $\pi^* g_0 = U^{\frac{4s}{n-2s}}(y)\, dy$, where g_0 is the standard metric on S^n and U is defined in (3). In S^n, the fractional Laplacian $(-\Delta)^s$ reduces to an elliptic pseudodifferential operator $P_s^{g_0}$ of order $2s$ with principal symbol $\sigma_{2s}(P_s^{g_0}) = |\xi|_{g_0}^2 s$. In [Chang and González 2011] a relation between this operator and a Dirichlet-to-Neumann operator of uniformly nondegenerate elliptic boundary value problems in the spirit of [Caffarelli and Silvestre 2007] is established. We have $\pi^*(L_s^{g_0} v) = U^{-(n+2s)/(n-2s)}(-\Delta)^s(U\pi^* v)$ for any v defined on S^n. Thus u is a solution to (1) if and only if w, defined by $u = U\pi^* w$, solves

$$\Delta_{g_0} w + \gamma(|w|^{\frac{4s}{n-2s}} w - w) = 0 \quad \text{in } S^n. \tag{4}$$

Positive solutions to (4) solve the so-called fractional Yamabe problem on the sphere S^n. We refer to [González and Qing 2013] for a general formulation of the fractional Yamabe problem and results concerning its solvability.

Finite energy sign-changing solutions to (1), or equivalently (4), are poorly understood.

The purpose of this paper is to give a first example of finite energy sign-changing solutions to (1), in all dimensions $n \geq 3$, and for $s \in \left(\frac{1}{2}, 1\right)$: we build a solution to (1) which looks like the solution U surrounded by k negative copies U properly scaled and distributed along the vertices of a regular polygon with radius 1. Our main result is the following theorem:

Theorem 1.1. *Let $n \geq 3$ and $s \in \left(\frac{1}{2}, 1\right)$. Write $\mathbb{R}^n = \mathbb{C} \times \mathbb{R}^{n-2}$ and let $\xi_j^k = (e^{2j\pi i/k}, 0)$, $j = 1, \dots, k$. Then for any sufficiently large k, there is a finite energy solution to Problem (1) of the form*

$$u_k(x) = U(x) - \sum_{j=1}^{k} \mu_k^{-\frac{n-2s}{2}} U\big(\mu_k^{-1}(x - \xi_j)\big) + o(1),$$

where

$$\mu_k = \left(k^2 2^{\frac{n-2s}{2}} \sum_{j=1}^{\infty} j^{2s-n} \right)^{-1} (1 + o(1))$$

Moreover,

$$J(u_k) = (k+1)J(U) + O(1). \tag{5}$$

Here $O(1)$ remains bounded and $o(1) \to 0$ uniformly as $k \to +\infty$.

The proof of the result consists in defining a first approximation and then showing that a small perturbation of this approximation provides an actual solution to the problem. This is done by linearizing the equation around the approximation and applying an invertibility theory for the linearized operator. In this step, we use the nondegeneracy property of U proved in [Dávila et al. 2013], which states that all bounded solutions of the linear problem

$$(-\Delta)^s\phi - \gamma\,\frac{n+2s}{n-2s}\,U^{\frac{4s}{n-2s}}\phi = 0$$

are linear combinations of

$$\partial_{x_j}U(x), \quad \text{for } j = 1, \dots, n,$$

and

$$\frac{n-2s}{2}U(x) + x \cdot \nabla U(x).$$

Indeed, the above functions belong to the kernel of the linearized operator, due to the corresponding rigid motion under which (1) is invariant. These are the only nontrivial elements of the kernel according to [Dávila et al. 2013].

A second ingredient we take advantage of to produce an invertibility theory is the symmetry of the configuration. This reflects into the fact that our approximation, as well as our final solution, satisfy the symmetries

$$u(\bar{y}, y') = u(e^{\frac{2\pi j}{k}\iota}\bar{y}, y'), \qquad j = 1, \dots, k-1, \tag{6}$$

$$u(y_1, y_2, \dots, y_j, \dots, y_n) = u(y_1, y_2, \dots, -y_j, \dots, y_n) \qquad j = 2, \dots, n. \tag{7}$$

Furthermore, they are invariant under Kelvin transform, namely

$$u(y) = |y|^{2s-n}u\left(\frac{y}{|y|^2}\right).$$

The final step in the proof consists in adjusting properly the parameter μ_k. A detailed description of the scheme of the proof is given in Section 2.

Let us mention that a very similar construction for finite energy, sign-changing solutions to the classic Yamabe-type problem in $\mathbb{R}^n$:

$$\Delta u + |u|^{\frac{4}{n-2}}u = 0 \quad \text{in } \mathbb{R}^n,$$

namely when $s = 1$ in (1), has been done in [del Pino et al. 2011; 2013]. Indeed, our result extends to the case $s \in \left(\frac{1}{2}, 1\right)$, the construction done in [del Pino et al. 2011], from which we are inspired.

We learned recently of [Fang 2014], where the author constructs solutions to (1) similar to ours, covering the whole range $s \in (0, 1)$. Nevertheless, in that case, the concentration parameter μ_k is of order k^{-3} [Fang 2014, (2.4)], while our concentration parameter is $\mu_k \sim k^{-2}$, as $k \to \infty$. It is not clear to us how this choice of the parameter's rate provides a real solution to (1). Indeed, it is this choice of the parameter's rate, in terms of k, that allows the author of [Fang 2014] to cover the whole range $s \in (0, 1)$.

Our restriction on s is consequence of two inequalities: we need a certain power of integrability q to be $q < n$ in order to have a good first approximation when estimated in proper norms, and at the same time we need $q > n/(2s)$ to guarantee enough regularity. These constraints restrict us to $s \in \left(\frac{1}{2}, 1\right)$. We believe that our construction should work in the whole range $s \in (0, 1)$, and in fact we think that $\mu_k \sim k^{-2}$, as $k \to \infty$, for the whole range $s \in (0, 1)$, but an invertibility theory on different weighted Sobolev spaces is needed. We will treat this problem in a forthcoming paper.

The rest of the paper will be devoted to the proof of Theorem 1.1.

2. Ansatz for the solution and scheme of the proof

This section is devoted to define a first approximation for a solution to (1) and to describe the scheme of the proof of our result.

We start reminding that U defined in (3) is invariant under Kelvin transform, namely

$$U(y) := |y|^{2s-n} U(|y|^{-2} y).$$

Even more, it can be proved that also the family of solutions

$$\mu^{-\frac{n-2s}{2}} \left(\frac{y-\xi}{\mu} \right)$$

is invariant under Kelvin transform if and only if

$$|\xi|^2 + \mu^2 = 1.$$

Let k be a positive integer and define, for any $j = 1, \dots, k$, the k points

$$\xi_j = \sqrt{1-\mu^2} \left(e^{2\pi i (j-1)/k}, 0, \dots, 0 \right) \in \mathbb{R}^2 \times \mathbb{R}^{n-2},$$

where $\mu > 0$ is a positive number of the form

$$\mu = \frac{\delta}{k^2}, \quad \text{with } c < \delta < c^{-1} \tag{8}$$

for a certain constant $c > 0$, independent of k, as $k \to \infty$. Define

$$U_*(y) = U(y) - \sum_{j=1}^{k} U_j(y), \quad \text{where } U_j(y) = \mu^{-\frac{n-2s}{2}} U(\mu^{-1}(y - \xi_j)). \tag{9}$$

For large values of k, which at the same time make the scaling parameters μ very small, we shall show that U_* is a good approximate solution for (1). Observe that the function U_* satisfies the symmetry properties (6) and (7). Furthermore, U_* is invariant under Kelvin transform

$$U_*(y) = |y|^{2-n} U_*\left(\frac{y}{|y|^2}\right).$$

This is consequence of a straightforward computation, using the fact that

$$\mu^2 + |\xi_j|^2 = 1 \quad \text{for any } j = 1, \ldots, k.$$

We will show that (1) admits a solution of the form

$$u(y) = U_*(y) + \phi(y)$$

where ϕ is small when compared with U_*. It satisfies the symmetry conditions (6) and (7), and it is invariant under Kelvin transform. Then (1) can be rewritten in terms of ϕ as

$$(-\Delta)^s \phi - p\gamma |U_*|^{p-1} \phi - E - \gamma N(\phi) = 0, \tag{10}$$

where E is

$$\gamma^{-1} E = \left|U - \sum U_j\right|^{p-1} \left(U - \sum U_j\right) - \left(U^p - \sum U_j^p\right) \tag{11}$$

and

$$N(\phi) = |U_* + \phi|^{p-1}(U_* + \phi) - |U_*|^{p-1} - |U_*|^{p-1} U_* - p|U_*|^{p-1} \phi. \tag{12}$$

The size of the error term E defined in (11) turns out to be relatively small, as the number k tends to infinity, when estimated with proper norms. Let us fix a number $q > \frac{n}{2s}$; we define the weighted L^q norm

$$\|h\|_{**} := \|(1 + |y|)^{n+2s-2n/q} h\|_{L^q(\mathbb{R}^n)}$$

Let $\eta > 0$ be a small and fixed number, independent of k. The error can be estimated separately in the *exterior region* $\bigcap_j \left\{|y - \xi_j| > \frac{\eta}{k}\right\}$ and then in each of the *inner regions* $\{|y - \xi_j| < \frac{\eta}{k}\}$. Indeed, we shall prove that there exists a constant C such that, for all k large enough,

$$\|(1 + |y|)^{n+2s-2n/q} E\|_{L^q(\bigcap_j \{|y-\xi_j| > \frac{\eta}{k}\})} \leq C k^{1-n/q}. \tag{13}$$

Observe that, in order to have a small (in k) size for the error in the exterior domain, we need $q < n$. On the other hand, for regularity issue we will discuss later, we assume that $q > \frac{n}{2s}$. The set of possible values for q, $\frac{n}{2s} < q < n$, is not empty since we are considering s in the range $s \in \left(\frac{1}{2}, 1\right)$.

If we change scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y)$, in $|y| < \eta/(\mu k)$, for any $j = 1, \ldots, k$, we have the following estimate for the error in each *interior domain*:

$$\|(1+|y|)^{n+2s-2n/q}\tilde{E}_j(y)\|_{L^q\{|y|<\frac{\eta}{k\mu}\}} \leq Ck^{-n/q}. \tag{14}$$

We shall prove the validity of estimates (13) and (14) at the end of this section.

In order to solve in ϕ the nonlinear Equation (10), we use a *gluing method*. Let ζ be a cutoff function defined as follows: $\zeta(t) = 1$ for $t < 1$ and $\zeta(t) = 0$ for $t > 2$. We also defined $\zeta^-(t) = \zeta(2t)$. Then we set

$$\zeta_j(y) = \begin{cases} \zeta\left(k\eta^{-1}|y|^{-2}\left|y - \xi_j|y|\right|\right) & \text{if } |y| > 1, \\ \zeta(k\eta^{-1}|y - \xi_j|) & \text{if } |y| \leq 1. \end{cases}$$

Observe that

$$\zeta_j(y) = \zeta_j(|y|^{-2}y)$$

A function ϕ of the form

$$\phi = \sum_{j=1}^{k} \tilde{\phi}_j + \psi. \tag{15}$$

is a solution of the problem (10), provided that we can solve the following coupled system of elliptic equation in $(\tilde{\phi}_1, \tilde{\phi}_2, \ldots, \tilde{\phi}_k)$ and ψ:

$$(-\Delta)^s(\tilde{\phi}_j) - p\gamma|U_*|\zeta_j\tilde{\phi}_j - \zeta_j\Big(p\gamma|U_*|^{p-1}\psi + E + \gamma N\Big(\tilde{\phi}_j + \sum_{i\neq j}\tilde{\phi}_i + \psi\Big)\Big) = 0, \tag{16}$$

where $j = 1, 2, \ldots, k$ and

$$\begin{aligned}(-\Delta)^s\psi - p\gamma U^{p-1}\psi &- \Big(p\gamma\big(|U_*|^{p-1} - U^{p-1}\big)\Big(1 - \sum_{j=1}^{k}\zeta_j\Big) + p\gamma U^{p-1}\sum_{j=1}^{k}\zeta_j\Big)\psi \\ &- p\gamma|U_*|^{p-1}\sum_j(1-\zeta_j)\tilde{\phi}_j \\ &- \Big(1 - \sum_{j=1}^{k}\zeta_j\Big)\Big(E + \gamma N\Big(\sum_{j=1}^{k}\tilde{\phi}_j + \psi\Big)\Big) = 0\end{aligned} \tag{17}$$

To solve the above coupled system, we follow the following strategy. First we solve (17) in the unknown ψ, assuming that $\tilde{\phi}_j$ are fixed functions satisfying

$$\tilde{\phi}_j(\overline{y}, y') = \tilde{\phi}_1(e^{\frac{2\pi j}{k}\imath}\overline{y}, y'), \qquad j = 1, 2, \ldots, k-1, \tag{18}$$

$$\tilde{\phi}_1(y_1, y_2, \ldots, y_j, \ldots, y_n) = \tilde{\phi}_1(y_1, y_2, \ldots, -y_j, \ldots, y_n) \qquad j = 2, \ldots, n, \tag{19}$$

and the invariant condition under Kelvin's transform,

$$\tilde{\phi}_1 = |y|^{2s-n}\tilde{\phi}_1(|y|^{-2}y). \tag{20}$$

Furthermore, we assume that

$$\|\phi_1\|_* < \rho \quad \text{where } \phi_1 = \mu^{\frac{n-2s}{2}}\tilde{\phi}_1(\xi_1 + \mu y). \tag{21}$$

We have the validity of the following result:

Proposition 2.1. *There exist constants k_0, C, ρ_0 such that for all $k \geq k_0$, the following holds: Suppose that $\tilde{\phi}_j$, $j = 1, 2, \dots, k$, satisfy conditions (18)–(21) with $\rho < \rho_0$. Then there exists a unique solution $\psi = \Psi(\phi_1)$ to (17) that satisfies the symmetries*

$$\psi(\bar{y}, y') = \psi(e^{\frac{2\pi j}{k}\imath}\bar{y}, y'), \qquad j = 1, 2, \dots, k-1,$$

$$\psi(\bar{y}, \dots, y_j, \dots, y_n) = \psi(\bar{y}, \dots, -y_j, \dots, y_n), \qquad j = 3, \dots, n,$$

$$\psi = |y|^{2s-n}\psi(|y|^{-2}y),$$

$$\|\psi\|_* \leq \frac{C}{k^{n/q-1}} + C\|\phi_1\|_*^2.$$

Moreover, the operator Ψ satisfies the Lipschitz condition

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \leq C\|\phi_1^1 - \phi_1^2\|_*.$$

Once we have the result of the above Proposition, under the assumption on $\tilde{\phi}_j$ we have that all equations (16) reduce to just one, say that for $\tilde{\phi}_1$. Then we will find a solution to our problem if we solve

$$(-\Delta)^s\tilde{\phi}_1 - p\gamma|U_1|^{p-1}\tilde{\phi}_1 - \zeta_1 E - \gamma\mathcal{N}(\phi_1) = 0 \quad \text{in } \mathbb{R}^n \tag{22}$$

where

$$\mathcal{N}(\phi_1) = p\big(|U_*|^{p-1}\zeta_1 - |U_1|^{p-1}\big)\phi_1 + \zeta_1\Big(p|U_*|^{p-1}\Psi(\phi_1) + N\Big(\tilde{\phi}_1 + \sum_{i\neq 1}\tilde{\phi}_i + \Psi(\phi_1)\Big)\Big)$$

Rather than solving (22) directly, we shall first solve the corresponding projected version of (22):

$$(-\Delta)^s\tilde{\phi}_1 - p\gamma|U_1|^{p-1}\tilde{\phi}_1 - \zeta_1 E + \gamma\mathcal{N}(\phi) = c_{n+1}U_1^{p-1}\tilde{Z}_{n+1} \quad \text{in } \mathbb{R}^n \tag{23}$$

where

$$c_{n+1} = -\frac{\displaystyle\int_{\mathbb{R}^n}(\zeta_1 E + \gamma\mathcal{N}(\phi))\tilde{Z}_{n+1}}{\displaystyle\int_{\mathbb{R}^n} U_1^{p-1}\tilde{Z}_{n+1}^2}. \tag{24}$$

and

$$\tilde{Z}_{n+1}(y) = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y-\xi_1)) \tag{25}$$

Proposition 2.2. *There exist constants k_0, C such that for all $k \geq k_0$, the following holds: Let $\Psi(\phi_1)$ the solution predicted by Proposition 2.1. Then there exists a unique solution $\psi_1 = \Phi(\delta)$, $c_{n+1} = c_{n+1}(\delta)$ to (23) and (24), which depends continuously on δ. Moreover,*

$$\|\Phi\|_* \leq Ck^{-\frac{n}{q}} \quad \text{and} \quad \|\mathcal{N}(\phi)\|_{**} \leq Ck^{-\frac{2n}{q}},$$

for some fixed positive constant C.

To conclude our argument, we shall show the existence of a number δ in the definition of μ in (8) so that the above constant c_{n+1} is equal to zero. In this way, we constructed a solution to (1) with the qualitative properties predicted by Theorem 1.1.

Scheme of the paper. In Section 3 we prove some basic results on linear problems in $\mathbb{R}^n$. These results will be applied to prove Propositions 2.1 and 2.2 in Section 4. Section 5 is dedicated to show the existence of $\delta > 0$ so that $c_{n+1} = 0$, concluding in this way the proof of our theorem.

We finish this section with the proof of estimates (13) and (14).

Proof of (13). This is in the region $\bigcap_j \{|y-\xi_j| > \frac{\eta}{k}\}$. For any y in this exterior region,

$$|E(y)| \leq C\left(\frac{1}{(1+|y|^2)^{2s}} + \left|\sum_{j=1}^{k} \frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}\right|^{\frac{4s}{n-2s}}\right)\left(\sum_{j=1}^{k} \frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}\right),$$

for some positive constant $C > 0$. Since for any j fixed and $|y-\xi_j| = \frac{\eta}{k}$ we have

$$\sum_{i=1}^{k} \frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_i|^{n-2s}} = \frac{1}{k^{n-2s}} k^{n-2s} + \sum_{i \neq j}^{k} \frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_i|^{n-2s}} \leq 1 + \frac{k-1}{ck^{n-2s}},$$

then we conclude that

$$|E| \leq C \frac{\mu^{\frac{n-2s}{2}}}{(1+|y|^2)^{2s}} \sum_{j=1}^{k} \frac{1}{|y-\xi_j|^{n-2s}}.$$

Thus a direct computation gives

$$
\begin{aligned}
\big\|(1+|y|)^{n+2s-2n/q}E\big\|_{L^q(\mathrm{Ext})} \\
&\le C\mu^{\frac{n-2s}{2}}\left\|\frac{(1+|y|)^{n+2s-2n/q})}{(1+|y|^2)^{2s}}\sum_{j=1}^{k}\frac{1}{|y-\xi_j|^{n-2s}}\right\|_{L^q(\mathrm{Ext})}\\
&\le C\mu^{\frac{n-2s}{2}}\sum_{j=1}^{k}\left(\int_{|y-\xi_j|>\frac{\eta}{k}}\frac{(1+|y|)^{(n+2s)q-2n}}{(1+|y|^2)^{2sq}}\frac{1}{|y-\xi_j|^{(n-2s)q}}\,dy\right)^{\frac{1}{q}}\\
&\le C\mu^{\frac{n-2s}{2}}k\left(\int_{\frac{\eta}{k}}^{1}\frac{t^{n-1}}{t^{(n-2s)q}}\,dt\right)^{\frac{1}{q}}\\
&= C\mu^{\frac{n-2s}{2}}k\big(k^{(n-2s)q-n}-1\big)^{\frac{1}{q}}\\
&\le C\mu^{\frac{n-2s}{2}}k^{(n-2s)+1-\frac{n}{q}} \qquad \square
\end{aligned}
$$

Proof of (14). This is in the inner region $|y-\xi_j|<\frac{\eta}{k}$, for some j fixed. Observe that if y is close to ξ_j, then

$$U_j \sim O(\mu^{-(n-2s)/2}).$$

For any y in this region, there exists $t\in(0,1)$ such that

$$E = p\Big(-U_j + t\Big(-\sum_{i\neq j}U_j + U\Big)\Big)^{p-1}\Big(-\sum_{i\neq j}U_j+U\Big) - U^p + \sum_{i\neq j}U_j.$$

We consider the change of scale $\tilde{E}_j(y) := \mu^{\frac{n+2s}{2}}E(\xi_j+\mu y)$, $|y|<\frac{\eta}{\mu k}$. Therefore, we obtain that for some $t\in(0,1)$

$$
\begin{aligned}
\tilde{E}_j(y) = p\Big(-U(y) + t\Big(\sum_{i\neq j}U(y-\mu^{-1}(\xi_i-\xi_j)) + \mu^{\frac{n-2s}{2}}U(\xi_j+\mu y)\Big)\Big)^{p-1}\\
\times\Big(-\sum_{i\neq j}U(y-\mu^{-1}(\xi_i-\xi_j)) + \mu^{\frac{n-2s}{2}}U(\xi_j+\mu y)\Big)\\
+\sum_{i\neq j}U^p(y-\mu^{-1}(\xi_i-\xi_j)) - \mu^{\frac{n+2s}{2}}U^p(\xi_j+\mu y).
\end{aligned}
$$

Taking into account the configuration of the points ξ_j, we have

$$|\xi_i-\xi_j| \sim \frac{|i-j|}{k}.$$

Furthermore, for $i \neq j$ and $|y| < \frac{\eta}{k\mu}$,

$$U(y-\mu^{-1}(\xi_i-\xi_j)) \le C\frac{\mu^{n-2s}}{|\xi_j-\xi_i|^{n-2s}}\left(\frac{|\xi_j-\xi_i|^2}{\mu^2+|\mu y-(\xi_j-\xi_i)|^2}\right)^{\frac{n-2s}{2}} \le C\frac{\mu^{n-2s}k^{n-2s}}{|i-j|^{n-2s}}.$$

Moreover,

$$\Big|\sum_{i\neq j} U(y-\mu^{-1}(\xi_i-\xi_j)\Big| \le Ck^{n-2s}\mu^{n-2s} \quad \text{and} \quad \mu^{\frac{n-2s}{2}}U(\xi_j+\mu y)) \le C\mu^{\frac{n-2s}{2}}$$

for some constant $C>0$. Thus we conclude that

$$|\tilde{E}_j(y)| \le C\left(\frac{k^{n-2s}\mu^{n-2s}}{1+|y|^{4s}}+\mu^{\frac{n+2s}{2}}\right),$$

and we have an estimate of the error in the inner region

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\tilde{E}_j(y)\right\|_{L^q\{|y|<\frac{\eta}{k\mu}\}} \le C\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left(\frac{k^{n-2s}\mu^{n-2s}}{1+|y|^{4s}}+\mu^{\frac{n+2s}{2}}\right)\right\|_{L^q\{|y|<\frac{\eta}{k\mu}\}}.$$

Since

$$\left\|(1+|y|)^{n-2s-\frac{2n}{q}}\right\|^q_{L^q\{|y|<\frac{\eta}{k\mu}\}} \le C\int_0^{\frac{\eta}{k\mu}}(1+r)^{(n-2s)q-n-1}\,dr \le C\Big(\frac{1}{k\mu}\Big)^{(n-2s)q-n}$$

and

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\right\|^q_{L^q\{|y|<\frac{\eta}{k\mu}\}} \le C\int_0^{\frac{\eta}{k\mu}}(1+r)^{(n+2s)q-n-1}\,dr \le C\Big(\frac{1}{k\mu}\Big)^{(n+2s)q-n}$$

it follows that

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\tilde{E}_j(y)\right\|_{L^q\{|y|<\frac{\eta}{k\mu}\}} \le Ck^{-\frac{n}{q}}(1+k^{-4s}).$$

This gives the proof of (14). □

3. Some linear problems

Let L_0 be the linear operator defined by

$$L_0(\phi) := (-\Delta)^s(\phi) - p\gamma U^{p-1}\phi \quad \text{in } \mathbb{R}^n.$$

As we know from [Dávila et al. 2013], the set of bounded solutions of the homogeneous equation $L_0(\phi) = 0$ is spanned by the $n + 1$ functions defined by

$$Z_i = \partial_{x_i} U, \quad i = 1, \ldots, n, \quad \text{and} \quad Z_{n+1} = \tfrac{1}{2}(n - 2s)U + x \cdot \nabla U.$$

We now establish a solvability result for the linear problem

$$L_0(\phi) = h \quad \text{in } \mathbb{R}^n,$$

under proper orthogonality conditions on h and ϕ. For this purpose, we introduce the norm

$$\|\phi\|_* := \|(1 + |y|^{n-2s})\phi\|_\infty. \tag{26}$$

Lemma 3.1. *Assume $q \in (\frac{n}{2s}, \frac{n}{s})$. Let h be such that $\|h\|_{**} < \infty$ and*

$$\int_{\mathbb{R}^n} U^{p-1} Z_l h \, dx = 0 \quad \text{for all } l = 1, 2, \ldots, n + 1.$$

Then the equation

$$(-\Delta)^s \phi - pU^{p-1}\phi = h \quad \text{in } \mathbb{R}^n \tag{27}$$

has a unique solution ϕ with $\|\phi\|_* < +\infty$ such that

$$\int_{\mathbb{R}^n} U^{p-1} Z_l \phi \, dx = 0 \quad \text{for all } l = 1, 2, \ldots, n + 1.$$

Furthermore, there exists a constant $C > 0$, depending only on q, s, and n, such

$$\|\phi\|_* \le C\|h\|_{**}. \tag{28}$$

Proof. Let H^s be the completion of $C_0^\infty(\mathbb{R}^n)$ equipped with the norm

$$\|\phi\|_{H^s} = \sqrt{\int_{\mathbb{R}^n} |\phi|^2 + \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|\phi(x) - \phi(y)|^2}{|x - y|^{n+2s}} \, dx \, dy},$$

and let $(H^s, \langle \cdot, \cdot \rangle_{H^s})$ be a Hilbert space with the product

$$\langle f, g \rangle_{H^s} = \int_{\mathbb{R}^{2n}} \frac{(f(x) - f(y))(g(x) - g(y))}{|x - y|^{n+2s}} \, dx \, dy.$$

Let us consider the subspace

$$H = \left\{ \phi \in H^s(\mathbb{R}^n) \text{ such that } \int U^{p-1} Z_l \phi \, dx = 0, \quad l = 1, 2, \ldots, n + 1 \right\}.$$

We consider the problem of finding $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}} \phi (-\Delta)^{\frac{s}{2}} \tau \, dx - p\gamma \int_{\mathbb{R}^n} U^{p-1} \phi \tau + \int_{\mathbb{R}^n} h\tau = 0 \quad \text{for all } \tau \in H;$$

this variational formulation makes sense if we consider for instance $h \in L^{\frac{2n}{n+2s}}$, since $H^s(\mathbb{R}^n) \hookrightarrow L^{\frac{2n}{n-2s}}(\mathbb{R}^n)$ continuously; see, for instance, [Di Nezza et al. 2012].

Let $f \in L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$. By Riesz's theorem there exist a unique $\phi \in H$ such that

$$\int_{\mathbb{R}^n} (-\Delta)^{\frac{s}{2}}\phi(-\Delta)^{\frac{s}{2}}\tau\,dx + \int_{\mathbb{R}^n} f\tau\,dx = 0 \quad \text{for all } \tau \in H.$$

Thus $A(f) = \phi$ defines a linear operator between $L^{\frac{2n}{n+2s}}(\mathbb{R}^n)$ and H. By the local compactness of Sobolev embedding [Di Nezza et al. 2012] and the decay at infinity of U^{p-1}, we have that the map $H \to L^{\frac{2n}{n+2s}}$, $\phi \mapsto U^{p-1}\phi$ is compact. Hence, Fredholm's alternative applies to the problem

$$\phi - A(p\gamma U^{p-1}\phi) = A(h). \tag{29}$$

For $h = 0$, we have $L_0(\phi) = 0$ and $\phi \in H$. Thus $(-\Delta)^s\phi = pU^{p-1}\phi$ in $\mathbb{R}^n$; hence,

$$\phi(x) = \sigma_{n,s}\,p\gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi(y)}{|x-y|^{n-2s}},$$

for some explicit positive constant $\sigma_{n,s}$. We claim that ϕ is bounded. Indeed, let $\delta > 0$ be a fixed positive small number and write

$$\int_{\mathbb{R}^n} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} = \int_{|x-y|<\delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} + \int_{|x-y|>\delta} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} := I_1 + I_2. \tag{30}$$

We have

$$I_1 \le C\|\phi\|_\infty \int_{|x-y|<\delta} \frac{1}{|x-y|^{n-2s}}\,dy \le C\delta^{2s}\|\phi\|_\infty \tag{31}$$

and, using the Holder inequality repeatedly,

$$\begin{aligned} I_2 &\le \left(\int_{|x-y|>\delta} \left(\frac{1}{|x-y|^{n-2s}}\right)^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{2n}} \left(\int_{|x-y|>\delta} \left(U^{p-1}\phi\right)^{\frac{2n}{n+2s}}\right)^{\frac{n+2s}{2n}} \\ &\le C\left(\int_{|x-y|>\delta} \phi^{\frac{2n}{n-2s}}\right)^{\frac{n-2s}{2n}} \left(\int_{|x-y|>\delta} U^{(p-1)\frac{2n}{4s}}\right)^{\frac{4s}{2n}} \le C\|\phi\|_{L^{2n/(n-2s)}} \end{aligned}$$

Choosing δ properly small, we obtain that ϕ is bounded. We can now apply the result in [Dávila et al. 2013] and conclude that ϕ is a linear combination of the functions Z_l, $l = 1, \dots, n+1$. Since $\phi \in H$ we have that $\phi \equiv 0$. Fredholm's alternative implies that, for any h satisfying the orthogonality condition, a function $\phi \in H$ solution to (29) exists.

Assume now that ϕ solves (27), we shall now show the a priori bound (28). We first show that ϕ is bounded. First we have

$$\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \le \|\phi\|_{H^s(\mathbb{R}^n)} \le \|h\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} \le \|(1+|y|)^{n+2s-2n/q}h\|_{L^q(\mathbb{R}^n)}.$$

Observe now that

$$\phi(x) = \sigma_{n,s}\, p \int_{\mathbb{R}^n} \frac{U^{p-1}\phi(y)}{|x-y|^{n-2s}} + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}}.$$

Fixing a small $\delta > 0$, we get

$$\int_{\mathbb{R}^n} \frac{h(y)}{|x-y|^{n-2s}}\, dy = \int_{|x-y|<\delta} \frac{h(y)}{|x-y|^{n-2s}}\, dy + \int_{|x-y|<\delta} \frac{h(y)}{|x-y|^{n-2s}}\, dy = J_1 + J_2$$

with

$$J_1 \le \int_{|x-y|<\delta} \left(\frac{1}{|x-y|^{(n-2s)q'}} \right)^{q'} \|h\|_{L^q(\mathbb{R}^n)} \le C \|h\|_{L^q(\mathbb{R}^n)}$$

since $q > \frac{n}{2s}$, and

$$J_2 \le \left(\int_{|x-y|>\delta} \frac{1}{|x-y|^{2n}} \right)^{\frac{n-2s}{2n}} \|h\|_{L^{2n/(n+2s)}} \le C \|h\|_{L^{2n/(n+2s)}}.$$

Thus, thanks also to (30) and (31), for all $x \in \mathbb{R}^n$,

$$|\phi(x)| \le C\delta^{2s} \|\phi\|_\infty + C\big(\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}} \big).$$

Choosing δ small, we conclude that ϕ is bounded since

$$\|\phi\|_\infty \le C\big(\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|h\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}} \big). \tag{32}$$

Next we show the decay rate at infinity of ϕ. Consider

$$\tilde{\phi}(y) = |y|^{2s-n} \phi(|y|^{-2} y) \quad \text{and} \quad \tilde{h}(y) = |y|^{-n-2s} h(|y|^{-2} y).$$

A direct computation shows that

$$(-\Delta)^s \tilde{\phi} - p\gamma U^{p-1}(y) \tilde{\phi} = \tilde{h} \quad \text{on } \mathbb{R}^n \setminus \{0\},$$

and

$$\begin{aligned} \|\tilde{\phi}\|_{H^s(\mathbb{R}^n)} + \|\tilde{\phi}\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} &= \|\phi\|_{H^s(\mathbb{R}^n)} + \|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)}, \\ \|\tilde{h}\|_{L^q(\mathbb{R}^n)} = \|(1+|y|)^{n+2s-2n/q} h\|_{L^q(\mathbb{R}^n)} &= \|h\|_{**}. \end{aligned}$$

Applying the estimate (32) to $\tilde{\phi}$, we get

$$\begin{aligned} \|\tilde{\phi}\|_{L^\infty(B(0,1))} \le \|\tilde{\phi}\|_{L^\infty(\mathbb{R}^n)} &\le C\big(\|\tilde{\phi}\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)} + \|\tilde{h}\|_{L^{2n/(n+2s)}} \big) \\ &\le C\big(\|\phi\|_{L^{2n/(n-2s)}(\mathbb{R}^n)} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)} + \|h\|_{L^{2n/(n+2s)}} \big) \\ &\le C\big(\|h\|_{**} + \|\tilde{h}\|_{L^q(\mathbb{R}^n)} \big) = C\|h\|_{**}. \end{aligned}$$

Since $\| |y|^{n-2s} \phi \|_{L^\infty\{|y|>1\}} = \|\tilde{\phi}\|_{L^\infty(B(0,1))}$, we conclude that $\|\phi\|_* \le C\|h\|_{**}$. □

Under further symmetry conditions on h and ϕ, (27) can be solved without the orthogonality conditions. For a general function ψ defined in $\mathbb{R}^n$, consider the symmetries

$$\psi(\bar{y}, y') = \psi(e^{\frac{2\pi j}{k}\imath}\bar{y}, y'), \quad j = 1, 2, \dots, k-1, \tag{33}$$

and

$$\psi(\bar{y}, \dots, y_j, \dots, y_n) = \psi(\bar{y}, \dots, -y_j, \dots, y_n), \quad j = 3, \dots, n, \tag{34}$$

together with invariance under the Kelvin transform

$$\psi(y) = |y|^{2s-n}\psi(|y|^{-2}y). \tag{35}$$

Lemma 3.2. *Assume that h satisfies (33), (34), and $\|h\|_{**} < \infty$. Furthermore, we assume that*

$$h(y) = |y|^{-n-2s}h(|y|^{-2}y).$$

Then (27) has a unique bounded solution $\phi = T(h)$ that satisfies symmetries (33), (34), and (35). Moreover, there exists C depending only on q, s, and n such that

$$\|\phi\|_* \le \|h\|_{**}.$$

The proof of this result is very close to the proof of [del Pino et al. 2011, (4.19)]. We refer the interested reader to that reference.

For a later purpose, we need to establish a result like the one in Lemma 3.1 for a linear operator more general then L_0.

Lemma 3.3. *Let $2s < \nu < n$. There exist numbers δ, C, depending on ν, n such that the following holds: If g, a, and ϕ are functions such that $\|(1+|y|^\nu)g\|_\infty < +\infty$, $\|(1+|y|^{\nu-2s})\phi\|_\infty < +\infty$, and $\|(1+|y|^{2s})a\|_\infty < \delta$, and*

$$L_0(\phi) + a(y)\phi = g(y) + \sum_{l=1}^{n+1} c_l U^{p-1} Z_l \quad \textit{in } \mathbb{R}^n, \tag{36}$$

where

$$\int_{\mathbb{R}^n} U^{p-1} Z_l \phi = 0 \quad \textit{for all } l = 1, \dots, n+1 \tag{37}$$

and

$$c_l \int_{\mathbb{R}^n} U^{p-1} Z_l^2 = \int_{R^n} (a(y)\phi - g(y)) Z_l \phi \quad \textit{for all } l = 1, \dots, n+1, \tag{38}$$

then

$$\|(1+|y|^{\nu-2s})\phi\|_\infty \le C\|(1+|y|^\nu)g\|_\infty. \tag{39}$$

Proof. By contradiction, let us assume the existence of functions ϕ_n, a_n, g_n and constants c_l^n such that (36)–(38) hold, and

$$\|(1+|y|^{\nu})g_n\|_\infty \to 0, \quad \|(1+|y|^{\nu-2s})\phi_n\|_\infty = 1, \quad \|(1+|y|^{2s})a_n\|_\infty \to 0. \tag{40}$$

Clearly, $\|(1+|y|^{\nu})a_n g_n\|_\infty \to 0$ and $c_l^n \to 0$, so without loss of generality we may assume that $a_n \equiv 0$ and $c_l^n = 0$. We claim first that

$$\|\phi_n\|_\infty \to 0.$$

Assume the opposite: there are numbers γ, $R > 0$ and points x_n such that

$$|\phi_n(x_n)| \geq \gamma, \quad |x_n| \leq R.$$

Passing to a subsequence, and arguing like in the proof of Lemma 3.1, we find that ϕ_n converges in the energy space and locally uniformly over compact sets to a bounded function $\phi_0 \neq 0$ with

$$L_0(\phi_0) = 0, \quad \text{and} \quad \int_{\mathbb{R}^n} U^{p-1}\phi Z_l = 0 \quad \text{for all } l,$$

which gives $\phi_0 = 0$. This is a contradiction due to the result in [Dávila et al. 2013]. Thus we have that $\|\phi_n\|_\infty \to 0$.

Next we shall show that $\|(1+|y|^{\nu-2s})\phi_n\|_\infty \to 0$, thus getting to a contradiction with (40), and the proof of the Lemma. Using the equation, we have that

$$\phi_n(x) = \sigma_{n,s}\, p\gamma \int_{\mathbb{R}^n} \frac{U^{p-1}(y)\phi_n(y)}{|x-y|^{n-2s}}\, dy + \sigma_{n,s} \int_{\mathbb{R}^n} \frac{g_n(y)}{|x-y|^{n-2s}}\, dy \tag{41}$$

for some explicit positive constant $\sigma_{n,s}$. Since $2s < \nu < n$, and taking into account that $\|(1+|y|^{\nu})g_n\|_\infty \to 0$, as well as the behavior of U^{p-1} at infinity, there exists a positive constant C, independent of n, such that

$$|\phi_n(x)| \leq C\left(\frac{\|\phi_n\|_\infty}{(1+|x|^{2s})} + \frac{o(1)}{(1+|x|^{\nu-2s})}\right)$$

for some $o(1) \to 0$, as $n \to \infty$. Replacing the above estimate in (41) and repeating the same procedure a finite number of times, we get that

$$|\phi_n(x)| \leq C\frac{\|\phi_n\|_\infty + o(1)}{(1+|x|^{\nu-2s})}. \qquad \square$$

4. Proof of Propositions 2.1 and 2.2

Proof of Proposition 2.1. Let us fix functions $\tilde{\phi}_j$ and we assume that they satisfy the symmetry assumptions (6), (7) and the invariance under Kelvin transform

$$\tilde{\phi}_1 = |y|^{2s-n}\tilde{\phi}_1(|y|^{-2}y).$$

Finally, we assume

$$\|\phi_1\|_* < \rho, \quad \text{where } \phi_1 = \mu^{\frac{n-2s}{2}} \tilde{\phi}_1(\xi_1 + \mu y). \tag{42}$$

for a small, fixed $\rho > 0$.

We next solve (17). To do so, we write it in the form

$$(-\Delta)^s(\psi) - p\gamma U^{p-1}(y)\psi \underbrace{- \gamma V(y)\psi - p\gamma |U_*|^{p-1} \sum_{j=1}^{k} (1-\zeta_j)\tilde{\phi}_j - M(\psi)}_{:=h} = 0,$$

where

$$V(y) := \underbrace{p(|U_*|^{p-1} - U^{p-1})\left(1 - \sum_{j=1}^{k} \zeta_j\right)}_{:=V_1} + \underbrace{pU^{p-1} \sum_{j=1}^{k} \zeta_j}_{:=V_2} := V_1 + V_2$$

and

$$M(\psi) := \left(1 - \sum_{j=1}^{k} \zeta_j\right)\left(E + \gamma N\left(\sum_{j=1}^{k} \tilde{\phi}_j + \psi\right)\right)$$

A basic observation is that the function h as defined above satisfies the conditions (33), (34), and $\|h\|_{**} < \infty$. Furthermore, we have that

$$h(y) = |y|^{-n-2s} h(|y|^{-2} y).$$

Hence, we can define the linear operator T in the Lemma 3.2 and we can write our problem (17) in fixed point as

$$\psi = -T\left(V\psi + p\gamma |U_*|^{p-1} \sum_j (1-\zeta)\tilde{\phi}_j + M(\psi)\right) =: \mathcal{M}(\psi) \tag{43}$$

We notice that $\mathcal{M}$ is well defined in space X of continuous functions ψ with $\|\psi\|_* \leq \infty$, and satisfying

$$\begin{aligned} \psi(\bar{y}, y') &= \psi(e^{\frac{2\pi j}{k}\imath} \bar{y}, y'), && j = 1, 2, \ldots, k-1, \\ \psi(\bar{y}, \ldots, y_j, \ldots, y_n) &= \psi(\bar{y}, \ldots, -y_j, \ldots, y_n), && j = 3, \ldots, n, \\ \psi &= |y|^{2s-n} \psi(|y|^{-2} y). \end{aligned}$$

We claim that

$$\|V\psi(y)\|_{**} \leq Ck^{1-\frac{n}{q}} \|\psi\|_* \tag{44}$$

and

$$\left\| p\gamma |U_*|^{p-1} \sum_{j=1}^{k} (1-\zeta_j)\tilde{\phi}_j \right\|_{**} \leq Ck^{1-\frac{n}{q}} \|\psi\|_*. \tag{45}$$

We claim that if

$$\|\psi\|_* + \|\phi_1\|_* \le 2\rho,$$

then

$$\|M(\psi)\|_{**} \le C\big(k^{1-\frac{n}{q}} + k^{1-\frac{n}{q}}\|\phi_1\|_*^2 + \|\psi\|_*^2\big). \tag{46}$$

Furthermore, for ψ_1, ψ_2 in X,

$$\|M(\psi_1) - M(\psi_2)\|_{**} \le C\rho\|\psi_1 - \psi_2\|_*$$

We can thus conclude that, for ρ small enough, the operator $\mathcal{M}$ defines a contraction map in the set of functions $\psi \in X$ with

$$\|\psi\|_* \le C\big(\|\phi_1\|_*^2 + k^{1-\frac{n}{q}}\big). \tag{47}$$

From the estimate (47), we get the Lipschitz dependence

$$\|\Psi(\phi_1^1) - \Psi(\phi_1^2)\|_* \le C\|\phi_1^1 - \phi_1^2\|_*.$$

We shall next show the validity of (44), (45), and (46).

Proof of (44). Consider

$$f(t) = \Big|U - t\sum_{j=1}^{k} U_j\Big|^{p-1}.$$

By the mean value theorem,

$$|V_1| \le p(p-1)\Big|U - s\sum_{j=1}^{k} U_j\Big|^{p-2}\Big(\sum_{j=1}^{k} U_j\Big) \le CU^{p-2}\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}.$$

Thus, if for all j, $|y-\xi_j| > \frac{\eta}{k}$, then

$$|V_1\psi(y)| \le C\|\psi\|_* U^{p-1}(y)\sum_{j=1}^{k}\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi_j|^{n-2s}}.$$

Since $\zeta_j \equiv 1$ on $|y-\xi_j| < \frac{\eta}{k}$,

$$\begin{aligned}
\big\|(1+|y|)^{n+2s-\frac{2n}{q}} V_1\psi\big\|_{L^q(\mathbb{R}^n)} &= \big\|(1+|y|)^{n+2s-\frac{2n}{q}} V_1\psi\big\|_{L^q(\mathbb{R}^n\setminus\bigcup_j B(\xi_j,\frac{\eta}{k}))}\\
&\le Ck\Big(\int_{B(\xi_1,\frac{\eta}{k})^c\cap B(0,2)}\frac{\mu^{\frac{(n-2s)q}{2}}}{|y-\xi_j|^{(n-2s)q}}\,dy\Big)^{\frac{1}{q}}\|\psi\|_*\\
&\le Ck\mu^{\frac{n-2s}{2}}k^{(n-2s)-\frac{n}{q}}\|\psi\|_*,
\end{aligned}$$

for some positive constant C. Thus $\|V_1\psi(y)\|_{**} \le Ck^{1-\frac{n}{q}}\|\psi\|_*$. On the other hand,

$$\begin{aligned}\|V_2\psi\|_{**} &= \|(1+|y|)^{n+2s-\frac{2n}{q}} pU^{p-1}\sum_{j=1}^k \zeta_j\psi\|_{L^q(\mathbb{R}^n)}\\ &\le C\Big(\int_{B(0,1)}\Big((1+|y|)^{n+2s-\frac{2n}{q}}U^{p-1}\sum_{j=1}^k\zeta_j\psi\Big)^q dy\Big)^{\frac{1}{q}}\end{aligned}$$

with

$$\begin{aligned}&\Big(\int_{B(0,1)}\Big((1+|y|)^{n+2s-\frac{2n}{q}}U^{p-1}\sum_{j=1}^k\zeta_j\psi\Big)^q dy\Big)^{\frac{1}{q}}\\ &\qquad\le C\sum_j\Big(\int_{B(\xi_j,\frac{2\eta}{k})}\frac{U^{(p-1)q}(1+|y|)^{(n+2s)q-2n}}{(1+|y|)^{(n-2s)q}}\,dy\Big)^{\frac{1}{q}}\|\psi\|_*\\ &\qquad\le Ck^{1-n}\|\psi\|_* \qquad\square\end{aligned}$$

Proof of (45). Estimate (45) can be obtained arguing as in the proof of estimate (44), after noticing that

$$|\tilde\phi_j(y)| \le CU(y)\|\phi_1\|_*\frac{\mu^{\frac{n-2s}{2}}}{|y-\xi|^{n-2s}}. \qquad\square$$

Proof of (46). For the moment we shall assume that

$$\|\psi\|_* + \|\phi_1\|_* \le 2\rho$$

for a ρ sufficiently small. Let us assume that $|y-\xi_j| > \frac{\eta}{k}$ for all j. First we recall that

$$\Big\|(1+|y|)^{n+2s-\frac{2n}{q}}\Big(1-\sum_{j=1}^k\zeta_j\Big)E\Big\|_{L^q(\mathbb{R}^n)} = \|(1+|y|)^{n+2s-\frac{2n}{q}}E\|_{L^q(\mathrm{Ext})} \le Ck^{1-\frac{n}{q}}$$

Then we find in this region

$$\Big|N\Big(\sum_{j=1}^k\tilde\phi_j+\psi\Big)\Big| \le CU^{p-2}\Big(\Big|\sum_{j=1}^k\tilde\phi_j\Big|^2+|\psi|^2\Big).$$

But

$$U^{p-2}\Big|\sum_{j=1}^k\tilde\phi_j\Big|^2 \le C\|\phi_1\|_*^2U^p\sum_{j=1}^k\frac{\mu^{n-2s}}{|y-\xi_j|^{2(n-2s)}}, \qquad U^{p-2}|\psi|^2 \le U^p\|\psi\|_*^2$$

Thus, we have

$$\begin{aligned}
&\left\|(1+|y|)^{n+2s-\frac{2n}{q}}\left(1-\sum_{j=1}^{k}\zeta_j\right)\left(\gamma N\left(\sum_{j=1}^{k}\tilde{\phi}_j+\psi\right)\right)\right\|_{L^q(\mathbb{R}^n)}\\
&\qquad=\left\|(1+|y|)^{n+2s-\frac{2n}{q}}(\gamma N(\phi))\right\|_{L^q(\mathrm{Ext})}\\
&\qquad\le C\|\phi_1\|_*^2\left\|(1+|y|)^{n+2s-\frac{2n}{q}}U^p\left(\sum_{j=1}^{k}\frac{\mu^{n-2s}}{|y-\xi_j|^{2(n-2s)}}+\psi\right)\right\|_{L^q(\mathrm{Ext})}\\
&\qquad\le\frac{C\mu^{n-2s}}{k^{-2(n-2s)+\frac{n}{q}-1}}\|\phi_1\|_*^2+C\|\psi\|_*^2
\end{aligned}$$

Using the above inequalities, we get

$$\|M(\psi)\|_{**}\le Ck^{1-\frac{n}{q}}+k^{1-\frac{n}{q}}\|\phi_1\|_*^2+C\|\psi\|_*^2, \qquad \square$$

This concludes the proof of Proposition 2.1.

Proof of Proposition 2.2. In order to prove Proposition 2.2, we need to consider the linear problem

$$(-\Delta)^s\tilde{\phi}_1-p\gamma U_1^{p-1}\tilde{\phi}-\tilde{h}(y)=c_{n+1}U_1^{p-1}\tilde{Z}_{n+1}\quad\text{in }\mathbb{R}^n \tag{48}$$

for a general function $\tilde{h}$, where

$$\tilde{Z}_{n+1}(y)=\mu^{-\frac{n-2s}{2}}Z_{n+1}(\mu^{-1}(y-\xi_1))\quad\text{and}\quad c_{n+1}=\frac{\int_{\mathbb{R}^n}\tilde{h}\tilde{Z}_{n+1}}{\int_{\mathbb{R}^n}U_1^{p-1}\tilde{Z}_{n+1}^2}.$$

Lemma 4.1. *Assume that $\tilde{h}$ is even with respect to each variable $y_2,\dots,y_n$ and it satisfies the invariance*

$$\tilde{h}(y)=|y|^{-n-2s}h(|y|^{-2}y)$$

Assume in addition that

$$h(y)=\mu^{\frac{n+2s}{2}}\tilde{h}(\xi_1+\mu y)$$

satisfies $\|h\|_{**}\le\infty$. Then (48) has a unique solution $\tilde{\phi}:=\tilde{T}(\tilde{h})$ that is even with respect to each of the variables $y_2,\dots,y_n$, invariant under Kelvin's transformations

$$\tilde{\phi}(y)=|y|^{2s-n}\tilde{\phi}(|y|^{-2}y),$$

where $\phi(y)=\mu^{\frac{n-2s}{2}}\tilde{\phi}(\xi_1+\mu y)$ and satisfies

$$\int_{\mathbb{R}^n}\phi U^{p-1}Z_{n+1}=0.$$

Moreover, there exists C such that

$$\|\phi\|_* \le C\|h\|_{**}.$$

Proof. We consider ϕ and h such that

$$(-\Delta)^s\phi - p\gamma|U|^{p-1}\phi = h(y) \quad \text{in } \mathbb{R}^n, \quad \text{and} \quad \int_{\mathbb{R}^n} \tilde{h}\tilde{Z}_{n+1} = 0.$$

The evenness of h in the last $(n-1)$ coordinates guarantees that

$$\int_{\mathbb{R}^n} hZ_l = 0, \quad l = 2, \dots, n, n+1.$$

We have that to prove that $\int_{\mathbb{R}^n} hZ_1 = 0$. Let

$$I(t) = \int_{\mathbb{R}^n} w_\mu(y - t\xi_1)\tilde{h}(y)\,dy.$$

We notice that

$$(\xi_1)_1 \int_{\mathbb{R}^n} hZ_1 = \partial_t I(t)\big|_{t=0} = -(\xi_1)_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(y - \xi_1)h(y)\,dy; \tag{49}$$

after a change of variable,

$$I(t) = \int_{\mathbb{R}^n} w_\mu(|y|^{-2}y - t\xi_1)\tilde{h}(|y|^{-2}y)|y|^{-2n} = \int_{\mathbb{R}^n} w_{\mu(t)}(y - a(t)\xi_1)\tilde{h}(y)\,dy$$

where

$$\mu(t) = \frac{\mu t}{\mu^2 + |\xi_1|2t^2} \quad \text{and} \quad s(t) = \frac{t}{\mu^2 + |\xi_1|^2t^2}.$$

Hence,

$$\begin{aligned} \partial_t I(t)\big|_{t=1} = \mu'(1)&\int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\big|_{\mu=1}\tilde{h}(y)\,dy \\ &- s'(1)\xi_1 \int_{\mathbb{R}^n} \partial_{y_1} w_\mu(t)(y - \xi_1)h(y)\,dy = 0. \end{aligned} \tag{50}$$

We can check that

$$\int_{\mathbb{R}^n} \partial_\mu w_\mu(y - \xi_1)\big|_{\mu=1}\tilde{h}(y)\,dy = \int_{\mathbb{R}^n} Z_{n+1}(y)h(y)\,dy = 0$$

and $s'(1) = 1 - 2|\xi_1|^2$. Hence, using (49) and (50), we obtain $\int_{\mathbb{R}^n} hZ_1 = 0$. It follows from Lemma 3.1 that there exists a unique solution ϕ_1 for (48) with

$$\int_{\mathbb{R}^n} hZ_l = 0, \quad l = 1, \dots, n+1 \quad \text{and} \quad \|\phi\|_* \le C\|h\|_{**}.$$

Arguing by uniqueness, as in proof of Lemma 3.2, we find that $\tilde{\phi}$ satisfies the corresponding symmetries. □

We use the above lemma to solve (23) and (24). We consider the operator $\tilde{T}$ defined in the lemma. We are going to prove the existence of a solution to (23) by a fixed point argument

$$\tilde{\phi}_1 = \tilde{T}(\zeta_1 + \gamma \mathcal{N}(\phi_1)) =: \mathcal{M}(\phi_1). \tag{51}$$

For any f we set $\bar{f}(y) = \mu^{\frac{n+2s}{2}} f(\xi + \mu y)$. Let

$$f_1(y) = p\zeta_1(|U_*|^{p-1} - |U_1|^{p-1})\tilde{\phi}_1.$$

For $|y| < \frac{\eta}{k\mu}$,

$$|\bar{f}_1(y)| \le C\left(\mu^{n-2s}k^{n-2s}\sum_{j=1}^{k-1}\frac{1}{j^{n-2s}} + \mu^{\frac{n-2s}{2}}\right)U^{p-1}\|\phi_1\|_*$$

and so

$$\|\bar{f}_1(y)\|_{**} \le C\big(\mu^{n-2s}k^{n-2s} + \mu^{\frac{n-2s}{2}}\big)(\mu k)^{-n+2s+\frac{n}{q}}\|\phi_1\|_* = C\mu^{\frac{2n}{q}}\|\phi_1\|_*.$$

Analogously for $f_2 = (\zeta_1 - 1)U_1^{p-1}\tilde{\phi}_1$ in the region $|y| < \frac{\eta}{\mu k}$,

$$|\bar{f}_2(y)| \le U^p\|\phi_1\|_*;$$

hence $\|\bar{f}_2\|_{**} \le Ck^{-\frac{n}{q}}\|\phi_1\|_*$. Now we consider $f_3 = \zeta_1 p|U_*|^{p-1}\Psi(\phi_1)$ on $|y| < \frac{n}{\mu k}$,

$$|\bar{f}_3| \le CU^{p-1}\mu^{\frac{n-2s}{2}}\|\Psi(\phi_1)\|_\infty \le CU^{p-1}\mu^{\frac{n-2s}{2}}(\|\phi_1\|_* + k^{1-\frac{n}{q}});$$

thus,

$$\|\bar{f}_3(y)\|_{**} \le C\mu^{\frac{n}{2q}}(\|\phi_1\|_* + k^{1-\frac{n}{q}}).$$

Now, for

$$f_4 = \zeta_1 N\left(\tilde{\phi}_1 + \sum_{i=2}\tilde{\phi}_i\right)\Psi(\phi_1)$$

we notice that

$$\overline{N}(\phi) = (V_* + \hat{\phi})^p - V_*^p - pV_*^{p-1}\hat{\phi}$$

where $\hat{\phi}(y) := \mu^{\frac{n-2s}{2}}\phi(\xi_1 + \mu y)$ and

$$V_*(y) = U(y) + \sum_{i=2}^{k} U(y + \mu^{-1}(\xi_1 - \xi_j)) - \mu^{\frac{n-2s}{2}}U(\xi_1 + \mu y)$$

with

$$\phi = \tilde{\phi}_1 + \sum_{i=2}^{k}\tilde{\phi}_i + \Psi(\phi_1).$$

Therefore

$$|\bar{f}_4| \le C\big(U^{p-1}\mu^{\frac{n-2s}{2}}\|\phi_1\|_* + U^{p-1}\mu^{\frac{n-2s}{2}}(\|\phi_1\|_* + k^{1-\frac{n}{q}})\big),$$

and hence,

$$\|\bar{f}_4\|_{**} \le C\big(\mu^{\frac{n}{2q}} \|\phi_1\|_* + \mu^{\frac{n}{2q}} (\|\phi_1\|_* + k^{1-\frac{n}{q}})^2\big).$$

Concerning $f_5 = \zeta_1 E$, we recall that

$$\|\bar{f}_5\|_{**} \le C\mu^{\frac{n}{2q}}.$$

The above estimates suggest that it is possible to apply a fixed point argument of contraction type in the set of all continuous functions $\phi_1 = \Phi(\delta)$ such that $\|\phi_1\|_* \le C\mu^{\frac{n}{2q}}$. This gives the existence and the estimate for ϕ_1, satisfying

$$\|\Phi\|_* \le Ck^{-\frac{n}{q}},$$

and

$$\|\mathcal{N}(\phi)\|_{**} \le Ck^{-\frac{2n}{q}}.$$

Straightforward computations shows also the continuous dependence of $\phi_1 = \Phi(\delta)$ and c_{n+1} on the parameter δ. This concludes the proof of Proposition 2.2.

5. Conclusion

In this section we show the existence of $\delta > 0$ such that $c_{n+1}(\delta) = 0$ in (23). Indeed this fact guarantees that the function

$$U_* + \phi,$$

where $U_* = U - \sum U_j$ is defined in (9) and $\phi = \sum_{j=1}^k \tilde{\phi}_j + \psi$ is defined in (15), is a solution for the original problem (1). Let

$$\tilde{Z}_{n+1} = \mu^{-\frac{n-2s}{2}} Z_{n+1}(\mu^{-1}(y - \xi_1)).$$

We recall that

$$Z_{n+1}(y) = y \cdot \nabla U + \frac{n-2s}{2} U.$$

We need the existence of a δ such that

$$c_{n+1} = \int_{\mathbb{R}^n} (\zeta_1 E + \gamma \mathcal{N}(\phi_1)) \tilde{Z}_{n+1} = 0. \tag{52}$$

Since we are assuming that $s > \frac{1}{2}$, we claim that

$$\int_{\mathbb{R}^n} \zeta_1 E \tilde{Z}_{n+1} = A\,\delta\, k^{2s-n} \Big(-2^{\frac{n-2s}{2}} \Big(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}} \Big) \delta + 1 \Big) + k^{1-n}\, \Theta_k(\delta) \tag{53}$$

and

$$\int_{\mathbb{R}^n} \gamma \mathcal{N}(\phi_1) \tilde{Z}_{n+1} = k^{-(n-2s)} k^{1-\frac{n}{q}}\, \Theta_k(\delta), \tag{54}$$

where $\Theta_k(\delta)$ denotes a continuous function of δ, which is uniformly bounded, as $k \to \infty$. Since $n - 2s > 1$ for any $s \in \left(\frac{1}{2}, 1\right)$, from (53) and (54) we obtain the existence of a unique δ solution to (52) with

$$\delta = \left(2^{\frac{n-2s}{2}}\left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}}\right)\right)^{-1}(1 + O(k^{1-2s})).$$

What is left of this section is devoted to the proof of (53) and (54).

Proof of (53). We write

$$\int_{\mathbb{R}^n} \zeta_1 E\tilde{Z}_{n+1} = \int_{\mathbb{R}^n} E\tilde{Z}_{n+1} + \int_{\mathbb{R}^n} (\zeta_1 - 1)E\tilde{Z}_{n+1}.$$

Expanding the first term, we get

$$\int_{\mathbb{R}^n} E\tilde{Z}_{n+1} = \int_{B_1} E\tilde{Z}_{n+1} + \int_{\mathbb{R}^n \setminus \bigcup B_j} E\tilde{Z}_{n+1} + \sum_{j \neq 1} \int_{B_j} E\tilde{Z}_{n+1} := I_1 + I_2 + I_3,$$

where $B_j = B(\xi_j, \frac{\eta}{k})$. With the scaling $x = \mu y + \xi_1$ and writing

$$\tilde{E}(y) = \mu^{\frac{n+2s}{2}} E(\xi_1 + \mu y),$$

we get

$$\int_{B_1} E\tilde{Z}_{n+1} = \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1(y) Z_{n+1}(y)\, dy.$$

Thus

$$\begin{aligned}
I_1 &= \int_{B(0, \frac{\eta}{\mu k})} \tilde{E}_1 Z_{n+1}(y)\, dy \\
&= -\gamma p \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\
&\quad + \gamma p \mu^{\frac{n-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1}\, dy \\
&\quad + \gamma p \int_{B(0, \frac{\eta}{\mu k})} \left((U(y) + sV(y))^{p-1} - U^{p-1}\right) V(y) Z_{n+1}\, dy \\
&\quad + \gamma \sum_{j \neq 1} \int_{B(0, \frac{\eta}{\mu k})} U^p(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\
&\quad - \mu^{\frac{n+2s}{2}} \gamma \int_{B(0, \frac{\eta}{\mu k})} U^p(\xi_j + \mu y) Z_{n+1}\, dy,
\end{aligned}$$

where

$$V(y) = \Big(- \sum_{j \neq 1} U(y - \mu^{-1}(\xi_j - \xi_1)) + \mu^{\frac{n-2s}{2}} U(\xi_1 + \mu y) \Big).$$

For $j \neq 1$, and by Taylor expansion,

$$U(y + \mu^{-1}(\xi_1 - \xi_j)) = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} (1 + O(\mu^2 k^2)),$$

where $\hat{\xi}_1 = (1, 0, \dots, 0)$ and

$$\hat{\xi}_j = e^{\frac{2\pi(j-1)}{k}} \hat{\xi}_1;$$

thus

$$\begin{aligned}
&\int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(y - \mu^{-1}(\xi_j - \xi_1)) Z_{n+1} \\
&\qquad = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} (1 + O(\mu^2 k^2)) Z_{n+1} \\
&\qquad = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \Big(\int_{\mathbb{R}^n} U^{p-1} Z_{n+1} - \int_{\mathbb{R}^n \setminus B(0, \frac{\eta}{\mu k})} U^{p-1} Z_{n+1} \\
&\qquad\qquad + O(\mu^2 k^2) \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} Z_{n+1} \Big) \\
&\qquad = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} \big(C_1 + O(\mu^{2s} k^{2s}) + O(\mu^2 k^2) \big) \\
&\qquad = \frac{2^{\frac{n-2s}{2}} \mu^{n-2s}}{|\hat{\xi}_j - \hat{\xi}_1|^{n-2s}} C_1 (1 + O(\mu^{2s} k^{2s})),
\end{aligned}$$

where

$$C_1 = \int_{\mathbb{R}^n} U^{p-1} Z_{n+1}.$$

For the second term,

$$\mu^{\frac{n-2s}{2}} \int_{B(0, \frac{\eta}{\mu k})} U^{p-1} U(\xi_1 + \mu y) Z_{n+1} \, dy = \mu^{\frac{n-2s}{2}} C_1 (1 + O(\mu^{2s} k^{2s})).$$

Furthermore,

$$\left|\int_{B(0,\frac{\eta}{\mu k})}\big((U(y)+sV(y))^{p-1}-U^{p-1}\big)V(y)Z_{n+1}\,dy\right|$$

$$\le \left|\sum_{i\neq 1}\int_{B(0,\frac{\eta}{\mu k})}U^p(y-\mu^{-1}(\xi_j-\xi_1))Z_{n+1}\right|$$

$$\le C\sum_{i\neq 1}\frac{\mu^{n+2s}}{|\hat{\xi}_1-\hat{\xi}_i|^{n+2s}}\int_{B(0,\frac{\eta}{\mu k})}\frac{1}{(1+|y|)^{n-2s}}$$

$$\le C(\mu k)^{-2s}\sum_{i\neq 1}\frac{\mu^{n+2s}}{|\hat{\xi}_1-\hat{\xi}_i|^{n+2s}}$$

and

$$\left|\mu^{\frac{n+2s}{2}}\gamma\int_{B(0,\frac{\eta}{\mu k})}U^p(\xi_j+\mu y)Z_{n+1}\,dy\right|$$

$$\le C\mu^{\frac{n+2s}{2}}\int_{B(0,\frac{\eta}{\mu k})}\frac{1}{(1+|y|)^{n-2s}}\,dy\le C\mu^{\frac{n-2s}{2}}k^{-2s}.$$

Therefore, we conclude that

$$I_1=A\delta k^{-(n-2s)}\left(-2^{\frac{n-2s}{2}}\left(\sum_{j=1}^{\infty}\frac{1}{j^{n-2s}}\right)\delta+1\right)+k^{-n}\Theta_k(\delta),$$

where $\Theta_k(\delta)$ is a smooth function of δ, which is uniformly bounded as $k\to\infty$.

