Pacific Journal of Mathematics

Volume 283 No. 2 August 2016

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2016 is US $440/year for the electronic version, and $600/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.257

THE FUNDAMENTAL THEOREM OF TROPICAL DIFFERENTIAL ALGEBRAIC GEOMETRY

FUENSANTA AROCA, CRISTHIAN GARAY AND ZEINAB TOGHANI

Let I be an ideal of the ring of Laurent polynomials $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ with coefficients in a real-valued field (K, v). The fundamental theorem of tropical algebraic geometry states the equality $\operatorname{trop}(V(I)) = V(\operatorname{trop}(I))$ between the tropicalization $\operatorname{trop}(V(I))$ of the closed subscheme $V(I) \subset (K^*)^n$ and the tropical variety $V(\operatorname{trop}(I))$ associated to the tropicalization of the ideal $\operatorname{trop}(I)$.

In this work we prove an analogous result for a differential ideal G of the ring of differential polynomials $K[[t]]\{x_1, \dots, x_n\}$, where K is an uncountable algebraically closed field of characteristic zero. We define the tropicalization $\operatorname{trop}(\operatorname{Sol}(G))$ of the set of solutions $\operatorname{Sol}(G) \subset K[[t]]^n$ of G, and the set of solutions $\operatorname{Sol}(\operatorname{trop}(G)) \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ associated to the tropicalization of the ideal $\operatorname{trop}(G)$. These two sets are linked by a tropicalization morphism $\operatorname{trop} : \operatorname{Sol}(G) \to \operatorname{Sol}(\operatorname{trop}(G))$.

We show the equality $\operatorname{trop}(\operatorname{Sol}(G)) = \operatorname{Sol}(\operatorname{trop}(G))$, answering a question recently raised by D. Grigoriev.

1. Introduction

The first proof of the fundamental theorem of tropical algebraic geometry appeared in 2003 in a preprint by Einsiedler, Kapranov and Lind [Einsiedler et al. 2006], and was limited to hypersurfaces. Later, the theorem was established in full generality in [Speyer and Sturmfels 2004]. Extensions to arbitrary codimension ideals and arbitrary valuations have been done subsequently; see, for example, [Aroca et al. 2010; Jensen et al. 2008; Aroca 2010].

The tropical variety of a hypersurface is dual to a subdivision of the Newton polyhedron of its defining function. The Newton polygon was introduced by Puiseux [1850] for plane algebraic curves and extended to differential polynomials by Fine [1889]. Both the extensions of the polygon and the polyhedron have served

This research was supported by PAPIIT IN104713 and IN108216, ECOS NORD M14M03 and CONACYT. Part of this work was done while Aroca and Toghani were at the CNRS unité UMR7373 12M in Marseille, Aroca with a CNRS contract and Toghani with the support of LAISLA.
MSC2010: 13N99, 13P15, 14T99.
Keywords: differential algebra, tropical geometry, arc spaces, power series solutions of ODEs.

to prove existence theorems and to construct algorithms that compute solutions; see for example [Grigoriev and Singer 1991; Cano 1993; Aroca and Cano 2001; Aroca et al. 2003].

Grigoriev [2015] introduces the notion of *tropical linear differential equations* in n variables and designs a polynomial complexity algorithm for solving systems of tropical linear differential equations in one variable. In the same preprint, Grigoriev suggests several lines for further research. One of his questions is whether a theorem such as the fundamental theorem of tropical algebraic geometry holds in this context.

More precisely, Grigoriev notes that, for a differential ideal G in n independent variables, we have the inclusion $\mathrm{trop}(\mathrm{Sol}(G)) \subset \mathrm{Sol}(\mathrm{trop}(G))$ and asks:

> Is it true that for any differential ideal G and a family $S_1, \ldots, S_n \subset \mathbb{Z}_{\geq 0}$ being a solution of the tropical differential equation $\mathrm{trop}(g)$ for any $g \in G$, there exists a power series solution of G whose tropicalization equals $S_1, \ldots, S_n$?

Here, we give a positive answer to this question when G is a differential ideal of differential polynomials over the ring of formal power series $K[\![t]\!]$, K being an uncountable algebraically closed field of characteristic zero. Our proof uses techniques developed in the theory of arc spaces; see [Nash 1995].

In Section 2, the basic definitions of differential algebraic geometry are recalled. In Sections 3, 4 and 5, we explain the tropicalization morphisms. Arc spaces and their connection with sets of solutions of differential ideals are discussed in Section 6. The main result is proved in the last two sections.

2. Differential algebraic geometry

We will begin by recalling some basic definitions of differential algebraic geometry. The reference for this section is the book by J. F. Ritt [1950].

Let R be a commutative ring with unity. A *derivation* on R is a map $d : R \to R$ that satisfies $d(a+b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b)$ for all $a, b \in R$. The pair (R, d) is called a *differential ring*. An ideal $I \subset R$ is said to be a *differential ideal* when $d(I) \subset I$.

Let (R, d) be a differential ring and let $R\{x_1, \ldots, x_n\}$ be the set of polynomials with coefficients in R in the variables $\{x_{ij} : i = 1, \ldots, n,\ j \geq 0\}$. The derivation d on R can be extended to a derivation d of $R\{x_1, \ldots, x_n\}$ by setting $d(x_{ij}) = x_{i(j+1)}$ for $i = 1, \ldots, n$ and $j \geq 0$. The pair $(R\{x_1, \ldots, x_n\}, d)$ is a differential ring called *the ring of differential polynomials in n variables with coefficients in R*.

A differential polynomial $P \in R\{x_1, \ldots, x_n\}$ induces a mapping from R^n to R given by

$$P : R^n \to R, \quad (\varphi_1, \ldots, \varphi_n) \mapsto P|_{x_{ij} = d^j \varphi_i}, \tag{2-1}$$

where $P|_{x_{ij}=d^j\varphi_i}$ is the element of R obtained by substituting $x_{ij} \mapsto d^j\varphi_i$ in the differential polynomial P.

The equality

$$d^k(P(\varphi)) = (d^kP)(\varphi) \tag{2-2}$$

holds for any $P \in R\{x_1, \ldots, x_n\}$ and $\varphi \in R^n$.

A *zero* or a *solution* of $P \in R\{x_1, \ldots, x_n\}$ is an n-tuple $\varphi \in R^n$ such that $P(\varphi) = 0$. An n-tuple $\varphi \in R^n$ is a *solution* of $\Sigma \subset R\{x_1, \ldots, x_n\}$ when it is a solution of every differential polynomial in Σ; that is,

$$\mathrm{Sol}(\Sigma) := \{\varphi \in R^n \,:\, P(\varphi) = 0 \text{ for all } P \in \Sigma\}.$$

The following result can be found in [Ritt 1950, p. 21].

Proposition 2.1. *The solution of any infinite system of differential polynomials*

$$\Sigma \subset F\{x_1, \ldots, x_n\},$$

where F is a differential field of characteristic zero, is the solution of some finite subset of the system.

A *differential monomial* in n independent variables of order less than or equal to r is an expression of the form

$$E_M := \prod_{\substack{1 \le i \le n \\ 0 \le j \le r}} x_{ij}^{M_{ij}}, \tag{2-3}$$

where $M = (M_{ij})_{1 \le i \le n,\, 0 \le j \le r}$ is a matrix in $\mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\ge 0})$.

With this notation, a differential polynomial $P \in R\{x_1, \ldots, x_n\}$ is an expression of the form

$$P = \sum_{M \in \Lambda \subset \mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\ge 0})} \psi_M E_M, \tag{2-4}$$

with $r \in \mathbb{Z}_{\ge 0}$, $\psi_M \in R$ and Λ finite.

The mapping induced by the monomial E_M is given by

$$E_M : R^n \to R, \quad (\varphi_1, \ldots, \varphi_n) \mapsto \prod_{\substack{1 \le i \le n \\ 0 \le j \le r}} (d^j\varphi_i)^{M_{ij}},$$

and the map (2-1) induced by the differential polynomial P in (2-4) is

$$P : R^n \to R, \quad \varphi = (\varphi_1, \ldots, \varphi_n) \mapsto \sum_{M \in \Lambda} \psi_M E_M(\varphi). \tag{2-5}$$

3. The differential ring of formal power series and tropicalization

In what follows, we work with the differential valued ring $R = K[\![t]\!]$ where K is an uncountable algebraically closed field of characteristic zero. We set $F = \mathrm{Frac}(R)$.

The elements of R are expressions of the form

$$\varphi = \sum_{j\in\mathbb{Z}_{\geq 0}} a_j t^j \tag{3-1}$$

with $a_j \in K$ for $j \in \mathbb{Z}_{\geq 0}$.

The *support* of φ is the set

$$\mathrm{Supp}(\varphi) := \{i \in \mathbb{Z}_{\geq 0} : a_i \neq 0\},$$

the valuation on R is given by

$$\mathrm{val}(\varphi) = \min \mathrm{Supp}(\varphi)$$

and the derivative of φ is the element

$$d\varphi = \sum_{j\in\mathbb{Z}_{\geq 0}} j a_j t^{j-1}$$

of R. The bijection

$$\Psi : K^{\mathbb{Z}_{\geq 0}} \to R, \quad \underline{a} = (a_j)_{j\geq 0} \mapsto \sum_{j\geq 0} \frac{1}{j!} a_j t^j$$

between $K^{\mathbb{Z}_{\geq 0}}$ and R allows us to identify points of R with points of $K^{\mathbb{Z}_{\geq 0}}$. Moreover, the mapping Ψ has the following property:

$$d^s\Psi(\underline{a}) = \sum_{j\geq 0} \frac{a_{s+j}}{j!} t^j, \tag{3-2}$$

which implies

$$d^s\Psi(\underline{a})|_{t=0} = a_s, \quad s \in \mathbb{Z}_{\geq 0}$$

and then

$$\underline{a} = \big(d^j\Psi(\underline{a})|_{t=0}\big)_{j\geq 0}. \tag{3-3}$$

The mapping that sends each series in R to its support set (a subset of $\mathbb{Z}_{\geq 0}$) will be called the *tropicalization* map

$$\mathrm{trop} : R \to \mathcal{P}(\mathbb{Z}_{\geq 0}), \quad \varphi \mapsto \mathrm{Supp}(\varphi)$$

where $\mathcal{P}(\mathbb{Z}_{\geq 0})$ denotes the power set of $\mathbb{Z}_{\geq 0}$.

For fixed n, the mapping from R^n to the n-fold product of $\mathcal{P}(\mathbb{Z}_{\geq 0})$ will also be denoted by trop:

$$\operatorname{trop}: R^n \to \mathcal{P}(\mathbb{Z}_{\geq 0})^n, \quad \varphi = (\varphi_1, \ldots, \varphi_n) \mapsto \operatorname{trop}(\varphi) = (\operatorname{Supp}(\varphi_1), \ldots, \operatorname{Supp}(\varphi_n)).$$

Given a subset T of R^n, the *tropicalization* T is its image under the map trop:

$$\operatorname{trop}(T) := \{\operatorname{trop}(\varphi) : \varphi \in T\} \subset \mathcal{P}(\mathbb{Z}_{\geq 0})^n.$$

Example 3.1. Set $T := \{(a + 5t + bt^2, 2 + at - 8t^2 + ct^3) : a, b, c \in K\} \subset K[\![t]\!]^2$. We have

$$\begin{aligned}\operatorname{trop}(T) = \big\{&(\{1\}, \{0, 2\}), (\{0, 1\}, \{0, 1, 2\}),\\ &(\{1, 2\}, \{0, 2\}), (\{1\}, \{0, 2, 3\}), (\{0, 1, 2\}, \{0, 1, 2\}),\\ &(\{0, 1\}, \{0, 1, 2, 3\}), (\{1, 2\}, \{0, 2, 3\}), (\{0, 1, 2\}, \{0, 1, 2, 3\})\big\}.\end{aligned}$$

Since K is of characteristic zero, for every $\varphi \in R$, we have

$$\operatorname{trop}(d^j\varphi) = \{i - j : i \in \operatorname{trop}(\varphi) \cap \mathbb{Z}_{\geq j}\}$$

then

$$\operatorname{val}(d^j\varphi) = \min(\operatorname{trop}(\varphi) \cap \mathbb{Z}_{\geq j}) - j.$$

The above equality justifies the following definition:

Definition. A subset $S \subseteq \mathbb{Z}_{\geq 0}$ induces a mapping $\operatorname{Val}_S : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\operatorname{Val}_S(j) := \begin{cases} s - j & \text{with } s = \min\{\alpha \in S : \alpha \geq j\}, \\ \infty & \text{when } S \cap \mathbb{Z}_{\geq j} = \varnothing. \end{cases} \tag{3-4}$$

Example 3.2. Consider the set $S := \{1, 3, 4\}$. We have

(1) $\operatorname{Val}_S(2) = \min\{s \in S : s \geq 2\} - 2 = 3 - 2 = 1$ and

(2) $\operatorname{Val}_S(5) = \infty$.

4. Tropical differential polynomials

We denote by $\mathbb{T}$ the (idempotent) semiring $\mathbb{T} = (\mathbb{Z}_{\geq 0} \cup \{\infty\}, \oplus, \odot)$, with $a \oplus b = \min\{a, b\}$ and $a \odot b = a + b$.

Definition. A *tropical differential monomial* in the variables $x_1, \ldots, x_n$ of order less than or equal to r is an expression of the form

$$\varepsilon_M := x^{\odot M} = \bigodot_{\substack{1 \leq i \leq n \\ 0 \leq j \leq r}} x_{ij}^{\odot M_{ij}}, \tag{4-1}$$

where $M = (M_{ij})_{1 \leq i \leq n,\, 0 \leq j \leq r}$ is a matrix in $\mathcal{M}_{n \times (r+1)}(\mathbb{Z}_{\geq 0})$.

Definition. A *tropical differential polynomial* in the variables $x_1, \dots, x_n$ of order less than or equal to r is an expression of the form

$$\phi = \phi(x_1, \dots, x_n) = \bigoplus_{M \in \Lambda \subset \mathcal{M}_{n\times(r+1)}(\mathbb{Z}_{\geq 0})} a_M \odot \varepsilon_M, \tag{4-2}$$

where $a_M \in \mathbb{T}$ and Λ is a finite set.

The set of tropical differential polynomials will be denoted by $\mathbb{T}\{x_1, \dots, x_n\}$. A tropical differential monomial ε_M induces a mapping from $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\varepsilon_M(S_1, \dots, S_n) := \bigodot_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} \mathrm{Val}_{S_i}(j)^{\odot M_{ij}} = \sum_{\substack{1\leq i\leq n\\ 0\leq j\leq r}} M_{ij} \cdot \mathrm{Val}_{S_i}(j),$$

where $\mathrm{Val}_{S_i}(j)$ is defined as in (3-4).

Remark 4.1. Note that $\varepsilon_M(S_1, \dots, S_n) = 0$ if and only if $j \in S_i$ for all i, j with $M_{ij} \neq 0$.

A tropical differential polynomial ϕ as in (4-2) induces a mapping from $\mathcal{P}(\mathbb{Z}_{\geq 0})^n$ to $\mathbb{Z}_{\geq 0} \cup \{\infty\}$ given by

$$\phi(S) = \bigoplus_{M \in \Lambda} a_M \odot \varepsilon_M(S) = \min_{M\in\Lambda}\{a_M + \varepsilon_M(S)\}.$$

Definition. An n-tuple $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is said to be a *solution* of the tropical differential polynomial ϕ in (4-2) if either

(1) there exist $M_1, M_2 \in \Lambda$ with $M_1 \neq M_2$ such that $\phi(S) = a_{M_1} \odot \varepsilon_{M_1}(S) = a_{M_2} \odot \varepsilon_{M_2}(S)$, or

(2) $\phi(S) = \infty$.

Let $H \subset \mathbb{T}\{x_1, \dots, x_n\}$ be a family of tropical differential polynomials. An n-tuple $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is a *solution* of H when it is a solution of every tropical polynomial in H; that is,

$$\mathrm{Sol}(H) := \{S \in (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n : S \text{ is a solution of } \phi \text{ for every } \phi \in H\}.$$

Example 4.2. Consider the tropical differential polynomial

$$\phi(x) := 1 \odot x' \oplus 2 \odot x^{(3)} \oplus 3.$$

Since $\phi(S) \neq \infty$ for every $S \subset \mathcal{P}(\mathbb{Z}_{\geq 0})$, the set S is a solution of ϕ if one of the following holds:

(1) $1 + \mathrm{Val}_S(1) = 3 \leq 2 + \mathrm{Val}_S(3)$,

(2) $1 + \mathrm{Val}_S(1) = 2 + \mathrm{Val}_S(3) \leq 3$,

(3) $2 + \mathrm{Val}_S(3) = 3 \leq 1 + \mathrm{Val}_S(1)$.

The first condition never holds. The second condition holds when $S = B \cup \{2, 3\} \cup C$ and $B \subset \{0\}$, $\min C \geq 4$. The third condition holds when $S = \{4\} \cup C \cup B$ with $\min C \geq 5$ and $B \subset \{0\}$. Thus,

$$\mathrm{Sol}(P) = \bigl\{B \cup \{2, 3\} \cup C : B \subset \{0\},\ \min C \geq 4\bigr\} \cup \bigl\{B \cup \{4\} \cup C : \min C \geq 5,\ B \subset \{0\}\bigr\}.$$

5. Tropicalization of differential polynomials

Let P be a differential polynomial as in (2-4). The *tropicalization* of P is the tropical differential polynomial

$$\mathrm{trop}(P) := \bigoplus_{M \in \Lambda} \mathrm{val}(\psi_M) \odot \varepsilon_M. \tag{5-1}$$

Remark 5.1. Let P be a differential polynomial in $R\{x_1, \ldots, x_n\}$. We have that $\mathrm{trop}(tP)(S) \geq 1$ for any $S \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$.

Definition. Let $G \subset R\{x_1, \ldots, x_n\}$ be a differential ideal. Its *tropicalization* $\mathrm{trop}(G)$ is the set of tropical differential polynomials $\{\mathrm{trop}(P) : P \in G\}$.

Proposition 5.2. *Let G be a differential ideal in the ring of differential polynomials* $R\{x_1, \ldots, x_n\}$. *If* $\varphi \in \mathrm{Sol}(G)$, *then* $\mathrm{trop}(\varphi) \in \mathrm{Sol}(\mathrm{trop}(G))$.

Proof. Given a differential monomial E_M and $\varphi \in R^n$, we have that

$$\mathrm{val}(E_M(\varphi)) = \varepsilon_M(\mathrm{trop}(\varphi)).$$

It follows that if $\varphi \in R^n$ is a solution to the differential polynomial

$$P = \sum_{M \in \Lambda} \psi_M E_M,$$

then $\mathrm{trop}(\varphi) \in (\mathcal{P}(\mathbb{Z}_{\geq 0}))^n$ is a solution to $\mathrm{trop}(P)$. So, if $\varphi \in R^n$ is a solution to every differential polynomial P in a differential ideal G, then $\mathrm{trop}(\varphi)$ is a solution to every tropical differential polynomial $\mathrm{trop}(P) \in \mathrm{trop}(G)$. □

We can now clearly state the question posed in [Grigoriev 2015]. The latter result allows us to define a mapping $\mathrm{trop} : \mathrm{Sol}(G) \to \mathrm{Sol}(\mathrm{trop}(G))$ for any differential ideal $G \subset R\{x_1, \ldots, x_n\}$. The question is whether or not this map is surjective.

Example 5.3. Let $P \in R\{x\}$ be the differential polynomial

$$P := x'' - t.$$

The set of solutions of P is the same as the set of solutions of the differential ideal generated by P:

$$\mathrm{Sol}(P) = \bigl\{c_1 + c_2 t + \tfrac{1}{6} t^3 : c_1, c_2 \in K\bigr\}.$$

The tropicalization of the set of solutions of P is

$$\mathrm{trop}(\mathrm{Sol}(P)) = \big\{\{0, 1, 3\}, \{0, 3\}, \{1, 3\}, \{3\}\big\}.$$

Now, the tropicalization of P induces the mapping

$$\mathrm{trop}(P) : \mathcal{P}(\mathbb{Z}_{\geq 0}) \to \mathbb{Z}_{\geq 0}, \quad S \mapsto \min\{\mathrm{Val}_S(2), 1\}.$$

Since $\mathrm{trop}(P)(S) \neq \infty$ for every $S \subset \mathcal{P}(\mathbb{Z}_{\geq} 0)$, the set of solutions of $\mathrm{trop}(P)$ is

$$\mathrm{Sol}(\mathrm{trop}(P)) = \{S \subset \mathcal{P}(\mathbb{Z}_{\geq} 0) \ : \ 2 \notin S \text{ and } 3 \in S\}.$$

Differentiating P, we have that $d^2P = x^{(4)}$ is in the differential ideal generated by P. Its tropicalization induces the mapping

$$\mathrm{trop}(d^2P) : \mathcal{P}(\mathbb{Z}_{\geq 0}) \to \mathbb{Z}_{\geq 0}, \quad S \mapsto \mathrm{Val}_S(4).$$

We have that $S \subset \mathcal{P}(\mathbb{Z}_{\geq} 0)$ is a solution of $\mathrm{trop}(d^2P)$ if and only if $S \subset \{0, 1, 2, 3\}$, i.e.,

$$\mathrm{Sol}(\mathrm{trop}(d^2P)) = \mathcal{P}(\{0, 1, 2, 3\}).$$

In this example,

$$\mathrm{Sol}(\mathrm{trop}(P)) \cap \mathrm{Sol}(\mathrm{trop}(d^2P)) = \mathrm{trop}(\mathrm{Sol}(P)).$$

6. Arc spaces and the set of solutions of a differential ideal

The natural inclusion $K[x_{1\,0}, \dots, x_{n\,0}] \subset R\{x_1, \dots, x_n\}$ lets us recognize the arc space of the variety defined by an ideal $I \subset K[x_1, \dots, x_n]$ as the space of solutions of the differential ideal generated by I in $R\{x_1, \dots, x_n\}$. In this section we extend some definitions and results developed in the theory of arc spaces; see for example [Nash 1995; Bruschek et al. 2013].

Consider the bijection

$$\Psi : \big(K^{\mathbb{Z}_{\geq 0}}\big)^n \to R^n, \quad \underline{a} = (a_{ij})_{1\leq i\leq n,\, j\geq 0} \mapsto \bigg(\sum_{j\geq 0} \frac{1}{j!} a_{1j} t^j, \dots, \sum_{j\geq 0} \frac{1}{j!} a_{nj} t^j\bigg).$$

Lemma 6.1. *Given $P \in R\{x_1, \dots, x_n\}$ and $\underline{a} \in (K^{\mathbb{Z}_{\geq 0}})^n$, we have*

$$P(\Psi(\underline{a})) = \sum_{k\geq 0} c_k t^k \tag{6-1}$$

with

$$c_k = \frac{1}{k!} (d^k(P))\big|_{t=0}(\underline{a}).$$

Proof. For $\underline{a} = (a_{ij})_{1\le i\le n,\, j\ge 0} \in \left(K^{\mathbb{Z}_{\ge 0}}\right)^n$, write $\Psi(\underline{a}) = (\Psi(\underline{a})_1, \dots, \Psi(\underline{a})_n)$ and $P(\Psi(\underline{a})) = \sum_{k\ge 0} c_k t^k$ for some $c_k \in K$, $k \ge 0$. Differentiating (6-1) and evaluating at zero, we have

$$c_k = \frac{1}{k!}\left[d^k(P(\Psi(\underline{a})))\right]_{t=0} \overset{(2\text{-}2)}{=} \frac{1}{k!}\left[(d^kP)(\Psi(\underline{a}))\right]_{t=0} \overset{(2\text{-}1)}{=} \frac{1}{k!}\left[(d^kP)|_{x_{ij}=\Psi(\underline{a})_i^{(j)}}\right]_{t=0}$$
$$= \frac{1}{k!}\left[(d^kP)|_{x_{ij}=\Psi(\underline{a})_i^{(j)}|_{t=0}}\right]_{t=0} \overset{(3\text{-}3)}{=} \frac{1}{k!}\left[(d^kP)|_{x_{ij}=a_{ij}}\right]_{t=0} = \frac{1}{k!}(d^kP)|_{t=0}(\underline{a}). \qquad \square$$

Let G be a differential ideal in $R\{x_1, \dots, x_n\}$. We can consider G as an infinite system of differential polynomials in $F\{x_1, \dots, x_n\}$, where $F = \mathrm{Frac}(R)$ is a field of characteristic zero. By Proposition 2.1, there exist $f_1, \dots, f_s \in G$ such that

$$\mathrm{Sol}(G) = \bigcap_{\ell=1}^{s} \mathrm{Sol}(f_\ell).$$

For $1 \le \ell \le s$ and $k \in \mathbb{Z}_{\ge 0}$, the $(d^k f_\ell)|_{t=0}$ are polynomials in the variables x_{ij} with coefficients in K. Set

$$F_{\ell k} := (d^k f_\ell)|_{t=0} \in K[x_{ij} \,:\, 1 \le i \le n,\ j \ge 0]$$

and

$$A_\infty := V\left(\{F_{\ell k}\}_{1\le \ell\le s,\, k\ge 0}\right) \subset \left(K^{\mathbb{Z}_{\ge 0}}\right)^n. \tag{6-2}$$

By Lemma 6.1,

$$\mathrm{Sol}(G) = \Psi(A_\infty).$$

We will now describe an extension to differential ideals of the definition of m-jet of arc spaces; see for example [Mourtada 2011].

For each $m \ge 0$, let N_m be the smallest positive integer such that

$$F_{\ell k} \in K[x_{ij} \,:\, 1 \le i \le n,\ 0 \le j \le N_m] \quad \text{for all } 1 \le \ell \le s,\ 0 \le k \le m \tag{6-3}$$

and set

$$A_m := V\left(\{F_{\ell k}\}_{1\le \ell\le s,\, 0\le k\le m}\right) \subset \left(K^{N_m+1}\right)^n. \tag{6-4}$$

For $m \ge m' \ge 0$, denote by $\pi_{(m,m')}$ the natural algebraic morphism

$$\pi_{(m,m')} : K^{n(N_m+1)} \to K^{n(N_{m'}+1)}.$$

Then

$$\pi_{(m,m')}(A_m) \subset A_{m'}$$

and A_∞ is the inverse limit of the system $((A_m)_{m\in\mathbb{Z}_{\ge 0}}, (\pi_{(m,m')})_{m\ge m'\in\mathbb{Z}_{\ge 0}})$:

$$A_\infty = \varprojlim A_m.$$

When $f_1, \dots, f_s$ are elements of $K[x_{10}, \dots, x_{n0}]$, the sets A_m are the m-jets of the space A_∞. Otherwise, note that the construction depends strongly on the choice of $f_1, \dots, f_s$.

7. Intersections with tori

Let $G \subset R\{x_1, \dots, x_n\}$ be a differential ideal, let $f_1, \dots, f_s \in G$ be such that $\mathrm{Sol}(G) = \bigcap_{\ell=1}^{s} \mathrm{Sol}(f_\ell)$, and let A_∞ be as in (6-2) and A_m as in (6-4).

An n-tuple $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ is in $\mathrm{trop}(\mathrm{Sol}(G))$ if and only if there exists $\underline{a} \in A_\infty$ with $\mathrm{trop}(\Psi(\underline{a})) = S$, i.e., if $S_i = \{j : a_{ij} \neq 0\}$ for $i = 1, \dots, n$. Set

$$\mathbb{V}_S^* := \left\{(x_{ij})_{1\leq i\leq n,\, j\geq 0} \in \left(K^{\mathbb{Z}_{\geq 0}}\right)^n : x_{ij} = 0 \text{ if and only if } j \notin S_i\right\},$$

then $S \in \mathrm{trop}(\mathrm{Sol}(G))$ if and only if

$$(A_\infty)_S := A_\infty \cap \mathbb{V}_S^*$$

is not empty.

For $m \geq 0$, consider the finite dimensional torus

$$(\mathbb{V}_m)_S^* := \left\{(x_{ij})_{1\leq i\leq n,\, 0\leq j\leq N_m} \in K^{n(N_m+1)} : x_{ij} = 0 \text{ if and only if } j \notin S_i\right\},$$

where N_m is the minimum such that (6-3) holds. We have $(\mathbb{V}_m)_S^* \simeq (K^*)^{L_m}$, with $L_m \leq n(N_m + 1)$. Set

$$(A_m)_S := A_m \cap (\mathbb{V}_m)_S^*.$$

For $m \geq m' \geq 0$, the inclusions

$$\pi_{(m,m')}((\mathbb{V}_m)_S^*) \subset (\mathbb{V}_{m'})_S^* \quad \text{and} \quad \pi_{(m,m')}((A_m)_S) \subset (A_{m'})_S$$

hold, and $(A_\infty)_S$ is the inverse limit of $\left(((A_m)_S)_{m\in\mathbb{Z}_{\geq 0}}, (\pi_{(m,m')})_{m\geq m'\in\mathbb{Z}_{\geq 0}}\right)$:

$$(A_\infty)_S = \varprojlim (A_m)_S.$$

Set

$$(B_m)_S := \bigcap_{i=m}^{\infty} \pi_{(i,m)}((A_i)_S);$$

then

$$(A_\infty)_S = \varprojlim (B_m)_S$$

and the projections

$$\pi_{(m,m')} : (B_m)_S \to (B_{m'})_S$$

are surjective. Then (see, for example, [Bourbaki 1968, Proposition 5, p. 198]), the set $\varprojlim (B_m)_S$ is nonempty if and only if $(B_0)_S$ is nonempty. In other words, we have the following remark.

Remark 7.1. The set $(A_\infty)_S$ is nonempty if and only if $\bigcap_{i=0}^{\infty} \pi_{(i,0)}((A_i)_S)$ is nonempty.

By Chevalley's theorem (see, for example, [Mumford 1999, p. 51]), each $\pi_{(m,0)}((A_m)_S)$ is a constructible set. A constructible set is, by definition, a finite union of locally closed sets. A set is locally closed when it is an open set of its closure. The constructible sets form a Boolean algebra.

We recall the following statement about nested sequences of constructible sets:

Proposition 7.2. *Let K be an uncountable algebraically closed field of characteristic zero. Let $\{E_\alpha\}_{\alpha=1}^{\infty}$ be an increasing family of constructible sets in K^n with $K^n = \bigcup_{\alpha=1}^{\infty} E_\alpha$. Then there exists α such that $K^n = E_\alpha$.*

We are now ready to prove the result that will allow us, in the next section, to work in the noetherian ring $K[x_{ij} : 1 \le i \le n,\ 0 \le j \le N_m]$ instead of the nonnoetherian $K[x_{ij} : 1 \le i \le n,\ 0 \le j]$.

Proposition 7.3. *The set $(A_\infty)_S$ is nonempty if and only if $(A_m)_S$ is nonempty for all $m \in \mathbb{Z}_{\ge 0}$.*

Proof. Since the constructible sets form a Boolean algebra, the nested sequence of constructible sets inside $(K^*)^{L_0} \simeq (\mathbb{V}_0)_S^*$,

$$\cdots \subset \pi_{(2,0)}((A_2)_S) \subset \pi_{(1,0)}((A_1)_S) \subset (A_0)_S \subset (K^*)^{L_0}, \tag{7-1}$$

induces an increasing family of constructible sets

$$\varnothing \subset (K^*)^{L_0} \backslash (A_0)_S \subset (K^*)^{L_0} \backslash \pi_{(1,0)}((A_1)_S) \subset (K^*)^{L_0} \backslash \pi_{(2,0)}((A_2)_S) \subset \cdots . \tag{7-2}$$

The set $\bigcap_{i=0}^{\infty} \pi_{(i,0)}((A_i)_S)$ is empty if and only if $(K^*)^{L_0} \setminus \bigcap_{i=0}^{\infty} \pi_{(i,0)}((A_i)_S)$ is $(K^*)^{L_0}$; that is, if and only if

$$(K^*)^{L_0} = \bigcup_{i=0}^{\infty} (K^*)^{L_0} \setminus \pi_{(i,0)}((A_i)_S).$$

Then, by Proposition 7.2, there exists m such that $(K^*)^{L_0} \setminus \pi_{(m,0)}((A_m)_S) = (K^*)^{L_0}$. That is, there exists m such that $(A_m)_S$ is empty.