Now we are going to estimate I_2. The Holder inequality gives

$$\left|\int_{\mathbb{R}^n\setminus\cup B_j}E\tilde{Z}_{n+1}\right|$$

$$\le C\|(1+|y|)^{n+2s-\frac{2n}{q}}E\|_{L^q(\mathbb{R}^n\setminus\cup B_j)}$$

$$\times\left\|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\right\|_{L^{q/(q-1)}(\mathbb{R}^n\setminus\cup B_j)}.$$

A direct computation gives that

$$\left\|(1+|y|)^{-n-2s+\frac{2n}{q}}\mu^{\frac{n-2s}{2}}Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\right\|_{L^{q/(q-1)}(\mathbb{R}^n\setminus\cup B_j)}\le Ck^{-n\frac{q-1}{q}}$$

for some constant $C>0$. Thus we conclude that

$$|I_2|\le Ck^{1-n}$$

since we have already proved — see (13) — that

$$\|(1+|y|)^{n+2s-\frac{2n}{q}}E\|_{L^q(\mathbb{R}^n\setminus\cup B_j)}\le Ck^{1-\frac{n}{q}}.$$

Let $j \neq 1$ be fixed and $\tilde{E}_j(y) = \mu^{\frac{n+2s}{2}} E(\xi_j + \mu y)$. After the change of variable $x = \mu y + \xi_j$, we obtain

$$\begin{aligned}
\left|\int_{B_j} E\tilde{Z}_{n+1}\right| &= \left|\mu^{\frac{n-2s}{2}} \int_{B(0,\frac{\eta}{\mu k})} \tilde{E}_j \tilde{Z}_{n+1}(\mu y + \xi_j)\right| \\
&\le C\mu^{\frac{n-2s}{2}} \|(1+|y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j\|_{L^q(B(0,\frac{\eta}{\mu k}))} \\
&\quad \times \left\|(1+|y|)^{-n-2s+\frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\right\|_{L^{q/(q-1)}(B(0,\frac{\eta}{\mu k}))}.
\end{aligned}$$

We have

$$\begin{aligned}
&\left\|(1+|y|)^{-n-2s+\frac{2n}{q}} \mu^{\frac{n-2s}{2}} Z_{n+1}(y+\mu^{-1}(\xi_j-\xi_1))\right\|_{L^{q/(q-1)}(B(0,\frac{\eta}{\mu k}))} \\
&\qquad \le C\frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}} \left(\int_1^{\frac{\eta}{\mu k}} \frac{t^{n-1}}{t^{(n+2s-\frac{2n}{q})\frac{q}{q-1}}}\, dt\right)^{\frac{q-1}{q}} \\
&\qquad \le C\frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}} (\mu k)^{2s-\frac{n}{q}}
\end{aligned}$$

and

$$\left\|(1+|y|)^{n+2s-\frac{2n}{q}} \tilde{E}_j\right\|_{L^q(B(0,\frac{\eta}{\mu k}))} \le (\mu k)^{\frac{n}{q}} \left(1 + k^{-(n+2s)} \mu^{-\frac{n-2s}{2}}\right).$$

Hence,

$$\begin{aligned}
|I_3| &= \left|\sum_{j\neq 1} \int_{B_j} E\tilde{Z}_{n+1}\right| \\
&\le \mu^{\frac{n-2s}{2}} (\mu k)^{\frac{n}{q}} \left(1 + k^{-(n+2s)} \mu^{-\frac{n-2s}{2}}\right) \sum_{j=1}^{k} \frac{\mu^{\frac{n-2s}{2}}}{|\xi_j-\xi_1|^{n-2s}} (\mu k)^{2s-\frac{n}{q}} \\
&\le C\mu^{\frac{n-2s}{2}} k^{-2s}
\end{aligned}$$

Finally, we conclude that

$$\int_{\mathbb{R}^n} E\tilde{Z}_{n+1} = A\delta k^{-(n-2s)} \left(-2^{\frac{n-2s}{2}} \left(\sum_{j=1}^{\infty} \frac{1}{j^{n-2s}}\right)\delta + 1\right) + k^{1-n}\Theta_k(\delta), \tag{55}$$

where $\Theta_k(\delta)$ is a smooth function of δ, which is uniformly bounded as $k \to \infty$.

In order to complete the proof of (53), we first estimate:

$$\left|\int_{\mathbb{R}^n} (\zeta_1 - 1) E\tilde{Z}_{n+1}\right| \le C\left|\int_{|y-\xi_1|>\frac{\eta}{k}} E\tilde{Z}_{n+1}\right|.$$

Then we split the domain of integration:

$$\int_{|y-\xi_1|>\frac{\eta}{k}} E\tilde{Z}_{n+1} = \int_{\bigcap_j |y-\xi_j|>\frac{\eta}{k}} E\tilde{Z}_{n+1} + \sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1}$$

In the exterior region, we already proved that

$$\int_{\bigcap_j |y-\xi_j|>\frac{\eta}{k}} E\tilde{Z}_{n+1} = k^{1-n}\Theta_k(\delta),$$

for some smooth function Θ_k of δ, which is uniformly bounded as $k\to\infty$. On the another hand, to estimate

$$\sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1}$$

we can argue like in the estimate of the term I_3 above, thus concluding that

$$\left|\sum_{j=2}^{k}\int_{|y-\xi_j|<\frac{\eta}{k}} E\tilde{Z}_{n+1}\right| \le Ck^{-n}$$

for some constant $C>0$. □

Proof of (54). It is convenient to decompose

$$\mathcal{N}(\phi_1) = \tilde{\mathcal{N}}(\phi_1) + N(\tilde{\phi}_1)$$

where

$$\tilde{\mathcal{N}}(\phi_1) = p\big(|U_*|^{p-1}\zeta_1 - U_1^{p-1}\big)\tilde{\phi}_1 + p\zeta_1|U_*|^{p-1}\Psi(\phi_1) + N\Big(\tilde{\phi}_1 + \sum_{j\neq 1}\tilde{\phi}_j + \Psi(\phi_1)\Big) - N(\tilde{\phi}_1)$$

and

$$N(\tilde{\phi}_1) = |U_* + \tilde{\phi}_1|^{p-1}(U_* + \tilde{\phi}_1) - |U_*|^{p-1}U_* - p|U_*|^{p-1}\tilde{\phi}_1$$

We have that

$$I := \int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)\tilde{Z}_{n+1} = \mu^{\frac{n+2}{2}}\int_{\mathbb{R}^n} \tilde{\mathcal{N}}(\phi_1)(\xi_1+\mu x)Z_{n+1}(x)\,dx$$

so that, from the estimates found, we readily check

$$|I| \le Ck^{2s-n}k^{1-\frac{n}{q}}\int_{R^n} U^{p-1}|Z_{n+1}|. \tag{56}$$

On the other hand, if we let

$$II := \int_{R^n} N(\tilde{\phi}_1)\tilde{Z}_{n+1},$$

we find that

$$|II| \le \|\phi_1\|_* \int_{R^n} U^{p-1}|\phi_1||Z_{n+1}|.$$

Now, we notice that from (23), we can write

$$L_0(\phi_1) + a\phi_1 = g + \sum_l c_l U^{p-1} Z_l, \quad \text{where } a = \mu^{\frac{n+2s}{2}} \gamma N(\tilde{\phi}_1)(\xi_1 + \mu y)$$

so that

$$|a| \le CU^{p-1}\|\phi_1\|_* \quad \text{and} \quad |g| \le C\mu^{\frac{n-2s}{2}}(1+|y|)^{-4s}.$$

Thus, applying Lemma 3.3 with $\nu = 4s$, we find

$$|\phi_1| \le C\mu^{\frac{n-2s}{2}}(1+|y|)^{-2s}$$

and we conclude that

$$|II| \le C\|\phi_1\|_* \mu^{\frac{n-2s}{2}} \le Ck^{2s-n-\frac{n}{q}}.$$

Combining this with (56), we find

$$\left| \int_{R^n} \mathcal{N}(\phi_1)\tilde{Z}_{n+1} \right| \le Ck^{2s-n}k^{1-\frac{n}{q}}. \qquad \square$$

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CALCULATION OF LOCAL FORMAL MELLIN TRANSFORMS

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Much recent work has been done on the local Fourier transforms for connections on the punctured formal disk. Specifically, the local Fourier transforms have been introduced, shown to induce certain equivalences of categories, and explicit formulas have been found to calculate them. In this paper, we prove analogous results in a similar situation, the local Mellin transforms for connections on the punctured formal disk. Specifically, we introduce the local Mellin transforms and show that they induce equivalences between certain categories of vector spaces with connection and vector spaces with invertible difference operators, as well as find formulas for explicit calculation in the same spirit as the calculations for the local Fourier transforms.

1. Introduction

Recently, much research has been done on local Fourier transforms for connections on the punctured formal disk. Namely, H. Bloch and H. Esnault [2004] and R. Garcia Lopez [2004] introduced and analyzed the local Fourier transforms. Explicit formulas for calculation of the local Fourier transforms were proved independently by J. Fang [2009] and C. Sabbah [2008] using different methods. D. Arinkin [2008] gave a different framework for the local Fourier transforms and also gave explicit calculation of the Katz–Radon transform. In [Graham-Squire 2013], we used Arinkin's techniques from [2008] to reproduce the calculations of [Fang 2009; Sabbah 2008]. The global Mellin transform for connections on a punctured formal disk was given by Laumon [1996] as well as Loeser and Sabbah [1991], but since that time little work has been done on the Mellin transform in this area. Arinkin [2008, Section 2.5] remarks that it would be interesting to apply his methods to other integral transforms such as the Mellin transform. This paper is the answer to that query. We introduce the *local* Mellin transforms on the punctured formal disk and prove results for them which are analogous to those of the local Fourier transforms. One main difference between the analysis of the local Fourier and local Mellin transforms is this: whereas the local Fourier transforms deal only with

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differential operators, the local Mellin transforms input a differential operator and output a difference operator.

The work done in this paper is as follows: after some preliminary definitions, we introduce the local Mellin transforms $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$ for connections on the punctured formal disk. Our construction of the local Mellin transforms is analogous to that of [Bloch and Esnault 2004; Arinkin 2008] for the local Fourier transforms. In particular, we mimic the framework given in the latter reference to define the local Mellin transforms, as Arinkin's construction lends itself most easily to calculation. We also show that the local Mellin transforms induce equivalences between certain categories of vector spaces with connection and categories of vector spaces with difference operators. Such equivalences could, in principle, reduce questions about difference operators to questions about (relatively more-studied) connections, although we do not do such an analysis in this work. We end by using the techniques of [Graham-Squire 2013] to give explicit formulas for calculation of the local Mellin transforms in the same spirit as the results of [Fang 2009; Graham-Squire 2013; Sabbah 2008]. An example of our main result is the following calculation of $\mathcal{M}^{(0,\infty)}$:

Let $\Bbbk$ be an algebraically closed field of characteristic zero. Definitions for R, S, E, and D can be found in the body of the paper.

Theorem 10.1. *Let s and r be positive integers, $a \in \Bbbk - \{0\}$, and $f \in R_r^\circ(z)$ with $f = az^{-s/r} + \underline{o}(z^{-s/r})$. Then*

$$\mathcal{M}^{(0,\infty)}(E_f) \simeq D_g,$$

where $g \in S_s^\circ(\theta)$ is determined by the following system of equations:

$$f = -\theta^{-1},$$
$$g = z - (-a)^{r/s}\,\frac{r+s}{2s}\,\theta^{1+(r/s)}.$$

A necessary tool for the calculation is the formal reduction of differential operators as well as the formal reduction of linear difference operators. There are considerable parallels between difference operators and connections, and we refer the reader to [van der Put and Singer 1997] for more details.

2. Connections and difference operators

Connections on the formal disk.

Definition 2.1. Let V be a finite-dimensional vector space over $K = \Bbbk((z))$. A *connection* on V is a $\Bbbk$-linear operator $\nabla : V \to V$ satisfying the Leibniz identity:

$$\nabla(fv) = f\nabla(v) + \frac{df}{dz}v$$

for all $f \in K$ and $v \in V$. A choice of basis in V gives an isomorphism $V \simeq K^n$; we can then write ∇ as $\frac{d}{dz} + A$, where $A = A(z) \in \mathfrak{gl}_n(K)$ is the *matrix* of ∇ with respect to this basis.

Definition 2.2. We write $\mathcal{C}$ for the *category of vector spaces with connections over* K. Its objects are pairs (V, ∇), where V is a finite-dimensional K-vector space and $\nabla : V \to V$ is a connection. Morphisms between (V_1, ∇_1) and (V_2, ∇_2) are K-linear maps $\phi : V_1 \to V_2$ that are *horizontal* in the sense that $\phi\nabla_1 = \nabla_2\phi$.

Properties of connections. We summarize below some well-known properties of connections on the formal disk. The results go back to Turrittin [1955] and Levelt [1975]; more recent references include [Babbitt and Varadarajan 1985; Beilinson et al. 2002, Sections 5.9 and 5.10; Malgrange 1991; van der Put and Singer 1997].

Let q be a positive integer and define $K_q := \Bbbk((z^{1/q}))$. Note that K_q is the unique extension of K of degree q. For every $f \in K_q$, we define an object $E_f \in \mathcal{C}$ by

$$E_f = E_{f,q} = \left(K_q, \tfrac{d}{dz} + z^{-1} f\right).$$

In terms of the isomorphism class of an object E_f, the reduction procedures of [Turrittin 1955; Levelt 1975] imply that we need only consider f in the quotient

$$\Bbbk((z^{1/q})) \Big/ \left(z^{1/q}\, \Bbbk[[z^{1/q}]] + \frac{\mathbb{Z}}{q} \right)$$

where $\Bbbk[[z]]$ denotes formal *power* series.

Let R_q be the set of orbits for the action of the Galois group $\mathrm{Gal}(K_q/K)$ on the quotient. Explicitly, the Galois group is identified with the group of degree q roots of unity $\eta \in \Bbbk$; the action on $f \in R_q$ is by $f(z^{1/q}) \mapsto f(\eta z^{1/q})$. Finally, denote by $R_q^\circ \subset R_q$ the set of $f \in R_q$ that cannot be represented by elements of K_r for any $0 < r < q$. Thus R_q° is the locus of R_q where $\mathrm{Gal}(K_q/K)$ acts freely.

Proposition 2.3. (1) *The isomorphism class of E_f depends only on the orbit of the image of f in R_q.*

(2) *E_f is irreducible if and only if the image of f in R_q belongs to R_q°. As q and f vary, we obtain a complete list of isomorphism classes of irreducible objects of $\mathcal{C}$.*

(3) *Every $E \in \mathcal{C}$ can be written as*

$$E \simeq \bigoplus_i (E_{f_i,q_i} \otimes J_{m_i}),$$

where the $E_{f,q}$ are irreducible, $J_m = \left(K^m, \frac{d}{dz} + z^{-1} N_m\right)$, and N_m is the nilpotent Jordan block of size m.

Proofs of the proposition are straightforward to construct; examples can be found in [Malgrange 1991; van der Put and Singer 1997; Beilinson et al. 2002].

Remark. We refer to the objects $(E_f \otimes J_m) \in \mathcal{C}$ as *indecomposable* objects in $\mathcal{C}$.

Difference operators on the formal disk. Vector spaces with difference operator and vector spaces with connection are defined in a similar fashion.

Definition 2.4. Let V be a finite-dimensional vector space over $K = \Bbbk((\theta))$. A *difference operator* on V is a $\Bbbk$-linear operator $\Phi : V \to V$ satisfying

$$\Phi(fv) = \varphi(f)\Phi(v)$$

for all $f \in K$ and $v \in V$, with $\varphi : K^n \to K^n$ as the $\Bbbk$-automorphism defined below. A choice of basis in V gives an isomorphism $V \simeq K^n$; we can then write Φ as $A\varphi$, where $A = A(\theta) \in \mathfrak{gl}_n(K)$ is the *matrix* of Φ with respect to this basis, and for $v(\theta) \in K^n$,

$$\varphi(v(\theta)) = v\left(\frac{\theta}{1+\theta}\right) = v\left(\sum_{i=1}^{\infty}(-1)^{i+1}\theta^i\right).$$

We follow the convention of [Praagman 1983, Section 1] to define φ over the extension $K_q = \Bbbk((\theta^{1/q}))$. Thus for all $q \in \mathbb{Z}^+$, φ extends to a $\Bbbk$-automorphism of K_q^n defined by

$$\varphi(v(\theta^{1/q})) = v\left(\theta^{1/q}\sum_{i=0}^{\infty}\binom{-1/q}{i}\theta^i\right).$$

Definition 2.5. We write $\mathcal{N}$ for the *category of vector spaces with invertible difference operator over K*. Objects are pairs (V, Φ), where V is a finite-dimensional K-vector space and $\Phi : V \to V$ is an invertible difference operator. Morphisms between (V_1, Φ_1) and (V_2, Φ_2) are K-linear maps $\phi : V_1 \to V_2$ such that $\phi\Phi_1 = \Phi_2\phi$.

Properties of difference operators. In [Chen and Fahim 1998; Praagman 1983], a canonical form for difference operators is constructed. We give an equivalent construction in the theorem below, which is a restatement of certain Praagman results with different notation to better fit our situation.

Theorem 2.6 [Praagman 1983, Theorem 8 and Corollary 9]. *Let $\Phi : V \to V$ be an invertible difference operator. Then there exists a finite* (*Galois*) *extension K_q of K and a basis of $K_q \otimes_K V$ such that Φ is expressed as a diagonal block matrix. Each block is of the form*

$$F_g = \begin{bmatrix} g & & \\ \theta^{\lambda+1} & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $g \in K_q$, $\lambda \in \frac{\mathbb{Z}}{q}$, $g = a_0\theta^\lambda + \cdots + a_q\theta^{\lambda+1}$, $a_0 \neq 0$, and a_q defined up to a shift by $\frac{a_0\mathbb{Z}}{q}\theta^{\lambda+1}$. The matrix is unique modulo the order of the blocks.

Remark. The F_g are the indecomposable components for the matrix of Φ.

Theorem 2.6 allows us to describe the category $\mathcal{N}$ in a fashion similar to our description of the category $\mathcal{C}$. For every $g \in K_q$, we define an object $D_g \in \mathcal{N}$ by

$$D_g = D_{g,q} := (K_q, g\varphi).$$

The canonical form given in Theorem 2.6 implies that we need only consider g in the following quotient of the multiplicative group $\Bbbk((\theta^{1/q}))^*$:

$$K_q^* \Big/ \left(1 + \frac{\mathbb{Z}}{q}\theta + \theta^{1+(1/q)}\Bbbk[[\theta^{1/q}]]\right). \tag{2-1}$$

Let S_q be the set of orbits for the action of the Galois group $\mathrm{Gal}(K_q/K)$ on the quotient given in (2-1). Denote by $S_q^\circ \subset S_q$ the set of $g \in S_q$ that cannot be represented by elements of K_r for any $0 < r < q$. As before, S_q° can be thought of as the locus where $\mathrm{Gal}(K_q/K)$ acts freely.

Proposition 2.7. (1) *The isomorphism class of D_g depends only on the orbit of the image of g in S_q.*

(2) *D_g is irreducible if and only if the image of g in S_q belongs to S_q°. As q and g vary, we obtain a complete list of isomorphism classes of irreducible objects of $\mathcal{N}$.*

(3) *Every $D \in \mathcal{N}$ can be written as*

$$D \simeq \bigoplus_i (D_{g_i,q_i} \otimes T_{m_i}),$$

where the $D_{g,q}$ are irreducible, $T_m = (K^m, U_m\varphi)$, and $U_m = I_m + \theta N_m$.

Notation. At times it is useful to keep track of the choice of local coordinate for $\mathcal{C}$ and $\mathcal{N}$, and we denote this with a subscript. To stress the coordinate, we write $\mathcal{C}_0$ to indicate the coordinate z at the point zero, $\mathcal{C}_x$ to indicate the coordinate $z - x := z_x$ at a point $x \neq 0$, and $\mathcal{C}_\infty$ to indicate the coordinate $\zeta = \frac{1}{z}$ at the point at infinity. Note that $\mathcal{C}_0$, $\mathcal{C}_x$, and $\mathcal{C}_\infty$ are all isomorphic to $\mathcal{C}$, but not canonically. Similarly, we can write $\mathcal{N}_\infty$ to indicate that we are considering $\mathcal{N}$ with local coordinate at infinity. Since we only work with the point at infinity for $\mathcal{N}$, though, we generally omit the subscript.

We also have a superscript notation for categories, but our conventions for the categories $\mathcal{C}$ and $\mathcal{N}$ are different and a potential source of confusion. Superscript notation for vector spaces with connection is well-established, and the superscript corresponds to *slope*; for a formal definition of slope, see [Katz 1987]. Thus, for example, we denote by $\mathcal{C}_\infty^{<1}$ the full subcategory of $\mathcal{C}_\infty$ of connections whose irreducible components all have slopes less than one; that is, E_f such that $-1 < \mathrm{ord}(f)$.

The correspondence to slope makes sense in the context of connections because all connections have nonnegative slope, i.e., for all E_f we have $\mathrm{ord}(f) \le 0$. For difference operators we have no such restriction on the order of f, though, and thus a correspondence to slope would be artificial. The superscripts for difference operators therefore refer to the *order of irreducible components* as opposed to the slope. Thus, for example, the notation $\mathcal{N}^{>0}$ indicates the full subcategory of $\mathcal{N}$ of difference operators whose irreducible components D_g have the property that $\operatorname{ord}(g) > 0$.

3. Tate vector spaces

The z-adic topology.

Definition 3.1. We define the z-adic topology on the vector space V as follows: a *lattice* is a $\Bbbk$-subspace $L \subset V$ that is of the form $L = \bigoplus_i \Bbbk[[z]] e_i$ for some basis e_i of V over K. Then the *z-adic topology on V* is defined by letting the basis of open neighborhoods of $v \in V$ be cosets $v + L$ for all lattices $L \subset V$.

Remark. An equivalent definition for the z-adic topology, without reference to choice of basis, is given in [Arinkin 2008, Section 4.2]. The z-adic topology is also equivalent to the topology induced by any norm, as described in Lemma 4.4.

For ease of explication, we copy the remaining definitions and results in this section from [op. cit., Section 5.3]. For more details on Tate vector spaces, see [Beilinson and Drinfeld 2004, Section 2.7.7].

Tate vector spaces.

Definition 3.2. Let V be a topological vector space over $\Bbbk$, where $\Bbbk$ is equipped with the discrete topology. V is *linearly compact* if it is complete, Hausdorff, and has a base of neighborhoods of zero consisting of subspaces of finite codimension. Equivalently, a linearly compact space is the topological dual of a discrete space. V is a *Tate space* if it has a linearly compact open subspace.

Definition 3.3. A $\Bbbk[[z]]$-module M is *of Tate type* if there is a finitely generated submodule $M' \subset M$ such that M/M' is a torsion module that is "cofinitely generated" in the sense that

$$\dim_{\Bbbk} \operatorname{Ann}_z(M/M') < \infty, \quad \text{where } \operatorname{Ann}_z(M/M') = \{m \in M/M' \mid zm = 0\}.$$

Lemma 3.4. (1) *Any finitely generated $\Bbbk[[z]]$-module M is linearly compact in the z-adic topology.*

(2) *Any $\Bbbk[[z]]$-module of Tate type is a Tate vector space in the z-adic topology.*

Proposition 3.5. *Let V be a Tate space. Suppose an operator $Z : V \to V$ satisfies the following conditions*:

(1) *Z is continuous, open, and (linearly) compact. In other words, if $V' \subset V$ is an open linearly compact subspace, then so are $Z(V')$ and $Z^{-1}(V')$.*

(2) *Z is contracting. In other words, $Z^n \to 0$ in the sense that for any linearly compact subspace $V' \subset V$ and any open subspace $U \subset V$, we have $Z^n(V') \subset U$ for $n \gg 0$.*

Then there exists a unique structure of a Tate type $\Bbbk[\![z]\!]$-module on V such that $z \in \Bbbk[\![z]\!]$ acts as Z and the topology on V coincides with the z-adic topology.

4. The norm and order of an operator

Definition of norm. In the discussion of norms in this subsection we primarily follow the conventions of [Cassels and Fröhlich 1967], though our presentation is self-contained. Similar treatments of norms can also be found in [André and Baldassarri 2001; Kedlaya 2010]. Fix a real number ϵ such that $0 < \epsilon < 1$. For $f = \sum_{i=k} c_i \theta^{i/q} \in K_q$ with $c_k \neq 0$, we define the order of f as $\operatorname{ord}(f) := k/q$.

Definition 4.1. Let $f \in K$. The valuation $|\cdot|$ on K is defined as

$$|f| = \epsilon^{\operatorname{ord}(f)}$$

with $|0| = 0$.

This is a nonarchimedean discrete valuation, and K is complete with respect to the topology induced by the valuation.

Definition 4.2. Let V be a vector space over K. A *nonarchimedean norm* on V is a real-valued function $\|\cdot\|$ on V such that the following hold:

(1) $\|v\| > 0$ for $v \in V - \{0\}$.

(2) $\|v + w\| \leq \max(\|v\|, \|w\|)$ for all $v, w \in V$.

(3) $\|f \cdot v\| = |f| \cdot \|v\|$ for $f \in K$ and $v \in V$.

Example 4.3. Let $f_i \in K$. The function

$$\|(f_1, \ldots, f_n)\| = \max|f_i|$$

is a norm on K^n, and K^n is complete with respect to this norm.

Lemma 4.4 [Cassels and Fröhlich 1967, lemma in Section 2.8]. *Any two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on a finite-dimensional vector space V over K are equivalent in the following sense*: *there exists a real number $C > 0$ such that*

$$\frac{1}{C}\|\cdot\|_1 \leq \|\cdot\|_2 \leq C\|\cdot\|_1.$$

It follows from Lemma 4.4 that all norms on a finite-dimensional vector space over K induce the same topology.

Definition 4.5. Let $A : V \to V$ be a $\Bbbk$-linear operator. We define the *norm of an operator* to be

$$\|A\| = \sup_{v \in V - \{0\}} \frac{\|A(v)\|}{\|v\|}.$$

Note that $\|A\| < \infty$ if and only if A is continuous [Kolmogorov and Fomin 1975, Chapter 6, Theorem 1].

Invariant norms. The norm of an operator given in Definition 4.5 depends on the choice of the nonarchimedean norm $\|\cdot\|$. To find an invariant for norms of operators, consider the following two norms:

Definition 4.6. The *infimum norm* is defined as

$$\|A\|_{\text{inf}} = \inf\{\|A\| : \|\cdot\| \text{ is a norm on } V\}$$

and the *spectral radius* of A is given by

$$\|A\|_{\text{spec}} = \lim_{n \to \infty} \sqrt[n]{\|A^n\|}.$$

Note that A must be continuous to guarantee that the limit defining the spectral radius exists. It follows from Lemma 4.4 that the spectral radius does not depend on the choice of norm $\|\cdot\|$. For operators in general the spectral radius is often the more useful invariant, but for the class of operators we consider (connections, difference operators, and their inverses) the two definitions coincide and we primarily use the infimum norm.

Norms of similitudes.

Proposition 4.7. *Let $\|\cdot\|_1$ and $\|\cdot\|_2$ be two norms on V. Then for any invertible $\Bbbk$-linear operator $A : V \to V$, we have $\|A\|_1 \cdot \|A^{-1}\|_2 \geq 1$.*

Corollary 4.8. *Let $A : V \to V$ be invertible and let $\|\cdot\|$ be a norm such that $\|A\| \cdot \|A^{-1}\| = 1$. Then $\|A\| = \|A\|_{\text{inf}}$.*

Definition 4.9. Let $\|\cdot\|$ be a norm on V. An operator $A : V \to V$ is a *similitude* (with respect to $\|\cdot\|$) if $\|Av\| = \lambda\|v\|$ for all $v \in V$. It follows that $\|A\| = \lambda$.

Claim 4.10. *Let $A : V \to V$ be an invertible similitude with $\|Av\| = \lambda\|v\|$. Then $\|A\|_{\text{inf}} = \lambda$ and $\|A^{-1}\| = 1/\lambda$.*

Properties of norms. Given the canonical form of a connection or difference operator, it is quite easy to calculate the norm. In particular we note that indecomposable connections with no horizontal sections and indecomposable invertible difference operators are similitudes.

Remark. We introduce here notation to clear up a potentially confusing situation. The issue is the notation $\nabla = \frac{d}{dz} + A$ for a connection. In particular, at the local coordinate $\zeta = \frac{1}{z}$ the change of variable gives us $\nabla = -\zeta^2 \frac{d}{d\zeta} + A(\zeta)$. To emphasize the local coordinate we will use the notation ∇_z (respectively ∇_ζ) to indicate that we are writing ∇ in terms of z (respectively ζ). In particular, we have the equalities $\nabla_z = -\zeta^2 \nabla_\zeta$ and $z\nabla_z = -\zeta\nabla_\zeta$.

Proposition 4.11. *Suppose that* $(V, \nabla) = (E_f \otimes J_m) \in \mathcal{C}$ *is indecomposable and that* ∇ *has no horizontal sections. Then*:

(1) $\|\nabla\|_{\inf} = \epsilon^{\operatorname{ord}(f)-1}$.

If ∇ *is invertible we also have*

(2) $\|\nabla^{-1}\|_{\inf} = \epsilon^{-\operatorname{ord}(f)+1}$.

(3) *For* $(V, \nabla) \in \mathcal{C}_0$, $\|(z\nabla)^{-1}\|_{\inf} = \epsilon^{-\operatorname{ord}(f)}$.

(4) *For* $(V, \nabla) \in \mathcal{C}_\infty$, $\|(z\nabla)^{-1}\|_{\inf} = \|(\zeta\nabla_\zeta)^{-1}\|_{\inf} = \epsilon^{-\operatorname{ord}(f)}$.

(5) *For* $(V, \nabla) \in \mathcal{C}_x$, $\|(z\nabla_{z_x})^{-1}\|_{\inf} = \epsilon^{1-\operatorname{ord}(f)}$.

Proposition 4.12. *For an indecomposable* $(V, \Phi) = (D_g \otimes T_m) \in \mathcal{N}$,

(1) $\|\Phi\|_{\inf} = \epsilon^{\operatorname{ord}(g)}$.

(2) $\|(\theta\Phi)^{-1}\|_{\inf} = \epsilon^{-\operatorname{ord}(g)-1}$.

Order of an operator. The *order* of an operator is a notion closely related to the norm of an operator. It is often more convenient to work with order as opposed to norm, so we give a brief introduction to order below.

Definition 4.13. Let $B : V \to V$ be a $\Bbbk$-linear operator and let $\|\cdot\|$ be a norm defined on V. Then the *order* of B is

$$\operatorname{Ord}(B) = \log_\epsilon \|B\|_{\mathrm{spec}},$$

with $\operatorname{Ord}(0) := \infty$.

Example 4.14. The term "order" is suggestive for the following reason. Given Definition 4.13, the properties of similitudes, and ∇ an indecomposable connection with no horizontal sections, the following property holds: $\operatorname{Ord}(\nabla) = \ell$ if and only if for all $n \in \mathbb{Q}$ we have $\nabla(z^n 1) = (*z^{n+\ell})1 + \text{higher order terms}$, where 1 is the identity element of V. Similarly for an indecomposable difference operator Φ, $\operatorname{Ord}(\Phi) = j$ if and only if $\Phi(\theta^n 1) = (*\theta^{n+j})1 + \text{higher order terms}$. Note that $* \in \Bbbk - \mathbb{Z}$ if $\ell = -1$ and $* \in \Bbbk$ otherwise.

In the context of the order of an operator, we can state the results of Propositions 4.11 and 4.12 as follows.

Corollary 4.15. (1) *For indecomposable* $(V,\nabla) = (E_f \otimes J_m)$, *with* $z\nabla$ *invertible, in either* $\mathcal{C}_0$ *or* $\mathcal{C}_\infty$,

$$\operatorname{Ord}(\nabla) = \operatorname{ord}(f) - 1, \quad \operatorname{Ord}(z\nabla) = \operatorname{ord}(f), \quad \textit{and} \quad \operatorname{Ord}((z\nabla)^{-1}) = -\operatorname{ord}(f).$$

(2) *For indecomposable* $(V,\nabla_{z_x}) = (E_f \otimes J_m) \in \mathcal{C}_x$, *with* $z\nabla_{z_x}$ *invertible,*

$$\operatorname{Ord}(z\nabla_{z_x}) = \operatorname{Ord}(\nabla_{z_x}) = \operatorname{ord}(f) - 1 \quad \textit{and} \quad \operatorname{Ord}((z\nabla_{z_x})^{-1}) = 1 - \operatorname{ord}(f).$$

(3) *For indecomposable* $(V,\Phi) = (D_g \otimes U_m) \in \mathcal{N}$,

$$\operatorname{Ord}(\Phi) = \operatorname{ord}(g) \quad \textit{and} \quad \operatorname{Ord}((\theta\Phi)^{-1}) = -\operatorname{ord}(g) - 1.$$

5. Lemmas

Fractional powers of an operator. The operator-root lemma below shows how to calculate the power of a sum of certain operators, even for fractional powers. The idea is that once a certain root $(1/p)$ of the operator is chosen, the fractional power is easily defined as an integer power of that root.

Lemma 5.1 (operator-root lemma). *Let* A *and* B *be the following* $\Bbbk$*-linear operators on* K_q: A *is multiplication by* $f = az^{p/q} + \underline{o}(z^{p/q})$, $0 \neq a \in \Bbbk$, *and* $B = z^n \frac{d}{dz}$ *with* $n \neq 0$, $p \neq 0$, *and* $q > 0$ *all integers. We have* $\operatorname{Ord}(A) = p/q$ *and* $\operatorname{Ord}(B) = n-1$, *and we assume that* $p/q < n-1$. *Then for any* p*-th root of* A *we can choose a* p*-th root of* $(A+B)$, *written* $(A+B)^{1/p}$, *such that*

$$(A+B)^m = A^m + mA^{(m-1)}B + \frac{m(m-1)}{2}A^{m-2}[B,A] + \underline{o}(z^{(p/q)(m-1)+n-1})$$

holds for all $m \in \frac{\mathbb{Z}}{p}$, *where* $(A+B)^m = ((A+B)^{1/p})^{pm}$.

Proof. A full proof is found in [Graham-Squire 2013, Lemma 4.4]. □

Tate vector space lemmas. We also need some lemmas describing our situation in the language of Tate vector spaces. The proofs are straightforward and are omitted.

Lemma 5.2. *Let* $Z : V \to V$ *be a* $\Bbbk$*-linear operator. If* $\operatorname{Ord}(Z) > 0$, *then* Z *is contracting.*

Lemma 5.3. *A* K*-vector space* V *is of Tate type if and only if it is finite dimensional.*

6. Global Mellin transform

The "classical" Mellin transform can be stated as follows: for an appropriate f the Mellin transform of f is given by

$$\tilde{f}(\eta) = \int_0^\infty z^{\eta-1} f(z)\, dz.$$

One can check that

$$\eta \tilde{f} = -(z\, df/dz)^{\sim} \quad \text{and} \quad \Phi \tilde{f} = \widetilde{(zf)},$$

where Φ is the difference operator taking $\tilde{f}(\eta)$ to $\tilde{f}(\eta+1)$.

This leads to the notion of the *global* Mellin transform for connections on a punctured formal disk, which was introduced by Laumon [1996] and also presented by Loeser and Sabbah [1991]. Below is our definition for the global Mellin transform, which is equivalent to Laumon's.

Definition 6.1. The global Mellin transform $\mathcal{M} : \Bbbk[z, z^{-1}]\langle\nabla\rangle \to \Bbbk[\eta]\langle\Phi, \Phi^{-1}\rangle$ is a homomorphism between algebras defined on its generators by $-z\nabla \mapsto \eta$ and $z \mapsto \Phi$. Note that we have $[\nabla, z] = 1$ for the domain and $[\Phi, \eta] = \Phi$ for the target space, and the homomorphism preserves these equalities.

As in the case of the Fourier transform, we derive our definition of the local Mellin transform from the global situation. In particular, the local Mellin transform has different "flavors" depending on the point of singularity, so we refer to them as *local Mellin transforms*.

7. Definitions of local Mellin transforms

Below we give definitions of the local Mellin transforms. To alleviate potential confusion, let us explain the format we will use for the definitions. We begin by stating the definition in its entirety, but it is not a priori clear that all statements of the definition are true. We then claim that the transform is in fact well-defined and give a proof to clear up the questionable parts of the definition.

Definition 7.1. Let $E = (V, \nabla) \in \mathcal{C}_0^{>0}$. Thus all indecomposable components of ∇ have slope greater than zero, so each indecomposable component $E_f \otimes J_m$ has $\operatorname{ord}(f) < 0$. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \quad \text{and} \quad \Phi := z : V \to V \tag{7-1}$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We write $V = V_\theta$ to denote that we are considering V as a $\Bbbk((\theta))$-vector space. We define *the local Mellin transform from zero to infinity of E* to be the object

$$\mathcal{M}^{(0,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Definition 7.2. Let $E = (V, \nabla) \in \mathcal{C}_x$ be such that ∇ has no horizontal sections. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \quad \text{and} \quad \Phi := z : V \to V \tag{7-2}$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We define *the local Mellin transform from x to infinity of E* to be the object

$$\mathcal{M}^{(x,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Remark. Since $E \in \mathcal{C}_x$, we are thinking of K as $\Bbbk((z_x))$. This emphasizes that we are localizing at a point $x \neq 0$ with local coordinate $z_x = z - x$.

Note that in the following definition we are thinking of K as $\Bbbk((\zeta))$, since we are localizing at the point at infinity $\zeta = \frac{1}{z}$.

Definition 7.3. Let $E = (V, \nabla) \in \mathcal{C}_\infty^{>0}$. Thus all irreducible components of ∇ have slope greater than zero. Consider on V the $\Bbbk$-linear operators

$$\theta := -(z\nabla)^{-1} : V \to V \quad \text{and} \quad \Phi := z : V \to V$$

Then θ extends to an action of $\Bbbk((\theta))$ on V, $\dim_{\Bbbk((\theta))} V < \infty$, and Φ is an invertible difference operator. We define *the local Mellin transform from infinity to infinity of E* to be the object

$$\mathcal{M}^{(\infty,\infty)}(E) := (V_\theta, \Phi) \in \mathcal{N}.$$

Claim 7.4. *$\mathcal{M}^{(0,\infty)}$ is well-defined.*

Proof. To prove the claim we must show the following:

(i) θ extends to an action of $\Bbbk((\theta))$ on V.

(ii) V_θ is finite-dimensional.

(iii) Φ is an invertible difference operator on V_θ.

We prove (i) with Lemma 7.5 below. In the proof of Lemma 7.5 we show that $(z\nabla)^{-1}$ satisfies the conditions of Proposition 3.5, and it follows that V_θ is of Tate type. Lemma 5.3 then implies that V_θ is finite-dimensional, proving (ii). To prove (iii), we first note that Φ is invertible by construction. To see that Φ is a difference operator, we need to show that $\Phi(fv) = \varphi(f)\Phi(v)$ for all $f \in K$ and $v \in V$. Since Φ is $\Bbbk$-linear and Laurent polynomials are dense in Laurent series, this reduces to showing that $\Phi(\theta^i) = \varphi(\theta^i)\Phi$, which can be proved by induction so long as you can show that

$$\Phi(\theta) = \frac{\theta}{1+\theta}\Phi.$$

This last equation is equivalent to $(\eta+1)\Phi = \Phi\eta$, which we now prove. Using the fact that $[\nabla, z] = 1$ and the definitions given in (7-1) we compute

$$(\eta+1)\Phi = -z\nabla z + z = -z(z\nabla + 1) + z = z(-z\nabla) = \Phi\eta. \qquad \square$$

Lemma 7.5. *The definition for θ, given in (7-1), extends to an action of $\Bbbk((\theta))$ on V.*

Proof. Since all indecomposable components of ∇ have positive slope, ∇ (and $z\nabla$) will be invertible and thus θ is well-defined. An action of $\Bbbk[\theta^{-1}] = \Bbbk[-z\nabla]$ on V is trivially defined. If $(z\nabla)^{-1} : V \to V$ satisfies the conditions of Proposition 3.5, we will also have an action of $\Bbbk[[\theta]]$ on V. This will give a well defined action of $\Bbbk((\theta))$ on V. Thus all we need to prove is that $(z\nabla)^{-1} : V \to V$ satisfies the conditions of Proposition 3.5.

We must show that $\theta = (z\nabla)^{-1} : V \to V$ is continuous, open, linearly compact, and contracting. Due to the canonical form for difference operators, we can assume without loss of generality that ∇ is indecomposable and $z\nabla$ is of the form

$$z\frac{d}{dz} + \begin{bmatrix} f & & \\ 1 & \ddots & \\ & \ddots & \ddots \end{bmatrix}$$

with $f \in \Bbbk[z^{-1/r}]$ and $\operatorname{ord}(f) = -m/r < 0$. Let $\{e_i\}$ be the canonical basis. Since lattices are linearly compact open subspaces, to prove that $(z\nabla)^{-1}$ is open, continuous, and linearly compact it suffices to show that $(z\nabla)$ and $(z\nabla)^{-1}$ map a lattice of the form $L_k = \bigoplus (z^{1/r})^k \Bbbk[[z^{1/r}]] e_i$ to a lattice of the same form.

We see that

$$z\nabla(L_k) = \bigoplus (z^{1/r})^{k-m} \Bbbk[[z^{1/r}]] e_i = L_{k-m},$$
$$(z\nabla)^{-1}(L_k) = \bigoplus (z^{1/r})^{k+m} \Bbbk[[z^{1/r}]] e_i = L_{k+m},$$

so $(z\nabla)^{-1}$ is open, continuous, and linearly compact.

To show that $(z\nabla)^{-1}$ is contracting, by Lemma 5.2 we only need to show that $\operatorname{Ord}((z\nabla)^{-1}) > 0$. By Corollary 4.15(1), then, it suffices to show that we have $\operatorname{ord}(f) < 0$ for the indecomposable $(V, \nabla) = E_f \otimes J_m$. This condition is fulfilled by assumption, since all indecomposable components have slope greater than zero. □

The proof that $\mathcal{M}^{(x,\infty)}$ is well-defined is similar to the proof above, with only one major caveat. In the proof of (i), we use the fact that the leading term of the operator is the only important term for the theoretical calculation. Thus one can think of z as $z_x + x$, and reduce to considering $z\nabla$ as merely $x\nabla$, from which the result readily follows. The proofs of (ii) and (iii) are identical. The proof that $\mathcal{M}^{(\infty,\infty)}$ is well-defined is virtually identical to the proof of Claim 7.4 once the change of variable from z to ζ is taken into consideration.

Remark. Note that the local Mellin transforms above give functors to apply to all connections except for certain connections with regular singularity. More precisely, the only invertible connections for which $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$ *cannot* be applied are those connections in $\mathcal{C}_0$ and $\mathcal{C}_\infty$ with slope zero. We conjecture that these connections with regular singularity will map to difference operators with singularity at a point $y \neq \infty$. This regular singular case is sufficiently small, and the techniques necessary to prove our conjecture sufficiently different from the situation described above, that we do not discuss it here.

8. Definition of local inverse Mellin transforms

Definition 8.1. Let $D = (V, \Phi) \in \mathcal{N}^{>0}$. Thus Φ is invertible and the irreducible components of Φ have order greater than zero. Consider on V the $\Bbbk$-linear operators

(8-1) $$z := \Phi : V \to V \quad \text{and} \quad \nabla := -(\theta\Phi)^{-1} : V \to V$$

Then z extends to an action of $\Bbbk((z))$ on V, $\dim_{\Bbbk((z))} V < \infty$, and ∇ is a connection. We write V_z for V to denote that we are considering V as a $\Bbbk((z))$-vector space. We define *the local inverse Mellin transform from zero to infinity of D* to be the object

$$\mathcal{M}^{-(0,\infty)}(D) := (V_z, \nabla) \in \mathcal{C}_0.$$

Definition 8.2. Let $D = (V, \Phi) \in \mathcal{N}^{=0}$ be such that all irreducible components of Φ have order zero with the same leading coefficient $x \neq 0$, and $\Phi - x$ is invertible.

Consider on V the $\Bbbk$-linear operators

$$z := \Phi : V \to V \quad \text{and} \quad \nabla := -(\theta\Phi)^{-1} : V \to V.$$

Then the action of $z - x = z_x$ is clearly defined, z_x extends to an action of $\Bbbk((z_x))$ on V, $\dim_{\Bbbk((z_x))} V < \infty$, and ∇ is a connection. We write V_{z_x} for V to denote that we are considering V as a $\Bbbk((z_x))$-vector space. We define *the local inverse Mellin transform from x to infinity of D* to be the object

$$\mathcal{M}^{-(x,\infty)}(D) := (V_{z_x}, \nabla) \in \mathcal{C}_x.$$

Definition 8.3. Let $D = (V, \Phi) \in \mathcal{N}^{<0}$. Thus Φ is invertible and the irreducible components of Φ have order less than zero. Consider on V the $\Bbbk$-linear operators

$$z := \Phi : V \to V \quad \text{and} \quad \nabla := -(\theta\Phi)^{-1} : V \to V$$

Then $\zeta = z^{-1}$ extends to an action of $\Bbbk((\zeta))$ on V and $\dim_{\Bbbk((\zeta))} V < \infty$. We write V_ζ for V to denote that we are considering V as a $\Bbbk((\zeta))$-vector space. We define *the local inverse Mellin transform from infinity to infinity of D* to be the object

$$\mathcal{M}^{-(\infty,\infty)}(D) := (V_\zeta, \nabla) \in \mathcal{C}_\infty.$$

Claim 8.4. *$\mathcal{M}^{-(0,\infty)}$ is well-defined.*

Proof. To prove the claim we must show the following:

(i) z extends to an action of $\Bbbk((z))$ on V.

(ii) V_z is finite-dimensional.

(iii) ∇ is a connection on V_z.

We prove (i) with Lemma 8.5 below. In the proof of Lemma 8.5 we show that V_z is of Tate type. Lemma 5.3 then implies that V_z is finite-dimensional, proving (ii). To prove (iii) we must show that $[\nabla, f] = f'$ for all $f \in \Bbbk((z))$. Since ∇ is $\Bbbk$-linear and Laurent polynomials are dense in Laurent series, to show that $[\nabla, f] = f'$ we merely need to show that $[\nabla, z^n] = nz^{n-1}$ for all $n \in \mathbb{Z}$. A straightforward calculation shows that $[\nabla, z] = 1$, though, and then $[\nabla, z^n] = nz^{n-1}$ follows by induction. □

Lemma 8.5. *The definition of z given in (8-1) extends to an action of $\Bbbk((z))$ on V.*

Proof. Since Φ is invertible, an action of $\Bbbk[z^{-1}]$ is defined. We prove that Φ satisfies the conditions of Proposition 3.5 to show that an action of $\Bbbk[[z]]$ is well-defined.

To apply Proposition 3.5, we need to show that $z = \Phi$ is continuous, open, linearly compact, and contracting. First we show that Φ is open, continuous, and linearly compact. We can assume that Φ is indecomposable, so in canonical form $(V, \Phi) = D_g \otimes T_m$ for some $g \in K_r$ with $\operatorname{ord}(g) = s/r$.

Let $\{e_i\}$ be the canonical basis. As in previous proofs, it suffices to show that Φ and Φ^{-1} map a lattice of the form $L_k = \bigoplus (\theta^{1/r})^k A e_i$ to a lattice of the same form; note that here we are using $A = \Bbbk[[\theta^{1/r}]]$. Calculation using the canonical form shows that $\Phi(L_k) = L_{k+s}$ and $\Phi^{-1}(L_k) = L_{k-s}$, so Φ is open, continuous, and linearly compact.

To show that an indecomposable Φ is contracting, by Lemma 5.2 we need to show that $\operatorname{Ord}(\Phi) > 0$. By Corollary 4.15(2), then, we simply need to show that for $(V, \Phi) = D_g \otimes T_m$ we have $\operatorname{ord}(g) > 0$. This follows from the assumption that all irreducible components of Φ have order greater than zero. □

The proofs that $\mathcal{M}^{-(x,\infty)}$ and $\mathcal{M}^{-(\infty,\infty)}$ are well-defined are similar and are omitted.

9. Equivalence of categories

Assuming that composition of the functors is defined, by inspection one can see that $\mathcal{M}^{(0,\infty)}$ and $\mathcal{M}^{-(0,\infty)}$ are inverse functors (and the same holds for the pairs $\mathcal{M}^{(x,\infty)}$, $\mathcal{M}^{-(x,\infty)}$ and $\mathcal{M}^{(\infty,\infty)}$, $\mathcal{M}^{-(\infty,\infty)}$).

Thus to show that the local Mellin transforms induce certain equivalences of categories, all we need is to confirm that the functors map into the appropriate subcategories. We first prove an important property of normed vector spaces which coincides with properties of Tate vector spaces. This will be useful in demonstrating the equivalence of categories.

Normed vector spaces. Our first goal is to prove the following lemma, which will greatly simplify the relationship between the norm of an operator and its local

Mellin transform. First we give some definitions related to infinite-dimensional vector spaces over $\Bbbk$.

Definition 9.1. Let V be an infinite-dimensional vector space over $\Bbbk$. A *norm* on V is a real-valued function $\|\cdot\|$ such that the following hold:

(1) $\|v\| > 0$ for $v \in V - \{0\}$, $\|0\| = 0$.

(2) $\|v + w\| \leq \max(\|v\|, \|w\|)$ for all $v, w \in V$.

(3) $\|c \cdot v\| = \|v\|$ for $c \in \Bbbk$ and $v \in V$.

Note that the above definition applies to an infinite-dimensional vector space over $\Bbbk$, as opposed to K. Thus it is similar to Definition 4.2, but not the same.

Definition 9.2. An infinite-dimensional vector space V over $\Bbbk$ is *locally linearly compact* if for any $r_1 > r_2 > 0$, $r_i \in \mathbb{R}$, the ball of radius r_2 has finite codimension in the ball of radius r_1.

Proposition 9.3. *Let V be an infinite-dimensional vector space over $\Bbbk$, equipped with a norm $\|\cdot\|$ such that V is complete in the induced topology. Let $0 < \epsilon < 1$ and $Y : V \to V$ be an invertible $\Bbbk$-linear operator such that $\|Y\| = \epsilon^\alpha < 1$ and $\|Y^{-1}\| = \epsilon^{-\alpha}$. Define $\hat{\epsilon} := \epsilon^\alpha$. Then*

(1) *V has a unique structure of a $K = \Bbbk((y))$-vector space such that y acts as Y and the norm $\|\cdot\|$ agrees with the valuation on K where $|f| = \hat{\epsilon}^{\operatorname{ord}(f)}$ for $f \in K$.*

(2) *V is finite-dimensional over K if and only if V is locally linearly compact.*

Remark. If V is a Tate vector space then the unique structure of Proposition 9.3(1) coincides with that of Proposition 3.5.

Corollary 9.4. *Let V be a $\Bbbk((y))$-vector space, $Z : V \to V$ a similitude, and $\|Z\| = \|Z\|_{\inf} = \epsilon^\alpha < 1$. Then V can be considered as a $\Bbbk((Z))$-vector space (in the spirit of Proposition 9.3) and for any similitude $A : V \to V$ we have $\|A\| = \|A\|_Z$. In particular, A will be a similitude when V is viewed as either a $\Bbbk((y))$- or a $\Bbbk((Z))$-vector space.*

Lemmas.

Lemma 9.5. *The local Mellin transforms map indecomposable objects to indecomposable objects.*

Proof. We give the proof for $\mathcal{M}^{(0,\infty)}$; the proofs for the others are identical. Suppose that $\mathcal{M}^{(0,\infty)}(V, \nabla) = (V_\theta, \Phi)$ and V_θ has a proper subspace W such that $\Phi(W) \subset W$. Since V_θ is a $\Bbbk((\theta))$-vector space we also trivially have that $\theta(W) \subset W$. By definition of $\mathcal{M}^{(0,\infty)}$, this means that $z(W) \subset W$ and $-(z\nabla)^{-1}(W) \subset W$. In particular, it follows that $\nabla(W) \subset W$, so W is a proper subspace of V which is ∇-invariant. This implies that if the local Mellin transform of an object is decomposable, the original object is decomposable as well, and the result follows. □

Lemma 9.6. *Let* $E = (V, \nabla) \in \mathcal{C}_0^{>0}$, θ, *and* Φ *be as in Definition 7.1. Then* $\mathcal{M}^{(0,\infty)}(E) \in \mathcal{N}^{>0}$.

Proof. Due to the canonical decomposition it suffices to prove the lemma when E is indecomposable. Then ∇ and z are similitudes, so by Corollary 9.4, θ and Φ are also similitudes. By Lemma 9.5, Φ is indecomposable, so to prove Lemma 9.6 it suffices to show that $\|\Phi\|_\theta < 1$.

By Corollary 9.4, $\|A\|_z = \|A\|_\theta$ for any similitude A, and it follows that $\|\Phi\|_\theta = \|z\|_z = (\epsilon)^1 < 1$. □

The next lemmas have proofs similar to the proof of Lemma 9.6; they are omitted.

Lemma 9.7. *If* $D = (V, \Phi) \in \mathcal{N}^{>0}$ *is as in Definition 8.1, then* $\mathcal{M}^{-(0,\infty)}(D) \in \mathcal{C}_0^{>0}$.

Lemma 9.8. *If* $E = (V, \nabla) \in \mathcal{C}_x$ *is as in Definition 7.2, then* $\mathcal{M}^{(x,\infty)}(E) \in \mathcal{N}^0$.

Lemma 9.9. *If* $E = (V, \nabla) \in \mathcal{C}_\infty^{>0}$ *is as in Definition 7.3, then* $\mathcal{M}^{(\infty,\infty)}(E) \in \mathcal{N}^{<0}$.

Lemma 9.10. *If* $D = (V, \Phi) \in \mathcal{N}^{<0}$ *is as in Definition 8.3, then* $\mathcal{M}^{-(\infty,\infty)}(D) \in \mathcal{C}_\infty^{>0}$.

Proofs for equivalence of categories.

Theorem 9.11. *The local Mellin transform* $\mathcal{M}^{(0,\infty)}$ *induces an equivalence of categories between* $\mathcal{C}_0^{>0}$ *and* $\mathcal{N}^{>0}$.

Proof. This follows from Lemmas 9.6 and 9.7, as well as the fact (stated above) that $\mathcal{M}^{(0,\infty)}$ and $\mathcal{M}^{-(0,\infty)}$ are inverse functors. □

Theorem 9.12. *The local Mellin transform* $\mathcal{M}^{(x,\infty)}$ *induces an equivalence of categories between the subcategory of* $\mathcal{C}_x$ *of connections with no horizontal sections and* $\mathcal{N}^0$.

Theorem 9.13. *The local Mellin transform* $\mathcal{M}^{(\infty,\infty)}$ *induces an equivalence of categories between* $\mathcal{C}_\infty^{>0}$ *and* $\mathcal{N}^{<0}$.

10. Explicit calculations of local Mellin transforms

In this section we give precise statements of explicit formulas for calculating the local Mellin transforms and their inverses. The results and proofs found in this chapter are analogous to those given for the local formal Fourier transforms in [Graham-Squire 2013]. Section 11 is devoted to proving the formulas given in Section 10.

Calculation of $\mathcal{M}^{(0,\infty)}$.

Theorem 10.1. *Let* s *and* r *be positive integers,* $a \in \Bbbk - \{0\}$, *and* $f \in R_r^\circ(z)$ *with* $f = az^{-s/r} + \underline{o}(z^{-s/r})$. *Then*

$$\mathcal{M}^{(0,\infty)}(E_f) \simeq D_g,$$

where $g \in S_s^\circ(\theta)$ *is determined by the following system of equations*:

$$f = -\theta^{-1}, \tag{10-1}$$

$$g = z - (-a)^{r/s}\left(\frac{r+s}{2s}\right)\theta^{1+(r/s)}. \tag{10-2}$$

Remark. We determine g using (10-1) and (10-2) as follows. One can think of (10-1) as an implicit definition for the variable z. Thus we first use (10-1) to give an explicit expression for z in terms of $\theta^{1/s}$. We then substitute this explicit expression into (10-2) to get an expression for $g(\theta)$ in terms of $\theta^{1/s}$. This same pattern for determining g holds for similar calculations in this section.

When we use (10-1) to write an expression for z in terms of $\theta^{1/s}$, the expression is not unique since we must make a choice of a root of unity. More concretely, let η be a primitive s-th root of unity. Then replacing $\theta^{1/s}$ with $\eta\theta^{1/s}$ in our explicit equation for z will yield another possible expression for z. This choice will not affect the overall result, however, since all such possible expressions will lie in the same Galois orbit. Thus by Proposition 2.7(1), any choice of root of unity will correspond to the same difference operator.

Corollary 10.2. *Let* E *be an object in* $\mathcal{C}_0^{>0}$. *By* Proposition 2.3(3), *let* E *have decomposition* $E \simeq \bigoplus_i (E_{f_i} \otimes J_{m_i})$ *where all* E_{f_i} *have positive slope. Then*

$$\mathcal{M}^{(0,\infty)}(E) \simeq \bigoplus_i (D_{g_i} \otimes T_{m_i})$$

where $D_{g_i} = \mathcal{M}^{(0,\infty)}(E_{f_i})$ *for all* i.

Sketch of proof. The equivalence of categories given in Theorem 9.11 implies that

$$\mathcal{M}^{(0,\infty)}\left(\bigoplus_i (E_{f_i} \otimes J_{m_i})\right) \simeq \bigoplus_i \mathcal{M}^{(0,\infty)}(E_{f_i} \otimes J_{m_i}).$$

The equivalence also implies that $\mathcal{M}^{(0,\infty)}$ will map the indecomposable object $E_f \otimes J_m$ (as the unique indecomposable in $\mathcal{C}_0$ formed by m successive extensions of E_f) to an indecomposable object $D_g \otimes T_m$ (as the unique indecomposable in $\mathcal{N}$ formed by m successive extensions of D_g). It follows that we only need to know how $\mathcal{M}^{(0,\infty)}$ acts on E_f, which is given by Theorem 10.1. ☐

Remark. Analogous corollaries hold for the calculation of the other local Mellin transforms, however we do not state them explicitly.

Calculation of $\boldsymbol{\mathcal{M}^{(x,\infty)}}$.

Theorem 10.3. *Let* s *be a nonnegative integer*, r *a positive integer*, *and* $a \in \Bbbk - \{0\}$. *Let* $f \in R_r^\circ(z_x)$ *with* $f = a z_x^{-s/r} + \underline{o}(z_x^{-s/r})$. *Then*

$$\mathcal{M}^{(x,\infty)}(E_f) \simeq D_g,$$

where $g \in S^{\circ}_{r+s}(\theta)$ *is determined by the following system of equations*:

$$f = -\frac{z_x}{z}\,\theta^{-1},$$
$$g = z + \frac{xs}{2(s+r)}\,\theta.$$

Calculation of $\mathcal{M}^{(\infty,\infty)}$**.**

Theorem 10.4. *Let s and r be positive integers and* $a \in \Bbbk - \{0\}$. *Then for* $f \in R^{\circ}_r(\zeta)$ *with* $f = a\zeta^{-s/r} + \underline{o}(\zeta^{-s/r})$,

$$\mathcal{M}^{(\infty,\infty)}(E_f) \simeq D_g,$$

where $g \in S^{\circ}_s(\theta)$ *is determined by the following system of equations*:

$$f = -\theta^{-1},$$
$$g = z - (-a)^{r/s}\,\frac{r+s}{2s}\,\theta^{1-(r/s)}.$$

In Section 9 we explained that $\mathcal{M}^{-(0,\infty)}$, $\mathcal{M}^{-(x,\infty)}$, and $\mathcal{M}^{-(\infty,\infty)}$ are inverse functors for $\mathcal{M}^{(0,\infty)}$, $\mathcal{M}^{(x,\infty)}$, and $\mathcal{M}^{(\infty,\infty)}$, respectively. It follows that explicit formulas for the local inverse Mellin transforms can be found merely by “inverting” the expressions found in Theorems 10.1, 10.3, and 10.4. We give an example below of what this would look like for $\mathcal{M}^{-(0,\infty)}$, the other local inverse Mellin transforms are similar. The proofs are omitted.

Theorem 10.5. *Let p and q be positive integers and let* $g \in S^{\circ}_q(\theta)$ *be given by* $g = a\theta^{p/q} + \underline{o}(\theta^{p/q})$, $a \neq 0$. *Then*

$$\mathcal{M}^{-(0,\infty)}(D_g) \simeq E_f,$$

where $f \in R^{\circ}_p(z)$ *is determined by the following system of equations*:

$$g + a\,\frac{p+q}{2q}\,\theta^{1+(p/q)} = z, \tag{10-3}$$

$$f = -\theta^{-1}. \tag{10-4}$$

Remark. We determine f using (10-3) and (10-4) as follows. First, using (10-3) we explicitly express θ in terms of $z^{1/p}$. We then substitute this explicit expression for θ into (10-4) and solve to get an expression for $f(z)$ in terms of $z^{1/p}$.

11. Proof of theorems

Outline. We begin with a brief outline of the proof for Theorem 10.1. Starting with Definition 8.1 of $\mathcal{M}^{(0,\infty)}$, we set $\theta = -(z\nabla)^{-1}$ and $\Phi = z$. For irreducible objects E_f and D_g we have $\nabla = \frac{d}{dz} + z^{-1}f$ and $\Phi = g\varphi$, and our goal is to use the given value of f to find the expression for g. Since $z = z(1) = \Phi(1) = g\varphi(1) = g$, this amounts to finding an expression for the operator z in terms of the operator θ. The equation $\theta = -(z\nabla)^{-1}$ gives an expression for θ in terms of z, and we use Lemma 5.1 (the operator-root lemma) to write an explicit expression for the operator z in terms of θ. The calculation primarily involves finding particular fractional powers of f, but we must also keep track of the interplay between the linear and differential parts of ∇ during the calculation; this interplay accounts for the subtraction of the term

$$(-a)^{r/s}\,\frac{r+s}{2s}\,\theta^{1+(r/s)}$$

from our expression for g.

The proofs for Theorems 10.3 and 10.4 are similar and thus outlines for their proofs are omitted. The only change of note is that in the proof of Theorem 10.3 we must also prove a separate case for when our connection is regular singular, i.e., when $\operatorname{ord}(f) = 0$.

Remark. We give a brief explanation regarding the origin of the system of equations found in Theorem 10.1. Consider the equations in (7-1). Let $\nabla = z^{-1}f$, i.e., as normally defined but without the differential part. Let $\Phi = g$, as normally defined but without the shift operator φ. Then the equations $f = -\theta^{-1}$ and $g = z$ fall out easily. The reason the extra term shows up in (10-2) is due to the interaction of the linear and differential parts of ∇, as described above in the outline.

Proof of Theorem 10.1. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = \dfrac{d}{dz} + z^{-1}f$, we find that

$$-\theta = \left(z\frac{d}{dz} + f\right)^{-1}. \tag{11-1}$$

We wish to express the operator z in terms of the operator θ.

Consider the equation

$$-\theta = f^{-1}, \tag{11-2}$$

which is (11-1) without the differential part. Equation (11-2) can be thought of as an implicit expression for the variable z in terms of the variable θ, which one can rewrite as an explicit expression $z = h(\theta) \in \Bbbk((\theta^{1/s}))$ for the variable z. Note that $h(\theta)$ is not the same as the operator z. The leading term of f is $az^{-s/r}$, so (11-2) implies that $h(\theta) = a^{r/s}(-\theta)^{r/s} + \underline{o}(\theta^{r/s})$. Similar reasoning and (11-1) indicate that the operator z will be of the form

$$z = h(\theta) + *(-\theta)^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}). \tag{11-3}$$

Here the $* \in \Bbbk$ represents the coefficient that will arise from the interaction of the linear and differential parts of the operator θ. We wish to find the value for $*$. Let

$A = f$ and $B = z\frac{d}{dz}$, then $[B, A] = zf'$. From (11-1) we have $-\theta = (A+B)^{-1}$, and we apply Lemma 5.1 (the operator-root lemma) to find

$$(11\text{-}4)\quad (-\theta)^{r/s}$$
$$= f^{-r/s} - \frac{r}{s} f^{-r/s-1} z\Big(\frac{\mathbb{Z}}{zr}\Big) - \frac{1}{2}\Big(\frac{r}{s}\Big)\Big(-\frac{r}{s}-1\Big) f^{-r/s-2} zf' + \underline{o}(z^{(r+s)/r})$$
$$= (a^{-r/s}z + \cdots) + a^{-(r+s)/s}\Big(\frac{-\mathbb{Z}}{s} + \frac{-(r+s)}{2s}\Big) z^{(r+s)/r} + \underline{o}(z^{(r+s)/r})$$
$$= a^{-r/s}\Big(z + \cdots + a^{-1}\Big(\frac{-\mathbb{Z}}{s} + \frac{-(r+s)}{2s}\Big) z^{1+(s/r)} + \underline{o}(z^{1+(s/r)})\Big)$$

and

$$(11\text{-}5)\qquad (-\theta)^{(r+s)/s} = a^{-1-(r/s)} z^{1+(s/r)} + \underline{o}(z^{1+(s/r)}).$$

Remark. We use the notation $\frac{\mathbb{Z}}{zr}$ to represent $\frac{d}{dz}$ since, for all $n \in \mathbb{Z}$, the operator $\frac{d}{dz} : K_r \to K_r$ acts on $z^{n/r}$ as multiplication by $\frac{n}{rz}$.

We can now find the value for $*$ as follows: Substituting the expressions from (11-4) and (11-5) into (11-3) and making a short calculation gives

$$* = a^{r/s}\Big(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\Big)$$

and thus

$$(11\text{-}6)\qquad z = h(\theta) + a^{r/s}\Big(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\Big)(-\theta)^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}).$$

According to (11-6), let us express $\hat{g}(\theta)$ as

$$(11\text{-}7)\qquad \hat{g}(\theta) = h(\theta) - (-a)^{r/s}\Big(\frac{\mathbb{Z}}{s} + \frac{r+s}{2s}\Big)\theta^{(r+s)/s} + \underline{o}(\theta^{(r+s)/s}).$$

Since $h(\theta) = z$, by Proposition 2.7(1), $M_{\hat{g}}$ will be isomorphic to M_g, where g is as given in Theorem 10.1. □

Proof of Theorem 10.3. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = \dfrac{d}{dz_x} + z_x^{-1} f$, we write $z = z_x + x$ and find that

$$(11\text{-}8)\qquad -\theta = \Big((x+z_x)\Big(\frac{d}{dz_x} + z_x^{-1}f\Big)\Big)^{-1} = \Big(zz_x^{-1}f + x\frac{d}{dz_x} + z_x\frac{d}{dz_x}\Big)^{-1}.$$

Thus in the expression for $-\theta^{-1}$ there are three terms. We handle the proof in two cases:

Regular singularity. In this case we have $f = \alpha \in \Bbbk - \{0\}$, $s = 0$, and $r = 1$. Because α is only defined up to a shift by $\mathbb{Z}$ we can ignore the d/dz_x term. The remaining portion of the proof is as described in the remark on page 134. Note that since $s = 0$, the extra θ term in the final equation in Theorem 10.3 will vanish.

Irregular singularity. In this situation we have $\operatorname{ord}(f) < 0$. As we shall see in the proof, the only terms in (11-8) that affect the final result are those of order less than or equal to -1 (with respect to z_x). Specifically, since $z_x\, d/dz_x$ has order zero, all terms derived from it in the course of the calculations will fall into the $\underline{o}(\theta)$ term. Thus we can safely ignore the term $z_x\, d/dz_x$ for the remainder of the proof and consider only

$$-\theta = \Big(z z_x^{-1} f + x \frac{d}{dz_x}\Big)^{-1}. \tag{11-9}$$

We wish to express the operator z in terms of the operator θ. The remainder of the proof is similar to the proof of Theorem 10.1, but we first solve for $z_x = z - x$ in terms of θ, then add x to both sides to get an equation for z alone. □

Proof of Theorem 10.4. Recall that $z = 1/\zeta$ and $f \in \Bbbk((\zeta))$. Given $\theta = -(z\nabla)^{-1}$ and $\nabla = -\zeta^2\, d/d\zeta + \zeta f$, we find that

$$-\theta = \Big(-\zeta \frac{d}{d\zeta} + f\Big)^{-1}. \tag{11-10}$$

We wish to express the operator z in terms of the operator θ. The proof is similar to that of Theorem 10.1, but first we find an expression for ζ in terms of θ, and then we will invert it. □

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THE UNTWISTING NUMBER OF A KNOT

KENAN INCE

Dedicated to Tim Cochran

The unknotting number of a knot is the minimum number of crossings one must change to turn that knot into the unknot. The algebraic unknotting number is the minimum number of crossing changes needed to transform a knot into an Alexander polynomial-one knot. We work with a generalization of unknotting number due to Mathieu and Domergue, which we call the untwisting number. The untwisting number is the minimum number (over all diagrams of a knot) of right- or left-handed twists on even numbers of strands of a knot, with half of the strands oriented in each direction, necessary to transform that knot into the unknot. We show that the algebraic untwisting number is equal to the algebraic unknotting number. However, we also exhibit several families of knots for which the difference between the unknotting and untwisting numbers is arbitrarily large, even when we only allow twists on a fixed number of strands or fewer.

1. Introduction

It is a natural knot-theoretic question to seek to measure "how knotted up" a knot is. One such "knottiness" measure is given by the *unknotting number* $u(K)$, the minimum number of crossings, taken over all diagrams of K, one must change to turn K into the unknot. By a *crossing change* we shall mean one of the two local moves on a knot diagram given in Figure 1.

This invariant is quite simple to define but has proven itself very difficult to master. Fifty years ago, Milnor conjectured that the unknotting number for the (p, q)-torus knot was $\frac{1}{2}(p-1)(q-1)$; only in 1993, in two celebrated papers, did Kronheimer and Mrowka [1993; 1995] prove this conjecture true. Hence, it is desirable to look at variants of unknotting number which may be more tractable. One natural variant (due to Murakami [1990]) is the *algebraic unknotting number* $u_a(K)$, the minimum number of crossing changes necessary to turn a given knot into an Alexander polynomial-one knot. Alexander polynomial-one knots are significant because they "look like the unknot" to *classical invariants*, knot invariants derived

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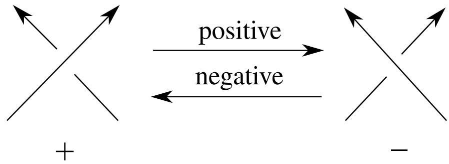

Figure 1. A positive and negative crossing change.

from the Seifert matrix. It is obvious that $u_a(K) \leq u(K)$ for any knot K, and there exist knots such that $u_a(K) < u(K)$ (for instance, any nontrivial knot with trivial Alexander polynomial).

Mathieu and Domergue [1988] defined another generalization of unknotting number. Livingston [2002] worked with this definition. He described it as follows:

> "One can think of performing a crossing change as grabbing two parallel strands of a knot with opposite orientation and giving them one full twist. More generally, one can grab $2k$ parallel strands of K with k of the strands oriented in each direction and give them one full twist."

Following Livingston, we call such a twist a *generalized crossing change*. We describe in Section 2A how a crossing change may be encoded as a ± 1-surgery on a nullhomologous unknot $U \subset S^3 - K$ bounding a disk D such that $D \cap K = 2$ points. From this perspective, a generalized crossing change is a relaxing of the previous definition to allow $D \cap K = 2k$ points for any k, provided $\text{lk}(K, U) = 0$ (see Figure 2). In particular, any knot can be unknotted by a finite sequence of generalized crossing changes.

One may then naturally define the *untwisting number* $tu(K)$ to be the minimum length, taken over all diagrams of K, of a sequence of generalized crossing changes beginning at K and resulting in the unknot. By $tu_p(K)$, we will denote the minimum number of twists on $2p$ or fewer strands needed to unknot K; notice that $tu_1(K) = u(K)$ and that

$$tu \leq \cdots \leq tu_{p+1} \leq tu_p \leq \cdots \leq tu_1 = u.$$

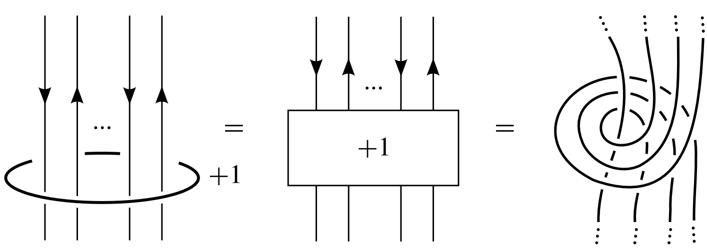

Figure 2. A right-handed, or positive, generalized crossing change.

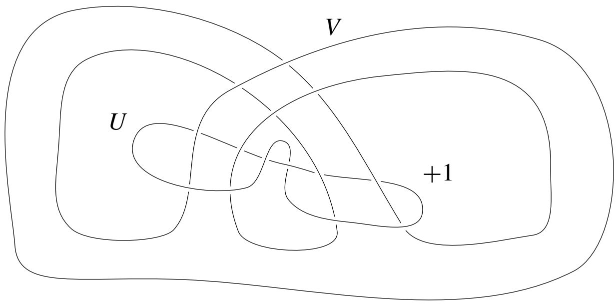

Figure 3. The generalized crossing change for $V \subset S^3$ which results in a knot $K \subset S^3$ with $tu(K) \neq u(K)$.

The *algebraic untwisting number* $tu_a(K)$ is the minimum number of generalized crossing changes, taken over all diagrams of K, needed to transform K into an Alexander polynomial-one knot. It is clear that $tu_a(K) \leq tu(K)$ for all knots K.

It is natural to ask how $tu(K)$ and $u(K)$ are related. We show that these invariants are "algebraically the same" in the following sense:

Theorem 1.1. *For any knot* $K \subset S^3$, $tu_a(K) = u_a(K)$.

Therefore, tu and u cannot be distinguished by classical invariants. By using the Jones polynomial, which is not a classical invariant, we can show that tu and u are not equal in general:

Theorem 1.2. *Let* K *be the image of* $V \subset S^3$ *in the manifold* $M \cong S^3$ *resulting from* $+1$*-surgery on the unknot* $U \subset S^3$ *shown in Figure 3. Then* $tu(K) = 1$ *but* $u(K) > 1$.

Furthermore, using the fact that the absolute value of the Ozsváth–Szabó τ invariant is a lower bound on unknotting number, we show in Section 5A that the difference $u - tu_p$ can be arbitrarily large, and thus so can the difference $u - tu$. Throughout this paper, $K_{p,q}$ will denote the (p, q)-cable of the knot K, where p denotes the longitudinal winding and q the meridional winding.

Theorem 1.3. *Let* K *be a knot in* S^3 *such that* $u(K) = 1$. *If* $\tau(K) > 0$ *and* $p, q > 0$, *then*

$$u(K_{p,q}) - tu_p(K_{p,q}) \geq p - 1.$$

In particular, if we take $q = 1$, *then* $tu_p(K_{p,q}) = 1$, *while* $u(K_{p,q}) \geq p$.

It may seem that the above examples are "cheating" in some sense, as in each of them the number of strands of K passing through the ± 1-framed unknot U in the generalized crossing change diagram is increasing along with $u(K)$. The following theorem shows that $u(K)$ can be arbitrarily larger than $tu(K)$ even when we restrict to doing q-generalized crossing changes for any fixed integer $q \geq 1$.

Theorem 1.4. *For any knot K with $u(K) = 1$ and $\tau(K) > 0$, the infinite family of knots $J_p^q := \#^p K_{q,1}$ satisfies*

$$u(J_p^q) - tu_q(J_p^q) \geq p$$

for any integers $p > 1$, $q > 0$.

So far, all of the families of knots we have worked with are quite complicated, in the sense that they are (p, q)-cables for large p or connected sums of such cables. One may wonder whether it is possible to find a "simpler" knot K for which $tu(K) < u(K)$. One measure of "knot simplicity" is *topological sliceness*; a knot K is *topologically slice* if there exists a locally flat disk $D \subset B^4$ such that $\partial D = K \subset S^3 = \partial B^4$.

Theorem 1.5. *For any knot K with $\tau(K) > 0$, let $D_+(K, 0)$ denote the positive-clasped, untwisted Whitehead double of K. Then the knots $S_p^q := \#^p (D_+(K, 0))_{q,1}$ are topologically slice and satisfy*

$$u(S_p^q) - tu_q(S_p^q) \geq p$$

for all integers $p > 0$, $q > 0$.

This paper is organized as follows. First, we will review the operations of Dehn surgery on knots and knot cabling and define the untwisting number more precisely. Next, we will give some background on the Blanchfield form which is necessary to prove that $tu_a = u_a$. Finally, we will prove that each of the above families of knots gives arbitrarily large gaps between u and tu.

Convention. In this paper, all manifolds are assumed to be compact, orientable, and connected.

2. Preliminaries

2A. ***Dehn surgery.*** In this section, we will describe the operation of Dehn surgery on knots.

Definition 2.1. Let $K \subset S^3$ be an oriented knot and $U \subset S^3$ be an unknot with $\operatorname{lk}(K, U) = 0$. Let W be a closed tubular neighborhood of U in S^3. Let λ be a longitude of W, and let μ be a meridian of W such that $\operatorname{lk}(\mu, \lambda) = 1$. The 3-manifold

$$M = (S^3 - \mathring{W}) \cup_h W,$$

where $h : \partial W \to \partial W$ is a homeomorphism taking a meridian of W onto $\pm\mu + \lambda \subset W$, is the *result of* ± 1*-surgery on* U, and U is said to be ± 1*-framed*. In this situation, we define a *generalized crossing change diagram for* K to be a diagram of the link $K \cup U$ with the number ± 1 written next to U, indicating that U is ± 1-framed. Figure 3 is an example of a generalized crossing change diagram for the unknot V.

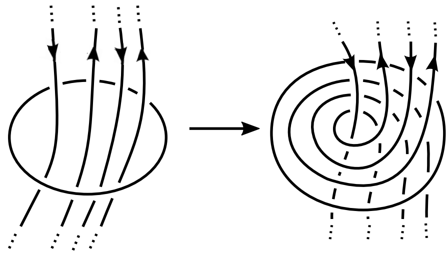

Figure 4. A right-handed twist about an unknotted component.

In the general case, note that the complement of $\mathring{N} \supset U$ in S^3 is a solid torus, which we may modify with a meridional twist. This alters K as follows: if D is a disk bounded by U such that k strands of K pass through D in straight segments, then each of the k straight pieces is replaced by a helix which screws through a neighborhood of D in the right-hand sense (see Figure 4).

If U is -1-framed, the knot obtained by erasing U and twisting the strands of K that pass through U as in Figure 4 represents the image of K under the -1-surgery on U [Rolfsen 1976]. If instead U has framing $+1$, the knot obtained by erasing U and giving K a left-handed meridional twist represents the image of K under the $+1$-surgery on U. The process of performing a $\mp$-meridional twist on the complement of a ± 1-framed unknot U, then erasing U from the resulting diagram, is called a *blow-down on U*. The inverse process of introducing an unknotted component U to a surgery diagram consisting of a knot K, then performing a $\pm$-meridional twist on the complement of U to link it with K, is known as a *blow-up on U* and results in a diagram consisting of K and the ∓ 1-framed unknot U, where $\operatorname{lk}(K, U) = 0$.

Now, it can be easily verified that blowing down the $+1$-framed unknot on the left side of Figure 5 transforms the crossing labeled $+$ into the crossing labeled $-$. The inverse process of introducing an unknot to the right side of Figure 5 and performing a $-$-meridional twist on its complement yields the positive crossing.

2B. ***Untwisting number.*** We define a *$\pm$-generalized crossing change on K* as the process of blowing down the ± 1-framed unknot in a generalized crossing change diagram for K. In this situation, K must pass through U an even number of times, for otherwise $\operatorname{lk}(K, U) \neq 0$. If at most $2p$ strands of K pass through U in a

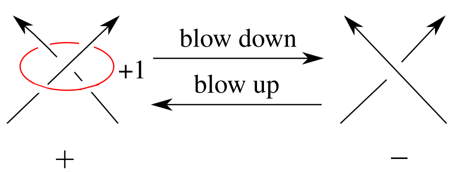

Figure 5. Crossing changes as blow-downs of ± 1-framed unknots.

generalized crossing change diagram, we may call the associated $\pm$-generalized crossing change a *$\pm p$-generalized crossing change on K*.

The *result of a $\pm$-generalized crossing change on K* is defined to be the image of K under the blow-down. The *untwisting number $tu(K)$ of K* is the minimum length of a sequence of generalized crossing changes on K such that the result of the sequence is the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes. Note that by the reasoning on page 58 of [Adams 1994], this definition is equivalent to taking the minimum length, over all diagrams of K, of a sequence of generalized crossing changes beginning with a fixed diagram of K such that the result of the sequence is the unknot, where we do not allow ambient isotopy of the diagram in between generalized crossing changes.

For $p = 1, 2, 3, \ldots$, we define the *p-untwisting number $tu_p(K)$* to be the minimum length of a sequence of $\pm p$-generalized crossing changes on K resulting in the unknot, where we allow ambient isotopy of the diagram in between generalized crossing changes.

It follows immediately that we have the chain of inequalities

$$tu(K) \leq \cdots \leq tu_{p+1}(K) \leq tu_p(K) \leq \cdots \leq tu_2(K) \leq tu_1(K) = u(K). \tag{2-1}$$

2C. *Cabling.* In this section, we define satellite and cable knots.

Definition 2.2. A closed subset X of a solid torus $V \cong S^1 \times D^2$ is called *geometrically essential* in V if X intersects every PL meridional disk in V.

Let $P \subset V \subset S^3$ be a knot which is geometrically essential in an unknotted solid torus V. Let $C \subset S^3$ be another knot and let V_1 be a tubular neighborhood of C in S^3. Let $h : V \to V_1$ be a homeomorphism and let K be $h(P)$. Then P is called the *pattern* for the knot K, C is the *companion* of K, and K is called a *satellite of C with pattern P*, or just a *satellite knot* for short.

If the homeomorphism h takes the preferred longitude and meridian of V, respectively, to the preferred longitude and meridian of V_1, then h is said to be *faithful*. If P is the (p, q)-torus knot just under ∂V and h is faithful, then K is called the *(p, q)-cable based on C*, denoted $C_{p,q}$, or simply a *cable knot*.

Throughout this paper, we will denote the (p, q)-torus knot by $U_{p,q}$ since it is the (p, q)-cable of the unknot U.

2D. *The Blanchfield form.* Let $K \subset S^3$ be a knot. By Λ we shall denote the ring $\mathbb{Z}[t^{\pm 1}]$, and by Ω we will denote the field $\mathbb{Q}(t)$.

Twisted homology, cohomology groups, and Poincaré duality. Following [Borodzik and Friedl 2014], let X be a manifold with infinite cyclic first homology, and fix a choice of isomorphism of $H_1(X)$ with the infinite cyclic group generated by the indeterminate t. Let $\pi : \widetilde{X} \to X$ be the infinite cyclic cover of X. Given a submanifold Y of X, let $\widetilde{Y} = \pi^{-1}(Y)$. Since $\mathbb{Z}$ is the deck transformation group of $\widetilde{X}$, Λ acts on the relative chain group $C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})$. If N is any Λ-module, we may define

$$H^*(X, Y; N) := H_*(\mathrm{Hom}_\Lambda(C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z}), N))$$

and

$$H_*(X, Y; N) := H_*\Big(\overline{C_*(\widetilde{X}, \widetilde{Y}; \mathbb{Z})} \otimes_\Lambda N\Big).$$

Here, if H is any Λ-module, $\overline{H}$ denotes the module with the involuted Λ-structure: multiplication by $p(t) \in \Lambda$ in $\overline{H}$ is the same as multiplication by $p(t^{-1})$ in H. When $Y = \varnothing$, we just write $H_*(X; N)$ or $H^*(X; N)$.

Since $\Omega := \mathbb{Q}(t)$ is flat over Λ, we have isomorphisms

$$H_*(X, Y; \Omega) \cong H_*(X, Y; \Lambda) \otimes_\Lambda \Omega$$

and

$$H^*(X, Y; \Omega) \cong H^*(X, Y; \Lambda) \otimes_\Lambda \Omega.$$

If X is an n-manifold, and N is a Λ-module, Poincaré duality gives Λ-module isomorphisms

$$H_i(X, \partial X; N) \cong H^{n-i}(X; N).$$

The Blanchfield form. As above, let $\Lambda = \mathbb{Z}[t, t^{-1}]$ and $\Omega = \mathbb{Q}(t)$. Let A be an $n \times n$ invertible hermitian matrix with entries in Λ. We define $\lambda(A)$ to be the pairing

$$\lambda(A) : \Lambda^n / A\Lambda^n \times \Lambda^n / A\Lambda^n \to \Omega/\Lambda$$

sending the pair of column vectors (a, b) to $\bar{a}^t A^{-1} b$. Note that $\lambda(A)$ is a nonsingular, hermitian pairing.

Let $X(K) = S^3 - N(K)$ denote the exterior of K. Consider the sequence of maps

$$\begin{aligned}\Phi : H_1(X(K); \Lambda) &\xrightarrow{\pi_*} H_1(X(K), \partial X(K); \Lambda)\\ &\xrightarrow{\mathrm{PD}} H^2(X(K); \Lambda) \xleftarrow{\delta} H^1(X(K); \Omega/\Lambda)\\ &\xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_\lambda(H_1(X(K); \Lambda), \Omega/\Lambda)}.\end{aligned}$$

Here π_* is induced by the quotient map $C(X) \to C(X)/C(\partial X)$, PD is the Poincaré duality map, δ is from the long exact sequence in cohomology obtained from the coefficients $0 \to \Lambda \to \Omega \to \Omega/\Lambda \to 0$, and ev is the Kronecker evaluation map. It is well known (see [Hillman 2012, Section 2] for details) that π_* and δ are isomorphisms, PD is the Poincaré duality isomorphism, and ev is also an isomorphism by the universal coefficient spectral sequence (see [Levine 1977, Theorem 2.3] for details on the universal coefficient spectral sequence). Thus, the above maps define a nonsingular pairing

$$\lambda(K) : H_1(X(K);\Lambda) \times H_1(X(K);\Lambda) \to \Omega/\Lambda,$$
$$(a,b) \mapsto \Phi(a)(b),$$

called the *Blanchfield pairing of* K. This pairing is hermitian.

Now, let V be any $2k \times 2k$ matrix which is S-equivalent to a Seifert matrix for K. Recall that $V - V^T$ is antisymmetric with determinant ± 1. It is well known that, perhaps after replacing V by PVP^T for some $P \in \mathrm{GL}_{2k}(\mathbb{Z})$,

$$V - V^T = \begin{pmatrix} 0 & -I_k \\ I_k & 0 \end{pmatrix}, \tag{2-2}$$

where I_k denotes the $k \times k$ identity matrix. We define $A_K(t)$ to be the matrix

$$\begin{pmatrix} (1-t^{-1})^{-1} I_k & 0 \\ 0 & I_k \end{pmatrix} V \begin{pmatrix} I_k & 0 \\ 0 & (1-t) I_k \end{pmatrix} + \begin{pmatrix} I_k & 0 \\ 0 & (1-t^{-1}) I_k \end{pmatrix} V^T \begin{pmatrix} (1-t)^{-1} I_k & 0 \\ 0 & I_k \end{pmatrix}.$$

Using (2-2), we can write

$$V = \begin{pmatrix} B & C+I \\ C^T & D \end{pmatrix}.$$

One may then compute, as in the proof of [Borodzik and Friedl 2015, Lemma 2.2], that

$$A_K(1) = \begin{pmatrix} B & -I_k \\ -I_k & 0 \end{pmatrix}.$$

Thus, the matrix $A_K(t)$ is a hermitian matrix defined over Λ, and $\det(A_K(1)) = (-1)^k$.

Proposition 2.3 [Borodzik and Friedl 2015, Proposition 2.1]. *Let K be a knot and $A_K(t)$ be as above. Then $\lambda(A_K(t))$ is isometric as a sesquilinear form to $\lambda(K)$.*

2E. *The twisted intersection pairing.* Let W be a topological 4-manifold with boundary M such that $\pi_1(W) = \mathbb{Z}$. Consider the maps

$$H_2(W;\Lambda) \xrightarrow{\pi_*} H_2(W,M;\Lambda) \xrightarrow{\mathrm{PD}} H^2(W;\Lambda) \xrightarrow{\mathrm{ev}} \overline{\mathrm{Hom}_\Lambda(H_2(W;\Lambda),\Lambda)},$$

where the first map is induced by the quotient, the second map is Poincaré duality, and the third map is the Kronecker evaluation map. The second and third maps are obviously isomorphisms, and the first map is an isomorphism by the long exact sequence of the pair (W, M). Hence this composition defines a pairing

$$H_2(W;\Lambda) \times H_2(W;\Lambda) \to \Lambda,$$

which we call the *twisted intersection pairing on* W.

3. Algebraic untwisting number equals algebraic unknotting number

Our proof that $tu_a(K) = u_a(K)$ generalizes [Borodzik and Friedl 2014; 2015]. Following [Borodzik and Friedl 2014], define a knot invariant $n(K)$ to be the minimum size of a square hermitian matrix $A(t)$ over $\mathbb{Z}[t^{\pm 1}]$ such that $\lambda(A)$ is isometric to $\lambda(K)$ and $A(1)$ is congruent over $\mathbb{Z}$ to a diagonal matrix which has only ± 1 entries. Borodzik and Friedl showed that $u_a(K) = n(K)$. Since $tu_a(K) \leq u_a(K)$, it is obvious that $tu_a(K) \leq n(K)$ as well. After stating Borodzik and Friedl's results, we will show that $n(K) \leq tu_a(K)$; hence $tu_a(K) = n(K) = u_a(K)$ for all knots K. In fact, we will show something stronger.

Theorem 3.1. *Let $K \subset S^3$ be a knot. For every algebraic unknotting sequence for K with u_+ positive crossing changes and u_- negative crossing changes, there exists an algebraic untwisting sequence for K with u_+ positive generalized crossing changes and u_- negative generalized crossing changes. In particular, $u_a(K) = tu_a(K)$.*

In order to prove Theorem 3.1, we must first recall some notation and results used in [Borodzik and Friedl 2015]. The main theorem of that paper implies that $n(K) \leq u_a(K)$:

Theorem 3.2 [Borodzik and Friedl 2015, Theorem 1.1]. *Let K be a knot which can be changed into an Alexander polynomial-one knot by a sequence of u_+ positive crossing changes and u_- negative crossing changes. Then there exists a hermitian matrix $A(t)$ of size $u_+ + u_-$ over Λ such that*

(1) *$\lambda(A(t))$ is isometric to $\lambda(K)$;*

(2) *$A(1)$ is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to 1.*

In particular, $n(K) \leq u_a(K)$.

We need one definition:

Definition 3.3. Let K be a knot and $M(K)$ the result of 0-surgery on K. A 4-manifold W *tamely cobounds* $M(K)$ if

(1) $\partial W = M(K)$;

(2) the inclusion induced map $H_1(M(K); \mathbb{Z}) \to H_1(W; \mathbb{Z})$ is an isomorphism;

(3) $\pi_1(W) = \mathbb{Z}$.

If, in addition, the intersection form on $H_2(W; \mathbb{Z})$ is diagonalizable, we say that W *strictly cobounds* $M(K)$.

Theorem 3.4 [Borodzik and Friedl 2015, Theorem 2.6]. *Let K be a knot and let W be a topological 4-manifold which tamely cobounds $M(K)$. Then $H_2(W; \Lambda)$ is free of rank $b_2(W)$. Moreover, if B is an integral matrix representing the ordinary* *intersection pairing of* W*, then there exists a basis* $\mathcal{B}$ *for* $H_2(W;\Lambda)$ *such that the matrix* $A(t)$ *representing the twisted intersection pairing with respect to* $\mathcal{B}$ *satisfies*

(1) $\lambda(A(t))$ *is isometric to* $\lambda(K)$*;*

(2) $A(1) = B$*.*

We generalize Theorem 3.2 as follows:

Theorem 3.5. *Let* K *be a knot which can be changed into an Alexander polynomial-one knot by a sequence of* u_+ *positive and* u_- *negative generalized crossing changes. Then there exists a hermitian matrix of size* $u_+ + u_-$ *over* Λ *with the properties*

(1) $\lambda(A(t))$ *is isometric to* $\lambda(K)$*;*

(2) $A(1)$ *is a diagonal matrix such that* u_+ *diagonal entries are equal to* -1 *and* u_- *diagonal entries are equal to* 1*.*

In particular, $n(K) \leq tu_a(K)$*.*

The proof of Theorem 3.5 is similar to that of Theorem 3.2. By Theorem 3.4, in order to prove Theorem 3.5, we only need to show the following proposition.

Proposition 3.6. *Let* K *be a knot such that* u_+ *positive generalized crossing changes and* u_- *negative generalized crossing changes turn* K *into an Alexander polynomial-one knot. Then there exists an oriented topological* 4*-manifold* W *which strictly cobounds* $M(K)$*. Moreover, the intersection pairing on* $H_2(W;\mathbb{Z})$ *is represented by a diagonal matrix of size* $u_+ + u_-$ *such that* u_+ *entries are equal to* -1 *and* u_- *entries are equal to* $+1$*.*

Proof. Let K be a knot such that u_+ positive generalized crossing changes and u_- negative generalized crossing changes turn K into an Alexander polynomial-one knot J. We write $s = u_+ + u_-$ and $n_i = -1$ for $i = 1, \dots, u_+$ and $n_i = 1$ for $i = u_+ + 1, \dots, u_+ + u_-$. Then there exist simple closed curves $c_1, \dots, c_s$ in $S^3 - N(J)$ such that

(1) $c_1 \cup \cdots \cup c_s$ is the unlink in S^3;

(2) the linking numbers $\operatorname{lk}(c_i, K)$ are zero for all i;

(3) the image of J under the n_i-surgeries is the knot K.

Note that the curves $c_1, \dots, c_s$ lie in $S^3 - N(J)$; hence we can view them as lying in $M(J)$. The manifold $M(K)$ is then the result of n_i-surgery on all the $c_i \subset M(J)$, where $i = 1, \dots, s$.

Since J is a knot with trivial Alexander polynomial, by Freedman's theorem [Freedman and Quinn 1990] J is topologically slice and there exists a locally flat slice disk $D \subset B^4$ for J such that $\pi_1(B^4 - D) = \mathbb{Z}$. Let $X := B^4 - N(D)$. Then X is an oriented topological 4-manifold such that

(1) $\partial X \cong M(J)$ as oriented manifolds;

(2) $\pi_1(X) \cong \mathbb{Z}$;

(3) the inclusion induced map $H_1(M(J);\mathbb{Z}) \to H_1(X;\mathbb{Z})$ is an isomorphism;

(4) $H_2(X;\mathbb{Z}) = 0$.

Let W be the 4-manifold which is obtained by adding 2-handles along $c_1, \ldots, c_s \subset M(J)$ with framings $n_1, \ldots, n_s$ to X. Then $\partial W \cong M(K)$ as oriented manifolds. From now on, we write $M := M(K)$. Since the curves $c_1, \ldots, c_s$ are nullhomologous, the map $H_1(M;\mathbb{Z}) \to H_1(W;\mathbb{Z})$ is an isomorphism and $\pi_1(W) \cong \mathbb{Z}$. It thus remains to prove the following lemma:

Lemma 3.7. *The ordinary intersection pairing on W is represented by a diagonal matrix of size $s = u_+ + u_-$ with u_+ diagonal entries equal to -1 and u_- diagonal entries equal to* 1.

Recall that the curves $c_1, \ldots, c_s$ form the unlink in S^3 and that the linking numbers $\mathrm{lk}(c_i, J)$ are zero. Therefore, the curves $c_1, \ldots, c_s$ are also nullhomologous in $M(J)$. Thus we can now find disjoint surfaces $F_1, \ldots, F_s$ in $M(J) \times [0, 1]$ such that $\partial F_i = c_i \times \{1\}$. By adding the cores of the 2-handles attached to the c_i, we obtain closed surfaces $C_1, \ldots, C_s$ in W. It is clear that $C_i \cdot C_j = 0$ for $i \neq j$ and $C_i \cdot C_i = n_i$.

We argue using Mayer–Vietoris that the surfaces $C_1, \ldots, C_s$ present a basis for $H_2(W;\mathbb{Z})$. Write $W := X \cup H$, where $H \cong \bigsqcup_{i=1}^{s}(B^2 \times B^2)$ is the set of 2-handles attached to $c_1, \ldots, c_s$. Then write $Y := X \cap H$, so that

$$Y = \bigsqcup_{i=1}^{s} N(c_i) \cong \bigsqcup_{i=1}^{s} (S^1 \times D^2).$$

We have the Mayer–Vietoris sequence

$$\cdots \to H_2(X) \oplus H_2(H) \xrightarrow{\psi_*} H_2(W) \xrightarrow{\partial_*} H_1(Y) \xrightarrow{\phi_*} H_1(X) \oplus H_1(H) \xrightarrow{\psi_*} H_1(W) \to 0.$$

Now, since $H_1(Y)$ is generated by all the S^1-factors, or the longitudes $c_1, \ldots, c_s$, and $H_1(H) = H_2(H) = H_2(X) = 0$, the sequence becomes

$$0 \to H_2(W) \xrightarrow{\partial_*} \langle c_1, \ldots, c_s \rangle \xrightarrow{i_*} H_1(X) \xrightarrow{\psi_*} H_1(W) \to 0.$$

From [Livingston 1993, Lemma 8.12], for example, we have:

Lemma 3.8. *Suppose that for some knot K in S^3, there is a locally flat surface F in B^4 with $F \cap S^3 = \partial F \cap S^3 = K$. Then the inclusion map induces an isomorphism $H_1(S^3 - K) \to H_1(B^4 - F) \cong \mathbb{Z}$.*

In our case, the inclusion $S^3 - K \hookrightarrow X$ induces an isomorphism $H_1(S^3 - K) \to H_1(X)$. Since i_* is induced by inclusion and the longitudes $c_1, \ldots, c_s$ are nullhomologous in $S^3 - K$, we see that i_* must be the zero map in X. Hence ∂_* is an isomorphism $H_2(W) \cong H_1(Y)$, and $H_2(W) = \langle C_1, \ldots, C_s \rangle$.

In particular, the intersection matrix on W with respect to this basis is given by $(C_i \cdot C_j)$, i.e., it is a diagonal matrix such that u_+ diagonal entries are equal to -1 and u_- diagonal entries are equal to $+1$. This concludes the proof of Lemma 3.7. Proposition 3.6 follows. Together with Theorem 3.4, this completes the proof of Theorem 3.5. □

We have shown that, for every untwisting sequence for K with u_+ positive generalized crossing changes and u_- negative generalized crossing changes, there exists a hermitian matrix $A(t)$ of size $u_+ + u_-$ such that $\lambda(A(t))$ is isometric to $\lambda(K)$ and $A(1)$ is diagonal with u_+ entries equal to -1 and u_- entries equal to 1. Borodzik and Friedl [2014] have already shown that, for every hermitian matrix $A(t)$ representing $\lambda(K)$ such that $A(1)$ is diagonal with u_+ -1's and u_- $+1$'s, there exists an algebraic unknotting sequence for K consisting of u_+ positive and u_- negative crossing changes. Theorem 3.1 follows.

4. Untwisting number does not equal unknotting number

Although the algebraic versions of tu and u are equal, $tu \neq u$ in general. We use a result of Miyazawa [1998] to give our first example of a knot K with $tu(K) = 1$ but $u(K) > 1$.

Theorem 4.1. *Let K be the knot resulting from blowing down the $+1$-framed unknot $U \subset S^3 \setminus V$ in Figure 3. Then $tu(K) = 1$ but $u(K) > 1$.*

From this point forward, we will denote the signature of any knot K by $\sigma(K)$. In order to analyze the unknotting number of K, we will use the following theorem:

Theorem 4.2 [Miyazawa 1998]. *If $u(K) = 1$ and $\sigma(K) = \pm 2$, then*

$$V_K^{(1)}(-1) \equiv 24a_4(K) - \tfrac{1}{8}\sigma(K)(\det K + 1)(\det K + 5) \pmod{48},$$

where $V_K^{(1)}$ denotes the first derivative of the Jones polynomial of K and a_4 is the coefficient of z^4 in the Conway polynomial $\nabla_K(z) = \sum_{n=0}^{\infty} a_{2n}(K) z^{2n}$.

We compute using the Mathematica package KnotTheory (http://katlas.org/wiki/The_Mathematica_Package_KnotTheory) that $\sigma(K) = 2$; therefore Theorem 4.2 applies. We also compute using the KnotTheory package that the Jones polynomial $V_K(q)$ for our knot K is

$$V_K(q) = q - q^2 + 2q^3 - q^4 + q^6 - q^7 + q^8 - q^9 - q^{12} + q^{13};$$

hence $V_K^{(1)}(-1) = 8$. The Conway polynomial of K is computed to be

$$\nabla_K(z) = \sum_{n=0}^{\infty} a_{2n}(K) z^{2n} = 1 + z^2$$

(hence $a_4 = 0$), and the determinant of K is 3. In our case, the right-hand side of the congruence in Theorem 4.2 becomes

$$0 - \tfrac{1}{4}(4)(8) = -8$$

and $8 \not\equiv -8 \pmod{48}$. Hence K cannot have unknotting number one, although it was constructed to have untwisting number one. Note that this also shows Miyazawa's Jones polynomial criterion does not extend to untwisting-number-one knots.

5. Arbitrarily large gaps between unknotting and untwisting numbers

5A. *Arbitrarily large gaps between u and tu_p.* Now that we have shown that there exists a knot K with $tu(K) < u(K)$, it is natural to ask how large the difference $u(K) - tu(K)$ can be. Recall that the (p,q)-cable of a knot K is denoted $K_{p,q}$; we denote the (p,q)-torus knot by $U_{p,q}$, the (p,q)-cable of the unknot. The knots we will be working with are (p,q)-cables of knots K with $u(K) = 1$ and $\tau(K) > 0$, where $p, q > 0$.

To get a lower bound on $u(K_{p,q})$ for such knots, we compute $\tau(K_{p,q})$ for all p, q. For cables of alternating (or more generally, "homologically thin") knots such as the trefoil, Petkova [2013] gives a formula for computing τ. However, since we will later compute τ for cables of nonalternating knots, we use a more general method of computing $\tau(K_{p,q})$ using the ϵ-invariant $\epsilon(K) \in \{-1, 0, 1\}$ introduced by Hom:

Theorem 5.1 [Hom 2014]. *Let $K \subset S^3$.*

(1) *If $\epsilon(K) = 1$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p-1)(q-1)$.*

(2) *If $\epsilon(K) = -1$, then $\tau(K_{p,q}) = p\tau(K) + \frac{1}{2}(p-1)(q+1)$.*

(3) *If $\epsilon(K) = 0$, then $\tau(K) = 0$ and*

$$\tau(K_{p,q}) = \tau(U_{p,q}) = \begin{cases} \frac{1}{2}(p-1)(q+1), & q < 0, \\ \frac{1}{2}(p-1)(q-1), & q > 0. \end{cases}$$

Theorem 5.2 [Ozsváth and Szabó 2003]. *For the (p,q)-torus knot $U_{p,q}$ with $p, q > 0$, τ equals the 3-sphere genus of $U_{p,q}$, denoted $g(U_{p,q})$:*

$$\tau(U_{p,q}) = g(U_{p,q}) = \tfrac{1}{2}(p-1)(q-1).$$

Proposition 5.3 [Hom 2014]. *Let $K \subset S^3$ be a knot. If $|\tau(K)| = g(K)$, then $\epsilon(K) = \operatorname{sgn}\tau(K)$.*

Theorem 5.4. *Let K be a knot in S^3 with unknotting number one. If $\tau(K) > 0$ and $p, q > 0$, then*

$$u(K_{p,q}) - tu_p(K_{p,q}) \geq p - 1.$$

In particular, $tu_p(K_{p,1}) = 1$, while $u(K_{p,1}) \geq p$.

Proof. Let V be the unknot that results from performing the unknotting crossing change on K. Consider a generalized crossing change diagram for V together with the ± 1-framed surgery curve U that transforms V back into K. Then take the (p, q)-cable $V_{p,q}$ of V in this diagram, leaving U alone. The resulting $V_{p,q}$ is the (p, q)-torus knot before performing the ± 1-surgery, but the image of V under ± 1-surgery on U is K; hence the image of $V_{p,q}$ under the ± 1-surgery on U is $K_{p,q}$. Therefore, blowing down the surgery curve U (through which $V_{p,q}$ passes $2p$ times) results in a diagram for $K_{p,q}$ in S^3. Since $K_{p,q}$ and $V_{p,q}$ differ by a single twist,

$$tu_p(K_{p,q}) \leq tu_p(V_{p,q}) + 1.$$

Since

$$tu_p(V_{p,q}) \leq u(V_{p,q}) = \tfrac{1}{2}(p-1)(q-1),$$

we get that

$$tu_p(K_{p,q}) \leq \tfrac{1}{2}(p-1)(q-1) + 1.$$

In particular, this inequality shows that $tu_p(K_{p,1}) = 1$. If $\tau(K) > 0$, then necessarily $\epsilon(K) \neq 0$ by (3) of Theorem 5.1, so that $\epsilon(K) = \pm 1$. In this case,

$$\tau(K_{p,q}) = p\tau(K) + \tfrac{1}{2}(p-1)(q \mp 1),$$

and thus

$$u(K_{p,q}) \geq |\tau(K_{p,q})| = p\tau(K) + \tfrac{1}{2}(p-1)(q \mp 1) \geq p + \tfrac{1}{2}(p-1)(q \mp 1).$$

When $q = 1$, we get that $u(K_{p,1}) \geq p$. Combining our estimates,

$$\begin{aligned} u(K_{p,q}) - tu_p(K_{p,q}) &\geq \big(p + \tfrac{1}{2}(p-1)(q \mp 1)\big) - \big(1 + \tfrac{1}{2}(p-1)(q-1)\big) \\ &\geq \big(p + \tfrac{1}{2}(p-1)(q-1)\big) - \big(1 + \tfrac{1}{2}(p-1)(q-1)\big) \\ &\geq p - 1. \end{aligned}$$

□

5B. ***Arbitrarily large gaps between u and tu_q.*** The above examples $\{K_{p,1}\}$ show that for every p there exists a knot $K_{p,1}$ with $u(K_{p,1}) \geq p$, even though $tu_p(K_{p,1}) = 1$. However, in order to untwist any such $K_{p,1}$, we must twist at least $2p$ strands at once. A natural follow-up question is whether there exists a knot K with $u(K) \geq p$ that can be untwisted by a single $\pm q$-generalized crossing change, where $q < p$. More generally, we may ask whether, for any fixed q, there is a family of knots which give us arbitrarily large gaps between u and tu_q. We answer this question in the affirmative.

Theorem 5.5. *Let K be a knot with $u(K) = 1$ and $\tau(K) > 0$, and let $J_p^q := \#^p K_{q,1}$. For any $p > 0$ and $q > 1$, we have $tu_q(J_p^q) \leq p$ and $u(J_p^q) - tu_q(J_p^q) \geq p$.*

Proof. First, we note that for any knot K, $J_p^q = \#^p K_{q,1}$ can be unknotted by performing p generalized crossing changes on at most $2q$ strands each, one generalized crossing change to unknot each copy of $K_{q,1}$. Therefore, $tu_q(J_p^q) \leq p$. Since τ is additive under connected sum,

$$\tau(J_p^q) = p \cdot \tau(K_{q,1}) \geq pq,$$

and hence $u(J_p^q) \geq pq$ for all p. Therefore,

$$u(J_p^q) - tu_q(J_p^q) \geq pq - p = p(q-1) \geq p, \qquad \square$$

Note. In the case where K has $\sigma(K) = \pm 2$, e.g., when K is a right-handed trefoil knot, we can do better by computing tu_q precisely. We use the fact that $\frac{1}{2}|\sigma(K)|$ is a lower bound for $tu_q(K)$ for any q. First, recall that the *Tristram–Levine signature function* of a knot K, $\sigma_\omega(K)$, is equal to the signature of the matrix $(1-\omega)V + (1-\overline{\omega})V^T$, where $\omega \in \mathbb{C}$ has norm 1 and V is a Seifert matrix for K. Note that

$$\sigma_{-1}(K) = \sigma(2(V + V^T)) = \sigma(V + V^T) = \sigma(K).$$

We use Litherland's formula [1979] for Tristram–Levine signatures of cable knots to compute that

$$\sigma_{-1}(K_{p,q}) = \sigma_{(-1)^p}(K) + \sigma_{-1}(U_{p,q})$$

and, since $\sigma_1 \equiv 0$, while $\sigma_{-1} = \sigma$,

$$\sigma(K_{q,1}) = \begin{cases} \sigma(K) + \sigma(U_{q,1}) = \sigma(K), & q \text{ odd}, \\ \sigma(U_{q,1}) = 0, & q \text{ even}, \end{cases}$$

since the $(q, 1)$-torus knot is the unknot for any q. Now, since the knot signature is additive over connected sum,

$$\sigma(J_p^q) = p\sigma(K_{q,1}) = \begin{cases} \sigma(K) \cdot p = \pm 2p, & q \text{ odd}, \\ 0, & q \text{ even}, \end{cases}$$

and therefore, when p is odd,

$$tu_q(J_p^q) \geq \tfrac{1}{2}|\sigma(\kappa_p^q)| = p.$$

Since we already know $tu_q(J_p^q) \leq p$, in fact we must have $tu_q(J_p^q) = p$ for odd $p \geq 1$.

5C. ***Arbitrarily large gaps between u and tu_q for topologically slice knots.*** Consider the diagram of an unknot $U(K)$ in Figure 6, where K is any knot with $\tau(K) > 0$. Let $p \geq 2$ be an integer.

We take the $(q, 1)$-cable of $U(K)$, which is still an unknot. Then, we perform a -1-twist on the $+1$-framed unknot, obtaining a knot S^q. Clearly, $tu_q(S^q) = 1$.

Furthermore, S^q is the $(q, 1)$-cable of the knot $D_+(K, 0)$, the untwisted Whitehead double of K. This is because $U(K)$ represents $D_+(K, 0)$ in the manifold obtained from the $+1$-surgery, and the cabling operation converts this knot into the

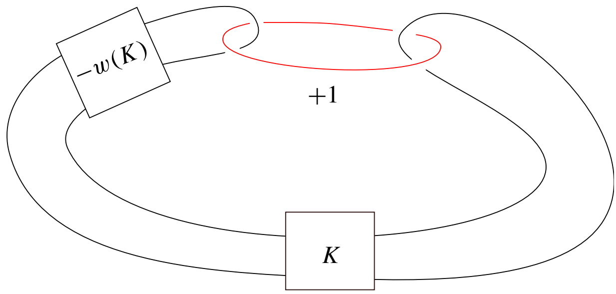

Figure 6. The knot $U(K)$ (an unknot), together with a $+1$-surgery curve.

$(q, 1)$-cable of $D_+(K, 0)$. Since untwisted Whitehead doubles are topologically (but not necessarily smoothly) slice [Freedman and Quinn 1990], $D_+(K, 0)$ is topologically concordant to the unknot. It is well known that, if K is concordant to J, then $K_{m,n}$ is concordant to $J_{m,n}$ for all integers m, n. Hence $S_{q,1}$ is also topologically concordant to the unknot $U_{q,1}$, and therefore S_p is topologically slice for all p.

Now, define $S_p^q := \#^p\, D_+(K, 0)$. Connected sums of topologically slice knots are topologically slice; hence S_p^q is topologically slice. Moreover, as above, we have that $tu_q(S_p^q) \le p \cdot tu_q(S^q) = p$.

We will now get a lower bound on $u(S_p^q)$ and thus show that $u(S_p^q) - tu_q(S_p^q)$ can be arbitrarily large. The Ozsváth–Szabó τ invariant gives such a lower bound. Thus, we need to compute $\tau(S_p^q)$ for all p, q.

We show that $\epsilon(D_+(K, 0)) = 1$ and hence, applying Theorem 5.1, that

$$\tau(S^q) = q\tau(D_+(K, 0)).$$

We first compute $\tau(D_+(K, 0))$.

Theorem 5.6 [Hedden 2007]. *Let $D_+(K, t)$ denote the positive t-twisted Whitehead double of a knot K. Then*

$$\tau(D_+(K, t)) = \begin{cases} 1, & t < 2\tau(K), \\ 0, & \textit{otherwise}. \end{cases}$$

Since $\tau(K) > 0$ in our case, $t = 0 < 2 \le 2\tau(K)$, and so $\tau(D_+(K, 0)) = 1$. Furthermore, as is the case with any Whitehead double, $g(D_+(K, 0)) = 1$, so $|\tau(D_+(K, 0))| = 1 = g(D_+(K, 0))$ and, by Proposition 5.3,

$$\epsilon(D_+(K, 0)) = \operatorname{sgn} \tau(D_+(K, 0)) = +1.$$

We then apply Theorem 5.1 to S^q to get that

$$\tau(S^q) = q\tau(D_+(K, 0)).$$

Since $\tau(D_+(K,0)) = 1$, we have that $\tau(S^q) = q$ and, hence, $\tau(S_p^q) = pq$. Thus, $u(S_p^q) \geq pq$. Therefore,

$$u(S_p) - tu_q(S_p) \geq pq - p = p(q-1) \geq p,$$

as desired.

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A PLANCHEREL FORMULA FOR $L^2(G/H)$ FOR ALMOST SYMMETRIC SUBGROUPS

BENT ØRSTED AND BIRGIT SPEH

We study the Plancherel formula for a new class of homogeneous spaces for real reductive Lie groups; these spaces are fibered over non-Riemannian symmetric spaces, and they exhibit a phenomenon of uniform infinite multiplicities. The proof for this is new but rather elementary, and we give all details. As an application we use several results from the recent literature studying possible nontemperedness of homogeneous spaces; thus we provide examples of nontempered representations of the group appearing in the Plancherel formula for our homogeneous spaces. Several classes of examples are given, each building on different techniques and new results from the theory of symmetric spaces.

I. Introduction

Considerable efforts have been devoted to obtaining the Plancherel formula for homogeneous spaces of the form G/H with G a real reductive Lie group and H a symmetric subgroup, a program completed by T. Oshima, P. Delorme, E. van den Ban, and H. Schlichtkrull. This is a central theme in harmonic analysis, and there are a number of natural ways to extend such a program. One is to consider spherical spaces, i.e., where the homogeneous space admits an open orbit of a parabolic subgroup. In this paper we shall rather extend the interest to

(1) square-integrable sections of homogeneous line bundles over symmetric spaces, and

(2) spaces fibered over symmetric spaces.

Of course, these two questions are related, and we shall find several classes of spaces where rather explicit answers can be found. As an example consider $G = \mathrm{SL}(2, \mathbb{R})$ with H the connected diagonal subgroup; for each unitary character of H we may consider the space (1) and the corresponding Plancherel formula: This turns out to be independent of the character, and hence the space as in (2)

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above (in our case just the group G) has the same L^2-content as the symmetric space, only with infinite multiplicity. It is perhaps a little surprising, that one may thus find embeddings of, e.g., the discrete series of G in a uniform way in each of the spaces of sections (1).

To be more specific, our motivation in undertaking this work was to understand the disintegration of the representation of a semisimple Lie group G on the space $L^2(G/H_{ss})$ where H_{ss} is a semisimple subgroup which differs from a symmetric subgroup by a noncompact central real factor. In this paper we study this situation for the simplest nonsymmetric subgroups H_{ss} from the point of view of harmonic analysis and obtain a Plancherel theorem for space $L^2(G/H_{ss})$ in terms of the one for G/H.

Recently Y. Benoist and T. Kobayashi [2015] proved general criteria to determine if for a semisimple subgroup H the spectrum of $L^2(G/H)$ contains nontempered representations; this they use to determine in many examples if $L^2(G/H)$ is tempered. Here a representation is called tempered if it appears in the usual Plancherel formula for $L^2(G)$. However these authors do not obtain any results concerning the multiplicity of the representations in the Plancherel formula. By obtaining a Plancherel formula for $L^2(G/H_{ss})$ we are in a position to determine exactly in our examples which nontempered representations appear in the spectrum, and also to show that they appear with infinite multiplicities.

We consider a noncompact subgroup $H = H_{ss} Z_H$ where H is a subgroup of finite index in the fixpoints of an involution of G and $Z_H \simeq \mathbb{R}$ is a subgroup of finite index of the center of H. Under these assumptions we show the following.

Theorem. *As a left regular representation of G*

$$L^2(G/H_{ss}) \simeq L^2(G/H) \otimes L^2(Z_H).$$

It is instructive to compare with the situation where the central subgroup is compact, e.g., the case of G a simple noncompact Lie group and K a maximal compact subgroup with a one-dimensional center Z. Here G/K is a noncompact Riemannian symmetric space of Hermitian type, and $L^2(G/K)$ has a different Plancherel decomposition than $L^2(G/K, \chi)$, the square-integrable sections of the line bundle induced from a nontrivial unitary character χ of Z. In particular the first space contains no discrete series representations, whereas the second space typically does. Compare with Proposition III.3 for our situation of a noncompact center.

A related problem for spherical varieties over non-Archimedian fields is discussed in [Sakellaridis and Venkatesh 2014, Section 9.5].

The paper is organized as follows: In Section II, we show that we can regard H as a subgroup of finite index in the Levi subgroup of a parabolic subgroup with abelian nilradical. In Section III we prove our main theorem above. In Section IV we discuss some examples. In particular we note that we find several examples of nontempered homogeneous spaces, some of them new; quite possibly our method could extend to other instances of Plancherel theorems, such as cases of vector bundles (as opposed to the cases of line bundles treated here).

II. Notation and preliminaries

We introduce the notation and prove some preliminary results.

Notation and assumptions. Let G be a real linear semisimple connected algebraic group with maximal compact subgroup K and complexification $G_{\mathbb{C}}$. We consider G and $K \subset G_{\mathbb{C}}$ as subgroups.

Proposition II.1. *Suppose that $P = LN$ is a maximal parabolic subgroup with an abelian nilradical N. Then L is the fixpoint set of an involution*

$$\tau : G \to G.$$

Proof (due to Dan Barbasch). We consider a maximal split Cartan subgroup and its corresponding complex Cartan subalgebra. A parabolic subalgebra is given by removing some simple roots from the diagram. The only way to get an abelian nilradical is to remove a single simple root which appears with coefficient at most one 1 if we write the roots as linear combinations of simple roots. The involution τ is then conjugation by $\exp(i\pi\varpi)$ where ϖ is the coroot of the simple root which was removed. □

Let H be a subgroup of the Levi subgroup L of P which contains the connected component L^0 of L. Then $H = H_{ss}Z_H$ where Z_H is a one dimensional connected subgroup in the center of H and H_{ss} is semisimple or discrete.

Example 1. $G = \mathrm{SL}(2, \mathbb{R})$, L diagonal matrices which are the fixed points under the conjugation by the diagonal matrix of order 2 and determinant -1. Alternatively we consider the adjoint representation. Then L is the stabilizer of a semisimple nontrivial element of order 2. It is also the fixed point set of the automorphism by the adjoint action of the matrix

$$\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} = \exp\left(\pi i \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}\right).$$

We have to consider two subgroups H and H_{ss}:

(a) $H = L,\ H_{ss} = \mathbb{Z}_2,\ Z_H = \mathbb{R}_+ . G/H_{ss} = \mathrm{PSL}(2, \mathbb{R})$,

(b) $H = L^0$, $H_{ss} = I$ and $G/H_{ss} = \mathrm{SL}(2, \mathbb{R})$.

Proposition II.2. *Suppose that F is the fixed point set of an involution $\tau : G \to G$. Assume in addition that it is a product $F = F_{ss}Z_F$ where Z_F is a subgroup of the* *center of F isomorphic to $\mathbb{R}^+$ and F_{ss} is a semisimple group. Then F is contained in the Levi subgroup of a maximal parabolic subgroup P with abelian nilradical N.*

Proof. We choose maximally split Cartan subgroup $C \subset F$ with complexified Lie algebra $\mathfrak{h}_{\mathbb{C}}$. We choose the simple roots of $\mathfrak{h}_{\mathbb{C}}, \mathfrak{g}_{\mathbb{C}}$ so that they are simple roots in $\mathfrak{h}_{\mathbb{C}}, \mathfrak{f}_{\mathbb{C}}$. (In the lexicographical order we let $\mathfrak{f}$ come before $\mathfrak{g}$, $\mathfrak{f}$ the Lie algebra of F.) Then $\mathfrak{f}$ is the Levi subalgebra of a maximal parabolic subalgebra $\mathfrak{p}_{\mathbb{C}} = \mathfrak{f}_{\mathbb{C}} \oplus \mathfrak{n}_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$.

It remains to show that N is abelian. Since τ leaves C invariant the induced homomorphism of $\tau : N \to N$ is equal to -1. Since τ induces a Lie algebra homomorphism and hence preserves the Lie bracket in $\mathfrak{n}_{\mathbb{C}}$, the results follows from the observation that $\tau(X) = -X$ and $\tau(Y) = -Y$ then $\tau([X, Y]) = [X, Y]$. □

Note that in the setting above, we have a direct product decomposition

$$G/H \times Z_H = G/H_{ss}.$$

This will be useful later in connection with integration over this space, and in considering the corresponding L^2-space.

About $L^2(G/H)$. Keep the assumptions on G, H, Z_H as above. We extend a unitary character $\chi \in \widehat{Z_H}$ to a character of H and consider the unitary induced representation $\mathrm{Ind}_H^G \chi$ on $L^2(G/H)_{\chi^{-1}}$. Normalize Plancherel measures on Z_H and its dual group in the usual way.

Proposition II.3. *As a representation of G*

$$L^2(G/H_{ss}) = \int_{\chi \in \widehat{Z_H}} L^2(G/H)_{\chi^{-1}} d\chi,$$

Proof. For $f \in L^2(G/H_{ss})$ and $\chi \in \widehat{Z_H}$ define

$$F(\chi, g) = \int_{Z_H} f(gz)\chi(z)^{-1}\, dz,$$

Then for $z_0 \in Z_H$

$$F(\chi, gz_0) = F(\chi, g)\chi^{-1}(z_0),$$

so $F(\chi) \in L^2(G/H)_{\chi^{-1}}$. By Fourier analysis on Z_H we have

$$\int_{\chi \in \widehat{Z_H}} |F(\chi, g)|^2 d\chi = \int_{z \in Z_H} |f(gz)|^2, dz.$$

So

$$\int_{G/H_{ss}} |f(g')|^2 dg' = \int_{G/H} \int_{Z_H} |f(g'z)|^2 dz dg'$$

completes the proof. □

III. Main results

In this section we relate the Plancherel formula for the left regular representation of G on $L^2(G/H_{ss})$ to the Plancherel formula for the left regular representation on $L^2(G/H)$. It turns out that these two spaces have the same content of unitary representations of G, only differing by their multiplicities.

Induction to the parabolic subgroup P.

Lemma III.1. *Let $\widehat{N}$ the dual group of N. There exist finitely many open H orbits $\mathbb{O}_i$ in $\widehat{N}$ so that $\widehat{N}$ is the closure of their union $\bigcup_i \mathbb{O}_i$.*

Proof. Here we refer to results by Wallach [2006]. Here he proves that our parabolic algebras are "very nice" since they have abelian nilradicals (see Corollary 6.4 of that reference). In particular there is only one open orbit of L on N.

Since our group H is a subgroup of finite index in L we will get a finite number of open orbits with dense union. Actually, the statement that "open orbit is generic" (i.e., "nice parabolic") would suffice for our purposes here. □

Let $\chi \in \widehat{Z_H}$. We consider again χ as a character of H and consider again the unitary induced representation $\operatorname{Ind}_H^P \chi$.

Proposition III.2. *Let χ and $\tilde{\chi}$ be unitary characters of Z_H considered as characters of H. Then we have (equivalence of representations)*

$$\operatorname{Ind}_H^P \chi = \operatorname{Ind}_H^P \tilde{\chi}.$$

Proof. We denote the induced representations acting on functions $F \in L^2(N)$ by

$$\rho_\chi(n_0)F(n) = F(n \cdot n_0),$$
$$\rho_\chi(h_0)F(n) = \chi(h_0)F(h_0^{-1} n h_0).$$

Using the Fourier transform we realize the representation $\operatorname{Ind}_H^P \chi$ on $L^2(\hat{N})$. It is a direct sum of irreducible representations on $L^2(\mathbb{O}_i)$ where

$$\hat{\rho}_\chi(n_0) \text{ is a multiplication operator,}$$
$$\hat{\rho}_\chi(h_0)\hat{F}(\xi) = \chi(h_0)J(h_0^t\xi)^{1/2}\hat{F}(h_0^t\xi).$$

The other representation on the orbit is obtained by multiplication of the right hand side of the second equation with a character $\chi_1 = \tilde{\chi}\chi^{-1}$ of H. In each orbit we fix an element ξ_i.

We get a intertwining operator on each of the irreducible representations by

$$I(F)(\xi) = \chi_1(\xi)F(\xi).$$

Here $\xi = h\xi_i$ and $\chi_1(\xi) := \chi_1(h)$. □

Example 2. Consider the group $P = HN$ with

$$H = \left\{ \begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix} \middle| \; a > 0 \right\}$$

and

$$N = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \middle| \; b \in \mathbb{R} \right\}.$$

We note that there are three orbits of H on

$$\hat{N} = \left\{ \xi_t \;\middle|\; \xi_t\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) = e^{it\cdot b} \right\},$$

namely, $\mathbb{O}^+ = \{\xi_t \mid t > 0\}$, $\mathbb{O}^- = \{\xi_t \mid t < 0\}$ and $\mathbb{O}^1 = \{\xi_0\}$. The unitary representation ρ_1 of P induced from the trivial representation of H acts on $L^2(N)$ by

$$\rho_1\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) F(x) = a^{1/2} F(ax)$$

and

$$\rho_1\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) F(x) = F(x+b).$$

To analyze this representation we consider the Fourier transform of $L^2(N)$. The representation is a direct sum of two unitary representations of functions whose Fourier transform has support in $\xi \in \mathbb{O}^+$ and in $\xi \in \mathbb{O}^-$.