The result follows from Remark 7.1. □

8. The fundamental theorem of differential tropical geometry

Theorem 8.1. *Let G be a differential ideal in $K[\![t]\!]\{x_1, \dots x_n\}$, where K is an uncountable algebraically closed field of characteristic zero. The equality*

$$\operatorname{Sol}(\operatorname{trop}(G)) = \operatorname{trop}(\operatorname{Sol}(G))$$

holds.

Proof. The inclusion $\mathrm{trop}(\mathrm{Sol}(G)) \subset \mathrm{Sol}(\mathrm{trop}(G))$ is just Proposition 5.2. Here we will prove

$$\mathrm{Sol}(\mathrm{trop}(G)) \subset \mathrm{trop}(\mathrm{Sol}(G)).$$

Let $S = (S_1, \dots, S_n) \in \mathcal{P}(\mathbb{Z}_{\geq 0})^n$ be such that there is no solution of G whose tropicalization is S. We will show that S cannot be a solution of the tropicalization of G.

Suppose that $\mathrm{Sol}(G) = \bigcap_{\ell=1}^{s} \mathrm{Sol}(f_\ell)$, for some $f_1, \dots f_s \in G$. For $1 \leq \ell \leq s$ and $k \in \mathbb{Z}_{\geq 0}$, we write $F_{\ell k} := (d^k f_\ell)|_{t=0}$.

As we have seen above, $S \notin \mathrm{trop}(\mathrm{Sol}(G))$ implies that $(A_\infty)_S$ is empty. Then, by Proposition 7.3 there exists $m \in \mathbb{N}$ such that $(A_m)_S$ is empty.

Take $m \in \mathbb{N}$ such that $(A_m)_S$ is empty. Set $\overline{F_{\ell k}}$ to be the image of $F_{\ell k}$ in the ring

$$K[x_{ij} \,:\, 1 \leq i \leq n,\ 0 \leq j \leq N_m]/\langle x_{ij} \,:\, j \notin S_i \rangle.$$

Since $(A_m)_S$ is empty we have

$$V(\overline{F_{\ell k}} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m) \subset V\Bigg(\prod_{\{0 \leq i \leq n,\ j \in S_i \,:\, j \leq N_m\}} x_{ij} \Bigg)$$

so by the Nullstellensatz, there exists $\alpha \geq 1$ such that

$$E_M = \Bigg(\prod_{\{0 \leq i \leq n,\ j \in S_i \,:\, j \leq N_m\}} x_{ij} \Bigg)^{\alpha} \in \langle \overline{F_{\ell k}} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m \rangle.$$

Here E_M is the differential monomial induced by the matrix $M \in \mathcal{M}_{n \times (N_m+1)}(\mathbb{Z}_{\geq 0})$ with entries $M_{ij} = 0$ for $j \notin S_i$ and $M_{ij} = \alpha$ for $j \in S_i$.

It follows that there exists

$$\{G_{\ell k} \,:\, 1 \leq \ell \leq s,\ 0 \leq k \leq m\} \subset K[x_{ij} \,:\, 1 \leq i \leq n,\ j \in S_i,\ j \leq N_m]$$

such that

$$\sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} \overline{F_{\ell k}} = E_M. \tag{8-1}$$

Then

$$\sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} F_{\ell k} = E_M + h \tag{8-2}$$

for some $h \in \langle x_{ij} : j \notin S_i,\ j \leq N_m \rangle \subset K[x_{ij} : 1 \leq i \leq n,\ 0 \leq j \leq N_m]$.

Now, by definition of $F_{\ell k}$, there exists λ in $K[\![t]\!]\{x_0, \dots, x_n\}$ such that

$$g := \sum_{\substack{1 \leq \ell \leq s \\ 0 \leq k \leq m}} G_{\ell k} d^k f_\ell = E_M + h + t\lambda. \tag{8-3}$$

Since G is a differential ideal and $f_1, \dots f_s \in G$, the differential polynomial g is in G.

We now have:

- By Remark 4.1, $\varepsilon_M(S) = 0$ and if $h \neq 0$, then $\operatorname{trop}(h)(S) \geq 1$.
- By Remark 5.1, if $t\lambda \neq 0$, then $\operatorname{trop}(t\lambda)(S) \geq 1$.

Thus, $(\operatorname{trop}(g))(S) = 0$ and the minimum is attained only at the monomial ε_M, and hence, S is not a solution of $\operatorname{trop}(g)$. So S is not a solution of the tropicalization of G, which is what we wanted to prove. □

Acknowledgments

We would like to thank José Manuel Aroca, Ana Bravo and Guillaume Rond for fruitful conversations during the preparation of this work. Garay would also like to thank Andrew Stout, and Toghani wishes to thank Pedro Luis de Angel.

References

[Aroca 2010] F. Aroca, "Krull-tropical hypersurfaces", *Ann. Fac. Sci. Toulouse Math.* (6) **19**:3-4 (2010), 525–538. MR 2790807 Zbl 1223.14069

[Aroca and Cano 2001] F. Aroca and J. Cano, "Formal solutions of linear PDEs and convex polyhedra: Effective methods in rings of differential operators", *J. Symbolic Comput.* **32**:6 (2001), 717–737. MR 1866713 Zbl 1077.35040

[Aroca et al. 2003] F. Aroca, J. Cano, and F. Jung, "Power series solutions for non-linear PDE's", pp. 15–22 in *ISSAC 2003: Proc. Int. Symposium on Symbolic and Algebraic Computation* (Philadelphia, 2003), edited by J. R. Sendra, ACM, New York, 2003. MR 2035190 Zbl 1072.68644

[Aroca et al. 2010] F. Aroca, G. Ilardi, and L. López de Medrano, "Puiseux power series solutions for systems of equations", *Internat. J. Math.* **21**:11 (2010), 1439–1459. MR 2747737 Zbl 1247.14066

[Bourbaki 1968] N. Bourbaki, *Theory of sets*, Hermann, Paris, 1968. MR 0237342 Zbl 1061.03001

[Bruschek et al. 2013] C. Bruschek, H. Mourtada, and J. Schepers, "Arc spaces and the Rogers–Ramanujan identities", *Ramanujan J.* **30**:1 (2013), 9–38. MR 3010462 Zbl 1281.14003

[Cano 1993] J. Cano, "On the series defined by differential equations, with an extension of the Puiseux polygon construction to these equations", *Analysis* **13**:1-2 (1993), 103–119. MR 1245746 Zbl 0793.34009

[Einsiedler et al. 2006] M. Einsiedler, M. Kapranov, and D. Lind, "Non-Archimedean amoebas and tropical varieties", *J. Reine Angew. Math.* **601** (2006), 139–157. MR 2289207 Zbl 1115.14051

[Fine 1889] H. B. Fine, "On the functions defined by differential equations, with an extension of the Puiseux polygon construction to these equations", *Amer. J. Math.* **11**:4 (1889), 317–328. MR 1505516 JFM 21.0302.01

[Grigoriev 2015] D. Grigoriev, "Tropical differential equations", preprint, 2015. arXiv 1502.08010

[Grigoriev and Singer 1991] D. Yu. Grigor'ev and M. F. Singer, "Solving ordinary differential equations in terms of series with real exponents", *Trans. Amer. Math. Soc.* **327**:1 (1991), 329–351. MR 1012519 Zbl 0758.12004

[Jensen et al. 2008] A. N. Jensen, H. Markwig, and T. Markwig, "An algorithm for lifting points in a tropical variety", *Collect. Math.* **59**:2 (2008), 129–165. MR 2414142 Zbl 1151.13021

[Mourtada 2011] H. Mourtada, "Jet schemes of toric surfaces", *C. R. Math. Acad. Sci. Paris* **349**:9-10 (2011), 563–566. MR 2802925 Zbl 1222.14031

[Mumford 1999] D. Mumford, *The red book of varieties and schemes*, Lecture Notes in Mathematics **1358**, Springer, Berlin, 1999. MR 1748380 Zbl 0945.14001

[Nash 1995] J. F. Nash, Jr., "Arc structure of singularities", *Duke Math. J.* **81**:1 (1995), 31–38. MR 1381967 Zbl 0880.14010

[Puiseux 1850] V. Puiseux, "Recherches sur les fonctions algébriques", *J. Math. Pures Appl.* (1850), 365–480.

[Ritt 1950] J. F. Ritt, *Differential Algebra*, American Mathematical Society Colloquium Publications **33**, Amer. Math. Soc., New York, 1950. MR 0035763 Zbl 0037.18402

[Speyer and Sturmfels 2004] D. Speyer and B. Sturmfels, "The tropical Grassmannian", *Adv. Geom.* **4**:3 (2004), 389–411. MR 2071813 Zbl 1065.14071

Received January 12, 2016. Revised March 9, 2016.

FUENSANTA AROCA
INSTITUTO DE MATEMÁTICAS, UNIDAD CUERNAVACA
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
AV. UNIVERSIDAD S/N. COL. LOMAS DE CHAMILPA
62210 CUERNAVACA
MEXICO
fuen@im.unam.mx

CRISTHIAN GARAY
INSTITUT DE MATHÉMATIQUES DE JUSSIEU
INSTITUT DE MATHÉMATIQUES DE JUSSIEU–PARIS RIVE GAUCHE
4 PLACE JUSSIEU, CASE 247
75252 PARIS CEDEX 5
FRANCE
cristhian.garay@imj-prg.fr

ZEINAB TOGHANI
INSTITUTO DE MATEMÁTICAS, UNIDAD CUERNAVACA
UNIVERSIDAD NACIONAL AUTÓNOMA DE MÉXICO
AV. UNIVERSIDAD S/N. COL. LOMAS DE CHAMILPA
62210 CUERNAVACA
MEXICO
zeinab.toghani@im.unam.mx

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.271

A SIMPLE SOLUTION TO THE WORD PROBLEM FOR VIRTUAL BRAID GROUPS

PAOLO BELLINGERI, BRUNO A. CISNEROS DE LA CRUZ AND LUIS PARIS

We show a simple and easily implementable solution to the word problem for virtual braid groups.

1. Introduction

Virtual braid groups were introduced by L. Kauffman [1999] in his seminal paper on virtual knots and links. They can be defined in several ways, such as in terms of Gauss diagrams [Bar-Natan and Dancso 2015; Cisneros de la Cruz 2015], in terms of braids in thickened surfaces [Cisneros de la Cruz 2015], and in terms of virtual braid diagrams. The latter will be our starting point of view.

A *virtual braid diagram* on n strands is an n-tuple $\beta = (b_1, \dots, b_n)$ of smooth paths in the plane $\mathbb{R}^2$ satisfying the following conditions:

(a) $b_i(0) = (i, 0)$ for all $i \in \{1, \dots, n\}$.

(b) There is a permutation $g \in \mathfrak{S}_n$ such that $b_i(1) = (g(i), 1)$ for all $i \in \{1, \dots, n\}$.

(c) $(p_2 \circ b_i)(t) = t$ for all $i \in \{1, \dots, n\}$ and all $t \in [0, 1]$, where $p_2 : \mathbb{R}^2 \to \mathbb{R}$ denotes the projection on the second coordinate.

(d) The b_i intersect transversely in a finite number of double points, called the *crossings* of the diagram.

Each crossing is endowed with one of the following attributes: positive, negative, virtual. In the figures they are generally indicated as in Figure 1. Let VBD_n be the set of virtual braid diagrams on n strands, and let $\sim$ be the equivalence relation on VBD_n generated by ambient isotopy and the virtual Reidemeister moves depicted in Figure 2. The concatenation of diagrams induces a group structure on $\mathrm{VBD}_n/\!\sim$. The latter is called the *virtual braid group* on n strands, and is denoted by VB_n.

MSC2010: primary 20F36; secondary 20F10, 57M25.
Keywords: word problem, virtual braid group.

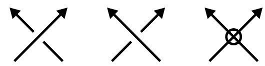


Figure 1. Positive, negative and virtual crossings in a virtual braid diagram.

It was observed in [Kamada 2007; Vershinin 2001] that VB_n has a presentation with generators $\sigma_1, \dots, \sigma_{n-1}, \tau_1, \dots, \tau_{n-1}$ and relations

$$\begin{aligned} &\tau_i^2 = 1 && \text{for } 1 \le i \le n-1, \\ &\sigma_i\sigma_j = \sigma_j\sigma_i,\ \ \sigma_i\tau_j = \tau_j\sigma_i,\ \ \tau_i\tau_j = \tau_j\tau_i && \text{for } |i-j| \ge 2, \\ &\sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j,\ \ \sigma_i\tau_j\tau_i = \tau_j\tau_i\sigma_j,\ \ \tau_i\tau_j\tau_i = \tau_j\tau_i\tau_j && \text{for } |i-j| = 1. \end{aligned}$$

A solution to the word problem for virtual braid groups was shown in [Godelle and Paris 2012]. However, this solution is quite theoretical and its understanding requires some heavy technical knowledge on Artin groups. Therefore, it is incomprehensible and useless for most of the potential users, including low-dimensional topologists. Moreover, its implementation would be difficult. Our aim here is to show a new solution, which is simpler and easily implementable, and whose understanding does not require any special technical knowledge. This new solution is in the spirit of the one shown in [Godelle and Paris 2012], in the sense that one of the main ingredients in its proof is the study of parabolic subgroups in Artin groups.

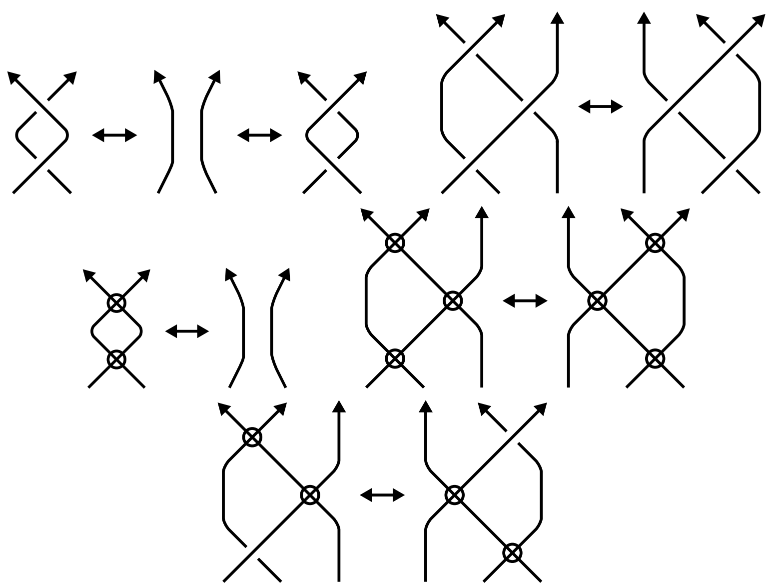


Figure 2. Virtual Reidemeister moves.

We have not calculated the complexity of this algorithm, as this is probably at least exponential because of the inductive step 3 (see Section 2). Nevertheless, it is quite efficient for a limited number of strands (see the example at the end of Section 2), and, above all, it should be useful to study theoretical questions on VB_n such as the faithfulness of representations of this group in automorphism groups of free groups and/or in linear groups. Note that the faithfulness of such a representation will immediately provide another, probably faster, solution to the word problem for VB_n.

The Burau representation easily extends to VB_n [Vershinin 2001], but the question whether VB_n is linear or not is still open. A representation of VB_n in $\mathrm{Aut}(F_{n+1})$ was independently constructed in [Bardakov 2005] and [Manturov 2003], but such a representation has recently been proven to be not faithful for $n \geq 4$ [Chterental 2015, Proposition 5.3] (see the example at the end of Step 1). So, we do not know yet any representation on which we can test our algorithm.

Chterental [2015] shows a faithful action of VB_n on a set of objects that he calls "virtual curve diagrams". We have some hope to use this action to describe another explicit solution to the word problem for VB_n. But, for now, we do not know any formal definition of this action, nor how it could be encoded in an algorithm.

2. The algorithm

Our solution to the word problem for VB_n is divided into four steps. In Step 1 we define a subgroup KB_n of VB_n and a generating set $\mathcal{S}$ for KB_n, and we show an algorithm (called Algorithm A) which decides whether an element of VB_n belongs to KB_n and, if yes, determines a word over $\mathcal{S}^{\pm 1}$ which represents this element. For $\mathcal{X} \subset \mathcal{S}$, we denote by $\mathrm{KB}_n(\mathcal{X})$ the subgroup of KB_n generated by $\mathcal{X}$. The other three steps provide a solution to the word problem for $\mathrm{KB}_n(\mathcal{X})$ which depends recursively on the cardinality of $\mathcal{X}$. Step 2 is the beginning of the induction. More precisely, the algorithm proposed in Step 2 (called Algorithm B) is a solution to the word problem for $\mathrm{KB}_n(\mathcal{X})$ when $\mathcal{X}$ is a full subset of $\mathcal{S}$ (the notion of "full subset" will be also defined in Step 2; for now, the reader just need to know that singletons are full subsets). In Step 3 we suppose given a solution to the word problem for $\mathrm{KB}_n(\mathcal{X})$, and, for a given subset $\mathcal{Y} \subset \mathcal{X}$, we show an algorithm which solves the membership problem for $\mathrm{KB}_n(\mathcal{Y})$ in $\mathrm{KB}_n(\mathcal{X})$ (called Algorithm C). In Step 4 we show an algorithm which solves the word problem for $\mathrm{KB}_n(\mathcal{X})$ when $\mathcal{X}$ is not a full subset, under the assumption that the group $\mathrm{KB}_n(\mathcal{Y})$ has a solvable word problem for any proper subset $\mathcal{Y}$ of $\mathcal{X}$ (called Algorithm D).

Step 1. Recall that $\mathfrak{S}_n$ denotes the group of permutations of $\{1, \dots, n\}$. We denote by $\theta : \mathrm{VB}_n \to \mathfrak{S}_n$ the epimorphism which sends σ_i to 1 and τ_i to $(i, i+1)$ for all $1 \leq i \leq n-1$, and by KB_n the kernel of θ. Note that θ has a section $\iota : \mathfrak{S}_n \to \mathrm{VB}_n$

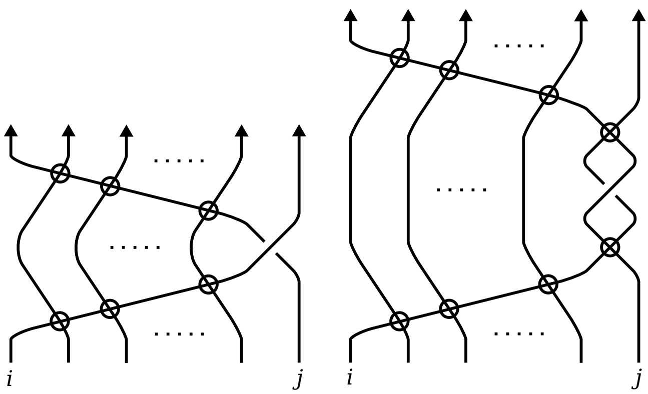


Figure 3. Generators for KB_n: $\delta_{i,j}$ (left) and $\delta_{j,i}$ (right)

which sends $(i, i+1)$ to τ_i for all $1 \le i \le n-1$, and therefore VB_n is a semidirect product $\mathrm{VB}_n = \mathrm{KB}_n \rtimes \mathfrak{S}_n$. The following proposition is proved in Rabenda's master's thesis, which, unfortunately, is not available anywhere. However, its proof can also be found in [Bardakov and Bellingeri 2009].

Proposition 2.1. *For* $1 \le i < j \le n$ *we set*

$$\delta_{i,j} = \tau_i \tau_{i+1} \cdots \tau_{j-2} \sigma_{j-1} \tau_{j-2} \cdots \tau_{i+1} \tau_i,$$
$$\delta_{j,i} = \tau_i \tau_{i+1} \cdots \tau_{j-2} \tau_{j-1} \sigma_{j-1} \tau_{j-1} \tau_{j-2} \cdots \tau_{i+1} \tau_i.$$

Then KB_n *has a presentation with generating set*

$$\mathcal{S} = \{\delta_{i,j} \mid 1 \le i \ne j \le n\}$$

and relations

$$\delta_{i,j}\delta_{k,l} = \delta_{k,l}\delta_{i,j} \qquad \textit{for } i, j, k, l \textit{ distinct},$$
$$\delta_{i,j}\delta_{j,k}\delta_{i,j} = \delta_{j,k}\delta_{i,j}\delta_{j,k} \quad \textit{for } i, j, k \textit{ distinct}.$$

The virtual braids $\delta_{i,j}$ and $\delta_{j,i}$ are depicted in Figure 3.

The following is an important tool in the forthcoming Algorithm A.

Lemma 2.2 [Bardakov and Bellingeri 2009]. *Let u be a word over* $\{\tau_1, \dots, \tau_{n-1}\}$, *let $\bar{u}$ be the element of* VB_n *represented by u, and let* $i, j \in \{1, \dots, n\}$, $i \ne j$. *Then* $\bar{u}\delta_{i,j}\bar{u}^{-1} = \delta_{i',j'}$, *where* $i' = \theta(\bar{u})(i)$ *and* $j' = \theta(\bar{u})(j)$.

Note that $\tau_i^{-1} = \tau_i$, since $\tau_i^2 = 1$, for all $i \in \{1, \dots, n-1\}$. Hence, the letters $\tau_1^{-1}, \dots, \tau_{n-1}^{-1}$ are not needed in the above lemma and below.

We give an algorithm which, given a word u over $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, \tau_1, \dots, \tau_{n-1}\}$, decides whether the element $\bar{u}$ of VB_n represented by u belongs to KB_n. If yes, it also determines a word u' over $\mathcal{S}^{\pm 1} = \{\delta_{i,j}^{\pm} \mid 1 \le i \ne j \le n\}$ which represents $\bar{u}$. The fact that this algorithm is correct follows from Lemma 2.2.

Algorithm A. Let u be a word over $\{\sigma_1^{\pm 1}, \dots, \sigma_{n-1}^{\pm 1}, \tau_1, \dots, \tau_{n-1}\}$. We write u in the form

$$u = v_0 \sigma_{i_1}^{\varepsilon_1} v_1 \cdots v_{l-1} \sigma_{i_l}^{\varepsilon_l} v_l,$$

where $v_0, v_1, \dots, v_l$ are words over $\{\tau_1, \dots, \tau_{n-1}\}$, and $\varepsilon_1, \dots, \varepsilon_l \in \{\pm 1\}$. On the other hand, for a word $v = \tau_{j_1} \cdots \tau_{j_k}$ over $\{\tau_1, \dots, \tau_{n-1}\}$, we set $\theta(v) = (j_1, j_1+1) \cdots (j_k, j_k+1) \in \mathfrak{S}_n$. Note that $\theta(\bar{u}) = \theta(v_0)\,\theta(v_1) \cdots \theta(v_l)$. If $\theta(\bar{u}) \neq 1$, then $\bar{u} \notin \mathrm{KB}_n$. If $\theta(\bar{u}) = 1$, then $\bar{u} \in \mathrm{KB}_n$, and $\bar{u}$ is represented by

$$u' = \delta_{a_1,b_1}^{\varepsilon_1} \delta_{a_2,b_2}^{\varepsilon_2} \cdots \delta_{a_l,b_l}^{\varepsilon_l},$$

where

$$a_k = \theta(v_0 \cdots v_{k-1})(i_k) \quad \text{and} \quad b_k = \theta(v_0 \cdots v_{k-1})(i_k + 1)$$

for all $k \in \{1, \dots, l\}$.

Example. Chterental [2015] proved that the Bardakov–Manturov representation of VB_n in $\mathrm{Aut}(F_{n+1})$ (see for instance [Bardakov 2005] for the definition) is not faithful, showing that the element $\omega = (\tau_3 \sigma_2 \tau_1 \sigma_2^{-1})^3$ is nontrivial in VB_4 while the corresponding automorphism of F_5 is trivial. In [Chterental 2015] the nontriviality of ω is shown by means of an action on some curve diagrams, but this fact can easily be checked with Algorithm A. Indeed, $\theta(\omega) = ((3,4)(1,2))^3 = (3,4)(1,2) \neq 1$, hence $\omega \neq 1$.

Step 2. Let S be a finite set. A *Coxeter matrix* over S is a square matrix $M = (m_{s,t})_{s,t \in S}$, indexed by the elements of S, such that $m_{s,s} = 1$ for all $s \in S$, and $m_{s,t} = m_{t,s} \in \{2, 3, 4, \dots\} \cup \{\infty\}$ for all $s, t \in S$, $s \neq t$. We represent this Coxeter matrix with a labeled graph $\Gamma = \Gamma_M$, called a *Coxeter diagram*. The set of vertices of Γ is S. Two vertices $s, t \in S$ are connected by an edge labeled by $m_{s,t}$ if $m_{s,t} \neq \infty$.

If a, b are two letters and m is an integer ≥ 2, we set $\langle a, b \rangle^m = (ab)^{m/2}$ if m is even, and $\langle a, b \rangle^m = (ab)^{(m-1)/2} a$ if m is odd. In other words, $\langle a, b \rangle^m$ denotes the word $aba \cdots$ of length m. The *Artin group* of Γ is the group $A = A(\Gamma)$ defined by the presentation

$$A = \big\langle S \mid \langle s, t \rangle^{m_{s,t}} = \langle t, s \rangle^{m_{s,t}} \text{ for all } s, t \in S,\ s \neq t \text{ and } m_{s,t} \neq \infty \big\rangle.$$

The *Coxeter group* of Γ, denoted by $W = W(\Gamma)$, is the quotient of A by the relations $s^2 = 1,\ s \in S$.

Example. Let $\mathrm{V}\Gamma_n$ be the Coxeter diagram defined as follows. The set of vertices of $\mathrm{V}\Gamma_n$ is $\mathcal{S}$. If $i, j, k, l \in \{1, \dots, n\}$ are distinct, then $\delta_{i,j}$ and $\delta_{k,l}$ are connected by an edge labeled by 2. If $i, j, k \in \{1, \dots, n\}$ are distinct, then $\delta_{i,j}$ and $\delta_{j,k}$ are connected by an edge labeled by 3. There is no other edge in $\mathrm{V}\Gamma_n$. Then, by Proposition 2.1, KB_n is isomorphic to $A(\mathrm{V}\Gamma_n)$.

Let Γ be a Coxeter diagram. For $X \subset S$, we denote by Γ_X the subdiagram of Γ spanned by X, by A_X the subgroup of $A = A(\Gamma)$ generated by X, and by W_X the subgroup of $W = W(\Gamma)$ generated by X. By [van der Lek 1983], A_X is the Artin group of Γ_X, and, by [Bourbaki 1968], W_X is the Coxeter group of Γ_X.

For $\mathcal{X} \subset \mathcal{S}$, we denote by $\mathrm{KB}_n(\mathcal{X})$ the subgroup of KB_n generated by $\mathcal{X}$. By the above, $\mathrm{KB}_n(\mathcal{X})$ has a presentation with generating set $\mathcal{X}$ and relations

- $st = ts$ if s and t are connected in $\mathrm{V}\Gamma_n$ by an edge labeled by 2,
- $sts = tst$ if s and t are connected in $\mathrm{V}\Gamma_n$ by an edge labeled by 3.

Definition. We say that a subset $\mathcal{X}$ of $\mathcal{S}$ is *full* if any two distinct elements s, t of $\mathcal{X}$ are connected by an edge of $\mathrm{V}\Gamma_n$. (Recall that the aim of Step 2 is to give a solution to the word problem for $\mathrm{KB}_n(\mathcal{X})$ when $\mathcal{X}$ is full.)

We denote by $F_n = F(x_1, \dots, x_n)$ the free group of rank n freely generated by $x_1, \dots, x_n$. For $i, j \in \{1, \dots, n\}$, $i \neq j$, we define $\varphi_{i,j} \in \mathrm{Aut}(F_n)$ by

$$\varphi_{i,j}(x_i) = x_i x_j x_i^{-1}, \quad \varphi_{i,j}(x_j) = x_i \quad \text{and} \quad \varphi_{i,j}(x_k) = x_k \ \text{ for } k \notin \{i, j\}.$$

It is easily seen from the presentation in Proposition 2.1 that the map $\mathcal{S} \to \mathrm{Aut}(F_n)$, $\delta_{i,j} \mapsto \varphi_{i,j}$, induces a representation $\varphi : \mathrm{KB}_n \to \mathrm{Aut}(F_n)$. For $\mathcal{X} \subset \mathcal{S}$, we denote by $\varphi_{\mathcal{X}} : \mathrm{KB}_n(\mathcal{X}) \to \mathrm{Aut}(F_n)$ the restriction of φ to $\mathrm{KB}_n(\mathcal{X})$. The following will be proved in Section 3;

Proposition 2.3. *If $\mathcal{X}$ is a full subset of $\mathcal{S}$, then $\varphi_{\mathcal{X}} : \mathrm{KB}_n(\mathcal{X}) \to \mathrm{Aut}(F_n)$ is faithful.*

Notation. From now on, if u is a word over $\mathcal{S}^{\pm 1}$, then $\bar{u}$ will denote the element of KB_n represented by u.

Algorithm B. Let $\mathcal{X}$ be a full subset of $\mathcal{S}$ and let $u = s_1^{\varepsilon_1} \cdots s_l^{\varepsilon_l}$ be a word over $\mathcal{X}^{\pm 1}$. We have $\varphi_{\mathcal{X}}(\bar{u}) = \varphi_{\mathcal{X}}(s_1)^{\varepsilon_1} \cdots \varphi_{\mathcal{X}}(s_l)^{\varepsilon_l}$. If $\varphi(\bar{u}) = \mathrm{Id}$, then $\bar{u} = 1$. Otherwise, $\bar{u} \neq 1$.

Step 3. Let G be a group, and let H be a subgroup of G. A solution to the *membership problem* for H in G is an algorithm which, given $g \in G$, decides whether g belongs to H or not. In the present step we will assume that $\mathrm{KB}_n(\mathcal{X})$ has a solution to the word problem, and, from this solution, we will give a solution to the membership problem for $\mathrm{KB}_n(\mathcal{Y})$ in $\mathrm{KB}_n(\mathcal{X})$ for $\mathcal{Y} \subset \mathcal{X}$. Furthermore, if the tested element belongs to $\mathrm{KB}_n(\mathcal{Y})$, then this algorithm will determine a word over $\mathcal{Y}^{\pm 1}$ which represents this element.

Let u be a word over $\mathcal{S}$. (Remark: here the alphabet is $\mathcal{S}$, and not $\mathcal{S}^{\pm 1}$.)

- Suppose that u is written in the form $u_1 s s u_2$, where u_1, u_2 are words over $\mathcal{S}$ and s is an element of $\mathcal{S}$. Then we say that $u' = u_1 u_2$ is obtained from u by an *M-operation of type* I.

- Suppose that u is written in the form u_1stu_2, where u_1, u_2 are words over $\mathcal{S}$ and s, t are two elements of $\mathcal{S}$ connected by an edge labeled by 2. Then we say that $u' = u_1tsu_2$ is obtained from u by an *M-operation of type* $\mathrm{II}^{(2)}$.
- Suppose that u is written in the form u_1stsu_2, where u_1, u_2 are words over $\mathcal{S}$ and s, t are two elements of $\mathcal{S}$ connected by an edge labeled by 3. Then we say that $u' = u_1tstu_2$ is obtained from u by an *M-operation of type* $\mathrm{II}^{(3)}$.

Let $\mathcal{Y}$ be a subset of $\mathcal{S}$.

- Suppose that u is written in the form tu', where u' is a word over $\mathcal{S}$ and t is an element of $\mathcal{Y}$. Then we say that u' is obtained from u by an *M-operation of type* $\mathrm{III}_{\mathcal{Y}}$.

We say that u is *M-reduced* (resp. *$M_{\mathcal{Y}}$-reduced*) if its length cannot be shortened by M-operations of type I, $\mathrm{II}^{(2)}$, $\mathrm{II}^{(3)}$ (resp. of type I, $\mathrm{II}^{(2)}$, $\mathrm{II}^{(3)}$, $\mathrm{III}_{\mathcal{Y}}$). An *$M$-reduction* (resp. *$M_{\mathcal{Y}}$-reduction*) of u is an M-reduced word (resp. $M_{\mathcal{Y}}$-reduced word) obtained from u by M-operations (resp. $M_{\mathcal{Y}}$-operations). We can easily enumerate all the words obtained from u by M-operations (resp. $M_{\mathcal{Y}}$-operations), hence we can effectively determine an M-reduction and/or an $M_{\mathcal{Y}}$-reduction of u.

Let $\mathcal{Y}$ be a subset of $\mathcal{S}$. From a word $u = s_1^{\varepsilon_1} \cdots s_l^{\varepsilon_l}$ over $\mathcal{S}^{\pm 1}$, we construct a word $\pi_{\mathcal{Y}}(u)$ over $\mathcal{Y}^{\pm 1}$ as follows:

- For $i \in \{0, 1, \dots, l\}$ we set $u_i^+ = s_1 \cdots s_i$ (as ever, u_0^+ is the identity).
- For $i \in \{0, 1, \dots, l\}$ we calculate an $M_{\mathcal{Y}}$-reduction v_i^+ of u_i^+.
- For a word $v = t_1 \cdots t_k$ over $\mathcal{S}$, we let $\mathrm{op}(v) = t_k \cdots t_1$. Let $i \in \{1, \dots, l\}$. If $\varepsilon_i = 1$, we set $w_i^+ = v_{i-1}^+ \cdot s_i \cdot \mathrm{op}(v_{i-1}^+)$. If $\varepsilon_i = -1$, we set $w_i^+ = v_i^+ \cdot s_i \cdot \mathrm{op}(v_i^+)$.
- For all $i \in \{1, \dots, l\}$ we calculate an M-reduction r_i of w_i^+.
- If r_i is of length 1 and $r_i \in \mathcal{Y}$, we set $T_i = r_i^{\varepsilon_i}$. Otherwise we set $T_i = 1$.
- We set $\pi_{\mathcal{Y}}(u) = T_1 T_2 \cdots T_l$.

The proof of the following is given in Section 4.

Proposition 2.4. *Let $\mathcal{Y}$ be a subset of $\mathcal{S}$. Let u, v be two words over $\mathcal{S}^{\pm 1}$. If $\bar{u} = \bar{v}$, then $\overline{\pi_{\mathcal{Y}}(u)} = \overline{\pi_{\mathcal{Y}}(v)}$. Moreover, we have $\bar{u} \in \mathrm{KB}_n(\mathcal{Y})$ if and only if $\bar{u} = \overline{\pi_{\mathcal{Y}}(u)}$.*

Algorithm C. Take two subsets $\mathcal{X}$ and $\mathcal{Y}$ of $\mathcal{S}$ such that $\mathcal{Y} \subset \mathcal{X}$, and assume given a solution to the word problem for $\mathrm{KB}_n(\mathcal{X})$. Let u be a word over $\mathcal{X}^{\pm 1}$. We calculate $v = \pi_{\mathcal{Y}}(u)$. If $\overline{uv^{-1}} \neq 1$, then $\bar{u} \notin \mathrm{KB}_n(\mathcal{Y})$. If $\overline{uv^{-1}} = 1$, then $\bar{u} \in \mathrm{KB}_n(\mathcal{Y})$ and v is a word over $\mathcal{Y}^{\pm 1}$ which represents the same element as u.

We can use Algorithm C to show that the representation $\varphi : \mathrm{KB}_n \to \mathrm{Aut}(F_n)$ of Step 2 is not faithful. Indeed, let $\alpha = \delta_{1,3}\delta_{3,2}\delta_{3,1}$ and $\beta = \delta_{2,3}\delta_{1,3}\delta_{3,2}$. A direct calculation shows that $\varphi(\alpha) = \varphi(\beta)$. Now, set $\mathcal{X} = \mathcal{S}$ and $\mathcal{Y} = \{\delta_{1,3}, \delta_{3,2}, \delta_{3,1}\}$. We have $\pi_{\mathcal{Y}}(\delta_{1,3}\delta_{3,2}\delta_{3,1}) = \delta_{1,3}\delta_{3,2}\delta_{3,1}$, hence $\alpha \in \mathrm{KB}_n(\mathcal{Y})$, and we have $\pi_{\mathcal{Y}}(\delta_{2,3}\delta_{1,3}\delta_{3,2}) = 1$ and $\beta \neq 1$, hence $\beta \notin \mathrm{KB}_n(\mathcal{Y})$. Thus $\alpha \neq \beta$.

Step 4. Now, we assume that $\mathcal{X}$ is a nonfull subset of $\mathcal{S}$, and that we have a solution to the word problem for $\mathrm{KB}_n(\mathcal{Y})$ for any proper subset $\mathcal{Y}$ of $\mathcal{X}$ (induction hypothesis). We can and do choose two proper subsets $\mathcal{X}_1, \mathcal{X}_2 \subset \mathcal{X}$ satisfying the following properties:

(a) $\mathcal{X} = \mathcal{X}_1 \cup \mathcal{X}_2$.

(b) Let $\mathcal{X}_0 = \mathcal{X}_1 \cap \mathcal{X}_2$. There is no edge in $\mathrm{V}\Gamma_n$ connecting an element of $\mathcal{X}_1 \setminus \mathcal{X}_0$ to an element of $\mathcal{X}_2 \setminus \mathcal{X}_0$.

It is easily seen from the presentations of the $\mathrm{KB}_n(\mathcal{X}_i)$ given in Step 2 that we have the amalgamated product

$$\mathrm{KB}_n(\mathcal{X}) = \mathrm{KB}_n(\mathcal{X}_1) *_{\mathrm{KB}_n(\mathcal{X}_0)} \mathrm{KB}_n(\mathcal{X}_2).$$

Our last algorithm is based on the following result. This is well known and can be found for instance in [Serre 1977, Chapitre 5.2].

Proposition 2.5. *Let $A_1 *_B A_2$ be an amalgamated product of groups. Let $g_1, \dots, g_l$ be a sequence of elements of $A_1 \sqcup A_2$ different from* 1 *and satisfying the following condition*:

if $g_i \in A_1$ (resp. $g_i \in A_2$) then $g_{i+1} \in A_2 \setminus B$ (resp. $g_{i+1} \in A_1 \setminus B$) for all $i \in \{1, \dots, l-1\}$.

Then $g_1 g_2 \cdots g_l$ is different from 1 *in $A_1 *_B A_2$.*

Algorithm D. Let u be a word over $\mathcal{X}^{\pm 1}$. We write u in the form $u_1 u_2 \cdots u_l$, where

- u_i is either a word over $\mathcal{X}_1^{\pm 1}$ or a word over $\mathcal{X}_2^{\pm 1}$,
- if u_i is a word over $\mathcal{X}_1^{\pm 1}$ (resp. over $\mathcal{X}_2^{\pm 1}$), then u_{i+1} is a word over $\mathcal{X}_2^{\pm 1}$ (resp. over $\mathcal{X}_1^{\pm 1}$).

We decide whether $\bar{u}$ is trivial by induction on l. Suppose that $l = 1$ and $u = u_1 \in \mathrm{KB}_n(\mathcal{X}_j)$ ($j \in \{1, 2\}$). Then we apply the solution to the word problem for $\mathrm{KB}_n(\mathcal{X}_j)$ to decide whether $\bar{u}$ is trivial or not. Suppose that $l \geq 2$. For all i we set $v_i = \pi_{\mathcal{X}_0}(u_i)$. If $\overline{u_i v_i^{-1}} \neq 1$ for all i, then $\bar{u} \neq 1$. Suppose there exists an integer $i \in \{1, \dots, l\}$ such that $\overline{u_i v_i^{-1}} = 1$. Let $u_i' = v_1 u_2$ if $i = 1$, $u_i' = u_{l-1} v_l$ if $i = l$, and $u_i' = u_{i-1} v_i u_{i+1}$ if $2 \leq i \leq l-1$. Set $v = u_1 \cdots u_{i-2} u_i' u_{i+2} \cdots u_l$. Then $\bar{u} = \bar{v}$ and, by induction, we can decide whether v represents 1 or not.

Example. In order to illustrate our solution to the word problem for KB_n, we turn now to give a more detailed and efficient version of the algorithm for the group KB_4. We start with the following observation:

Remark. For $\mathcal{X} \subset \mathcal{S}$, we denote by $\mathrm{V}\Gamma_n(\mathcal{X})$ the full subgraph of $\mathrm{V}\Gamma_n$ spanned by $\mathcal{X}$. Let $\mathcal{X}$, $\mathcal{Y}$ be two subsets of $\mathcal{S}$. Note that an injective morphism of Coxeter graphs $\mathrm{V}\Gamma_n(\mathcal{Y}) \hookrightarrow \mathrm{V}\Gamma_n(\mathcal{X})$ induces an injective homomorphism $\mathrm{KB}_n(\mathcal{Y}) \hookrightarrow \mathrm{KB}_n(\mathcal{X})$. So,

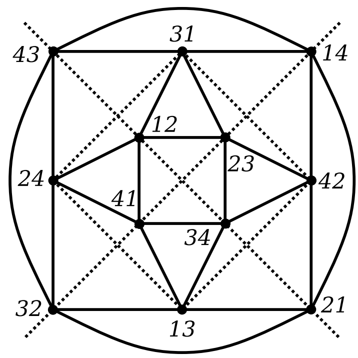


Figure 4. Coxeter graph $V\Gamma_4$.

if we had a solution to the word problem for $KB_n(\mathcal{X})$, then such a morphism would determine a solution to the word problem for $KB_n(\mathcal{Y})$.

The Coxeter graph $V\Gamma_4$ is depicted in Figure 4. Our convention in this figure is that a full edge is labeled by 3 and a dotted edge is labeled by 2. Note that there are two edges that go through "infinity", one connecting $\delta_{2,1}$ to $\delta_{4,3}$, and one connecting $\delta_{1,4}$ to $\delta_{3,2}$.

Consider the following subsets of $\mathcal{S}$:

$\mathcal{X}^{(1)}=\{\delta_{1,2},\delta_{2,3},\delta_{3,4},\delta_{4,1},\delta_{3,1}\}$, $\mathcal{X}_1^{(1)}=\{\delta_{1,2},\delta_{2,3},\delta_{3,4},\delta_{4,1}\}$, $\mathcal{X}_2^{(1)}=\{\delta_{1,2},\delta_{2,3},\delta_{3,1}\}$.

$\mathcal{X}^{(2)}=\mathcal{X}^{(1)}\cup\{\delta_{4,2}\}$, $\mathcal{X}_1^{(2)}=\mathcal{X}^{(1)}$, $\mathcal{X}_2^{(2)}=\{\delta_{4,2},\delta_{3,4},\delta_{2,3},\delta_{3,1}\}$.

$\mathcal{X}^{(3)}=\mathcal{X}^{(2)}\cup\{\delta_{1,3}\}$, $\mathcal{X}_1^{(3)}=\mathcal{X}^{(2)}$, $\mathcal{X}_2^{(3)}=\{\delta_{1,3},\delta_{4,1},\delta_{3,4},\delta_{4,2}\}$.

$\mathcal{X}^{(4)}=\mathcal{X}^{(3)}\cup\{\delta_{2,4}\}$, $\mathcal{X}_1^{(4)}=\mathcal{X}^{(3)}$, $\mathcal{X}_2^{(4)}=\{\delta_{2,4},\delta_{1,3},\delta_{4,1},\delta_{1,2},\delta_{3,1}\}$.

$\mathcal{X}^{(5)}=\mathcal{X}^{(4)}\cup\{\delta_{1,4}\}$, $\mathcal{X}_1^{(5)}=\mathcal{X}^{(4)}$, $\mathcal{X}_2^{(5)}=\{\delta_{1,4},\delta_{4,2},\delta_{2,3},\delta_{3,1}\}$.

$\mathcal{X}^{(6)}=\mathcal{X}^{(5)}\cup\{\delta_{2,1}\}$, $\mathcal{X}_1^{(6)}=\mathcal{X}^{(5)}$, $\mathcal{X}_2^{(6)}=\{\delta_{2,1},\delta_{1,3},\delta_{3,4},\delta_{4,2},\delta_{1,4}\}$.

$\mathcal{X}^{(7)}=\mathcal{X}^{(6)}\cup\{\delta_{3,2}\}$, $\mathcal{X}_1^{(7)}=\mathcal{X}^{(6)}$, $\mathcal{X}_2^{(7)}=\{\delta_{3,2},\delta_{2,4},\delta_{4,1},\delta_{1,3},\delta_{2,1},\delta_{1,4}\}$.

$\mathcal{X}^{(8)}=\mathcal{X}^{(7)}\cup\{\delta_{4,3}\}=\mathcal{S}$, $\mathcal{X}_1^{(8)}=\mathcal{X}^{(7)}$, $\mathcal{X}_2^{(8)}=\{\delta_{4,3},\delta_{3,2},\delta_{2,4},\delta_{1,2},\delta_{3,1},\delta_{1,4},\delta_{2,1}\}$.

Let $k\in\{1,\dots,8\}$. Note that $\mathcal{X}^{(k)}=\mathcal{X}_1^{(k)}\cup\mathcal{X}_2^{(k)}$. The Coxeter graph $V\Gamma_4(\mathcal{X}^{(k)})$ is depicted in Figure 5. In this figure the elements of $\mathcal{X}_1^{(k)}$ are represented by punctures, while the elements of $\mathcal{X}_2^{(k)}$ are represented by small circles.

We solve the word problem for $KB_4(\mathcal{X}^{(k)})$ successively for $k=1,2,\dots,8$, thanks to the following observations. Since $\mathcal{X}^{(8)}=\mathcal{S}$, this will provide a solution to the word problem for KB_4.

(1) Let $k\in\{1,\dots,8\}$. Set $\mathcal{X}_0^{(k)}=\mathcal{X}_1^{(k)}\cap\mathcal{X}_2^{(k)}$. Observe that there is no edge in $V\Gamma_4$ connecting an element of $\mathcal{X}_1^{(k)}\setminus\mathcal{X}_0^{(k)}$ to an element of $\mathcal{X}_2^{(k)}\setminus\mathcal{X}_0^{(k)}$.

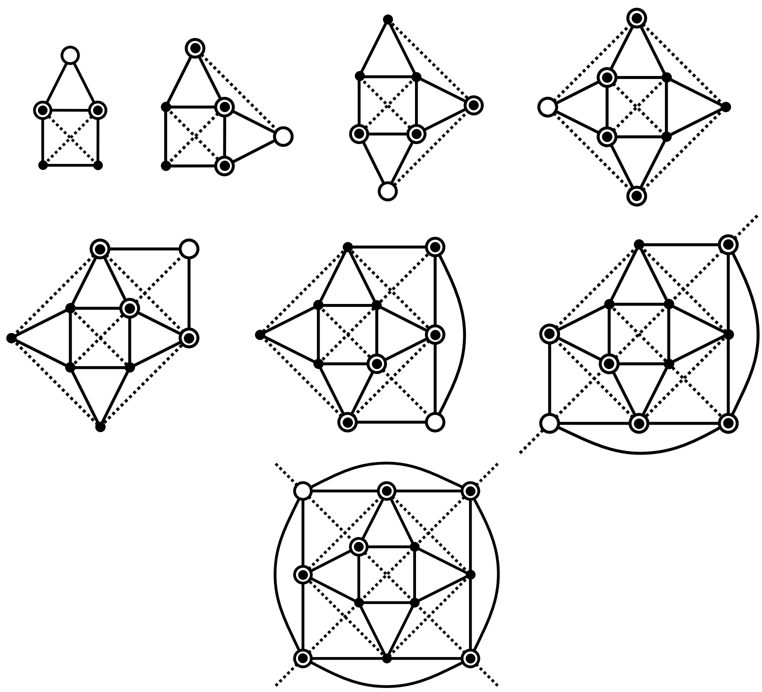


Figure 5. Coxeter graphs $\mathrm{V}\Gamma_4(\mathcal{X}^{(k)})$ for $k = 1, \dots, 8$ from left to right.

Hence, we can solve, using Algorithm D, the word problem for $\mathrm{KB}_4(\mathcal{X}^{(k)})$ from solutions to the word problem for $\mathrm{KB}_4(\mathcal{X}_1^{(k)})$ and for $\mathrm{KB}_4(\mathcal{X}_2^{(k)})$.

(2) The subsets $\mathcal{X}_1^{(1)}$ and $\mathcal{X}_2^{(1)}$ are full, hence we can solve the word problem for $\mathrm{KB}_4(\mathcal{X}_1^{(1)})$ and for $\mathrm{KB}_4(\mathcal{X}_2^{(1)})$ with Algorithm B.

(3) Let $k \geq 2$. On the one hand, we have $\mathcal{X}_1^{(k)} = \mathcal{X}^{(k-1)}$. On the other hand, it is easily seen that there is an injective morphism $\mathrm{V}\Gamma_4(\mathcal{X}_2^{(k)}) \hookrightarrow \mathrm{V}\Gamma_4(\mathcal{X}^{(k-1)})$. Hence, by the remark given at the beginning of the subsection, we can solve the word problem for $\mathrm{KB}_4(\mathcal{X}_1^{(k)})$ and for $\mathrm{KB}_4(\mathcal{X}_2^{(k)})$ from a solution to the word problem for $\mathrm{KB}_4(\mathcal{X}^{(k-1)})$.

3. Proof of Proposition 2.3

Recall that $F_n = F(x_1, \dots, x_n)$ denotes the free group of rank n freely generated by $x_1, \dots, x_n$, and that we have a representation $\varphi : \mathrm{KB}_n \to \mathrm{Aut}(F_n)$ which sends $\delta_{i,j}$ to $\varphi_{i,j}$, where

$$\varphi_{i,j}(x_i) = x_i x_j x_i^{-1}, \quad \varphi_{i,j}(x_j) = x_i \quad \text{and} \quad \varphi_{i,j}(x_k) = x_k \text{ for } k \notin \{i, j\}.$$

For $\mathcal{X} \subset \mathcal{S}$, we denote by $\varphi_{\mathcal{X}} : \mathrm{KB}_n(\mathcal{X}) \to \mathrm{Aut}(F_n)$ the restriction of φ to $\mathrm{KB}_n(\mathcal{X})$. In this section we prove that $\varphi_{\mathcal{X}}$ is faithful if $\mathcal{X}$ is a full subset of $\mathcal{S}$.

Consider the groups

$$B_n = \left\langle \sigma_1, \dots, \sigma_{n-1} \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } |i-j| = 1 \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } |i-j| \geq 2 \end{array} \right\rangle,$$

$$\tilde{B}_n = \left\langle \sigma_1, \dots, \sigma_n \;\middle|\; \begin{array}{ll} \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j & \text{if } i \equiv j \pm 1 \bmod n \\ \sigma_i\sigma_j = \sigma_j\sigma_i & \text{if } i \neq j \text{ and } i \not\equiv j \pm 1 \bmod n \end{array} \right\rangle, \quad n \geq 3.$$

The group B_n is the classical *braid group*, and $\tilde{B}_n$ is the *affine braid group*.

We define representations $\psi_n : B_n \to \mathrm{Aut}(F_n)$ and $\tilde{\psi}_n : \tilde{B}_n \to \mathrm{Aut}(F_n)$ in the same way as φ, as follows:

$$\psi_n(\sigma_i)(x_i) = x_i x_{i+1} x_i^{-1}, \quad \psi_n(\sigma_i)(x_{i+1}) = x_i, \quad \psi_n(\sigma_i)(x_k) = x_k \text{ if } k \notin \{i, i+1\},$$
$$\tilde{\psi}_n(\sigma_i) = \psi_n(\sigma_i) \text{ for } i < n,$$
$$\tilde{\psi}_n(\sigma_n)(x_n) = x_n x_1 x_n^{-1}, \quad \tilde{\psi}_n(\sigma_n)(x_1) = x_n, \quad \tilde{\psi}_n(\sigma_n)(x_k) = x_k \text{ if } k \notin \{1, n\},$$

The key of the proof of Proposition 2.3 is the following:

Theorem 3.1 [Artin 1947; Bellingeri and Bodin 2016]. *The representations $\psi_n : B_n \to \mathrm{Aut}(F_n)$ and $\tilde{\psi}_n : \tilde{B}_n \to \mathrm{Aut}(F_n)$ are faithful.*

The *support* of a generator $\delta_{i,j}$ is defined to be $\mathrm{supp}(\delta_{i,j}) = \{i, j\}$. The *support* of a subset $\mathcal{X}$ of $\mathcal{S}$ is $\mathrm{supp}(\mathcal{X}) = \bigcup_{s \in \mathcal{X}} \mathrm{supp}(s)$. We say that two subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{S}$ are *perpendicular*[1] if $\mathrm{supp}(\mathcal{X}_1) \cap \mathrm{supp}(\mathcal{X}_2) = \varnothing$. Note that this condition implies that $\mathcal{X}_1 \cap \mathcal{X}_2 = \varnothing$. More generally, we say that a family $\mathcal{X}_1, \dots, \mathcal{X}_l$ of subsets of $\mathcal{S}$ is *perpendicular* if $\mathrm{supp}(\mathcal{X}_i) \cap \mathrm{supp}(\mathcal{X}_j) = \varnothing$ for all $i \neq j$. In that case we write $\mathcal{X}_1 \cup \cdots \cup \mathcal{X}_l = \mathcal{X}_1 \boxplus \cdots \boxplus \mathcal{X}_l$. We say that a subset $\mathcal{X}$ of $\mathcal{S}$ is *indecomposable* if it is not the union of two perpendicular nonempty subsets. The next observations will be of importance in what follows.

Remarks. Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two perpendicular subsets of $\mathcal{S}$, and let $\mathcal{X} = \mathcal{X}_1 \boxplus \mathcal{X}_2$.

(1) $\mathcal{X}$ is a full subset if and only if $\mathcal{X}_1$ and $\mathcal{X}_2$ are both full subsets.

(2) $\mathrm{KB}_n(\mathcal{X}) = \mathrm{KB}_n(\mathcal{X}_1) \times \mathrm{KB}_n(\mathcal{X}_2)$.

Indeed, if $\delta_{i,j} \in \mathcal{X}_1$ and $\delta_{k,l} \in \mathcal{X}_2$, then i, j, k, l are distinct, and therefore $\delta_{i,j}$ and $\delta_{k,l}$ are connected by an edge labeled by 2, and $\delta_{i,j}\delta_{k,l} = \delta_{k,l}\delta_{i,j}$.

Lemma 3.2. *Let $\mathcal{X}_1$ and $\mathcal{X}_2$ be two perpendicular subsets of $\mathcal{S}$, and let $\mathcal{X} = \mathcal{X}_1 \boxplus \mathcal{X}_2$. Then $\varphi_{\mathcal{X}} : \mathrm{KB}_n(\mathcal{X}) \to \mathrm{Aut}(F_n)$ is faithful if and only if $\varphi_{\mathcal{X}_1} : \mathrm{KB}_n(\mathcal{X}_1) \to \mathrm{Aut}(F_n)$ and $\varphi_{\mathcal{X}_2} : \mathrm{KB}_n(\mathcal{X}_2) \to \mathrm{Aut}(F_n)$ are both faithful.*

[1] This terminology is derived from the theory of Coxeter groups.

Proof. For $X \subset \{x_1, \dots, x_n\}$, we denote by $F(X)$ the subgroup of F_n generated by X. There is a natural embedding $\iota_X : \mathrm{Aut}(F(X)) \hookrightarrow \mathrm{Aut}(F_n)$ defined by

$$\iota_X(\alpha)(x_i) = \begin{cases} \alpha(x_i) & \text{if } x_i \in X, \\ x_i & \text{otherwise.} \end{cases}$$

Moreover, if X_1 and X_2 are disjoint subsets of $\{x_1, \dots, x_n\}$, then the homomorphism

$$\begin{aligned} (\iota_{X_1} \times \iota_{X_2}) : \mathrm{Aut}(F(X_1)) \times \mathrm{Aut}(F(X_2)) &\to \mathrm{Aut}(F_n), \\ (\alpha_1, \alpha_2) &\mapsto \iota_{X_1}(\alpha_1)\, \iota_{X_2}(\alpha_2), \end{aligned}$$

is well-defined and injective. From now on, we will assume $\mathrm{Aut}(F(X))$ to be embedded in $\mathrm{Aut}(F_n)$ via ι_X for all $X \subset \{x_1, \dots, x_n\}$.

By an abuse of notation, for $\mathcal{X} \subset \mathcal{S}$ we will also denote by $\mathrm{supp}(\mathcal{X})$ the set $\{x_i \mid i \in \mathrm{supp}(\mathcal{X})\}$. Set $X_1 = \mathrm{supp}(\mathcal{X}_1)$ and $X_2 = \mathrm{supp}(\mathcal{X}_2)$. We have $\mathrm{Im}(\varphi_{\mathcal{X}_i}) \subset \mathrm{Aut}(F(X_i))$ for $i = 1, 2$, $X_1 \cap X_2 = \varnothing$, and $\mathrm{KB}_n(\mathcal{X}) = \mathrm{KB}_n(\mathcal{X}_1) \times \mathrm{KB}_n(\mathcal{X}_2)$. Hence, Lemma 3.2 follows from the following claim, whose proof is left to the reader:

Let $f_1 : G_1 \to H_1$ and $f_2 : G_2 \to H_2$ be two group homomorphisms. Let $(f_1 \times f_2) : (G_1 \times G_2) \to (H_1 \times H_2)$ be the homomorphism defined by $(f_1 \times f_2)(u_1, u_2) = (f_1(u_1), f_2(u_2))$. Then $(f_1 \times f_2)$ is injective if and only if f_1 and f_2 are both injective. $\square$

For $2 \le m \le n$ we set

$$\mathcal{Z}_m = \{\delta_{1,2}, \dots, \delta_{m-1,m}\}, \quad \tilde{\mathcal{Z}}_m = \{\delta_{1,2}, \dots, \delta_{m-1,m}, \delta_{m,1}\}.$$

Note that the map $\{\sigma_1, \dots, \sigma_{m-1}\} \to \mathcal{Z}_m$, $\sigma_i \mapsto \delta_{i,i+1}$, induces an isomorphism $f_m : B_m \to \mathrm{KB}_n(\mathcal{Z}_m)$. This follows from the presentation of $\mathrm{KB}_n(\mathcal{Z}_m)$ given in Step 2 of Section 2. Similarly, for $m \ge 3$ the map $\{\sigma_1, \dots, \sigma_m\} \to \tilde{\mathcal{Z}}_m$, $\sigma_i \mapsto \delta_{i,i+1}$ for $1 \le i \le m-1$, $\sigma_m \mapsto \delta_{m,1}$, induces an isomorphism $\tilde{f}_m : \tilde{B}_m \to \mathrm{KB}_n(\tilde{\mathcal{Z}}_m)$.

Recall that the symmetric group $\mathfrak{S}_n$ acts on $\mathcal{S}$ by $g\delta_{i,j} = \delta_{g(i),g(j)}$, and that this action induces an action of $\mathfrak{S}_n$ on KB_n. On the other hand, there is a natural embedding $\mathfrak{S}_n \hookrightarrow \mathrm{Aut}(F_n)$, where $g \in \mathfrak{S}_n$ sends x_i to $x_{g(i)}$ for all $i \in \{1, \dots, n\}$, and this embedding induces by conjugation an action of $\mathfrak{S}_n$ on $\mathrm{Aut}(F_n)$. It is easily seen that the homomorphism $\varphi : \mathrm{KB}_n \to \mathrm{Aut}(F_n)$ is equivariant under these actions of $\mathfrak{S}_n$.

Lemma 3.3. *If $\mathcal{X}$ is a full and indecomposable nonempty subset of $\mathcal{S}$, then there exist $g \in \mathfrak{S}_n$ and $m \in \{2, \dots, n\}$ such that either $\mathcal{X} = g\mathcal{Z}_m$, or $\mathcal{X} = g\tilde{\mathcal{Z}}_m$ and $m \ge 3$.*

Proof. An *oriented graph* Υ is the data of two sets, $V(\Upsilon)$, called the *set of vertices*, and $E(\Upsilon)$, called the *set of arrows*, together with two maps $\mathrm{sou}, \mathrm{tar} : E(\Upsilon) \to V(\Upsilon)$. We associate an oriented graph $\Upsilon_{\mathcal{X}}$ to any subset $\mathcal{X}$ of $\mathcal{S}$ as follows. The set of vertices is $V(\Upsilon_{\mathcal{X}}) = \mathrm{supp}(\mathcal{X})$, the set of arrows is $E(\Upsilon_{\mathcal{X}}) = \mathcal{X}$, and, for $\delta_{i,j} \in \mathcal{X}$, we set $\mathrm{sou}(\delta_{i,j}) = i$ and $\mathrm{tar}(\delta_{i,j}) = j$. Assume that $\mathcal{X}$ is a full and indecomposable

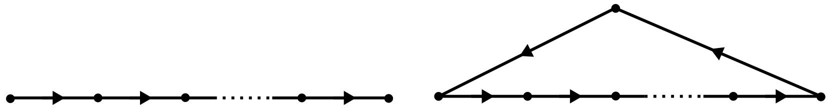


Figure 6. Oriented segment and oriented cycle.

nonempty subset of $\mathcal{S}$. Since $\mathcal{X}$ is indecomposable, $\Upsilon_{\mathcal{X}}$ must be connected. Since $\mathcal{X}$ is full, if $s, t \in \mathcal{X}$ are two different arrows of $\Upsilon_{\mathcal{X}}$ with a common vertex, then there exist $i, j, k \in \{1, \dots, n\}$ distinct such that either $s = \delta_{j,i}$ and $t = \delta_{i,k}$, or $s = \delta_{i,j}$ and $t = \delta_{k,i}$. This implies that $\Upsilon_{\mathcal{X}}$ is either an oriented segment, or an oriented cycle with at least 3 vertices (see Figure 6). If $\Upsilon_{\mathcal{X}}$ is an oriented segment, then there exist $g \in \mathfrak{S}_n$ and $m \in \{2, \dots, n\}$ such that $\mathcal{X} = g\mathcal{Z}_m$. If $\Upsilon_{\mathcal{X}}$ is an oriented cycle, then there exist $g \in \mathfrak{S}_n$ and $m \in \{3, \dots, n\}$, such that $\mathcal{X} = g\tilde{\mathcal{Z}}_m$. □

Proof of Proposition 2.3. Let $\mathcal{X}$ be a full nonempty subset of $\mathcal{S}$. Write $\mathcal{X} = \mathcal{X}_1 \boxplus \cdots \boxplus \mathcal{X}_l$, where $\mathcal{X}_j$ is an indecomposable nonempty subset. As observed above, each $\mathcal{X}_j$ is also a full subset. Moreover, by Lemma 3.2, in order to show that $\varphi_{\mathcal{X}}$ is faithful, it suffices to show that $\varphi_{\mathcal{X}_j}$ is faithful for all $j \in \{1, \dots, l\}$. So, we can assume that $\mathcal{X}$ is a full and indecomposable nonempty subset of $\mathcal{S}$. By Lemma 3.3, there exist $g \in \mathfrak{S}_n$ and $m \in \{2, \dots, n\}$ such that either $\mathcal{X} = g\mathcal{Z}_m$, or $\mathcal{X} = g\tilde{\mathcal{Z}}_m$ and $m \geq 3$. Since φ is equivariant under the actions of $\mathfrak{S}_n$, upon conjugating by g^{-1} we can assume that either $\mathcal{X} = \mathcal{Z}_m$ or $\mathcal{X} = \tilde{\mathcal{Z}}_m$. Set $Z_m = \{x_1, \dots, x_m\} = \operatorname{supp}(\mathcal{Z}_m) = \operatorname{supp}(\tilde{\mathcal{Z}}_m)$, and identify F_m with $F(Z_m)$. Then $\varphi_{\mathcal{Z}_m} = \psi_m \circ f_m^{-1}$ and $\varphi_{\tilde{\mathcal{Z}}_m} = \tilde{\psi}_m \circ \tilde{f}_m^{-1}$, hence $\varphi_{\mathcal{X}}$ is faithful by Theorem 3.1. □

4. Proof of Proposition 2.4

The proof of Proposition 2.4 is based on some general results on Coxeter groups and Artin groups. Recall that the definitions of Coxeter diagram, Artin group and Coxeter group are given at the beginning of Step 2 in Section 2. Recall also that, if Y is a subset of the set S of vertices of Γ, then Γ_Y denotes the full subdiagram spanned by Y, A_Y denotes the subgroup of $A = A(\Gamma)$ generated by Y, and W_Y denotes the subgroup of $W = W(\Gamma)$ generated by Y.