We consider $\chi_s : a \to a^{is}$ as a character of H. After applying the Fourier transform the representation $\widehat{\rho_s}$ induced from χ_s has the form

$$\widehat{\rho_s}\left(\begin{pmatrix} a & 0 \\ 0 & 1 \end{pmatrix}\right) \hat{F}(\xi) = a^{-1/2} a^{is} \hat{F}(a^{-1}\xi)$$

and

$$\widehat{\rho_t}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right) \hat{F}(\xi) = e^{ib\xi} \hat{F}(\xi).$$

The equivalence of the representations ρ_s and ρ_1 follows from the intertwining operator

$$\mathcal{I}_s : \rho_0 \to \rho_s \quad \text{defined by } \mathcal{I}_s \hat{F}(\xi) = \xi^{is} \hat{F}(\xi).$$

Induction to G.

Proposition III.3. *Let χ and $\tilde{\chi}$ be characters of Z_H considered as characters of H. As representations of G we have (equivalence)*

$$\mathrm{Ind}_H^G \chi = \mathrm{Ind}_H^G \tilde{\chi}.$$

Proof. By induction by stages (Proposition III.2) we have

$$\mathrm{Ind}_H^G \chi = \mathrm{Ind}_P^G \mathrm{Ind}_H^P \chi = \mathrm{Ind}_P^G \mathrm{Ind}_H^P \tilde{\chi} = \mathrm{Ind}_H^G \tilde{\chi}. \qquad \square$$

Example 3. $G = \mathrm{SU}(1,1)$ and

$$H = A = \exp \mathbb{R} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We identify G/K with the complex unit disk $\mathcal{D}$. We realize the discrete series representations D_n in the holomorphic functions on $\mathcal{D}$. Then we have the H-invariant distribution vector, giving the embedding into $L^2(G/H)$,

$$v^* = (1+z^2)^{-n/2} \in D_n^{-\infty,H},$$

and similarly the distribution vector

$$v^* = (1+z^2)^{-n/2} \left(\frac{1-z}{1+z}\right)^{i\lambda} \in D_n^{-\infty,H,\chi_\lambda},$$

transforming by the character χ_λ of H. So indeed every discrete series representation occurs in every $L^2(G/H)_{\chi_\lambda}$.

Theorem III.4. *As a left regular representation of G*

$$L^2(G/H_{ss}) \simeq (\mathrm{Ind}_H^G 1) \otimes L^2(Z_H) \simeq L^2(G/H) \otimes L^2(Z_H).$$

Proof. This follows from Propositions II.3 and III.3. □

Corollary III.5. *All irreducible representations in the discrete spectrum of*

$$L^2(G/H_{ss})$$

have infinite multiplicity.

Definition. Following Benoist and Kobayashi we say that $L^2(G/H_{ss})$ is not tempered if the representations in the Plancherel formula for the right regular representation of G on $L^2(G/H_{ss})$ are not a subset of the representations of the Plancherel formula for G.

Corollary III.6. *$L^2(G/H_{ss})$ is tempered if and only if $L^2(G/H)$ is tempered.*

Example 1 (continued). $G = \mathrm{SL}(2,\mathbb{R})$, H diagonal matrices, Then $X = G/H$ is a hyperboloid and

$$L^2(G/H) = \oplus_{\nu \in 2\mathbb{N}} D_\nu \oplus 2 \int_0^\infty \pi_{it},$$

where D_ν are the discrete series representations with parameter ν and π_{it} are the tempered spherical principal series representations with parameter it. Here $H_{ss} = \mathbb{Z}_2$, then $L^2(G/H_{ss}) = L^2(\mathrm{PSL}(2,\mathbb{R}))$ and so the left regular representation contains the even discrete series representations with ∞ multiplicity.

If H is connected, then $L^2(G/H)$ contains all discrete series representations and so does the left regular representation of G on $L^2(G)$.

IV. More examples

We discuss in this section some interesting examples of groups G and H_{ss}, illustrating our results; one aspect is to find reductive spaces that are not tempered.

We use the Plancherel formula to determine if $L^2(G/H_{ss})$ is tempered. Some of our examples are also contained in [Benoist and Kobayashi 2015], where they are obtained with a different technique; others are new.

E. van den Ban and H. Schlichtkrull [2005] proved a Plancherel formula for $L^2(G/H)$ for a fixed point set H of an involution τ of G. They showed that only discrete series representations of $L^2(G/H)$ and principal series representations unitarily induced from a $\theta\tau$ invariant parabolic MAN, a discrete series representation π of $M/M \cap H$ and a unitary character of A contribute to the Plancherel formula. On the other hand the work of M. Flensted-Jensen and Oshima and Matsuki shows that the discrete spectrum of G/H is nontrivial if and only if

$$\operatorname{rank} G/H = \operatorname{rank} K/K \cap H.$$

A parametrization of the representations in the discrete spectrum was obtained by T. Matsuki and T. Oshima [1984]. See also [Schlichtkrull 1983]. We will make extensive use of these results in the proofs of our examples.

Remark 1. Induction by stages enlarges the set of pairs G, $\tilde{H}$ for which $L^2(G/\tilde{H})$ is tempered. (See [Fell 1962, Theorem 4.2]; here the point is that induction preserves weak containment, so if we have groups $H \subset \tilde{H} \subset G$ so that $L^2(G/H)$ is tempered and we know that $L^2(\tilde{H}/H)$ contains the trivial representation weakly, then also $L^2(G/\tilde{H})$ is tempered.)

Remark 2. The nontempered representations in the discrete spectrum of $L^2(G/H_{ss})$ are automorphic representations [Burger and Sarnak 1991]. Most of these automorphic representations are known and have been constructed using other techniques for example in [Kudla and Rallis 1990; Howe and Piatetski-Shapiro 1979; Schlichtkrull 1983; Mœglin and Waldspurger 1989].

Example 4. Let $G = \mathrm{SL}(2n, \mathbb{R})$, We take H as the connected component of $S\big(\mathrm{GL}(p, \mathbb{R}) \times \mathrm{GL}(q, \mathbb{R})\big)$. Then $H_{ss} = \mathrm{SL}(p, \mathbb{R}) \times \mathrm{SL}(q, \mathbb{R})$ where $p + q = 2n$. and

$$\operatorname{rank} G/H = \operatorname{rank} K/K \cap H = \min(p, q).$$

The results of van den Ban and Schlichtkrull show that all the representations in the continuous spectrum are unitarily induced from $\theta\tau$-stable parabolic subgroups. It is easy to see that these parabolic subgroups are all cuspidal and thus the representations in the discrete spectrum determine whether $L^2(\mathrm{SL}(n, \mathbb{R})/H)$ is tempered.

We recall the parametrization of the representations in the discrete spectrum. Using the decomposition $\mathfrak{h}\otimes\mathbb{C}\oplus\mathfrak{q}\otimes\mathbb{C}$ of $\mathfrak{gl}(2n,\mathbb{C})$ we conclude that the skew diagonal matrices in $\mathfrak{q}\otimes\mathbb{C}$ are a maximal abelian subspace of $\mathfrak{so}(2n,\mathbb{C})\cap\mathfrak{q}\otimes\mathbb{C}$. By [Ōshima and Matsuki 1984] their centralizer L is the Levi subgroup of a θ stable parabolic subgroup. The representations in the discrete spectrum are cohomologically induced from a character of the subgroup L. If the commutator subgroup L does not contain a noncompact semisimple subgroup then the representations are tempered. (For this, see [Knapp and Vogan 1995, Chapter XI] or [Vogan and Zuckerman 1984, Theorem 6.16]). Thus we conclude:

- If $p=q=n$ the subgroup $[L,L]$ is a product of n compact tori. Hence all representation $L^2(\mathrm{SL}(2n,\mathbb{R})/(\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R}))$ in the discrete spectrum are tempered and thus $L^2(\mathrm{SL}(2n,\mathbb{R})/H_{ss})$ is tempered.
- If $p-q\geq 2$ then L has a noncompact subgroup and hence the representations in the discrete spectrum of $L^2(G/H_{ss})$ are the Langlands subquotient of representations which is not unitarily induced. Hence
$$L^2(\mathrm{SL}(2n,\mathbb{R})/(\mathrm{SL}(n,\mathbb{R})\times\mathrm{SL}(n,\mathbb{R}))$$
is not tempered.
- Using Remark 2 we can construct a large number of additional semisimple subgroups H_{ss} so that $L^2(\mathrm{SL}(2n,\mathbb{R})/H_{ss})$ is tempered.

C. Mœglin and J. L. Waldspurger [1989] show that these representations are in the residual spectrum of a congruence subgroup of $\mathrm{GL}(n,\mathbb{R})$. Similar considerations for general linear groups can be found in [Venkatesh 2005].

Example 5. $G=\mathrm{SO}(p,q)$, $p+q=2n\geq 4$ with $p\geq q>2$ and
$$H=\mathrm{SO}(1,1)\times\mathrm{SO}(p-1,q-1)\quad\text{and}\quad H_{ss}=\mathrm{SO}(p-1,q-1).$$

Claim. $L^2\big(\mathrm{SO}(p,q)/\,\mathrm{SO}(p-1,q-1)\big)$ *is not tempered.*

We have
$$\operatorname{rank} G/H=\operatorname{rank} K/K\cap H=2.$$
We argue as in Example 2. The group $[L,L]$ has a factor isomorphic to
$$\mathrm{SO}(p-2,q-2),$$
and is hence is not compact. So there are nontempered representations in the discrete spectrum.

T. Kobayashi [1992] considered the case G/H_0 where $H=H_c\times H_0$. Here H_c is a compact orthogonal group and H_0 is a noncompact orthogonal group. He determined the parameter of the representations in the discrete spectrum of $L^2(G/H_0)$ and their multiplicities.

Example 6. $G = \mathrm{Sp}(n, \mathbb{R})$, $H = \mathrm{GL}(n, \mathbb{R})$, $H_{ss} = \mathrm{SL}(n, \mathbb{R})$.

Claim. $L^2(\mathrm{Sp}(n, \mathbb{R})/\,\mathrm{SL}(n, \mathbb{R}))$ *is tempered.*

The proof proceeds as follows:

Step 1: All the representations in the discrete spectrum are tempered.

Step 2: Each conjugacy class of parabolic subgroups contains a $\theta\tau$-invariant parabolic subgroup MAN.

Step 3: All discrete series representations of $M/M \cap H$ are tempered.

For simplicity assume that the symplectic group is defined by the quadratic form defined by the matrix

$$\begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix},$$

where I is the identity matrix. The subgroup $H = \mathrm{GL}(n, \mathbb{R})$ of G is the fixed point set of the automorphism τ defined by conjugation with

$$\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

The maximal compact subgroup K_H of H is $K \cap H = O(n)$. Furthermore

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p},$$
$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}.$$

The one-dimensional torus T_0 in the center of K also defines a torus on $K/H \cap K$. Its Lie algebra $\mathfrak{t}_0$ is direct summand of the maximal abelian subalgebra $\mathfrak{a}_k$ of $\mathfrak{q}_k = \mathfrak{k} \cap \mathfrak{q}$. Since T_0 defines the complex structure on the symmetric space G/K the centralizer of $\mathfrak{a}_k$ in G is contained in K. Thus every representation in the discrete spectrum is tempered.

The $\theta\tau$-stable parabolic subalgebras are determined by maximal abelian subspaces $\mathfrak{i}$ in $\mathfrak{p} \cap \mathfrak{q}$. Now

$$\tilde{\mathfrak{h}} := \mathfrak{k} \cap \mathfrak{h} \oplus \mathfrak{p} \cap \mathfrak{q} = \mathfrak{gl}(n, \mathbb{R})$$

is the fixed point set of the involution $\theta\tau$ since the fixed point set of $\theta\tau$ is conjugate in $\mathrm{GL}(2n, \mathbb{R})$ to $\mathrm{GL}(n, \mathbb{R})$. This implies that there is a n-dimensional abelian split subalgebra $\tilde{\mathfrak{a}}_H$ in $p \cap q$ consisting of the matrices

$$\begin{pmatrix} 0 & D \\ D & 0 \end{pmatrix},$$

where D is a real diagonal matrix. Hence every conjugacy classes of parabolic subgroups contains a $\theta\tau$-stable parabolic $P_s = M_s A_s N_s$ whose Levi subgroup is a centralizer of $\tilde{\mathfrak{a}}_H$.

Next we have to determine $M_s \cap H$. Note that M_s is a product of general linear groups and a symplectic group. The factors isomorphic to general linear groups are subgroups of $\tilde{H}$. Since $\tilde{H} \cap H = K \cap H$ is an orthogonal group, the intersection of the general linear subgroups of M_s with H are orthogonal groups and hence the corresponding symmetric space has no discrete spectrum. Thus we may assume that $M_s = \mathrm{Sp}(m, \mathbb{R})$ with $m < n$. In this case $\theta\tau$ is an involution of M_s with fixed points $\tilde{H} \cap M_s = \mathrm{GL}(m, \mathbb{R})$. Furthermore since θ and τ commute their restriction to M_s also defines an automorphisms of M_s. So the fixed point set of $\theta\tau_{|M_s}$ is conjugate to the fixpoint set of $\tau_{|M_s}$ in $\mathrm{GL}(2n, \mathbb{R})$. Hence we conclude that $M_s \cap H$ is isomorphic to $\mathrm{GL}(m, \mathbb{R})$. By Step 1 the representations in the discrete spectrum of $\mathrm{Sp}(m, \mathbb{R})/\mathrm{GL}(m, \mathbb{R})$ are tempered and thus by [van den Ban and Schlichtkrull 2005] all the representations in the continuous spectrum of $\mathrm{Sp}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{R})$ are tempered.

Example 7. Cayley-type spaces are considered in [Ólafsson and Ørsted 1999; Faraut and Korányi 1994]. These are

(a) $G = \mathrm{Sp}(n, \mathbb{R})$, $H = \mathrm{GL}(n, R)$ and $H_{ss} = \mathrm{SL}(n, \mathbb{R})$, $n > 1$;

(b) $G = \mathrm{SO}(2, n)$, $H = \mathrm{SO}(1, 1)\,\mathrm{SO}(1, n-1)$ and $H_{ss} = \mathrm{SO}(1, n)$, $n > 2$;

(c) $G = \mathrm{SU}(n, n)$, $H = \mathrm{SL}(n, C)\mathbb{R}^+$ and $H_{ss} = \mathrm{SL}(n, \mathbb{C})$;

(d) $G = O^*(2n)$, $H = \mathbb{R}^+\,\mathrm{SU}^*(2n)$ and $H_{ss} = \mathrm{SU}^*(2n)$;

(e) $G = E_{7(-25)}$, $H = E_{6(-26)}\mathbb{R}^+$ and $H_{ss} = E_{6(-26)}$.

Claims.

- *In Example 7*(b)–(d) *with n large enough $L^2(G/H_{ss})$ is not tempered.*
- *In Example 7*(a) $L^2(G/H_{ss})$ *is tempered.*
- *We expect that in Example 7(e) $L^2(G/H_{ss})$ is tempered.*

Proof. The proof is based on case by case considerations of the spectrum of $L^2(G/H)$. Ólafsson and Ørsted [1999] proved that all these spaces are of equal rank and hence $L^2(G/H)$ has a discrete spectrum.

Case (b). The arguments in Example 5 show that the representations in the discrete spectrum of $L^2(\mathrm{SO}(n, 2)/\mathrm{SO}(n-1, 1))$ are tempered if and only if $n \le 2$. So we can conclude that $L^2(\mathrm{SO}(n, 2)/\mathrm{SO}(n-1, 1))$ is not tempered if $3 \le n$.

Case (c). It was proved in [Ólafsson and Ørsted 1988] that the discrete spectrum for $\mathrm{SU}(n, n)/H$ contains some nontempered highest weight representations. Hence $L^2(\mathrm{SU}(n, n)/\mathrm{SL}(n, \mathbb{C}))$ is not tempered.

Case (a). This was proved in Example 6.

Case (d). We have $\mathrm{rank}(G/H) = n$. The Levi of the θ-stable parabolic subgroup also contains a subgroup of type A_{2n-1}. Since it is not the maximal compact subgroup, L has a noncompact subgroup. This implies that the discrete spectrum of $L^2(O^*(2n)/\operatorname{SU}^*(2n))$ is not tempered.

Case (e). We only prove that the discrete spectrum is tempered. The arguments are the same as in Example 4. Recall that

(1) the rank of G/H is 3;

(2) the maximal compact subgroup K of G is $E_6\operatorname{SO}(2)$;

(3) the maximal compact subgroup K_H of H is F_4;

(4) $K/H\cap K$ has a one-dimensional compact torus T_0 as factor.

The centralizer of this torus T_0 is K. Its Lie algebra is a direct summand of the maximal abelian subalgebra $\mathfrak{a}_k$ of $\mathfrak{q}_k=\mathfrak{k}\cap\mathfrak{q}$. Since T_0 defines the complex structure on the symmetric space G/K the centralizer of $\mathfrak{a}_k$ in G is contained in K. Thus every representation in the discrete spectrum is tempered.

As in Example 6 we conclude that the fixed point set of $\theta\tau$ is a subgroup isomorphic to $H=E_{6(-26)}\mathbb{R}^+$, which has real rank 3. Hence every conjugacy classes of parabolic subgroups contains a $\theta\tau$ -stable parabolic $P_s=M_sA_sN_s$ whose Levi subgroup is a centralizer of $\tilde{\mathfrak{a}}_H$. □

Example 8. $G=\operatorname{SL}(2n,\mathbb{C})$ and H_{ss} has a covering $T^1\operatorname{SL}(p,\mathbb{C})\times\operatorname{SL}(q,\mathbb{C})$, p+q = 2n for a one dimensional torus T^1. Then

$$L^2(\operatorname{SL}(n,\mathbb{C})/\operatorname{SL}(p,\mathbb{C})\times\operatorname{SL}(q,\mathbb{C}))=\oplus_{\delta\in\hat{T}}L^2(\operatorname{SL}(n,\mathbb{C})/H_{ss},\delta),$$

where $L^2(\operatorname{SL}(n,\mathbb{C})/H_{ss},\delta)$ are the L^2-sections of the line bundle defined by the character δ of H_{ss}. As in Example 2 we are in the equal rank case.

The same arguments as in Example 4 show:

- If $p=q=n$ the subgroup $[L,L]$ is compact. Hence all representations in the discrete spectrum of $L^2(\operatorname{SL}(2n,\mathbb{C})/H)$ are tempered, which implies that $L^2(\operatorname{SL}(2n,\mathbb{C})/H_{ss})$ is tempered.
- If $p-q\geq 2$ then $[L,L]$ is not compact and hence the representations in the discrete spectrum of $L^2(G/H_{ss})$ are the Langlands subquotients of representations which are not unitarily induced. Hence $L^2(\operatorname{SL}(2n,\mathbb{C})/\operatorname{SL}(n,\mathbb{C})\times\operatorname{SL}(n,\mathbb{C}))$ is not tempered.

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MULTIPLICATIVE REDUCTION AND THE CYCLOTOMIC MAIN CONJECTURE FOR GL_2

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We show that the cyclotomic Iwasawa–Greenberg main conjecture holds for a large class of modular forms with multiplicative reduction at p, extending previous results for the good ordinary case. In fact, the multiplicative case is deduced from the good case through the use of Hida families and a simple Fitting ideal argument.

1. Introduction

The cyclotomic Iwasawa–Greenberg main conjecture was established in [Skinner and Urban 2014], in combination with work of Kato [2004], for a large class of newforms $f \in S_k(\Gamma_0(N))$ that are ordinary at an odd prime $p \nmid N$, subject to $k \equiv 2 \pmod{p-1}$ and certain conditions on the mod p Galois representation associated with f. The purpose of this note is to extend this result to the case where $p \mid N$ (in which case k is necessarily equal to 2).

Recall that the coefficients a_n of the q-expansion $f = \sum_{n=1}^{\infty} a_n q^n$ of f at the cusp at infinity (equivalently, the Hecke eigenvalues of f) are algebraic integers that generate a finite extension $\mathbb{Q}(f) \subset \mathbb{C}$ of $\mathbb{Q}$. Let p be an odd prime and let L be a finite extension of the completion of $\mathbb{Q}(f)$ at a chosen prime above p (equivalently, let L be a finite extension of $\mathbb{Q}_p$ in a fixed algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$ that contains the image of a chosen embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$). Suppose that f is ordinary at p with respect to L in the sense that a_p is a unit in the ring of integers $\mathcal{O}$ of L. Then the p-adic L-function $\mathcal{L}_f$ of f is an element of the Iwasawa algebra $\Lambda_{\mathcal{O}} = \mathcal{O}[\![\Gamma]\!]$, where $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q})$ is the Galois group of the cyclotomic $\mathbb{Z}_p$-extension $\mathbb{Q}_\infty$ of $\mathbb{Q}$. A defining property of $\mathcal{L}_f$ is that it interpolates normalized special values of the L-function of f twisted by Dirichlet characters associated with finite-order characters of Γ. The Iwasawa–Greenberg Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty,L}(f)$, defined with respect to the p-adic Galois representation V_f of f over L — a two-dimensional L-vector space — and a Galois-stable $\mathcal{O}$-lattice $T_f \subset V_f$, is a discrete, cofinite $\Lambda_{\mathcal{O}}$-module, and the Iwasawa–Greenberg characteristic ideal $\mathrm{Ch}_L(f) \subset \Lambda_{\mathcal{O}}$ is the characteristic $\Lambda_{\mathcal{O}}$-ideal of the Pontryagin dual $X_{\mathbb{Q}_\infty,L}(f)$ of $\mathrm{Sel}_{\mathbb{Q}_\infty,L}(f)$. The

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Iwasawa–Greenberg main conjecture for f then asserts that there is an equality of ideals $\mathrm{Ch}_L(f) = (\mathcal{L}_f)$ in $\Lambda_{\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and even in $\Lambda_{\mathcal{O}}$ if T_f is residually irreducible.

Theorem A. *Let $p \geq 3$ be a prime. Let $f \in S_k(\Gamma_0(N))$ be a newform and let L and $\mathcal{O}$ be as above and suppose f is ordinary at p with respect to L. If*

(i) $k \equiv 2 \pmod{p-1}$;

(ii) *the reduction $\bar{\rho}_f$ of the representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathcal{O}}(T_f)$ modulo the maximal ideal of $\mathcal{O}$ is irreducible*;

(iii) *there exists a prime $q \neq p$ such that $q \parallel N$ and $\bar{\rho}_f$ is ramified at q*,

then $\mathrm{Ch}_L(f) = (\mathcal{L}_f)$ in $\Lambda_{\mathcal{O}}$. That is, the Iwasawa–Greenberg main conjecture is true.

When $p \nmid N$ this is just Theorem 1 of [Skinner and Urban 2014][1]. When $p \mid N$, in which case the ordinary hypothesis forces $p \parallel N$ and $k = 2$, this is not an immediate consequence of the results in [Skinner and Urban 2014], as this case is excluded from Kato's divisibility theorem [2004, Theorem 17.4], which is a crucial ingredient in the deduction of the main conjecture from the main results in [Skinner and Urban 2014]. However, as we explain in this note, the main conjecture in the case $p \mid N$ can be deduced from knowing it when $p \nmid N$.

Having the cyclotomic main conjecture in hand, one obtains results toward special value formulas. For example:

Theorem B. *Let $p \geq 3$ be a prime. Let $f \in S_2(\Gamma_0(N))$ be a newform and let L and $\mathcal{O}$ be as above and suppose f is ordinary. Suppose also that*

(i) *the reduction $\bar{\rho}_f$ of the representation $\rho_f : \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathcal{O}}(T_f)$ modulo the maximal ideal of $\mathcal{O}$ is irreducible*;

(ii) *there exists a prime $q \neq p$ such that $q \parallel N$ and $\bar{\rho}_f$ is ramified at q*;

(iii) *if $p \mid N$ and $a_p = 1$, then the $\mathfrak{L}$-invariant $\mathfrak{L}(V_f) \in L$ is nonzero.*

Let

$$L^{\mathrm{alg}}(f, 1) = \frac{L(f, 1)}{-2\pi i \Omega_f^+}.$$

Then

$$\#\mathcal{O}/(L^{\mathrm{alg}}(f, 1)) = \#\mathrm{Sel}_L(f) \cdot \prod_\ell c_\ell(T_f).$$

In particular, if $L(f, 1) = 0$, then $\mathrm{Sel}_L(f)$ has $\mathcal{O}$-corank at least one.

[1] In order to conclude that the equality holds in $\Lambda_{\mathcal{O}}$ and not just $\Lambda_{\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, Theorem 1 in [Skinner and Urban 2014] requires that ρ_f have an $\mathcal{O}$-basis with respect to which the image contains $\mathrm{SL}_2(\mathbb{Z}_p)$. But as we explain in Section 2.5, hypotheses (ii) and (iii) of Theorem A are enough for the arguments. We also explain that the reference to [Vatsal 2003] in [Skinner and Urban 2014] should have been augmented with a reference to [Chida and Hsieh 2016].

Here Ω_f^+ is one of two canonical periods associated with f as in [Skinner and Urban 2014, §3.3.3] (and well-defined up to an element of $\mathcal{O}^\times \cap \mathbb{Q}(f)$), $\mathrm{Sel}_L(f)$ is the Selmer group associated by Bloch–Kato to the Galois lattice T_f, $c_\ell(T_f)$ is the Tamagawa factor at ℓ of T_f (and equals 1 unless $\ell \mid N$), and $\mathfrak{L}(V_f)$ is the $\mathcal{L}$-invariant of a modular form f (or of V_f) with split multiplicative reduction at p introduced by Mazur, Tate, and Teitelbaum [1986] (see also [Greenberg and Stevens 1993, §3]). It is conjectured that $\mathfrak{L}(V_f)$ is always nonzero; this is known if f is the modular form associated to an elliptic curve, but in general it is an open question.

As a special case of Theorem B, obtained by taking f to be the newform associated with an elliptic curve E over $\mathbb{Q}$, we have:

Theorem C. *Let E be an elliptic curve over $\mathbb{Q}$ with good ordinary or multiplicative reduction at a prime $p \geq 3$. Suppose that*

(i) *$E[p]$ is an irreducible $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation;*

(ii) *there exists a prime $q \neq p$ at which E has multiplicative reduction and $E[p]$ is ramified.*

If $L(E, 1) \neq 0$ then

$$\mathrm{ord}_p\left(\frac{L(E,1)}{\Omega_E}\right) = \mathrm{ord}_p\left(\#\text{Ш}(E)\prod_\ell c_\ell(E)\right),$$

and if $L(E, 1) = 0$ then $\mathrm{Sel}_{p^\infty}(E)$ has $\mathbb{Z}_p$-corank at least one.

Here, Ω_E is the Néron period of E, Ш(E) is the Tate–Shafarevich group of $E/\mathbb{Q}$, and the $c_\ell(E)$ are the Tamagawa numbers of E. In particular, $c_\ell(E)$ is the order of the group of irreducible components of the special fiber of the Néron model of E over $\mathbb{Z}_\ell$.

Our proof of Theorem A is relatively simple. Let $N = pM$. We first make two reductions: (1) it suffices to prove the theorem with the field L replaced by any finite extension, and (2) it suffices to prove the equality $\mathrm{Ch}_L^\Sigma(f) = (\mathcal{L}_f^\Sigma)$, where Σ is any finite set of primes containing all $\ell \mid N$, $\mathcal{L}_f^\Sigma$ is the incomplete p-adic L-function with the Euler factors at primes in Σ different from p removed, and $\mathrm{Ch}_L^\Sigma(f)$ is the characteristic ideal of the Pontryagin dual $X_{\mathbb{Q}_\infty,L}^\Sigma(f)$ of the Iwasawa–Greenberg Selmer group $\mathrm{Sel}_{\mathbb{Q}_\infty,L}^\Sigma(f)$ with all conditions at primes in Σ different from p relaxed. Then we exploit Hida theory to deduce that one can choose L so that for each integer $m > 0$ there exists a newform $f_m \in S_{k_m}(\Gamma_0(M))$ with $k_m \equiv k \pmod{(p-1)p^m}$, $\mathbb{Q}(f_m) \subset L$ and f_m ordinary at p with respect to L, and the ordinary p-stabilization f_m^* of f_m satisfies $f_m^* \equiv f \pmod{p^m}$ in the sense that the q-expansions (which have coefficients in $\mathcal{O}$) are congruent modulo p^m. Furthermore, as a consequence of the existence of the "two-variable" p-adic L-function associated to a Hida family we also have $\mathcal{L}_{f_m}^\Sigma \equiv \mathcal{L}_f^\Sigma \pmod{p^m\Lambda_\mathcal{O}}$. Kato

[2004] has proved that $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f_m)$ is a torsion $\Lambda_{\mathcal{O}}$-module, and an argument of Greenberg then shows that it has no nonzero finite-order $\Lambda_{\mathcal{O}}$-submodules. From this it follows that $\mathrm{Ch}^{\Sigma}_L(f_m)$ equals the $\Lambda_{\mathcal{O}}$-Fitting ideal $F^{\Sigma}_L(f_m)$ of $X^{\Sigma}_L(f_m)$. The congruence $f^*_m \equiv f \pmod{p^m}$ implies that $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f)[p^m] \cong \mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f_m)[p^m]$, so comparing Fitting ideals yields

$$(F^{\Sigma}_L(f), p^m) = (F^{\Sigma}_L(f_m), p^m) = (\mathrm{Ch}^{\Sigma}_L(f_m), p^m) \subset \Lambda_{\mathcal{O}}.$$

From the main conjecture for f_m (the congruence $f^*_m \equiv f \pmod{p}$ ensures that the hypotheses of Theorem A also hold for f_m) and the congruence modulo p^m of p-adic L-functions we then have

$$(F^{\Sigma}_L(f), p^m) = (\mathrm{Ch}^{\Sigma}_L(f_m), p^m) = (\mathcal{L}^{\Sigma}_{f_m}, p^m) = (\mathcal{L}^{\Sigma}_f, p^m) \subset \Lambda_{\mathcal{O}}$$

for all integers $m > 0$. This, together with the nonvanishing of the p-adic L-function $\mathcal{L}^{\Sigma}_f$, implies that $F^{\Sigma}_L(f) \neq 0$ and hence that $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module. Then $\mathrm{Ch}^{\Sigma}_L(f) = F^{\Sigma}_L(f)$, by the earlier argument of Greenberg, and so $(\mathrm{Ch}^{\Sigma}_L(f), p^m) = (\mathcal{L}^{\Sigma}(f), p^m) \subset \Lambda_{\mathcal{O}}$ for all $m > 0$. As $\mathrm{Ch}^{\Sigma}_L(f) \subset \Lambda_{\mathcal{O}}$ is a principal ideal, it then easily follows that $\mathrm{Ch}^{\Sigma}_L(f) = (\mathcal{L}^{\Sigma}_f)$, proving Theorem A.

If the analytic or algebraic μ-invariant for some f_m (the power of the uniformizer of L dividing $\mathcal{L}^{\Sigma}_{f_m}$ or $\mathrm{Ch}^{\Sigma}_L(f_m)$) — or even for some other ordinary eigenform suitably congruent to f for which the main conjecture holds — were known to be zero, then Theorem A would follow from the main results of [Emerton et al. 2006]. However, presently little is known about the vanishing of these μ-invariants.

Theorem B is deduced from Theorem A via an argument of Greenberg [1999].

In addition to extending the main conjecture to the case of multiplicative reduction, our motivation for writing this note was in part to provide an explicit reference for the expression for the special value $L^{\mathrm{alg}}(f, 1)$ in terms of the size of Selmer groups that is required for the arguments in [Zhang 2014] and, by including the multiplicative reduction case, also to provide an important ingredient for the extension of the main results of [Zhang 2014] to cases of multiplicative reduction. Additional motivation for the latter stems from the author's collaboration with Manjul Bhargava and Wei Zhang to provide lower bounds on the proportion of elliptic curves that satisfy the rank part of the Birch–Swinnerton-Dyer conjecture.

While preparing this note the author learned of Olivier Fouquet's [2014] work on the equivariant Tamagawa number conjecture for motives of modular forms. That work should provide another means for deducing Theorem B in the case $p \mid N$ from the main results[2] in [Skinner and Urban 2014] as well as some additional weakening of the conditions on primes away from p. The deduction of Theorem A

[2]But see also note 1, especially as the main results in [Fouquet 2014] rely on Theorem A as stated, at least for the $p \nmid N$ case.

for $p \mid N$ in this paper uses no more machinery than already developed in [Skinner and Urban 2014] or than is required for our deduction of Theorem B.

2. Gathering the pieces

In this section we recall the various objects that go into the Iwasawa–Greenberg main conjecture for modular forms, some of their properties, and some useful relations. Throughout p is a fixed odd prime.

Let $\overline{\mathbb{Q}} \subset \mathbb{C}$ be the algebraic closure of $\mathbb{Q}$ and let $G_{\mathbb{Q}} = \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime ℓ, let $\overline{\mathbb{Q}}_\ell$ be a fixed algebraic closure of $\overline{\mathbb{Q}}_\ell$. For each ℓ we also fix an embedding $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_\ell$, which identifies $G_{\mathbb{Q}_\ell} = \mathrm{Gal}(\overline{\mathbb{Q}}_\ell/\overline{\mathbb{Q}})$ with a decomposition subgroup in $G_{\mathbb{Q}}$; let $I_\ell \subset G_{\mathbb{Q}_\ell}$ be the inertia subgroup. Let $\mathrm{frob}_\ell \in G_{\mathbb{Q}_\ell}$ be (a lift of) an arithmetic Frobenius element.

Let $\epsilon : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ be the p-adic cyclotomic character. This is just the projection to $\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})$, the latter being canonically isomorphic to $\mathbb{Z}_p^\times$. Similarly, let $\omega : G_{\mathbb{Q}} \to \mathbb{Z}_p^\times$ be the mod p Teichmüller character. This is just the composition of the reduction of ϵ mod p and the multiplicative homomorphism $(\mathbb{Z}/p\mathbb{Z})^\times \hookrightarrow \mathbb{Z}_p^\times$ defined by the Teichmüller lifts.

Let $\mathbb{Q}_\infty \subset \mathbb{Q}[\mu^{p^\infty}] \subset \overline{\mathbb{Q}}$ be the cyclotomic $\mathbb{Z}_p$-extension of $\mathbb{Q}$. That is, $\mathbb{Q}_\infty$ is the unique abelian extension of $\mathbb{Q}$ such that $\Gamma = \mathrm{Gal}(\mathbb{Q}_\infty/\mathbb{Q}) \cong \mathbb{Z}_p$. Let $\gamma \in \Gamma$ be a fixed topological generator. As $\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q}) \xrightarrow{\sim} \mathrm{Gal}(\mathbb{Q}[\mu_p]/\mathbb{Q}) \times \Gamma$, there is a lift $\tilde{\gamma}$ of γ to $\mathrm{Gal}(\mathbb{Q}[\mu_{p^\infty}]/\mathbb{Q})$ identified with $(1, \gamma)$, and we let $u = \epsilon(\tilde{\gamma}) \in \mathbb{Z}_p^\times$.

2.1. ***Galois representations and (ordinary) newforms.*** Let $f \in S_k(\Gamma_0(N))$ be a newform. Let $\mathbb{Q}(f) \subset \mathbb{C}$ be the finite extension of $\mathbb{Q}$ generated by the Fourier coefficients $a_n(f)$ of the q-expansion $f = \sum_{n=0}^\infty a_n(f)q^n$ of f at the cusp at infinity (equivalently, the field obtained by adjoining the eigenvalues of the action of the usual Hecke operators on f). Fix an embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$ and let $L \subset \overline{\mathbb{Q}}_p$ be a finite extension of $\mathbb{Q}_p$ containing the image of $\mathbb{Q}(f)$. Let $\mathcal{O}$ be the ring of integers of L (the valuation ring), let $\mathfrak{m}$ be its maximal ideal, and let $\kappa = \mathcal{O}/\mathfrak{m}$ be its residue field.

Associated with f and L (and the embedding $\mathbb{Q}(f) \hookrightarrow L$) is a two-dimensional L-space V_f and an absolutely irreducible continuous $G_{\mathbb{Q}}$-representation $\rho_f : G_{\mathbb{Q}} \to \mathrm{Aut}_L(V_f)$ such that ρ_f is unramified at all primes $\ell \nmid Np$ and $\det(1 - X \cdot \rho_f(\mathrm{frob}_\ell)) = 1 - a_\ell(f)X + \ell^{k-1}X^2$ for such ℓ. In particular, trace $\rho_f(\mathrm{frob}_\ell) = a_\ell(f)$ if $\ell \nmid pN$, and $\det \rho_f = \epsilon^{k-1}$.

Let $T, T' \subset V_f$ be two $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattices. Let $\bar{\rho}$ and $\bar{\rho}'$ denote, respectively, the two-dimensional κ-representations $T/\mathfrak{m}T$ and $T'/\mathfrak{m}T'$. The following lemma is well known, but we include it for later reference.

Lemma 2.1.1. (a) *If $\bar{\rho}$ or $\bar{\rho}'$ is irreducible, then $\bar{\rho}$ and $\bar{\rho}'$ are equivalent as κ-representations. In particular, $\bar{\rho}$ is irreducible if and only if $\bar{\rho}'$ is irreducible.*

(b) *If $\bar{\rho}$ or $\bar{\rho}'$ is irreducible, then there exists $a \in L^\times$ such that $T = aT'$.*

Proof. Replacing T' with some $\mathcal{O}$-multiple, we may assume that T' is a sublattice of T. Then $T/T' \cong \mathcal{O}/\mathfrak{m}^n \times \mathcal{O}/\mathfrak{m}^m$ with $n \le m$. Let ϖ be a uniformizer of $\mathcal{O}$ (a generator of $\mathfrak{m}$). Then $\varpi^n T/(T' + \varpi^{n+1}T) \cong \mathcal{O}/\mathfrak{m}^{\min(1,m-n)}$ is a $G_{\mathbb{Q}}$-stable quotient of $T/\mathfrak{m}T \cong \varpi^n T/\varpi^{n+1}T$ of at most one-dimension over k. If $\bar{\rho}$ is irreducible, then this quotient must be trivial and so $m - n = 0$ and $T' = \varpi^n T$, in which case $T'/\mathfrak{m}T' \cong \varpi^n T/\varpi^{n+1}T \cong T/\mathfrak{m}T$ as $G_{\mathbb{Q}}$-representations over κ. Reversing the roles of T and T' in this argument then yields the lemma. □

We then define $\bar{\rho}_f$ to be the κ-representation $T/\mathfrak{m}T$ of $G_{\mathbb{Q}}$ for a Galois-stable $\mathcal{O}$-lattice $T \subset V_f$. By the above lemma, if $\bar{\rho}_f$ is irreducible for some choice of T, then it is irreducible for any choice of T, and the equivalence class of $\bar{\rho}_f$ is independent of T. Of course, it is not difficult to show that the semisimplification of $\bar{\rho}_f$ is independent of T even when $\bar{\rho}_f$ is not irreducible, but will not need this.

Suppose $k \ge 2$ and f is ordinary with respect to the embedding $\mathbb{Q}(f) \hookrightarrow L$. That is, $a_p(f) \in \mathcal{O}^\times$. As proved in general by Wiles [1988, Theorem 2.2.2], in this case V_f has a unique $G_{\mathbb{Q}_p}$-stable L-line $V_f^+ \subset V_f$ such that $G_{\mathbb{Q}_p}$ acts on V_f^+ via the character $\alpha_f^{-1}\epsilon^{k-1}$, where $\alpha_f : G_{\mathbb{Q}_p} \to \mathcal{O}^\times$ is the unique unramified character such that $\alpha_f(\mathrm{frob}_p)$ equals the (unit) root α_p in $\mathcal{O}^\times$ of the polynomial $x^2 - a_p(f)x + p^{k-1}$ if $p \nmid N$ and $\alpha_f(\mathrm{frob}_p) = a_p(f)$ if $p \mid N$. (Note that the reduction of the polynomial $x^2 - a_p(f)x + p^{k-1}$ modulo $\mathfrak{m}$ is $x(x - \bar{a}_p(f))$ and so, by Hensel's lemma, $\bar{a}_p(f)$ lifts to a root in $\mathcal{O}^\times$.) The action of $G_{\mathbb{Q}_p}$ on the quotient $V_f^- = V_f/V_f^+$ is via α_f. Given any $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice $T \subset V_f$ we let $T^+ = T \cap V_f^+$ and $T^- = T/T^+$. Then T^+ is the unique $G_{\mathbb{Q}_p}$-stable free $\mathcal{O}$-summand of rank one on which $G_{\mathbb{Q}_p}$ acts via $\alpha_f^{-1}\epsilon^{k-1}$, and T^- is the unique $G_{\mathbb{Q}_p}$-stable free $\mathcal{O}$-module quotient of rank one on which $G_{\mathbb{Q}_p}$ acts via α_f.

The following lemma is also well known, but we also include it for completeness.

Lemma 2.1.2. *Suppose $a_p(f) \in \mathcal{O}^\times$. If $p \mid N$, then $p \parallel N$, $k = 2$, and $a_p(f) = \pm 1$.*

Proof. If $f \in S_k(\Gamma_0(N))$ is a newform with trivial Nebentypus such that $p \mid N$, then $a_p(f) \neq 0$ if and only if $p \parallel N$, in which case $a_p(f)^2 = p^{k-2}$ (see [Miyake 1989, Theorem 4.6.17]). If $a_p(f) \in \mathcal{O}^\times$, then it follows that $k = 2$ and $a_p(f)^2 = 1$, so $a_p(f) = \pm 1$. □

Note that if f is a newform with $p \mid N$ that is ordinary with respect to some embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$, then, since $a_p(f) = \pm 1$ by the lemma, it is ordinary with respect to all such embeddings. Also, as noted in the proof of the lemma, if $f \in S_2(\Gamma_0(N))$ is a newform with $p \parallel N$ then $a_p(f) = \pm 1$ and so f is ordinary with respect to any embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$.

In keeping with the terminology for elliptic curves, we say that a newform $f \in S_2(\Gamma_0(N))$ has *multiplicative reduction at p* if $p \parallel N$ and that it has *good reduction* *at p* if $p \nmid N$. Additionally, we say f has *split* (resp. *nonsplit*) multiplicative reduction at p if $p \parallel N$ and $a_p(f) = 1$ (resp. $a_p = -1$).

2.2. $\mathfrak{L}$-*invariants*. Suppose $f \in S_2(\Gamma_0(N))$ is a newform with split multiplicative reduction at p. The Galois representation V_f restricted to $G_{\mathbb{Q}_p}$ is an extension

$$0 \to V_f^+ \cong L(1) \to V_f \to V_f^- \cong L \to 0.$$

This extension is known to be nonsplit and semistable but not crystalline. This follows for example from the main result[3] of [Saito 1997]. Let $\pi_{V_f} : H^1(\mathbb{Q}_p, V_f) \to H^1(\mathbb{Q}_p, L)$ be the induced map on cohomology. As the extension is nonsplit, the image of π_{V_f} is a one-dimensional L-space. As explained in [Greenberg and Stevens 1993, §3], the $\mathfrak{L}$-invariant $\mathfrak{L}(V_f)$ of V_f is the negative of the "slope" of the line $\mathrm{im}(\pi_{V_f})$ with respect to a particular basis of the two-dimensional L-space $H^1(\mathbb{Q}_p, L)$.

We have

$$H^1(\mathbb{Q}_p, L) = \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}_p}, L) = \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}_p}^{\mathrm{ab},p}, L),$$

where $G_{\mathbb{Q}_p}^{\mathrm{ab},p}$ is the maximal abelian pro-p quotient of $G_{\mathbb{Q}_p}$. Local class field theory gives an identification[4]

$$\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n} \xrightarrow{\sim} G^{\mathrm{ab},p}.$$

From the decomposition $\mathbb{Q}_p^\times = p^{\mathbb{Z}} \times \mathbb{Z}_p^\times$ we obtain an L-basis $\{\psi_{\mathrm{ur}}, \psi_{\mathrm{cyc}}\}$ of $H^1(\mathbb{Q}_p, L) = \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}_p}^{\mathrm{ab},p}, L)$, with

$$\psi_{\mathrm{ur}}(p) = 1 = (\log_p u)^{-1} \cdot \psi_{\mathrm{cyc}}(u) \quad \text{and} \quad \psi_{\mathrm{ur}}(u) = 0 = \psi_{\mathrm{cyc}}(p).$$

Recall that $u = \epsilon(\tilde{\gamma})$ is a topological generator of $1 + p\mathbb{Z}_p$. The condition that V_f is not crystalline is equivalent to $\mathrm{im}(\pi_{V_f}) \not\subset L \cdot \psi_{\mathrm{ur}}$. Let $0 \neq \lambda \in \mathrm{im}(\pi_{V_f})$ and write $\lambda = x \cdot \psi_{\mathrm{cyc}} + y \cdot \psi_{\mathrm{ur}}$. Then $x \neq 0$, and the $\mathfrak{L}$-invariant $\mathfrak{L}(V_f)$ of the extension V_f is defined to be

$$\mathfrak{L}(V_f) = -x^{-1} y \in L.$$

[3] In [Saito 1997] it is proved that the Frobenius semisimplification of the Weil–Deligne representation attached by Fontaine to the dual representation $V_f^\vee$ is just the Weil–Deligne representation attached by the local Langlands correspondence to the p-component π_p of the automorphic representation $\pi = \otimes_v \pi_v$ of $\mathrm{GL}_2(\mathbb{A})$ corresponding to the newform f. If f has split (resp. nonsplit) multiplicative reduction at p, then another way to state Lemma 2.1.2 is that π_p is the special representation (resp. the twist of the special representation by the unramified quadratic character). The local Langlands correspondence attaches to a (twist of a) special representation a Weil–Deligne representation with nontrivial monodromy (in particular, one that is not split).

[4] To be precise, we normalize the reciprocity law so that uniformizers are taken to arithmetic Frobenius elements.

This is independent of the choice of λ.

The nonsplit extension V_f also defines a line $\ell_{V_f} \in H^1(\mathbb{Q}_p, L(1))$ (the image of the boundary map $L = H^0(\mathbb{Q}_p, L) \to H^1(\mathbb{Q}_p, L(1))$. Under the perfect pairing $\langle\cdot,\cdot\rangle : H^1(\mathbb{Q}_p, L) \times H^1(\mathbb{Q}_p, L(1)) \to H^2(\mathbb{Q}_p, L(1)) = L$ of Tate local duality, the lines $\mathrm{im}(\pi_{V_f})$ and ℓ_{V_f} are mutual annihilators. So $\mathfrak{L}(V_f)$ can also be expressed in terms of $\langle\psi_{\mathrm{ur}}, c\rangle$ and $\langle\psi_{\mathrm{cyc}}, c\rangle$ for $0 \neq c \in \ell_{V_f}$.

The Kummer isomorphism yields an identification

$$(\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L \xrightarrow{\sim} H^1(\mathbb{Q}_p, L(1)).$$

Then, together with the above identification of $H^1(\mathbb{Q}_p, L)$, the pairing $\langle\cdot,\cdot\rangle$ of local Tate duality is identified with the usual L-linear pairing

$$\mathrm{Hom}_{\mathbb{Z}_p}\big((\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}), L\big) \times (\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L \to L.$$

So if $0 \neq c \in \ell_{V_f}$, then

$$\mathfrak{L}(V_f) = \psi_{\mathrm{ur}}(c)^{-1}\psi_{\mathrm{cyc}}(c).$$

Let $H^1_f(\mathbb{Q}_p, L(1))$ be the local Bloch–Kato Selmer group [Bloch and Kato 1990, (3.7.2)]. Essentially by definition, $H^1_f(\mathbb{Q}_p, L(1))$ is the subgroup of $H^1(\mathbb{Q}_p, L(1))$ that classifies crystalline extensions of L by $L(1)$ (see [loc. cit., p. 354]). The condition that V_f not be crystalline is therefore equivalent to $\ell_{V_f} \not\subset H^1_f(\mathbb{Q}_p, L(1))$, and so, as $H^1_f(\mathbb{Q}_p, L(1))$ is identified with $(\varprojlim_n \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} L$ (see [loc. cit., Example 3.9]),

$$\psi_{\mathrm{ur}}(c) \neq 0,$$

which explains why the preceding formula for $\mathfrak{L}(V_f)$ is well defined.

Example. Suppose f is associated with an elliptic curve $E/\mathbb{Q}$ with split multiplicative reduction at p and let $q_E \in \mathbb{Q}_p^\times$ be the Tate period of E. Then $V_f = T_pE \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is the $G_{\mathbb{Q}_p}$-extension associated to the image of q_E in $H^1(\mathbb{Q}_p, \mathbb{Q}_p(1))$ under the Kummer map. That is, $\ell_{V_f} = \mathbb{Q}_p \cdot q_E \in (\varprojlim_n \mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$, and so $\mathfrak{L}(V_f) = \log_p q_E/\mathrm{ord}_p(q_E)$. As the j-invariant $j(q_E) = j(E) \in \mathbb{Q}$ of E is algebraic, q_E is transcendental by a theorem of Barré-Sirieix, Diaz, Gramain, and Philibert [1996], and so $\log_p q_E \neq 0$. Therefore, $\mathfrak{L}(V_f) \neq 0$.

2.3. ***Iwasawa–Greenberg Selmer groups.*** Let $f \in S_k(\Gamma_0(N))$ be a newform that is ordinary with respect to an embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$. Let $L \subset \overline{\mathbb{Q}}_p$ be any finite extension of $\mathbb{Q}_p$ containing the image of $\mathbb{Q}(f)$ and let $\mathcal{O}$ be the ring of integers of L. Let $T_f \subset V_f$ be a fixed $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice.

Let $\Lambda_{\mathcal{O}} = \mathcal{O}[\![\Gamma]\!]$. Let $\Psi : G_{\mathbb{Q}} \twoheadrightarrow \Gamma \subset \Lambda_{\mathcal{O}}^\times$ be the natural projection. This is a continuous $\Lambda_{\mathcal{O}}$-valued character that is unramified away from p and totally ramified at p. Let $\Lambda_{\mathcal{O}}^* = \mathrm{Hom}_{\mathrm{cts}}(\Lambda_{\mathcal{O}}, \mathbb{Q}_p/\mathbb{Z}_p)$ be the Pontryagin dual of $\Lambda_{\mathcal{O}}$. This is a discrete $\Lambda_{\mathcal{O}}$-module via $r \cdot \varphi(x) = \varphi(rx)$, for $r, x \in \Lambda_{\mathcal{O}}$ and $\varphi \in \Lambda_{\mathcal{O}}^*$. We similarly define a $\Lambda_{\mathcal{O}}$-module structure on the Pontryagin dual of any $\Lambda_{\mathcal{O}}$-module.

Put $\mathcal{M} = T_f \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}^*$, with $G_{\mathbb{Q}}$-action given by $\rho_f \otimes \Psi^{-1}$. Let $\mathcal{M}^+ = T_f^+ \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}^*$ and $\mathcal{M}^- = \mathcal{M}/\mathcal{M}^+$. Let Σ be any finite set of primes containing p, and let $S = \Sigma \cup \{\ell \mid N\}$. Let $\mathbb{Q}_S$ be the maximal extension of $\mathbb{Q}$ unramified outside S and ∞, and let $G_S = \mathrm{Gal}(\mathbb{Q}_S/\mathbb{Q})$. Following Greenberg, we define a Selmer group $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ by

$$\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f) = \ker\left\{ H^1(G_S, \mathcal{M}) \to H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}} \times \prod_{\ell \in S\setminus\Sigma} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}} \right\}.$$

This is a discrete, cofinite $\Lambda_{\mathcal{O}}$-module (see [Greenberg 2006, Proposition 3.2]). Its Pontryagin dual $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a finite $\Lambda_{\mathcal{O}}$-module. We denote by $\mathrm{Ch}^{\Sigma}_L(f)$ the $\Lambda_{\mathcal{O}}$-characteristic ideal of $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$; this is a principal ideal. In general, these all depend on the choice of T_f, but if $\bar{\rho}_f$ is irreducible, then Lemma 2.1.1 shows that $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is independent of T_f up to isomorphism, and hence so is $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$. In particular, if $\bar{\rho}_f$ is irreducible, then the ideal $\mathrm{Ch}^{\Sigma}_L(f)$ does not depend on the choice of T_f.

Furthermore, if $L_1 \supset L$ is a finite extension with ring of integers $\mathcal{O}_1 \supset \mathcal{O}$, then $T_{f,1} = T_f \otimes_{\mathcal{O}} \mathcal{O}_1$ is a $G_{\mathbb{Q}}$-stable $\mathcal{O}_1$-lattice in $V_1 = V_f \otimes_L L_1$ and $T^+_{f,1} = T_f^+ \otimes_{\mathcal{O}} \mathcal{O}_1$. Hence $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L_1}(f)$, the Selmer group defined with respect to the lattice $T_{f,1}$, is canonically isomorphic to $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f) \otimes_{\mathcal{O}} \mathcal{O}_1$ as a $\Lambda_{\mathcal{O}_1} = \Lambda_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}_1$-module, from which it follows that its Pontryagin dual $X^{\Sigma}_{\mathbb{Q}_\infty,L_1}(f)$ is isomorphic to $X^{\Sigma}_{\mathbb{Q}_\infty,L} \otimes_{\mathcal{O}} \mathcal{O}_1$ as a $\Lambda_{\mathcal{O}_1}$-module and therefore

$$\mathrm{Ch}^{\Sigma}_{L_1}(f) = \mathrm{Ch}^{\Sigma}_L(f) \cdot \Lambda_{\mathcal{O}_1}. \tag{2-3-1}$$

The relation between the Selmer groups $\mathrm{Sel}^{\Sigma_1}_{\mathbb{Q}_\infty,L}(f)$ and $\mathrm{Sel}^{\Sigma_2}_{\mathbb{Q}_\infty,L}(f)$ with $\Sigma_1 \subset \Sigma_2$ is clear:

$$\mathrm{Sel}^{\Sigma_1}_{\mathbb{Q}_\infty,L}(f) = \ker\left\{ \mathrm{Sel}^{\Sigma_2}_{\mathbb{Q}_\infty,L}(f) \xrightarrow{\mathrm{res}} \prod_{\ell \in S_2\setminus S_1} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}} \right\}.$$

Each $H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}}$, $\ell \neq p$, is a cotorsion $\Lambda_{\mathcal{O}}$-module, and the $\Lambda_{\mathcal{O}}$-characteristic ideal of its Pontryagin dual is generated by $P_\ell(\Psi^{-1}\epsilon^{-1}(\mathrm{frob}_\ell))$, where

$$P_\ell(X) = \det(1 - X \cdot \rho_f(\mathrm{frob}_\ell) \mid V_{f,I_\ell})$$

with V_{f,I_ℓ} being the space of I_ℓ-coinvariants of the representation V_f. In particular, $X^{\Sigma_2}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module if and only if $X^{\Sigma_1}_{\mathbb{Q}_\infty,L}(f)$ is, and

$$\mathrm{Ch}^{\Sigma_2}_L(f) \supseteq \mathrm{Ch}^{\Sigma_1}_L(f) \cdot \prod_{\ell \in \Sigma_2\setminus\Sigma_1} (P_\ell(\Psi^{-1}\epsilon^{-1}(\mathrm{frob}_\ell)).$$

Later, we shall see that this last inclusion is often an equality.

If $\Sigma = \{p\}$ then we will omit it from our notation, writing $\mathrm{Sel}_{\mathbb{Q}_\infty,L}(f)$, $X_{\mathbb{Q}_\infty,L}(f)$, and $\mathrm{Ch}_L(f)$ instead.

The following lemma shows that if Σ is large enough and that if $\bar\rho_f$ is irreducible, then $\mathrm{Sel}^\Sigma_{\mathbb{Q}_\infty,L}[p^m]$ and $X^\Sigma_{\mathbb{Q}_\infty,L}(f)/p^m X^\Sigma_{\mathbb{Q}_\infty,L}(f)$ depend only on the pair $(T_f/p^mT_f, T_f^+/p^mT_f^+)$ (up to isomorphism).

Lemma 2.3.1. *Suppose $\Sigma \supset \{\ell \mid N\}$ and that $\bar\rho_f$ is irreducible. Then the inclusion $\mathcal{M}[p^m] \subset \mathcal{M}$ induces an identification*

$$\mathrm{Sel}^\Sigma_{\mathbb{Q}_\infty,L}(f)[p^m] = \ker\big\{H^1(G_S, \mathcal{M}[p^m]) \xrightarrow{\mathrm{res}} H^1(I_p, \mathcal{M}^-[p^m])^{G_{\mathbb{Q}_p}}\big\}.$$

Since $\mathcal{M}[p^m] \cong T_f/p^mT_f \otimes_{\mathcal{O}} \Lambda^*_{\mathcal{O}}[p^m]$, $\mathcal{M}^+[p^m] \cong T_f^+/p^mT_f^+ \otimes_{\mathcal{O}} \Lambda^*_{\mathcal{O}}[p^m]$, and $\mathcal{M}^-[p^m] = \mathcal{M}[p^m]/\mathcal{M}^+[p^m]$, it follows that the dependence is only on the pair $(T_f/p^mT_f, T_f^+/p^mT_f^+)$.

Proof. Since $\bar\rho_f$ is irreducible, the inclusion $\mathcal{M}[p^m] \hookrightarrow \mathcal{M}$ induces an identification $H^1(G_\Sigma, \mathcal{M}[p^m]) = H^1(G_\Sigma, \mathcal{M})[p^m]$. So $\mathrm{Sel}^\Sigma_{\mathbb{Q}_\infty,L}(f)[p^m]$ is the kernel of the restriction map $H^1(G_\Sigma, \mathcal{M}[p^m]) \to H^1(I_p, \mathcal{M}^-)$, which factors through the restriction map $H^1(G_\Sigma, \mathcal{M}[p^m]) \to H^1(I_p, \mathcal{M}^-[p^m])$. The kernel of the natural map $H^1(I_p, \mathcal{M}^-[p^m]) \to H^1(I_p, \mathcal{M}^-)$ is the image of $(\mathcal{M}^-)^{I_p}/p^m(\mathcal{M}^-)^{I_p}$ via the boundary map. But $(\mathcal{M}^-)^{I_p} \cong \mathrm{Hom}_{\mathrm{cts}}(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p)$ since I_p acts via Ψ^{-1} on $\mathcal{M}^- \cong \Lambda^*_{\mathcal{O}}$, and so $(\mathcal{M}^-)^{I_p}/p^m(\mathcal{M}^-)^{I_p} = 0$ as $\mathrm{Hom}_{\mathrm{cts}}(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p)$ is p-divisible. □

The key to our proofs of both Theorems A and B is an understanding of the images of the restriction maps

$$H^1(G_S, \mathcal{M}) \xrightarrow{\mathrm{res}} H^1(\mathbb{Q}_p, \mathcal{M}^-) \times \prod_{\ell\in S, \ell\neq p} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}} \tag{2-3-2}$$

and

$$H^1(G_S, \mathcal{M}) \xrightarrow{\mathrm{res}} H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}} \times \prod_{\ell\in S, \ell\neq p} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}}, \tag{2-3-3}$$

where $S \supset \{\ell \mid Np\}$ is any finite set of primes. The kernel of (2-3-3) is, of course, just $\mathrm{Sel}_{\mathbb{Q}_\infty,L}(f)$. We denote the kernel of (2-3-2) by $\mathcal{S}$ (it is independent of S as $H^1(G_S, \mathcal{M}) = \ker\{H^1(G_{S\sqcup\{\ell\}}, \mathcal{M}) \xrightarrow{\mathrm{res}} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}}\}$ if $\mathcal{M}$ is unramified at ℓ) and let $\mathcal{X}$ be its Pontryagin dual. As $\mathcal{S}$ is a submodule of each $\mathrm{Sel}^\Sigma_{\mathbb{Q}_\infty,L}(f)$, $\mathcal{X}$ is a quotient of each $X^\Sigma_{\mathbb{Q}_\infty,L}(f)$.

The next two propositions record some properties of the above restriction maps. The ideas behind the proofs of these propositions are due to Greenberg (see especially [1999, §§3,4; 2010b; 2010a]). As there is not a convenient reference for the exact case considered here, we have included the details of the arguments.

Proposition 2.3.2. *Suppose $k \equiv 2 \pmod{p-1}$, $\bar{\rho}_f$ is irreducible, and $X_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module. The restriction maps (2-3-2) and (2-3-3) are surjective.*

Proof. As $H^1(\mathbb{Q}_p, \mathcal{M}^-) \twoheadrightarrow H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}}$, (2-3-3) is surjective if (2-3-2) is. That is, to prove the proposition it suffices to prove surjectivity of (2-3-2). To establish this surjectivity we introduce some auxiliary Selmer groups.

Let $\mathcal{N} = \mathrm{Hom}_{\mathcal{O}}(T_f, \mathcal{O}(1)) \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$, with $G_{\mathbb{Q}}$-action given by $\epsilon\rho_f^\vee \otimes \Psi$, and let $\mathcal{N}^+ = \mathrm{Hom}_{\mathcal{O}}(T_f/T_f^+, \mathcal{O}(1)) \otimes_{\mathcal{O}} \Lambda_{\mathcal{O}}$, which is $G_{\mathbb{Q}_p}$-stable with $G_{\mathbb{Q}_p}$ acting via $\alpha_f^{-1}\epsilon \otimes \Psi$. These are free $\Lambda_{\mathcal{O}}$-modules, and $\mathcal{N}^+$ is a $\Lambda_{\mathcal{O}}$-direct summand of $\mathcal{N}$. Let $\mathcal{N}^- = \mathcal{N}/\mathcal{N}^+$. The pairing

$$(\cdot,\cdot) : \mathcal{M} \times \mathcal{N} \to \mathbb{Q}_p/\mathbb{Z}_p, \quad (t \otimes \varphi, \phi \otimes r) = \varphi(\phi(t)\cdot r),$$

is a $G_{\mathbb{Q}}$-equivariant perfect pairing under which $\mathcal{M}^+$ and $\mathcal{N}^+$ are mutual annihilators. Under the induced (perfect) local Tate pairing

$$H^i(\mathbb{Q}_p, \mathcal{M}) \otimes H^{2-i}(\mathbb{Q}_p, \mathcal{N}) \to \mathbb{Q}_p/\mathbb{Z}_p,$$

the images $L_p(\mathcal{Y}) = \mathrm{im}\{H^1(\mathbb{Q}_p, \mathcal{Y}^+) \to H^1(\mathbb{Q}_p, \mathcal{Y})\}$ for $\mathcal{Y} = \mathcal{M}, \mathcal{N}$ are also mutual annihilators. Let

$$\mathrm{Sel}^S(\mathcal{N}) = \ker\big\{H^1(G_S, \mathcal{N}) \xrightarrow{\text{res}} H^1(\mathbb{Q}_p, \mathcal{N})/L_p(\mathcal{N}) \hookrightarrow H^1(\mathbb{Q}_p, \mathcal{N}^-)\big\}.$$

Let $\text{Ш}^1(\mathbb{Q}, S, \mathcal{N}) \subseteq \mathrm{Sel}^\Sigma(\mathcal{N})$ consist of those classes that are trivial at all places in S.

For $\ell \neq p$, $H^1(\mathbb{F}_\ell, \mathcal{M}^{I_\ell}) = 0$ and so $H^1(\mathbb{Q}_\ell, \mathcal{M}) \xrightarrow{\sim} H^1(I_\ell, \mathcal{M})^{G_{\mathbb{Q}_\ell}}$. Also, $H^2(\mathbb{Q}_p, \mathcal{M}^+) = 0$ as its dual is $H^0(\mathbb{Q}_p, \mathcal{N}^-) = 0$, so $H^1(\mathbb{Q}_p, \mathcal{M})/L_p(\mathcal{M}) \xrightarrow{\sim} H^1(\mathbb{Q}_p, \mathcal{M}^-)$. Global Tate duality then identifies the dual of the cokernel of (2-3-2) with $\mathrm{Sel}^S(\mathcal{N})/\text{Ш}^1(\mathbb{Q}, S, \mathcal{N})$ (see [Greenberg 2010b, Proposition 3.1]). To show that this last group is trivial, we will prove that $\mathrm{Sel}^S(\mathcal{N})$ is $\Lambda_{\mathcal{O}}$-torsion-free if nonzero and also prove that $\mathrm{Sel}^S(\mathcal{N})$ is a torsion $\Lambda_{\mathcal{O}}$-module.

Suppose $H^1(G_S, \mathcal{N})$ has nontrivial $\Lambda_{\mathcal{O}}$-torsion: $H^1(G_S, \mathcal{N})[x] \neq 0$ for some $0 \neq y \in \Lambda_{\mathcal{O}}$. Let $\Lambda_x = \Lambda_{\mathcal{O}}/x\Lambda_{\mathcal{O}}$ and $N_x = \mathcal{N}/x\mathcal{N}$. It follows from the long exact cohomology sequence associated with the short exact sequence

$$0 \longrightarrow \mathcal{N} \xrightarrow{n \mapsto xn} \mathcal{N} \longrightarrow N_x \longrightarrow 0$$

that $H^1(G_S, \mathcal{N})[x]$ is the image of $N_x^{G_S}$ under the boundary map. Let $0 \neq y \in N_x^{G_S}$. Let $\mathfrak{n} \subset \Lambda_x$ be the maximal ideal, and let $r \geq 0$ be the largest integer such that $y \in \mathfrak{n}^r N_x$. Since $\mathfrak{n}^r N_x/\mathfrak{n}^{r+1} N_x \cong \mathcal{N} \otimes_{\Lambda_{\mathcal{O}}} \mathfrak{n}^r/\mathfrak{n}^{r+1}$, the $k[G_S]$-module $N_x' = \mathfrak{n}^r N_x/\mathfrak{n}^{r+1} N_x$ is just the sum of $\dim_k(\mathfrak{n}^r/\mathfrak{n}^{r+1})$ copies of $\bar{\rho}_f$. As $\bar{\rho}_f$ is irreducible, it follows that $(N_x')^{G_S} = 0$. But by the choice of r, y has nontrivial image in $\mathcal{N}_x'$ and is fixed by G_S. From this contradiction we conclude[5] that $H^1(G_S, \mathcal{N})$

[5] See also [Greenberg 2006, Proposition 2.25] for another proof.

has no nonzero $\Lambda_{\mathcal{O}}$-torsion. The same is then true of the submodules $\mathrm{Sel}^S(\mathcal{N})$ and $Ш^1(\mathbb{Q}, S, \mathcal{N})$.

We prove that $\mathrm{Sel}^S(\mathcal{N})$ is torsion by exhibiting elements x in the maximal ideal of $\Lambda_{\mathcal{O}}$ such that $\mathrm{Sel}^S(\mathcal{N})/x\mathrm{Sel}^S(\mathcal{N})$ has finite order. Let $x = \gamma - u^m \in \Lambda_{\mathcal{O}}$ with m an integer. Let $N_x = \mathcal{N}/x\mathcal{N}$, $N_x^+ = \mathcal{N}^+/x\mathcal{N}^+$, and $N_x^- = N_x/N_x^+$. These are free $\mathcal{O}$-modules. If $p \nmid N$ or $m \neq 0$, then the natural injection

$$H^1(G_S, \mathcal{N})/xH^1(G_S, \mathcal{N}) \hookrightarrow H^1(G_S, N_x)$$

induces an injection

$$\text{(2-3-4)}\quad \mathrm{Sel}^S(\mathcal{N})/x\mathrm{Sel}^S(\mathcal{N}) \hookrightarrow \mathrm{Sel}^S(N_x) = \ker\{H^1(G_S, N_x) \to H^1(\mathbb{Q}_p, N_x^-)\}.$$

For this, we first note that the image of the induced map from $\mathrm{Sel}^S(\mathcal{N})/x\mathrm{Sel}^S(\mathcal{N})$ to $H^1(G_S, N_x)$ lies in $\mathrm{Sel}^S(N_x)$. It remains to prove injectivity. Let $c \in \mathrm{Sel}^S(\mathcal{N})$ be such that it has trivial image in $\mathrm{Sel}^S(N_x)$. Then $c = xd$ for some $d \in H^1(G_S, \mathcal{N})$ such that $xd = 0$ in $H^1(\mathbb{Q}_p, \mathcal{N}^-)$. The kernel of multiplication by x on $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ is the image of $H^0(\mathbb{Q}_p, N_x^-)$. But N_x^- is a free $\mathcal{O}$-module with $G_{\mathbb{Q}_p}$ acting via the character $\alpha_f \epsilon^{2-k+m}\omega^{-m}$, and so $H^0(\mathbb{Q}_p, N_x^+) = 0$ unless $m = k-2$ and $\alpha_f = 1$. But $\alpha_f = 1$ only if $p \,\|\, N$ and $k = 2$. It follows that if $p \nmid N$ or $m \neq 0$, then multiplication by x is injective on $H^1(\mathbb{Q}_p, \mathcal{N}^-)$ and, therefore, $d \in \mathrm{Sel}^S(\mathcal{N})$, proving the injectivity in (2-3-4).

From (2-3-4) it follows that to prove $\mathrm{Sel}^S(\mathcal{N})$ is torsion it suffices to show that there is some $m \neq 0$ such that $\mathrm{Sel}^S(N_x)$ has finite order. As $\mathrm{Sel}^S(N_x)$ has finite order if and only if $\mathrm{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ has finite order — in which case it must be trivial — it suffices to prove the latter. Furthermore, as $\bar{\rho}_f$ is irreducible and so $H^1(G_S, N_x)$ — and hence also $\mathrm{Sel}^S(N_x)$ — is a torsion-free $\mathcal{O}$-module and therefore free, it would then follow that $\mathrm{Sel}^S(N_x) = 0$.

Let $M_x = N_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$ and $M_x^- = N_x^- \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$. From the long exact cohomology sequence associated with the short exact sequence

$$0 \to N_x = N_x \otimes_{\mathbb{Z}_p} \mathbb{Z}_p \to N_x \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \to M_x = N_x \otimes_{\mathbb{Z}_p} \mathbb{Q}/\mathbb{Z}_p \to 0$$

we deduce an injection $H^1(G_S, N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \hookrightarrow H^1(G_S, M_x)$. Under this injection the image of the canonical map

$$\mathrm{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to H^1(G_S, N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$$

maps into

$$\mathrm{Sel}^S(M_x) = \ker\{H^1(G_S, M_x) \xrightarrow{\text{res}} H^1(\mathbb{Q}_p, M_x^-)\}.$$

The kernel of the induced map $\mathrm{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to \mathrm{Sel}^S(M_x)$ is then just the kernel of $\mathrm{Sel}^S(N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \to H^1(G_S, N_x) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p$, which is finite (having order at most that of the torsion subgroup of the quotient $H^1(G_S, N_x)/\mathrm{Sel}^S(N_x)$).

So to prove that there is an $m \neq 0$ such that $\mathrm{Sel}^S(N_x)$ has finite order, it suffices to find such an m for which $\mathrm{Sel}^S(M_x)$ has finite order.

Let $m \neq 0$ be an integer such that $m \equiv 0 \pmod{p-1}$. Let $y = \gamma - u^{k-2-m}$. Then, as $k \equiv 2 \pmod{p-1}$, $M_x \cong \mathcal{M}[y]$ as $\mathcal{O}[G_{\mathbb{Q}}]$-modules, and the isomorphism can be chosen so that M_x^- is identified with $\mathcal{M}^-[y]$. It follows that

$$\mathrm{Sel}^S(M_x) = \mathrm{Sel}^S(\mathcal{M}[y]) \hookrightarrow \mathrm{Sel}^S_{\mathbb{Q}_\infty,L}(f)[y], \tag{2-3-5}$$

where $\mathrm{Sel}^S(\mathcal{M}[y])$ is defined just as $\mathrm{Sel}^S(M_x)$, and where the injection is induced by the natural identification $H^1(G_S, \mathcal{M}[y]) \xrightarrow{\sim} H^1(G_S, \mathcal{M})[y]$ (which is injective as $\bar{\rho}_f$ is irreducible).

As $X_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module, so is $X^S_{\mathbb{Q}_\infty,L}(f)$. Therefore, for all but finitely many integers m, $X^S_{\mathbb{Q}_\infty,L}(f)/yX^S_{\mathbb{Q}_\infty,L}(f)$ has finite order. As the latter is dual to $\mathrm{Sel}^S_{\mathbb{Q}_\infty,L}(f)[y]$, it follows from (2-3-5) that there is an $m \neq 0$ with $m \equiv 0 \pmod{p-1}$ such that $\mathrm{Sel}^S(M_x)$ has finite order. As explained above, the existence of such an x implies the desired surjectivity of (2-3-2). □

Proposition 2.3.3. *Suppose $k \equiv 2 \pmod{p-1}$, $\bar{\rho}_f$ is irreducible, and $X_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module.*

(i) *The $\Lambda_{\mathcal{O}}$-module $\mathcal{X}$ has no nonzero finite-order $\Lambda_{\mathcal{O}}$-submodules.*

(ii) *Let Σ be any finite set of primes containing p. The $\Lambda_{\mathcal{O}}$-module $X^\Sigma_{\mathbb{Q}_\infty,L}(f)$ has no nonzero finite-order $\Lambda_{\mathcal{O}}$-submodules.*

Proof. To prove part (i), let $S \supset \{\ell \mid Np\}$ be any finite set of primes and let

$$\mathcal{P}_S = H^1(\mathbb{Q}_p, \mathcal{M}^-) \times \prod_{\ell \in S, \ell \neq p} H^1(\mathbb{Q}_\ell, \mathcal{M}).$$

For $x = \gamma - u^m \in \Lambda_{\mathcal{O}}$, $\mathcal{P}_S[x]$ is a quotient of

$$P_{S,x} = H^1(\mathbb{Q}_p, \mathcal{M}[x])/L_p(\mathcal{M}[x]) \times \prod_{\ell \in S, \ell \neq p} H^1(\mathbb{Q}_\ell, \mathcal{M}[x]),$$

where $L_p(\mathcal{M}[x]) = \mathrm{im}\{H^1(\mathbb{Q}_p, \mathcal{M}^+[x]) \to H^1(\mathbb{Q}_p, \mathcal{M}[x])\}$. Therefore the cokernel of the restriction map $H^1(G_S, \mathcal{M}[x]) = H^1(G_S, \mathcal{M})[x] \to \mathcal{P}_S[x]$ is a quotient of the cokernel of the restriction map $H^1(G_S, \mathcal{M}[x]) \to P_{S,x}$. By global Tate duality, the Pontryagin dual of the latter is a subquotient of $\mathrm{Sel}^S(N_x)$, where N_x and $\mathrm{Sel}^S(N_x)$ are as in (2-3-4). But, as shown in the proof of Proposition 2.3.2, m can be chosen so that $\mathrm{Sel}^S(N_x) = 0$ and hence so that $H^1(G_S, \mathcal{M})[x] \twoheadrightarrow \mathcal{P}_S[x]$. It then follows from an application of the snake lemma to multiplication by x of the short exact sequence

$$0 \to \mathcal{S} \to H^1(G_S, \mathcal{M}) \to \mathcal{P}_S \to 0$$

that, for such a choice of m,

$$\mathcal{S}/x\mathcal{S} \hookrightarrow H^1(G_S, \mathcal{M})/xH^1(G_S, \mathcal{M}). \tag{2-3-6}$$

However, as shown in both [Skinner and Urban 2014, Lemma 3.3.18] and [Greenberg 2010a, Proposition 2.6.1], the right-hand side of (2-3-6) is trivial for all but finitely many m, so the m can also be chosen so that $\mathcal{S}/x\mathcal{S} = 0$. Let $X \subseteq \mathcal{X}$ be a sub-$\Lambda_{\mathcal{O}}$-module of finite order, and let X^* be its Pontryagin dual. Then X^*/xX^* is a quotient of $\mathcal{S}/x\mathcal{S}$ and so is 0. By Nakayama's lemma $X^* = 0$, hence $X = 0$. This proves (i).

To prove part (ii), let $S \supset \Sigma \cup \{\ell \mid Np\}$ and let

$$\mathcal{P}_{S,\Sigma} = H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}} \times \prod_{\ell \in S\setminus\Sigma} H^1(\mathbb{Q}_\ell, \mathcal{M})$$

and

$$\mathcal{P}_{S,\Sigma,x} = H^1(\mathbb{Q}_p, \mathcal{M}[x])/L_p(\mathcal{M}[x]) \times \prod_{\ell \in S\setminus\Sigma} H^1(\mathbb{Q}_\ell, \mathcal{M}[x]).$$

We may then argue as in the proof of part (i) but with $\mathcal{P}_S$ replaced by $\mathcal{P}_{S,\Sigma}$. Then $\mathcal{S}$ is replaced by $\mathrm{Sel}^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$. Furthermore, as $\mathcal{P}_{S,\Sigma,x}$ is a quotient of $\mathcal{P}_{S,x}$, the surjectivity of the restriction map $H^1(G_S, \mathcal{M}[x]) \to \mathcal{P}_{S,\Sigma,x}$, and hence of the restriction map $H^1(G_S, \mathcal{M}[x]) \to \mathcal{P}_{S,\Sigma}[x]$, follows for a suitable $x = \gamma - u^m \in \Lambda_{\mathcal{O}}$ from the surjectivity of the restriction map onto $\mathcal{P}_{S,x}$ established in the proof of part (i). □

Let $F^{\Sigma}_L(f)$ be the $\Lambda_{\mathcal{O}}$-Fitting ideal of $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$. The following is a straightforward consequence of the preceding propositions.

Lemma 2.3.4. *Suppose $k \equiv 2 \pmod{p-1}$ and $\bar{\rho}_f$ is irreducible.*

(i) $\mathrm{Ch}^{\Sigma}_L(f) = \mathrm{Ch}_L(f) \cdot \prod_{\ell\in\Sigma,\ell\neq p} P_\ell(\Psi^{-1}\epsilon^{-1}(\mathrm{frob}_\ell))$.

(ii) $F^{\Sigma}_L(f) = \mathrm{Ch}^{\Sigma}_L(f)$.

Proof. If $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is not a torsion $\Lambda_{\mathcal{O}}$-module (equivalently, $X_{\mathbb{Q}_\infty,L}(f)$ is not a torsion $\Lambda_{\mathcal{O}}$-module), then $\mathrm{Ch}_L(f)$, $\mathrm{Ch}^{\Sigma}_L(f)$, and $F^{\Sigma}_L(f)$ are all zero, so there is nothing to prove. We suppose then that $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module.

Part (i) is immediate from Proposition 2.3.2 and the definition of characteristic ideals. For part (ii), we first note that $F^{\Sigma}_L(f) \subset \mathrm{Ch}^{\Sigma}_L(f)$. Let $\mathfrak{a}$ be the kernel of the quotient $\Lambda_{\mathcal{O}}/F^{\Sigma}_L(f) \twoheadrightarrow \Lambda_{\mathcal{O}}/\mathrm{Ch}^{\Sigma}_L(f)$. Since $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module and $\mathrm{Ch}^{\Sigma}_L(f)$ is a principal ideal, there exists $\lambda = \gamma - u^m \in \Lambda_{\mathcal{O}}$ such that λ is not a zero-divisor in $\Lambda_{\mathcal{O}}/\mathrm{Ch}^{\Sigma}_L(f)$ and $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)/\lambda X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}/\lambda\Lambda_{\mathcal{O}} = \mathcal{O}$-module. The size of this module is then equal to the size of both $\Lambda_{\mathcal{O}}/(\lambda, F^{\Sigma}_L(f))$

and $\Lambda_{\mathcal{O}}/(\lambda, \mathrm{Ch}_L^\Sigma(f))$ (which are necessarily finite), the first by basic properties[6] of Fitting ideals and the second by Proposition 2.3.3(ii) and a standard argument[7] from Iwasawa theory. It follows that the natural projection $\Lambda_{\mathcal{O}}/(\lambda, F_L^\Sigma(f)) \twoheadrightarrow \Lambda_{\mathcal{O}}/(\lambda, \mathrm{Ch}_L^\Sigma(f))$ is an isomorphism. Applying the snake lemma to the diagram obtained by multiplying the short exact sequence

$$0 \to \mathfrak{a} \to \Lambda_{\mathcal{O}}/F_L^\Sigma(f) \to \Lambda_{\mathcal{O}}/\mathrm{Ch}_L^\Sigma(f) \to 0$$

by λ then yields an exact sequence

$$0 \to \mathfrak{a}/\lambda\mathfrak{a} \to \Lambda_{\mathcal{O}}/(\lambda, F_L^\Sigma(f)) \xrightarrow{\sim} \Lambda_{\mathcal{O}}/(\lambda, \mathrm{Ch}_L^\Sigma(f)) \to 0.$$

Therefore $\mathfrak{a}/\lambda\mathfrak{a}$, and hence $\mathfrak{a}$, is 0. □

2.4. ***p-adic L-functions.*** Let f, L, $\mathcal{O}$, and $\Lambda_{\mathcal{O}}$ be as in the preceding section, with the assumption that $k \geq 2$ and f is ordinary with respect to L. Amice and Vélu [1975] and Vishik [1976] (see also [Mazur et al. 1986]) constructed a p-adic L-function for f. This is a power series $\mathcal{L}_f \in \Lambda_{\mathcal{O}}$ with the property that if $\phi : \Lambda_{\mathcal{O}} \to \overline{\mathbb{Q}}_p$ is a continuous $\mathcal{O}$-homomorphism such that $\phi(\gamma) = \zeta u^m$ with ζ a primitive $p^{t_\phi - 1}$-th root of unity and $0 \leq m \leq k-2$ an integer, then[8]

(2-4-1)
$$\mathcal{L}_f(\phi) := \phi(\mathcal{L}_f) = e(\phi)\frac{p^{t'_\phi(m+1)} m! L(f, \chi_\phi^{-1}\omega^{-m}, m+1)}{(-2\pi i)^{m+1} G(\chi_\phi^{-1}\omega^{-m})\Omega_f^{\mathrm{sgn}((-1)^m)}},$$
$$e(\phi) = \alpha_p^{-t_\phi}\left(1 - \frac{\omega^{-m}\chi_\phi^{-1}p^{k-2-m}}{\alpha_p}\right)\left(1 - \frac{\omega^m\chi_\phi(p)p^m}{\alpha_p}\right),$$

[6]Suppose R is a Noetherian ring and M is a finite R-module (hence finitely presented). Let $F_R(M)$ be the R-Fitting ideal of M. These basic properties are: (i) for any ideal $I \subset R$, $F_{R/I}(M/IM) = F_R(M) \mod I$; (ii) if $M = R/\alpha_1 \times \cdots \times R/\alpha_m$, then $F_R(M) = \alpha_1 \cdots \alpha_m$; and (iii) if R is a PID, then $\mathrm{length}_R(M) = \mathrm{length}_R(R/F_R(M))$. For properties (i) and (ii), see [Mazur and Wiles 1984, Appendix A]. Property (iii) follows from (ii).

[7]The argument: A finitely generated torsion $\Lambda_{\mathcal{O}}$-algebra X admits a $\Lambda_{\mathcal{O}}$-homomorphism $X \to Y = \prod_{i=1}^r \Lambda_{\mathcal{O}}/(f_i)$ with finite-order kernel $\mathfrak{a}$ and cokernel $\mathfrak{b}$ and such that the $\Lambda_{\mathcal{O}}$-characteristic ideal of X is $(f_1 \cdots f_r)$. Let $f = f_1 \cdots f_r$. If X has no finite-order $\Lambda_{\mathcal{O}}$-submodules, then the map to Y is an injection. Multiplying the short exact sequence $0 \to X \to Y \to \mathfrak{b} \to 0$ by $\lambda = \gamma - u^m$ and applying the snake lemma is easily seen to give

$$\#X/\lambda X = \#Y/\lambda Y = \prod \#\Lambda_{\mathcal{O}}/(\lambda, f_i) = \prod \#\mathcal{O}/(f_i(u^m - 1)) = \#\mathcal{O}/(f(u^m-1)) = \#\Lambda_{\mathcal{O}}/(\lambda, f),$$

where we have written $f_i(u^m - 1)$ and $f(u^m - 1)$ for the respective images of f_i and f under the continuous $\mathcal{O}$-algebra homomorphism $\Lambda_{\mathcal{O}} \to \mathcal{O}$ sending γ to u^m.

[8]The power of $-2\pi i$ in the denominator of this formula is incorrectly given as $(-2\pi i)^m$ in some of the formulas in [Skinner and Urban 2014], namely in the introduction, in §3.4.4, and in Theorem 3.26 of [loc. cit.]. In these cases the correct factor is $(-2\pi i)^{m+1}$. This error originates in the difference between $\Omega_f^\pm$ as defined in [loc. cit., §3.3.3] and the $\Omega^\pm$ in [Mazur et al. 1986, I.9]: $\Omega^\pm = -2\pi i \Omega_f^\pm$. The exponents of $-2\pi i$ are correct in the formulas in [Skinner and Urban 2014] for the L-function of f twisted by a Hecke character of the imaginary quadratic field $\mathcal{K}$.

where α_p is the unique (unit) root in $\mathcal{O}^\times$ of $x^2 - a_p(f)x + p^{k-1}$ if $p \nmid N$ and $\alpha_p = a_p(f)$ if $p \mid N$, $t'_\phi = 0$ if $t_\phi = 1$ and $p-1 \mid m$ and otherwise $t'_\phi = t_\phi$, χ_ϕ is the primitive Dirichlet character of p-power order and conductor (which can be viewed as a finite-order character of $\mathbb{Z}_p^\times$) such that $\chi_\phi(u) = \zeta^{-1}$, $G(\chi_\phi^{-1}\omega^{-m})$ is the usual Gauss sum (and so equals 1 if $t'_\phi = 0$), and $\Omega_f^\pm$ are the canonical periods of f (these are well defined up to a unit in $\mathcal{O}$; see [Skinner and Urban 2014, §3.3.3]).

Let Σ be a finite set of primes. We define an incomplete p-adic L-function $\mathcal{L}_f^\Sigma \in \Lambda_{\mathcal{O}}$ by

$$\mathcal{L}_f^\Sigma = \mathcal{L}_f \cdot \prod_{\ell \in \Sigma, \ell \neq p} P_\ell(\Psi^{-1}\epsilon^{-1}(\mathrm{frob}_\ell)). \tag{2-4-2}$$

Note that

$$P_\ell(\Psi^{-1}\epsilon^{-1}(\mathrm{frob}_\ell)) = \begin{cases} 1 - a_\ell(f)\ell^{-1}\Psi^{-1}(\mathrm{frob}_\ell) + \ell^{k-3}\Psi^{-2}(\mathrm{frob}_\ell), & \ell \nmid N, \\ 1 - a_\ell(f)\ell^{-1}\Psi^{-1}(\mathrm{frob}_\ell), & \ell \mid N. \end{cases}$$

In particular, the value of $\mathcal{L}_f^\Sigma$ under a continuous $\mathcal{O}$-algebra homomorphism $\phi : \Lambda_{\mathcal{O}} \to \overline{\mathbb{Q}}_p$ such that $\phi(\gamma) = \zeta u^m$, $0 \le m \le k-2$, can be expressed in terms of a special value of an incomplete L-function:

$$\mathcal{L}_f^\Sigma(\phi) = e(\phi) \frac{p^{t'_\phi(m+1)} m! L^{\Sigma \setminus \{p\}}(f, \chi_\phi^{-1}\omega^{-m}, m+1)}{(-2\pi i)^{m+1} G(\chi_\phi^{-1}\omega^{-m}) \Omega_f^{\mathrm{sgn}((-1)^m)}}.$$

Remark 2.4.1. Let $\mathbb{Z}(f)$ be the ring of integers of $\mathbb{Q}(f)$ and let $\mathfrak{p}$ be the prime of $\mathbb{Z}(f)$ determined by the chosen embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$. Then $\Omega_f^\pm$ is well defined up to a unit in the localization $\mathbb{Z}(f)_{(\mathfrak{p})}$ of $\mathbb{Z}(f)$, and the value of the p-adic L-function under a homomorphism ϕ as above lies in a finite extension of $\mathbb{Z}(f)_{(\mathfrak{p})}$. It is in this way that period-normalized values of the L-function $L(f, s)$ and its twists, which *a priori* are complex values, can be viewed as being in $\overline{\mathbb{Q}}_p$ without fixing an isomorphism $\overline{\mathbb{Q}}_p \cong \mathbb{C}$.

Suppose f has split multiplicative reduction at p. Then it follows easily from (2-4-1) that if $\phi_0 : \Lambda_{\mathcal{O}} \to \overline{\mathbb{Q}}_p$ is the $\mathcal{O}$-algebra homomorphism such that $\phi_0(\gamma) = 1$, then $\mathcal{L}_f(\phi_0) = 0$. In particular, $\mathcal{L}_f = (\gamma - 1) \cdot \mathcal{L}'_f$ for some $\mathcal{L}'_f \in \Lambda_{\mathcal{O}}$. Greenberg and Stevens [1993, Theorem 7.1] proved that $\mathcal{L}'_f(\phi_0) = \phi_0(\mathcal{L}'_f)$ is related to the $\mathfrak{L}$-invariant of V_f by the formula

$$\mathcal{L}'_f(\phi_0) = (\log_p u)^{-1} \mathfrak{L}(V_f) \frac{L(f, 1)}{-2\pi i \Omega_f^+}. \tag{2-4-3}$$

More precisely, if we identify $\Lambda_{\mathcal{O}}$ with the power-series ring $\mathcal{O}[\![T]\!]$ by sending γ to $1 + T$, and if we let $L_p(f, s) = \mathcal{L}_f(u^{s-1} - 1)$, $s \in \mathbb{Z}_p$, then Greenberg and

Stevens proved that

$$\frac{d}{ds}L_p(f,s)|_{s=1} = \mathfrak{L}(V_f)\frac{L(f,1)}{-2\pi i\Omega_f^+}.$$

This is easily seen to be equivalent to (2-4-3). This formula was conjectured by Mazur, Tate, and Teitelbaum [1986, §13].

2.5. *The Iwasawa–Greenberg main conjecture.* Let f, L, $\mathcal{O}$, $\Lambda_{\mathcal{O}}$, $\mathcal{L}_f$, etc., be as in the preceding sections. Along the lines of Iwasawa's original main conjecture for totally real number fields, Mazur and Swinnerton-Dyer (for modular elliptic curves) and Greenberg (more generally) made the following conjecture.

Conjecture 2.5.1. *If Σ is any finite set of primes containing p, then $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module and $\mathrm{Ch}^{\Sigma}_L(f) = (\mathcal{L}^{\Sigma}_f)$ in $\Lambda_{\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and even in $\Lambda_{\mathcal{O}}$ if $\bar{\rho}_f$ is irreducible.*

It follows easily from Lemma 2.3.4(i) and (2-4-2) that if this conjecture holds for one set Σ then it holds for all sets Σ. Also, the conjecture with L replaced by any finite extension implies the conjecture for L, as can be seen by the observations in Section 2.3 on the relation (2-3-1) between $\mathrm{Ch}^{\Sigma}_L(f)$ and $\mathrm{Ch}^{\Sigma}_{L_1}(f)$ for a finite extension $L_1 \supset L$.

In [Skinner and Urban 2014] the following theorem was proved, in combination with results of Kato [2004], which established this conjecture for a large class of modular forms.

Theorem 2.5.2. *Suppose*

(i) $k \equiv 2 \pmod{p-1}$;

(ii) $\bar{\rho}_f$ *is irreducible*;

(iii) *there exists a prime $q \neq p$ such that $q \parallel N$ and $\bar{\rho}_f$ is ramified at q*;

(iv) $p \nmid N$ (*this is automatic if $k \neq 2$*).

Then for any finite set of primes Σ, $X^{\Sigma}_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module and $\mathrm{Ch}^{\Sigma}_L(f) = (\mathcal{L}^{\Sigma}_f)$ in $\Lambda_{\mathcal{O}}$.

In [Skinner and Urban 2014] an additional hypothesis is required to conclude equality in $\Lambda_{\mathcal{O}}$ and not just in $\Lambda_{\mathcal{O}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$:

($*$) There exists an $\mathcal{O}$-basis of T_f such that the image of ρ_f contains $\mathrm{SL}_2(\mathbb{Z}_p)$.

This hypothesis was included because it is part of the statement of [Kato 2004, Theorem 17.4]. However, a closer reading of the proof of [loc. cit.] shows that all that is necessary is that (a) $\bar{\rho}_f$ be irreducible and (b) there exist an element $g \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])$ such that $T_f/(\rho_f(g)-1)T_f$ is a free $\mathcal{O}$-module of rank one, as we explain in the following paragraph. All references to theorems or sections in the following paragraph are to [Kato 2004] unless otherwise indicated.

Hypothesis $(*)$ intervenes in the proof of Theorem 17.4 through Theorem 15.5(4), which is proved in §13.14. Hypothesis (a) together with Lemma 2.1.1 of this paper implies that, in the notation of [Kato 2004], the conclusion in §13.14 that $T_f = a \cdot V_{O_\lambda}(f)$ for some $a \in F_\lambda^\times$ holds; Lemma 2.1.1 of this paper can replace the reference to Lemma 14.7 in §13.14, which is the only explicit use of a basis with an image containing $\mathrm{SL}_2(\mathbb{Z}_p)$ in the proof of Theorem 15.5(4). Hypothesis (a) also, of course, ensures that the hypotheses of Theorem 12.4(3) hold, as needed in §13.14. Hypothesis (b) ensures that the hypotheses of Theorem 13.4(3) hold. The proof of Theorem 15.5(4) in §13.14 then holds with $(*)$ replaced by the hypotheses (a) and (b) above.

We now check that (a) and (b) hold under the hypotheses of Theorem 2.5.2. Hypothesis (a) is just hypothesis (ii) of the theorem. Hypothesis (b) is satisfied in light of hypothesis (iii) of the theorem: As $q \parallel N$, the action of I_q on V_f is nontrivial and unipotent and in particular factors through the tame quotient (this is a consequence of the "local-global" compatibility of the Galois representation ρ_f [Carayol 1986, Theorem A]). It follows that $\rho_f(\tau)$ is unipotent for any $\tau \in I_q$ projecting to a topological generator of the tame quotient and, since $\bar{\rho}_f$ is ramified at q, $\bar{\rho}_f(\tau) \neq 1$, hence $T_f/(\rho_f(\tau) - 1)T_f$ is a free $\mathcal{O}$-module of rank one. As $\tau \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}[\mu_{p^\infty}])$, condition (b) holds for $g = \tau$.

We also take this opportunity to note that the reference to [Vatsal 2003] in the proof of [Skinner and Urban 2014, Proposition 12.3.6] is not sufficient. It may be that the weight two specialization of the Hida family in [loc. cit.] that has trivial character also has multiplicative reduction at p. This case is excluded in [Vatsal 2003], though the ideas in that paper can be extended to this case, as is explained in [Chida and Hsieh 2016]. The reference to [Vatsal 2003, Theorem 1.1] must be augmented by a reference to [Chida and Hsieh 2016, Theorem C].

The purpose of this paper is, of course, to show that hypothesis (iv) can be removed from Theorem 2.5.2.

The main results of [Skinner and Urban 2014] show that for a suitable imaginary quadratic field K and a large enough set Σ, the equality $\mathrm{Ch}_L^\Sigma(f)\,\mathrm{Ch}_L^\Sigma(f \otimes \chi_K) = (\mathcal{L}_f^\Sigma \mathcal{L}_{f\otimes\chi_K}^\Sigma)$ holds, where $f \otimes \chi_K$ is the newform associated with the twist of f by the primitive quadratic Dirichlet character corresponding to K. When $p \nmid N$, this equality can be refined to an equality of the individual factors via the inclusions $\mathcal{L}_f^\Sigma \in \mathrm{Ch}_L^\Sigma(f)$ and $\mathcal{L}_{f\otimes\chi_K}^\Sigma \in \mathrm{Ch}_L^\Sigma(f \otimes \chi_K)$, which are proved in [Kato 2004]. When $p \mid N$, these inclusions do not follow directly from [Kato 2004]; additional arguments are required.

2.6. ***Hida families.*** Let $f \in S_k(\Gamma_0(N))$ be a newform that is ordinary with respect to an embedding $\mathbb{Q}(f) \hookrightarrow \overline{\mathbb{Q}}_p$. Write $N = p^r M$ with $p \nmid M$ (so $r = 0$ or 1 by

Lemma 2.1.2). Let $L \subset \overline{\mathbb{Q}}_p$ be any finite extension of $\mathbb{Q}_p$ containing the image of $\mathbb{Q}(f)$ and let $\mathcal{O}$ be the ring of integers of L. Let $R_0 = \mathcal{O}[\![X]\!]$. Hida (see especially [1986; 1988]) proved that there is a finite, local R_0-domain R and a formal q-expansion

$$\mathbb{f} = \sum_{n=1}^{\infty} a_n q^n \in R[\![q]\!], \quad a_1 = 1,$$

satisfying

- $R = R_0[\{a_\ell \ : \ \ell = \text{prime}\}]$;
- if $\phi : R \to \overline{\mathbb{Q}}_p$ is a continuous $\mathcal{O}$-algebra homomorphism such that $\phi(1+X) = (1+p)^{k'}$, with $k' > 2$ and $k' \equiv k \pmod{p-1}$, then $\sum_{n=1}^{\infty} \phi(a_n) q^n$ is the q-expansion of a p-stabilized newform, in the sense that there is a newform $f_\phi \in S_{k'}(\Gamma_0(M))$ and an embedding $\mathbb{Q}(f_\phi) \hookrightarrow \overline{\mathbb{Q}}_p$ such that $\phi(a_\ell) = a_\ell(f_\phi)$ for all primes $\ell \neq p$ and $\phi(a_p)$ is the unit root of the polynomial $x^2 - a_p(f_\phi)x + p^{k'-1}$;
- there is a continuous $\mathcal{O}$-algebra homomorphism $\phi_0 : R \to \mathcal{O}$ such that $\phi_0(1+X) = (1+p)^k$ and $\phi_0(a_\ell) = a_\ell(f)$, $\ell \neq p$, and $\phi_0(a_p)$ is the unit root of $x^2 - a_p(f)x + p^{k-1}$ if $r = 0$ and $\phi_0(a_p) = a_p(f)$ if $r = 1$.

Furthermore, after possibly replacing L with a finite extension, we may assume

- $\mathcal{O}$ is integrally closed in R.

Then, as explained by Greenberg and Stevens [1993] (see also [Nekovář and Plater 2000, (1.4.7)]),

- there is an integer c and an $\mathcal{O}$-algebra embedding

$$R \hookrightarrow R_c = \left\{ \sum_{i=0}^{\infty} u_i (x-k)^i \ : \ u_i \in L, \ \lim_{i \to \infty} \operatorname{ord}_p(u_i) + ci = +\infty \right\} \subset L[\![x]\!]$$

  such that the induced embedding of R_0 sends $1 + X$ to the power series expansion of $(1+p)^x$ about $x = k$ and ϕ_0 is the homomorphism induced by evaluating at $x = k$.

Then evaluating at $x = k'$ for an integer $k' > 2$ with $k' \equiv k \pmod{(p-1)p^c}$ defines a continuous $\mathcal{O}$-algebra homomorphism $\phi_{k'} : R \to L$ such that $\phi_k(1+X) = (1+p)^{k'}$ with corresponding newform $f_{\phi_{k'}} \in S_{k'}(\Gamma_0(M))$. Furthermore, it is clear that given any integer $m > 0$, there is an integer $r_m > 0$ such that if $k' \equiv k \pmod{(p-1)p^{r_m}}$, then $\phi_{k'} \equiv \phi_0 \pmod{p^m \mathcal{O}}$; in particular, for all primes $\ell \neq p$

$$a_\ell(f_{\phi_{k'}}) \equiv a_\ell(f) \pmod{p^m \mathcal{O}}.$$

For each integer m we choose such a $k' = k_m$ and write f_m for the corresponding $f_{\phi_{k_m}}$. Note that we have chosen $k_m > 2$ so that f_m is a newform of level not divisible by p, though p might divide the level of f.

Suppose that $\bar{\rho}_f$ is irreducible. Then there is a free rank two R-module $\mathbb{T}$ and a continuous Galois representation

$$\rho_R : G_{\mathbb{Q}} \to \mathrm{Aut}_R(\mathbb{T})$$

that is unramified at each $\ell \nmid pN$ and such that for any such prime trace $\rho_R(\mathrm{frob}_\ell) = a_\ell \in R$. In particular for $\phi : R \to \mathcal{O}$ being ϕ_0 or one of the homomorphisms ϕ_{k_m}, $T_{f_\phi} = \mathbb{T} \otimes_{R,\phi} \mathcal{O}$ is a $G_{\mathbb{Q}}$-stable $\mathcal{O}$-lattice in $\mathbb{T} \otimes_{R,\phi} L \cong V_{f_\phi}$. Let $T_f = T_{f_{\phi_0}}$ and $T_{f_m} = T_{f_{\phi_{k_m}}}$. Since ϕ_0 and ϕ_m agree modulo p^m, reduction modulo p^m induces identifications

$$T_f/p^m T_f = \mathbb{T} \otimes_{R,\phi_0} \mathcal{O}/p^m\mathcal{O} = \mathbb{T} \otimes_{R,\phi_m} \mathcal{O}/p^m\mathcal{O} = T_{f_m}/p^m T_{f_m} \tag{2-6-1}$$

as $\mathcal{O}[G_{\mathbb{Q}}]$-modules.

Suppose also that

$$\alpha_f^{-1}\epsilon^{k-1} \not\equiv \alpha_f \pmod{\mathfrak{m}}.$$

This ensures that there is a free rank-one $G_{\mathbb{Q}_p}$-stable R-summand $\mathbb{T}^+ \subset \mathbb{T}$ such that for any of the ϕ as before, $\mathbb{T}^+ \otimes_{R,\phi} \mathcal{O} = T^+_{f_\phi}$. The identification $T_f/p^m T_f = T_{f_m}/p^m T_{f_m}$ induces an identification

$$T_f^+/p^m T_f^+ = T_{f_m}^+/p^m T_{f_m}^+. \tag{2-6-2}$$

Greenberg and Stevens [1993] and Kitagawa [1994] and others have shown that the p-adic L-functions $\mathcal{L}_{f_\phi}$ for the forms f_ϕ arising from a Hida family fit into a "two-variable" p-adic L-function. In particular, following Emerton, Pollack, and Weston, we have the following.

Proposition 2.6.1 [Emerton et al. 2006, §3 especially Proposition 3.4.3]. *Let Σ be a finite set of primes containing p. If $\bar{\rho}_f$ is irreducible, then there exists $\mathcal{L}_{\mathbb{F}}^\Sigma \in R[\![\Gamma]\!]$ such that for each continuous $\mathcal{O}$-algebra homomorphism $\phi : R \to \overline{\mathbb{Q}}_p$ as above, the image of $\mathcal{L}_{\mathbb{F}}^\Sigma$ in $R[\![\Gamma]\!] \otimes_{R,\phi} \phi(R)' = \Lambda_{\phi(R)'}$ is a multiple of the p-adic L-function $\mathcal{L}_{f_\phi}^\Sigma$ by a unit in $\phi(R)'$.*

Here $\phi(R)'$ is the integral closure of $\phi(R)$ in its field of fractions (which is a finite extension of L). In particular, as $\phi_{k_m}(R) = \mathcal{O}$, the image of $\mathcal{L}_{\mathbb{F}}^\Sigma$ in $R[\![\Gamma]\!] \otimes_{R,\phi_{k_m}} \mathcal{O} = \Lambda_{\mathcal{O}}$ is just $u_m \mathcal{L}_{f_m}^\Sigma$ for some $u_m \in \mathcal{O}^\times$. Assuming that $\bar{\rho}_f$ is irreducible, for each m we then have an equality of $\Lambda_{\mathcal{O}}$-ideals

$$(\mathcal{L}_f^\Sigma, p^m) = (\mathcal{L}_{f_m}^\Sigma, p^m) \subseteq \Lambda_{\mathcal{O}}. \tag{2-6-3}$$

3. Assembling the pieces

We can now put together the various objects and results from Section 2 to prove Theorems A and B as indicated in the introduction. We will freely use the notation introduced in Section 2.

3.1. ***Proof of Theorem A.*** Let f, L, $\mathcal{O}$ be as in the statement of Theorem A. In particular, $f \in S_k(N)$ is a newform of some weight $k \geq 2$ that is congruent to 2 modulo $p-1$ and some level N. Furthermore, if $f = \sum_{n=1}^{\infty} a_n(f)q^n$ is the q-expansion of f, then $a_p(f) \in \mathcal{O}^\times$. If $p \nmid N$, then by Theorem 2.5.2 the Iwasawa–Greenberg main conjecture is true: for any finite set of primes Σ containing p, $\mathrm{Ch}_L^\Sigma(f) = (\mathcal{L}_f^\Sigma)$ in $\Lambda_\mathcal{O}$. So we assume that $p \mid N$. By Lemma 2.1.2 we then have $N = pM$ with $p \nmid M$ and $k = 2$. Let $T_f \subset V_f$ be a $G_\mathbb{Q}$-stable $\mathcal{O}$-lattice. By Lemma 2.1.1 this lattice is unique up to $L^\times$-multiple since $\bar{\rho}_f$ is assumed irreducible.

Let $\Sigma \supset \{\ell \mid N\}$ be a finite set of primes. After possibly replacing L with a finite extension, for each integer $m > 0$ there exists

(a) a newform $f_m \in S_{k_m}(\Gamma_0(M))$ with $\mathbb{Q}(f_m) \subset L$, $k_m > 2$, and $k_m \equiv 2 \pmod{p-1}$ and such that $a_p(f_m) \in \mathcal{O}^\times$;

(b) a $G_\mathbb{Q}$-stable $\mathcal{O}$-lattice $T_{f_m} \subset V_{f_m}$ and an isomorphism $T_f/p^m T_f \cong T_{f_m}/p^m T_{f_m}$ as $\mathcal{O}[G_\mathbb{Q}]$-modules that identifies $T_f^+/p^m T_f^+$ with $T_{f_m}^+/p^m T_{f_m}^+$ as $\mathcal{O}[G_{\mathbb{Q}_p}]$-modules;

(c) an equality of ideals $(\mathcal{L}_f^\Sigma, p^m) = (\mathcal{L}_{f_m}^\Sigma, p^m) \subseteq \Lambda_\mathcal{O}$.

The forms f_m in (a) are just those defined in the discussion of Hida families in Section 2.6. Then (b) is just (2-6-1) and (2-6-2), and (c) is (2-6-3). Furthermore, we also have

(d) $\bar{\rho}_{f_m} \cong \bar{\rho}_f$ is irreducible and ramified at some $q \neq p$ such that $q \parallel M$;

(e) $X_{\mathbb{Q}_\infty,L}^\Sigma(f_m)$ is a torsion $\Lambda_\mathcal{O}$-module and $\mathrm{Ch}_L^\Sigma(f_m) = (\mathcal{L}_{f_m}^\Sigma) \subseteq \Lambda_\mathcal{O}$;

(f) $X_{\mathbb{Q}_\infty,L}^\Sigma(f_m)$ has no nonzero finite-order $\Lambda_\mathcal{O}$-submodules, so $F_L^\Sigma(f_m) = \mathrm{Ch}^\Sigma(f_m)$.

Note that (d) follows from (b) and the hypotheses on N and $\bar{\rho}_f$ in Theorem A, while (e) and (f) follow from the Iwasawa–Greenberg main conjecture for f_m (which holds by (a), (d), and Theorem 2.5.2 since f_m is of level M and $p \nmid M$) together with Proposition 2.3.3 and Lemma 2.3.4.

From (b) together with Lemma 2.3.1 we conclude that there is a $\Lambda_\mathcal{O}$-isomorphism

$$\mathrm{Sel}_{\mathbb{Q}_\infty,L}^\Sigma(f)[p^m] \cong \mathrm{Sel}_{\mathbb{Q}_\infty,L}^\Sigma(f_m)[p^m]$$

of $\Lambda_\mathcal{O}$-modules, and hence, upon taking Pontryagin duals, also a $\Lambda_\mathcal{O}$-isomorphism

$$X_{\mathbb{Q}_\infty,L}^\Sigma(f)/p^m X_{\mathbb{Q}_\infty,L}^\Sigma(f) \cong X_{\mathbb{Q}_\infty,L}^\Sigma(f_m)/p^m X_{\mathbb{Q}_\infty,L}^\Sigma(f_m).$$

From basic properties of Fitting ideals we then conclude that there is an equality of $\Lambda_\mathcal{O}$-ideals

$$(F_L^\Sigma(f), p^m) = (F_L^\Sigma(f_m), p^m).$$

Together with (c), (e), and (f) we then have

$$(F_L^\Sigma(f), p^m) = (\mathcal{L}_f^\Sigma, p^m) \subseteq \Lambda_\mathcal{O}. \tag{3-1-1}$$

As $\mathcal{L}_f$, and hence $\mathcal{L}_f^\Sigma$, is nonzero by a well-known theorem of Rohrlich [1988, Theorem 1], if m is large enough then $(\mathcal{L}_f^\Sigma, p^m) \neq p^m\Lambda_{\mathcal{O}}$. From this and (3-1-1) it then follows that if m is large enough, then $(F_L^\Sigma(f), p^m) \neq p^m\Lambda_{\mathcal{O}}$ and hence $F_L^\Sigma(f) \neq 0$. As $F_L^\Sigma(f) \neq 0$, $X_{\mathbb{Q}_\infty,L}^\Sigma(f)$ must be a torsion $\Lambda_{\mathcal{O}}$-module. It then follows from Proposition 2.3.3(ii) and Lemma 2.3.4(ii) that $\mathrm{Ch}_L^\Sigma(f) = F_L^\Sigma(f)$. Combining this with (3-1-1) we then conclude that for all integers m

$$(\mathrm{Ch}_L^\Sigma(f), p^m) = (\mathcal{L}_f^\Sigma, p^m) \subseteq \Lambda_{\mathcal{O}}. \tag{3-1-2}$$

The characteristic ideal $\mathrm{Ch}_L^\Sigma(f)$ is a principal ideal. Let $\mathcal{C}_f^\Sigma$ be a generator. From (3-1-2) it follows that for each integer m there is a $u_m \in \Lambda_{\mathcal{O}}$ such that

$$\mathcal{C}_f^\Sigma - u_m\mathcal{L}_f^\Sigma \in p^m\Lambda_{\mathcal{O}}. \tag{3-1-3}$$

Let ϖ be a uniformizer of $\mathcal{O}$ and let e be such that $(p) = (\varpi^e)$. As $\mathcal{L}_f^\Sigma \neq 0$, there exists an integer $m_0 \geq 0$ such that $\mathcal{L}_f^\Sigma(f) \in \varpi^{m_0}\Lambda_{\mathcal{O}}$, but $\mathcal{L}_f^\Sigma(f) \notin \varpi^{m_0+1}\Lambda_{\mathcal{O}}$. It then follows from (3-1-3) that

$$u_{m'} - u_m \in \varpi^{me-m_0}\Lambda_{\mathcal{O}}, \quad m' \geq m.$$

Therefore the sequence $\{u_m\}$ converges in $\Lambda_{\mathcal{O}}$ to an element $u \in \Lambda_{\mathcal{O}}$ such that for all m, $u - u_m \in \varpi^{me-m_0}\Lambda_{\mathcal{O}}$. From this and (3-1-3) it follows that

$$\mathcal{C}_f^\Sigma - u\mathcal{L}_f^\Sigma \in \varpi^{me-m_0} \quad \text{for all } m \geq 0,$$

whence $\mathcal{C}_f^\Sigma = u\mathcal{L}_f^\Sigma$. That is $\mathcal{C}_f^\Sigma \in (\mathcal{L}_f^\Sigma)$.

Since $X_{\mathbb{Q}_\infty,L}^\Sigma(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module, $\mathrm{Ch}_L^\Sigma(f)$ is nonzero, and so $\mathcal{C}_f^\Sigma \neq 0$. We may then reverse the roles of $\mathcal{C}_f^\Sigma$ and $\mathcal{L}_f^\Sigma$ in the above argument to show that $\mathcal{L}_f^\Sigma \in (\mathcal{C}_f^\Sigma)$. From the two inclusions we then conclude

$$(\mathcal{L}_f^\Sigma) = (\mathcal{C}_f^\Sigma) = \mathrm{Ch}_L^\Sigma \subseteq \Lambda_{\mathcal{O}}.$$

This proves the desired equality, at least for the chosen L and for Σ containing all primes $\ell \mid N$. But, as observed in Section 2.5, this implies the desired equality for all sets Σ and all possible L. That is, the Iwasawa–Greenberg main conjecture holds for f: Theorem 2.5.2 holds without hypothesis (iv).

3.2. *Proof of Theorem B.* Let f, L, $\mathcal{O}$ be as in the statement of Theorem B. As these also satisfy the hypotheses of Theorem A, $X_{\mathbb{Q}_\infty,L}(f)$ is a torsion $\Lambda_{\mathcal{O}}$-module and its $\Lambda_{\mathcal{O}}$-characteristic ideal $\mathrm{Ch}_L(f)$ is generated by the p-adic L-function $\mathcal{L}_f$. Furthermore, by Proposition 2.3.3, neither $X_{\mathbb{Q}_\infty,L}(f)$ nor $\mathcal{X}$ have a nonzero finite-order $\Lambda_{\mathcal{O}}$-submodule. To deduce the conclusions of Theorem B from this, we make a close study of $\mathrm{Sel}_{\mathbb{Q}^\infty}(f)[\gamma-1]$ and $\mathcal{S}[\gamma-1]$, following Greenberg [1999].

Since $H^1(\mathbb{F}_p, (\mathcal{M}^-)^{I_p}) = \ker\{H^1(\mathbb{Q}_p, \mathcal{M}^-) \twoheadrightarrow H^1(I_p, \mathcal{M}^-)^{G_{\mathbb{Q}_p}}\}$, it follows from Proposition 2.3.2 — specifically the surjectivity of (2-3-2) — that there is an exact sequence

$$0 \to \mathcal{S} \to \mathrm{Sel}_{\mathbb{Q}_\infty,L}(f) \to H^1(\mathbb{F}_p, (\mathcal{M}^-)^{I_p}) \to 0.$$

As $G_{\mathbb{Q}_p}$ acts on $\mathcal{M}^- \cong \Lambda^*$ through the character $\alpha_f \Psi^{-1}$, $(\mathcal{M}^-)^{I_p} \cong \Lambda^*[\gamma - 1] = \mathrm{Hom}_{\mathbb{Z}_p}(\mathcal{O}, \mathbb{Q}_p/\mathbb{Z}_p) \cong L/\mathcal{O}$, with $G_{\mathbb{Q}_p}$ acting through the unramified character α_f. Let

$$\alpha_p = \alpha_f(\mathrm{frob}_p).$$

Then $H^1(\mathbb{F}_p, (\mathcal{M}^-)^{I_p}) = 0$ unless $\alpha_p = 1$ (i.e., unless f has split multiplicative reduction at p), in which case it is isomorphic to $L/\mathcal{O}$. Letting $\mathrm{Ch}_L(f)'$ be the $\Lambda_{\mathcal{O}}$-characteristic ideal of $\mathcal{X}$, it follows that

$$\mathrm{Ch}_L(f) = \mathrm{Ch}_L(f)' \cdot \begin{cases} (\gamma - 1), & f \text{ has split multiplicative reduction at } p, \\ 1, & \text{otherwise.} \end{cases}$$

This reflects the "extra zero" phenomenon in the split multiplicative case observed at the end of Section 2.4. In fact, we then have

$$\mathrm{Ch}_L(f)' = \begin{cases} (\mathcal{L}'_f), & f \text{ has split multiplicative reduction at } p, \\ (\mathcal{L}_f), & \text{otherwise.} \end{cases}$$

As $\mathcal{X}$ has no nonzero finite-order $\Lambda_{\mathcal{O}}$-submodules, a standard result[9] in Iwasawa theory gives $\#\mathcal{X}/(\gamma-1)\mathcal{X} = \#\Lambda_{\mathcal{O}}/(\gamma-1, \mathrm{Ch}_L(f)')$. As $\#\mathcal{S}[\gamma-1] = \#\mathcal{X}/(\gamma-1)\mathcal{X}$, we then find

$$\#\mathcal{S}[\gamma - 1] = \begin{cases} \#\mathcal{O}/(\mathcal{L}'_f(\phi_0)), & f \text{ has split multiplicative reduction at } p, \\ \#\mathcal{O}/(\mathcal{L}_f(\phi_0)), & \text{otherwise,} \end{cases} \tag{3-2-1}$$

where $\phi_0 : \Lambda_{\mathcal{O}} \to \mathcal{O}$ is the continuous $\mathcal{O}$-algebra homomorphism sending γ to 1.

Let $\Sigma = \{\ell \mid Np\}$. Let

$$W = \mathcal{M}[\gamma - 1] \cong T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p \quad \text{and} \quad W^\pm = \mathcal{M}[\gamma - 1]^\pm \cong T_f^\pm \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p.$$

Let

$$\mathcal{P}_\Sigma = H^1(\mathbb{Q}_p, \mathcal{M}^-) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, \mathcal{M})$$

and

$$P_\Sigma = H^1(\mathbb{Q}_p, W)/L_p(W) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, W),$$

where $L_p(W) = \mathrm{im}\{H^1(\mathbb{Q}_p, W^+) \to H^1(\mathbb{Q}_p, W)\}$. Let P_Σ^{div} be defined just as P_Σ but with $L_p(W)$ replaced by its maximal divisible subgroup $L_p(W)^{\mathrm{div}}$. The usual

[9] See note 5.

(torsion) Bloch–Kato Selmer group for T_f is just

$$\mathrm{Sel}_L(f) = \ker\{H^1(G_\Sigma, W) \xrightarrow{\text{res}} P_\Sigma^{\text{div}}\}.$$

As the restriction map $H^1(G_\Sigma, \mathcal{M}) \to \mathcal{P}_\Sigma$ is surjective by Proposition 2.3.2, we conclude that there is a short exact sequence

$$0 \to \mathrm{Sel}_L(f) \to \mathcal{S}[\gamma-1] \to \mathrm{im}\{H^1(G_\Sigma, W) \xrightarrow{\text{res}} P_\Sigma^{\text{div}}\} \cap \ker\{P_\Sigma^{\text{div}} \to \mathcal{P}_\Sigma[\gamma-1]\} \to 0.$$

Let $K = \ker\{P_\Sigma^{\text{div}} \to \mathcal{P}_\Sigma[\gamma-1]\}$. We claim that

$$\#\mathcal{S}[\gamma-1] = \#\mathrm{Sel}_L(f) \cdot \#K. \tag{3-2-2}$$

If $\mathrm{Sel}_L(f)$ is infinite, there is nothing to prove since $\mathrm{Sel}_L(f) \subset \mathcal{S}[\gamma-1]$. Suppose then that $\mathrm{Sel}_L(f)$ is finite. We will show that the restriction map $H^1(G_\Sigma, W) \xrightarrow{\text{res}} P_\Sigma^{\text{div}}$ is surjective, from which the claim follows.

By global duality, the cokernel of the restriction map $H^1(G_\Sigma, W) \to P_\Sigma^{\text{div}}$ is dual to a subquotient of

$$\mathrm{Sel}^\Sigma(T_f)^{\text{sat}} = \ker\{H^1(G_\Sigma, T_f) \to H^1(\mathbb{Q}_p, T_f)/L_p(T_f)^{\text{sat}}\},$$

where $L_p(T_f) = \mathrm{im}\{H^1(\mathbb{Q}_p, T_f^+) \to H^1(\mathbb{Q}_p, T_f)\}$ and

$$L_p(T_f)^{\text{sat}} = \{x \in H^1(\mathbb{Q}_p, T_f) : p^n x \in L_p(T_f) \text{ for some } n \geq 0\}.$$

Here we have used that $T_f \cong \mathrm{Hom}_{\mathbb{Z}_p}(W, \mathbb{Q}_p/\mathbb{Z}_p(1))$ as an $\mathcal{O}[G_\mathbb{Q}]$-module and that such an isomorphism identifies $L_p(T_f)^{\text{sat}}$ and $L_p(W)^{\text{div}}$ as mutual annihilators under local Tate duality. Then $\mathrm{Sel}^\Sigma(T_f)^{\text{sat}}$ is a torsion-free $\mathcal{O}$-module (as $\bar{\rho}_f$ is irreducible) and its $\mathcal{O}$-rank equals the $\mathcal{O}$-corank of $\mathrm{Sel}_L(f)$. In fact, $\mathrm{Sel}^\Sigma(T_f)^{\text{sat}} = H^1(G_\Sigma, T_f) \cap H^1_f(\mathbb{Q}, V_f)$, where

$$H^1_f(\mathbb{Q}, V_f) = \ker\left\{H^1(G_\Sigma, V_f) \xrightarrow{\text{res}} H^1(\mathbb{Q}_p, V_f)/L_p(V_f) \times \prod_{\ell \in \Sigma, \ell \neq p} H^1(\mathbb{Q}_\ell, V_f)\right\}$$

and $L_p(V_f) = \mathrm{im}\{H^1(\mathbb{Q}_p, V_f^+) \to H^1(\mathbb{Q}_p, V_f)\}$. (So $H^1_f(\mathbb{Q}, V_f)$ is just the usual characteristic zero Bloch–Kato Selmer group of V_f.) In particular, the $\mathcal{O}$-rank of $\mathrm{Sel}^\Sigma(T_f)^{\text{sat}}$ is the L-dimension of $H^1_f(\mathbb{Q}, V_f)$. The image of $H^1_f(\mathbb{Q}, V_f)$ in $H^1(G_\Sigma, T_f \otimes_{\mathbb{Z}_p} \mathbb{Q}_p/\mathbb{Z}_p) \cong H^1(G_\Sigma, W)$ is the maximal divisible submodule of $\mathrm{Sel}_L(f)$. However, the latter is assumed to be of finite order, so its maximal divisible subgroup is trivial. This proves that $\mathrm{Sel}^\Sigma(T_f)^{\text{sat}} = 0$ and hence that the restriction map $H^1(G_\Sigma, W) \xrightarrow{\text{res}} P_{\Sigma,x}^{\text{div}}$ is a surjection. The equality (3-2-2) follows.

Put

$$L^{\text{alg}}(f, 1) = \frac{L(f, 1)}{-2\pi i \Omega_f^+}.$$

Combining (3-2-1) with (3-2-2), the Greenberg–Stevens formula (2-4-3), and the specialization formula for $\mathcal{L}_f$ yields

$$\#\mathrm{Sel}_L(f)\cdot\#K=\begin{cases}\#\mathcal{O}/(\frac{1}{\log_p u}\cdot\mathfrak{L}(V_f)\cdot L^{\mathrm{alg}}(f,1)), & \alpha_p=1,\\ \#\mathcal{O}((1-\alpha_p)^2\cdot L^{\mathrm{alg}}(f,1)), & \text{otherwise.}\end{cases}\tag{3-2-3}$$

Therefore, to complete the proof Theorem B it remains to express $\#K$ in terms of Tamagawa factors and the L-invariant $\mathfrak{L}(V_f)$.

From the definition of K,

$$K=\prod_{\ell\in\Sigma}K_\ell,\tag{3-2-4}$$

with

$$K_\ell=\begin{cases}\ker\{H^1(\mathbb{Q}_\ell,W)\to H^1(\mathbb{Q}_\ell,\mathcal{M})\}, & \ell\neq p,\\ \ker\{H^1(\mathbb{Q}_p,W)/L_p(W)^{\mathrm{div}}\to H^1(\mathbb{Q}_p,\mathcal{M}^-)\}, & \ell=p.\end{cases}$$

If $\ell\neq p$, then $\mathcal{M}^{I_\ell}$ is $(\gamma-1)$-divisible and so $H^1(I_\ell,W)\hookrightarrow H^1(I_\ell,\mathcal{M})$ and

$$K_\ell=\ker\{H^1(\mathbb{F}_\ell,W^{I_\ell})\to H^1(\mathbb{F}_\ell,\mathcal{M}^{I_\ell})=0\}=H^1(\mathbb{F}_\ell,W^{I_\ell}).$$

Therefore

$$\#K_\ell=\#H^1(\mathbb{F}_\ell,W^{I_\ell})=c_\ell(T_f),\tag{3-2-5}$$

where $c_\ell(T_f)=\#H^1(\mathbb{F}_\ell,W^{I_\ell})$ is just the Tamagawa number at $\ell\neq p$ defined by Bloch and Kato for the p-adic representation T_f. Note that $c_\ell(T_f)=1$ if $\ell\nmid N$ (i.e., if T_f is unramified at ℓ). Hence to complete the proof of Theorem B it remains to express $\#K_p$ in terms of α_p if f does not have split multiplicative reduction at p (equivalently $\alpha_p\neq 1$) and in terms of $\mathfrak{L}(V_f)$ and the Tamagawa number at p of T_f otherwise.