Let Γ be a Coxeter diagram, let S be its set of vertices, let A be the Artin group of Γ, and let W be its Coxeter group. Since we have $s^2 = 1$ in W for all $s \in S$, every element g in W can be represented by a word over S. Such a word is called an *expression* of g. The minimal length of an expression of g is called the *length* of g and is denoted by $\lg(g)$. An expression of g of length $\lg(g)$ is a *reduced expression* of g. Let Y be a subset of S, and let $g \in W$. We say that g is *Y-minimal* if it is of minimal length among the elements of the coset $W_Y g$. The first ingredient in our proof of Proposition 2.4 is the following:

Proposition 4.1 [Bourbaki 1968, Chapitre IV, Exercice 3]. *Let $Y \subset S$ and let $g \in W$. There exists a unique Y-minimal element lying in the coset $W_Y g$. Moreover, the following conditions are equivalent*:

(a) *g is Y-minimal.*

(b) $\lg(sg) > \lg(g)$ *for all $s \in Y$.*

(c) $\lg(hg) = \lg(h) + \lg(g)$ *for all $h \in W_Y$.*

Remark. For $g \in W$ and $s \in S$, we always have either $\lg(sg) = \lg(g) + 1$, or $\lg(sg) = \lg(g) - 1$. This is a standard fact on Coxeter groups that can be found for instance in [Bourbaki 1968]. So, the inequality $\lg(sg) > \lg(g)$ means $\lg(sg) = \lg(g) + 1$ and the inequality $\lg(sg) \le \lg(g)$ means $\lg(sg) = \lg(g) - 1$.

Let u be a word over S.

- Suppose that u is written in the form $u_1 ss u_2$, where u_1, u_2 are words over S and s is an element of S. Then we say that $u' = u_1 u_2$ is obtained from u by an *M-operation of type* I.
- Suppose that u is written in the form $u = u_1 \langle s, t\rangle^{m_{s,t}} u_2$, where u_1, u_2 are words over S and s, t are two elements of S connected by an edge labeled by $m_{s,t}$. Then we say that $u' = u_1 \langle t, s\rangle^{m_{s,t}} u_2$ is obtained from u by an *M-operation of type* II.

We say that a word u is *M-reduced* if its length cannot be shortened by M-operations of types I or II. The second ingredient in our proof is the following.

Theorem 4.2 [Tits 1969]. *Let $g \in W$.*

(1) *An expression w of g is a reduced expression if and only if w is M-reduced.*

(2) *Any two reduced expressions w and w' of g are connected by a finite sequence of M-operations of type* II.

Let Y be a subset of S. The third ingredient is a set retraction $\rho_Y : A \to A_Y$ to the inclusion map $\iota_Y : A_Y \to A$, constructed in [Godelle and Paris 2012; Charney and Paris 2014]. This is defined as follows. Let α be an element of A.

- Choose a word $\hat{\alpha} = s_1{}^{\varepsilon_1} \cdots s_l{}^{\varepsilon_l}$ over $S^{\pm 1}$ which represents α.
- Let $i \in \{0, 1, \dots, l\}$. Set $g_i = s_1 s_2 \cdots s_i \in W$, and write g_i in the form $g_i = h_i k_i$, where $h_i \in W_Y$ and k_i is Y-minimal.
- Let $i \in \{1, \dots, l\}$. If $\varepsilon_i = 1$, set $z_i = k_{i-1} s_i k_{i-1}^{-1}$. If $\varepsilon_i = -1$, set $z_i = k_i s_i k_i^{-1}$.
- Let $i \in \{1, \dots, l\}$. We set $T_i = z_i^{\varepsilon_i}$ if $z_i \in Y$. Otherwise we set $T_i = 1$.
- Set $\hat{\rho}_Y(\alpha) = T_1 T_2 \cdots T_l$.

Proposition 4.3 [Godelle and Paris 2012; Charney and Paris 2014]. *Let $\alpha \in A$. The element $\rho_Y(\alpha) \in A_Y$ represented by the word $\hat{\rho}_Y(\alpha)$ defined above does not depend on the choice of the representative $\hat{\alpha}$ of α. Furthermore, the map $\rho_Y : A \to A_Y$ is a set retraction to the inclusion map $\iota_Y : A_Y \hookrightarrow A$.*

We turn now to apply these three ingredients to our group KB_n and prove Proposition 2.4. Let KW_n denote the quotient of KB_n by the relations $\delta_{i,j}^2 = 1$, $1 \le i \ne j \le n$. Note that KW_n is the Coxeter group of the Coxeter diagram $\mathrm{V}\Gamma_n$. For $\mathcal{Y} \subset \mathcal{X}$, we denote by $\mathrm{KW}_n(\mathcal{Y})$ the subgroup of KW_n generated by $\mathcal{Y}$.

Lemma 4.4. *Let $g \in \mathrm{KW}_n$.*

(1) *An expression w of g is a reduced expression if and only if w is M-reduced.*

(2) *Any two reduced expressions w and w' of g are connected by a finite sequence of M-operations of types $\mathrm{II}^{(2)}$ and $\mathrm{II}^{(3)}$.*

(3) *Let $\mathcal{Y}$ be a subset of $\mathcal{S}$, and let w be a reduced expression of g. Then g is $\mathcal{Y}$-minimal (in the sense given above) if and only if w is $M_{\mathcal{Y}}$-reduced.*

Proof. Parts (1) and (2) are Theorem 4.2 applied to KW_n. So, we only need to prove (3).

Suppose that g is not $\mathcal{Y}$-minimal. By Proposition 4.1, there exists $s \in \mathcal{Y}$ such that $\lg(sg) \le \lg(g)$, that is, $\lg(sg) = \lg(g) - 1$. Let w' be a reduced expression of sg. The word sw' is an expression of g and $\lg(sw') = \lg(w) = \lg(g)$, hence sw' is a reduced expression of g. By Theorem 4.2, w and sw' are connected by a finite sequence of M-operations of types $\mathrm{II}^{(2)}$ and $\mathrm{II}^{(3)}$. On the other hand, w' is obtained from sw' by an M-operation of type $\mathrm{III}_{\mathcal{Y}}$. So, w' is obtained from w by M-operations of types I, $\mathrm{II}^{(2)}$, $\mathrm{II}^{(3)}$ and $\mathrm{III}_{\mathcal{Y}}$, and we have $\lg(w') < \lg(w)$, hence w is not $M_{\mathcal{Y}}$-reduced.

Suppose that w is not $M_{\mathcal{Y}}$-reduced. Let w' be an $M_{\mathcal{Y}}$-reduction of w, and let g' be the element of KW_n represented by w'. Since w' is an $M_{\mathcal{Y}}$-reduction of w, the element g' lies in the coset $\mathrm{KW}_n(\mathcal{Y})\, g$. Moreover, $\lg(g') = \lg(w') < \lg(w) = \lg(g)$, hence g is not $\mathcal{Y}$-minimal. □

Proof of Proposition 2.4. Let $\mathcal{Y}$ be a subset of $\mathcal{S}$. Consider the retraction $\rho_{\mathcal{Y}} : \mathrm{KB}_n \to \mathrm{KB}_n(\mathcal{Y})$ constructed in Proposition 4.3. We shall prove that, if u is a word over $\mathcal{S}^{\pm 1}$, then $\overline{\pi_{\mathcal{Y}}(u)} = \rho_{\mathcal{Y}}(\bar{u})$. This will prove Proposition 2.4. Indeed, if $\bar{u} = \bar{v}$, then $\overline{\pi_{\mathcal{Y}}(u)} = \rho_{\mathcal{Y}}(\bar{u}) = \rho_{\mathcal{Y}}(\bar{v}) = \overline{\pi_{\mathcal{Y}}(v)}$. Moreover, since $\rho_{\mathcal{Y}} : \mathrm{KB}_n \to \mathrm{KB}_n(\mathcal{Y})$ is a retraction to the inclusion map $\mathrm{KB}_n(\mathcal{Y}) \hookrightarrow \mathrm{KB}_n$, we have $\rho_{\mathcal{Y}}(\bar{u}) = \bar{u}$ if and only if $\bar{u} \in \mathrm{KB}_n(\mathcal{Y})$, hence $\overline{\pi_{\mathcal{Y}}(u)} = \bar{u}$ if and only if $\bar{u} \in \mathrm{KB}_n(\mathcal{Y})$.

Let $u = s_1^{\varepsilon_1} \cdots s_l^{\varepsilon_l}$ be a word over $\mathcal{S}^{\pm 1}$. Let α be the element of KB_n represented by u.

- For $i \in \{0, 1, \dots, l\}$, we set $u_i^+ = s_1 \cdots s_i$, and we denote by g_i the element of KW_n represented by u_i^+.

- Let $i \in \{0, 1, \dots, l\}$. We write $g_i = h_i k_i$, where $h_i \in \mathrm{KW}_n(\mathcal{Y})$, and k_i is $\mathcal{Y}$-minimal. Let v_i^+ be an $M_{\mathcal{Y}}$-reduction of u_i^+. Then, by Lemma 4.4, v_i^+ is a reduced expression of k_i.
- Let $i \in \{1, \dots, l\}$. If $\varepsilon_i = 1$, we set $z_i = k_{i-1} s_i k_{i-1}^{-1}$ and $w_i^+ = v_{i-1}^+ \cdot s_i \cdot \mathrm{op}(v_{i-1}^+)$. If $\varepsilon_i = -1$, we set $z_i = k_i s_i k_i^{-1}$ and $w_i^+ = v_i^+ \cdot s_i \cdot \mathrm{op}(v_i^+)$. Note that w_i^+ is an expression of z_i.
- Let $i \in \{1, \dots, l\}$. Let r_i be an M-reduction of w_i^+. By Lemma 4.4, r_i is a reduced expression of z_i. Note that we have $z_i \in \mathcal{Y}$ if and only if r_i is of length 1 and $r_i \in \mathcal{Y}$.
- Let $i \in \{1, \dots, l\}$. If r_i is of length 1 and $r_i \in \mathcal{Y}$, we set $T_i = r_i^{\varepsilon_i}$. Otherwise we set $T_i = 1$.
- By construction, we have $\hat{\rho}_{\mathcal{Y}}(\alpha) = \pi_{\mathcal{Y}}(u) = T_1 T_2 \cdots T_l$. □

Acknowledgment

The research of Bellingeri was partially supported by the French government grant ANR-11-JS01-002-01.

References

[Artin 1947] E. Artin, "Theory of braids", *Ann. of Math.* (2) **48** (1947), 101–126. MR 0019087 Zbl 0030.17703

[Bar-Natan and Dancso 2015] D. Bar-Natan and Z. Dancso, "Finite type invariants of w-knotted objects, I: w-knots and the Alexander polynomial", preprint, 2015. arXiv 1405.1956

[Bardakov 2005] V. G. Bardakov, "Virtual and welded links and their invariants", *Sib. Èlektron. Mat. Izv.* **2** (2005), 196–199. MR 2177995 Zbl 1100.57006

[Bardakov and Bellingeri 2009] V. G. Bardakov and P. Bellingeri, "Combinatorial properties of virtual braids", *Topology Appl.* **156**:6 (2009), 1071–1082. MR 2493369

[Bellingeri and Bodin 2016] P. Bellingeri and A. Bodin, "The braid group of a necklace", *Math. Z.* (online publication February 2016).

[Bourbaki 1968] N. Bourbaki, *Éléments de mathématique, Fasc. XXXIV: Groupes et algèbres de Lie. Chapitre IV: Groupes de Coxeter et systèmes de Tits. Chapitre V: Groupes engendrés par des réflexions. Chapitre VI: Systèmes de racines*, Actualités Scientifiques et Industrielles **1337**, Hermann, Paris, 1968. MR 0240238 Zbl 0186.33001

[Charney and Paris 2014] R. Charney and L. Paris, "Convexity of parabolic subgroups in Artin groups", *Bull. Lond. Math. Soc.* **46**:6 (2014), 1248–1255. MR 3291260 Zbl 1308.20037

[Chterental 2015] O. Chterental, "Virtual braids and virtual curve diagrams", *J. Knot Theory Ramifications* **24**:13 (2015), art. ID # 1541001. MR 3434540 Zbl 06532326

[Cisneros de la Cruz 2015] B. A. Cisneros de la Cruz, "Virtual braids from a topological viewpoint", *J. Knot Theory Ramifications* **24**:6 (2015), art. ID # 1550033. MR 3358444 Zbl 1327.57004

[Godelle and Paris 2012] E. Godelle and L. Paris, "$K(\pi, 1)$ and word problems for infinite type Artin–Tits groups, and applications to virtual braid groups", *Math. Z.* **272**:3-4 (2012), 1339–1364. MR 2995171 Zbl 1300.20045

[Kamada 2007] S. Kamada, "Braid presentation of virtual knots and welded knots", *Osaka J. Math.* **44**:2 (2007), 441–458. MR 2351010 Zbl 1147.57008

[Kauffman 1999] L. H. Kauffman, "Virtual knot theory", *European J. Combin.* **20**:7 (1999), 663–690. MR 1721925 Zbl 0938.57006

[van der Lek 1983] H. van der Lek, *The homotopy type of complex hyperplane complements*, Ph.D. thesis, Radboud University Nijmegen, 1983.

[Manturov 2003] V. O. Manturov, "On the recognition of virtual braids", *Zap. Nauchn. Sem. S.-Peterburg. Otdel. Mat. Inst. Steklov.* (*POMI*) **299**:Geom. i Topol. 8 (2003), 267–286, 331–332. In Russian; translated in *J. Math. Sci.* **131**:1 (2005), 5409–5419. MR 2038828 Zbl 1144.20310

[Serre 1977] J.-P. Serre, *Arbres, amalgames,* SL_2, Astérisque **46**, Société Mathématique de France, Paris, 1977. MR 0476875 Zbl 0369.20013

[Tits 1969] J. Tits, "Le problème des mots dans les groupes de Coxeter", pp. 175–185 in *Symposia Mathematica* (Rome, 1967/68), vol. 1, Academic Press, London, 1969. MR 0254129 Zbl 0206.03002

[Vershinin 2001] V. V. Vershinin, "On homology of virtual braids and Burau representation", *J. Knot Theory Ramifications* **10**:5 (2001), 795–812. MR 1839703 Zbl 0997.57020

Received June 17, 2015. Revised February 18, 2016.

PAOLO BELLINGERI
LABORATOIRE NICOLAS ORESME, ESPE DE BASSE NORMANDIE
UNIVERSITÉ DE CAEN
14032 CAEN
FRANCE
paolo.bellingeri@unicaen.fr

BRUNO A. CISNEROS DE LA CRUZ
INSTITUTO DE MATEMÁTICAS DE LA UNAM
OAXACA DE JUÁREZ
OAXACA 68000
MEXICO
brunoc@matem.unam.mx

LUIS PARIS
IMB UMR5584, CNRS
UNIV. BOURGOGNE FRANCHE-COMTÉ
21000 DIJON
FRANCE
lparis@u-bourgogne.fr

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.289

COMPLETELY CONTRACTIVE PROJECTIONS ON OPERATOR ALGEBRAS

DAVID P. BLECHER AND MATTHEW NEAL

The main goal of this paper is to find operator algebra variants of certain deep results of Størmer, Friedman and Russo, Choi and Effros, Effros and Størmer, Robertson and Youngson, Youngson, and others, concerning projections on C^*-algebras and their ranges. In particular, we investigate the "bicontractive projection problem" and related questions in the category of operator algebras. To do this, we will add the ingredient of "real positivity" from recent papers of the first author with Read.

1. Introduction

In previous papers (listed in the bibliography) both authors separately studied projections (that is, idempotent linear maps) and conditional expectations on unital operator algebras (that is, closed algebras of operators on a Hilbert space that contain the identity operator) and other classes of Banach spaces. Results were proved such as Corollary 4.2.9 in [Blecher and Le Merdy 2004]: A completely contractive projection P on such an algebra A which is unital (that is, $P(1) = 1$) and whose range is a subalgebra is a *conditional expectation* (that is, $P(P(a)bP(c)) = P(a)P(b)P(c)$ for $a, b, c \in A$). This is an analogue of the matching theorem due to Tomiyama for C^*-algebras.

The main goal of our paper is to find variants, valid for operator algebras which are unital or which have an approximate identity, of certain deeper results in the C^*-algebra case found in [Størmer 1982; Friedman and Russo 1984; 1985; Effros and Størmer 1979; Robertson and Youngson 1982], and elsewhere, concerning projections and their ranges. In particular we wish to investigate the "bicontractive projection problem" and related problems (such as the "symmetric projection problem" and the "contractive projection problem") in the category of operator algebras. To do this, we will add the ingredient of *real positivity* from recent papers

Blecher was partially supported by a grant from the National Science Foundation. Neal was supported by Denison University.
MSC2010: primary 46L05, 46L07, 47L07, 47L30; secondary 46H10, 46B40, 46L30.
Keywords: operator algebra, completely contractive map, projection, conditional expectation, bicontractive projection, real positive, noncommutative Banach–Stone theorem.

of Blecher and Read [2011; 2013; 2014]; see also [Blecher and Neal 2012a; 2012b; Bearden et al. 2014; Blecher and Ozawa 2015; Blecher 2015]. A key idea in those papers is that real positivity is often the right replacement in general algebras for positivity in C^*-algebras. This will be our guiding principle here too.

We now discuss the structure of our paper. In Section 2 we discuss completely contractive projections on operator algebras. A well-known and lovely result of Choi and Effros [1977, Theorem 3.1] (or rather, its proof) shows that the range of a completely positive projection $P : B \to B$ on a C^*-algebra B, is again a C^*-algebra with product $P(xy)$. A quite deep theorem of Friedman and Russo [1985], or a simpler variant of it by Youngson [1983], shows that something similar is true if P is simply contractive, or if B is replaced by a ternary ring of operators. The analogous result for unital completely contractive projections on unital operator algebras is true too, and is implicit in the proof of the result quoted in the first paragraph. However, there seems to be no analogous result for (nonunital) completely contractive projections on nonunital operator algebras without adding extra hypotheses on P. The "guiding principle" in the previous paragraph suggests to add the condition that P is also "real completely positive" (we define this below). Then the question does make good sense, and we are able to prove the desired result. Thus the range of a real completely positive completely contractive projection $P : A \to A$ on an operator algebra with approximate identity is again an operator algebra with product $P(xy)$. We also have a converse and several complements to this result in Section 2, as well as some other facts about completely contractive projections, such as how one is often able to reduce the problem to algebras which have an identity. We also show that for algebras with no kind of approximate identity, there is a biggest "nice part" on which completely contractive projections (and the other classes of projections discussed below) work well.

In Sections 3, 4, and 5, we turn from the "contractive projection problem" to the "bicontractive projection problem" and related questions. A projection P is *bicontractive* if both P and $I - P$ are contractive. By the bicontractive projection problem for a Banach space X, one usually means the characterization of all bicontractive projections $P : X \to X$, or the characterization of the ranges of the bicontractive projections (or both). On a unital C^*-algebra B it is known, by work of some of the authors mentioned above, that the unital bicontractive projections are precisely $\frac{1}{2}(I + \theta)$, for a period-2 $*$-automorphism $\theta : B \to B$. The possibly nonunital bicontractive projections P on B are of a similar form, and indeed if P is also positive then $q = P(1)$ is a central projection in the multiplier algebra $M(B)$ with respect to which P decomposes into a direct sum of 0 and a projection of the above form $\frac{1}{2}(I + \theta)$, for a period-2 $*$-automorphism θ of qB. (See Theorem 3.2 for the idea of the proof of this.) Conversely, note that a map P of the latter form is automatically *completely bicontractive* (that is, is bicontractive at each matrix level), indeed is *completely symmetric* (that is, $I - 2P$ is completely contractive), and the range of P is a C^*-subalgebra, and P is a conditional expectation.

One may ask: what from the last paragraph is true for general operator algebras? Again the guiding principle referred to earlier leads us to use in place of the positivity in that result, the real positivity in the sense of [Blecher and Read 2013; 2014]. The next thing to note is that now "completely bicontractive" is no longer the same as "completely symmetric" for projections. The "completely symmetric" case works beautifully, and the solution to our "symmetric projection problem" is presented in Theorem 3.7. This result is one somewhat satisfactory generalization (we shall see others later) to operator algebras of the C^*-algebraic theorem in the last paragraph. For the more general class of completely bicontractive projections, as seems to be often the case in generalizing C^*-algebraic theory to more general algebras, a first look is disappointing. Indeed most of the last paragraph no longer works in general. One does not always get an associated completely isometric automorphism θ with $P = \frac{1}{2}(I + \theta)$, and $q = P(1)$ need not be a central projection. Indeed we have solved here and elsewhere (see, e.g., [Blecher and Labuschagne 2003; Blecher 2004; Blecher and Magajna 2005a, p. 92–93; 2005b]) many of the obvious questions about contractive projections, completely contractive projections, and conditional expectations, on operator algebras. Unfortunately many of the answers are counterexamples. However, as also seems to be often the case, a closer look at examples reveals an interesting question. Namely, given a real completely positive projection $P : A \to A$ which is completely bicontractive, when is the range of P an (approximately unital) subalgebra of A, so that P is a conditional expectation? For operator algebras we consider this to be the correct version of the bicontractive projection problem. In Sections 3 and 4 we elucidate this question. We remark that in [Blecher et al. $\geq$ 2016] we study the "Jordan algebra" variants of many of the results in Sections 2 and 3 of the present paper.

In Section 4 we discuss the completely bicontractive projection problem, construct some interesting examples, and give some reasonable conditions under which $P(A)$ is a subalgebra and P is a conditional expectation. In particular we solve in full generality our version of the bicontractive projection problem for uniform algebras (that is, closed subalgebras of $C(K)$), and indeed for any algebra satisfying a condition related to semisimplicity. Theorem 4.9 is one of the main results of the paper, giving a very general condition for $P(A)$ being a subalgebra in terms of certain support projections. In fact, at the time of writing, for all we know the condition in Theorem 4.9 is necessary and sufficient; at least we have no examples to the contrary. In Section 5 we discuss another condition that completely bicontractive projections may satisfy, and examine some consequences of this. In Section 6 we discuss Jordan homomorphisms on operator algebras, and among other things, solve two natural "completely isometric problems", for Jordan subalgebras of operator algebras and for operator algebras, related to the noncommutative Banach–Stone theorem.

We now turn to precise definitions and notation. Any unexplained terms below can probably be found in [Blecher and Le Merdy 2004; Paulsen 2002; Blecher and Read 2011; 2013] or any of the other books on operator spaces. All vector spaces are over the complex field $\mathbb{C}$. The letters K, H denote Hilbert spaces. If X is an operator space we often write X_+ for the elements in X which are positive (i.e., at least 0) in the usual C^*-algebraic sense. We write $\mathrm{Ball}(X)$ for the set $\{x \in X : \|x\| \leq 1\}$. By an *operator algebra* we mean a not necessarily selfadjoint closed subalgebra of $B(H)$, the bounded operators on a Hilbert space H. We write $C^*(A)$ for the C^*-algebra generated by A, that is, the smallest C^*-subalgebra containing A. A *unital operator space* is a subspace X of $B(H)$ or a unital C^*-algebra containing the identity (operator). We often write this identity as 1_X. A map $T : X \to Y$ is *unital* if $T(1_X) = 1_Y$. We say that an algebra is *approximately unital* if it has a contractive approximate identity (cai). For us a *projection in* an operator algebra A is always an orthogonal projection lying in A, whereas a *projection on* A is a linear idempotent map $P : A \to A$. If A is a nonunital operator algebra represented (completely) isometrically on a Hilbert space H then one may identify the unitization A^1 with $A + \mathbb{C}I_H$. The second dual A^{**} is also an operator algebra with its (unique) Arens product; this is also the product inherited from the von Neumann algebra B^{**} if A is a subalgebra of a C^*-algebra B. Note that A has a cai if and only if A^{**} has an identity $1_{A^{**}}$ of norm-1, and then A^1 is sometimes identified with $A + \mathbb{C}1_{A^{**}}$. The multiplier algebra $M(A)$ of such A may be taken to be the *idealizer* of A in A^{**}, that is, $\{\eta \in A^{**} : \eta A + A\eta \subset A\}$.

If A is an approximately unital operator algebra or unital operator space then $I(A)$ denotes the injective envelope, an injective unital C^*-algebra containing A. It contains A as a subalgebra if A is approximately unital [Blecher and Le Merdy 2004, Corollary 4.2.8]. For us the most important properties of $I(A)$ are: first, that it is injective in the category of operator spaces, so that any completely contractive map from a subspace of an operator space Y into $I(A)$ extends to a complete contraction from Y to $I(A)$. Second, $I(A)$ is *rigid*, so that the identity map on $I(A)$ is the only complete contraction $I(A) \to I(A)$ extending the identity map on A. The *C^*-envelope* $C^*_e(A)$ is the C^*-subalgebra of $I(A)$ generated by A. If A is unital it has the property that given any unital complete isometry $T : A \to B(K)$, there exists a unique $*$-homomorphism $\pi : C^*(A) \to C^*_e(A)$ with $\pi \circ T$ equal to the inclusion map of A in $C^*_e(A)$.

We recall that a contractive completely positive map on a C^*-algebra is completely contractive. A unital linear map between operator systems is positive and $*$-linear if it is contractive; and it is completely positive if and only if it is completely contractive. See, e.g., [Blecher and Le Merdy 2004, Section 1.3] for these.

A *hereditary subalgebra*, or HSA, in an operator algebra A is an approximately unital subalgebra with $DAD \subset D$. See [Blecher et al. 2008] for their basic theory. The support projection of an HSA in A is the identity of its bidual, viewed within A^{**}.

We write $\mathfrak{r}_A = \{x \in A : x + x^* \geq 0\}$, and call these the *real positive elements*. This space may be defined purely internally without using the "star", as the *accretive* elements, which have several purely metric definitions; see, e.g., [Blecher 2015, Lemma 2.4]. Also $\mathfrak{r}_A$ is the closure of the positive real multiples of $\mathfrak{F}_A = \{a \in A : \|1 - a\| \leq 1\}$ [Blecher and Read 2013]. Read's theorem states that any operator algebra with cai has a real positive cai; see [Blecher 2013] for a proof of this. Since $M_n(A)^{**} \cong M_n(A^{**})$ [Blecher and Le Merdy 2004, Theorem 1.4.11], we have that $\mathfrak{r}_{M_n(A^{**})}$ is the weak-$*$ closure of $\mathfrak{r}_{M_n(A)}$, and $\mathfrak{r}_{M_n(A)} = M_n(A) \cap \mathfrak{r}_{M_n(A^{**})}$ for each n.

A linear map $T : A \to B$ between operator algebras or unital operator spaces is *real positive* if $T(\mathfrak{r}_A) \subset \mathfrak{r}_B$. It is *real completely positive*, or RCP for short, if T_n is real positive on $M_n(A)$ for all $n \in \mathbb{N}$. One may also define these maps in terms of the set $\mathfrak{F}_A$ above, as in [Blecher and Read 2013], but the definitions are shown to be equivalent in [Bearden et al. 2014, Section 2]. From the latter reference: a linear map $T : A \to B(H)$ on an approximately unital operator algebra or unital operator space A is RCP if and only if T has a completely positive (in the usual sense) extension $\tilde{T} : C^*(A) \to B(H)$. Here $C^*(A)$ is a C^*-algebra generated by A. We call this the generalized Arveson extension theorem. Thus real complete positivity on A is equivalent to P extending to a completely positive map on a containing C^*-algebra. A unital completely contractive map on a unital operator space is RCP, since it extends to a completely contractive map on a containing unital C^*-algebra, and such maps are completely positive, as we said above.

A *ternary ring of operators*, or TRO, is a subspace Z of $B(K, H)$ such that $ZZ^*Z \subset Z$. A *WTRO* is a weak-$*$ closed TRO. The second dual of a TRO is a WTRO; see [Blecher and Le Merdy 2004, Chapter 8] for this and the next several facts. We write $L(Z)$ for the *linking C^*-algebra* of a TRO; this has "four corners": ZZ^*, Z, Z^*, and Z^*Z. Here ZZ^* is the closure of the linear span of products zw^* with $z, w \in Z$, and similarly for Z^*Z. One gets a similar von Neumann algebra for WTROs. A *ternary morphism* on a TRO Z is a linear map T such that $T(xy^*z) = T(x)T(y)^*T(z)$ for all $x, y, z \in Z$. A *tripotent* is an element $u \in Z$ such that $uu^*u = u$. We order tripotents by $u \leq v$ if and only if $uv^*u = u$. This turns out to be equivalent to $u = vu^*u$, or to $u = uu^*v$, and implies that $u^*u \leq v^*v$ and $uu^* \leq vv^*$ [Battaglia 1991]. If $x \in \mathrm{Ball}(Z)$, define $u(x)$ to be the weak-$*$ limit of the sequence $(x(x^*x)^n)$ in Z^{**}. This is the largest tripotent in B^{**} satisfying $vv^*x = v$ [Edwards and Rüttimann 1996]. If $x \geq 0$ or if $u(x)$ is a projection, then $u(x)$ is also the weak-$*$ limit of powers x^n as $n \to \infty$; see, e.g., [Blecher and Neal 2012b; Blecher 2013].

We will say that an idempotent linear $P : X \to X$ is a *symmetric* projection if

$\|I-2P\| \leq 1$, and *completely symmetric* one if $\|I-2P\|_{\mathrm{cb}} \leq 1$. This is related to the notion of u-ideal [Godefroy et al. 1993], but we will not need anything from that theory. Such are automatically bicontractive or completely bicontractive, respectively. We say that P is *completely hermitian* if P is hermitian in $CB(X)$. Note that since $\exp(itP) = I - P + e^{it}P$ it follows that P is completely hermitian if and only if $\|I - P + e^{it}P\|_{\mathrm{cb}} \leq 1$ for all real t. This is essentially the notion of being (completely) *bicircular*. Clearly if P is completely hermitian then it is completely symmetric. We will not discuss (completely) hermitian projections much in this paper; these seem much less interesting.

2. Completely contractive projections on approximately unital operator algebras

Looking at examples it becomes clear that projections on operator algebras with no kind of approximate identity can be very badly behaved. Hence we will say little in our paper about such algebras. However, it is worth mentioning that we can pick out a "good part" of such a projection. This is the content of our first result.