Let

$$c_p'=\#\ker\{H^1(\mathbb{Q}_p,W)/L_p(W)^{\mathrm{div}}\twoheadrightarrow H^1(\mathbb{Q}_p,W)/L_p(W)\}$$

and

$$c_p''=\#\ker\{H^1(\mathbb{Q}_p,W)/L_p(W)\to H^1(\mathbb{Q}_p,\mathcal{M})/L_p(\mathcal{M})\}.$$

Then

$$\#K_p=c_p'c_p''.$$

By Tate local duality, $L_p(W)$ is dual to $H^1(\mathbb{Q}_p,T_f)/L_p(T_f)$ and $L_p(W)^{\mathrm{div}}$ is dual to $H^1(\mathbb{Q}_p,T_f)/L_p(T_f)^{\mathrm{sat}}$. Therefore

$$c_p'=\#(L_p(W)/L_p(W)^{\mathrm{div}})=\#(L_p(T_f)^{\mathrm{sat}}/L_p(T_f)).$$

Since

$$H^1(\mathbb{Q}_p,T_f)/L_p(T_f)\hookrightarrow H^1(\mathbb{Q}_p,T_f^-)$$

and

$$H^1(\mathbb{Q}_p, T_f)/L_p(T_f)^{\mathrm{sat}} \hookrightarrow H^1(\mathbb{Q}_p, V_f^-),$$

we find that the (injective) image of $L_p(T_f)^{\mathrm{sat}}/L_p(T_f)$ in $H^1(\mathbb{Q}_p, T_f^-)$ is just

$$\mathrm{im}\{H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-)\} \cap \ker\{H^1(\mathbb{Q}_p, T_f^-) \to H^1(\mathbb{Q}_p, V_f^-)\}.$$

But $H^1(I_p, T_f^-) \hookrightarrow H^1(I_p, V_f^-)$, so

$$\ker\{H^1(\mathbb{Q}_p, T_f^-) \to H^1(\mathbb{Q}_p, V_f^-)\} = H^1(\mathbb{F}_p, T_f^-).$$

On the other hand, the boundary map injects the cokernel of $H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-)$ into $H^2(\mathbb{Q}_p, T_f^+)$ but sends the subgroup $H^1(\mathbb{F}_p, T_f^-)$ to zero (since $\mathrm{Gal}(\overline{\mathbb{F}}_p/\mathbb{F}_p)$ has cohomological dimension one). Hence $H^1(\mathbb{F}_p, T_f^-)$ is contained in the image of $H^1(\mathbb{Q}_p, T_f)/L_p(T_f) \hookrightarrow H^1(\mathbb{Q}_p, T_f^-)$. It then follows that

$$L_p(T_f)^{\mathrm{sat}}/L_p(T_f) \xrightarrow{\sim} H^1(\mathbb{F}_p, T_f^-) \cong \begin{cases} 0, & \alpha_p = 1, \\ \mathcal{O}/(\alpha_p - 1), & \text{otherwise.} \end{cases}$$

In particular,

$$c_p' = \begin{cases} 1, & \alpha_p = 1, \\ \#(\mathcal{O}/(\alpha_p - 1)), & \text{otherwise.} \end{cases}$$

It remains to deduce the desired expression for c_p''. By definition c_p'' equals

$$\#\Big(\mathrm{im}\{H^1(\mathbb{Q}_p, W)/L_p(W) \hookrightarrow H^1(\mathbb{Q}_p, W^-)\} \\ \cap \ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\}\Big).$$

Since $H^2(\mathbb{Q}_p, W^+)$ is dual to $H^0(\mathbb{Q}_p, T_f^-)$ and the latter is 0 if $\alpha_p \neq 1$, we have $H^1(\mathbb{Q}_p, W)/L_p(W) \xrightarrow{\sim} H^1(\mathbb{Q}_p, W^-)$ if $\alpha_p \neq 1$. It follows that in this case

$$c_p'' = \#\ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\} = \#(\mathcal{M}^-)^{G_{\mathbb{Q}_p}}/(\gamma - 1) \cdot (\mathcal{M}^-)^{G_{\mathbb{Q}_p}}.$$

As I_p acts on $\mathcal{M}^-$ through the character Ψ^{-1} and frob_p acts on $(\mathcal{M}^-)^{I_p} = \mathcal{M}^-[\gamma - 1] \cong L/\mathcal{O}$ as multiplication by α_p, we find

$$c_p'' = \#L/\mathcal{O}[\alpha_p - 1] = \#\mathcal{O}/(\alpha_p - 1), \quad \alpha_p \neq 1.$$

Suppose then that $\alpha_p = 1$. It follows from local duality that c_p'' equals the index of the $\mathcal{O}$-submodule of $H^1(\mathbb{Q}_p, T_f^+)$ generated by $\ker\{H^1(\mathbb{Q}_p, T_f^+) \hookrightarrow H^1(\mathbb{Q}_p, T_f)\}$ and the annihilator of $\ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\}$. The first is just the image of $\mathcal{O} \cong H^0(\mathbb{Q}_p, T_f^-) \to H^1(\mathbb{Q}_p, T_f^+)$ determined by the $G_{\mathbb{Q}_p}$-extension T_f. Let c_{V_f} be an $\mathcal{O}$-generator; this is a nonzero element in ℓ_{V_f} in the notation of Section 2.2. On the other hand, as $H^1(\mathbb{Q}_p, W^-) \cong \mathrm{Hom}_{\mathrm{cts}}(G_{\mathbb{Q}_p}^{\mathrm{ab},p}, L/\mathcal{O})$, the kernel

$\ker\{H^1(\mathbb{Q}_p, W^-) \to H^1(\mathbb{Q}_p, \mathcal{M}^-)\}$ is readily seen to be $\mathrm{Hom}_{\mathrm{cts}}(\Gamma, L/\mathcal{O})$ — those homomorphisms that factor through Γ. Then, under the identification

$$H^1(\mathbb{Q}_p, T_f^+) = H^1(\mathbb{Q}_p, \mathcal{O}(1)) = (\varprojlim_n \mathbb{Q}_p{}^\times/(\mathbb{Q}_p{}^\times)^{p^n}) \otimes_{\mathbb{Z}_p} \mathcal{O},$$

the annihilator of $\mathrm{Hom}_{\mathrm{cts}}(\Gamma, L/\mathcal{O})$ is identified with the $\mathcal{O}$-module $p \otimes \mathcal{O}$ generated by the image of $p^{\mathbb{Z}}$. The index of $\mathcal{O} \cdot c_{V_f} + p \otimes \mathcal{O}$ is just the index of the projection of c_{V_f} to $(\varprojlim_n \mathbb{Z}_p^\times/(\mathbb{Z}_p^\times)^{p^n}) \otimes_{\mathbb{Z}_p} \mathcal{O}$. From the definition of ψ_{cyc} in Section 2.2, this index is just $\#\mathcal{O}/((1/\log_p u) \cdot \psi_{\mathrm{cyc}}(c_{V_f}))$. So by the definition of ψ_{ur} (which is nonzero on c_{V_f} as $0 \neq c_{V_f} \in \ell_{V_f}$) and the definition of $\mathfrak{L}(V_f)$,

$$c_p'' = \#\mathcal{O}/((1/\log_p u) \cdot \psi_{\mathrm{cyc}}(c_{V_f})) = \#\mathcal{O}/((1/\log_p u) \cdot \mathfrak{L}(V_f) \cdot \psi_{\mathrm{ur}}(c_{V_f})), \quad \alpha_p = 1.$$

Combining the formulas for c_p'' in the two cases with those for c_p' we find

$$\#K_p = \begin{cases} \#\mathcal{O}/((1/\log_p u) \cdot \mathfrak{L}(V_f) \cdot \psi_{\mathrm{ur}}(c_{V_f})), & \alpha_p = 1, \\ \#\mathcal{O}/(\alpha_p - 1)^2, & \alpha_p \neq 1. \end{cases} \tag{3-2-6}$$

Suppose $\mathfrak{L}(V_f) \neq 0$ if $\alpha_p = 1$. Then combining (3-2-3) with (3-2-4), (3-2-5), and (3-2-6) yields

$$\#\mathcal{O}/(L^{\mathrm{alg}}(f, 1)) = \#\mathrm{Sel}_L(f) \cdot \prod_{\ell \neq p} c_\ell(T_f) \cdot \begin{cases} \#\mathcal{O}/(\psi_{\mathrm{ur}}(c_{V_f})), & \alpha_p = 1, \\ 1, & \alpha_p \neq 1. \end{cases}$$

That the final term is just the Bloch–Kato Tamagawa number at p of the representation T_f, which we denote $c_p(T_f)$, can be shown as in [Dummigan 2005] (in that paper $c_p(T_f)$ is denoted $\mathrm{Tam}_M^0(T_f)$). The only significant change is the need to include the $\mathcal{O}$-action, but this is a straightforward modification. In the $p \nmid N$ case — that is, the case where V_f is a crystalline representation of $G_{\mathbb{Q}_p}$ — the fact that $c_p(T_f) = 1$ follows by the arguments used to prove [Dummigan 2005, Theorem 5.1]. The $p \| N$ case — in which case V_f is a semistable representation of $G_{\mathbb{Q}_p}$ — follows as in [Dummigan 2005, §7] from the arguments used to prove [Dummigan 2005, Theorem 6.1]. We therefore have the formula asserted in Theorem B:

$$\#\mathcal{O}/(L^{\mathrm{alg}}(f, 1)) = \#\mathrm{Sel}_L(f) \cdot \prod_{\ell} c_\ell(T_f). \tag{3-2-7}$$

This completes the proof of Theorem B.

3.3. *Proof of Theorem C.* Theorem C is just a special case of Theorem B. To see this, let E be as in Theorem C and let $f \in S_2(\Gamma_0(N))$ be the newform associated with E, so N is the conductor of E and $L(E, s) = L(f, s)$. For Theorem C to follow from Theorem B, it suffices to have that under the hypotheses of Theorem C, hypotheses (i), (ii), and (iii) of Theorem B hold for f and Ω_E is a $\mathbb{Z}_{(p)}^\times$-multiple of $-2\pi i \Omega_f^+$.

That hypotheses (i) and (ii) of Theorem C imply hypotheses (i) and (ii) of Theorem B is immediate. Furthermore, as noted in the example at the end of Section 2.2, if E has split multiplicative reduction at p then the $\mathfrak{L}$-invariant $\mathfrak{L}(V_f)$ of f is nonzero, hence hypothesis (iii) of Theorem B also holds.

To compare periods, we first recall that if ω_E is a Néron differential of E then

$$\Omega_E = \int_{c^+} \omega_E \in \mathbb{C}^\times,$$

where c^+ is a generator of the submodule $H_1(E(\mathbb{C}), \mathbb{Z})^+ \subset H_1(E(\mathbb{C}), \mathbb{Z})$ that is fixed by the action of $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$; this is well defined up to multiplication by ± 1. Now let

$$\phi : X_1(N) \to E^{\mathrm{opt}}$$

be an optimal parametrization for the $\mathbb{Q}$-isogeny class of E as in [Stevens 1989, Proposition (1.4)]. Then, as demonstrated in the proof of [Greenberg and Vatsal 2000, Proposition (3.1)], $\Omega_{E^{\mathrm{opt}}}$ equals $-2\pi i \Omega_f^+$ up to a $\mathbb{Z}_{(p)}^\times$-multiple[10]. Let

$$\beta : E^{\mathrm{opt}} \to E$$

be a $\mathbb{Q}$-isogeny. Since $E[p]$ is an irreducible $G_{\mathbb{Q}}$-representation, β can be chosen so that its degree is prime to p. Then $\beta^*\omega_E$ is a $\mathbb{Z}_{(p)}^\times$-multiple of $\omega_{E^{\mathrm{opt}}}$, and so Ω_E is a $\mathbb{Z}_{(p)}^\times$-multiple of $\Omega_{E^{\mathrm{opt}}}$ and hence also of $-2\pi i \Omega_f^+$.

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COMMENSURATORS OF SOLVABLE S-ARITHMETIC GROUPS

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We show that the abstract commensurator of an S-arithmetic subgroup of a solvable algebraic group over $\mathbb{Q}$ is isomorphic to the $\mathbb{Q}$-points of an algebraic group, and compare this with examples of nonlinear abstract commensurators of S-arithmetic groups in positive characteristic. In particular, we include a description of the abstract commensurator of the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$.

1. Introduction

Overview. In this paper we show that the abstract commensurator of an S-arithmetic subgroup of a solvable $\mathbb{Q}$-group is isomorphic to the $\mathbb{Q}$-points of an algebraic group. We then include examples to show that the analogous result in positive characteristic does not hold. As part of these examples, we provide a description of the abstract commensurator of the lamplighter group.

Background. A $\mathbb{Q}$-*group* $\boldsymbol{G}$ is a linear algebraic group defined over $\mathbb{Q}$. For S any finite set of prime numbers, let $\boldsymbol{G}(S)$ denote the set of S-*integer points* of G, that is, those matrices in $\boldsymbol{G}(\mathbb{Q})$ whose entries have denominators with prime divisors belonging to S. A subgroup of $\boldsymbol{G}(\mathbb{Q})$ is S-*arithmetic* if it is commensurable with $\boldsymbol{G}(S)$. When $S = \varnothing$, an S-arithmetic group is called an *arithmetic* group.

Remark. Beware of our unconventional choice of notation for S, which by definition includes only *non-Archimedean* valuations on $\mathbb{Q}$.

The *abstract commensurator* of a group Γ, denoted $\mathrm{Comm}(\Gamma)$, is the group of equivalence classes of isomorphisms between finite-index subgroups of Γ, where two isomorphisms are equivalent if they agree on a finite-index subgroup of Γ.

The starting point for our work is the following result, immediate from the fact that S-arithmetic subgroups of $\mathbb{Q}$-groups are preserved by isomorphism of their ambient $\mathbb{Q}$-groups; see [Platonov and Rapinchuk 1994, Theorem 5.9, p. 269]. Let $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ denote the group of $\mathbb{Q}$-defined automorphisms of $\boldsymbol{G}$.

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Proposition 1.1. *Suppose $\boldsymbol{G}$ is any $\mathbb{Q}$-group. For any finite set of primes S, there is a natural map* $\Theta : \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \to \mathrm{Comm}(\boldsymbol{G}(S))$.

In the case that $\boldsymbol{G}$ is a higher-rank, connected, adjoint, semisimple linear algebraic group that is simple over $\mathbb{Q}$, rigidity theorems of Margulis [1991] imply that the map Θ of Proposition 1.1 is an isomorphism. Similarly, if $\boldsymbol{G}$ is unipotent then Θ is an isomorphism by Mal'cev rigidity; see Theorem 3.3. Moreover, in each of these cases the group $\mathrm{Aut}(\boldsymbol{G})$ has the structure of a $\mathbb{Q}$-group such that $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \cong \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q})$.

Main result. When $\boldsymbol{G}$ is solvable and not unipotent the group $\boldsymbol{G}(S)$ is not rigid in the above sense. One approach to remedying this lack of rigidity is taken in [Witte 1997], where solvable S-arithmetic groups are shown to satisfy a form of Archimedean superrigidity. For solvable arithmetic groups, another study of this failure of rigidity appears in [Grunewald and Platonov 1999]. Extending these methods, we prove the main theorem of this paper:

Theorem 1.2. *Let $\boldsymbol{G}$ be a solvable $\mathbb{Q}$-group and let S be a finite set of primes. Then there is a finite-index subgroup* $\mathrm{Comm}^0(\boldsymbol{G}(S)) \leq \mathrm{Comm}(\boldsymbol{G}(S))$ *and a $\mathbb{Q}$-group $\boldsymbol{D}$ such that*

$$\mathrm{Comm}^0(\boldsymbol{G}(S)) \cong \boldsymbol{D}(\mathbb{Q}).$$

The group $\boldsymbol{D}$ is constructed explicitly as a quotient of an iterated semidirect product of groups. See Section 3C for proof and details.

When $S = \varnothing$ the arithmetic group $\boldsymbol{G}(S) = \boldsymbol{G}(\mathbb{Z})$ is virtually polycyclic, and hence virtually a lattice in a connected, simply connected solvable Lie group. In [Studenmund 2015] it was shown that the abstract commensurator of a lattice in a connected, simply connected solvable Lie group is isomorphic to the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. Therefore the $S = \varnothing$ case of Theorem 1.2 is a consequence of [Studenmund 2015].

When $S \neq \varnothing$ the group $\boldsymbol{G}(S)$ is no longer necessarily polycyclic, so different methods are necessary. When $\boldsymbol{U}$ is a unipotent group, for any set of primes S we have

$$\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q}).$$

In particular the abstract commensurator is independent of S. For example, we have $\mathrm{Comm}(\mathbb{Z}[1/2]) \cong \mathrm{Comm}(\mathbb{Z}[1/3]) \cong \mathbb{Q}^*$. Note that for each nontrivial unipotent group this provides an infinite family of pairwise non-abstractly-commensurable groups with isomorphic abstract commensurator.

When $\boldsymbol{G}$ contains a torus, the abstract commensurator of an S-arithmetic subgroup may depend on S. For example, let $\boldsymbol{T}$ be the Zariski-closure of the cyclic subgroup generated by the matrix $\left(\begin{smallmatrix} 2 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Note that $\boldsymbol{T}$ is diagonalizable over $\mathbb{R}$ and over $\mathbb{Q}_{11}$ since 5 has an 11-adic square root, while $\boldsymbol{T}$ is not diagonalizable over either $\mathbb{Q}$

or $\mathbb{Q}_3$. It follows from Theorem 2.1 below that

$$\boldsymbol{T}(\varnothing) \doteq \mathbb{Z}, \quad \boldsymbol{T}(\{3\}) \doteq \mathbb{Z}, \quad \boldsymbol{T}(\{11\}) \doteq \mathbb{Z}^2, \quad \text{and} \quad \boldsymbol{T}(\{3, 11\}) \doteq \mathbb{Z}^2,$$

where we write $G \doteq H$ if G and H contain isomorphic subgroups of finite index. Then $\mathrm{Comm}(\boldsymbol{T}(\{11\}))$ and $\mathrm{Comm}(\boldsymbol{T}(\{3, 11\}))$ are each isomorphic to $\mathrm{GL}_2(\mathbb{Q})$, but neither is isomorphic to $\mathrm{Comm}(\boldsymbol{T}(\{3\})) \cong \mathbb{Q}^*$. This dependence on S appears even for groups whose maximal torus acts faithfully on the unipotent radical; see Theorem 1.3.

Explicit description of commensurator. A key case is when the action of any maximal torus of $\boldsymbol{G}$ on the unipotent radical of $\boldsymbol{G}$ is faithful. Such a solvable algebraic group is said to be *reduced*. When $\boldsymbol{G}$ is reduced, we have the following explicit statement whether or not $S = \varnothing$.

Theorem 1.3. *Let $\boldsymbol{G}$ be a connected and reduced solvable $\mathbb{Q}$-group, let S be a finite set of primes, and let Δ be an S-arithmetic subgroup of $\boldsymbol{G}$. Suppose $\boldsymbol{G}(S)$ is Zariski-dense in $\boldsymbol{G}$. There is an isomorphism of abstract groups*

$$\mathrm{Comm}(\Delta) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})) \rtimes \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}), \tag{1}$$

where N is the maximum rank of any torsion-free, free abelian subgroup of $\boldsymbol{T}(S)$ for any maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \leq \boldsymbol{G}$ and $\mathrm{Hom}_{\mathbb{Q}}$ denotes the group of $\mathbb{Q}$-vector space homomorphisms under addition. There is a subgroup $\mathrm{Comm}^0(\Delta) \leq \mathrm{Comm}(\Delta)$ of finite index which has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group.

Note that the semidirect product appearing in (1) is a semidirect product of abstract groups. However, there is a subgroup of finite index which has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. See Section 3 for details.

Remark. In the case $S = \varnothing$, Theorem 1.2 follows from Theorem 1.3 by the fact that any solvable arithmetic group Γ is abstractly commensurable with an arithmetic subgroup of a *reduced* solvable group. See [Grunewald and Platonov 1999, Theorem 3.4] for a proof of this fact. This is possible because arithmetic subgroups of tori are abstractly commensurable with arithmetic subgroups of abelian unipotent groups; both are virtually free abelian. The same method does not work when S is nonempty: S-arithmetic subgroups of tori are virtually free abelian while S-arithmetic subgroups of unipotent groups are not.

Remark. Bogopolski [2012] has computed abstract commensurators of the solvable Baumslag–Solitar groups to be

$$\mathrm{Comm}(\mathrm{BS}(1, n)) \cong \mathbb{Q} \rtimes \mathbb{Q}^*.$$

Theorem 1.3 recovers Bogopolski's result in the case that n is a prime power, since $\mathrm{BS}(1, p^2)$ is isomorphic to the group $\boldsymbol{G}(S)$, where $S = \{p\}$ and $\boldsymbol{G} = \boldsymbol{B}_2/Z(\boldsymbol{B}_2)$ for

$$\boldsymbol{B}_2 = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \,\middle|\, xy = 1 \right\} \subseteq \mathrm{GL}_2(\mathbb{C}).$$

Note that $\mathrm{BS}(1, n^k)$ is a finite-index subgroup of $\mathrm{BS}(1, n)$; hence the two groups have isomorphic abstract commensurators.

When n is not a prime power, $\mathrm{BS}(1, n)$ is no longer commensurable with an S-arithmetic group. However, $\mathrm{BS}(1, n^2)$ embeds as a Zariski-dense subgroup of

$$(\boldsymbol{B}_2/Z(\boldsymbol{B}_2))(S),$$

where S consists of the prime factors of n. It may be possible to modify the proof of Theorem 1.3 to compute $\mathrm{Comm}(\mathrm{BS}(1, n))$ for any n from this embedding.

Number fields. Above we have defined S-arithmetic subgroups only of $\mathbb{Q}$-groups, but S-arithmetic groups may be defined over any global field. Our methods fail to prove any obvious analog of Theorem 1.2 for S-arithmetic groups over general number fields. In particular, if Γ is an S-arithmetic subgroup of a unipotent group $\boldsymbol{U}$ defined over K then $\mathrm{Comm}(\Gamma)$ may depend on S, in contrast with the case of $K = \mathbb{Q}$. This is explained in more detail in Section 4.

Despite this difference, the conclusion of Theorem 1.2 holds for unipotent groups $\boldsymbol{G}$ and may hold for general solvable $\boldsymbol{G}$. The difficulty in finding a proof lies in finding an alternative to the use of Proposition 1.1; see the remarks at the end of Section 4.

Function fields and the lamplighter group. In contrast to the case of S-arithmetic groups over number fields, Theorem 1.2 has no obvious analog for S-arithmetic groups over global fields of positive characteristic. Section 5 includes examples demonstrating this failure.

A well-known example of a solvable S-arithmetic group in characteristic 2 is the lamplighter group $(\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}$. Section 6 describes the abstract commensurator of the lamplighter group, with the following main result.

Theorem 1.4. *Using the definitions in Equations (6) and (7) of Section 6, there is an isomorphism*

$$\mathrm{Comm}((\mathbb{Z}/2\mathbb{Z}) \wr \mathbb{Z}) \cong (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

Using this decomposition we show, for example, that the abstract commensurator of the lamplighter group contains every finite group as a subgroup.

2. Background and definitions

For any group Γ, a *partial automorphism* of Γ is an isomorphism between finite-index subgroups of Γ. Two partial automorphisms ϕ_1 and ϕ_2 are *equivalent* if there is some finite index $\Delta \leq \Gamma$ such that $\phi_1\big|_\Delta = \phi_2\big|_\Delta$; an equivalence class of partial automorphisms is a *commensuration* of Γ. The *abstract commensurator* $\mathrm{Comm}(\Gamma)$ is the group of commensurations of Γ. If Γ_1 and Γ_2 are abstractly commensurable groups then $\mathrm{Comm}(\Gamma_1) \cong \mathrm{Comm}(\Gamma_2)$. We will implicitly use this fact often.

A subgroup $\Delta \leq \Gamma$ is *commensuristic* if $\phi(\Delta \cap \Gamma_1)$ is commensurable with Δ for every partial automorphism $\phi : \Gamma_1 \to \Gamma_2$ of Γ. Say that Δ is *strongly commensuristic* if $\phi(\Delta \cap \Gamma_1) = \Delta \cap \Gamma_2$ for every such ϕ. If Δ is commensuristic, restriction induces a map $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Delta)$. If Δ is strongly commensuristic, then there is a natural map $\mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/\Delta)$.

A group Γ *virtually* has a property P if there is a subgroup $\Delta \leq \Gamma$ of finite index with property P. For any Λ, a *virtual homomorphism* $\Gamma \to \Lambda$ is a homomorphism from a finite-index subgroup of Γ to Λ. Two such virtual homomorphisms are *equivalent* if they agree on a finite-index subgroup of Γ.

By a *$\mathbb{Q}$-defined linear algebraic group*, or *$\mathbb{Q}$-group*, we mean a subgroup $\boldsymbol{G} \leq \mathrm{GL}_n(\mathbb{C})$ for some n that is closed in the Zariski topology and whose defining polynomials may be chosen to have coefficients in $\mathbb{Q}$. The *$\mathbb{Q}$-points* of $\boldsymbol{G}$ are $\boldsymbol{G}(\mathbb{Q}) = \boldsymbol{G} \cap \mathrm{GL}_n(\mathbb{Q})$. If S is a finite set of prime numbers, we define the group of *S-integer points* of $\boldsymbol{G}$, denoted $\boldsymbol{G}(S)$, to be the subgroup of elements of $\boldsymbol{G}(\mathbb{Q})$ with matrix coefficients having denominators divisible only by elements of S. A subgroup of $\boldsymbol{G}(\mathbb{Q})$ is *S-arithmetic* if it is commensurable with $\boldsymbol{G}(S)$. An abstract group Γ is *S-arithmetic* if it is abstractly commensurable with an S-arithmetic subgroup of some $\mathbb{Q}$-group $\boldsymbol{G}$.

Now let $\boldsymbol{G}$ be a solvable $\mathbb{Q}$-group, S be a finite set of primes, and $\Gamma = \boldsymbol{G}(S)$. Since $[\boldsymbol{G} : \boldsymbol{G}^0] < \infty$, we will assume $\boldsymbol{G}$ is connected. The subgroup $\boldsymbol{U} \leq \boldsymbol{G}$ consisting of all unipotent elements of $\boldsymbol{G}$ is connected, is defined over $\mathbb{Q}$, and is called the *unipotent radical*. For any maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \leq \boldsymbol{G}$, there is a semidirect product decomposition $\boldsymbol{G} = \boldsymbol{U} \rtimes \boldsymbol{T}$.

For any $\mathbb{Q}$-defined torus $\boldsymbol{T}$ and any field extension F of $\mathbb{Q}$, the *F-rank* of $\boldsymbol{T}$, denoted $\mathrm{rank}_F(\boldsymbol{T})$, is the dimension of any maximal subtorus of $\boldsymbol{T}$ diagonalizable over F. We will use the following special case of [Platonov and Rapinchuk 1994, Theorem 5.12, p. 276].

Theorem 2.1. *Let $\boldsymbol{T}$ be a torus defined over $\mathbb{Q}$ and S a finite set of prime numbers. Then $\boldsymbol{T}(S)$ is isomorphic to the product of a finite group and a free abelian group of rank*

$$N = \mathrm{rank}_{\mathbb{R}}(\boldsymbol{T}) - \mathrm{rank}_{\mathbb{Q}}(\boldsymbol{T}) + \sum_{p \in S} \mathrm{rank}_{\mathbb{Q}_p}(\boldsymbol{T}).$$

If $\boldsymbol{U}$ is a connected unipotent $\mathbb{Q}$-group, then $\mathrm{Aut}(\boldsymbol{U})$ may be identified with the automorphism group of the Lie algebra of $\boldsymbol{U}$ and thus has the structure of a $\mathbb{Q}$-group. This structure is such that $\mathrm{Aut}(\boldsymbol{U})(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{U})$, where $\mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{U})$ is the group of $\mathbb{Q}$-defined automorphisms of $\boldsymbol{U}$.

A solvable $\mathbb{Q}$-group $\boldsymbol{G}$ is said to be *reduced*, or to have *strong unipotent radical*, if the action of any maximal $\mathbb{Q}$-defined torus on the unipotent radical is faithful. If $\boldsymbol{G}$ is reduced then $\mathrm{Aut}(\boldsymbol{G})$ naturally has the structure of a $\mathbb{Q}$-group such that $\mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}) = \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ (see [Grunewald and Platonov 1999, Section 4] or [Baues and Grunewald 2006, Section 3]) and the identity component $\mathrm{Aut}^0(\boldsymbol{G})$ is a finite-index subgroup of $\mathrm{Aut}(\boldsymbol{G})$ that acts trivially on the quotient of $\boldsymbol{G}$ by its unipotent radical.

3. Proof of main theorems

3A. *Setup.* In this section we begin the work necessary to prove Theorem 1.2, by way of Theorem 1.3. Let $\boldsymbol{G}$ be a connected solvable $\mathbb{Q}$-group, let S be a finite set of prime numbers, and let $\Gamma \leq \boldsymbol{G}(\mathbb{Q})$ be an S-arithmetic subgroup. Replacing $\boldsymbol{G}$ by the Zariski-closure of Γ, we will assume going forward that Γ is Zariski-dense in $\boldsymbol{G}$.

Write $\boldsymbol{G} = \boldsymbol{U} \rtimes \boldsymbol{T}$ as above. We will assume without loss of generality that Γ decomposes as $\Gamma = \boldsymbol{U}(S) \rtimes \Gamma_{\boldsymbol{T}}$ for some finitely generated, torsion-free, free abelian S-arithmetic subgroup $\Gamma_{\boldsymbol{T}} \leq \boldsymbol{T}(S)$; see [Platonov and Rapinchuk 1994, Lemma 5.9] and Theorem 2.1.

A group Γ is *uniquely p-radicable* if for every $\gamma \in \Gamma$ there is a unique element $\delta \in \Gamma$ such that $\delta^p = \gamma$.

Lemma 3.1. *Suppose Δ is any finite-index subgroup of Γ and $p \in S$. Then $\Delta \cap \boldsymbol{U}(S)$ is the unique maximal uniquely p-radicable subgroup of Δ.*

Proof. Since $\Gamma_{\boldsymbol{T}}$ is isomorphic to $\mathbb{Z}^N$ for some N, it suffices to show that $\boldsymbol{U}(S) \cap \Delta$ is uniquely p-radicable. Moreover, because the property of being uniquely p-radicable is inherited by subgroups of finite index, it suffices to check that $\boldsymbol{U}(S)$ is uniquely p-radicable. It is a standard fact that $\boldsymbol{U}$ is $\mathbb{Q}$-isomorphic to a subgroup of the group of $n \times n$ matrices with 1's on the diagonal, which we denote $\boldsymbol{U}_n$. Therefore $\boldsymbol{U}(S)$ is commensurable with a subgroup of $\boldsymbol{U}_n(S)$. The desired property is preserved by commensurability of torsion-free groups, so it suffices to show that $\boldsymbol{U}_n(S)$ is uniquely p-radicable. This may easily be done by induction on n. □

Corollary 3.2. *If $S \neq \varnothing$, then $\boldsymbol{U}(S)$ is strongly commensuristic in Γ.*

Remark. If $S = \varnothing$ then Corollary 3.2 is still true when $\boldsymbol{G}$ is reduced. This follows from the fact that $\Gamma \cap \boldsymbol{U}$ is the Fitting subgroup of Γ for any arithmetic subgroup $\Gamma \leq \boldsymbol{G}(\mathbb{Q})$; see [Grunewald and Platonov 1999, Lemma 2.6] for a proof.

Theorem 3.3. *There is an isomorphism* $\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q})$.

Proof. Since $\boldsymbol{U}(S)$ has the property that for each $u \in \boldsymbol{U}(\mathbb{Q})$ there is some number k such that $u^k \in \boldsymbol{U}(\mathbb{Z})$, any partial automorphism ϕ of $\boldsymbol{U}(S)$ is determined by its values on $\boldsymbol{U}(\mathbb{Z})$. The resulting map $\phi\big|_{\boldsymbol{U}(\mathbb{Z})} : \boldsymbol{U}(\mathbb{Z}) \to \boldsymbol{U}(\mathbb{Q})$ uniquely extends to a $\mathbb{Q}$-defined homomorphism $\tilde{\phi} : \boldsymbol{U} \to \boldsymbol{U}$ by a theorem of Mal'cev (see, for example, the proof of [Raghunathan 1972, Theorem 2.11, p. 33].) Since the dimension of the Zariski-closure of $\phi(\boldsymbol{U}(\mathbb{Z}))$ is equal to the dimension of $\boldsymbol{U}$ by [Raghunathan 1972, Theorem 2.10, p. 32], the map $\hat{\phi}$ is an automorphism of $\boldsymbol{U}$.

The assignment $[\phi] \mapsto \tilde{\phi}$ gives a well-defined mapping $\xi : \mathrm{Comm}(\boldsymbol{U}(S)) \to \mathrm{Aut}(\boldsymbol{U})(\mathbb{Q})$. We see that ξ is injective because $\boldsymbol{U}(S)$ is Zariski-dense in $\boldsymbol{U}$, and ξ is surjective because every $\mathbb{Q}$-defined automorphism of $\boldsymbol{U}$ induces a commensuration of $\boldsymbol{U}(S)$ by Proposition 1.1. □

3B. *Reduced case.* Now assume that $\boldsymbol{G}$ is reduced. We prove Theorem 1.3 using methods following those used to prove Theorems A and C of [Grunewald and Platonov 1999].

Proof of Theorem 1.3. Let $\boldsymbol{U}$ be the unipotent radical of $\boldsymbol{G}$ and fix a maximal $\mathbb{Q}$-defined torus $\boldsymbol{T} \leq \boldsymbol{G}$. We assume without loss of generality that $\Delta = (\Delta \cap \boldsymbol{U}) \rtimes (\Delta \cap \boldsymbol{T})$.

Suppose $\phi : \Delta_1 \to \Delta_2$ is a partial automorphism of Δ. By Corollary 3.2 and Theorem 3.3, ϕ induces a $\mathbb{Q}$-defined automorphism $\Phi_{\boldsymbol{U}} \in \mathrm{Aut}(\boldsymbol{U})$. Define $\alpha : \boldsymbol{G} \to \mathrm{Aut}(\boldsymbol{U})$ to be the map induced by conjugation. Note that $\alpha\big|_{\boldsymbol{T}}$ is injective since $\boldsymbol{G}$ is reduced.

It is straightforward to check that for any $\delta \in \Delta_1$ we have

$$\Phi_{\boldsymbol{U}} \circ \alpha(\delta) \circ \Phi_{\boldsymbol{U}}^{-1} = \alpha(\phi(\delta)).$$

It follows that conjugation by $\Phi_{\boldsymbol{U}}$ preserves $\alpha(\boldsymbol{G})$ inside $\mathrm{Aut}(\boldsymbol{U})$. Conjugation by $\Phi_{\boldsymbol{U}}$ therefore induces an isomorphism between $\alpha(\boldsymbol{T})$ and $\alpha(\boldsymbol{T}')$ for some maximal $\mathbb{Q}$-defined torus $\boldsymbol{T}' \leq \boldsymbol{G}$, and hence an isomorphism $\Phi_{\boldsymbol{T}} : \boldsymbol{T} \to \boldsymbol{T}'$. Note that $\Phi_{\boldsymbol{T}}$ is defined to satisfy the relation

$$\Phi_{\boldsymbol{U}} \circ \alpha(t) \circ \Phi_{\boldsymbol{U}}^{-1} = \alpha(\Phi_{\boldsymbol{T}}(t)) \tag{2}$$

for all $t \in \boldsymbol{T}$.

The maps $\Phi_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}}$ determine a self-map of $\boldsymbol{G}$: for each $g \in \boldsymbol{G}$, write $g = ut$ for $u \in \boldsymbol{U}$ and $t \in \boldsymbol{T}$ and set

$$\Phi_0(g) := \Phi_{\boldsymbol{U}}(u)\Phi_{\boldsymbol{T}}(t).$$

Equation (2) implies that Φ_0 is a $\mathbb{Q}$-defined automorphism of $\boldsymbol{G}$. However, the map $\mathrm{Comm}(\Delta) \to \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ defined by $[\phi] \mapsto \Phi_0$ is *not* necessarily a well-defined homomorphism of groups. We will show that Φ_0 can be modified in a unique way to produce an automorphism Φ so that $\Phi(\delta)\phi(\delta)^{-1} \in Z(\boldsymbol{G})$ for all $\delta \in \Delta_1$. This condition will guarantee the map $[\phi] \mapsto \Phi$ defines a homomorphism.

It is straightforward to check from our definitions that $\alpha(\Phi_0(\delta)\phi(\delta)^{-1})$ is trivial for all $\delta \in \Delta_1$. Therefore $v(\delta) := \Phi_0(\delta)\phi(\delta)^{-1}$ defines a function $v : \Delta_1 \to Z(\boldsymbol{U})(\mathbb{Q})$. One can check that

$$v(\delta_1\delta_2) = v(\delta_1)\phi(\delta_1)v(\delta_2)\phi(\delta_1)^{-1}.$$

That is, ϕ is a *derivation* when $Z(\boldsymbol{U})(\mathbb{Q})$ is given the structure of a left Δ_1-module by $\delta \cdot z = \phi(\delta)z\phi(\delta)^{-1}$ for $\delta \in \Delta_1$ and $z \in Z(\boldsymbol{U})(\mathbb{Q})$.

The derivation v is trivial on $\Delta_1 \cap \boldsymbol{U}$, and therefore descends to a derivation $\bar{v} : \Delta_1 \cap \boldsymbol{T} \to Z(\boldsymbol{U})(\mathbb{Q})$. Now decompose $Z(\boldsymbol{U})(\mathbb{Q})$ as a direct sum of weight spaces for the action of $\boldsymbol{T}$ and let V be the sum of all weight spaces with nontrivial weights. Let $v^{\perp}$ be the component of the derivation $\bar{v}$ in the submodule V. Since $C_V(\boldsymbol{T})$ is trivial, it follows from a standard cohomological fact (see [Segal 1983, Chapter 3, Theorem 2**, p. 44]) that $v^{\perp}$ is an inner derivation. That is, there is some $x \in V$ such that $v^{\perp}(\delta) = \phi(\delta)x\phi(\delta)^{-1}x^{-1}$ for all $\delta \in \Delta \cap \boldsymbol{T}$. It follows that

$$v(\delta)x\phi(\delta)x^{-1}\phi(\delta)^{-1} \in Z(\boldsymbol{G})(\mathbb{Q}).$$

When x is viewed as an element of $Z(\boldsymbol{U})(\mathbb{Q})$, the choice of x is unique up to $Z(\boldsymbol{G})(\mathbb{Q})$.

Given Φ_0 and x as above, the assignment $\mu(\phi) = c_x \circ \Phi_0$, where $c_x(g) = xgx^{-1}$ for all $g \in \boldsymbol{G}$, determines a well-defined map

$$\mu : \mathrm{Comm}(\Delta) \to \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}).$$

One can check using an obvious modification of [Grunewald and Platonov 1999, Lemma 2.9] that μ is a homomorphism. Because Γ is Zariski-dense in $\boldsymbol{G}$, the map

$$\Theta : \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G}) \to \mathrm{Comm}(\boldsymbol{G}(S))$$

of Proposition 1.1 is injective. In fact Θ is a section of μ; to see this, note that if $\phi = \Theta(\Phi)$ then the associated maps $\Phi_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}}$ are $\Phi_{\boldsymbol{U}} = \Phi\big|_{\boldsymbol{U}}$ and $\Phi_{\boldsymbol{T}} = \Phi\big|_{\boldsymbol{T}}$, which clearly satisfy (2), and moreover the associated derivation ν is trivial. It follows that there is an isomorphism

$$\mathrm{Comm}(\Delta) \cong \ker(\mu) \rtimes \mathrm{Aut}(\boldsymbol{G})(\mathbb{Q}).$$

Now suppose that $[\phi] \in \ker(\mu)$. It follows from the above that ϕ is a virtual homomorphism $\Delta \to Z(\boldsymbol{G})(\mathbb{Q})$ trivial on $\Delta \cap \boldsymbol{U}$. We can view ϕ as a virtual homomorphism $\Delta \cap \boldsymbol{T} \to Z(\boldsymbol{G})(\mathbb{Q})$. Since $\Delta \cap \boldsymbol{T}$ is virtually $\mathbb{Z}^N$, the group of equivalence classes of such virtual homomorphisms is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$.

We therefore have a well-defined map

$$\xi : \ker(\mu) \to \mathrm{Hom}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})).$$

Clearly ξ is injective. On the other hand, suppose that $[\Delta \cap \boldsymbol{T} : \Lambda] < \infty$ and that $f : \Lambda \to Z(\boldsymbol{G})(\mathbb{Q})$ is a homomorphism. There is a finite-index subgroup $\widetilde{\Lambda} \le \Lambda$ such that $f(\widetilde{\Lambda}) \le Z(\boldsymbol{G})(S)$. The map

$$\phi : \boldsymbol{U}(S) \rtimes \widetilde{\Lambda} \to \boldsymbol{U}(S) \rtimes \widetilde{\Lambda}$$

defined by $\phi(u, \lambda) = (u \cdot f(\lambda), \lambda)$ induces a commensuration of Δ mapping to f under ξ; hence ξ is surjective. This completes the proof that $\mathrm{Comm}(\Delta)$ has the desired semidirect product decomposition.

Let

$$\mathrm{Comm}^0(\Delta) = \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q})) \rtimes \mathrm{Aut}^0(\boldsymbol{G})(\mathbb{Q}).$$

Clearly $\mathrm{Comm}^0(\Delta)$ has finite index in $\mathrm{Comm}(\Delta)$. We will show that $\mathrm{Comm}^0(\Gamma)$ has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. We first understand the action of $\mathrm{Aut}(\boldsymbol{G})$ on $\mathrm{Hom}(\mathbb{Q}^N, Z(\boldsymbol{G}))$. Any $\Phi \in \mathrm{Aut}_{\mathbb{Q}}(\boldsymbol{G})$ induces a commensuration of Δ virtually preserving $\boldsymbol{U}(S)$, hence induces a commensuration of $\boldsymbol{T}(S)$. Let $\widetilde{\Phi}_{\boldsymbol{T}} \in \mathrm{GL}_N(\mathbb{Q})$ be the automorphism corresponding to the induced commensuration of $\boldsymbol{T}(S)$. Then the action is given by

$$(\Phi \cdot \alpha)(t) = \Phi_{\boldsymbol{U}}(\alpha(\widetilde{\Phi}_{\boldsymbol{T}}^{-1} t)).$$

Note that if $\Phi \in \mathrm{Aut}^0(\boldsymbol{G})$ then Φ acts trivially on the quotient $\boldsymbol{G}/\boldsymbol{U}$; hence the induced map $\widetilde{\Phi}_{\boldsymbol{T}}$ is trivial.

The group $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$ is isomorphic to the $\mathbb{Q}$-points of $(\boldsymbol{G}_a)^{Nd}$, a product of additive groups defined over $\mathbb{Q}$, where d is the dimension of $Z(\boldsymbol{G})$. Under this identification, the action of $\mathrm{Aut}(Z(\boldsymbol{G}))(\mathbb{Q})$ by postcomposition on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^N, Z(\boldsymbol{G})(\mathbb{Q}))$ corresponds to the diagonal linear action of $\mathrm{Aut}(Z(\boldsymbol{G}))$ on $(\boldsymbol{G}_a)^{Nd}$. Since the restriction map $\mathrm{Aut}(\boldsymbol{G}) \to \mathrm{Aut}(Z(\boldsymbol{G}))$ is defined over $\mathbb{Q}$ by definition of the algebraic structure on $\mathrm{Aut}(\boldsymbol{G})$, the action map

$$\mathrm{Aut}^0(\boldsymbol{G}) \times (\boldsymbol{G}_a)^{Nd} \to (\boldsymbol{G}_a)^{Nd}$$

is defined over $\mathbb{Q}$. Hence the semidirect product $(\boldsymbol{G}_a)^{Nd} \rtimes \mathrm{Aut}^0(\boldsymbol{G})$ is an algebraic group whose $\mathbb{Q}$-points are identified with $\mathrm{Comm}^0(\Delta)$. □

3C. ***Nonreduced case.*** Now consider the case that $\boldsymbol{G}$ is a connected solvable group, not necessarily reduced. As above we will assume without loss of generality that Γ is Zariski-dense in $\boldsymbol{G}$ and decomposes as $\Gamma = \boldsymbol{U}(S) \rtimes \Gamma_{\boldsymbol{T}}$. Assume for the rest of this section that $S \neq \varnothing$. (The case that $S = \varnothing$ is addressed by the remarks following the statement of Theorem 1.2.) Our primary goal is to reduce to a situation where Theorem 1.3 can be applied. This reduction will occur over several steps.

Define $\boldsymbol{T}_0 \leq \boldsymbol{T}$ to be the centralizer of $\boldsymbol{U}$ in $\boldsymbol{T}$, a $\mathbb{Q}$-defined subgroup of $\boldsymbol{T}$. There is a $\mathbb{Q}$-defined subgroup $\boldsymbol{T}_1 \leq \boldsymbol{T}$ such that $\boldsymbol{T} = \boldsymbol{T}_0\boldsymbol{T}_1$ and $\boldsymbol{T}_0 \cap \boldsymbol{T}_1$ is finite. Without loss of generality we replace $\boldsymbol{G}$ by $\boldsymbol{G}/(\boldsymbol{T}_0 \cap \boldsymbol{T}_1)$ and henceforth assume that $\boldsymbol{T}_0 \cap \boldsymbol{T}_1 = \{1\}$. Note that now $\boldsymbol{U} \rtimes \boldsymbol{T}_1$ is a reduced solvable $\mathbb{Q}$-group. Moreover, without loss of generality we replace $\Gamma_{\boldsymbol{T}}$ with $\Gamma_0 \times \Gamma_1$, where $\Gamma_i \cong \mathbb{Z}^{N_i}$ is an S-arithmetic subgroup of $\boldsymbol{T}_i$ for each $i = 0, 1$. See Theorem 2.1 for the formula used to determine N_i.

From the semidirect product decomposition $\Gamma = (\boldsymbol{U}(S) \times \Gamma_0) \rtimes \Gamma_1$, let us denote elements of Γ by triples (u, γ_0, γ_1), where $u \in \boldsymbol{U}(S)$ and $\gamma_i \in \Gamma_i$ for $i = 0, 1$.

Define $Z_{\boldsymbol{U}}(\Gamma) = Z(\Gamma) \cap \boldsymbol{U}$. Clearly we have

$$Z(\Gamma) = Z_{\boldsymbol{U}}(\Gamma) \times \Gamma_0.$$

If Δ is any finite-index subgroup of Γ, then $Z(\Delta) = \Delta \cap Z(\boldsymbol{G})$ by the Zariski-density of Δ. It follows that $Z(\Gamma)$ is strongly commensuristic in Γ. Moreover, since $\boldsymbol{U}(S)$ is strongly commensuristic in Γ it follows that $Z_{\boldsymbol{U}}(\Gamma)$ is strongly commensuristic in Γ.

Any virtual homomorphism $\alpha : \Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$ determines a partial automorphism ψ_α of Γ defined on an appropriate subgroup of Γ by

$$\psi_\alpha(u, \gamma_0, \gamma_1) := (u + \alpha(\gamma_0, \gamma_1), \gamma_0, \gamma_1).$$

Let $\mathcal{W}$ denote the subgroup of $\operatorname{Comm}(\Gamma)$ arising in this way from equivalence classes of virtual homomorphisms $\Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$. There is an isomorphism

$$\mathcal{W} \cong \operatorname{Hom}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d),$$

where d is the dimension of $Z(\boldsymbol{G}) \cap \boldsymbol{U}$.

Let

$$\operatorname{Comm}_{\Gamma_0}(\Gamma) = \{[\phi : H \to K] \in \operatorname{Comm}(\Gamma) \mid \phi(H \cap \Gamma_0) = K \cap \Gamma_o\}.$$

Lemma 3.4. $\mathcal{W} \cdot \operatorname{Comm}_{\Gamma_0}(\Gamma) = \operatorname{Comm}(\Gamma)$.

Proof. We first show that $\mathcal{W}$ is a normal subgroup of $\operatorname{Comm}(\Gamma)$ so that the product $\mathcal{W} \cdot \operatorname{Comm}_{\Gamma_0}(\Gamma)$ is well defined. To see this, take any $\phi \in \operatorname{Comm}(\Gamma)$. Since $\boldsymbol{U}(S)$ is commensuristic in Γ and is fixed by any $\psi_\alpha \in \mathcal{W}$ we see that $\phi \circ \psi_\alpha \circ \phi^{-1}$ is trivial on $\boldsymbol{U}(S)$. It follows by direct computation that

$$\phi \circ \psi_\alpha \circ \phi^{-1} = \psi_{\phi_{\boldsymbol{U}} \circ \alpha \circ \phi_{\boldsymbol{T}}^{-1}},$$

where $\phi_{\boldsymbol{U}}$ is the restriction of ϕ to $Z_{\boldsymbol{U}}(\Gamma)$ and $\phi_{\boldsymbol{T}}$ is the commensuration of $\Gamma_0 \times \Gamma_1$ induced by ϕ under the quotient map $\Gamma \to \Gamma/\boldsymbol{U}(S)$. The map $\phi_{\boldsymbol{U}} \circ \alpha \circ \phi_{\boldsymbol{T}}^{-1}$ is a virtual homomorphism from $\Gamma_o \times \Gamma_1$ to $Z_{\boldsymbol{U}}(\Gamma)$ because $Z_{\boldsymbol{U}}(\Gamma)$ is commensuristic in Γ. This shows that $\mathcal{W}$ is normal in $\operatorname{Comm}(\Gamma)$.

Suppose $\phi : H \to K$ is a partial automorphism of Γ. Since $\boldsymbol{U}(S)$ is strongly commensuristic, ϕ induces a commensuration $[\nu] \in \mathrm{Comm}(\Gamma_0 \times \Gamma_1)$. There is a function $\alpha : H \cap (\Gamma_0 \times \Gamma_1) \to K \cap Z_{\boldsymbol{U}}(\Gamma)$ such that

$$\phi(0, \gamma_0, \gamma_1) = (\alpha(\gamma_0), \nu(\gamma_0, \gamma_1))$$

for all $(\gamma_0, \gamma_1) \in H \cap (\Gamma_0 \times \Gamma_1)$. In fact the function α is a virtual homomorphism $\Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$.

Define a virtual homomorphism $\beta : \Gamma_0 \times \Gamma_1 \to Z_{\boldsymbol{U}}(\Gamma)$ by $\beta = -\alpha \circ \nu^{-1}$. A straightforward computation shows that

$$(\psi_\beta \circ \phi)(0, \gamma_0, \gamma_1) = (0, \nu(\gamma_0, \gamma_1))$$

for all $(\gamma_0, \gamma_1) \in H \cap (\Gamma_0 \times \Gamma_1)$. Since $Z(\Gamma)$ is commensuristic in Γ, it follows that $(\psi_\beta \circ \phi)(0, \gamma_0, 0) = (0, \nu(\gamma_0), 0)$ for all $\gamma_0 \in H \cap \Gamma_0$. This means that $\psi_\beta \circ \phi \in \mathrm{Comm}_{\Gamma_0}(\Gamma)$, which completes the proof. □

We now turn to the task of elucidating the structure of $\mathrm{Comm}_{\Gamma_0}(\Gamma)$. There is a natural map

$$\xi : \mathrm{Comm}_{\Gamma_0}(\Gamma) \to \mathrm{Comm}(\Gamma/\Gamma_0).$$

Define $\mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$ to be the kernel of ξ. Because Γ/Γ_0 is naturally identified with the subgroup $\boldsymbol{U}(S) \rtimes \Gamma_1 \leq \Gamma$, it is easy to see that ξ is surjective. Therefore there is a short exact sequence

$$1 \to \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{Comm}_{\Gamma_0}(\Gamma) \to \mathrm{Comm}(\Gamma/\Gamma_0) \to 1. \tag{3}$$

Because Γ decomposes as a direct product $\Gamma = (\boldsymbol{U}(S) \rtimes \Gamma_1) \times \Gamma_0$, the sequence (3) splits and we can identify $\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Comm}(\boldsymbol{U}(S) \rtimes \Gamma_1)$. By Theorem 1.3 there is an isomorphism

$$\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})) \rtimes \mathrm{Aut}(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}).$$

Note that $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1) = Z(\boldsymbol{G}) \cap \boldsymbol{U}$, so recalling that d is the dimension of $Z(\boldsymbol{G}) \cap \boldsymbol{U}$ we may write

$$\mathrm{Comm}(\Gamma/\Gamma_0) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d) \rtimes \mathrm{Aut}(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}).$$

Lemma 3.5. *Let $\Gamma_i \cong \mathbb{Z}^{N_i}$ for $i = 0, 1$ be as above. There is an isomorphism*

$$\mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0}) \rtimes \mathrm{GL}_{N_0}(\mathbb{Q}),$$

where the action is by postcomposition.

Proof. There is a homomorphism $\Psi : \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{GL}_{N_0}(\mathbb{Q})$ given by restriction to Γ_0. Because Γ_0 splits off as a direct product factor, Ψ is surjective and the following exact sequence splits:

$$1 \to \ker(\Psi) \to \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \to \mathrm{GL}_{N_0}(\mathbb{Q}) \to 1.$$

The kernel of Ψ is given by equivalence classes of virtual homomorphisms $\boldsymbol{U}(S) \rtimes \Gamma_1 \to \Gamma_0$. There are no virtual homomorphisms $\boldsymbol{U}(S) \to \Gamma_0$ because Γ_0 is free abelian and every finite-index subgroup of $\boldsymbol{U}(S)$ is p-radicable for any $p \in S$. Therefore the kernel of Ψ may be identified with equivalence classes of virtual homomorphisms from Γ_1 to Γ_0, which form a group isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$. The fact that the action is by postcomposition is immediate. $\square$

Define

$$\mathrm{Comm}^0_{\Gamma_0}(\Gamma) = \mathrm{Comm}_{\boldsymbol{T}}(\Gamma) \rtimes \mathrm{Comm}^0(\Gamma/\Gamma_0),$$

where $\mathrm{Comm}^0(\Gamma/\Gamma_0)$ is as defined in Theorem 1.3. This is a finite-index subgroup of $\mathrm{Comm}_{\Gamma_0}(\Gamma)$. Note that the subgroup $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d) \leq \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts trivially on $\mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$, and the subgroup $\mathrm{GL}_{N_0}(\mathbb{Q}) \leq \mathrm{Comm}_{\boldsymbol{T}}(\Gamma)$ is centralized by the action of $\mathrm{Comm}^0(\Gamma/\Gamma_0)$. There is therefore a normal subgroup of $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ isomorphic to

$$\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0}) \times \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{d}),$$

which is isomorphic to $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$. So we may write

$$\mathrm{Comm}^0_{\Gamma_0}(\Gamma) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d}) \rtimes (\mathrm{GL}_{N_0}(\mathbb{Q}) \times \mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})), \tag{4}$$

where the commuting actions of $\mathrm{GL}_{N_0}(\mathbb{Q})$ and $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ are each by postcomposition.

Lemma 3.6. *There is a* $\mathbb{Q}$*-group* $\boldsymbol{C}$ *such that* $\mathrm{Comm}^0_{\Gamma_0}(\Gamma) \cong \boldsymbol{C}(\mathbb{Q})$.

Proof. For each $i = 1, \dots, N_1$ and $j = 1, \dots, N_0 + d$, let $\boldsymbol{A}_{i,j}$ be a copy of the 1-dimensional additive $\mathbb{Q}$-group $\boldsymbol{G}_a$. Define

$$\boldsymbol{C}_{\boldsymbol{T}} = \prod_{i=1}^{N_1} \prod_{j=1}^{N_0+d} \boldsymbol{A}_{i,j}.$$

Fix bases $\{v_i\}_{i=1}^{N_1}$ for $\mathbb{Q}^{N_1}$, and $\{w_i\}_{i=1}^{N_0}$ for $\mathbb{Q}^{N_0}$, and $\{w_i\}_{i=N_0+1}^{N_0+d}$ for $\mathbb{Q}^d$, so that $\{w_i\}_{i=1}^{N_0+d}$ is a basis for $\mathbb{Q}^{N_0+d}$. Let $e_{i,j}$ be the element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ that sends v_i to w_j and each v_k to zero for $k \neq i$. Then the collection of $\{e_{i,j}\}$ are a basis for $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$. Fix an isomorphism $\boldsymbol{C}_{\boldsymbol{T}}(\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ that takes a generator of $\boldsymbol{A}_{i,j}$ to $e_{i,j}$ for each pair i, j.

The algebraic group GL_{N_0} acts on $\boldsymbol{C}_{\boldsymbol{T}}$ by acting in the standard way on each group $\prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$ for fixed i and trivially on each factor $\boldsymbol{A}_{i,j}$ for $j > N_o$. This action is defined over $\mathbb{Q}$. The restriction of this action to the group action of $\mathrm{GL}_{N_0}(\mathbb{Q})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$ inside $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ is the action in (4).

Identify each group $\prod_{j=N_0+1}^{N_0+d} \boldsymbol{A}_{i,j}$ with $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$. This determines an action of the group $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$ on each group $\prod_{j=N_0+1}^{N_0+d} \boldsymbol{A}_{i,j}$ for fixed i, hence an action on all of $\boldsymbol{C}_{\boldsymbol{T}}$. This action is defined over $\mathbb{Q}$, and its restriction to $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ agrees with the action in (4).

Using the actions defined above, the algebraic group

$$\boldsymbol{C} = (\boldsymbol{G}_a)^{N_1(N_0+d)} \rtimes (\mathrm{GL}_{N_0} \times \mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1))$$

is a $\mathbb{Q}$-group with $\boldsymbol{C}(\mathbb{Q}) = \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$. □

The group $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts on $\mathcal{W}$ by conjugation. Under the identification $\mathcal{W} \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ and the decomposition of (4), this gives actions of each of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$, $\mathrm{GL}_{N_0}(\mathbb{Q})$, and $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$. We record here some facts about these actions that are straightforward to verify.

Lemma 3.7. *The action of* $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ *on* $\mathcal{W}$ *is given by the following*:

(1) *the action of* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ *on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *factors through the quotient* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0})$ *acting on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *by precomposition by the inverse*;

(2) *the action of* $\mathrm{GL}_{N_0}(\mathbb{Q})$ *is by precomposition by the inverse acting on* $\mathbb{Q}^{N_0} \leq \mathbb{Q}^{N_0+N_1}$;

(3) *the group* $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ *acts on* $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ *by postcomposition, where* $\mathbb{Q}^d$ *is identified with* $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$.

We now complete the proof of the main theorem of this paper in the case $S \neq \varnothing$.

Proof of Theorem 1.2. Continue using the notation of Section 3C and Lemmas 3.4–3.7. We will define a $\mathbb{Q}$-group $\boldsymbol{D}$ so that $\boldsymbol{D}(\mathbb{Q}) \cong \mathcal{W} \cdot \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$. Because $\mathcal{W} \cdot \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ is a subgroup of finite index in $\mathrm{Comm}(\Gamma)$, this is the desired result.

Because $\mathcal{W}$ is normal in $\mathrm{Comm}(\Gamma)$, the group $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ acts on $\mathcal{W}$ by conjugation. This determines an action of $\boldsymbol{C}(\mathbb{Q})$ on $\mathcal{W}$. We will show there is an algebraic group $\boldsymbol{W}$ with $\boldsymbol{W}(\mathbb{Q}) \cong \mathcal{W}$ and an algebraic action of $\boldsymbol{C}$ on $\boldsymbol{W}$ such that the induced action of $\boldsymbol{C}(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ agrees with the action of $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ on $\mathcal{W}$ under our identifications.

Consider indexed copies of the additive group $\boldsymbol{G}_a^{i,j}$ for $i = 1, \ldots, N_0 + N_1$ and $j = 1, \ldots, d$. Let

$$\boldsymbol{W} = \prod_{i=1}^{N_0+N_1} \prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}.$$

Fix bases $\{x_i\}_{i=1}^{N_0}$ for $\mathbb{Q}^{N_0}$, and $\{x_i\}_{i=N_0+1}^{N_0+N_1}$ for $\mathbb{Q}^{N_1}$, and $\{y_i\}_{i=1}^{d}$ for $\mathbb{Q}^d$, so that $\{x_i\}_{i=1}^{N_0+N_1}$ is a basis for $\mathbb{Q}^{N_0+N_1}$. Let $f_{i,j}$ be the element of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$ that sends x_i to y_j and each x_k to zero for $k \neq i$. Then the collection of $\{f_{i,j}\}$ are a basis for $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$. Fix an isomorphism $\boldsymbol{W}(\mathbb{Q}) \cong \mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$

that takes a generator of $\boldsymbol{G}_a^{i,j}$ to $f_{i,j}$ for each pair i, j. This gives an isomorphism $\boldsymbol{W}(\mathbb{Q}) \cong \mathcal{W}$.

For each fixed i we may identify the group $\prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}$ with $Z(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$. This identification determines an action of $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)$ on each group $\prod_{j=1}^{d} \boldsymbol{G}_a^{i,j}$, hence an action on all of $\boldsymbol{W}$ which is defined over $\mathbb{Q}$. This action restricts to an action of $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ which agrees under our identifications with the action of the subgroup $\mathrm{Aut}^0(\boldsymbol{U} \rtimes \boldsymbol{T}_1)(\mathbb{Q}) \leq \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ on $\mathcal{W}$.

For each fixed j, the algebraic group GL_{N_0} acts on $\prod_{i=1}^{N_0} \boldsymbol{G}_a^{i,j}$ by the dual (inverse transpose) of the standard action. Letting GL_{N_0} act trivially on each $\boldsymbol{G}_a^{i,j}$ for $i > N_0$, this induces an action of GL_{N_0} on $\boldsymbol{W}$. The restriction of this action to $\mathrm{GL}_{N_0}(\mathbb{Q})$ on $\boldsymbol{W}(\mathbb{Q})$ agrees with the action of the subgroup $\mathrm{GL}_{N_0}(\mathbb{Q}) \leq \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ on $\mathcal{W}$.

Finally, the group $\prod_{i=1}^{N_1} \prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$ embeds as a unipotent subgroup of $\mathrm{GL}_{N_0+N_1}$, and through this embedding acts by the inverse transpose on $\prod_{i=1}^{N_0+N_1} \boldsymbol{G}_a^{i,j}$ for each fixed j. There is a natural quotient map $\boldsymbol{C_T} \to \prod_{i=1}^{N_1} \prod_{j=1}^{N_0} \boldsymbol{A}_{i,j}$, and through this map $\boldsymbol{C_T}$ acts on $\boldsymbol{W}$ in such a way that the restriction to the $\mathbb{Q}$-points agrees with the action of $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^{N_0+d})$ on $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_0+N_1}, \mathbb{Q}^d)$.

In total these define an action of $\boldsymbol{C}$ on $\boldsymbol{W}$ which is defined over $\mathbb{Q}$. Therefore $\boldsymbol{W} \rtimes \boldsymbol{C}$ has the structure of a $\mathbb{Q}$-group.

The unipotent group $(\boldsymbol{G}_a)^{N_1 d}$ embeds in $\boldsymbol{W}$ and $\boldsymbol{C_T}$, via maps $\alpha : (\boldsymbol{G}_a)^{N_1 d} \to \boldsymbol{W}$ and $\beta : (\boldsymbol{G}_a)^{N_1 d} \to \boldsymbol{C_T}$, such that the image of $(\boldsymbol{G}_a)^{N_1 d}(\mathbb{Q})$ is identified with $\mathrm{Hom}_{\mathbb{Q}}(\mathbb{Q}^{N_1}, \mathbb{Q}^d)$ under each of α and β. Let $\Theta \leq \boldsymbol{W} \rtimes \boldsymbol{C}$ be the embedding of $(\boldsymbol{G}_a)^{N_1 d}$ under the product map $(-\alpha) \times \beta$. Note that Θ is a normal unipotent subgroup of $\boldsymbol{W} \rtimes \boldsymbol{C}$, so the quotient $\boldsymbol{D} = (\boldsymbol{W} \rtimes \boldsymbol{C})/\Theta$ is a $\mathbb{Q}$-group with $\boldsymbol{D}(\mathbb{Q}) = (\boldsymbol{W}(\mathbb{Q}) \rtimes \boldsymbol{C}(\mathbb{Q}))/\Theta(\mathbb{Q})$.

There are isomorphisms $\boldsymbol{W}(\mathbb{Q}) \to \mathcal{W}$ and $\boldsymbol{C}(\mathbb{Q}) \to \mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ which induce a surjective map

$$\Phi : \boldsymbol{W}(\mathbb{Q}) \rtimes \boldsymbol{C}(\mathbb{Q}) \to \mathrm{Comm}^0(\Gamma)$$

because the action of $\boldsymbol{C}$ on $\boldsymbol{W}$ is compatible with the action of $\mathrm{Comm}^0_{\Gamma_0}(\Gamma)$ on $\mathcal{W}$. The kernel of Φ is precisely the subgroup $\Theta(\mathbb{Q})$, so Φ descends to an isomorphism $\boldsymbol{D}(\mathbb{Q}) \cong \mathrm{Comm}^0(\Gamma)$. □

4. Number fields

Linear algebraic groups can be defined over arbitrary fields. Let K be a global field and S a set of multiplicative valuations of K. The ring of S-*integral* elements of K, denoted $K(S)$, is the ring of $x \in K$ such that $v(x) \leq 1$ for each non-Archimedean valuation $v \notin S$. If $\boldsymbol{G}$ is a linear algebraic group defined over K, let $\boldsymbol{G}(K(S))$ denote the group of matrices in $\boldsymbol{G}$ with entries in $K(S)$. See [Margulis 1991, Chapter I] for details.

The following example shows that if $\boldsymbol{U}$ is a unipotent group defined over a number field K and S is a set of multiplicative valuations, then $\mathrm{Comm}(\boldsymbol{U}(K(S)))$ may depend on S. This stands in contrast with Theorem 3.3, which directly implies that $\mathrm{Comm}(\boldsymbol{U}(K(S)))$ is independent of S when $K = \mathbb{Q}$. The author is grateful to Dave Morris for suggesting this example.