Proposition 2.1. *Let $P : A \to A$ be a real completely positive completely contractive map* (*resp. projection*) *on an operator algebra A* (*possibly with no kind of approximate identity*). *There exists a largest approximately unital subalgebra D of A, and it is an HSA* (*hereditary subalgebra*) *of A. Moreover, $P(D) \subset D$, and the restriction P' of P to D is a real completely positive completely contractive map* (*resp. projection*) *on D. In addition, P' is completely bicontractive* (*resp. completely symmetric*) *if P has the same property.*

Proof. By [Blecher and Read 2014, Corollary 2.2], $D = \mathfrak{r}_A - \mathfrak{r}_A$ is the largest approximately unital subalgebra of A. This algebra is written as A_H there, and was first introduced in [Blecher and Read 2013, Section 4]. Clearly $P(D) \subset D$. The rest is obvious. $\square$

Remark. The last result is also true with the word "completely" removed throughout, with the same proof.

Remark. Letting p be the support projection of the HSA D above, if P extends to a completely positive complete contraction on a containing C^*-algebra (as in our generalized Arveson extension theorem mentioned in the introduction; see also [Bearden et al. 2014, Theorem 2.6]), then one can show that $P^{**}(pa) = pP(a)$ and $P^{**}(ap) = P(a)p$ for $a \in A$. It follows that P may be pictured as a 2×2 matrix with its "good part" above in the $(1,1)$ corner. However, in general it seems one can say little about the other corners; they can be quite messy. This is why we focus on algebras with approximate identities in our paper.

Proposition 2.2. *A real completely positive completely contractive map* (*resp. projection*) *on an approximately unital operator algebra A extends to a unital real completely positive completely contractive map* (*resp. projection*) *on the unitization A^1.* (*If A is unital then we define $A^1 = A \oplus^\infty \mathbb{C}$ here.*)

Proof. Suppose that $P : A \to A \subset B(H)$ is the map. By [Bearden et al. 2014, Theorem 2.6], P extends uniquely to a completely positive completely contractive map $C^*(A) \to B(H)$. By [Choi and Effros 1976, Lemma 3.9] it extends further to a unital completely positive map $C^*(A) + \mathbb{C}I_H \to B(H)$. The restriction of the latter map to $A + \mathbb{C}I_H$ may be viewed as a unital (real completely positive) completely contractive map on the unitization $A^1 \to A^1$, and it is evidently a projection if P was a projection. □

The previous result gives a way to reduce to the unital case. However, this method does not seem to be helpful later in our paper when dealing with bicontractive or symmetric projections, and we will need a different reduction to the unital case.

Lemma 2.3. *Let $P : A \to A$ be a real positive, contractive map on a unital operator algebra. Then* $0 \leq P(1) \leq 1$.

Proof. The restriction of P to $\Delta(A) = A \cap A^*$ is real positive. Hence it is positive by the proof of [Bearden et al. 2014, Theorem 2.4]. So $0 \leq P(1) \leq 1$. □

Lemma 2.4. *Suppose that E is a completely contractive completely positive projection on an operator system X. Then the range of E, with its usual matrix norms, is an operator system with matrix cones $E_n(M_n(X)_+) = M_n(X)_+ \cap \mathrm{Ran}(E_n)$, and unit $E(1)$.*

Proof. We will use the Choi–Effros characterization of operator systems [1977]. Because $\mathrm{Ran}(E_n)$ is a $*$-subspace of $M_n(X)$, with the inherited cone from $M_n(X)_+$, it is a partially ordered, matrix ordered, Archimedean $*$-vector space with proper cones. If $x = x^*$ there exists a positive scalar t with $-t1 \leq x \leq t1$, so that $-tE(1) \leq x \leq tE(1)$. So $E(1)$ is an order unit. If $x \in \mathrm{Ran}(E)$ with $\|x\|_X \leq 1$ then

$$\begin{bmatrix} 1 & x \\ x^* & 1 \end{bmatrix} \geq 0$$

in $M_2(X)$. Applying E_2, we deduce that

$$\begin{bmatrix} E(1) & x \\ x^* & E(1) \end{bmatrix} \geq 0,$$

so that the norm of x is ≤ 1 in the new *order unit norm*; see [Choi and Effros 1977, p. 179]. Conversely, if the last norm is ≤ 1, or equivalently if the last centered equation holds, then it is a simple exercise in operator theory that $\|x\|_X \leq 1$, since $\|E(1)\| \leq 1$ and $E(1) \geq 0$. Thus the order unit norm coincides with the old norm.

The computation is similar at each matrix level. By the Choi–Effros characterization of operator systems, (Ran(E), $E(1)$) is an operator system with the given matrix cones, and the order-unit matrix norms are the usual norms. $\square$

The following generalization of the Choi–Effros result referred to in the introduction solves the completely contractive projection problem in the category of approximately unital operator algebras and real completely positive projections. We remark that the case when also $P(1_A) = 1_A$ is implicit in the proof of [Blecher and Le Merdy 2004, Corollary 4.2.9].

Theorem 2.5. *Let A be an approximately unital operator algebra, and $P : A \to A$ a completely contractive projection which is also real completely positive. Then the range $B = P(A)$ is an approximately unital operator algebra with product $P(xy)$. We have*

$$P(P(a)b) = P(P(a)P(b)) = P(aP(b)), \quad a, b \in A.$$

In particular $P(P(1)^n)) = P(1)$ for all $n \in \mathbb{N}$, if A is unital. With respect to the "multiplication" $P(xy)$, we conclude that A is a bimodule over B, and P, when viewed as a map $A \to B$, is a B-bimodule map (with B equipped with its new product). If A is unital then $P(1)$ is the identity for the latter product. Moreover (not assuming A is unital), P extends to a completely positive completely contractive projection on the injective envelope $I(A)$.

Proof. We give two proofs, since they both use techniques the reader will need to be familiar with in the rest of our paper.

Set $B = P(A)$. Extend P to a unital completely contractive projection P^1 on A^1 by Proposition 2.2. We may then use the proof of [Blecher and Le Merdy 2004, Corollary 4.2.9], which proceeds by extending P to a unital (completely positive and) completely contractive projection E on $I(A^1)$. It follows from the Choi–Effros result mentioned early in our introduction, that Ran(E) is a unital C^*-algebra with product $E(xy)$, and B with product $P(xy)$ is a unital subalgebra of this C^*-algebra. We also have by the same Choi–Effros result (or its proof) that $E(E(a)b) = E(E(a)E(b)) = E(aE(b))$ for all $a, b \in A$, giving the centered equation in the theorem statement. This gives the first several assertions of our theorem. Note that

$$P(P(e_t)P(a)) = P(e_t P(a)) \to P(P(a)) = P(a), \quad a \in A,$$

if (e_t) is a cai for A, and similarly on the right, so that $(P(e_t))$ is a cai for $P(A)$ in its new product. That A is actually a B-bimodule follows from the centered equation in the theorem statement; for example, because

$$\begin{aligned} P(P(P(a)P(b))c) &= P(P(P(a)P(b))P(c)) = P(P(a)P(b)P(c)) \\ &= P(P(a)P(P(b)P(c))) = P(P(a)P(P(b)c)). \end{aligned}$$

The centered equation in the theorem statement is just saying that P is a B-bimodule map for the given products. The final assertion about extending to $I(A)$ is easy from the above in the case that A is unital; the other case we will do below.

For the second proof, first suppose that A is unital. Let $B = P(A)$, and set $X = A + A^*$, $Y = B + B^*$, and $v = P(1)$. By [Bearden et al. 2014, Theorem 2.6], P extends uniquely to a completely positive completely contractive map P' on X. Since $X = A + A^*$ this map is uniquely determined; it must be $a_1 + a_2^* \mapsto P(a_1) + P(a_2)^*$, a projection on X with range Y. By Lemma 2.4 (Y, v) is an operator system with positive cone $P(X_+)$. Let $j : Y \to X$ be the inclusion map. Then extend P' to a completely positive complete contraction $\tilde{P} : I(A) \to I(B)$ [loc. cit.]. Extend j to a completely positive complete contraction $\tilde{j} : I(B) \to I(A)$ by Arveson's extension theorem [1969]. Then $\tilde{P} \circ \tilde{j}$ equals the identity map on $I(B)$ by rigidity of the injective envelope, since $P \circ j = I_B$. Thus $E = \tilde{j} \circ \tilde{P}$ is a completely positive completely contractive projection on $I(A)$ extending P. We deduce just as in the last paragraph that $\mathrm{Ran}(E)$ is a unital C^*-algebra with product $E(xy)$, B with product $P(xy)$ is a unital subalgebra of this C^*-algebra, and $P(P(a)b) = P(P(a)P(b)) = P(aP(b))$.

Finally, if A is nonunital but approximately unital, then P^{**} is a completely contractive projection on A^{**}, which is also real positive by the proof of the main theorem in [Bearden et al. 2014, Section 2]. By the unital case, $P^{**}(xy)$ is an operator algebra product on $P^{**}(A^{**})$, with unit $v = P^{**}(1)$. Hence by restriction $P(xy)$ is an operator algebra product on $P(A)$, and the centered equation in the theorem statement holds on A, as does the assertions about bimodules. Note that $P^{**}(A^{**}) = (P(A))^{\perp\perp}$, so that $P^{**}(A^{**}) \cong P(A)^{**}$. So $P(A)^{**}$ is unital, and hence $P(A)$ is approximately unital (or this may be seen directly using the centered equation in the theorem statement). Also, since $v = P^{**}(1)$ acts as an identity on $P(A)$ in the new product, we can identify $P(A) + \mathbb{C}v$, as a unital operator space, with the unitization of $P(A)$ with its new operator algebra product. Then the restriction r of P^{**} to $A + \mathbb{C}1_{A^{**}}$ can be viewed as a real completely positive completely contractive projection on the unitization A^1. By the last paragraph, r extends to a completely positive completely contractive projection on $I(A^1)$. However, $I(A^1) = I(A)$ by, e.g., [Blecher and Le Merdy 2004, Corollary 4.2.8]. $\square$

Remark. Thus the category of approximately unital operator algebras and real completely positive projections forms a projectively stable category in the sense of Friedman and Russo; see [Neal and Russo 2011, p. 295–296] and, e.g., [Friedman and Russo 1985]. Namely, if $\mathcal{B}$ is the category of Banach spaces with morphisms being the contractive projections, a subcategory $\mathcal{S}$ of $\mathcal{B}$ is *projectively stable* if $\mathcal{S}$ is closed under images of morphisms. That is, for an object E and morphism $\varphi : E \to E$ in $\mathcal{S}$, the image $\varphi(E)$ is again an object in $\mathcal{S}$ (although not necessarily a subobject with respect to the full structure of objects in $\mathcal{S}$). For example, the subcategory of unital C^*-algebras and completely positive unital projections is projectively stable, by the theorem of Choi and Effros used earlier. Other projectively stable categories are listed in the last references; e.g., the subcategory of TROs and completely contractive projections is projectively stable by Youngson's theorem [1983]. In the cited pages of [Neal and Russo 2011] the concept of a projectively rigid category is discussed. The associated question for us would be if the preduals of dual operator algebras are projectively (completely) rigid in their sense. However, the answer to this is in the negative, since the category of Banach or operator spaces is not projectively (completely) rigid, and then one can play the $\mathcal{U}(X)$ trick (described above Proposition 3.3 below) to answer the operator algebra question.

Remark. If A is unital and C is the C^*-subalgebra of $I(A)$ generated by $P(A)$, then the map $\tilde{P}$ in the proof restricts to a $*$-homomorphism from C onto $C^*_{\mathrm{e}}(B)$, the latter viewed as a subalgebra of $I(B)$ (or as a C^*-subalgebra of the space $\mathrm{Ran}(E)$ in the proof, with its "Choi–Effros product"). See, e.g., [Blecher and Le Merdy 2004, Theorem 1.3.14(3)].

Lemma 2.6. *Let A be a unital operator algebra, and let $P : A \to A$ be a contractive projection, such that* $\mathrm{Ran}(P)$ *contains an orthogonal projection q with $P(A) = qP(A)q$. Then $q = P(1_A)$.*

Proof. We have $\|q \pm (1-q)\| \leq 1$ so that $\|q \pm P(1-q)\| \leq 1$. Since $P(A) = qP(A)q$, and q is an extreme point of the unit ball of qAq (the identity is an extreme point of the unit ball of any unital Banach algebra), $P(1-q) = 0$. Thus $P(1) = q$. $\square$

The following reduction to the case of unital maps works under a certain condition which will be seen to be automatic in the setting found in the next sections of the paper.

Proposition 2.7. *Let A be an approximately unital operator algebra, and $P : A \to A$ a completely contractive projection. Then* $\mathrm{Ran}(P^{**})$ *contains an orthogonal projection q such that $P(A) = qP(A)q$ if and only if $P^{**}(1)$ is a projection. In this case, $q = P^{**}(1)$ and* $\mathrm{Ran}(P)$ *is an approximately unital operator algebra with product $P(xy)$, and its bidual has identity q. Also, P is real completely positive, all the conclusions of Theorem 2.5 hold, q is an open projection for A^{**} in the sense of* [Blecher et al. 2008], *and*

$$P(a) = qP(a)q = P^{**}(qaq), \quad a \in A$$

(*and we can replace P^{**} by P here if A is unital*). *Hence, $P(A) = qP(A)q = P^{**}(qAq)$, and P splits as the sum of the zero map on $q^\perp A + Aq^\perp + q^\perp Aq^\perp$, and a real completely positive completely contractive projection P' on qAq with range equal to $P(A)$. This projection P' on qAq is unital if A is unital.*

Proof. Let $Q = P^{**}$, a completely contractive projection on A^{**}. We can replace Q by P below if A is unital. If $P(A) = qP(A)q$ for a projection q then $Q(A^{**}) = qQ(A^{**})q$ by standard weak-$*$ approximation arguments, so by the lemma, $Q(1) = q$. Conversely, suppose that $Q(1) = q$ is a projection. Then $Q(q^{\perp}) = 0$. Note that $\mathrm{Ran}(Q)$ is a unital operator space (in $qA^{**}q$). So Q, and hence also P, is real completely positive by [Bearden et al. 2014, Lemma 2.2], since it extends by, e.g., [Blecher and Le Merdy 2004, Lemma 1.3.6] to a completely positive unital map from $X + X^*$ onto $Y + Y^*$, where $X = A^{**}$ and $Y = Q(A^{**})$. By extending Q further to a completely positive completely contractive map on a containing C^*-algebra, and using the Kadison–Schwarz inequality,

$$Q(aq^{\perp})^* Q(aq^{\perp}) \le Q(q^{\perp}a^*aq^{\perp}) \le Q(q^{\perp}) = 0, \quad a \in \mathrm{Ball}(A^{**}).$$

Thus $Q(a) = Q(aq)$ for all $a \in A^{**}$, and similarly $Q(a) = Q(qa)$. Also $Q(q)^2 = Q(q)$, and so $P(A) = Q(qAq) = qP(A)q$ by Choi's multiplicative domain trick. (The latter is usually stated for unital maps, but the general case may be reduced to this using [Choi and Effros 1976, Lemma 3.9].)

The rest follows from Theorem 2.5 and its proof, with the exception of q being an open projection for A^{**}. To see this, if (e_t) is a cai for $P(A)$ with its new product then using some of the facts here and in Theorem 2.5, we have $e_t = P(e_t q) = P(e_t)q \to q$ in the weak-$*$ topology. □

Remark. Note that even a completely contractive completely positive projection on a unital C^*-algebra need not have $P(1)$ a projection. To see this, choose a norm-1 element $x \neq 1$ in A_+ and a state φ with $\varphi(x) = 1$, and consider $P = \varphi(\,\cdot\,)x$.

Remark. Unfortunately the projection q here need not be central, even if P is completely bicontractive. See the next example.

Example 2.8. Consider the canonical projection of the upper triangular matrices A onto $\mathbb{C}E_{11}$. This is a real completely positive completely bicontractive projection (which is also completely bicontractive, completely hermitian, etc.), but it does not extend to a positive bicontractive projection on its C^*-envelope (or injective envelope) M_2. In this case note that $A + A^* = C^*(A) = C^*_{\mathrm{e}}(A) = I(A)$. On the positive side, the range of this projection is a subalgebra of A.

Corollary 2.9. *Let A be an approximately unital operator algebra with an approximately unital subalgebra B which is the range of a completely contractive projection P on A. Then P is real completely positive, and all the conclusions of Theorem 2.5 hold. Hence P is a conditional expectation: $P(a)b = P(ab)$ and $bP(a) = P(ba)$ for all $b \in B = P(A)$ and $a \in A$. It follows that $(P(e_t))$ is a cai for B for any cai (e_t) of A.*

Proof. Consider P^{**}, a completely contractive projection on A^{**} with range B^{**}. Of course B^{**} has an identity of norm-1 as we said in the introduction. By

Proposition 2.7, P^{**} is real completely positive, and hence so is its restriction P. The remaining assertions follow, e.g., from Proposition 2.7, except for the last assertion, which is an easy consequence of the second to last assertion. □

The last result, which may be seen as a converse to Theorem 2.5, generalizes the fact from [Blecher and Le Merdy 2004] mentioned at the start of the introduction. We showed in [Blecher 2004] that this is all false with the word "completely" removed; however, see [Lau and Loy 2008] for some later variants valid for certain Banach algebras.

We will need the following results later. If $P : M \to M$ is a unital completely contractive projection on a von Neumann algebra, there exists a *support projection* e, the perp of the supremum of all projections in $\mathrm{Ker}(P)$, as in [Effros and Størmer 1979, p. 129]. We have $e \in P(M)'$, and $P(x) = P(ex) = P(xe)$ for all $x \in M$, and if $x \in M_+$ then $P(x) = 0$ if and only if $xe = 0$ if and only if $ex = 0$; see, e.g., around Lemma 1.2 in that reference. Following the idea in the proof of part (3) of that lemma, we have:

Proposition 2.10. *Let $P : M \to M$ a weak-$*$ continuous unital completely contractive projection on a von Neumann algebra M. Let e be the support projection of P on M discussed above. Let N be the von Neumann algebra generated by $P(M)$. Then $P(x)e = eP(x)e = exe = xe$ for all $x \in N$.*

Proof. For $n = 0, 1, \ldots,$ let A_n be the span of products of 2^n elements from $P(M)$. Then A_n is a unital $*$-subspace of M. Suppose that $eP(x)e = exe$ for all $x \in A_n$. Then for such x, set $z = e(P(x^*x) - x^*ex)e$. Following the steps in the proof of [Effros and Størmer 1979, Lemma 1.2(3)] with minor modifications, we have $P(z) = 0$ and $z \geq 0$, so that by the facts above the present proposition we obtain $z = eze = 0$ and $eP(x^*x)e = ex^*xe$. By the polarization identity, $eP(y^*x)e = ey^*xe$ for $x, y \in A_n$. So $eP(x)e = exe$ for all $x \in A_{n+1}$, and hence for all $x \in N$. □

Corollary 2.11. *Let $P : A \to A$ be a unital completely contractive projection on an operator algebra. If $P(A)$ generates A as an operator algebra, then $(I - P)(A) =$ $\mathrm{Ker}(P)$ is an ideal in A. In any case, if D is the closed algebra generated by $P(A)$ then $(I - P)(D)$ is an ideal in D.*

Proof. We may assume that $P(A)$ generates A. As above we extend P to a unital completely contractive projection $\tilde{P}$ on a C^*-algebra B, which may be taken to be $I(A)$. The second adjoint of this is a weak-$*$ continuous unital completely contractive projection on a von Neumann algebra M, and we continue to write this projection as $\tilde{P}$. Let $\tilde{P}$ also denote the restriction of the latter projection to the von Neumann algebra N generated by $P(A)$ inside M. If $x \in (I - P)(A)$, then $\tilde{P}(x) = 0$, and so by Proposition 2.10 we have $ex = xe = 0$. Thus $x \in e^\perp M e^\perp$ (and is also in $e^\perp N e^\perp$). So for $y \in A$ we have $P(xy) = P(exy) = 0$, and similarly $yx \in (I - P)(A)$, so the latter is an ideal. □

For later use we record that if $P : A \to A$ is a unital completely contractive projection on an operator algebra, then in the language of the last proof,

$$D \cap e^{\perp} M e^{\perp} = D \cap e^{\perp} N e^{\perp} = \mathrm{Ker}(P|_D) = (I - P)(D).$$

Indeed since we said above Proposition 2.10 that $P(x) = P(exe)$ for any $x \in M$, we have $D \cap e^{\perp} M e^{\perp} \subset \mathrm{Ker}(P|_D)$. Moreover, if $d \in D$ with $P(d) = 0$ then the argument in the last proof shows that $d \in D \cap e^{\perp} N e^{\perp} \subset D \cap e^{\perp} M e^{\perp}$. So

$$D \cap e^{\perp} M e^{\perp} = D \cap e^{\perp} N e^{\perp} = \mathrm{Ker}(P|_D).$$

3. The symmetric projection problem and the bicontractive projection problem

It turns out that the variant of the bicontractive projection problem for symmetric projections works out perfectly. This is the question of characterizing (completely) symmetric projections in the categories we are interested in, and their ranges. Notice that if $P : X \to X$ is a projection on a normed space and if we let $\theta = 2P - I$, so that $P = \frac{1}{2}(I + \theta)$, then $\mathrm{Ran}(P)$ is exactly the set of fixed points of θ, and $\theta \circ \theta = I$. Note too that θ is contractive if P is symmetric. From the latter facts we deduce that θ is a bijective isometry whose inverse is itself. Also $\theta(1) = 1$ if X is a unital algebra and $P(1) = 1$. Applying the same argument at each matrix level we see:

Lemma 3.1. *A projection $P : X \to X$ on an operator space is completely symmetric* (*resp. symmetric*) *if and only if $\theta = 2P - I$ is a complete isometry* (*resp. an isometry*), *and in this case θ is also a surjection. If X is also an algebra* (*resp. Jordan algebra*) *and if θ is a homomorphism* (*resp. Jordan homomorphism*), *then the range of P is a subalgebra* (*resp. Jordan subalgebra*).

Proof. For the last part, $\mathrm{Ran}(P)$ is exactly the fixed points of θ. □

Thus the (completely) symmetric projection problem is in some sense a special case of the (complete) isometry problem: namely characterizing the linear (complete) isometries between the objects in our category. That is, the key to solving the (completely) symmetric projection problem is proving a Banach–Stone-type theorem in our category. The original Banach–Stone theorem characterizes unital isometries between $C(K)$ spaces, and in particular shows that such are $*$-isomorphisms. Putting this together with the last assertion of the last lemma, we see that one of the hoped-for conclusions of the (completely) symmetric projection problem, and by extension the (completely) bicontractive projection problem, is that the range of the projection is a subalgebra. We will also show in the completely symmetric case that if A is unital or approximately unital then so is $P(A)$.

Let us examine what this all looks like in a C^*-algebra, where as predicted in the last paragraph, much hinges on the known Banach–Stone-type theorem for C^*-algebras, due mainly to Kadison. The following is essentially well known (see, e.g., [Friedman and Russo 1984; Størmer 1982]), but we do not know of a reference which has all of these assertions, or is in the formulation we give.

Theorem 3.2. *If $P : A \to A$ is a projection on a C^*-algebra A, then P is bicontractive if and only if P is symmetric. Then P is bicontractive and completely positive if and only if there exists a central projection $q \in M(A)$, such that $P = 0$ on $q^\perp A$, and there exists a period-2 $*$-automorphism θ of qA so that $P = \frac{1}{2}(I + \theta)$ on qA.*

Proof. Clearly symmetric projections are bicontractive. Conversely, if P is bicontractive, then by [Friedman and Russo 1984, Theorem 2], $\theta = 2P - I_A$ is a linear surjective isometric selfmap of A preserving the Jordan triple product and satisfying $\theta \circ \theta = I_A$ and $P = \frac{1}{2}(I_A + \theta)$. So P is symmetric. If also P is positive then P, and hence also θ, is $*$-linear. Let Q be the extension of P to the second dual.

Suppose that $\theta : A \to B$ is a linear isometric surjection between C^*-algebras. By a result of Kadison [1951], $u = \theta^{**}(1)$ is unitary in B^{**}. Suppose now further that θ is $*$-linear. Then u is selfadjoint, and $u\theta^{**}(\,\cdot\,)$ is a unital isometry, so selfadjoint. Thus $u\theta(a) = \theta(a^*)^* u = \theta(a)u$, for $a \in A$, so u is central. If $a \in A_{\mathrm{sa}}$ then

$$u\theta(a^2) = (u\theta(a))^2 = \theta(a)^2 \in B,$$

since $u\theta(\,\cdot\,)$ is a Jordan morphism. Thus $uB = u\theta(A) \subset B$. So $u \in M(B)$.

Returning to our situation, let $q = Q(1)$. This is a central projection in $M(A)$, since $u = 2q - 1$. Since $Q(q^\perp) = 0$, if $a \in \mathrm{Ball}(A)_+$ then

$$P(q^\perp a) = P(q^\perp a q^\perp) \le Q(q^\perp) = 0,$$

and so $P = 0$ on $q^\perp A$. Also, since $\theta(q) = q$ and θ preserves Jordan triple products, it follows that $P(qa) = \frac{1}{2}(qa + \theta(qa)) = qP(a)$. Thus, $P(qA) = qP(A)$, and the restriction of P to qA is a unital bicontractive positive projection on a unital C^*-algebra. Also $\theta(qa) = q\theta(a)$ for $a \in A$, as we had above, so $\theta(qA) = qA$. Hence $\theta' = \theta|_{qA}$ is a unital isometric isomorphism of qA. □

Remark. The example $P(x) = \frac{1}{2}(x + x^T)$ on M_2 shows the necessity of the completely positive hypothesis in the part it pertains to. Note in this example P is positive and contractive, and $I - P$ is completely contractive.

We will generalize the bulk of the last result and its proof in Theorem 3.7.

We next show that unlike in the C^*-algebra case, for projections on operator algebras (completely) bicontractive is not the same as (completely) symmetric. Also, these both also differ from the notion of (completely) hermitian. We recall that any operator space $X \subset B(H)$ may be unitized to become an operator algebra as follows. We define

$$\mathcal{U}(X) = \left\{ \begin{bmatrix} \lambda_1 I_H & x \\ 0 & \lambda_2 I_H \end{bmatrix} : x \in X,\ \lambda_1, \lambda_2 \in \mathbb{C} \right\} \subset B(H \oplus H).$$

By definition, $\mathcal{U}(X)$ may be regarded as a subspace of the Paulsen system; see [Blecher and Le Merdy 2004, 1.3.14 and 2.2.10] or [Paulsen 2002]. It follows from Paulsen's lemma [op. cit., Lemma 8.1] or [Blecher and Le Merdy 2004, Lemma 1.3.15] that if $v : X \to X$ is a linear contraction (resp. complete contraction), then the mapping θ_v on $\mathcal{U}(X)$ defined by

$$\theta_v\left(\begin{bmatrix} \lambda_1 & x \\ 0 & \lambda_2 \end{bmatrix} \right) = \begin{bmatrix} \lambda_1 & v(x) \\ 0 & \lambda_2 \end{bmatrix}, \quad x \in X,\ \lambda_1, \lambda_2 \in \mathbb{C},$$

is a contractive (resp. completely contractive) homomorphism.

Proposition 3.3. *Suppose that X is an operator space, and that $P : X \to X$ is a linear idempotent map. Then P is completely contractive* (*resp. completely bicontractive, completely symmetric, completely hermitian*) *if and only if the induced map $\tilde{P} : \mathcal{U}(X) \to \mathcal{U}(X)$ is a real completely positive and completely contractive* (*resp. completely bicontractive, completely symmetric, completely hermitian*) *projection. In particular these hold* (*and with the word "completely" removed everywhere if one wishes*), *if $P : X \to X$ is a linear idempotent map on a Banach space, when we give X its minimal or maximal operator space structure*; *see, e.g.*, [Blecher and Le Merdy 2004; Paulsen 2002].

Proof. If P is completely contractive then by Paulsen's lemma referred to above, the unique $*$-linear unital extension of $\tilde{P}$ to $\mathcal{U}(X)+\mathcal{U}(X)^*$ is completely contractive and completely positive. Thus by [Bearden et al. 2014, Section 2], $\tilde{P}$ is real completely positive. Clearly $\tilde{P}$ is a projection. Conversely, if $\tilde{P}$ is completely contractive then so is P. We note that $I - \tilde{P}$ annihilates the diagonal, and acts as $I - P$ in the $(1, 2)$ entry. Thus $I - P$ is completely contractive if and only if $I - \tilde{P}$ is completely contractive. Also note that $2\tilde{P} - I$ does nothing to the diagonal, and acts as $2P - I$ in the $(2,1)$ entry; that is, $2\tilde{P} - I = (2P - I)\tilde{}$. By Paulsen's lemma again, as above, $2\tilde{P} - I$ is completely contractive if and only if $2P - I$ is completely contractive. Finally, $I - \tilde{P} + e^{it}\tilde{P}$ multiplies each of the two diagonal entries by e^{it}, and acts as $I - P + e^{it}P$ in the $(1, 2)$ entry. Multiplying by e^{-it}, we see again by Paulsen's lemma that this is completely contractive if and only if P is completely hermitian.

For the last assertion, if P is contractive (resp. bicontractive, symmetric, hermitian), then it is completely contractive (resp. bicontractive, symmetric, hermitian) on X with its minimal or maximal operator space structure by, e.g., [Blecher and Le Merdy 2004, 1.10 and 1.12]. We may then apply the case in the previous paragraph to obtain the result. Conversely, if $\tilde{P}$ is contractive (resp. bicontractive, etc.), then so is P since P is the $(1, 2)$ corner of $\tilde{P}$. The rest is clear. $\square$

The previous result provides examples of real completely positive unital completely bicontractive (resp. completely contractive, completely symmetric) projections which are not completely symmetric (resp. completely bicontractive, completely hermitian). As we said, this is in contrast to the C^*-algebra case where (complete) bicontractivity is equivalent to being (completely) symmetric [Størmer 1982; Friedman and Russo 1984].

Example 3.4. A more specific example of a real completely positive unital completely bicontractive projection on an operator algebra which is not symmetric arises by the last construction, from the following explicit completely bicontractive projection which is not completely symmetric. Let Y be $\mathbb{R}^2$, the latter viewed as a real Banach space whose unit ball is the unit ball of ℓ^∞_2 in the first quadrant and the unit ball of ℓ^1_2 in the second quadrant. Let X be the standard complexification of Y. Take $P : X \to X$ to be the usual complexification of the projection onto the first coordinate on Y. We thank Asvald and Vegard Lima for this example, which is a bicontractive projection on a Banach space which is not symmetric. Giving X its minimal or maximal operator space structure makes X an operator space, and makes P (by, e.g., [Blecher and Le Merdy 2004, 1.10 or 1.12]) a completely bicontractive projection which is not symmetric. Hence, by Proposition 3.3, we get a real completely positive unital completely bicontractive projection on an operator algebra which is not symmetric.

Example 2.8 shows that one cannot extend completely symmetric real completely positive projections to a positive bicontractive projection on a containing C^*-algebra. Things are much better if P is a unital map:

Lemma 3.5. *Let A be a unital operator algebra, and let $P : A \to A$ be a completely symmetric unital projection. Then P is real completely positive, and the range of P is a subalgebra of A. Moreover, P extends to a completely symmetric unital projection on $C^*_e(A)$ (or on $I(A)$).*

Proof. Any completely contractive unital map on A is real completely positive as we said in the introduction. Let $\theta = 2P - I$, so that $P = \frac{1}{2}(I + \theta)$. By Lemma 3.1, θ is a unital complete isometry. So in fact θ is a homomorphism, by the Banach–Stone theorem for operator algebras [Blecher and Le Merdy 2004]. This implies, as we said in Lemma 3.1, that $\mathrm{Ran}(P)$ is a subalgebra of A. We can extend θ uniquely to a unital $*$-isomorphism $\pi : C^*_e(A) \to C^*_e(A)$, with $\pi \circ \pi = I$, and it follows that $\tilde{P} = \frac{1}{2}(I + \pi)$ is a completely symmetric extension of P. It is also completely positive.

Similarly, since one may extend π further to a unital $*$-automorphism of $I(A)$, there is a completely symmetric unital projection on $I(A)$ extending P. $\square$

Lemma 3.6. *Let A be a unital operator algebra, and let $P : A \to A$ be a completely contractive real positive projection which is bicontractive. Then $P(1) = q$ is a* *projection* (*not necessarily central*), *and all the conclusions of Theorem 2.5 and Proposition 2.7 hold. Also*, *there exists a unital completely bicontractive* (*real completely positive*) *projection* $P' : qAq \to qAq$ *such that* P *is the zero map on* $q^\perp A + Aq^\perp$ *and* $P = P'$ *on* qAq. *We have* $\mathrm{Ran}(P) = \mathrm{Ran}(P')$.

Proof. Suppose that $q = P(1)$. Then $q \geq 0$ by Lemma 2.3, so that the closed algebra B generated by q and 1 is a C^*-algebra. Note that $P(q^n) = q$ by Theorem 2.5, so that $P(B) \subset B$. Let $Q = P|_B$; this is a bicontractive projection on B, and it is positive by the proof of Lemma 2.3. By [Friedman and Russo 1984, Theorem 2] and its proof, $q = P(1)$ is a partial isometry in B, hence a projection. Therefore, all the conclusions of Proposition 2.7 and Theorem 2.5 hold. So $P(a) = qP(a)q = P(qaq)$ for $a \in A$, and $P(A) = qP(A)q = P(qAq)$, so P splits as the sum of the zero map on $q^\perp A + Aq^\perp$ and a unital projection E on qAq. □

Remark. We give a direct proof (which also can be tweaked to work only assuming that P is a bicontractive real positive projection on A [Blecher et al. ≥ 2016]) that $P(1)$ is a projection in the case of the lemma above: Let $P(1) = q$. As we saw in Theorem 2.5, $P(q^n) = q$. Thus $P(u(q)) = q$, (we defined $u(\cdot)$ in the introduction). Suppose that c is a positive scalar with $c(q - u(q))$ of norm 1. Then

$$(I-P)\big(c(q-u(q))-u(q)\big) = c(q-u(q))-u(q)-(c(q-q)-q) = (1+c)(q-u(q)),$$

which has norm greater than 1. By the contractivity of $I - P$ we have a contradiction, unless $q = u(q)$. So $q = P(1)$ is a projection.

The following is the solution to the symmetric projection problem in the category of approximately unital operator algebras.

Theorem 3.7. *Let* A *be an approximately unital operator algebra*, *and let* $P : A \to A$ *be a completely symmetric real completely positive projection. Then the range of* P *is an approximately unital subalgebra of* A. *Moreover*, $P^{**}(1) = q$ *is a projection in the multiplier algebra* $M(A)$ (*so is both open and closed*).

Set $D = qAq$, *a hereditary subalgebra of* A *containing* $P(A)$. *There exists a period-*2 *surjective completely isometric homomorphism* $\theta : A \to A$ *such that* $\theta(q) = q$, *so that* θ *restricts to a period-*2 *surjective completely isometric homomorphism* $D \to D$. *Also*, P *is the zero map on* $q^\perp A + Aq^\perp + q^\perp Aq^\perp$, *and*

$$P = \tfrac{1}{2}(I + \theta) \quad \text{on } D.$$

In fact,

$$P(a) = \tfrac{1}{2}(a + \theta(a)(2q - 1)), \quad a \in A.$$

The range of P *is exactly the set of fixed points of* $\theta|_D$ *in* D.

Conversely, *any map of the form in the last equation*, *for a period-*2 *surjective completely isometric homomorphism* $\theta : A \to A$ *and a projection* $q \in M(A)$ *with* $\theta^{**}(q) = q$, *is a completely symmetric real completely positive projection.*

Proof. Applying Lemma 3.6 to P^{**}, we see that $P^{**}(1) = q$ is a projection, and all the conclusions of Proposition 2.7 and Theorem 2.5 are true for us. We will silently be using facts from these results below. In particular q is an open projection, so it supports an approximately unital subalgebra D of A with $D^{\perp\perp} = qA^{**}q$; see [Blecher et al. 2008]. Then $\theta = 2P - I$ is a linear completely isometric surjection on A by Lemma 3.1. So by the Banach–Stone theorem for operator algebras [Blecher and Le Merdy 2004, Theorem 4.5.13], there exists a completely isometric surjective homomorphism $\pi : A \to A$ and a unitary u with $u, u^{-1} \in M(A)$ with $\theta = \pi(\cdot)u$. We have

$$\theta^{**}(1) = 2P^{**}(1) - 1 = 2q - 1 = \pi^{**}(1)u = u,$$

so that u is a selfadjoint unitary (a symmetry), and $q \in M(A)$. So $qAq = D \subset A$. Since $P(A) = qP(A)q$, the range of P is contained in D, and the range of P is exactly the set of fixed points of θ, which all lie in D. This implies that $\mathrm{Ran}(P)$ is a subalgebra of A. It is approximately unital and P is real completely positive by Corollary 2.9.

We have $\theta^{**}(q) = q$ and $\pi^{**}(q) = \theta^{**}(q)u = qu = q$. Then $2P(a) - a = \pi(a)(2q - 1)$ and $P(a) = \frac{1}{2}(a + \pi(a)(2q - 1))$. Indeed

$$\pi(a) = (2P(a) - a)u = (2P(a) - a)(2q - 1) = 2P(a) - 2aq + a, \quad a \in A.$$

From this, or otherwise, one sees that π equals θ on D, and

$$\pi(D) = (2P - I)(D) \subset D.$$

However, $D = \theta^2(D) \subset \theta(D) = \pi(D)$, so $\pi(D) = D$. This completes the main part of the theorem.

For the converse, note that $P(a) = \frac{1}{2}(a + \pi(a)(2q - 1))$ is clearly completely symmetric on A, and

$$\begin{aligned} P(P(a)) &= P\big(\tfrac{1}{2}(a + \pi(a)(2q - 1))\big) \\ &= \tfrac{1}{4}\big(a + 2\pi(a)(2q - 1) + \pi(\pi(a)(2q - 1)\big)(2q - 1) = P(a), \end{aligned}$$

since π is period 2, $\pi^{**}(q) = q$, and $2q - 1$ is a symmetry. We have

$$P^{**}(1) = \tfrac{1}{2}\big(1 + \pi^{**}(1)(2q - 1)\big) = q,$$

so that P is real completely positive by Proposition 2.7. $\square$

It follows easily from the last theorem that a completely symmetric real completely positive projection P on A extends to a completely symmetric projection $\tilde{P}$ on the C^*-envelope of A. Moreover, $\tilde{P}(x) = \frac{1}{2}(a + \tilde{\pi}(a)(2q - 1))$, for a $*$-automorphism $\tilde{\pi}$. However, this extension will not in general be positive.

In the work [Blecher et al. ≥ 2016] in progress, we prove the Jordan algebra variant of the last result.

4. The bicontractive projection problem

The following could be compared with Corollary 2.11.

Lemma 4.1. *Let A be a unital operator algebra, and $P : A \to A$ a unital projection with $I - P$ completely contractive. Let $C = (I - P)(A) = \mathrm{Ker}(P)$. Then $C^2 \subset P(A)$. We also have*:

(1) *C is a subalgebra of A if and only if $C^2 = (0)$.*

(2) *If P is completely bicontractive* (*or more generally*, *if $P(aP(b)) = P(P(a)P(b))$ for $a, b \in A$*) *then $P(A)C + CP(A) \subset C$, and $C^3 \subset C$.*

(3) *If the conditions in* (1) *hold*, *and if P is completely contractive* (*or more generally*, *if $P(aP(b)) = P(P(a)P(b))$ for $a, b \in A$*) *then C is actually an ideal in A.*

(4) *If $P(A)C + CP(A) \subset C$* (*see* (2)), *then $\theta = 2P - I_A$ is a homomorphism if and only if $P(A)$ is a subalgebra of A, and then the range of P is the set of fixed points of this automorphism θ.*

Proof. Of course $A = C + B$, where $B = P(A)$. By Youngson's result [1983] applied to an extension Q of $I - P$ to a completely contractive projection on $I(A)$ (which exists by an easier variant of the proof in Theorem 2.5), we have $(I - P)(wz) = Q(w(I - P)(1)^* z) = 0$ for $z, w \in C$. So $P(wz) = wz \in B$, and this is zero if the kernel is a subalgebra. In any case, $C^2 \subset B$. Assuming that P is completely contractive (or that $P(aP(b)) = P(P(a)P(b))$ for $a, b \in A$, which is a weaker condition by, e.g., the fact in the first paragraph of our paper), if $z \in B$ and $w \in C$, then $P(wz) = P(P(w)z) = 0$, so $wz \in C$. So $CB \subset C$ and similarly $BC \subset C$. Thus $C^3 \subset BC \subset C$. This proves (2). If also C is a subalgebra, then it is an ideal, proving (3).

For (4), we may decompose $A = C \oplus B$, where $1_A \in B = P(A)$, and we have the relations $C^2 \subset B$, $CB + BC \subset C$. Using the latter it is a simple computation that the period-2 map $\theta : x + y \mapsto x - y$ for $x \in B$, $y \in C$ is a homomorphism (indeed an automorphism) on A if and only if $P(A)$ is a subalgebra of A; clearly $P(A)$ is the set of fixed points of θ. $\square$

Remark. Replacing P by $I - P$ we see that if $P : A \to A$ is a completely contractive projection with $P(1) = 0$, then $P(A)$ is a subalgebra if and only if $P(A)^2 = (0)$.

Remark. The kernel of a bicontractive projection need not be a subalgebra. For example, consider $P(f)(x) = \frac{1}{2}(f(x) + f(-x))$ for $f \in C([-1, 1])$.

The following clarifies the unital version of the bicontractive projection problem in relation to the existence of an associated period-2 automorphism.

Corollary 4.2. *If $P : A \to A$ is a unital idempotent on a unital operator algebra, let $\theta : A \to A$ be the associated linear period-2 automorphism $x + y \mapsto x - y$ for $x \in \mathrm{Ran}(P)$, $y \in \mathrm{Ker}(P)$. Then P is completely bicontractive if and only if $\|I \pm \theta\|_{\mathrm{cb}} \leq 2$. If these hold then the range of P is a subalgebra if and only if θ is also a homomorphism, and then the range of P is the set of fixed points of this automorphism θ. Also, P is completely symmetric if and only if θ is completely contractive.*

Proof. The first and last assertions are obvious, and the second assertion follows from Lemma 4.1(4). ☐

For us, the bicontractive projection problem is whether the range of a completely bicontractive real completely positive projection on an approximately unital operator algebra A is an approximately unital subalgebra of A. This is not obvious, although there are easy counterexamples if one drops some of the hypotheses. For example, consider the projection P on the upper triangular 2×2 matrices given by

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{2}(a-c) & 0 \\ 0 & \frac{1}{2}(c-a) \end{bmatrix}.$$

This is completely contractive and extends to a completely contractive projection on the containing C^*-algebra, and can be shown to be bicontractive with $P(1) = 0$, and its range is not a subalgebra.

We now show that the bicontractive projection problem can be reduced to the case that A is unital and $P(1) = 1$, by a sequence of three reductions. First, if $P : A \to A$ is a completely bicontractive real completely positive projection on an approximately unital operator algebra A, then P^{**} is a completely bicontractive real completely positive projection on a unital operator algebra. Thus, henceforth in this section and in the next, we assume that A is unital. Second, Lemma 3.6 allows us to reduce further to the case that $P(1) = 1$: it asserted that $P(1) = q$ is a projection (not necessarily central), all the conclusions of Theorem 2.5 and Proposition 2.7 hold, and there exists a unital completely bicontractive (real completely positive) projection $P' : qAq \to qAq$ such that P is the zero map on $q^\perp A + Aq^\perp$, $P = P'$ on qAq, and $\mathrm{Ran}(P) = \mathrm{Ran}(P')$.

Then the third of our reductions of the completely bicontractive projection problem puts the problem in a standard position. Of course, $P(A)$ is a subalgebra if and only if $Q(D)$ is a subalgebra, where D is the closed subalgebra of A generated by $P(A)$, and $Q = P|_D$, which is a completely bicontractive unital projection on D. That is, we may as well replace A by the closed subalgebra generated by $P(A)$.

Thus by these three steps above, we have reduced the completely bicontractive projection problem on approximately unital operator algebras to the standard position of a unital projection on a unital operator algebra, whose range generates A. In this situation we obtain the following structural result.

Corollary 4.3. *Let P be a completely bicontractive unital projection on a unital operator algebra A. Let D be the algebra generated by $P(A)$. Then $(I-P)(D) = \mathrm{Ker}(P|_D)$ is an ideal in D and the product of any two elements in this ideal is zero.*

Proof. We saw at the end of Section 2 that $(I-P)(D) = \mathrm{Ker}(P|_D)$ is an ideal in D, and in the notation there it equals $D \cap e^{\perp}Me^{\perp}$. Since $D \cap e^{\perp}Me^{\perp}$ is a subalgebra of D the result follows from Lemma 4.1. □

Remark. Note that $(I-P)(D) \subset e^{\perp}A$, but $P(D)$ is not a subset of eA.

The above shows that we can also solve the bicontractive projection problem in the affirmative for real completely positive completely bicontractive projections P on a unital operator algebra A such that the closed algebra generated by A is semiprime (that is, it has no nontrivial square-zero ideals):

Corollary 4.4. *Let $P : A \to A$ be a real completely positive completely bicontractive projection on a unital operator algebra. If A is an operator algebra containing no nonzero nilpotents, then $P(A)$ is a subalgebra of A. Also if the closed algebra D generated by $P(A)$ is semiprime, then $P(A)$ is a subalgebra of A.*

Proof. By the second reduction above, we may assume that P is unital. By Corollary 4.3, $(I-P)(D)$ is an ideal in D with square zero, and so is (0) in these cases. So $P(A) = P(D) = D$, a subalgebra. □

For the subcategory of uniform algebras (that is, closed subalgebras of $C(K)$, for compact K) which are unital or approximately unital, the bicontractive projection problem coincides with the symmetric projection problem, and again there is a complete solution:

Theorem 4.5. *Let $P : A \to A$ be a real positive bicontractive projection on a uniform algebra A, and suppose that A is unital or approximately unital. Then P is (completely) symmetric, and so we have all the conclusions of Theorem 3.7. In particular, $P(A)$ is a subalgebra of A, and P is a conditional expectation.*

Proof. Here bicontractive projections are the same as completely bicontractive projections [Blecher and Le Merdy 2004, 1.10]. By the obvious variant of the usual proof that positive maps into a $C(K)$ space are completely positive, we have that real positive maps into a uniform algebra are real completely positive. By the first two reductions described above we can assume that A and P are unital. We also know that $B = P(A)$ is a subalgebra by Corollary 4.4, since, e.g., nonzero nilpotents cannot exist in a function algebra. Thus by Corollary 4.2 the map $\theta(x+y) = x-y$

described there is an algebra automorphism of A, hence a (completely) isometric isomorphism (since norm equals spectral radius). So $P = \frac{1}{2}(I + \theta)$ is (completely) symmetric, and Theorem 3.7 applies. $\square$

Remark. The idea in the last proof that θ is automatically isometric, since it is an algebraic automorphism of a uniform algebra, and that this implies that P is symmetric, was found together with Joel Feinstein after submission.

By, e.g., Corollary 4.2, to find a counterexample to the conjecture that all (completely real positive) completely bicontractive unital projections have range which is a subalgebra, we need a unital operator algebra $D = C \oplus E$ where

$$C \neq (0), \quad C^2 = 0, \quad CE + EC \subset C, \quad 1_D \in E, \quad \text{and } E \text{ generates } D,$$

so in particular E^2 is not a subset of E, and with the projection maps onto C and E completely contractive. This is easy in a Banach algebra; one may equip ℓ^1_3, with the standard basis identified with symbols $1, a, b$ satisfying relations like $b^2 = 0$, $a^2 = b$, etc. (setting $C = \{b\}$, $E = \mathrm{Span}\{1, a\}$). To find an operator algebra example, we make a general construction.

Example 4.6. Let V be a closed subspace of $B(H) \oplus B(H)$, viewed as elements of $B(H^{(2)})$ supported on the $(1,1)$ and $(2,2)$ entries. Write v_1 and v_2 for the two "parts" of an element $v \in V$. Let C be the closed span of $\{v_1 w_2 : v, w \in V\}$, and we will assume that $C \neq (0)$. Let B be the set of elements of $B(H^{(3)})$ of the form

$$\begin{bmatrix} \lambda I & v_1 & c \\ 0 & \lambda I & v_2 \\ 0 & 0 & \lambda I \end{bmatrix}, \quad \lambda \in \mathbb{C},\ v \in V,\ c \in C.$$

Then the copy of C is an ideal in B with square zero, and it is generated by the copy of V in B. If E is the sum of this copy of V and $\mathbb{C} I_{H^{(3)}}$, then all the conditions in the last paragraph needed for a counterexample hold, with the exception of the projection onto E being completely contractive. We remark that one may also describe B more abstractly as a set $B = \mathbb{C}1 + V + C$ in $B(K)$, where V and C are closed subspaces of $B(K)$ with the properties that

$$(0) \neq C = V^2, \quad V^3 = C^2 = (0),$$

plus one more condition ensuring that $p_1 V(1 - p) = (0)$, where p_1 is the left support projection of C and p is the left support projection of $C + V$.

To fix the exception noted in the last paragraph, we consider the subalgebra $A = A(V)$ of $B(H^{(7)})$ consisting of all elements of form

$$\begin{bmatrix} \lambda I & v_1 & c & 0 & 0 & 0 & 0 \\ 0 & \lambda I & v_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & \lambda I & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \lambda I & 0 & 2v_1 & 0 \\ 0 & 0 & 0 & 0 & \lambda I & 0 & 2v_2 \\ 0 & 0 & 0 & 0 & 0 & \lambda I & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \lambda I \end{bmatrix}, \quad \lambda \in \mathbb{C},\ v \in V,\ c \in C.$$

We will abusively write the square-zero ideal in A consisting of the copy of C again as C, and again let E be the sum of $\mathbb{C}I_{H^{(7)}}$ and the isomorphic copy of V in $A(V)$, so that $A = C \oplus E$. Here

$$C \neq (0), \quad C^2 = 0, \quad CE + EC \subset C, \quad 1_D \in E, \quad \text{and } E \text{ generates } A,$$

as desired. The canonical projection map from A onto C is obviously completely contractive.

A particularly simple case is when H is one-dimensional, so that $A \subset M_7$, and where $V = \mathbb{C}I_2$. This algebra is obviously essentially just (i.e., is completely isometrically isomorphic to) the subalgebra of M_5 of matrices

$$\begin{bmatrix} \lambda & \nu & c & 0 & 0 \\ 0 & \lambda & \nu & 0 & 0 \\ 0 & 0 & \lambda & 0 & 0 \\ 0 & 0 & 0 & \lambda & 2\nu \\ 0 & 0 & 0 & 0 & \lambda \end{bmatrix}, \quad \lambda, \nu, c \in \mathbb{C},$$

with the projection being replacing the $(1, 3)$ entry by 0. Thus 5×5 matrices of scalars will suffice to give an interesting example. However, we will need the more general construction later to produce a more specialized counterexample.

Consider the algebra $\mathcal{U}(V)$ constructed from V as described in the paragraph above Proposition 3.3. We define $\mathcal{U}_0(V)$ to be the subalgebra of $\mathcal{U}(V)$ consisting of elements of $\mathcal{U}(V)$ with the two diagonal entries identical. It is a subalgebra of $B(H^{(4)})$.

Lemma 4.7. *In the situation of Example 4.6, the map $j : \mathcal{U}_0(V) \to B(H^{(3)})$ given by*

$$\begin{bmatrix} \lambda I & v \\ 0 & \lambda I \end{bmatrix} \mapsto \begin{bmatrix} \lambda I & \frac{1}{2}v_1 & 0 \\ 0 & \lambda I & \frac{1}{2}v_2 \\ 0 & 0 & \lambda I \end{bmatrix}$$

is completely contractive and unital.

Proof. To prove that it is completely contractive simply notice that j may be viewed as the composition of the canonical map $\mathcal{U}(V) \to \mathcal{U}(W)$, where W is the copy of V in the algebra B above, and the map $M_2(B) \to B$ given by pre- and postmultiplying by $\left[\frac{1}{\sqrt{2}} I_{H^{(3)}} \;\; \frac{1}{\sqrt{2}} I_{H^{(3)}}\right]$ and its transpose. □

Corollary 4.8. *If $A = A(V)$ is the unital operator algebra above in $B(H^{(7)})$ then the canonical projection $P : A \to A$ which replaces the* $(1, 3)$ *entry by* 0, *whose kernel is a nontrivial square-zero ideal C generated by $P(A)$, is a* (*real completely positive*) *completely bicontractive and unital projection*, *but its range is not a subalgebra*, *and it need not even be a Jordan subalgebra. A particularly simple case is the algebra* (*completely isometrically isomorphic to the algebra of*) *of* 5×5 *scalar matrices described above Lemma 4.7.*

Proof. In the 5×5 matrix example it is easily checked that $P(A)$ is not closed under squares, hence is not a Jordan subalgebra. It remains to prove that P is completely contractive. However, P is the composition of the canonical map $B(H^{(3)} \oplus H^{(4)}) \to B(H^{(4)})$ restricted to A, and the map $x \mapsto j(x) \oplus x$ on $\mathcal{U}_0(V)$, where j is as in Lemma 4.7. □

The following is another rather general condition under which the completely bicontractive projection problem is soluble. Indeed as we said in the introduction, all examples known to us of real completely positive completely bicontractive projections on unital operator algebras, whose range is a subalgebra, do satisfy the criterion in Theorem 4.9.

Theorem 4.9. *Let A be a unital operator algebra in a von Neumann algebra M* (*which could be taken to be $B(H)$, or $I(A)^{**}$ as above*) *and let $P : A \to A$ be a unital completely bicontractive projection. Let D be the closed algebra generated by $P(A)$, and let $C = (I - P)(D)$.*

Suppose further that $CP(A)^* \subset \overline{MC}^{w*}$, the weak-$*$ closed left ideal in M generated by C. (*This is equivalent to saying that the left support projection of $P(A)C^*$ is dominated by the right support projection of C.*) *Alternatively*, *assume that $P(A)^*C$ is contained in the weak-$*$ closed right ideal in M generated by C* (*or equivalently that the right support projection of $C^*P(A)$ is dominated by the left support projection of C*). *Then $P(A)$ is a subalgebra of A.*

Proof. We assume the left ideal condition; the other case is similar and left to the reader (or can be seen by looking at the opposite algebra A^{op}). By replacing A by D we may assume that $P(A)$ generates A. Suppose that M is a von Neumann algebra on a Hilbert space H. We set $p_1 = \bigvee_{z \in C} r(z)r(z)^*$ to be the left support projection of C and $p_2 = \bigvee_{z \in C} r(z)^*r(z)$ to be the right support projection of C. Note that $zp_1 = 0$ for $z \in C$, which implies that $p_2 p_1 = 0$. Let $p = p_1 + p_2$, a projection. Our hypothesis is equivalent to saying that

$$CP(A)^* = CP(A)^* p_2.$$

We may write

$$M = (1-p)M(1-p) + (1-p)Mp + pM(1-p) + pMp.$$

Thus M may be pictured as the direct sum of a von Neumann algebra with four corners (thus having a 2×2 matrix form). Let $y \in P(A)$. Then $yC \subset C \subset pM$, so that $(1-p)yp_1 = 0$. On the other hand, by hypothesis, $p_2 y^*(1-p) = p_2 y^* p_2(1-p) = 0$. Thus $(1-p)yp_2 = 0$ and so $(1-p)yp = 0$. Therefore,

$$y = (1-p)y(1-p) + py(1-p) + pyp = y(1-p) + pyp.$$

Furthermore $p_2 yC \subset p_2 C = p_2 p_1 C = (0)$, and so $p_2 y p_1 = 0$. And by hypothesis, $p_1 y p_2 = p_1(p_2 y^*)^* = p_1(p_2 y^* p_2)^* = 0$. So

$$p_1 P(A) p_2 = (0).$$

Thus

$$y = y(1-p) + p_1 y p_1 + p_2 y p_2.$$

If also $x \in P(A)$ then $x = x(1-p) + p_1 x p_1 + p_2 x p_2$, and so

$$xy = x(1-p)y(1-p) + p_1 x p_1 y p_1 + p_2 x p_2 y p_2,$$

and so again $p_1 xy p_2 = 0$. Since $C = (I-P)(A)$ we see that

$$(I-P)(xy) = p_1(I-P)(xy)p_2 = p_1 xy p_2 - p_1 P(xy) p_2 = 0,$$

so that $xy = P(xy) \in P(A)$. So $P(A)$ is a subalgebra. $\square$

Remark. If $CP(A)^* \subset [BC]$ where $B = C^*_{\mathrm{e}}(A)$ then the first hypothesis in the previous result holds.

Remark. If A is the counterexample algebra of 5×5 scalar matrices described above Lemma 4.7, it is very illustrative to compute the various associated objects of interest in our paper. We leave the details to the reader as an exercise. Here $C^*(A) = C^*_{\mathrm{e}}(A) = I(A) = M_3 \oplus M_2 \subset M_5$. If P is the projection in that example, namely the map that replaces the $(1,3)$ entry with 0, then $C^*(P(A)) = I(P(A)) = 0 \oplus M_2 \subset M_5$. A completely contractive completely positive projection $\tilde{P}$ on $C^*(A) = I(A)$ that extends P is the map

$$x \oplus \begin{bmatrix} a & b \\ c & d \end{bmatrix} \mapsto \begin{bmatrix} \frac{1}{2}(a+d) & \frac{1}{2}b & 0 & 0 & 0 \\ \frac{1}{2}c & \frac{1}{2}(a+d) & \frac{1}{2}b & 0 & 0 \\ 0 & \frac{1}{2}c & \frac{1}{2}(a+d) & 0 & 0 \\ 0 & 0 & 0 & a & b \\ 0 & 0 & 0 & c & d \end{bmatrix}, \quad x \in M_3.$$

To see that this is completely contractive is a tiny modification of the proof of Lemma 4.7. A completely contractive projection extending $I - P$ is the projection onto the $(1, 3)$ coordinate. The support projection of $\tilde{P}$ defined just before Proposition 2.10 is $e = 0 \oplus I_2$, which has complement $r = I_3 \oplus 0$. The projections p_1, p_2, p from the proof of Theorem 4.9 are $p_1 = E_{11},\ p_2 = E_{33},\ p = E_{11} + E_{33}$, and $1 - p = E_{22} + E_{44} + E_{55}$. Also, $r - p = E_{22}$. Note that $CP(A)^* p_2 \neq CP(A)^*$, of course. Indeed this example is an excellent illustration of what is going on in the proof of Theorem 4.9. Note that if we change the definition of A by replacing either the $(2, 3)$ entry or the $(3, 2)$ entry then the hypotheses of Theorem 4.9 *are* satisfied.

Examining why the general example described in Example 4.6 does not satisfy the hypotheses of Theorem 4.9 is illustrative: it is not hard to see that if it did then $v_1 w_2 z_2^* = 0$ for all $v, w, z \in V$. However, if $0 \neq \sum_{k=1}^n v_1^k w_2^k \in C$ then we obtain the contradiction

$$0 \neq \Big(\sum_{k=1}^n v_1^k w_2^k\Big)\Big(\sum_{k=1}^n v_1^k w_2^k\Big)^* = 0.$$

It would be interesting to investigate other conditions that might imply that $P(A)$ is a subalgebra, particularly when in the standard position (namely $P : A \to A$ is a unital completely bicontractive projection whose range generates A as an operator algebra). Some which might be worth investigating are if the algebra C in Theorem 4.9 is a maximal ideal in A, or if C contains the radical of A. Note that any one of these conditions rules out our counterexamples above.

5. Another condition

We now look at another condition on a completely contractive projection P which is automatic for bicontractive projections in the C^*-algebra case, namely that the induced projection on $\mathrm{Re}(A)$ is bicontractive. We will not assume that the induced projection on $\mathrm{Re}(A)$ has *completely* contractive complementary projection $I - P$. We are not able to solve the problem yet, but have made some progress towards the solution.

Lemma 5.1. *Let A be a unital operator algebra, and let $P : A \to A$ be a unital completely contractive projection such that the induced projection on* $\mathrm{Re}(A)$ *is bicontractive. We also write P for an extension to a unital completely contractive weak-$*$ continuous projection on the von Neumann algebra B^{**}, where B is a C^*-algebra containing A as a unital subalgebra* (*see the argument in the proof of Corollary 2.11*). *Let e be the support projection of P on B as in* [Effros and Størmer 1979, p. 129]. *If $x \in A \cap e^\perp B e^\perp$ and $x = a + ib$ with $a = a^*$, $b = b^*$, then* $\|a_+\| = \|a_-\| = \|b_+\| = \|b_-\|$.

Proof. Suppose that $\|a\| \leq 1$. By the Kadison–Schwarz inequality,

$$P(a)^* P(a) \leq P(a^* a) = P(e^\perp a^* a e^\perp) \leq P(e^\perp) = 0.$$

So $P(a) = 0$, and similarly $P(b) = 0$. Suppose that $\|a_+\| > \|a_-\|$. Then by the spectral theorem for a, there exists $\epsilon > 0$ with

$$\|a - \epsilon 1\| = \|a_+ - a_- - \epsilon 1\| < \|a_+ - a_-\| = \|a\|.$$

Thus $\|a - \epsilon 1\| < \|(I - P)(a - \epsilon 1)\|$, a contradiction. So $\|a_+\| \leq \|a_-\|$. A similar argument shows that $\|a_-\| \leq \|a_+\|$; therefore, $\|a_+\| = \|a_-\|$. Similarly (or by replacing x by ix), $\|b_+\| = \|b_-\|$.