Example 4.1. Take $\boldsymbol{U}$ to be the additive group $\boldsymbol{G}_a$ defined over $K = \mathbb{Q}(i)$. On the one hand, we have $\boldsymbol{U}(K(\varnothing)) = \mathbb{Z}[i]$ and so

$$\mathrm{Comm}(\boldsymbol{U}(K(\varnothing))) \cong \mathrm{GL}_2(\mathbb{Q}).$$

On the other hand, let $p = 5$ and write $p = ab$ for $a = 2+i$ and $b = 2-i$. Let v_a and v_b be the valuations corresponding to the distinct prime ideals (a) and (b) of $\mathbb{Z}[i]$, respectively. Set $S = \{v_a\}$ and $\Gamma = \boldsymbol{U}(K(S))$. Note that $\Gamma = \mathbb{Z}[i, 1/a]$. We will show that $\mathrm{Comm}(\Gamma)$ is much smaller than $\mathrm{GL}_2(\mathbb{Q})$.

Let K_b be the Cauchy completion of K with respect to the valuation v_b, and let $\mathcal{O}_b$ be the ring of integers of K_b. Note that K_b is a finite extension of $\mathbb{Q}_5$, and that Γ is a dense subgroup of $\mathcal{O}_b$. Any commensuration $[\phi] \in \mathrm{Comm}(\Gamma)$ induces a map $\Phi : K_b \to K_b$ that is continuous and $\mathbb{Q}$-linear, hence K_b-linear. Therefore Φ is multiplication by some nonzero $x \in K_b$. In fact it follows that $x \in K$ since Γ is virtually preserved and Zariski-dense in K. Every element of $K^\times$ induces a nontrivial commensuration, so we have

$$\mathrm{Comm}(\Gamma) \cong \mathbb{Q}(i)^\times.$$

In this example, $\mathrm{Comm}(\Gamma)$ has the structure of the $\mathbb{Q}$-points of a $\mathbb{Q}$-group. Hence the conclusion of Theorem 1.2 holds even though the method of proof does not.

Dave Morris has pointed out that the arguments of Example 4.1 extend to prove the following:

Proposition 4.2. *Let $\boldsymbol{U}$ be a unipotent group defined over a number field K. For every finite set S of valuations of K, there is a subfield $L \leq K$ such that*

$$\mathrm{Comm}(\boldsymbol{U}(S)) \cong \mathrm{Aut}(R_{K/L}\boldsymbol{U})(L),$$

where $R_{K/L}$ is the restriction of scalars operator.

With this, much of the proof of Theorem 1.2 still applies. For example, Theorem 2.1 generalizes to tori $\boldsymbol{T}$ defined over number fields K to show that $\boldsymbol{T}(K(S))$ is virtually a finitely generated, free abelian group for any finite S. However, there is an obstruction to extending the proof of Theorem 1.2: Proposition 1.1 no longer applies on passage to the restriction of scalars over L.

5. Function fields

In this section we provide examples of S-arithmetic groups over a global field of positive characteristic for which no obvious analog of Theorem 1.2 holds.

In what follows we use the global field $K = \mathbb{F}_q(t)$, the field of rational functions in one variable over the finite field with q elements. Choose $S = \{v_t, v_\infty\}$, where the valuations v_∞ and v_t are defined as follows. Given any $r \in \mathbb{F}_q(t)$, write $r(t) = t^k(f(t)/g(t))$, where f and g are polynomials with nontrivial constant term and $k \in \mathbb{Z}$. Then define

$$v_t(r) = q^{-k} \quad \text{and} \quad v_\infty(r) = q^{\deg(f)+k-\deg(g)}.$$

In this case, $K(S)$ is the ring of Laurent polynomials over $\mathbb{F}_q$, denoted $\mathbb{F}_q[t, t^{-1}]$.

Example 5.1. Consider the 1-dimensional additive algebraic group

$$\boldsymbol{G}_a = \left\{ \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \right\} \subseteq \mathrm{GL}_2 .$$

Then $\boldsymbol{G}_a(K(S)) \cong K(S)$ is an S-arithmetic group. There is an isomorphism of abstract groups

$$K(S) \cong \bigoplus_{k=-\infty}^{\infty} \mathbb{F}_q .$$

Proposition 5.2. *For any field F and any linear algebraic group $\boldsymbol{G}$ over F, there is no embedding* $\mathrm{Comm}(K(S)) \to \boldsymbol{G}(F)$.

Proof. It suffices to treat the case that $\boldsymbol{G} = \mathrm{GL}_d$ for some d. We will show that $\mathrm{Comm}(K(S))$ contains $\mathrm{GL}_n(\mathbb{F}_q)$ for every n, which implies that $\mathrm{Comm}(K(S))$ contains every finite group. This completes the proof, since $\mathrm{GL}_d(F)$ does not contain every finite group. (See, for example, [Serre 2007, Theorem 5].)

For each $n \in \mathbb{N}$, embed $\mathrm{GL}_n(\mathbb{F}_q)$ into $\mathrm{Comm}(K(S))$ "diagonally" as follows: Let $V = \bigoplus_{k=-\infty}^{\infty} \mathbb{F}_q$, and for each $\ell \in \mathbb{Z}$ define a subgroup $V_\ell \leq V$ by $V_\ell = \bigoplus_{k=n\ell}^{n(\ell+1)-1} \mathbb{F}_q$. Given any automorphism $\phi \in \mathrm{GL}_n(\mathbb{F}_q)$, define an automorphism $\Phi \in \mathrm{Aut}(V)$ piecewise by $\Phi\big|_{V_\ell} = \phi$. In this way every nontrivial element of $\mathrm{GL}_n(\mathbb{F}_q)$ determines a nontrivial commensuration of $V \cong K(S)$. □

In particular, Proposition 5.2 implies that Theorem 1.2 does not hold when $\mathbb{Q}$ is replaced by a global field of positive characteristic.

Example 5.3 (lamplighter group). Consider the algebraic group

$$\boldsymbol{B}_2 = \left\{ \begin{pmatrix} x & z \\ 0 & y \end{pmatrix} \middle|\, xy = 1 \right\} \subseteq \mathrm{GL}_2 .$$

Set $q = 2$. The S-arithmetic group $\boldsymbol{B}_2(\mathbb{F}_2[t, t^{-1}])$ is isomorphic to the (restricted) wreath product $\mathbb{F}_2^2 \wr \mathbb{Z}$, which is an index-2 subgroup of the *lamplighter group* $\mathbb{F}_2 \wr \mathbb{Z}$. The lamplighter group is isomorphic to the semidirect product

$$\left(\bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}\right) \rtimes \mathbb{Z},$$

where the $\mathbb{Z}$ acts by permutation of the $\mathbb{Z}/2\mathbb{Z}$ factors through the usual left action on the index set.

The abstract commensurator of $\mathbb{F}_2 \wr \mathbb{Z}$ is fairly complicated, and has not been well studied. See Section 6 for a more detailed discussion of $\mathrm{Comm}(\mathbb{F}_2 \wr \mathbb{Z})$. For now we use the fact that $\mathrm{Comm}(\mathbb{F}_2 \wr \mathbb{Z})$ contains the direct limit

$$\varinjlim_{n \in \mathbb{N}} \mathrm{Aut}(\mathbb{F}_2^n),$$

where the maps are the diagonal inclusions of $\mathrm{Aut}(\mathbb{F}_2^n)$ into $\mathrm{Aut}(\mathbb{F}_2^m)$ whenever $n \mid m$. It follows now as in Proposition 5.2 that $\mathrm{Comm}(\boldsymbol{B}_2(\mathbb{F}_2[t, t^{-1}]))$ is not a linear group over any field. This shows that Theorem 1.2 does not apply in positive characteristic even in the presence of a nontrivial action by a torus.

6. Commensurations of the lamplighter group

Define K to be the direct product

$$K := \bigoplus_{\mathbb{Z}} \mathbb{Z}/2\mathbb{Z}.$$

The group of integers $\mathbb{Z}$ acts on itself by left-translation, inducing an action on K by permutation of indices. The *lamplighter group*, which we will denote by Γ throughout this section, is the semidirect product $\Gamma = K \rtimes \mathbb{Z}$. The goal of this section is to show that $\mathrm{Comm}(\Gamma)$ admits the following decomposition.

Theorem 1.4. *Using the definitions in (6) and (7) below, there is an isomorphism*

$$\mathrm{Comm}(\Gamma) \cong (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}). \tag{5}$$

See [Houghton 1962] for an analogous description of automorphism groups of unrestricted wreath products.

Let $e_i \in \Gamma$ be the element of the direct sum subgroup which is nontrivial only at the i-th index and let $t \in \Gamma$ be a generator for $\mathbb{Z}$. By definition we have the relation $t^m e_i t^{-m} = e_{i+m}$. Then Γ is generated by the set $\{e_0, t\}$ and has the presentation

$$\Gamma = \langle e_0, t \mid e_0^2 = 1 \text{ and } [t^k e_0 t^{-k}, t^\ell e_0 t^{-\ell}] = 1 \text{ for all } k, \ell \in \mathbb{Z}\rangle.$$

Lemma 6.1. *The quotient map $\Gamma \to \Gamma/K$ induces a surjective homomorphism $\Theta : \mathrm{Comm}(\Gamma) \to \mathbb{Z}/2\mathbb{Z}$.*

Proof. The subgroup $K \leq \Gamma$ is equal to the set of torsion elements of Γ, and is therefore strongly commensuristic. It follows that there is a homomorphism $\Theta : \mathrm{Comm}(\Gamma) \to \mathrm{Comm}(\Gamma/K) \cong \mathrm{Comm}(\mathbb{Z})$. The nontrivial automorphism of $\mathbb{Z}$ induces an automorphism, hence a commensuration, of Γ by $t \mapsto t^{-1}$ and $e_i \mapsto e_{-i}$ for each $i \in \mathbb{Z}$. It remains to show that the image of Θ is in $\mathrm{Aut}(\mathbb{Z}) \leq \mathrm{Comm}(\mathbb{Z})$.

Suppose $\phi : \Delta_1 \to \Delta_2$ is a partial automorphism of Γ. In what follows, let $i = 1, 2$. Let $K_i = K \cap \Delta_i$. Choose $g_i \in \Delta_i$ so that its equivalence class $[g_i]$ generates the image of the quotient map $\Delta_i \to \Delta_i/K_i$. Let $G_i = \langle g_i \rangle$. Note that Δ_i admits a product decomposition $\Delta_i = K_i G_i$.

Let m_i be the integer such that $g_i = at^{m_i}$ for some $a \in K_i$. Replacing g_i with its inverse if necessary, assume that $m_i > 0$. Each group G_i naturally acts on K/K_i. Since K/K_i is finite, after replacing g_i with a power if necessary we assume that the action of G_i on K/K_i is trivial for $i = 1, 2$. Our goal is to prove $m_1 = m_2$.

One can check that ϕ induces an isomorphism $[K_1, G_1] \cong [K_2, G_2]$, where $[K_i, G_i]$ is the group generated by commutators of the form $[a, g] := aga^{-1}g^{-1}$ for $a \in K_i$ and $g \in G_i$. (In fact, in this case we know $[K_i, G_i]$ is equal to the *set* of elements of the form $[a, g_i]$, which is equal to $[a, t^{m_i}]$, for some $a \in K_i$. This is helpful in understanding the proof of the claim below.) Since ϕ induces an isomorphism

$$K_1/[K_1, G_1] \cong K_2/[K_2, G_2],$$

the desired result is apparent from the following claim.

Claim. *There are isomorphisms* $K_i/[K_i, G_i] \cong (\mathbb{Z}/2\mathbb{Z})^{m_i}$ *for* $i = 1, 2$.

Proof of claim. Let $H_{m_i} \leq K$ be the subgroup generated by the set $\{e_0, e_1, \ldots, e_{m_i-1}\}$. Clearly H_{m_i} is isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{m_i}$. Let $P_i = K_i \cap H_{m_i}$, and let $Q_i \leq H_{m_i}$ be a complement to P_i so that $H_{m_i} = P_i \oplus Q_i$. Consider the subset $S_i \subseteq K_i$ defined by

$$S_i = \{g \in K \mid g = p[q, g_i] \text{ for some } p \in P_i \text{ and } q \in Q_i\}.$$

The condition that G_i act trivially on K/K_i ensures that $[a, g_i] \in K_i$ for any $a \in K$, and so $S_i \subseteq K_i$. By construction S_i is in bijection with H_{m_i}, hence has cardinality 2^{m_i}. Consider the map of sets $\rho_i : S_i \to K_i/[K_i, G_i]$ sending an element to its equivalence class. Since $[K_i, G_i]$ consists of elements of the form $[a, g_i]$ for some $a \in K_i$, it is not hard to see from the construction of S_i that ρ_i is injective. We leave as an exercise to check that ρ_i is surjective. □

Let Θ be the surjection of Lemma 6.1. The short exact sequence

$$1 \to \ker(\Theta) \to \mathrm{Comm}(\Gamma) \to \mathbb{Z}/2\mathbb{Z} \to 1$$

splits, so that $\mathrm{Comm}(\Gamma) \cong \ker(\Theta) \rtimes (\mathbb{Z}/2\mathbb{Z})$. Since K is strongly commensuristic, there is a natural map $\Phi : \ker(\Theta) \to \mathrm{Comm}(K)$. We describe first the kernel of Φ and then the image of Φ.

If G is a group and A is a left G-module, then $\tau : G \to A$ is a *derivation* if $\tau(g_1 g_2) = \tau(g_1) + g_1 \cdot \tau(g_2)$ for all $g_1, g_2 \in G$. The set of derivations from G to A forms an abelian group, denoted $\mathrm{Der}(G, A)$. A *virtual derivation* from G to A is a derivation from a finite-index subgroup of G to A. Two virtual derivations are *equivalent* if they agree on a finite-index subgroup of G. The set of equivalence classes of virtual derivations forms a group

$$\mathrm{VDer}(G, A) := \varinjlim_{[G:H]<\infty} \mathrm{Der}(H, A). \tag{6}$$

Lemma 6.2. *There is an isomorphism* $\ker(\Phi) \cong \mathrm{VDer}(\mathbb{Z}, K)$.

Proof. Given any $[\phi] \in \ker(\Phi)$, find $m \in \mathbb{Z}$ so that $\phi(t^m)$ is defined. Then define a map $\tau : m\mathbb{Z} \to K$ by $\tau(t^k) = \phi(t^k)t^{-k}$ for any $k \in m\mathbb{Z}$. It is easy to check that τ is a derivation from $m\mathbb{Z}$ to K, and that the assignment $[\phi] \mapsto \tau$ gives a homomorphism $\mathrm{Comm}(\Gamma) \to \mathrm{VDer}(\mathbb{Z}, K)$. This assignment is clearly injective. On the other hand, if $\tau \in \mathrm{Der}(m\mathbb{Z}, K)$ then setting $\phi(xt^\ell) = x\tau(t^\ell)t^\ell$ for $x \in K$ defines an automorphism ϕ of $\Gamma_m \le \Gamma$. □

Let $\mathrm{Comm}(K)^{m\mathbb{Z}}$ denote the group of $m\mathbb{Z}$-equivariant commensurations of K. There are natural inclusions $\mathrm{Comm}(K)^{m\mathbb{Z}} \to \mathrm{Comm}(K)^{n\mathbb{Z}}$ whenever $m \mid n$. Define

$$\mathrm{Comm}_\infty(K) := \varinjlim_{m} \mathrm{Comm}(K)^{m\mathbb{Z}}. \tag{7}$$

Lemma 6.3. *There is an isomorphism* $\Phi(\ker(\Theta)) \cong \mathrm{Comm}_\infty(K)$.

Proof. Suppose $\alpha = \Phi([\phi])$ for some partial automorphism ϕ of Γ. Find $m \in \mathbb{Z}$ so that t^m is in the domain of ϕ. Define $x_0 = \phi(t^m)t^{-m} \in K$. Then given any $x \in K$, we have

$$\phi(t^m x t^{-m}) = x_0 t^m \phi(x) t^{-m} x_0^{-1} = t^m \phi(x) t^{-m}.$$

From this we see that any $\alpha \in \Phi(\ker(\Theta))$ is $m\mathbb{Z}$-equivariant for some m.

On the other hand, suppose $\beta : H_1 \to H_2$ is any partial automorphism of K that is $m\mathbb{Z}$-equivariant. Define $\Gamma_m = K \rtimes \langle t^m \rangle$, an index-$m$ subgroup of Γ. The formula $\phi(xt^\ell) = \alpha(x)t^\ell$ defines an automorphism $\phi \in \mathrm{Aut}(\Gamma_m)$. Hence $[\phi]$ is a commensuration of Γ which evidently satisfies $\Phi([\phi]) = \beta$. □

Proof of Theorem 1.4. It is clear from the proof of Lemma 6.3 that the short exact sequence

$$1 \to \mathrm{VDer}(\mathbb{Z}, K) \to \ker(\Theta) \to \mathrm{Comm}_\infty(K) \to 1$$

splits. Putting together Lemmas 6.1, 6.2, and 6.3, we have the semidirect product description of (5):

$$\mathrm{Comm}(\Gamma) = (\mathrm{VDer}(\mathbb{Z}, K) \rtimes \mathrm{Comm}_\infty(K)) \rtimes (\mathbb{Z}/2\mathbb{Z}).$$

The action of $\mathrm{Comm}_\infty(K)$ on $\mathrm{VDer}(\mathbb{Z}, K)$ is the action by postcomposition. The factor of $\mathbb{Z}/2\mathbb{Z}$ preserves $\mathrm{VDer}(\mathbb{Z}, K)$ and $\mathrm{Comm}_\infty(K)$, and acts on $\mathrm{VDer}(\mathbb{Z}, K)$ by precomposition. □

It is not clear whether a more explicit description of $\mathrm{Comm}_\infty(K)$ exists, but we can describe some subgroups. For example, the "diagonal embedding" construction of Proposition 5.2 shows that $\mathrm{Comm}_\infty(K)$ contains the direct limit

$$\varinjlim_m \mathrm{GL}_m(\mathbb{F}_2),$$

where $\mathrm{GL}_m(\mathbb{F}_2)$ includes into $\mathrm{GL}_n(\mathbb{F}_2)$ diagonally whenever $m \mid n$. So $\mathrm{Comm}_\infty(K)$ contains every finite group.

Note that $\mathrm{VDer}(\mathbb{Z}, K)$ contains every commensuration induced by conjugation by some $a \in K$. However, some elements of $\mathrm{VDer}(\mathbb{Z}, K)$ do not arise in this way. For example, any virtual derivation $\tau : m\mathbb{Z} \to K$ such that $\tau(t^m)$ is nontrivial in an odd number of coordinates cannot arise from conjugation.

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GERSTENHABER BRACKETS ON HOCHSCHILD COHOMOLOGY OF QUANTUM SYMMETRIC ALGEBRAS AND THEIR GROUP EXTENSIONS

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We construct chain maps between the bar and Koszul resolutions for a quantum symmetric algebra (skew polynomial ring). This construction uses a recursive technique involving explicit formulae for contracting homotopies. We use these chain maps to compute the Gerstenhaber bracket, obtaining a quantum version of the Schouten–Nijenhuis bracket on a symmetric algebra (polynomial ring). We compute brackets also in some cases for skew group algebras arising as group extensions of quantum symmetric algebras.

1. Introduction

Hochschild [1945] introduced homology and cohomology for algebras. Gerstenhaber [1963] studied extensively the algebraic structure of Hochschild cohomology — its cup product and graded Lie bracket (or Gerstenhaber bracket) — and consequently algebras with such structure are generally termed Gerstenhaber algebras. Many mathematicians have since investigated Hochschild cohomology for various types of algebras, and it has proven useful in many settings, including algebraic deformation theory [Gerstenhaber 1964] and support variety theory [Erdmann et al. 2004; Snashall and Solberg 2004].

The graded Lie bracket on Hochschild cohomology remains elusive in contrast to the cup product. The latter may be defined via any convenient projective resolution. The former is defined on the bar resolution, which is useful theoretically but not computationally, and one typically computes graded Lie brackets by translating to another more convenient resolution via explicit chain maps. Such

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chain maps are not always easy to find. One would like to define the graded Lie structure directly on another resolution or to find efficient techniques for producing chain maps.

In this paper, we begin in Section 2 by promoting a recursive technique for constructing chain maps. The technique is not new; for example it appears in a book of Mac Lane [1975]. See also [Le and Zhou $\geq$ 2016] for a more general setting. We first use this technique to construct chain maps between the bar and Koszul resolutions for symmetric algebras, reproducing in Theorem 3.5 the chain maps of [Shepler and Witherspoon 2011] that had been obtained via ad hoc methods. We then construct new chain maps more generally for quantum symmetric algebras (skew polynomial rings) in Theorem 4.6. We generalize an alternative description, due to Carqueville and Murfet [2016], of these chain maps for symmetric algebras to quantum symmetric algebras in (4.8). We use these chain maps to compute the Gerstenhaber bracket on quantum symmetric algebras, generalizing the Schouten–Nijenhuis bracket on the Hochschild cohomology of polynomial rings (Theorem 5.1). We then investigate the Hochschild cohomology of a group extension of a quantum symmetric algebra, obtaining results on brackets in the special cases that the action is diagonal (Theorem 7.1) or that the Hochschild cocycles have minimal degree as maps on tensor powers of the algebra (Corollary 7.4). In the latter case, we thereby obtain a new proof that all such Hochschild 2-cocycles are noncommutative Poisson structures (cf. [Naidu and Witherspoon 2016], in which algebraic deformation theory was used instead). Some results on brackets for group extensions of polynomial rings were given in [Halbout and Tang 2010] and [Shepler and Witherspoon 2012].

Let $\Bbbk$ be a field. All algebras will be associative $\Bbbk$-algebras with unity and tensor products will be taken over $\Bbbk$ unless otherwise indicated.

2. Construction of comparison morphisms

Let A be a ring and let M and N be two left A-modules. Let $P_\bullet$ (respectively, $Q_\bullet$) be a projective resolution of M (respectively, N). It is well known that given a homomorphism of A-modules $f : M \to N$, there exists a chain map $f_\bullet : P_\bullet \to Q_\bullet$ lifting f (and different lifts are equivalent up to homotopy). Sometimes in practice we need an explicit construction of such a chain map, called a comparison morphism, to perform computations. In this section, we recall a method to construct chain maps under the condition that $P_\bullet$ is a free resolution (see [Mac Lane 1975, Chapter IX, Theorem 6.2]). A method for arbitrary projective resolutions will be presented in [Le and Zhou $\geq$ 2016].

Let us fix some notation and assumptions. Suppose that

$$\cdots \longrightarrow P_n \xrightarrow{d_n^P} P_{n-1} \xrightarrow{d_{n-1}^P} \cdots \xrightarrow{d_1^P} P_0 \left(\xrightarrow{d_0^P} M \longrightarrow 0\right)$$

is a free resolution of M, that is, for each $n \geq 0$, $P_n = A^{(X_n)}$ for some set X_n. (The module $A^{(X_n)}$ is a direct sum of copies of A indexed by X_n. We identify each element of X_n with the identity 1_A in the copy of A indexed by that element.) Suppose that a projective resolution of N,

$$\cdots \longrightarrow Q_n \xrightarrow{d_n^Q} Q_{n-1} \xrightarrow{d_{n-1}^Q} \cdots \xrightarrow{d_1^Q} Q_0 \left(\xrightarrow{d_0^Q} N \longrightarrow 0\right),$$

comes equipped with a *chain contraction*: a collection of set maps $t_n : Q_n \to Q_{n+1}$ for each $n \geq 0$ and $t_{-1} : N \to Q_0$ such that for $n \geq 0$, we have $t_{n-1}d_n^Q + d_{n+1}^Q t_n = \mathrm{Id}_{Q_n}$ and $d_0^Q t_{-1} = \mathrm{Id}_N$. We use these next to construct a chain map, $f_n : P_n \to Q_n$ for $n \geq 0$, lifting $f_{-1} := f$. As P_n is free, we need only specify the values of f_n on elements of X_n, the generating set of P_n.

At first glance, it may appear that f_n defined below will be the zero map, since it is defined recursively by applying the differential more than once. However, the maps t_n are not in general A-module homomorphisms. The formula (2.1) is used only to define f_n on free basis elements, and f_n is then extended to an A-module map. In our examples the maps t_n will be $\Bbbk$-linear, but for the construction, they are only required to be maps of sets, since we apply them only to basis elements. In this weaker setting, such a collection of maps may be called a *weak self-homotopy* as in [Bian et al. 2009].

For $n = 0$, given $x \in X_0$, define $f_0(x) = t_{-1} f d_0^P(x)$. Then $d_0^Q f_0(x) = d_0^Q t_{-1} f d_0^P(x) = f d_0^P(x)$.

Suppose that we have constructed $f_0, \dots, f_{n-1}$ such that for $0 \leq i \leq n-1$, $d_i^Q f_i = f_{i-1} d_i^P$. For $x \in X_n$, define

$$f_n(x) = t_{n-1} f_{n-1} d_n^P(x). \tag{2.1}$$

Then

$$\begin{aligned} d_n^Q f_n(x) &= d_n^Q t_{n-1} f_{n-1} d_n^P(x) \\ &= f_{n-1} d_n^P(x) - t_{n-2} d_{n-1}^Q f_{n-1} d_n^P(x) \\ &= f_{n-1} d_n^P(x) - t_{n-2} f_{n-2} d_{n-1}^P d_n^P(x) \\ &= f_{n-1} d_n^P(x). \end{aligned}$$

This proves the following.

Proposition 2.2. *The maps f_n defined in (2.1) form a chain map from $P_\bullet$ to $Q_\bullet$ lifting $f : M \to N$.*

In the next two sections, we use this formula (2.1) to find explicit chain maps for symmetric and quantum symmetric algebras, and in the rest of this article we use the chain maps thus found in computations of Gerstenhaber brackets for these algebras and their group extensions.

3. Chain contractions and comparison maps for polynomial algebras

Let N be a positive integer. Let V be a vector space over the field $\Bbbk$ with basis $x_1, \dots, x_N$, and let

$$S(V) := \Bbbk[x_1, \dots, x_N]$$

be the polynomial algebra in N indeterminates. This is a Koszul algebra, so there is a standard complex $K_\bullet(S(V))$ that is a free resolution of $A := S(V)$ as an A-bimodule (equivalently as an A^e-module where $A^e = A \otimes A^{\mathrm{op}}$). We recall this complex next: for each p, let $\bigwedge^p(V)$ denote the p-th exterior power of V. Then $K_\bullet(S(V))$ is the complex

$$\cdots \longrightarrow A \otimes \textstyle\bigwedge^2(V) \otimes A \xrightarrow{d_2} A \otimes \bigwedge^1(V) \otimes A \xrightarrow{d_1} A \otimes A \left(\xrightarrow{d_0} A \longrightarrow 0\right);$$

that is, for $0 \le p \le N$, the degree p term is $K_p(S(V)) := A \otimes \bigwedge^p(V) \otimes A$. The differential d_p is defined by

$$\begin{aligned} d_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) &= \sum_{i=1}^{p} (-1)^{i+1} x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes 1 \\ &\qquad - \sum_{i=1}^{p} (-1)^{i+1} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes x_{j_i} \end{aligned}$$

whenever $1 \le j_1 < \cdots < j_p \le N$ and $p > 0$; the notation $\hat{x}_{j_i}$ indicates that the factor x_{j_i} is deleted. The map d_0 is multiplication.

From now on, we will write $\underline{\ell} = (\ell_1, \dots, \ell_N)$, an N-tuple of nonnegative integers, $\underline{x} = (x_1, \dots, x_N)$ and $\underline{x}^{\underline{\ell}} = x_1^{\ell_1} \cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_\bullet(S(V))$ consisting of maps $t_{-1} : A \to A \otimes A$ and

$$t_p : A \otimes \textstyle\bigwedge^p(V) \otimes A \to A \otimes \bigwedge^{p+1}(V) \otimes A$$

for $p \ge 0$. These maps will be left A-module homomorphisms, and thus we need only define them on choices of free basis elements of these free left A-modules.

To define t_{-1}, it suffices to specify $t_{-1}(1) = 1 \otimes 1$ and extend it A-linearly. If $p = 0$ and $\underline{\ell} \in \mathbb{N}^N$, define

$$t_0(1 \otimes \underline{x}^{\underline{\ell}}) = -\sum_{j=1}^{N} \sum_{r=1}^{\ell_j} (x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N}) \otimes x_j \otimes (x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}).$$

If $p \ge 1$, it suffices to give

$$t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{x}^{\underline{\ell}})$$

for $\underline{\ell} \in \mathbb{N}^N$ and $1 \leq j_1 < \cdots < j_p \leq N$, and we set

$$t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{x}^{\underline{\ell}})$$
$$= (-1)^{p+1} \sum_{j_{p+1}=j_p+1}^{N} \sum_{r=1}^{\ell_{j_{p+1}}} \big(x_{j_{p+1}}^{\ell_{j_{p+1}}-r} x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}} \cdots x_N^{\ell_N}\big) \otimes (x_{j_1} \wedge \cdots \wedge x_{j_{p+1}}) \otimes \big(x_1^{\ell_1} \cdots x_{j_{p+1}-1}^{\ell_{j_{p+1}-1}} x_{j_{p+1}}^{r-1}\big).$$

In the case $j_p = N$, the sum is empty, and so the value of t_p on such an element is 0.

Proposition 3.1. *The above-defined maps t_p, $p \geq -1$, form a chain contraction for the resolution $K_\bullet(S(V))$.*

Proof. It is easy to verify that $d_0 t_{-1} = \mathrm{Id}$. We need to show that for $p \geq 0$, $t_{p-1}d_p + d_{p+1}t_p = \mathrm{Id}$. We first let $p = 0$, and show that $t_{-1}d_0 + d_1 t_0 = \mathrm{Id}$.

For $\underline{\ell} \in \mathbb{N}^N$, we have $t_{-1}d_0(1 \otimes \underline{x}^{\underline{\ell}}) = t_{-1}(\underline{x}^{\underline{\ell}}) = \underline{x}^{\underline{\ell}} \otimes 1$, and

$$\begin{aligned}
d_1 t_0(1 \otimes \underline{x}^{\underline{\ell}}) &= d_1\Bigg(-\sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_j \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}\Bigg) \\
&= -\sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j - r+1} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1} \\
&\qquad + \sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r} \\
&= -\sum_{j=1}^{N}\sum_{r=0}^{\ell_j-1} x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r} \\
&\qquad + \sum_{j=1}^{N}\sum_{r=1}^{\ell_j} x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r} \\
&= -\sum_{j=1}^{N} x_j^{\ell_j} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} \\
&\qquad + \sum_{j=1}^{N} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{\ell_j} \\
&= -\sum_{j=1}^{N} x_j^{\ell_j} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} + \sum_{j=2}^{N+1} x_j^{\ell_j} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} \\
&= -\underline{x}^{\underline{\ell}} \otimes 1 + 1 \otimes \underline{x}^{\underline{\ell}}.
\end{aligned}$$

We thus obtain $(t_{-1}d_0 + d_1t_0)(1 \otimes \underline{x}^{\ell}) = \underline{x}^{\ell} \otimes 1 - \underline{x}^{\ell} \otimes 1 + 1 \otimes \underline{x}^{\ell} = 1 \otimes \underline{x}^{\ell}$ and therefore confirm the equality. Note that in the above proof, there are many terms which cancel one another.

The proof of the equality $t_{p-1}d_p + d_{p+1}t_p = \mathrm{Id}$ for $p \geq 1$ is similar to the above case $p = 0$, but is much more complicated. As in the case $p = 0$, for the cases $p \geq 1$ we must change indices several times in order to cancel many terms. $\square$

Now we can use the chain contraction of Proposition 3.1 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution. Such comparison morphisms were found in [Shepler and Witherspoon 2011] by ad hoc methods.

For any $\Bbbk$-algebra A associative with unity, denote by $\overline{A} = A/(\Bbbk \cdot 1)$ a $\Bbbk$-vector space. The *normalized bar resolution* of A has p-th term $B_p(A) = A \otimes \overline{A}^{\otimes p} \otimes A$ and differentials $\delta_p : A \otimes \overline{A}^{\otimes p} \otimes A \to A \otimes \overline{A}^{\otimes (p-1)} \otimes A$ given by

$$\delta_p(a_0 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) = \sum_{i=0}^{p} (-1)^i a_0 \otimes \cdots \otimes \overline{a_i}\,\overline{a_{i+1}} \otimes \cdots \otimes a_{p+1}$$

for $a_0, \ldots, a_{p+1} \in A$, where an overline indicates an image in $\overline{A}$. We shall see that this resolution is suitable for computation using the method from Section 2.

There is a standard chain contraction of the normalized bar resolution,

$$s_p : A \otimes \overline{A}^{\otimes p} \otimes A \to A \otimes \overline{A}^{\otimes (p+1)} \otimes A,$$

given by

$$s_p(1 \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes a_{p+1}) = (-1)^{p+1} \otimes \overline{a_1} \otimes \cdots \otimes \overline{a_p} \otimes \overline{a_{p+1}} \otimes 1. \tag{3.2}$$

Each s_p is then extended to a left A-module homomorphism. For convenience, we shall from now on abuse notation and write a_i in place of $\overline{a_i}$.

A chain map from the Koszul resolution to the normalized bar resolution is given by the standard embedding: for $p \geq 0$, define

$$\Phi_p : A \otimes \textstyle\bigwedge^p(V) \otimes A \to A \otimes \overline{A}^{\otimes p} \otimes A$$

by

$$\Phi_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) = \sum_{\pi \in \mathrm{Sym}_p} \operatorname{sgn}\pi \otimes x_{j_{\pi(1)}} \otimes \cdots \otimes x_{j_{\pi(p)}} \otimes 1 \tag{3.3}$$

for $1 \leq j_1 < \cdots < j_p \leq N$, where Sym_p denotes the symmetric group on p symbols.

The other direction is much more complicated. We shall define $\Psi_p : A \otimes \overline{A}^{\otimes p} \otimes A \to A \otimes \bigwedge^p(V) \otimes A$ for each $p \geq 0$. Let Ψ_0 be the identity map. For $p \geq 1$, define Ψ_p by

(3.4) $$\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes 1) = \sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{0\le r_s\le \ell^s_{j_s}-1\\ s=1,\dots,p}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}}\otimes (x_{j_1}\wedge\cdots\wedge x_{j_p}) \otimes \underline{x}^{\underline{\widehat{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}},$$

where, as in [Shepler and Witherspoon 2011], the N-tuple $\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}$ is defined by

$$\big(\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}\big)_j = \begin{cases} r_j+\ell^1_j+\cdots+\ell^{s-1}_j & \text{if } j=j_s,\\ \ell^1_j+\cdots+\ell^s_j & \text{if } j_s<j<j_{s+1},\end{cases}$$

and where the N-tuple $\underline{\widehat{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}$ is defined to be complementary to $\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}$ in the sense that

$$\underline{x}^{\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}}\ \underline{x}^{\underline{\widehat{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\,j_1,\dots,j_p)}_{(r_1,\dots,r_p)}}\ x_{j_1}\cdots x_{j_p} = \underline{x}^{\underline{\ell}^1}\cdots \underline{x}^{\underline{\ell}^p}\in \Bbbk[x_1,\dots,x_N].$$

Theorem 3.5 [Shepler and Witherspoon 2011]. *Let $\Phi_\bullet$ and $\Psi_\bullet$ be as defined in (3.3) and (3.4). Then*

(i) *the map $\Phi_\bullet$ is a chain map from the Koszul resolution to the normalized bar resolution*;

(ii) *the map $\Psi_\bullet$ is a chain map from the normalized bar resolution to the Koszul resolution*;

(iii) *the composition $\Psi_\bullet\circ\Phi_\bullet$ is the identity map.*

Proof. (i) We check that this standard map follows from the method in Section 2, in order to illustrate the method. We proceed by induction, applying (2.1) to the chain contraction $s_\bullet$ of the normalized bar resolution defined in (3.2).

The case $p=0$ is trivial. Now suppose that for $p\ge 0$, $\Phi_p : A\otimes \bigwedge^p(V)\otimes A\to A\otimes \overline{A}^{\otimes p}\otimes A$ is given by (3.3). We compute $\Phi_{p+1}(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1)$, where Φ_{p+1} is defined by (2.1) in terms of Φ_p. We have

$$\begin{aligned}\Phi_{p+1}&(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1)\\ &= s_p\Phi_p d_{p+1}(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1)\\ &= s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}x_{j_i}\otimes(x_{j_1}\wedge\cdots\wedge \hat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\Big)\\ &\qquad - s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}\otimes(x_{j_1}\wedge\cdots\wedge \hat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes x_{j_i}\Big).\end{aligned}$$

Notice that the value of s_p on

$$\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}x_{j_i}\otimes(x_{j_1}\wedge\cdots\wedge\hat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1\Big)$$

is 0, since the rightmost tensor factor is 1, and we work with the normalized bar resolution. For a permutation $\pi\in\mathrm{Sym}_p$ that fixes some letter i, $1\le i\le p+1$, consider the permutation $\hat{\pi}$ of the set $\{1,\dots,i-1,\hat{i},i+1,\dots,p+1\}$ corresponding to π via the bijection

$$\{1,\dots,i-1,i,i+1,\dots,p\}\simeq\{1,\dots,i-1,\hat{i},i+1,\dots,p+1\}$$

sending j to j for $1\le j\le i-1$ and to $j+1$ for $i\le j\le p$.

Define a new permutation $\tilde{\pi}\in S_{p+1}$ by imposing

$$\tilde{\pi}(j)=\begin{cases}\hat{\pi}(j) & \text{for } j<i,\\ \hat{\pi}(j+1) & \text{for } i\le j<p+1,\\ i & \text{for } j=p+1.\end{cases}$$

Then we have $\operatorname{sgn}\tilde{\pi}=(-1)^{p-i+1}\operatorname{sgn}\pi$, and so

$$\begin{aligned}
&\Phi_{p+1}(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_{p+1}})\otimes 1)\\
&\quad=-s_p\Phi_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}\otimes(x_{j_1}\wedge\cdots\wedge\hat{x}_{j_i}\wedge\cdots\wedge x_{j_{p+1}})\otimes x_{j_i}\Big)\\
&\quad=-s_p\Big(\sum_{i=1}^{p+1}(-1)^{i+1}\sum_{\substack{\tilde{\pi}\in S_{p+1}\\ \tilde{\pi}(p+1)=i}}(-1)^{p-i+1}\operatorname{sgn}\tilde{\pi}\otimes x_{j_{\tilde{\pi}(1)}}\otimes\cdots\otimes x_{j_{\tilde{\pi}(p)}}\otimes x_{j_{\tilde{\pi}(p+1)}}\Big)\\
&\quad=-(-1)^{p+1}\sum_{i=1}^{p+1}(-1)^{i+1}\sum_{\substack{\tilde{\pi}\in S_{p+1}\\ \tilde{\pi}(p+1)=i}}(-1)^{p-i+1}\operatorname{sgn}\tilde{\pi}\otimes x_{j_{\tilde{\pi}(1)}}\\
&\qquad\qquad\otimes\cdots\otimes x_{j_{\tilde{\pi}(p)}}\otimes x_{j_{\tilde{\pi}(p+1)}}\otimes 1\\
&\quad=\sum_{\tilde{\pi}\in S_{p+1}}\operatorname{sgn}\tilde{\pi}\otimes x_{j_{\tilde{\pi}(1)}}\otimes\cdots\otimes x_{j_{\tilde{\pi}(p)}}\otimes x_{j_{\tilde{\pi}(p+1)}}\otimes 1.
\end{aligned}$$

This completes the proof of (i).

(ii) As in (i), we apply the method in Section 2 to the chain contraction $t_\bullet$ of Proposition 3.1 to show that $\Psi_\bullet$ as defined in (3.4) is indeed the resulting chain map. We proceed by induction on p.

Suppose that Ψ_p is given by (3.4). Let us apply (2.1) and show that Ψ_{p+1} results. First notice that we can write

$$t_p(1\otimes(x_{j_1}\wedge\cdots\wedge x_{j_p})\otimes \underline{x}^{\underline{\ell}}) = (-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\sum_{r=1}^{\ell_{j_{p+1}}} \underline{x}^{\underline{Q}_r^{(\underline{\ell};\, j_{p+1})}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\wedge x_{j_{p+1}}\otimes \underline{x}^{\widehat{\underline{Q}}_r^{(\underline{\ell};\, j_{p+1})}}.$$

We have

$$d_{p+1}(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1) = \underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1+\sum_{i=1}^{p}(-1)^p\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^i+\underline{\ell}^{i+1}}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1 + (-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}}.$$

Now consider

$$\Psi_p(\underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1) = \sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{1\le r_s\le \ell_{j_s}^{s+1}\\ 1\le s\le p}} \underline{x}^{\underline{\ell}^1}\underline{x}^{\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^2,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_p)}} \otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^2,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_p)}}.$$

However, $\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^2,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_p)}$, by definition, has no terms of the form x_u^v with $u > j_p$. Thus, we have $t_p\Psi_p(\underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1)=0$.

Similarly we can prove that for $1\le i\le p$,

$$t_p\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^i+\underline{\ell}^{i+1}}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1)=0.$$

The only term left is $t_p\Psi_p((-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}})$. We obtain

$$t_p\Psi_p((-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes \underline{x}^{\underline{\ell}^2}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}}) = (-1)^{p+1}\sum_{1\le j_1<\cdots<j_p\le N}\ \sum_{\substack{1\le r_s\le \ell_{j_s}^{s}\\ 1\le s\le p}} t_p\Big(\underline{x}^{\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p} \otimes \underline{x}^{\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}}\underline{x}^{\underline{\ell}^{p+1}}\Big)$$

$$= \sum_{1\le j_1<\cdots<j_p\le N} \sum_{\substack{1\le r_s\le \ell^s_{j_s}\\ 1\le s\le p}} \sum_{j_{p+1}=j_p+1}^{N} \sum_{r=1}^{\ell^{p+1}_{j_{p+1}}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)}} \underline{x}^{\underline{Q}^{(\underline{\ell};\, j_{p+1})}_{r_{p+1}}} \otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}} \otimes \underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell};\, j_{p+1})}_{r_{p+1}}},$$

where

$$\underline{\ell} = \widehat{\underline{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)} + \underline{\ell}^{p+1}.$$

Now notice that

$$\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)} + \underline{Q}^{(\underline{\ell};\, j_{p+1})}_{r_{p+1}} = \underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_{p+1})}_{(r_1,\dots,r_{p+1})}$$

and

$$\widehat{\underline{Q}}^{(\underline{\ell};\, j_{p+1})}_{r_{p+1}} = \widehat{\underline{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_{p+1})}_{(r_1,\dots,r_{p+1})}.$$

We have the desired result:

$$\begin{aligned}
t_p\Psi_p d_{p+1}&(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1)\\
&= t_p\Psi_p((-1)^{p+1}\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes \underline{x}^{\underline{\ell}^{p+1}})\\
&= \sum_{1\le j_1<\cdots<j_{p+1}\le N} \sum_{\substack{1\le r_s\le \ell^s_{j_s}\\ 1\le s\le p+1}} \underline{x}^{\underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_{p+1})}_{(r_1,\dots,r_{p+1})}} \otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}} \otimes \underline{x}^{\widehat{\underline{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^{p+1};\, j_1,\dots,j_{p+1})}_{(r_1,\dots,r_{p+1})}}\\
&= \Psi_{p+1}(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^{p+1}}\otimes 1).
\end{aligned}$$

(iii) For $1\le i_1<\cdots<i_p\le N$, we have

$$\begin{aligned}
\Psi_p\Phi_p&(1\otimes(x_{i_1}\wedge\cdots\wedge x_{i_p})\otimes 1)\\
&= \Psi_p\Big(\sum_{\pi\in \mathrm{Sym}_p} \operatorname{sgn}\pi\otimes x_{i_{\pi(1)}}\otimes\cdots\otimes x_{i_{\pi(p)}}\otimes 1\Big)\\
&= \sum_{\pi\in\mathrm{Sym}_p}\operatorname{sgn}\pi \sum_{1\le j_1<\cdots<j_p\le N} \sum_{\substack{0\le r_s\le (e_{i_{\pi(s)}})_{j_s}-1\\ s=1,\dots,p}} \underline{x}^{\underline{Q}^{(e_{i_{\pi(1)}},\dots,e_{i_{\pi(p)}};\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)}} \otimes x_{j_1}\wedge\cdots\wedge x_{j_p}\otimes \underline{x}^{\widehat{\underline{Q}}},
\end{aligned}$$

where e_u is the u-th canonical basis vector $(0,\dots,0,1,0,\dots,0)$, the 1 in the u-th position, and

$$\widehat{\underline{Q}} = \widehat{\underline{Q}}^{(e_{i_{\pi(1)}},\dots,e_{i_{\pi(p)}};\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)}.$$

Notice that $\underline{Q}_{(r_1,\dots,r_p)}^{(e_{i_{\pi(1)}},\dots,e_{i_{\pi(p)}};j_1,\dots,j_p)}$ occurs in the sum only if $(i_{\pi(1)},\dots,i_{\pi(p)}) = (j_1,\dots,j_p)$. Here, π is the identity, $r_1=\dots=r_p=0$ and $\underline{Q}_{(r_1,\dots,r_p)}^{(e_{i_{\pi(1)}},\dots,e_{i_{\pi(p)}};j_1,\dots,j_p)}$ is the zero vector. Therefore,

$$\Psi_p\Phi_p(1\otimes x_{i_1}\wedge\dots\wedge x_{i_p}\otimes 1) = 1\otimes x_{i_1}\wedge\dots\wedge x_{i_p}\otimes 1. \qquad \square$$

For comparison, we give an alternative description of the maps Ψ_p due to Carqueville and Murfet [2016]: for each i, let $\tau_i : S(V)^e \to S(V)^e$ be the $\Bbbk$-linear map that is defined on monomials as follows. (We denote application of the map τ_i by a left superscript.)

$$\begin{aligned} &{}^{\tau_i}(x_1^{j_1}\cdots x_N^{j_N}\otimes x_1^{l_1}\cdots x_N^{l_N}) \\ &\qquad = x_1^{j_1}\cdots x_{i-1}^{j_{i-1}}x_{i+1}^{j_{i+1}}\cdots x_N^{j_N}\otimes x_1^{l_1}\cdots x_{i-1}^{l_{i-1}}x_i^{j_i+l_i}x_{i+1}^{l_{i+1}}\cdots x_N^{l_N}. \end{aligned}$$

Define difference quotient operators $\partial_{[i]} : S(V)\to S(V)^e$ for each i, $1\le i\le N$, as in [Carqueville and Murfet 2016, (2.12)] by

$$\partial_{[i]}(f) := \frac{{}^{\tau_1\cdots\tau_{i-1}}(f\otimes 1) - {}^{\tau_1\cdots\tau_i}(f\otimes 1)}{x_i\otimes 1 - 1\otimes x_i}.$$

For example, ${}^{\tau_1}(x_1^2x_2\otimes 1) = x_2\otimes x_1^2$, so that

$$\partial_{[1]}(x_1^2x_2) = \frac{x_1^2x_2\otimes 1 - x_2\otimes x_1^2}{x_1\otimes 1 - 1\otimes x_1} = x_1x_2\otimes 1 + x_2\otimes x_1.$$

Similarly, $\partial_{[2]}(x_1^2x_2) = 1\otimes x_1^2$.

Identify elements in $S(V)^e\otimes\bigwedge^p(V)$ with elements in $S(V)\otimes\bigwedge^p(V)\otimes S(V)$ via the canonical isomorphism between these two spaces. Then Ψ_p may be expressed as in [Carqueville and Murfet 2016, (2.22)]:

$$\Psi_p(1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}\otimes 1) = \sum_{1\le j_1<\dots<j_p\le N}\left(\prod_{s=1}^{p}\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\right)\otimes x_{j_1}\wedge\dots\wedge x_{j_p}.$$

For example, if $N=2$, then

$$\Psi_1(1\otimes x_1^2x_2\otimes 1) = x_1x_2\otimes 1\otimes x_1 + x_2\otimes x_1\otimes x_1 + 1\otimes x_1^2\otimes x_2.$$

We may similarly express the chain contraction t_p as

$$t_p(1\otimes x_{j_1}\wedge\dots\wedge x_{j_p}\otimes \underline{x}^{\underline{\ell}}) = (-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})\otimes x_{j_1}\wedge\dots\wedge x_{j_{p+1}}.$$

4. Chain contractions and comparison maps for quantum symmetric algebras

Let N be a positive integer, and for each pair $i, j \in \{1, 2, \dots, N\}$, let $q_{i,j}$ be a nonzero scalar in the field $\Bbbk$ such that $q_{i,i} = 1$ and $q_{j,i} = q_{i,j}^{-1}$ for all i, j. Denote by $\boldsymbol{q}$ the corresponding tuple of scalars, $\boldsymbol{q} := (q_{i,j})_{1\le i,j\le N}$. Let V be a vector space with basis $x_1, \dots, x_N$, and let

$$S_{\boldsymbol{q}}(V) := k\langle x_1, \dots, x_N \mid x_i x_j = q_{i,j} x_j x_i \text{ for all } 1 \le i, j \le N\rangle, \tag{4.1}$$

be the quantum symmetric algebra determined by $\boldsymbol{q}$. This is a Koszul algebra, and there is a standard complex $K_\bullet(S_{\boldsymbol{q}}(V))$ that is a free resolution of $S_{\boldsymbol{q}}(V)$ as an $S_{\boldsymbol{q}}(V)$-bimodule (see, e.g., [Wambst 1993, Proposition 4.1(c)]). Setting $A = S_{\boldsymbol{q}}(V)$, the complex is

$$\cdots \longrightarrow A \otimes \textstyle\bigwedge^2(V) \otimes A \xrightarrow{d_2} A \otimes \bigwedge^1(V) \otimes A \xrightarrow{d_1} A \otimes A \left(\xrightarrow{d_0} A \longrightarrow 0\right),$$

with differential d_p defined by

$$\begin{aligned} d_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) &= \sum_{i=1}^{p} (-1)^{i+1} \left(\prod_{s=1}^{i} q_{j_s, j_i}\right) x_{j_i} \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes 1 \\ &\quad - \sum_{i=1}^{p} (-1)^{i+1} \left(\prod_{s=i}^{p} q_{j_i, j_s}\right) \otimes (x_{j_1} \wedge \cdots \wedge \hat{x}_{j_i} \wedge \cdots \wedge x_{j_p}) \otimes x_{j_i} \end{aligned}$$

whenever $1 \le j_1 < \cdots < j_p \le N$ and $p > 0$; the map d_0 is multiplication.

As in the previous section, we write $\underline{\ell} = (\ell_1, \dots, \ell_N)$, $\underline{x} = (x_1, \dots, x_N)$ and $\underline{x}^{\underline{\ell}} = x_1^{\ell_1} \cdots x_N^{\ell_N}$. We shall give a chain contraction of $K_\bullet(S_{\boldsymbol{q}}(V))$,

$$t_p : A \otimes \textstyle\bigwedge^p(V) \otimes A \to A \otimes \bigwedge^{p+1}(V) \otimes A$$

for $p \ge 0$ and $t_{-1} : A \to A \otimes A$, which are moreover left A-module homomorphisms (cf. [Wambst 1993]).

Let $t_{-1}(1) = 1 \otimes 1$ and extend t_{-1} to be left A-linear. For $p \ge 0$, $\underline{\ell} \in \mathbb{N}^N$, and $1 \le j_1 < \cdots < j_p \le N$, let

$$\begin{aligned} t_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes \underline{x}^{\underline{\ell}}) &= (-1)^{p+1} \sum_{j_{p+1}=j_p+1}^{N} \sum_{r=1}^{\ell_{j_{p+1}}} \lambda_{j_{p+1},r}^{(\underline{\ell};\, j_1,\dots,j_p)} x_{j_{p+1}}^{\ell_{j_{p+1}}-r} x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}} \cdots x_N^{\ell_N} \\ &\qquad \otimes x_{j_1} \wedge \cdots \wedge x_{j_{p+1}} \otimes x_1^{\ell_1} \cdots x_{j_{p+1}-1}^{\ell_{j_{p+1}-1}} x_{j_{p+1}}^{r-1}, \end{aligned}$$

where

$$\lambda^{(\underline{\ell};\, j_1,\dots,j_p)}_{j_{p+1},r} = \Bigg(\prod_{s=1}^{j_{p+1}-1}\prod_{t=j_{p+1}}^{N} q_{s,t}^{\ell_s\ell_t}\Bigg)\Bigg(\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}\Bigg)^{r-1}\Bigg(\prod_{t=1}^{p} q_{j_t,j_{p+1}}^{\ell_{j_{p+1}}-r}\Bigg)\Bigg(\prod_{s=1}^{p+1}\prod_{t=j_{p+1}+1}^{N} q_{j_s,t}^{\ell_t}\Bigg).$$

Compared with the maps in the previous section for polynomial algebras, the only difference is that now there is a new coefficient. This (rather complicated) coefficient $\lambda^{(\underline{\ell};\, j_1,\dots,j_p)}_{j_{p+1},r}$ can be obtained as follows: in the right side of the formula for t_p, in comparison to its argument $1 \otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes \underline{x}^{\underline{\ell}}$ on the left side, whenever a factor x_i of $\underline{x}^{\underline{\ell}}$ has changed positions so that it is now to the left of a factor x_j with $i > j$ (including factors of the exterior product), one should include one factor of $q_{j,i}$. One can verify easily that $\lambda^{(\underline{\ell};\, j_1,\dots,j_p)}_{j_{p+1},r}$ has the given form. We shall call this rule the *twisting principle* and we use it several times later.

Proposition 4.2. *The above-defined maps t_p, $p \geq -1$, form a chain contraction over the resolution $K_\bullet(S_{\boldsymbol{q}}(V))$.*

Proof. One needs to verify that for $n \geq 0$, we have $t_{n-1}d_n + d_{n+1}t_n = \mathrm{Id}$ and $d_0 t_{-1} = \mathrm{Id}$. Notice that the computation used in the above equalities is the same as that for polynomial algebras, except that now for quantum symmetric algebras, we have some extra coefficients. One needs to show that these extra coefficients do not cause any problem.

Recall that in the proof of Proposition 3.1, the concrete computation is simplified by many terms which cancel one another. For example, this occurs in the verification of the equation $t_{-1}d_0 + d_1 t_0 = \mathrm{Id}$ in the proof of Proposition 3.1. For polynomial algebras, the proof works due to these cancelling terms.

For quantum symmetric algebras, things are not so easy. However, the twisting principle always holds; that is, when we apply a differential or chain contraction, once we produce a monomial (always in lexicographical order) or tensor of monomials, we need to include a coefficient before this monomial according to the twisting principle. Thus, if two terms cancel each other for polynomial algebras, as we have included the same coefficient, they still cancel for quantum symmetric algebras. □

Now we can use (2.1) and the chain contraction of Proposition 4.2 to give formulae for comparison morphisms between the normalized bar resolution and the Koszul resolution.

A chain map from the Koszul resolution to the normalized bar resolution is induced from the standard embedding of the Koszul resolution into the (unnormalized) bar resolution. See also [Wambst 1993, Lemma 5.3 and Theorem 5.4] for a more general setting. We give the formula as it appears in [Naidu et al. 2011,

§2.2(3)]. For $p \geq 0$, we define

$$\Phi_p : A \otimes \bigwedge^p(V) \otimes A \to A \otimes \bar{A}^{\otimes p} \otimes A$$

by

$$\Phi_p(1 \otimes (x_{j_1} \wedge \cdots \wedge x_{j_p}) \otimes 1) = \sum_{\pi \in \mathrm{Sym}_p} \operatorname{sgn} \pi \, q_\pi^{j_1,\dots,j_p} \otimes x_{j_{\pi(1)}} \otimes \cdots \otimes x_{j_{\pi(p)}} \otimes 1 \tag{4.3}$$

for $1 \leq j_1 < \cdots < j_p \leq N$. In the above formula, the coefficients $q_\pi^{j_1,\dots,j_p}$ are the scalars obtained from the twisting principle, that is,

$$q_\pi^{j_1,\dots,j_p} x_{j_{\pi(1)}} \cdots x_{j_{\pi(p)}} = x_{j_1} \cdots x_{j_p}. \tag{4.4}$$

The other direction is much more complicated. We shall see that for quantum symmetric algebras, the comparison morphism is a twisted version of that for a polynomial ring given in the previous section, with certain coefficients included according to the twisting principle.

We define the maps

$$\Psi_p : A \otimes \bar{A}^{\otimes p} \otimes A \to A \otimes \bigwedge^p(V) \otimes A$$

as follows. Let Ψ_0 be the identity map. For $p \geq 1$, define Ψ_p by

$$\begin{aligned} \Psi_p(1 \otimes \underline{x}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{x}^{\underline{\ell}^p} \otimes 1) &= \sum_{1 \leq j_1 < \cdots < j_p \leq N} \sum_{\substack{0 \leq r_s \leq \ell^s_{j_s} - 1 \\ s=1,\dots,p}} \mu_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)} \underline{x}^{\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}} \\ &\qquad \otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes \underline{x}^{\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}}, \end{aligned} \tag{4.5}$$

where, as before, we define the N-tuple $\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ by

$$\big(\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}\big)_j = \begin{cases} r_j + \ell^1_j + \cdots + \ell^{s-1}_j & \text{if } j = j_s, \\ \ell^1_j + \cdots + \ell^s_j & \text{if } j_s < j < j_{s+1}, \end{cases}$$

and where the N-tuple $\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ and scalar $\mu_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ are defined (uniquely) by

$$\mu_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)} \underline{x}^{\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}} x_{j_1} \cdots x_{j_p} \underline{x}^{\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}} = \underline{x}^{\underline{\ell}^1} \cdots \underline{x}^{\underline{\ell}^p} \in S_{\boldsymbol{q}}(V).$$

The coefficient $\mu_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ is obtained using the twisting principle in the right side of the formula for Ψ_p, and $\underline{Q}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ and $\widehat{\underline{Q}}_{(r_1,\dots,r_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}$ are the same as in the case of the polynomial algebra $\Bbbk[x_1, \dots, x_n]$. For comparison, we note that Wambst [1993, Lemma 6.7] gave such a chain map in degree 1.

Theorem 4.6. *Let $\Phi_\bullet$ and $\Psi_\bullet$ be as defined in (4.3) and (4.5). Then*

(i) *the map $\Phi_\bullet$ is a chain map from the Koszul resolution to the normalized bar resolution*;

(ii) *the map $\Psi_\bullet$ is a chain map from the normalized bar resolution to the Koszul resolution*;

(iii) *the composition $\Psi_\bullet \circ \Phi_\bullet$ is the identity map.*

Proof. (i) One direct proof was given in [Naidu et al. 2011, Lemma 2.3]. (The characteristic of $\Bbbk$ was assumed to be 0 in that paper; however, this assumption is not needed in that proof.) Another proof can be given by applying (2.1) to a chain contraction $s_\bullet$ over the normalized bar resolution as in the proof of Theorem 3.5(i). The twisting principle gives the coefficients.

(ii) One direct computational proof can be given by applying (2.1) to the chain contraction $t_\bullet$ of Proposition 4.2, as in the proof of Theorem 3.5(ii). Thus the same proof as that of Theorem 3.5(ii) works, taking care with the coefficients, by the twisting principle.

(iii) The same proof as that of Theorem 3.5(iii) works; by the twisting principle, the coefficients on both sides of the equation coincide. □

We now give alternative descriptions of the maps t_p and Ψ_p in this case of a quantum symmetric algebra. The description of Ψ_p will generalize that of Carqueville and Murfet [2016] from $S(V)$ to $S_{\boldsymbol{q}}(V)$. To this end, it is convenient to replace each term $S_{\boldsymbol{q}}(V) \otimes \bigwedge^p(V) \otimes S_{\boldsymbol{q}}(V)$ of the Koszul resolution by $S_{\boldsymbol{q}}(V) \otimes S_{\boldsymbol{q}}(V) \otimes \bigwedge^p(V)$, using the canonical isomorphism

$$\sigma_p : S_{\boldsymbol{q}}(V) \otimes S_{\boldsymbol{q}}(V) \otimes \textstyle\bigwedge^p(V) \to S_{\boldsymbol{q}}(V) \otimes \bigwedge^p(V) \otimes S_{\boldsymbol{q}}(V)$$

in which coefficients are inserted according to the twisting principle. For example, for $\underline{x}^{\underline{\ell}} \in S_{\boldsymbol{q}}(V)$ and $1 \le j_1 < \cdots < j_p \le N$,

$$\sigma_p(1 \otimes \underline{x}^{\underline{\ell}} \otimes x_{j_1} \wedge \cdots \wedge x_{j_p}) = \left(\prod_{s=1}^{N} \prod_{t=1}^{p} q_{s,j_t}^{\ell_s} \right) \otimes x_{j_1} \wedge \cdots \wedge x_{j_p} \otimes \underline{x}^{\underline{\ell}}.$$

Via this isomorphism, consider t_p as a map from $S_{\boldsymbol{q}}(V) \otimes S_{\boldsymbol{q}}(V) \otimes \bigwedge^p(V)$ to $S_{\boldsymbol{q}}(V) \otimes S_{\boldsymbol{q}}(V) \otimes \bigwedge^{p+1}(V)$. By abuse of notation, we still denote by t_p this new map; the same rule applies to Ψ_p.

For $1 \le j \le N$, define $\tau_j : S_{\boldsymbol{q}}(V)^e \to S_{\boldsymbol{q}}(V)^e$ to be the operator that replaces all factors of the form $x_j \otimes 1$ with $1 \otimes x_j$, but with coefficient inserted according to the twisting principle. For example, if $\underline{x}^{\underline{\ell}} \in S_{\boldsymbol{q}}(V)$, then

$$\tau_j(\underline{x}^{\underline{\ell}} \otimes 1) = \left(\prod_{s=j+1}^{N} q_{j,s}^{\ell_j \ell_s} \right) x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_j^{\ell_j}.$$

It is not difficult to see that for $1 \le i \ne j \le N$, $\tau_i\tau_j = \tau_j\tau_i$. Define quantum difference quotient operators $\partial_{[i]} : S_{\boldsymbol{q}}(V) \to S_{\boldsymbol{q}}(V) \otimes S_{\boldsymbol{q}}(V)$ for each i, $1 \le i \le N$, by

$$\partial_{[i]}(f) := (x_i \otimes 1 - 1 \otimes x_i)^{-1}({}^{\tau_1\cdots\tau_{i-1}}(f \otimes 1) - {}^{\tau_1\cdots\tau_i}(f \otimes 1)). \tag{4.7}$$

This definition should be understood as follows: by writing f as a linear combination of monomials, it suffices to define $\partial_{[i]}$ on each monomial $\underline{x}^{\underline{\ell}}$. The difference ${}^{\tau_1\cdots\tau_{i-1}}(\underline{x}^{\underline{\ell}} \otimes 1) - {}^{\tau_1\cdots\tau_i}(\underline{x}^{\underline{\ell}} \otimes 1)$ may be divided by $x_i \otimes 1 - 1 \otimes x_i$ on the left, by first factoring out $x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i}$ on the left. Applying the twisting principle, one sees that this is indeed always a factor. One must include a coefficient given by the twisting principle, then use the identity

$$(x_i \otimes 1 - 1 \otimes x_i)^{-1}(x_i^{\ell_i} \otimes 1 - 1 \otimes x_i^{\ell_i}) = \sum_{r=1}^{\ell_i} x_i^{\ell_i - r} \otimes x_i^{r-1}.$$

For example, for $f = x_1x_2^2$, let us compute $\partial_{[2]}(f)$. We have

$$ {}^{\tau_1}(x_1x_2^2 \otimes 1) = q_{1,2}^2 x_2^2 \otimes x_1 = q_{1,2}^2(x_2^2 \otimes 1)(1 \otimes x_1),$$

$$ {}^{\tau_1\tau_2}(x_1x_2^2 \otimes 1) = 1 \otimes x_1x_2^2 = q_{1,2}^2(1 \otimes x_2^2)(1 \otimes x_1),$$

and so

$$ {}^{\tau_1}(x_1x_2^2 \otimes 1) - {}^{\tau_1\tau_2}(x_1x_2^2 \otimes 1) = q_{1,2}^2(x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1).$$

We obtain thus

$$\begin{aligned}
\partial_{[2]}(f) &= (x_2 \otimes 1 - 1 \otimes x_2)^{-1}({}^{\tau_1}(x_1x_2^2 \otimes 1) - {}^{\tau_1\tau_2}(x_1x_2^2 \otimes 1)) \\
&= (x_2 \otimes 1 - 1 \otimes x_2)^{-1}(q_{1,2}^2(x_2^2 \otimes 1 - 1 \otimes x_2^2)(1 \otimes x_1)) \\
&= q_{1,2}^2(x_2 \otimes 1 + 1 \otimes x_2)(1 \otimes x_1) \\
&= q_{1,2}^2 x_2 \otimes x_1 + q_{1,2} \otimes x_1x_2.
\end{aligned}$$

In general, we have

$$\begin{aligned}
\partial_{[j]}(\underline{x}^{\underline{\ell}}) = \Big(\prod_{s=1}^{j-1} q_{s,j}^{\ell_s}\Big) \sum_{r=1}^{\ell_j} \Big(\prod_{s=1}^{j-1} \prod_{t=j+1}^{N} q_{s,t}^{\ell_s\ell_t}\Big) \\
\times \Big(\prod_{t=j+1}^{N} q_{j,t}^{\ell_t(r-1)}\Big) x_j^{\ell_j - r} x_{j+1}^{\ell_{j+1}} \cdots x_N^{\ell_N} \otimes x_1^{\ell_1} \cdots x_{j-1}^{\ell_{j-1}} x_j^{r-1}.
\end{aligned}$$

That is, one has an extra coefficient $\big(\prod_{s=1}^{j-1} q_{s,j}^{\ell_s}\big)$ as well as the coefficient included according to the twisting principle.