Let $y = \mathrm{Re}(2x - x^2)$. The last paragraph shows that $\|y\| = \|y_+\| = \|y_-\|$. Now assume that $\|a\| = 1 \geq \|b\|$. So $\|a_\pm\| = 1$. Let ψ be a state with $\psi(a_-) = 1$. Then $\psi(a_+) = 0$ or else $\psi(|a|) = \psi(a_+) + \psi(a_-) > 1$, which is impossible. By standard arguments these imply that $\psi(a_-^2) = 1$ and $\psi(a_+^2) = 0$. Since

$$y = 2a_+ - 2a_- - (a_+^2 + a_-^2) + (b_+^2 + b_-^2),$$

we have

$$\psi(y) = -3 + \psi(b_+^2 + b_-^2).$$

It is well known that for a selfadjoint operator $T = T_+ - T_- = R - S$ with $R, S \geq 0$, we have $\|T_+\| \leq \|R\|$. Thus

$$\|y\| = \|y_+\| \leq \|2a_+ - a_+^2 + (b_+^2 + b_-^2)\| \leq 2.$$

Since $\psi(y) = -3 + \psi(b_+^2 + b_-^2)$ we must have $\psi(b_+^2 + b_-^2) = 1$, so that $\|b^2\| = 1 = \|b\|$. Replacing x by ix, we see that $\|a\| = \|b\|$. $\square$

Lemma 5.2. *Let A be a unital operator algebra, and let $P : A \to A$ be a unital completely contractive projection such that the induced projection on* $\mathrm{Re}(A)$ *is bicontractive. We also write P for an extension to a unital completely contractive weak-$*$ continuous projection on the von Neumann algebra B^{**}, where B is a C^*-algebra containing A as a unital subalgebra* (*as in the last result*). *Let e be the support projection of P on B as in* [Effros and Størmer 1979, p. 129]. *If $x \in A \cap e^\perp B e^\perp$ and $x = a + ib$ with $a = a^*$, $b = b^*$, and $\|a\| = 1$, then $u(a)^2 = u(b)^2$.*

Proof. Since $b = b^*$, we know that $u(b)$ is a selfadjoint tripotent and $u(b)^2$ is a projection. It is well known that $u(x)^* u(x) = u(x^* x)$ (to see this, note that $x(x^* x)^n \to x u(x^* x)$, so that $x u(x^* x) = u(x)$ from which the relation is easy). Hence $u(b)^2 = u(b^2) = u(b_+^2 + b_-^2)$. As we saw in the last proof, if ψ is a state with $\psi(a_-) = 1$ then $\psi(b_+^2 + b_-^2) = 1$. Now $\psi(a_-) = 1$ if and only if $\psi(u(a_-)) = 1$, and $\psi(b_+^2 + b_-^2) = 1$ if and only if $\psi(u(b)^2) = 1$; see [Edwards and Rüttimann 1996, Lemma 3.3(i)]. So $\{u(a_-)\}' \cap S(B) \subset \{u(b)^2\}' \cap S(B)$, where $S(B)$ is the state space and the "prime" is as in [op. cit.]. From this, as is well known (and simple to prove), we have that $u(a_-) \leq u(b)^2$. Similarly, $u(a_+) \leq u(b)^2$, so that $u(a_-) + u(a_+) = u(a)^2 \leq u(b)^2$. Similarly, $u(b)^2 \leq u(a)^2$, so $u(a)^2 = u(b)^2$. $\square$

Lemma 5.3. *Let A be a unital operator algebra, and let $P : A \to A$ be a unital completely contractive projection such that the induced projection on* $\mathrm{Re}(A)$ *is bicontractive. We also write P for an extension to a unital completely contractive weak-$*$ continuous projection on the von Neumann algebra B^{**}, where B is a C^*-algebra containing A as a unital subalgebra* (*as in the last results*). *Let e be the support projection of P on B as in* [Effros and Størmer 1979, p. 129]. *Suppose that $x \in A \cap e^\perp B e^\perp$ has norm-*1. *Then $u(x)^2 = 0$ and $x = u(x) + y$ for some $y \in B^{**}$ with $u(x)y = u(x)y^* = yu(x) = y^*u(x) = 0$. Finally*, $\|\mathrm{Re}\ x\| = \frac{1}{2}$.

Proof. Suppose that $x \in A \cap e^\perp B e^\perp$, and choose an angle θ so that $\|\mathrm{Re}(e^{i\theta}x)\|$ is maximized. By Lemma 5.1 this equals $\|\mathrm{Im}(e^{i\theta}x)\|$. Write $z = e^{i\theta}x = a + ib$ with $a = a^*$, $b = b^*$. Scale z so that $\|a\| = 1$ (so $\|b\| = 1$ by Lemma 5.1). Write $a = u(a) + a_\perp$ and $b = u(b) + b_\perp$. Note that $u(a)a_\perp = u(a)(a - u(a)) = 0$, since $u(a)a = u(a)^3 a = u(a)^2$. Similarly $a_\perp u(a) = 0$, and $b_\perp u(b) = u(b)b_\perp = 0$. Since $u(b)^2 = u(a)^2$ by Lemma 5.2, we have $u(a)b_\perp = u(a)u(b)^2 b_\perp = 0$, and similarly $b_\perp u(a) = a_\perp u(b) = u(b)a_\perp = 0$. Hence $a_\perp$ and $b_\perp$ are contractions by the orthogonality of $u(a)$ and $a_\perp$, and of $u(b)$ and $b_\perp$. Consider

$$\frac{1+i}{\sqrt{2}}(a+ib) = \frac{a-b}{\sqrt{2}} + i\frac{a+b}{\sqrt{2}}.$$

By the maximality property of θ we have $\|(a-b)/\sqrt{2}\| = \|(a+b)/\sqrt{2}\| \le 1$. Now

$$\frac{u(a)-u(b)}{\sqrt{2}} = u(a)^2\,\frac{a-b}{\sqrt{2}} \quad \text{and} \quad \frac{u(a)+u(b)}{\sqrt{2}} = u(a)^2\,\frac{a+b}{\sqrt{2}},$$

so these are contractions. Squaring each of these we see that

$$u(a)^2 - \tfrac{1}{2}\big(u(a)u(b) + u(b)u(a)\big) \quad \text{and} \quad u(a)^2 + \tfrac{1}{2}\big(u(a)u(b) + u(b)u(a)\big)$$

are contractions. Since $u(a)^2$ is a projection, hence an extreme point, we deduce that $u(a)u(b) + u(b)u(a) = 0$, or $u(a)u(b) = -u(b)u(a)$. Using this, if $w_1 = \frac{1}{2}(u(a) + iu(b))$ then a simple computation shows that $w_1 w_1^* w_1 = w_1$, so that w_1 is a partial isometry. Let $w = z/2$. Clearly $\|w\| \le 1$, but now we see that

$$1 = \|w_1\| = \|u(a)^2 w\| \le \|w\|.$$

So $\|w\| = 1$ and $\|z\| = \|x\| = 2$. This proves the last assertion of the theorem, since $\|\mathrm{Re}(x)\| = \|\mathrm{Im}(x)\| \le 1$ by the maximality property of θ, but they clearly cannot be strict contractions since $\|x\| = 2$. So henceforth we may assume that $\theta = 0$ and $z = x$.

Let $w_2 = \frac{1}{2}(a_\perp + ib_\perp)$, so that $w = w_1 + w_2$, and $w_1 w_2 = w_2 w_1 = w_1 w_2^* = w_2^* w_1 = 0$. Note that $u(w) = w_1 + u(w_2) \neq 0$, and $u(e^{-i\theta}w) = e^{-i\theta}u(w)$, so $\|x\| = 2$. Also $ww^* = w_1 w_1^* + w_2 w_2^*$. Suppose ψ is a state with $\psi(w_2 w_2^*) = 1$. Then since $\|ww^*\| \le 1$ we must have $\psi(w_1 w_1^*) = 0$, which forces $\psi(u(a)^2) = \psi(u(b)^2) = 0$.

Thus $\psi \notin \{u(a^2)\}' = \{a^2\}'$ [Edwards and Rüttimann 1996, Lemma 3.3(i)], so $\psi(a_\perp^2) \neq 1$ since $a^2 = u(a^2) + a_\perp^2$. On the other hand,

$$1 = \psi\left(\frac{a_\perp^2}{4} + \frac{b_\perp^2}{4} + i\left(\frac{b_\perp a_\perp}{4} - \frac{a_\perp b_\perp}{4}\right)\right).$$

We deduce the contradiction that $\psi(a_\perp^2) = 1$. This contradiction shows that $1 - w_2 w_2^*$ is strictly positive so that $u(w_2 w_2^*) = 0$. Hence $u(ww^*) = u(w_1 w_1^*) = w_1 w_1^*$, and so $u(w) = \lim_{n\to\infty}(ww^*)^n w = w_1 w_1^* w = w_1$.

Finally, suppose that $x \in A \cap e^\perp B e^\perp$ has norm-1 (so that x may be taken to be our previous w). Then $u(x)^2 = w_1^2 = 0$. That $x = u(x) + y$, where $u(x)$ is orthogonal to y and y^*, follows because $w = u(w) + w_2$ and

$$u(w)w_2 = u(w)w_2^* = w_2 u(w) = w_2^* u(w) = 0,$$

the latter because $u(w)$ is a linear combination of the selfadjoint $u(a)$, $u(b)$, which are each orthogonal to $a_\perp$ and $b_\perp$. □

Corollary 5.4. *If the conditions of the previous lemmas hold and* $A \cap e^\perp B e^\perp = (0)$, *then* $P(A)$ *is a subalgebra of* A.

Proof. For $x, y \in P(A)$ we have $exye = eP(xy)e$ by Proposition 2.10. Thus $xy - P(xy) \in e^\perp B^{**} e^\perp \cap A = (0)$, showing that $P(A)$ is closed under products. □

Corollary 5.5. *If the conditions of the previous lemmas hold and* B *is commutative, then* $A \cap e^\perp B e^\perp = (0)$ *and* $P(A)$ *is a subalgebra of* A.

Proof. By the proof of Lemma 5.3, if $x \in A \cap e^\perp B e^\perp$ and $e^{i\theta} x = a + ib$ with $a = a^*$, $b = b^*$, and $\|a\| = 1$, we obtained $u(a)u(b) = -u(b)u(a) = 0$ and $u(a) = u(a)u(b)^2 = 0$. This is impossible since $\|a\| = 1$, so $A \cap e^\perp B e^\perp = (0)$. Then apply Corollary 5.4. □

As in Section 4, to show $P(A)$ is a subalgebra of A, we may replace A by D, the closed algebra generated by $P(A)$. After this is done, in the previous lemmas $A \cap e^\perp B e^\perp$ becomes $(I - P)(D)$.

Theorem 5.6. *Let* P *be a unital completely contractive projection on* A *such that* $I - P$ *is contractive on* $\mathrm{Re}(A)$. *Suppose that* A *is a subalgebra of* M_N *for some* $N \in \mathbb{N}$ *and let* D *be the closed algebra generated by* $P(A)$. *Then every element of* $(I - P)(D)$ *is nilpotent. Furthermore, if* D *is semisimple then the range of* P *is a subalgebra of* A.

Proof. Note that $(I - P)(D)$ is an ideal of D by Corollary 2.11. Suppose that $x \in (I - P)(D)$ has norm-1 and is not nilpotent. Set $y_1 = x$. By Lemma 5.3, $x = u(x) + x_\prime$ where $u(x)^2 = 0$ and $x_\prime \perp u(x)$. Furthermore $x^2 = (x_\prime)^2$ lies in D and $x^2 \perp u(x)$. Similarly, since $x^2 \neq 0$ we set $y_2 = x^2 / \|x^2\|$. Then $y_2 = u(y_2) + (y_2)_\prime$ where $u(y_2)$ and $(y_2)_\prime$ are perpendicular to each other, and $u(y_2)$ is perpendicular to $u(x)$. (This is because $u(y_2)$ is a limit of products beginning and ending with x^2, and, e.g., $x^2u(x) = (x_\prime)^2u(x) = 0$.) Continuing in this way, we obtain an infinite sequence of norm-1 elements y_k such that $u(y_n) \perp u(y_k)$ for $k \leq n$. It is well known that $u(y) \neq 0$ if $\|y\| = 1$. This contradicts finite-dimensionality. So x is nilpotent. Since $(I-P)(D)$ is an ideal of D consisting of nilpotents, it follows that it lies in the Jacobson radical of D. Thus if D is semisimple then $(I-P)(D) = (0)$, so that $P(A) = D$ as before. □

For the following we no longer assume A is finite-dimensional but retain the other assumptions of the above theorem.

Lemma 5.7. *Let P be a unital completely contractive projection on A such that $I-P$ is contractive on* $\mathrm{Re}(A)$, *and let D be the closed algebra generated by $P(A)$. If $x \in (I-P)(D)$ and $\|x\| = 1$, then $\|x^{2^n}\| \leq 2/2^{2^n}$. Also, x is quasiregular* (*that is*, *quasi-invertible*).

Proof. Let $x = a+ib$ as in previous lemmas. By Lemmas 5.1 and 5.2, $\|a\| = \|b\| = \frac{1}{2}$. Hence $\|\mathrm{Re}(x^2)\| = \|a^2 - b^2\| \leq \frac{1}{4}$. Again from Lemma 5.1, $\|x^2\| \leq \frac{1}{2}$. The first result now follows by considering normalizations of further powers of 2, and using mathematical induction. It is easily seen that

$$\sum_{k=1}^{\infty} \|x^k\| \leq 1 + \sum_{m=1}^{\infty} 2^m \, \|x^{2^m}\| \leq 1 + \sum_{m=1}^{\infty} 2^m \, \frac{2}{2^{2^m}} < \infty.$$

It follows that $\sum_{k=1}^{\infty} x^k$ converges, so that $1-x$ is invertible. □

Remark. It is still open whether the ideal $(I-P)(D)$ above consists entirely of quasiregular elements. If this is the case, then the above Theorem 5.6 holds for arbitrary unital operator algebras. Note too that the assertion about quasiregulars in Lemma 5.7 does follow from Lemma 5.1. That result shows that the ideal $(I-P)(D)$ in D has no nonzero real positive elements (for, in the language of that result, if $a_- = 0$ then $a_+ = b_+ = b_- = 0$). The ideas in the proof of [Blecher and Read 2013, Corollary 6.9] then also show that if $x \in \mathrm{Ball}((I-P)(D))$ then x is quasiregular.

6. Jordan morphisms and Jordan subalgebras of operator algebras

We recall that a *Jordan homomorphism* $T : A \to B$ is a linear map satisfying $T(ab+ba) = T(a)T(b) + T(b)T(a)$ for $a, b \in A$, or equivalently, that $T(a^2) = T(a)^2$ for all $a \in A$ (the equivalence follows by applying T to $(a+b)^2$). By a *Jordan operator algebra* we shall simply mean a norm-closed *Jordan subalgebra* A of an operator algebra, namely a norm-closed subspace closed under the Jordan product $\frac{1}{2}(ab+ba)$, or equivalently with $a^2 \in A$ for all $a \in A$ (that is, A is closed under squares).

It is natural to ask if the completely bicontractive algebra problem studied in Section 4 becomes simpler if the range of the projection $P : A \to A$ is also a Jordan subalgebra (that is, $P(a)^2 \in P(A)$) for all $a \in A$. We next dispose of this question:

Example 6.1. Let $y = E_{21} \oplus E_{12} \in M_4$, and let

$$x = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then $xy = -yx$, so that if F is the span of x and y then F is closed under squares. However, F is not an algebra since $xy \notin F$. Let $V = \{z \oplus z \in M_8 : z \in F\}$, and form the algebra $A = A(V)$ described in Example 4.6. By Corollary 4.8 the canonical projection $P : A \to A$ which replaces the $(1, 3)$ entry of a matrix in $A(V)$ by 0, is a (real completely positive) completely bicontractive and unital projection, but its range is not a subalgebra. However, its range is a Jordan subalgebra; $P(A)$ is closed under squares since $z^2 = 0$ for $z \in F$. Thus, the completely bicontractive algebra problem does not become simpler if the range of the projection $P : A \to A$ is also a Jordan subalgebra.

The following variant of the Banach–Stone theorem for C^*-algebras will be evident to "JB-experts".

Lemma 6.2. *Let A be a unital C^*-algebra, and $T : A \to B(H)$ a unital complete isometry such that $T(A)$ is closed under taking squares (thus, $T(A)$ is a Jordan algebra). Then $T(A)$ is a C^*-subalgebra of $B(H)$, and T is a $*$-homomorphism.*

Proof. Since such T is necessarily $*$-linear as we said in the introduction, $T(A)$ is a JB*-algebra, hence a selfadjoint JB*-triple; see, e.g., [Cabrera García and Rodríguez Palacios 2014]. By the theory of JB*-triples T is a Jordan homomorphism. (Two other proofs of this: look at the selfadjoint part and use the fact that isometries in that category are Jordan homomorphisms [Isidro and Rodríguez-Palacios 1995]; or it can be deduced using the C^*-envelope as in the next proof). In particular for each $x \in A_{\mathrm{sa}}$ we have $T(x^2) = T(x)^2$, so by Choi's multiplicative domain result (see, e.g., [Blecher and Le Merdy 2004, Proposition 1.3.11]) we have $T(xy) = T(x)T(y)$ for all $y \in A$. So T is a homomorphism and $T(A)$ is a C^*-subalgebra. $\square$

It is natural to ask if the analogous result is true for operator algebras. That is, if B is a closed unital Jordan subalgebra of an operator algebra A, and if B is unitally and linearly completely isometric to another unital operator algebra, then is B actually a subalgebra of A? If the algebra is also commutative this is true and follows from the next result.

Lemma 6.3. *Let A be a unital operator algebra, and let $T : A \to B$ be a unital complete isometry onto a unital Jordan operator algebra. Then T is a Jordan homomorphism, and $T(a^n) = T(a)^n$ for every $n \in \mathbb{N}$ and $a \in A$.*

Proof. Note that $T(a)^3$ is the Jordan product of $T(a)$ and $T(a)^2$, so $T(A)$ is closed under cubes. Similarly it is closed under every power. By the property of the C^*-envelope mentioned in the introduction, there exists a $*$-homomorphism $\pi : C^*(T(A)) \to C^*_{\mathrm{e}}(A)$ with $\pi \circ T = I_A$. So $a^n = \pi(T(a)^n) = \pi(T(a^n))$. Since $\pi|_{T(A)}$ is one-to-one, the results follow. □

Remark. In the proof of the last result one could have instead used [Arazy and Solel 1990, Corollary 2.8].

We now answer the question above Lemma 6.3 in the negative:

Example 6.4. Let $P : A \to A$ be a completely contractive projection on an operator algebra A on H whose kernel is an ideal I; see, e.g., Corollary 2.11 or Lemma 4.1. Then it is known that $B = A/I$ is an operator algebra [Blecher and Le Merdy 2004, Proposition 2.3.4], and the induced map $\tilde{P} : B \to P(A)$ is a completely isometric isomorphism, and $\tilde{P}$ will be unital if A and P are unital. If these hold, and in addition $P(A)$ is a Jordan subalgebra of A which is not a subalgebra, then $T = \tilde{P} : B \to A \subset B(H)$ is a unital complete isometry such that $T(B)$ is closed under taking squares (thus, $T(B)$ is a unital Jordan subalgebra), but $T(B)$ is not a subalgebra, and T is not an algebra homomorphism. In particular, we can take A to be the algebra in Example 6.1.

We finish our paper with another Banach–Stone-type theorem for operator algebras:

Proposition 6.5. *Suppose that $T : A \to B$ is a completely isometric surjection between approximately unital operator algebras. Then T is real* (*completely*) *positive if and only if T is an algebra homomorphism.*

Proof. If T is an algebra homomorphism then by Meyer's theorem [Blecher and Le Merdy 2004, Theorem 2.1.13], T extends to a unital completely isometric surjection between the unitizations, which then extends by Wittstock's extension theorem to a unital completely contractive, hence completely positive, map on a generated C^*-algebra. So T is real completely positive.

Conversely, suppose that T is real positive. We may assume that A and B are unital, since T^{**} is a real positive completely isometric surjection between unital operator algebras. By the Banach–Stone theorem for operator algebras [op. cit., Theorem 4.5.13], there exists a completely isometric surjective homomorphism $\pi : A \to B$ and a unitary u with $u, u^{-1} \in B$ with $T = \pi(\cdot)u$. The restriction of T to $\mathbb{C}1$ is real positive, hence positive; see [Bearden et al. 2014, Section 2]. Thus $u \geq 0$, and so $u = (u^2)^{1/2} = 1$. Hence T is an algebra homomorphism. □

Remark. One may also prove a limited version of this result for algebras with no kind of approximate identity by using the ideas in the proof of Proposition 2.1.

There is also a Jordan variant of the last result, a simple adaption of the main theorem in [Arazy and Solel 1990]. Here we just state the unital case; see [Blecher et al. $\geq$ 2016] for more on this topic.

Proposition 6.6. *Suppose that $T : A \to B$ is an isometric surjection between unital Jordan operator algebras. Then T is real positive if and only if T is a Jordan algebra homomorphism.*

Proof. If T is a Jordan algebra homomorphism and $u = T(1_A)$ and $T(v) = 1_B$ then $2u = u1_B + 1_B u = 2T(1_A \cdot v) = 2T(v) = 2 \cdot 1_B$. So $u = 1_B$. However, a unital contractive map is real positive [Bearden et al. 2014, Section 2].

The converse follows by the same proof as for Proposition 6.5, but using the form of the Banach–Stone theorem for operator algebras in [Arazy and Solel 1990, Corollary 2.10]. By that result $T(1)$ is a unitary u with $u, u^{-1} \in B$. Moreover $T(\cdot)u^{-1}$ is an isometric surjection onto B, so by the same result it is a Jordan homomorphism π. We have $T = \pi(\cdot)u$, and we finish as before. □

Acknowledgments

Some of this material was presented at an AMS special session in January 2015 (Operator Algebras and Their Applications: A Tribute to Richard V. Kadison), as briefly surveyed in the article [Blecher 2015], which will appear in the proceedings of that meeting. Corollary 4.2 and part of Theorem 4.5 were first stated in that article, and later added to the present paper. We thank Joel Feinstein for a conversation during which we together arrived at an insight needed in these results. We are also indebted to the referee for suggesting that we consider the Jordan algebra version of the topics of this paper; this insight led to the work [Blecher et al. $\geq$ 2016] in progress.

References

[Akemann 1970] C. A. Akemann, "Left ideal structure of C^*-algebras", *J. Functional Analysis* **6** (1970), 305–317. MR 0275177 Zbl 0199.45901

[Arazy and Solel 1990] J. Arazy and B. Solel, "Isometries of nonselfadjoint operator algebras", *J. Funct. Anal.* **90**:2 (1990), 284–305. MR 1052336 Zbl 0713.46043

[Arveson 1969] W. B. Arveson, "Subalgebras of C^*-algebras", *Acta Math.* **123** (1969), 141–224. MR 0253059 Zbl 0194.15701

[Barton and Timoney 1986] T. Barton and R. M. Timoney, "Weak*-continuity of Jordan triple products and its applications", *Math. Scand.* **59**:2 (1986), 177–191. MR 884654

[Battaglia 1991] M. Battaglia, "Order theoretic type decomposition of JBW*-triples", *Quart. J. Math. Oxford Ser.* (2) **42**:166 (1991), 129–147. MR 1107278 Zbl 0736.46052

[Bearden et al. 2014] C. A. Bearden, D. P. Blecher, and S. Sharma, "On positivity and roots in operator algebras", *Integral Equations Oper. Theory* **79**:4 (2014), 555–566. MR 3231244 Zbl 1311.46051

[Blecher 2004] D. P. Blecher, "Are operator algebras Banach algebras?", pp. 53–58 in *Banach algebras and their applications*, edited by A. T.-M. Lau and V. Runde, Contemporary Mathematics **363**, American Mathematical Society, Providence, RI, 2004. MR 2097949 Zbl 1078.46040

[Blecher 2013] D. P. Blecher, "Noncommutative peak interpolation revisited", *Bull. Lond. Math. Soc.* **45**:5 (2013), 1100–1106. MR 3105002 Zbl 1284.46053

[Blecher 2015] D. P. Blecher, "Generalization of C^*-algebra methods via real positivity for operator and Banach algebras", preprint, 2015. To appear in *Operator algebras and their applications: A tribute to Richard V. Kadison*, edited by R. S. Doran and E. Park, Contemporary Mathematics **671**, American Mathematical Society. arXiv 1508.03059

[Blecher and Labuschagne 2003] D. P. Blecher and L. E. Labuschagne, "Logmodularity and isometries of operator algebras", *Trans. Amer. Math. Soc.* **355**:4 (2003), 1621–1646. MR 1946408 Zbl 1026.46046

[Blecher and Le Merdy 2004] D. P. Blecher and C. Le Merdy, *Operator algebras and their modules: an operator space approach*, London Mathematical Society Monographs. New Series **30**, Oxford University Press, 2004. MR 2111973 Zbl 1061.47002

[Blecher and Magajna 2005a] D. P. Blecher and B. Magajna, "Duality and operator algebras: automatic weak* continuity and applications", *J. Funct. Anal.* **224**:2 (2005), 386–407. MR 2146046

[Blecher and Magajna 2005b] D. P. Blecher and B. Magajna, "Duality and operator algebras, II: Operator algebras as Banach algebras", *J. Funct. Anal.* **226**:2 (2005), 485–493. MR 2160105 Zbl 1083.46033

[Blecher and Neal 2012a] D. P. Blecher and M. Neal, "Open projections in operator algebras, I: Comparison theory", *Studia Math.* **208**:2 (2012), 117–150. MR 2910983 Zbl 1259.46045

[Blecher and Neal 2012b] D. P. Blecher and M. Neal, "Open projections in operator algebras, II: Compact projections", *Studia Math.* **209**:3 (2012), 203–224. MR 2944468 Zbl 1259.46046

[Blecher and Ozawa 2015] D. P. Blecher and N. Ozawa, "Real positivity and approximate identities in Banach algebras", *Pacific J. Math.* **277**:1 (2015), 1–59. MR 3393680 Zbl 06479248

[Blecher and Read 2011] D. P. Blecher and C. J. Read, "Operator algebras with contractive approximate identities", *J. Funct. Anal.* **261**:1 (2011), 188–217. MR 2785898 Zbl 1235.47087

[Blecher and Read 2013] D. P. Blecher and C. J. Read, "Operator algebras with contractive approximate identities, II", *J. Funct. Anal.* **264**:4 (2013), 1049–1067. MR 3004957 Zbl 1270.47067

[Blecher and Read 2014] D. P. Blecher and C. J. Read, "Order theory and interpolation in operator algebras", *Studia Math.* **225**:1 (2014), 61–95. MR 3299396 Zbl 06390240

[Blecher et al. 2008] D. P. Blecher, D. M. Hay, and M. Neal, "Hereditary subalgebras of operator algebras", *J. Oper. Theory* **59**:2 (2008), 333–357. MR 2411049 Zbl 1164.46018

[Blecher et al. $\geq$ 2016] D. P. Blecher, M. Neal, and Z. Wang, "Jordan operator algebras and their contractive projections", in preparation (tentative title).

[Cabrera García and Rodríguez Palacios 2014] M. Cabrera García and Á. Rodríguez Palacios, *Non-associative normed algebras, 1: The Vidav–Palmer and Gelfand–Naimark theorems*, Encyclopedia of Mathematics and its Applications **154**, Cambridge University Press, 2014. MR 3242640 Zbl 1322.46003

[Choi and Effros 1976] M.-D. Choi and E. G. Effros, "The completely positive lifting problem for C^*-algebras", *Ann. of Math.* (2) **104**:3 (1976), 585–609. MR 0417795 Zbl 0361.46067

[Choi and Effros 1977] M.-D. Choi and E. G. Effros, "Injectivity and operator spaces", *J. Funct. Anal.* **24**:2 (1977), 156–209. MR 0430809 Zbl 0341.46049

[Chu et al. 2004] C.-H. Chu, M. Neal, and B. Russo, "Normal contractive projections preserve type", *J. Operator Theory* **51**:2 (2004), 281–301. MR 2074182 Zbl 1113.46066

[Edwards and Rüttimann 1996] C. M. Edwards and G. T. Rüttimann, "Compact tripotents in bi-dual JB*-triples", *Math. Proc. Cambridge Philos. Soc.* **120**:1 (1996), 155–173. MR 1373355 Zbl 0853.46070

[Effros 1963] E. G. Effros, "Order ideals in a C^*-algebra and its dual", *Duke Math. J.* **30** (1963), 391–411. MR 0151864 Zbl 0117.09703

[Effros and Størmer 1979] E. G. Effros and E. Størmer, "Positive projections and Jordan structure in operator algebras", *Math. Scand.* **45**:1 (1979), 127–138. MR 567438 Zbl 0455.46059

[Friedman and Russo 1982] Y. Friedman and B. Russo, "Contractive projections on $C_0(K)$", *Trans. Amer. Math. Soc.* **273**:1 (1982), 57–73. MR 664029 Zbl 0534.46037

[Friedman and Russo 1984] Y. Friedman and B. Russo, "Conditional expectation without order", *Pacific J. Math.* **115**:2 (1984), 351–360. MR 765191 Zbl 0563.46039

[Friedman and Russo 1985] Y. Friedman and B. Russo, "Solution of the contractive projection problem", *J. Funct. Anal.* **60**:1 (1985), 56–79. MR 780104 Zbl 0558.46035

[Godefroy et al. 1993] G. Godefroy, N. J. Kalton, and P. D. Saphar, "Unconditional ideals in Banach spaces", *Studia Math.* **104**:1 (1993), 13–59. MR 1208038 Zbl 0814.46012

[Hamana 1999] M. Hamana, "Triple envelopes and Šilov boundaries of operator spaces", *Math. J. Toyama Univ.* **22** (1999), 77–93. MR 1744498 Zbl 0948.46045

[Harris 1981] L. A. Harris, "A generalization of C^*-algebras", *Proc. London Math. Soc.* (3) **42**:2 (1981), 331–361. MR 607306 Zbl 0476.46054

[Isidro and Rodríguez-Palacios 1995] J. M. Isidro and Á. Rodríguez-Palacios, "Isometries of JB-algebras", *Manuscripta Math.* **86**:3 (1995), 337–348. MR 1323796 Zbl 0834.17048

[Kadison 1951] R. V. Kadison, "Isometries of operator algebras", *Ann. of Math.* (2) **54** (1951), 325–338. MR 0043392 Zbl 0045.06201

[Kadison 1952] R. V. Kadison, "A generalized Schwarz inequality and algebraic invariants for operator algebras", *Ann. of Math.* (2) **56** (1952), 494–503. MR 0051442 Zbl 0047.35703

[Lau and Loy 2008] A. T.-M. Lau and R. J. Loy, "Contractive projections on Banach algebras", *J. Funct. Anal.* **254**:10 (2008), 2513–2533. MR 2406685 Zbl 1149.46040

[Neal 2000] M. Neal, "Inner ideals and facial structure of the quasi-state space of a JB-algebra", *J. Funct. Anal.* **173**:2 (2000), 284–307. MR 1760616 Zbl 0959.46039

[Neal and Russo 2000] M. Neal and B. Russo, "Contractive projections and operator spaces", *C. R. Acad. Sci. Paris Sér. I Math.* **331**:11 (2000), 873–878. MR 1806425 Zbl 0973.47052

[Neal and Russo 2003a] M. Neal and B. Russo, "Contractive projections and operator spaces", *Trans. Amer. Math. Soc.* **355**:6 (2003), 2223–2262. MR 1973989 Zbl 1088.46030

[Neal and Russo 2003b] M. Neal and B. Russo, "Operator space characterizations of C^*-algebras and ternary rings", *Pacific J. Math.* **209**:2 (2003), 339–364. MR 1978376 Zbl 1064.46060

[Neal and Russo 2006] M. Neal and B. Russo, "Representation of contractively complemented Hilbertian operator spaces on the Fock space", *Proc. Amer. Math. Soc.* **134**:2 (2006), 475–485. MR 2176016 Zbl 1089.46034

[Neal and Russo 2011] M. Neal and B. Russo, "Existence of contractive projections on preduals of JBW*-triples", *Israel J. Math.* **182** (2011), 293–331. MR 2783974 Zbl 1232.46063

[Neal et al. 2006] M. Neal, É. Ricard, and B. Russo, "Classification of contractively complemented Hilbertian operator spaces", *J. Funct. Anal.* **237**:2 (2006), 589–616. MR 2230351 Zbl 1106.46043

[Paulsen 2002] V. I. Paulsen, *Completely bounded maps and operator algebras*, Cambridge Studies in Advanced Mathematics **78**, Cambridge University Press, 2002. MR 1976867 Zbl 1029.47003

[Pedersen 1979] G. K. Pedersen, *C^*-algebras and their automorphism groups*, London Mathematical Society Monographs **14**, Academic Press, London, 1979. MR 548006 Zbl 0416.46043

[Robertson and Youngson 1982] A. G. Robertson and M. A. Youngson, "Positive projections with contractive complements on Jordan algebras", *J. London Math. Soc.* (2) **25**:2 (1982), 365–374. MR 653394 Zbl 0481.46032

[Russo 1994] B. Russo, "Structure of JB*-triples", pp. 209–280 in *Jordan algebras* (Oberwolfach, 1992), edited by W. Kaup and K. McCrimmon, de Gruyter, Berlin, 1994. MR 1293321 Zbl 0805.46072

[Størmer 1982] E. Størmer, "Positive projections with contractive complements on C^*-algebras", *J. London Math. Soc.* (2) **26**:1 (1982), 132–142. MR 667251 Zbl 0501.46048

[Youngson 1983] M. A. Youngson, "Completely contractive projections on C^*-algebras", *Quart. J. Math. Oxford Ser.* (2) **34**:136 (1983), 507–511. MR 723287 Zbl 0542.46029

Received August 6, 2015. Revised January 11, 2016.