The chain contraction

$$t_p : S_{\boldsymbol{q}}(V)\otimes S_{\boldsymbol{q}}(V)\otimes \textstyle\bigwedge^p(V) \to S_{\boldsymbol{q}}(V)\otimes S_{\boldsymbol{q}}(V)\otimes \textstyle\bigwedge^{p+1}(V)$$

may be expressed as

$$t_p(1\otimes \underline{x}^{\underline{\ell}}\otimes x_{j_1}\wedge\cdots\wedge x_{j_p})$$
$$=(-1)^{p+1}\sum_{j_{p+1}=j_p+1}^{N}\Big(\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}\Big)\Big(\prod_{t=1}^{p} q_{j_{p+1},j_t}\Big)\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})\otimes x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}.$$

This is justified by the fact that the coefficient in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$ is nearly the coefficient needed by the twisting principle. The discrepancy is that $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$ has an extra factor $\prod_{t=1}^{j_{p+1}-1} q_{t,j_{p+1}}^{\ell_t}$, and we still need to insert $\prod_{t=j_{p+1}+1}^{N} q_{j_{p+1},t}^{\ell_t}$ and $\prod_{t=1}^{p} q_{j_{p+1},j_t}$ because the last factor in $x_{j_1}\wedge\cdots\wedge x_{j_{p+1}}$ lies to the right of $x_{j_1}\wedge\cdots\wedge x_{j_p}$ and of $x_{j_{p+1}+1}^{\ell_{j_{p+1}+1}}\cdots x_N^{\ell_N}$ in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$. Altogether then, we need to include an extra factor of $\big(\prod_{t=1}^{N} q_{j_{p+1},t}^{\ell_t}\big)\big(\prod_{t=1}^{p} q_{j_{p+1},j_t}\big)$ in the coefficient in $\partial_{[j_{p+1}]}(\underline{x}^{\underline{\ell}})$.

The chain map $\Psi_p : S_{\boldsymbol{q}}(V)\otimes S_{\boldsymbol{q}}(V)\otimes \overline{S_{\boldsymbol{q}}(V)}^{\otimes p}\to S_{\boldsymbol{q}}(V)\otimes S_{\boldsymbol{q}}(V)\otimes \bigwedge^p(V)$ may be expressed as

$$\text{(4.8)}\quad \Psi_p(1\otimes 1\otimes \underline{x}^{\underline{\ell}^1}\otimes\cdots\otimes \underline{x}^{\underline{\ell}^p}) = \sum_{1\le j_1<\cdots<j_p\le N}\mu_{(j_1,\dots,j_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p)}\Big(\prod_{s=1}^{p}\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\Big)\otimes x_{j_1}\wedge\cdots\wedge x_{j_p},$$

where the scalar is defined according to the twisting principle by

$$\text{(4.9)}\qquad \underline{x}^{\underline{\ell}^1}\cdots \underline{x}^{\underline{\ell}^p} = \mu_{(j_1,\dots,j_p)}^{(\underline{\ell}^1,\dots,\underline{\ell}^p)}\Big(\prod_{s=1}^{p}\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\Big)' x_{j_1}\cdots x_{j_p}\in S_{\boldsymbol{q}}(V).$$

Here the factor $\big(\prod_{s=1}^{p}\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\big)'$ is understood as follows: if $\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s}) = a_s\otimes b_s$ (symbolically), then the product $\big(\prod_{s=1}^{p}\partial_{[j_s]}(\underline{x}^{\underline{\ell}^s})\big)'$ is $\big(\prod_s a_s\big)\big(\prod_s b_s\big)\in A$.

5. Gerstenhaber brackets for quantum symmetric algebras

The Schouten–Nijenhuis (Gerstenhaber) bracket on Hochschild cohomology of the symmetric algebra $S(V)$ is well known. In this section, we generalize it to the quantum symmetric algebras $S_{\boldsymbol{q}}(V)$. First we recall the definition of the Gerstenhaber bracket on Hochschild cohomology as defined on the normalized bar resolution of any $\Bbbk$-algebra A (associative with unity).

Let $f \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes p} \otimes A, A)$ and $f' \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{\otimes q} \otimes A, A)$. Define their bracket, $[f, f'] \in \mathrm{Hom}_{A^e}(A \otimes \overline{A}^{(p+q-1)} \otimes A, A)$, by

$$[f, f'] = \sum_{k=1}^{p} (-1)^{(q-1)(k-1)} f \circ_k f' - (-1)^{(p-1)(q-1)} \sum_{k=1}^{q} (-1)^{(p-1)(k-1)} f' \circ_k f,$$

where

$$\begin{aligned}(f \circ_k f')&(1 \otimes a_1 \otimes \cdots \otimes a_{p+q-1} \otimes 1)\\ &= f\big(1 \otimes a_1 \otimes \cdots \otimes a_{k-1} \otimes f'(1 \otimes a_k \otimes \cdots \otimes a_{k+q-1} \otimes 1)\\ &\qquad\qquad \otimes a_{k+q} \otimes \cdots \otimes a_{p+q-1} \otimes 1\big).\end{aligned}$$

In the above definition, the image of an element under f or f' is understood in $\overline{A}$, whenever required.

Let $\bigwedge_{\boldsymbol{q}^{-1}}(V^*)$ be the quantum exterior algebra defined by the tuple $\boldsymbol{q}^{-1}$; that is, $\bigwedge_{\boldsymbol{q}^{-1}}(V^*)$ is the algebra generated by the dual basis $\{dx_1, \dots, dx_N\}$ of V^* with respect to the basis $\{x_1, \dots, x_N\}$ of V, subject to the relations $(dx_i)^2 = 0$ and $dx_i\, dx_j = -q_{i,j}^{-1}\, dx_j\, dx_i$ for all i, j. We denote the product on $\bigwedge_{\boldsymbol{q}^{-1}}(V^*)$ by $\wedge$. It is convenient to use abbreviated notation for monomials in this algebra: if I is the p-tuple $I = (i_1, \dots, i_p)$, denote by dx_I the element $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$ of $\bigwedge_{\boldsymbol{q}^{-1}}(V^*)$. We also write $\underline{x}^{\wedge I}$ for $x_{i_1} \wedge \cdots \wedge x_{i_p}$. Another notation we shall use is $dx_{\underline{b}}$, defined for any $\underline{b}$ in $\{0,1\}^N$ to be $dx_{i_1} \wedge \cdots \wedge dx_{i_p}$, where $i_1, \dots, i_p$ are the positions of the entries 1 in $\underline{b}$, all other entries being 0. In this case we say the *length* of $\underline{b}$ is p, and write $|\underline{b}| = p$.

In [Naidu et al. 2011, Corollary 4.3], the Hochschild cohomology of $S_{\boldsymbol{q}}(V)$ is given as the graded vector subspace of $S_{\boldsymbol{q}}(V) \otimes \bigwedge_{\boldsymbol{q}^{-1}}(V^*)$ that in degree m is

$$\mathrm{HH}^m(S_{\boldsymbol{q}}(V)) = \bigoplus_{\substack{\underline{b} \in \{0,1\}^N \\ |\underline{b}| = m}} \bigoplus_{\substack{\underline{a} \in \mathbb{N}^N \\ \underline{a} - \underline{b} \in C}} \mathrm{Span}_k\{\underline{x}^{\underline{a}} \otimes dx_{\underline{b}}\},$$

where

$$C = \Big\{ \gamma \in (\mathbb{N} \cup \{-1\})^N \;\Big|\; \text{for each } i \in \{1, \dots, N\},\ \prod_{s=1}^{N} q_{is}^{\gamma_s} = 1 \text{ or } \gamma_i = -1 \Big\}.$$

We wish to compute the bracket of two elements

$$\alpha = \underline{x}^{\underline{a}} \otimes dx_J \quad \text{and} \quad \beta = \underline{x}^{\underline{b}} \otimes dx_L,$$

where $J = (j_1, \dots, j_p)$ and $L = (l_1, \dots, l_q)$. We fix some notations. We denote by $J \sqcup L$ the reordered disjoint union of J and L (multiplicities counted if there are equal indices), so $dx_{J \sqcup L} = 0$ if $J \cap L \neq \varnothing$ and the entries of $J \sqcup L$ are in increasing order. For $1 \le k \le p$, set

$$I_k := (j_1, \dots, j_{k-1}, l_1, \dots, l_q, j_{k+1}, \dots, j_p),$$

although we do not have $j_1 < \dots < j_{k-1} < l_1 < \dots < l_q < j_{k+1} < \dots < j_p$ in general. So we have $dx_{I_k} = \operatorname{sgn} \pi \, q_\pi^{I_k} \, dx_{J_k \sqcup L}$, where $J_k = (j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_p)$. Similarly for $1 \le k \le q$, set

$$I'_k := (l_1, \dots, l_{k-1}, j_1, \dots, j_p, l_{k+1}, \dots, l_q).$$

Once we know the bracket of two elements of this form, others may be computed by extending bilinearly. The scalars arising in each term from the twisting principle are potentially different, so it is more convenient to express brackets in terms of these basis elements of Hochschild cohomology.

Theorem 5.1. *The graded Lie bracket of $\alpha = \underline{x}^{\underline{a}} \otimes dx_J$ and $\beta = \underline{x}^{\underline{b}} \otimes dx_L$ is*

$$\begin{aligned}[\alpha, \beta] = \sum_{1 \le k \le p} &(-1)^{(q-1)(k-1)} \rho_k^{\underline{b};J,L} (\partial_{[j_k]}(\underline{x}^{\underline{b}})) \cdot \underline{x}^{\underline{a}} \otimes dx_{J_k \sqcup L} \\ &- (-1)^{(p-1)(q-1)} \sum_{1 \le k \le q} (-1)^{(p-1)(k-1)} \rho_k^{\underline{a};L,J} (\partial_{[j_k]}(\underline{x}^{\underline{a}})) \cdot \underline{x}^{\underline{b}} \otimes dx_{J \sqcup L_k},\end{aligned}$$

for certain scalars $\rho_k^{\underline{b};J,L}$ and $\rho_k^{\underline{a};L,J}$, where $\partial_{[j_k]}(\underline{x}^{\underline{b}})$ is defined in (4.7) and $\partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}}$ is given by the A^e-module structure over A, that is, if $\partial_{[j_k]}(\underline{x}^{\underline{b}}) = \sum_i u_i \otimes v_i \in A \otimes A$, then $\partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} = \sum_i u_i \underline{x}^{\underline{a}} v_i$.

Proof. We denote by $\cdot$ the composition of two maps instead of $\circ$, in order to avoid confusion with the circle product. We compute the bracket using the formula

$$[\alpha, \beta] = [\alpha \cdot \Psi_p, \beta \cdot \Psi_q] \cdot \Phi_{p+q-1}.$$

The element $\alpha = \underline{x}^{\underline{a}} \otimes dx_J$ as a map from $A \otimes A \otimes \bigwedge^p(V)$ to A sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{IJ} \underline{x}^{\underline{a}}$ for $I = (i_1, \dots, i_p)$; similarly the element $\beta = \underline{x}^{\underline{b}} \otimes dx_L$ as a map from $A \otimes A \otimes \bigwedge^q(V)$ to A sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{IL} \underline{x}^{\underline{b}}$. By formula (4.8) for Ψ_p, the map $\alpha \cdot \Psi_p : A \otimes A \otimes \bar{A}^{\otimes p} \to A \otimes A \otimes \bigwedge^p(V) \to A$ is given by

$$\alpha \cdot \Psi_p(1 \otimes 1 \otimes \underline{x}^{\underline{m}^1} \otimes \dots \otimes \underline{x}^{\underline{m}^p}) = \mu_{(j_1,\dots,j_p)}^{(\underline{m}^1,\dots,\underline{m}^p)} \left(\prod_{s=1}^{p} (\partial_{[j_s]}(\underline{x}^{\underline{m}^s})) \right) \cdot \underline{x}^{\underline{a}},$$

where the scalar coefficient is defined by (4.9). We have a similar formula for $\beta \cdot \Psi_q$.

For $1 \le k \le p$, the map $(\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q) : A \otimes A \otimes \bar{A}^{\otimes p+q-1} \to A$ sends $1 \otimes 1 \otimes \underline{x}^{\underline{m}^1} \otimes \dots \otimes \underline{x}^{\underline{m}^{p+q-1}}$ to

$$\begin{aligned}&\mu_k \mu_J^{(\underline{m}^1,\dots,\underline{m}^{k-1},\tilde{\underline{m}}^k,\underline{m}^{k+q},\dots,\underline{m}^{p+q-1})} \mu_L^{(\underline{m}^k,\dots,\underline{m}^{k+q-1})} \\ &\cdot \big(\partial_{[j_1]}(\underline{x}^{\underline{m}^1}) \cdots \partial_{[j_{k-1}]}(\underline{x}^{\underline{m}^{k-1}}) \partial_{[j_k]}(\underline{x}^{\tilde{\underline{m}}^k}) \partial_{[j_{k+1}]}(\underline{x}^{\underline{m}^{k+q}}) \cdots \partial_{[j_p]}(\underline{x}^{\underline{m}^{p+q-1}}) \big) \cdot \underline{x}^{\underline{a}},\end{aligned}$$

where μ_k and $\underline{\widetilde{m}}^k$ are defined by $\mu_k \underline{x}^{\underline{\widetilde{m}}^k} = \left(\prod_{t=1}^q (\partial_{[l_t]} \underline{x}^{\underline{m}^{t+k-1}})\right) \cdot \underline{x}^{\underline{b}}$.

For $I = (i_1, \dots, i_{p+q-1})$ with $1 \le i_1 < \cdots < i_{p+q-1} \le N$, let us compute $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}(1 \otimes 1 \otimes \underline{x}^{\wedge I})$. Indeed, by (4.3) and our identifications,

$$\Phi_{p+q-1}(1 \otimes 1 \otimes \underline{x}^{\wedge I}) = \sum_{\pi \in \mathrm{Sym}_{p+q-1}} \operatorname{sgn} \pi \, q_\pi^I \otimes 1 \otimes x_{i_{\pi(1)}} \otimes \cdots \otimes x_{i_{\pi(p+q-1)}}.$$

Now for a fixed $\pi \in \mathrm{Sym}_{p+q-1}$, as input into the formula of the previous paragraph, we have

$$\underline{m}^1 = e_{i_{\pi(1)}}, \quad \dots, \quad \underline{m}^{p+q-1} = e_{i_{\pi(p+q-1)}},$$

where $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, with the 1 in the i-th position, and since $\partial_{[j]}(x_i) = \delta_{ij} \otimes 1$, the factor

$$\left(\partial_{[j_1]}(\underline{x}^{\underline{m}^1}) \cdots \partial_{[j_{k-1}]}(\underline{x}^{\underline{m}^{k-1}}) \partial_{[j_k]}(\underline{x}^{\underline{\widetilde{m}}^k}) \partial_{[j_{k+1}]}(\underline{x}^{\underline{m}^{k+q}}) \cdots \partial_{[j_p]}(\underline{x}^{\underline{m}^{p+q-1}})\right) \cdot \underline{x}^{\underline{a}}$$

vanishes unless

$$j_1 = i_{\pi(1)}, \quad \dots, \quad j_{k-1} = i_{\pi(k-1)},$$
$$l_1 = i_{\pi(k)}, \quad \dots, \quad l_q = i_{\pi(k+q-1)},$$
$$j_{k+1} = i_{\pi(k+q)}, \quad \dots, \quad j_p = i_{\pi(p+q-1)},$$

that is, when $I_k = \pi(I) := (i_{\pi(1)}, \dots, i_{\pi(p+q-1)})$ or equivalently $I = J_k \sqcup L$. As long as $J_k \cap L = \varnothing$, there exist a unique I and permutation $\pi_k \in \mathrm{Sym}_{p+q-1}$ satisfying this property. In this case,

$$\mu_k \underline{x}^{\underline{\widetilde{m}}^k} = \left(\prod_{t=1}^q \partial_{[l_t]}(\underline{x}^{\underline{m}^{t+q-1}})\right) \cdot \underline{x}^{\underline{b}} = \underline{x}^{\underline{b}},$$

so that $\mu_k = 1$ and $\underline{\widetilde{m}}^k = \underline{b}$. Consequently, the map $((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1}$ sends $1 \otimes 1 \otimes \underline{x}^{\wedge I}$ to $\delta_{I, J_k \sqcup L} \, \rho_k^{\underline{b};J,L} \, \partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}}$, where

$$\rho_k^{\underline{b};J,L} = \operatorname{sgn} \pi_k \, q_{\pi_k}^I \, \mu_J^{(e_{j_1}, \dots, e_{j_{k-1}}, \underline{b}, e_{j_{k+1}}, \dots, e_{j_p})} \, \mu_L^{(e_{\ell_1}, \dots, e_{\ell_q})}$$

is determined by the permutation π_k as described above and the scalars defined by (4.4) and (4.9). Therefore,

$$((\alpha \cdot \Psi_p) \circ_k (\beta \cdot \Psi_q)) \cdot \Phi_{p+q-1} = \rho_k^{\underline{b};J,K} \, \partial_{[j_k]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \otimes dx_{J_k \sqcup L}.$$

The formula in the statement can be obtained accordingly. □

6. Gerstenhaber brackets for group extensions of quantum symmetric algebras

Let G be a finite group for which $|G| \neq 0$ in $\Bbbk$, acting linearly on a finite dimensional vector space V, thus inducing an action on the symmetric algebra $S(V)$ by automorphisms. When the action preserves the relations on the quantum symmetric algebra $S_{\boldsymbol{q}}(V)$ as defined by (4.1), there is also an action on this algebra. This is always the case, for example, if G acts diagonally on the chosen basis $x_1, \ldots, x_N$ of V. We shall first recall the definition of a group extension, $S_{\boldsymbol{q}}(V) \rtimes G$, of $S_{\boldsymbol{q}}(V)$, and explain how the Koszul resolution of $S_{\boldsymbol{q}}(V) \rtimes G$ is related to that of $S_{\boldsymbol{q}}(V)$. In fact this works for an arbitrary Koszul algebra, as we shall explain next. Although this is well known, we include details for completeness.

Let $R \subseteq V \otimes V$ be a G-invariant subspace. Let $T_{\Bbbk}(V)$ denote the tensor algebra of V over $\Bbbk$. Suppose that $A = T_{\Bbbk}(V)/(R)$ is a *Koszul algebra* over $\Bbbk$, with the induced action of G. That is, the complex $K_{\bullet}(A)$ in which $K_0(A) = A \otimes A$, $K_1(A) = A \otimes V \otimes A$, and

$$K_i(A) = \bigcap_{j=0}^{i-2} (A \otimes V^{\otimes j} \otimes R \otimes V^{\otimes(i-2-j)} \otimes A)$$

for $i \geq 2$ is a free A-bimodule resolution of A under the differential from the bar resolution. In the case $A = S_{\boldsymbol{q}}(V)$, this can be shown to be equivalent to the Koszul resolution given in Section 4. The group extension $A \rtimes G$ of A, or *skew group algebra*, is the tensor product $A \otimes \Bbbk G$ as a vector space, with multiplication given by $(a \otimes g)(b \otimes h) = a({}^{g}b) \otimes gh$ for all $a, b \in A$ and $g, h \in G$ (where we have used a left superscript to denote the group action). We shall denote elements of $A \rtimes G$ by $a \sharp g$, in place of $a \otimes g$, for $a \in A$ and $g \in G$, to indicate that they are elements of this skew group algebra. In this section we adapt and generalize the techniques of [Halbout and Tang 2010; Shepler and Witherspoon 2012] from $S(V) \rtimes G$ to $S_{\boldsymbol{q}}(V) \rtimes G$, explaining how to compute the Gerstenhaber bracket via the Koszul resolution and our chain maps from Section 4. In the next section we focus on some special cases to give explicit results.

We know that $A \rtimes G$ is a Koszul ring over $\Bbbk G$ (see [Beilinson et al. 1996, Definition 1.1.2 and Section 2.6]). In fact let $V \otimes \Bbbk G$ be the $\Bbbk G$-bimodule under the actions $g \cdot (v \otimes h) = {}^{g}v \otimes gh$ and $(v \otimes h) \cdot g = v \otimes hg$ for all $v \in V$ and $g, h \in G$. Then there is an algebra isomorphism

$$T_{\Bbbk G}(V \otimes \Bbbk G) \simeq T_{\Bbbk}(V) \rtimes G$$

sending $(v_1 \otimes g_1) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (v_{m-1} \otimes g_{m-1}) \otimes_{\Bbbk G} (v_m \otimes g_m)$ to $(v_1 \otimes {}^{g_1}v_2 \otimes \cdots \otimes {}^{g_1 \cdots g_{m-1}}v_m) \sharp g_1 \cdots g_m$, and the inverse isomorphism sends $(v_1 \otimes \cdots \otimes v_m) \sharp g$ to $(v_1 \otimes e_G) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (v_{m-1} \otimes e_G) \otimes_{\Bbbk G} (v_m \otimes g)$, where we write e_G or e for the unit element of G. Via this isomorphism, $R \otimes \Bbbk G$ becomes a $\Bbbk G$-subbimodule of $(V \otimes \Bbbk G) \otimes_{\Bbbk G} (V \otimes \Bbbk G) \simeq V \otimes V \otimes \Bbbk G$, and it induces an isomorphism of algebras, $A \rtimes G \simeq T_{\Bbbk G}(V \otimes \Bbbk G)/(R \otimes \Bbbk G)$.

The Koszul resolution $K_\bullet(A \rtimes G)$ of $A \rtimes G$ as a Koszul ring over $\Bbbk G$ is related to the Koszul resolution of A as follows:

$$\begin{aligned} K_0(A \rtimes G) &= (A \rtimes G) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq A \otimes A \otimes \Bbbk G \\ &= K_0(A) \otimes \Bbbk G, \end{aligned}$$

$$\begin{aligned} K_1(A \rtimes G) &= (A \rtimes G) \otimes_{\Bbbk G} (V \otimes \Bbbk G) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq A \otimes V \otimes A \otimes \Bbbk G \\ &= K_1(A) \otimes \Bbbk G, \end{aligned}$$

and for $i \geq 2$,

$$\begin{aligned} K_i(A \rtimes G) &= (A \rtimes G) \otimes_{\Bbbk G} \bigcap_{j=0}^{i-2} \big((V \otimes \Bbbk G)^{\otimes_{\Bbbk G} j} \otimes_{\Bbbk G} (R \otimes \Bbbk G) \\ &\qquad\qquad \otimes_{\Bbbk G} (V \otimes \Bbbk G)^{\otimes_{\Bbbk G} (i-2-j)}\big) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq (A \rtimes G) \otimes_{\Bbbk G} \left(\bigcap_{j=0}^{i-2} (V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)}) \otimes \Bbbk G \right) \otimes_{\Bbbk G} (A \rtimes G) \\ &\simeq \left(A \otimes \bigcap_{j=0}^{i-2} (V^{\otimes j} \otimes R \otimes V^{\otimes (i-2-j)}) \otimes A \right) \otimes \Bbbk G \\ &\simeq K_i(A) \otimes \Bbbk G. \end{aligned}$$

Notice that the above isomorphism is induced by the map sending

$$(a_0 \sharp g_0) \otimes_{\Bbbk G} ((a_1 \otimes g_1) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (a_p \otimes g_p)) \otimes_{\Bbbk G} (a_{p+1} \sharp g_{p+1})$$

to

$$(a_0 \otimes ({}^{g_0}a_1 \otimes \cdots \otimes {}^{g_0 \cdots g_{p-1}}a_p) \otimes {}^{g_0 \cdots g_p}a_{p+1}) \otimes (g_0 \cdots g_{p+1}).$$

The inverse isomorphism sends $(a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \sharp g$ to

$$(a_0 \sharp e) \otimes_{\Bbbk G} ((a_1 \otimes e) \otimes_{\Bbbk G} \cdots \otimes_{\Bbbk G} (a_p \otimes e)) \otimes_{\Bbbk G} (a_{p+1} \sharp g).$$

One may check that this isomorphism commutes with the differentials. Therefore as complexes of $A \rtimes G$-bimodules,

$$K_\bullet(A \rtimes G) \simeq K_\bullet(A) \otimes \Bbbk G.$$

Under this isomorphism, the $A \rtimes G$-bimodule structure of $K_p(A) \otimes \Bbbk G$, for each $p \geq 0$, is given by

$$(b \sharp h)\big((a_0 \otimes (a_1 \otimes \cdots \otimes a_p) \otimes a_{p+1}) \otimes g\big)(c \sharp k) \\ = \big(b\, {}^h a_0 \otimes ({}^h a_1 \otimes \cdots \otimes {}^h a_p) \otimes {}^h a_{p+1}\, {}^{hg} c\big) \otimes hgk.$$

Similar statements apply to the normalized bar resolution:

$$B_\bullet(A \rtimes G) \simeq B_\bullet(A) \otimes \Bbbk G,$$

where the former involves tensor products over $\Bbbk G$, and the latter over $\Bbbk$.

Now we consider the case of $A := S_{\boldsymbol{q}}(V)$, under the condition that the action of G on V preserves the relations of $S_{\boldsymbol{q}}(V)$. The differentials on $K_\bullet(A \rtimes G)$ (respectively, $B_\bullet(A \rtimes G)$) are those induced by the Koszul resolution (respectively, bar resolution) of $S_{\boldsymbol{q}}(V)$, under the exact functor $- \otimes \Bbbk G$. Therefore the contracting homotopy and chain maps for $S_{\boldsymbol{q}}(V)$ may be extended to the corresponding complexes for $S_{\boldsymbol{q}}(V) \rtimes G$:

$$\Phi_\bullet \otimes \Bbbk G : K_\bullet(A \rtimes G) \simeq K_\bullet(A) \otimes \Bbbk G \to B_\bullet(A) \otimes \Bbbk G \simeq B_\bullet(A \rtimes G)$$

and

$$\Psi_\bullet \otimes \Bbbk G : B_\bullet(A \rtimes G) \simeq B_\bullet(A) \otimes \Bbbk G \to K_\bullet(A) \otimes \Bbbk G \simeq K_\bullet(A \rtimes G).$$

However, since $\Phi_\bullet$ and $\Psi_\bullet$ are in general not G-invariant, there is no reason to expect that $\Phi_\bullet \otimes \Bbbk G$ and $\Psi_\bullet \otimes \Bbbk G$ should be chain maps of complexes of $(A \rtimes G)^e$-modules. Since $|G|$ is invertible in $\Bbbk$, we can apply the Reynolds operator (that averages over images of group elements) to obtain chain maps of complexes of $(A \rtimes G)^e$-modules, which are denoted by $\widetilde{\Phi}_\bullet$ and $\widetilde{\Psi}_\bullet$ respectively. We have thus quasi-isomorphisms

$$\operatorname{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) \underset{\widetilde{\Phi}^\bullet}{\overset{\widetilde{\Psi}^\bullet}{\rightleftarrows}} \operatorname{Hom}_{(A \rtimes G)^e}(B_\bullet(A) \otimes \Bbbk G, A \rtimes G).$$

We shall use the complex on the left side to compute Lie brackets, via the chain maps $\widetilde{\Psi}^\bullet$ and $\widetilde{\Phi}^\bullet$. Notice that for $A = S_{\boldsymbol{q}}(V)$, we have

$$\begin{aligned} \operatorname{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) &\simeq \operatorname{Hom}_{\Bbbk G^e}\big(\textstyle\bigwedge^\bullet(V) \otimes \Bbbk G, A \rtimes G\big) \\ &\simeq \operatorname{Hom}_{\Bbbk G}\big(\textstyle\bigwedge^\bullet(V), A \rtimes G\big) \\ &\simeq \big(A \rtimes G \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G. \end{aligned}$$

We wish to express the Lie bracket at the chain level, on elements of $\big(A \rtimes G \otimes \bigwedge^\bullet(V^*)\big)^G$. The method consists of the following steps (see [Halbout and Tang 2010; Shepler and Witherspoon 2012]).

(i) Compute the cohomology groups of the complexes $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$. In the case where the action of G on V is diagonal, this computation is done in [Naidu et al. 2011, Section 4].

(ii) Give a precise formula for the chain map Θ that is the composition

$$\begin{aligned}\Theta : \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G &\xrightarrow{\simeq} \mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)\\ &\xrightarrow{\tilde{\Psi}^\bullet} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A) \otimes \Bbbk G, A \rtimes G)\\ &\xrightarrow{\simeq} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A \rtimes \Bbbk G), A \rtimes G).\end{aligned}$$

(iii) Give a precise formula for the chain map Γ that is the composition

$$\begin{aligned}\Gamma : \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A \rtimes \Bbbk G), A \rtimes G) &\xrightarrow{\simeq} \mathrm{Hom}_{(A\rtimes G)^e}(B_\bullet(A) \otimes \Bbbk G, A \rtimes G)\\ &\xrightarrow{\tilde{\Phi}^\bullet} \mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)\\ &\xrightarrow{\simeq} \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.\end{aligned}$$

(iv) Use the formulae in the previous two steps to compute the Lie bracket of two cocycles given by Step (i).

We obtain thus:

Theorem 6.1. *Let $\alpha, \beta \in \big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$ be two cocycles. Then the Lie bracket of the two corresponding cohomological classes is represented by the cocycle*

$$[\alpha, \beta] = \Gamma([\Theta(\alpha), \Theta(\beta)]).$$

We see that the actual computations are rather hard and we shall perform these computations for the diagonal action case in the next section.

7. Diagonal actions

Assume now that G acts diagonally on the basis $\{x_1, \dots, x_N\}$ of V, in which case the action extends to an action of G on $S_{\boldsymbol{q}}(V)$ by automorphisms. Let $\chi_i : G \to \Bbbk^\times$ be the character of G corresponding to its action on x_i, that is,

$$g \cdot x_i = \chi_i(g) x_i$$

for all $g \in G$ and $i = 1, \dots, N$. For $I = (i_1, \dots, i_p)$ with $1 \le i_1 < \cdots < i_p \le N$, define $\chi_I(g) = \prod_{j=1}^p \chi_{i_j}(g)$, and for $\underline{\ell} \in \mathbb{N}^N$, define $\chi_{\underline{\ell}}(g) = \prod_{1 \le i \le N} \chi_i^{\ell_i}(g)$ for $g \in G$.

Let us make precise the action of G on $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ occurring in the isomorphism of the previous section,

$$\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) \simeq \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.$$

Letting $g, h \in G$, $\underline{\ell} \in \mathbb{N}^N$, and $I = (i_1 < \cdots < i_p)$, we have

$$^h(\underline{x}^{\underline{\ell}} \sharp g \otimes dx_I) = {}^h(\underline{x}^{\underline{\ell}}) \sharp {}^h g \otimes {}^h(dx_I) = \chi_{\underline{\ell}}(h) \chi_I(h^{-1}) \underline{x}^{\underline{\ell}} \sharp hgh^{-1} \otimes dx_I.$$

In [Naidu et al. 2011, Section 4], the authors compute homology of this chain complex $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ with the differential

$$d_p(\underline{x}^{\ell} \sharp g \otimes dx_I) = \sum_{i \notin I} (-1)^{\#\{s : i_s < i\}} \left(\left(\prod_{s : i_s < i} q_{i_s, i} \right) x_i \underline{x}^{\ell} - \left(\prod_{s : i_s > i} q_{i, i_s} \right) \underline{x}^{\ell}\, {}^g x_i \right) \sharp g \otimes dx_{I + e_i},$$

where e_i is the i-th element of the canonical basis of $\mathbb{N}^N$, and $I + e_i$ is the sequence of $p+1$ integers obtained by inserting 1 in the i-th position. Since the action of G is diagonal, this differential is G-equivariant. So the Reynolds operator is a chain map from $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ to $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$ which realizes $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$ as a direct summand of $(A \rtimes G) \otimes \bigwedge^\bullet(V^*)$ as complexes. We shall see that in fact, the induced structure of $\big((A \rtimes G) \otimes \bigwedge^\bullet(V^*)\big)^G$, as a complex, is the same as the one induced from the isomorphism

$$\mathrm{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) \simeq \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G.$$

We shall prove this fact in the first step below.

We follow the step-by-step outline given towards the end of Section 6. As we shall use the result of the second step in the first one, we begin with the second step.

Step (ii). As shown in the previous section, we have a series of isomorphisms:

$$\begin{aligned} \mathrm{Hom}_{(A \rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G) &\simeq \mathrm{Hom}_{(\Bbbk G)^e}\big(\textstyle\bigwedge^\bullet(V) \otimes \Bbbk G, A \rtimes G\big) \\ &\simeq \mathrm{Hom}_{\Bbbk G}\big(\textstyle\bigwedge^\bullet(V), A \rtimes G\big) \\ &\simeq \big((A \rtimes G) \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G. \end{aligned}$$

A map $f \in \mathrm{Hom}_{(A \rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ corresponds to $f_1 \in \mathrm{Hom}_{\Bbbk G^e}\big(\bigwedge^p V \otimes \Bbbk G, A \rtimes G\big)$ via

$$f_1(\underline{x}^{\wedge I} \otimes g) = f(1 \otimes \underline{x}^{\wedge I} \otimes 1 \otimes g)$$

and

$$f(a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes g) = (a_0 \sharp e) f_1(\underline{x}^{\wedge I} \otimes g)({}^{g^{-1}} a_{p+1} \sharp e).$$

The map $f_1 \in \mathrm{Hom}_{\Bbbk G^e}\big(\bigwedge^p V \otimes \Bbbk G, A \rtimes G\big)$ corresponds to $f_2 \in \mathrm{Hom}_{\Bbbk G}\big(\bigwedge^p V, A \rtimes G\big)$ via

$$f_2(\underline{x}^{\wedge I}) = f_1(\underline{x}^{\wedge I} \otimes e)$$

and

$$f_1(\underline{x}^{\wedge I} \otimes g) = f_2(\underline{x}^{\wedge I})(1 \sharp g).$$

Finally, $f_2 \in \mathrm{Hom}_{\Bbbk G}\big(\bigwedge^p V, A \rtimes G\big)$ corresponds to $f_3 \in \big((A \rtimes G) \otimes \bigwedge^p(V^*)\big)^G$ via

$$f_3 = \sum_{|I| = p} f_2(\underline{x}^{\wedge I}) \otimes dx_I,$$

and for $f_3 = \sum_{|J|=p} \sum_{g\in G}(a_{J,g} \sharp g) \otimes dx_J \in \big(A \rtimes G \otimes \bigwedge^p(V^*)\big)^G$, the corresponding $f_2 \in \mathrm{Hom}_{\Bbbk G}\big(\bigwedge^p V, A \rtimes G\big)$ sends $\underline{x}^{\wedge I}$ to $\sum_{g\in G} a_{I,g} \sharp g$.

Altogether then, $f \in \mathrm{Hom}_{(A\rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ corresponds to $f_3 \in \big(A \rtimes G \otimes \bigwedge^p V^*\big)^G$ via

$$f_3 = \sum_{|I|=p} f(1 \otimes \underline{x}^{\wedge I} \otimes 1 \otimes e) \otimes dx_I,$$

and for $f_3 = \sum_{|J|=p} \sum_{g\in G} a_{J,g} \sharp g \otimes dx_J \in \big(A \rtimes G \otimes \bigwedge^p(V^*)\big)^G$,

$$\begin{aligned} f(a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes g) &= \sum_{h\in G} (a_0 \sharp e)(a_{I,h} \sharp h)(1 \sharp g)({}^{g^{-1}}a_{p+1} \sharp e) \\ &= \sum_{h\in G} a_0 a_{I,h} {}^{h}(a_{p+1}) \sharp hg. \end{aligned}$$

Now for $\alpha = a \sharp g \otimes dx_J \in A \rtimes G \otimes \bigwedge^p(V^*)$, the Reynolds operator

$$\mathcal{R} : A \rtimes G \otimes \textstyle\bigwedge^p(V^*) \to \big(A \rtimes G \otimes \bigwedge^p(V^*)\big)^G$$

gives

$$f_3 = \frac{1}{|G|} \sum_{h\in G} \chi_J(h^{-1}) {}^{h}a \sharp\ hgh^{-1} \otimes dx_J,$$

and thus α corresponds to the map $f \in \mathrm{Hom}_{(A\rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ sending $a_0 \otimes \underline{x}^{\wedge I} \otimes a_{p+1} \otimes k$ to

$$\delta_{IJ} \frac{1}{|G|} \sum_{h\in G} \chi_J(h^{-1}) a_0 ({}^{h}a)({}^{hgh^{-1}}a_{p+1}) \sharp hgh^{-1}k.$$

We shall compute $\Theta\mathcal{R}(\alpha) \in \mathrm{Hom}_{\Bbbk}((A \rtimes G)^{\otimes p}, A \rtimes G)$ corresponding to f with $a = \underline{x}^{\underline{\ell}}$, which is the composition

$$\begin{aligned} &\underline{x}^{\underline{\ell}^1} \sharp g_1 \otimes \cdots \otimes \underline{x}^{\underline{\ell}^p} \sharp g_p \\ &\quad \mapsto \underline{x}^{\underline{\ell}^1} \otimes {}^{g_1}(\underline{x}^{\underline{\ell}^2}) \otimes \cdots \otimes {}^{g_1\cdots g_{p-1}}(\underline{x}^{\underline{\ell}^p}) \sharp g_1 \cdots g_p \\ &\quad = \chi_{\underline{\ell}^2}(g_1) \cdots \chi_{\underline{\ell}^p}(g_1 \cdots g_{p-1}) \underline{x}^{\underline{\ell}^1} \otimes \cdots \otimes \underline{x}^{\underline{\ell}^p} \sharp g_1 \cdots g_p \\ &\quad \mapsto \chi_{\underline{\ell}^2}(g_1) \cdots \chi_{\underline{\ell}^p}(g_1 \cdots g_{p-1}) \sum_{|I|=p} \sum_{\substack{0 \le r_s \le \ell^s_{i_s}-1 \\ s=1,\dots,p}} \mu \underline{x}^{\underline{Q}} \otimes x^{\wedge I} \otimes \underline{x}^{\underline{\hat{Q}}} \otimes g_1 \cdots g_p \qquad \text{(use } \Psi_\bullet) \\ &\quad \mapsto \frac{1}{|G|} \chi_{\underline{\ell}^2}(g_1) \cdots \chi_{\underline{\ell}^p}(g_1 \cdots g_{p-1}) \sum_{h\in G} \sum_{\substack{0 \le r_s \le \ell^s_{j_s}-1 \\ s=1,\dots,p}} \lambda\mu \\ &\qquad\qquad \times \chi_J(h^{-1}) \chi_{\underline{\ell}}(h) \chi_{\underline{\hat{Q}}}(hgh^{-1}) \underline{x}^{\underline{\ell}^1+\cdots+\underline{\ell}^p+\underline{\ell}-J} \sharp hgh^{-1} g_1 \cdots g_p, \end{aligned}$$

where, as in (4.5),

$$\mu = \mu^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)},$$

$$\underline{Q} = \underline{Q}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)},$$

$$\underline{\widehat{Q}} = \underline{\widehat{Q}}^{(\underline{\ell}^1,\dots,\underline{\ell}^p;\, j_1,\dots,j_p)}_{(r_1,\dots,r_p)},$$

$$\lambda \underline{x}^{\underline{Q}} \underline{x}^{\underline{\ell}} \underline{x}^{\underline{\widehat{Q}}} = \underline{x}^{\underline{\ell}^1+\cdots+\underline{\ell}^p+\underline{\ell}-I} \in S_{\boldsymbol{q}}(V).$$

This completes the second step.

Step (i). We identify the cohomology groups of the complexes $\big(A \rtimes G \otimes \bigwedge^{\bullet}(V^*)\big)^G$ with the computation in [Naidu et al. 2011, Section 4]. It suffices to see that the map

$$A \rtimes G \otimes {\textstyle\bigwedge^{\bullet}}(V^*) \xrightarrow{\mathcal{R}} \big(A \rtimes G \otimes {\textstyle\bigwedge^{\bullet}}(V^*)\big)^G \xrightarrow{\sim} \operatorname{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)$$

is a chain map, where $A \rtimes G \otimes \bigwedge^{\bullet}(V^*)$ is endowed with the differential given in [Naidu et al. 2011, Section 4] and $\operatorname{Hom}_{(A\rtimes G)^e}(K_\bullet(A) \otimes \Bbbk G, A \rtimes G)$ with the differential induced from that of $K_\bullet(A)$. We shall use the computations in the second step to prove this statement.

In fact, given $a \,\sharp\, g \otimes dx_I \in A \rtimes G \otimes \bigwedge^p(V^*)$, by the second step, it corresponds to the map $f \in \operatorname{Hom}_{(A\rtimes G)^e}(K_p(A) \otimes \Bbbk G, A \rtimes G)$ sending $a_0 \otimes \underline{x}^{\wedge J} \otimes a_{p+1} \otimes k$ to

$$\delta_{IJ} \frac{1}{|G|} \sum_{h\in G} \chi_I(h^{-1}) a_0 ({}^h a)({}^{hgh^{-1}} a_{p+1}) \,\sharp\, hgh^{-1}k.$$

Now df is the composition (for $k \in G$ and $L = (l_1, \dots, l_{p+1})$)

$$\begin{aligned} 1 \otimes \underline{x}^{\wedge L} \otimes 1 \otimes k \mapsto \sum_{j=1}^{p+1} (-1)^{j-1} \Bigg(\Big(\prod_{s=1}^{j} q_{l_s,l_j} \Big) x_{l_j} \otimes \underline{x}^{\wedge (L-e_{l_j})} \otimes 1 \otimes k \\ - \Big(\prod_{s=j}^{p+1} q_{l_j,l_s} \Big) 1 \otimes \underline{x}^{\wedge (L-e_{l_j})} \otimes x_{l_j} \otimes k \Bigg) \\ \mapsto \frac{1}{|G|} \sum_{h\in G} \sum_{j=1}^{p+1} (-1)^{j-1} \delta_{I,L-e_{l_j}} \chi_I(h^{-1}) \Bigg(\Big(\prod_{s=1}^{j} q_{l_s,l_j} \Big) x_{l_j} {}^h a \\ - \Big(\prod_{s=j}^{p+1} q_{l_j,l_s} \Big) \chi_{l_j}(hgh^{-1}) {}^h a x_{l_j} \Bigg) \,\sharp\, hgh^{-1}k. \end{aligned}$$

On the other hand, by [Naidu et al. 2011, Section 4],

$$d_p(\underline{x}^{\underline{\ell}} \sharp g \otimes dx_I) = \sum_{i \notin I} (-1)^{\#\{s : i_s < i\}} \Biggl(\Bigl(\prod_{s : i_s < i} q_{i_s, i} \Bigr) x_i \underline{x}^{\underline{\ell}} - \Bigl(\prod_{s : i_s > i} q_{i, i_s} \Bigr) \underline{x}^{\underline{\ell}}\, {}^g x_i \Biggr) \sharp g \otimes dx_{I + e_i},$$

which corresponds to the map sending $1 \otimes \underline{x}^{\wedge L} \otimes 1 \otimes k$ to

$$\frac{1}{|G|} \sum_{h \in G} \sum_{i \notin I} (-1)^{\#\{s : i_s < i\}} \Biggl(\Bigl(\prod_{s : i_s < i} q_{i_s, i} \Bigr) \chi_L(h^{-1}) \delta_{L, I + e_i} \chi_i(h) x_i\, {}^h a - \Bigl(\prod_{s : i_s > i} q_{i, i_s} \Bigr) \chi_i(hg)\, {}^h a x_i \Biggr) \sharp h g h^{-1} k.$$

One sees readily that these two expressions are the same.

Let us recall the result of [Naidu et al. 2011, Section 4]. For $g \in G$, define

$$C_g = \Bigl\{ \underline{c} \in (\mathbb{N} \cup \{-1\})^N \;\Big|\; \text{for each } i \in \{1, \dots, N\},\ \prod_{s=1}^{N} q_{i,s}^{c_s} = \chi_i(g) \text{ or } c_i = -1 \Bigr\}.$$

For $g \in G$ and $\gamma \in (\mathbb{N} \cup \{-1\})^N$, Naidu et al. introduced certain subcomplexes $K^\bullet_{g,\gamma}$ of $(A \rtimes G) \otimes \bigwedge^p(V^*)$ with $(A \rtimes G) \otimes \bigwedge^p(V^*) = \bigoplus_{g,\gamma} K^\bullet_{g,\gamma}$. They also proved that if $\gamma \in C_g$, the subcomplex $K^\bullet_{g,\gamma}$ has zero differential, and if $\gamma \notin C_g$, the subcomplex $K^\bullet_{g,\gamma}$ is acyclic. (We do not define $K^\bullet_{g,\gamma}$ here as we shall not need the details.) Using this information, for $m \in \mathbb{N}$, [Naidu et al. 2011, Theorem 4.1] gives

$$\mathrm{H}^m\bigl((A \rtimes G) \otimes \textstyle\bigwedge^p(V^*)\bigr) \simeq \mathrm{HH}^m(A, A \rtimes G) \simeq \displaystyle\bigoplus_{g \in G} \bigoplus_{\substack{\underline{b} \in \{0,1\}^N \\ |\underline{b}| = m}} \bigoplus_{\substack{\underline{a} \in \mathbb{N}^N \\ \underline{a} - \underline{b} \in C_g}} \mathrm{Span}_{\Bbbk}\{\underline{x}^{\underline{a}} \sharp g \otimes dx_{\underline{b}}\}.$$

We shall use these notations when expressing the Lie bracket of two cohomological classes. This completes the first step.

Step (iii). Now given a map $f \in \mathrm{Hom}_{\Bbbk}((A \rtimes G)^{\otimes \bullet}, A \rtimes G)$, we compute the corresponding $\Gamma(f) \in \bigl((A \rtimes G) \otimes \bigwedge^p(V^*)\bigr)^G$. Direct inspection gives

$$\Gamma(f) = \sum_{|I| = p} \sum_{\pi \in \mathrm{Sym}_p} \operatorname{sgn} \pi\, q_\pi^I f(x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p)}} \sharp e) \otimes dx_I,$$

where $q_\pi^I = q_\pi^{i_1, \dots, i_p}$ is defined in (4.4), and e denotes the identity group element.

Step (iv). We can now compute the Lie bracket of two cohomological classes.

Let

$$\alpha = \underline{x}^{\underline{a}} \sharp g \otimes dx_J \quad \text{and} \quad \beta = \underline{x}^{\underline{b}} \sharp h \otimes dx_L$$

for some group elements $g, h \in G$, where $J = (j_1, \ldots, j_p)$ and $L = (l_1, \ldots, l_q)$ and such that $\underline{a} - J \in C_g$ and $\underline{b} - K \in C_h$. Then α and β are cocycles for the complex $A \rtimes G \otimes \bigwedge^\bullet(V^*)$, because the subcomplex $K^\bullet_{g,\gamma}$ of $\mathrm{Hom}_{A^e}(K_\bullet(A), A \rtimes G)$ is a complex with zero differential whenever $\gamma \in C_g$ (for details, see [Naidu et al. 2011, Section 4]). Consequently, $\mathcal{R}\alpha$ and $\mathcal{R}\beta$ are G-invariant cocycles where, as before, $\mathcal{R}$ is the Reynolds operator. The bracket operation on Hochschild cohomology is determined by its values on cocycles of this form.

Theorem 7.1. *In the case where G acts diagonally on the basis $x_1, \ldots, x_N$, the graded Lie bracket of $\mathcal{R}\alpha$ and $\mathcal{R}\beta$, where $\alpha = \underline{x}^{\underline{a}} \sharp g \otimes dx_J$ and $\beta = \underline{x}^{\underline{b}} \sharp h \otimes dx_L$, is*

$$
\begin{aligned}
&[\mathcal{R}\alpha, \mathcal{R}\beta] \\
&= \sum_{1 \le s \le p} (-1)^{(q-1)(s-1)} \frac{1}{|G|^2} \sum_{k,\ell \in G} \rho_s^{\alpha,\beta} \partial_{[j_s]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \sharp kgk^{-1}\ell h\ell^{-1} \otimes dx_{J_s \sqcup L} \\
&\qquad - (-1)^{(p-1)(q-1)} \sum_{1 \le s \le q} (-1)^{(p-1)(s-1)} \frac{1}{|G|^2} \\
&\qquad\qquad \times \sum_{k,\ell \in G} \rho_s^{\beta,\alpha} \partial_{[l_s]}(\underline{x}^{\underline{a}}) \cdot \underline{x}^{\underline{b}} \sharp \ell h\ell^{-1} kgk^{-1} \otimes dx_{J \sqcup L_s}
\end{aligned}
$$

for certain coefficients $\rho_s^{\alpha,\beta}$ and $\rho_s^{\beta,\alpha}$.

Remark 7.2. This formula generalizes Theorem 5.1 (the case $G = 1$) and [Shepler and Witherspoon 2012, Corollary 7.3] (the case $q_{i,j} = 1$ for all i, j).

Proof. We may compute $[\mathcal{R}(\alpha), \mathcal{R}(\beta)]$ as $\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)])$.

Now by the third step,

$$
\begin{aligned}
\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)]) = \sum_{|I|=p+q-1} \sum_{\pi \in \mathrm{Sym}_{p+q-1}} &\operatorname{sgn}\pi \, q_\pi^I [\Theta(\mathcal{R}\alpha), \Theta(\mathcal{R}\beta)] \\
&\times (x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p+q-1)}} \sharp e) \otimes dx_I.
\end{aligned}
$$

Note that Ψ_p, when applied to an element of the form $1 \otimes x_{c_1} \otimes \cdots \otimes x_{c_p} \otimes 1$, is $1 \otimes x_{c_1} \wedge \cdots \wedge x_{c_p} \otimes 1$ if $1 \le c_1 < \cdots < c_p \le N$, and is 0 otherwise. This simplifies considerably the computation of $[\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)](x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p+q-1)}} \sharp e)$. For $1 \le s \le p$, we have

$$
\begin{aligned}
&(\Theta\mathcal{R}(\alpha) \circ_s \Theta\mathcal{R}(\beta))(x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{j_{\pi(p)}} \sharp e) = \\
&\Theta\mathcal{R}(\alpha)\big(x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes \Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(s+q-1)}} \sharp e) \otimes \cdots \otimes x_{i_{\pi(p+q-1)}} \sharp e\big).
\end{aligned}
$$

By Step (ii), a simple computation shows that $\Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(s+q-1)}} \sharp e)$ is nonzero only when

$$
i_{\pi(s)} = l_1, \quad \ldots, \quad i_{\pi(s+q-1)} = l_q,
$$

in which case it is equal to $\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\underline{x}^{\underline{b}}\sharp\,\ell h\ell^{-1}$. Therefore, when

$$i_{\pi(s)}=l_1,\quad \dots,\quad i_{\pi(s+q-1)}=l_q,$$

we have

$$\begin{aligned}
&\Theta\mathcal{R}(\alpha)\big(x_{i_{\pi(1)}}\sharp\, e\otimes\cdots\otimes\Theta\mathcal{R}(\beta)(x_{i_{\pi(s)}}\sharp\, e\otimes\cdots\otimes x_{i_{\pi(s+q-1)}}\sharp\, e)\\
&\qquad\qquad\otimes x_{i_{\pi(s+q)}}\sharp\, e\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp\, e\big)\\
&=\Theta\mathcal{R}(\alpha)\Bigg(x_{i_{\pi(1)}}\sharp e\otimes\cdots\otimes\bigg(\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\underline{x}^{\underline{b}}\sharp\ell h\ell^{-1}\otimes x_{i_{\pi(s+q)}}\sharp e\bigg)\\
&\qquad\qquad\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp\, e\Bigg)\\
&=\frac{1}{|G|}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\Theta\mathcal{R}(\alpha)\big(x_{i_{\pi(1)}}\sharp\, e\otimes\cdots\otimes\underline{x}^{\underline{b}}\sharp\,\ell h\ell^{-1}\otimes x_{i_{\pi(s+q)}}\sharp\, e\\
&\qquad\qquad\otimes\cdots\otimes x_{i_{\pi(p+q-1)}}\sharp\, e\big).
\end{aligned}$$

Applying Step (ii), in order that the above expression be nonzero, we must have

$$j_1=i_{\pi(1)},\quad\dots,\quad j_{s-1}=i_{\pi(s-1)},\quad j_{s+1}=i_{\pi(s+q)},\quad\dots,\quad j_p=i_{\pi(p+q-1)}.$$

When

$$\begin{aligned}
&i_{\pi(s)}=l_1,\quad\dots,\quad i_{\pi(s+q-1)}=l_q,\\
&\qquad j_1=i_{\pi(1)},\quad\dots,\quad j_{s-1}=i_{\pi(s-1)},\\
&\qquad\qquad j_{s+1}=i_{\pi(s+q)},\quad\dots,\quad j_p=i_{\pi(p+q-1)},
\end{aligned}$$

we have

$$\begin{aligned}
&(\Theta\mathcal{R}(\alpha)\circ_s\Theta\mathcal{R}(\beta))(x_{i_{\pi(1)}}\sharp\, e\otimes\cdots\otimes x_{j_{\pi(p)}}\sharp\, e)\\
&\quad=\frac{1}{|G|^2}\sum_{k\in G}\sum_{\ell\in G}\chi_L(\ell^{-1})\chi_{\underline{b}}(\ell)\chi_{j_{s+1}}(\ell h\ell^{-1})\cdots\chi_{j_p}(\ell h\ell^{-1})\\
&\qquad\times\sum_{0\le r\le b_{j_s}-1}\lambda\mu\chi_J(k^{-1})\chi_{\underline{a}}(k)\chi_{\underline{\widehat{Q}}}(kgk^{-1})\underline{x}^{\underline{a}+\underline{b}-e_{j_s}}\sharp\, kgk^{-1}\ell h\ell^{-1},
\end{aligned}$$

where

$$\begin{aligned}
\underline{x}^{\underline{Q}}&=x_{j_s}^r x_{j_s+1}^{b_{j_s+1}}\cdots x_N^{b_N},\\
\underline{x}^{\underline{\widehat{Q}}}&=x_1^{b_1}\cdots x_{j_s-1}^{b_{j_s-1}}x_{j_s}^{b_{j_s}-r+1},\\
\mu\underline{x}^{\underline{Q}}\underline{x}^{\underline{\widehat{Q}}}&=x_{j_1}\cdots x_{j_{s-1}}\underline{x}^{\underline{b}}x_{j_{s+1}}\cdots x_{j_p}\in S_{\boldsymbol{q}}(V),\\
\lambda\underline{x}^{\underline{Q}}\underline{x}^{\underline{a}}\underline{x}^{\underline{\widehat{Q}}}&=\underline{x}^{\underline{a}+\underline{b}-e_{j_s}}\in S_{\boldsymbol{q}}(V).
\end{aligned}$$

We see that in this case we have $I = J_s \sqcup L$. Furthermore, if this is the case, there is a unique permutation $\pi_s \in \mathrm{Sym}_{p+q-1}$ such that

$$j_1 = i_{\pi_s(1)}, \quad \dots, \quad j_{s-1} = i_{\pi_s(s-1)},$$
$$l_1 = i_{\pi_s(s)}, \quad \dots, \quad l_q = i_{\pi_s(s+q-1)},$$
$$j_{s+1} = i_{\pi_s(s+q)}, \quad \dots, \quad j_p = i_{\pi_s(p+q-1)},$$

that is, $\pi_s(I) = J_s \sqcup L$ as introduced before Theorem 5.1. We obtain that when $I = J_s \sqcup L$ and $\pi = \pi_s$ for $1 \le s \le p$,

$$(\Theta\mathcal{R}(\alpha) \circ_s \Theta\mathcal{R}(\beta))(x_{i_{\pi_s(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi_s(p+q-1)}} \sharp e) = \frac{1}{|G|^2} \sum_{k,\ell \in G} \rho_s^{\alpha,\beta} \partial_{[j_s]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \sharp kgk^{-1}\ell h\ell^{-1}$$

for a certain coefficient $\rho_s^{\alpha,\beta}$ determined by the above data.

Finally

$$\begin{aligned}
&\Gamma([\Theta\mathcal{R}(\alpha), \Theta\mathcal{R}(\beta)]) \\
&= \sum_{|I|=p+q-1} \sum_{\pi \in \mathrm{Sym}_{p+q-1}} \operatorname{sgn}\pi\, q_\pi^I \, [\Theta(\mathcal{R}\alpha),\Theta(\mathcal{R}\beta)](x_{i_{\pi(1)}} \sharp e \otimes \cdots \otimes x_{i_{\pi(p+q-1)}} \sharp e) \otimes dx_I \\
&= \frac{1}{|G|^2} \sum_{k,\ell \in G} \sum_{1 \le s \le p} (-1)^{(q-1)(s-1)} \rho_s^{\alpha,\beta} \partial_{[j_s]}(\underline{x}^{\underline{b}}) \cdot \underline{x}^{\underline{a}} \sharp kgk^{-1}\ell h\ell^{-1} \otimes dx_I \\
&\quad - (-1)^{(p-1)(q-1)} \frac{1}{|G|^2} \sum_{k,\ell \in G} \sum_{1 \le s \le q} (-1)^{(p-1)(s-1)} \\
&\quad \times \rho_s^{\beta,\alpha} \partial_{[\ell_s]}(\underline{x}^{\underline{a}}) \cdot \underline{x}^{\underline{b}} \sharp \ell h\ell^{-1} kgk^{-1} \otimes dx_I. \quad \square
\end{aligned}$$

In this diagonal case, the following corollary is immediate, since the difference operators in the bracket formula take 1 to 0. It generalizes [Shepler and Witherspoon 2012, Theorem 8.1].

Corollary 7.3. *Assume G acts diagonally on the chosen basis $x_1, \dots, x_N$ of V, and let $\alpha = 1 \sharp g \otimes dx_J$ and $\beta = 1 \sharp h \otimes dx_L$. Then $[\mathcal{R}\alpha, \mathcal{R}\beta] = 0 \in \mathrm{HH}^\bullet(A \rtimes G)$.*

In fact, this result can be seen to hold in the nondiagonal case as well, even without an explicit description of Hochschild cocycles in that case. Nonetheless we may still use a general argument for those cocycles having a particular form.

Corollary 7.4. *Assume G acts on V, not necessarily diagonally. Let α and β be cocycles in $\big(A \rtimes G \otimes \textstyle\bigwedge^\bullet(V^*)\big)^G$ for which α (respectively, β) is a linear combination of elements of the form $1 \sharp g \otimes dx_J$ (respectively, $1 \sharp h \otimes dx_L$). Then $[\alpha, \beta] = 0 \in \mathrm{HH}^\bullet(A \rtimes G)$. In particular, if α is a 2-cocycle, then it is a noncommutative Poisson structure.*

Proof. The proof is similar to that of Theorem 7.1. However, rather than computing explicitly, we shall only explain why the bracket is 0.

We compute $[\alpha, \beta]$ using Theorem 6.1. Consider α as a homomorphism in $\mathrm{Hom}_{(A\rtimes G)^e}(K_\bullet(A)\otimes \Bbbk G, A\rtimes G)$; then it maps into $\Bbbk\otimes\Bbbk G\subset A\rtimes G$. By Theorem 6.1

$$[\alpha, \beta] = [\alpha\cdot\widetilde{\Psi}_\bullet, \beta\cdot\widetilde{\Psi}_\bullet]\cdot\widetilde{\Phi}_\bullet.$$

Here $\widetilde{\Phi}_\bullet$ and $\widetilde{\Psi}_\bullet$ are chain maps of complexes of $(A\rtimes G)^e$-modules obtained by applying the Reynolds operator (that averages over images of group elements) to $\Phi_\bullet$ and $\Psi_\bullet$ respectively. So one needs to consider certain terms like $(\alpha\cdot {}^a\Psi)\circ_k(\beta\cdot {}^b\Psi)$ applied to ${}^c\Phi(1\otimes 1\otimes \underline{x}^{\wedge I})$ for $k\geq 1$, and $a, b, c\in G$.

Recall that, if $I=(i_1,\dots,i_p)$, then

$$\Phi(1\otimes 1\otimes \underline{x}^{\wedge I}) = \sum_{\pi\in\mathrm{Sym}_p} \operatorname{sgn}\pi\; q_\pi^{i_1,\dots,i_p}\otimes x_{i_{\pi(1)}}\otimes\cdots\otimes x_{i_{\pi(p)}}\otimes 1.$$

So ${}^c\Phi(1\otimes 1\otimes\underline{x}^{\wedge I})$ is a linear combination of terms of the form $1\otimes x_{j_1}\otimes\cdots\otimes x_{j_p}\otimes 1$ for $1\leq j_1,\dots,j_p\leq N$. In applying $(\alpha\cdot {}^a\Psi)\circ_k(\beta\cdot {}^b\Psi)$ to each term above, one first applies ${}^b\Psi$ to $1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1$, if the degree of β is m. By (4.5),

$$\Psi_m(1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1) = \mu\otimes x_{j_k}\wedge\cdots\wedge x_{j_{k+m-1}}\otimes 1$$

for some scalar μ and so ${}^b\Psi_m(1\otimes x_{j_k}\otimes\cdots\otimes x_{j_{k+m-1}}\otimes 1)$ is a linear combination of terms of the form $1\otimes x_{\ell_1}\wedge\cdots\wedge x_{\ell_m}\otimes 1$ with $1\leq \ell_1<\cdots<\ell_m\leq N$.

Applying β to the result, we obtain 0 unless $L=(\ell_1,\dots,\ell_m)$ for some L for which $1\,\sharp\, h\otimes dx_L$ has a nonzero coefficient in the expression β, in which case we obtain a nonzero scalar multiple of $1\,\sharp\, h$ for that term. After factoring h to the right, this becomes 0 as an element of the normalized bar resolution. The same argument applies to each term in $[\alpha, \beta]$, and so $[\alpha, \beta]=0$.

For the last statement, recall that a noncommutative Poisson structure is simply a Hochschild 2-cocycle whose square bracket is a coboundary. □

Compare to the proof of [Naidu and Witherspoon 2016, Theorem 4.6], of which the above corollary is a consequence via the alternative route of algebraic deformation theory.

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