DAVID P. BLECHER
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF HOUSTON
HOUSTON, TX 77204-3008
UNITED STATES
dblecher@math.uh.edu

MATTHEW NEAL
DEPARTMENT OF MATHEMATICS
DENISON UNIVERSITY
GRANVILLE, OH 43023
UNITED STATES
nealm@denison.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.325

INVARIANTS OF SOME COMPACTIFIED PICARD MODULAR SURFACES AND APPLICATIONS

AMIR DŽAMBIĆ

We investigate certain numerical invariants of compactified Picard modular surfaces by principal congruence subgroups of Picard modular groups. Applications to surface classification and modular forms are discussed.

1. Introduction

This article should be considered as a supplement to the work of R.-P. Holzapfel on invariants of Picard modular surfaces which are quotients of the two-dimensional complex ball by principal congruence subgroups.

Holzapfel [1980; 1998b] develops concrete formulas for the Chern numbers and related invariants of compactified ball quotients by principal congruence subgroups $\Gamma_{K(N)}$ of Picard modular groups $\Gamma_K = \mathrm{SU}((2,1), \mathcal{O}_K)$, where $K = \mathbb{Q}(\sqrt{-d})$ is an imaginary quadratic field and N is a positive integer. There, he uses results which are mainly developed in [Holzapfel 1998a]. Using Riemann–Roch theory in combination with proportionality theorems, he also gets information on the classification of these compactified ball quotients and dimensions of spaces of cusp forms relative to the congruence subgroups.

In this article we slightly extend Holzapfel's results, considering not only principal congruence subgroups by natural numbers, but also general integral ideals $\mathfrak{a}$ of the quadratic field $K = \mathbb{Q}(\sqrt{-d})$. Including some technical, number theoretical details, Holzapfel's arguments can also be applied to this slightly more general and larger class of congruence subgroups. One technical result is a formula for the index $[\Gamma_K : \Gamma_K(\mathfrak{a})]$. The final results in principle do not differ from those obtained in [Holzapfel 1980;1998b]. We characterize the class of principal congruence subgroups — excluding on technical grounds a few possible "exceptional cases" — for which the (smooth) compactified ball quotient is a surface of general type. For these surfaces the "coordinates" (c_2, c_1^2) in the surface geography are explicitly known. The dimensions of spaces of cusp forms can also be computed explicitly. Besides the application to the geography of surfaces of general type let us also mention the very recent application of this geometric property in arithmetic geometry,

---

MSC2010: 11F06, 14G35, 14J29.
Keywords: Picard modular surfaces, surfaces of general type, congruence subgroups.

as an ingredient in the proof of the mordellicity of the compactified congruence Picard modular surfaces by Dimitrov–Ramakrishnan [2015].

2. Picard modular groups and their congruence subgroups

Let $K = \mathbb{Q}(\sqrt{-d})$ be an imaginary quadratic field, and denote by $\mathbb{O}_K$ the ring of integers of K. Let V be a three-dimensional K-vector space equipped with a hermitian form $h : V \times V \to K$. Let us assume that the signature of h is $(2, 1)$, i.e., that h has two positive eigenvalues and one negative eigenvalue. Such h can, for instance, be represented by the diagonal matrix $\mathrm{diag}(1, 1, -1)$, but often it is useful to use another hermitian form. We consider the special unitary group

$$\boldsymbol{G} := \mathrm{SU}(h) = \{g \in \mathrm{SL}_3(\mathbb{C}) \mid h(gv, gw) = h(v, w) \text{ for all } v, w \in V_{\mathbb{C}}\}$$

as an algebraic group defined over $\mathbb{Q}$. Its group of $\mathbb{Q}$-rational points is

$$\boldsymbol{G}(\mathbb{Q}) := \mathrm{SU}(h, K) = \{g \in \mathrm{SL}_3(K) \mid h(gv, gw) = h(v, w) \text{ for all } w \in V\}.$$

Often the interpretation of $\boldsymbol{G}$ as a group corresponding to an involution is useful. For this, we remark that the map $\iota = \iota_h$, which associates with each matrix $g \in M_3(K)$ the matrix $g^{\iota_h} := M_h \bar{g}^t M_h^{-1}$, where M_h denotes the matrix which represents h with respect to a suitable basis ($\mathrm{diag}(1, 1, -1)$, say), defines an involution of the second kind on the matrix algebra $M_3(K)$. This means that ι is an antiautomorphism of $M_3(K)$ which acts as the complex conjugation, when restricted to the diagonal matrices $\mathrm{diag}(\alpha, \alpha, \alpha)$. Under this interpretation, the group $\boldsymbol{G}$ appears as the group of all matrices $g \in \mathrm{SL}_3$ such that $gg^{\iota_h} = 1_3$.

We define the so-called *full Picard modular group* as

$$\Gamma_K = \mathrm{SU}(h, \mathbb{O}_K) = \mathrm{SL}_3(\mathbb{O}_K) \cap \boldsymbol{G}(\mathbb{Q}).$$

Γ_K is obviously an arithmetic subgroup in $\boldsymbol{G}(\mathbb{Q})$ as well as its subgroups of finite index, which we simply call *Picard modular groups*. If $\mathfrak{a}$ is an ideal of $\mathbb{O}_K$, let $\Gamma_K(\mathfrak{a})$ denote the *principal congruence subgroup* of Γ_K with respect to $\mathfrak{a}$. It is defined as the subgroup of Γ_K consisting of all elements $\gamma \in \Gamma_K$ such that $\gamma - 1_3 \in M_3(\mathfrak{a})$. In other words, $\Gamma_K(\mathfrak{a})$ can be defined as the kernel of the canonical reduction map

$$\rho : \Gamma_K \to \mathrm{SL}_3(\mathbb{O}_K/\mathfrak{a}).$$

We define the *level* of $\Gamma_K(\mathfrak{a})$ as the absolute norm $N(\mathfrak{a}) = |\mathbb{O}_K/\mathfrak{a}|$. There are finitely many principal congruence subgroups of fixed level. By definition, a *congruence subgroup* of Γ_K is a group which contains a principal congruence subgroup. Since every principal congruence subgroup has a finite index in Γ_K (which is obvious by definition), all congruence subgroups are Picard modular groups. In this section we will treat the following two technical problems:

- computing the index $[\Gamma_K : \Gamma_K(\mathfrak{a})]$,
- determining $\mathfrak{a}$ for which the principal congruence subgroup $\Gamma_K(\mathfrak{a})$ is a neat subgroup.

Note that we say that an arithmetic group is *neat* if the subgroup of $\mathbb{C}^*$ generated by the eigenvalues of all elements in Γ is torsion free. A neat group is obviously torsion free.

Index computations. In order to compute the index $[\Gamma_K : \Gamma_K(\mathfrak{a})]$, we make use of the local-to-global principle which is applicable in the present case.

For a prime number p the group of $\mathbb{Q}_p$-rational points $\boldsymbol{G}(\mathbb{Q}_p)$ is

$$\boldsymbol{G}(\mathbb{Q}_p) = \{g \in \mathrm{SL}_3(K \otimes_{\mathbb{Q}} \mathbb{Q}_p) \mid g g^{\iota_p} = 1_3\}, \tag{2-1}$$

where we write ι_p for the natural extension of ι_h to the algebra $M_3(K \otimes_{\mathbb{Q}} \mathbb{Q}_p)$. The group $\boldsymbol{G}(\mathbb{Z}_p)$ is defined in an obvious manner. We can also define principal congruence subgroups $\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{a})$ of $\boldsymbol{G}(\mathbb{Z}_p)$ in the same way as in the case of Γ_K, now taking two-sided ideals $\mathfrak{a}$ of the order $\mathbb{O}_K \otimes \mathbb{Z}_p$ into account. The following lemma is the starting point of the index calculation. It is an important consequence of strong approximation.

Lemma 2.1. *Let $\mathfrak{a} \subset \mathbb{O}_K$ be an integral ideal of K, let $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ be the prime ideal decomposition of $\mathfrak{a}$, and denote by $p_i = \mathfrak{p}_i \cap \mathbb{Z}$ the unique integer prime over which $\mathfrak{p}$ lies. Then there is an isomorphism*

$$\Gamma_K / \Gamma_K(\mathfrak{a}) \cong \prod_{i=1}^{t} \boldsymbol{G}(\mathbb{Z}_{p_i}) / \boldsymbol{G}(\mathbb{Z}_{p_i})(\mathfrak{p}^{e_i}).$$

In particular,

$$[\Gamma_K : \Gamma_K(\mathfrak{a})] = \prod_{i=1}^{t} [\boldsymbol{G}(\mathbb{Z}_{p_i}) : \boldsymbol{G}(\mathbb{Z}_{p_i})(\mathfrak{p}^{e_i})].$$

Proof. Let λ be the map

$$\lambda : \Gamma_K \longrightarrow \prod_{i=1}^{t} \boldsymbol{G}(\mathbb{Z}_{p_i}) / \boldsymbol{G}(\mathbb{Z}_{p_i})(\mathfrak{p}^{e_i})$$

given by

$$\gamma \mapsto (\gamma \mod \mathfrak{p}_1^{e_1}, \dots, \gamma \mod \mathfrak{p}_t^{e_t}).$$

The kernel of λ is obviously $\Gamma_K(\mathfrak{a})$, by definition. To show surjectivity we recall the strong approximation property: for any finite set of primes $p_1, \dots, p_t$ and elements $g_{p_1} \in \boldsymbol{G}(\mathbb{Z}_{p_1}), \dots, g_{p_t} \in \boldsymbol{G}(\mathbb{Z}_{p_t})$ as well as exponents $e_1, \dots, e_t$, one can find an element $g \in \boldsymbol{G}(\mathbb{Z}) = \Gamma_K$ such that $g \equiv g_{p_i} \mod p_i^{e_i}$ for $i = 1, \dots, t$. But as $\mathfrak{p}_i \mid p_i$ this g satisfies $g \equiv g_{p_i} \mod \mathfrak{p}_i^{e_i}$. This proves the surjectivity of λ. □

In order to compute the local indices, we first need to know more about the local groups $\boldsymbol{G}(\mathbb{Q}_p)$. Looking at their definition (2-1), we see that their structure highly depends on the structure of $K \otimes \mathbb{Q}_p$, which itself is determined by the decomposition behavior of the prime p. Essentially there are two cases:

(i) *p is decomposed in $K = \mathbb{Q}(\sqrt{-d})$.* In this case $K \otimes \mathbb{Q}_p \cong K_{\mathfrak{p}} \oplus K_{\bar{\mathfrak{p}}}$ with two conjugate prime ideals $\mathfrak{p} \neq \bar{\mathfrak{p}}$ in $\mathbb{O}_K$ such that $p = \mathfrak{p}\bar{\mathfrak{p}}$. Therefore

$$M_3(K \otimes \mathbb{Q}_p) \cong M_3(K_{\mathfrak{p}}) \oplus M_3(K_{\bar{\mathfrak{p}}}).$$

Since p is decomposed in K, $\left(\frac{-d}{p}\right) = 1$ and $-d$ is a square in $\mathbb{Q}_p$. Therefore $K_{\mathfrak{p}} \cong K_{\bar{\mathfrak{p}}} \cong \mathbb{Q}_p$. The extension of the field automorphism $\bar{\cdot}: K \to K$ to $K \otimes \mathbb{Q}_p$ must be an involution on $K_{\mathfrak{p}} \oplus K_{\bar{\mathfrak{p}}} \cong \mathbb{Q}_p \oplus \mathbb{Q}_p$ and $\overline{(x, y)} = (y, x)$ is the only possibility. Therefore the extension ι_p of $\iota = \iota_h$ is given by changing the summands in $M_3(\mathbb{Q}_p) \oplus M_3(\mathbb{Q}_p)$. Now it is easy to see that the projection on one of the summands gives an isomorphism between $\boldsymbol{G}(\mathbb{Q}_p)$ and $\mathrm{SL}_3(\mathbb{Q}_p)$, since $\boldsymbol{G}(\mathbb{Q}_p)$ is defined as the group consisting of those pairs $(g, h) \in \mathrm{SL}_3(\mathbb{Q}_p) \oplus \mathrm{SL}_3(\mathbb{Q}_p)$ with $(g^{-1}, h^{-1}) = (h, g)$.

(ii) *p is nondecomposed in K.* In this case there is a unique prime ideal $\mathfrak{p}$ lying over p, such that $K \otimes \mathbb{Q}_p = K_{\mathfrak{p}}$, which is a quadratic extension of $\mathbb{Q}_p$. The extended involution ι_p is an involution of the second kind on $M_3(K_{\mathfrak{p}})$ given by $g \mapsto M_h \bar{g} M_h^{-1}$ and $\boldsymbol{G}(\mathbb{Q}_p) = \mathrm{SU}(h, K_{\mathfrak{p}})$.

Lemma 2.2. *If p is nondecomposed, $\boldsymbol{G}(\mathbb{Q}_p) \cong \mathrm{SU}_3(K_{\mathfrak{p}})$.*

Proof. By Landherr's theorem, the isometry class of a hermitian form h in n variables associated with a quadratic extension E/F of local fields is uniquely determined by its discriminant $\boldsymbol{d}(h) = [\det(h)] \in F^*/N_{E/F}(E^*) \cong \mathbb{Z}/2\mathbb{Z}$. On the other hand, the isomorphism class of the associated unitary group $\mathrm{SU}(h)$ only depends on h up to a multiplicative constant. Hence, the unitary group associated with $\mathrm{diag}(1, 1, -1)$ is isomorphic to the unitary group associated with $\mathrm{diag}(-1, -1, 1)$. Since the discriminant of the latter is 1, the result follows. $\square$

Now we compute the local indices:

Lemma 2.3. *Let $\mathfrak{p}$ be a prime ideal of $K = \mathbb{Q}(\sqrt{-d})$ such that $p = \mathfrak{p} \cap \mathbb{Z}$ is decomposed in K. Then*

$$[\boldsymbol{G}(\mathbb{Z}_p) : \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)] = p^{8n}(1 - p^{-3})(1 - p^{-2})$$

for any positive integer n.

Proof. By our discussion above, in the decomposed case $\boldsymbol{G}(\mathbb{Z}_p) = \mathrm{SL}_3(\mathbb{Z}_p)$ and $\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n) = \mathrm{SL}_3(\mathbb{Z}_p)(p^n)$. We note that the sequence

$$1 \longrightarrow \mathrm{SL}_n(\mathbb{Z}_p)(p^n) \longrightarrow \mathrm{SL}_n(\mathbb{Z}_p) \longrightarrow \mathrm{SL}_n(\mathbb{Z}_p/p^n\mathbb{Z}_p) \longrightarrow 1 \tag{2-2}$$

is exact. In fact, (2-2) is exact even if we replace $\mathbb{Z}_p$ and p^n by any commutative ring R and ideal $\mathfrak{I}$ such that $R/\mathfrak{I}$ is finite (see [Bass 1964], Corollary 5.2). Therefore

$$[\boldsymbol{G}(\mathbb{Z}_p) : \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)] = |\mathrm{SL}_3(\mathbb{Z}_p/p^n\mathbb{Z}_p)| = |\mathrm{SL}_3(\mathbb{Z}/p^n\mathbb{Z})|.$$

The latter number can be computed in an elementary way: Let $\rho : \mathrm{GL}_3(\mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})$ be the canonical projection map which sends every residue mod p^n to its residue mod p. The kernel $\ker(\rho)$ consists of $p^{9(n-1)}$ elements. With the well-known number $|\mathrm{GL}_3(\mathbb{Z}/p\mathbb{Z})| = p^9(1-p^{-3})(1-p^{-2})(1-p^{-1})$, we get $|\mathrm{GL}_3(\mathbb{Z}/p^n\mathbb{Z})| = p^{9n}(1-p^{-3})(1-p^{-2})(1-p^{-1})$. Dividing this number by $\varphi(p^n)$, which is the order of the multiplicative group $(\mathbb{Z}/p^n\mathbb{Z})^*$, we get the desired result. □

For some technical reasons, from now on we exclude from consideration the prime 2.

Lemma 2.4. *Let $\mathfrak{p}$ be a prime ideal in $\mathbb{O}_K$ and $p = \mathfrak{p} \cap \mathbb{Z} \neq 2$ nondecomposed. For every $n \geq 1$ the following hold*:

(1) *If p is inert*,

$$[\boldsymbol{G}(\mathbb{Z}_p) : \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)] = p^{8n}(1+p^{-3})(1-p^{-2}).$$

(2) *If p is ramified*,

$$[\boldsymbol{G}(\mathbb{Z}_p) : \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)] = p^{\epsilon_n}(1-p^{-2}),$$

where ϵ_n is defined as

$$\epsilon_n = \begin{cases} 4n & \textit{if } n \equiv 0 \bmod 2, \\ 4n-1 & \textit{if } n \equiv 1 \bmod 2. \end{cases}$$

Proof. First, we identify the desired index with the order of the finite group $\mathrm{SU}(\bar{h}, \mathbb{O}_\mathfrak{p}/\mathfrak{p}^n)$, where $\mathbb{O}_\mathfrak{p} = \mathbb{O}_{K_\mathfrak{p}}$ denotes the ring of integers in $K_\mathfrak{p}$ and $\bar{h}$ the restriction of h to $\mathbb{O}_\mathfrak{p}/\mathfrak{p}^n$. To do so, we have to know that the canonical reduction map $\boldsymbol{G}(\mathbb{Z}_p) = \mathrm{SU}(h, \mathbb{O}_\mathfrak{p}) \to \mathrm{SU}(\bar{h}, \mathbb{O}_\mathfrak{p}/\mathfrak{p}^n)$ is surjective. But this is a rather general fact (see for instance [Baeza 1973, Bemerkung (4.3)]). Moreover, we can extract the above local index formulas again from [Holzapfel 1998a, Corollary 5A.1.3] for the case in which p is inert, or p is ramified and n is an even positive integer. According to this, the only nontrivial case is the situation where p is ramified and n is odd. Let us from now on assume this situation, that is, p is ramified and n is odd.

By [Holzapfel 1998a, Corollary 5A.1.3] again we know that

$$\boldsymbol{G}(\mathbb{Z}_p)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^{n+1}) = |\boldsymbol{G}(\mathbb{Z}_p)/\boldsymbol{G}(\mathbb{Z}_p)(p^{(n+1)/2})| = p^{4(n+1)}(1-p^{-2}).$$

By the multiplicativity of the index we obtain

$$\boldsymbol{G}(\mathbb{Z}_p)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n) = p^{4(n+1)}(1-p^{-2})/[\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^{n+1})]. \tag{2-3}$$

In order to compute the index in the denominator, we first recall the isomorphism $\boldsymbol{G}(\mathbb{Q}_p) \cong \mathrm{SU}_3(K_\mathfrak{p})$ from Lemma 2.2. Let π be a generator of the valuation ideal $\mathfrak{p}$ of $\mathbb{O}_\mathfrak{p}$. As $p \neq 2$, the extension $K_\mathfrak{p}/\mathbb{Q}_p$ is tamely ramified and we can choose π such that $\bar{\pi} = -\pi$, where $\bar{\cdot}$ denotes the conjugation in $K_\mathfrak{p}$ (compare [Lang 1970, Proposition II.5.12]).

As a matrix in $M_3(K_\mathfrak{p})$, every $g \in \boldsymbol{G}(\mathbb{Z}_p)$ has a $\mathfrak{p}$-adic representation:

$$g = g_0 + \pi g_1 + \pi^2 g_2 + \pi^3 g_3 + \cdots,$$

for some matrices $g_j \in M_3(\mathbb{O}_\mathfrak{p}/\mathfrak{p}) = M_3(\mathbb{Z}/p\mathbb{Z})$, where the latter equality holds since p is ramified. We note that a matrix $g \in \boldsymbol{G}(\mathbb{Z}_p)$ lies in $\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)$ if and only if its $\mathfrak{p}$-adic representation starts with

$$g = 1_3 + \pi^n g_n + \pi^{n+1} g_{n+1} + \cdots .$$

For this reason, the map $g \mapsto g_n$ gives a map

$$\boldsymbol{G}(\mathbb{Z}_p)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n) \longrightarrow M_3(\mathbb{Z}/p\mathbb{Z}).$$

Note that, as p is ramified, the conjugation acts as the identity on $\mathbb{O}_\mathfrak{p}/\mathfrak{p}^n$. Consider a representative $g \in \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)$ of the class $[g] \in \boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^{n+1})$. Two equations characterize g as an element of $\mathrm{SU}_3(K_\mathfrak{p})$. First, g is hermitian, that is,

$$g\bar{g}^t = (1_3 + \pi^n g_n + \cdots)(1_3 + \bar{\pi}^n \bar{g}_n + \cdots)^t = 1_3$$

or, equivalently,

$$(1_3 + \pi^n g_n + \cdots)(1_3 - \pi^n \bar{g}_n^t + \cdots) = 1_3.$$

In other words, $g_n - g_n^t = 0$, that is, g_n is symmetric. On the other hand, as

$$\det(g) = \det(1_3 + \pi^n g_n + \cdots) = 1 \equiv 1 + \pi^n \operatorname{Tr}(g_n) \bmod \pi^{n+1},$$

g_n has trace 0. By this, the factor group $\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^{n+1})$ is characterized as the set of symmetric matrices in $M_3(\mathbb{Z}/p\mathbb{Z})$ with trace equal to 0. There are exactly p^5 such matrices, and (2-3) gives for n odd and p ramified

$$\boldsymbol{G}(\mathbb{Z}_p)/\boldsymbol{G}(\mathbb{Z}_p)(\mathfrak{p}^n) = p^{4n-1}(1 - p^{-2}),$$

from which the final formula follows. □

Now, summarizing results from Lemma 2.3 and Lemma 2.4, we get:

Theorem 2.5. *Let $\mathfrak{a}$ be an integral ideal in $K = \mathbb{Q}(\sqrt{-d})$ and let $\mathfrak{a} = \mathfrak{p}_1^{e_1} \cdots \mathfrak{p}_t^{e_t}$ be the prime ideal decomposition of $\mathfrak{a}$ with $p_i = \mathfrak{p}_i \cap \mathbb{Z}$. Assume that $p_i \neq 2$ for all $i = 1, \ldots, t$, if 2 is nondecomposed in K. Then it holds that*

$$[\Gamma_K : \Gamma_K(\mathfrak{a})] = \prod_{i=1}^{t} p^{\epsilon_i}(1 - p_i^{-2})(1 - \chi_D(p_i) p_i^{-3}),$$

where D is the discriminant of K, $\chi_D(\,)$ is the Jacobi symbol $\chi_D(p) = \left(\frac{D}{p}\right)$ and ϵ_i is defined by

$$\epsilon_i = \begin{cases} 8e_i & \text{if } \chi_D(p_i) \neq 0, \\ 4e_i & \text{if } \chi_D(p_i) = 0 \text{ and } e_i \equiv 0 \bmod 2, \\ 4e_i - 1 & \text{if } \chi_D(p_i) = 0 \text{ and } e_i \equiv 1 \bmod 2. \end{cases}$$

We remark that this result generalizes [Holzapfel 1998a, Theorem 5A.2.14].

Determination of neat subgroups. In this section we discuss the question, for which integral ideals $\mathfrak{a}$ the principal congruence subgroup $\Gamma_K(\mathfrak{a})$ is a neat subgroup. This rather technical property will be used in our consideration of Picard modular surfaces.

Lemma 2.6. *Assume that $\mathfrak{a}$ is an ideal in $\mathbb{O}_K$ such that $\mathfrak{a} \cap \mathbb{Z}$ and 2 are coprime. Then the principal congruence subgroup $\Gamma_K(\mathfrak{a})$ is neat if $N(\mathfrak{a}) > 3$.*

Proof. (Compare [Holzapfel 1980, Lemma 4.3].) Assume that $\Gamma_K(\mathfrak{a})$ is not neat. Then there are eigenvalues ζ of elements $\gamma \in \Gamma_K(\mathfrak{a})$ which generate a torsion subgroup in $\mathbb{C}^*$. One concludes that such eigenvalues must be roots of unity [Holzapfel 1980, proof of Lemma 4.3]. Now, on the one hand, such an eigenvalue ζ is a root of unity, n-th root of unity, say, and on the other hand, ζ is a zero of the characteristic polynomial of γ which has degree 3 over K. Therefore $[\mathbb{Q}(\zeta) : \mathbb{Q}] = \varphi(n) \leq 6$. Without loss of generality we can assume that $n = p$ is a prime number. So, in fact we have to check that for $n = 2, 3, 5, 7$ the primitive n-th root of unity does not appear as the eigenvalue of elements in $\Gamma_K(\mathfrak{a})$ for stated ideals $\mathfrak{a}$. For this, we first observe that there is a relation between the eigenvalues ζ of elements in $\Gamma_K(\mathfrak{a})$ and the ideal $\mathfrak{a}$, namely

$$\mathfrak{a}\mathbb{Z}[\zeta] \,\big|\, (\zeta - 1)\mathbb{Z}[\zeta]$$

Taking norms on both sides we obtain

$$N(\mathfrak{a}\mathbb{Z}[\zeta]) = N(\mathfrak{a})^{\varphi(n)/2} \,\big|\, N_{\mathbb{Q}(\zeta)/\mathbb{Q}}(\zeta - 1) = n.$$

It is easily checked that none of the above primes satisfies this relation for given $\mathfrak{a}$ with $(2, \mathfrak{a} \cap \mathbb{Z}) = 1$ and $N(\mathfrak{a}) > 3$. □

3. Proportionality principle for compactified Picard modular surfaces

As a noncompact almost simple Lie group, $\boldsymbol{G}(\mathbb{R}) = \mathrm{SU}(h, \mathbb{C})$ has a naturally associated irreducible simply connected hermitian symmetric domain $\mathbb{B} = \mathrm{SU}(h)/C$, with a maximal compact subgroup C. Taking h to be represented by $\mathrm{diag}(1, 1, -1)$, this symmetric domain is exactly the two-dimensional complex unit ball

$$\mathbb{B} = \mathbb{B}_2 = \big\{(z_1, z_2) \,\big|\, |z_1|^2 + |z_2|^2 < 1\big\}.$$

It is more natural to consider $\mathbb{B}$ embedded in $\mathbb{P}_2(\mathbb{C})$, which is interpreted as the compact dual symmetric space of $\mathbb{B}$. From the construction of $\mathbb{B}$, it is clear that the group $\mathrm{SU}(h)$ acts on $\mathbb{B}$ as a group of biholomorphic transformations. According to the theorem of Borel and Harish-Chandra (see [Borel and Harish-Chandra 1962]), the group Γ_K, and consequently every group Γ commensurable to Γ_K, is a discrete subgroup of finite covolume in the Lie group $\boldsymbol{G}(\mathbb{R}) = \mathrm{SU}(h, \mathbb{C})$ corresponding to $\boldsymbol{G}$. Therefore, the action of each Γ is properly discontinuous, and it is reasonable to consider the quotient space $Y(\Gamma) := \Gamma\backslash\mathbb{B}$. Now we recall some basic properties of $\overline{Y}(\Gamma)$.

Theorem 3.1. *Let $\Gamma \subset \Gamma_K$ be a Picard modular group and $Y(\Gamma) = \Gamma\backslash\mathbb{B}$ the corresponding locally symmetric space. Then the following hold*:

(1) $Y(\Gamma)$ *is not compact.*

(2) *There exists a compactification $\overline{Y}(\Gamma)$ of $Y(\Gamma)$* (*the so-called Baily–Borel compactification*), *which has a structure of a normal projective surface.*

(3) *Let $\widetilde{Y}(\Gamma)$ be the minimal resolution of the singularities of $\overline{Y}(\Gamma)$. We call, for brevity, $\widetilde{Y}(\Gamma)$ the smooth compactification of $Y(\Gamma)$.*

The proofs of the above statements, which are in fact special cases of theorems that are valid in much greater generality, can be found in [Holzapfel 1998a]. Let us roughly sketch the main steps of the proofs, also in order to introduce some notions which will be used later. The noncompactness follows from [Borel and Harish-Chandra 1962, Theorem 11.8]. By that theorem, $Y(\Gamma)$ is compact if and only if there are no nontrivial unipotent elements in $\boldsymbol{G}(\mathbb{Q})$. But there are many unipotent elements and they all lie in the unipotent radicals of minimal rational parabolic subgroups of $\boldsymbol{G}$, which themselves fix maximal h-isotropic subspaces of $V = K^3$. The compactification $\overline{Y}(\Gamma)$ of $Y(\Gamma)$ is obtained by taking the quotient $\overline{Y}(\Gamma) = \Gamma\backslash\mathbb{B}^*$, where $\mathbb{B}^* = \mathbb{B} \cup \partial_{\mathbb{Q}}\mathbb{B}$ with $\partial_{\mathbb{Q}}\mathbb{B} = \boldsymbol{G}(\mathbb{Q})/\boldsymbol{P}$ for some fixed rational parabolic subgroup $\boldsymbol{P} \subset \boldsymbol{G}$. By a suitable choice of $\boldsymbol{P}$ one can assume that $\partial_{\mathbb{Q}}\mathbb{B} = \{[v] \in \mathbb{P}_2(K) \mid h(v, v) = 0\}$. The set $\partial_{\Gamma}\mathbb{B} = \Gamma\backslash\boldsymbol{G}(\mathbb{Q})/\boldsymbol{P}$ is finite, so $\overline{Y}(\Gamma) = Y(\Gamma) \cup \{\text{finitely many points}\}$. By the theorem of Baily–Borel, $\overline{Y}(\Gamma)$ is a normal projective variety. The singularities come from nontrivial torsion elements in Γ and possibly from *cusps*, that is, points of $\overline{Y}(\Gamma) - Y(\Gamma)$. If we assume that Γ is neat, the theory of toroidal compactifications (see [Holzapfel 1998a, Chapter 4] for the special case of arithmetic lattices in $\mathrm{SU}(2, 1)$) provides a smooth compactification by replacing every cusp by a smooth elliptic curve. Note that, by a theorem of Borel, neat normal subgroups $\Gamma' \triangleleft \Gamma_K$ of finite index exist. If Γ is not neat, the toroidal compactification provides a surface with at most quotient singularities. They can be resolved by known methods. This resolution gives a map $\widetilde{Y}(\Gamma) \to \overline{Y}(\Gamma)$, with smooth $\widetilde{Y}(\Gamma)$.

Chern numbers of compactified Picard modular surfaces. The two Chern numbers c_2 and c_1^2 of a smooth compact complex surface are important numerical invariants. The famous proportionality theorem of F. Hirzebruch states that the Chern numbers of a (two-dimensional) ball quotient by a torsion free cocompact discrete subgroup satisfy $c_1^2 = 3c_2$. D. Mumford generalized Hirzebruch's theorem also to noncompact quotients. Before we state the proportionality result for Picard modular groups, as described in [Holzapfel 1998a], we need to introduce some notions.

Let Γ be a neat Picard modular group. The quotient $Y(\Gamma)$ is not compact, but one can still define Chern numbers of $Y(\Gamma)$ as the volumes

$$c_2(Y(\Gamma)) = \int_{F(\Gamma)} \eta_2 \quad \text{and} \quad c_1^2(Y(\Gamma)) = \int_{F(\Gamma)} \eta_1^2,$$

where $F(\Gamma)$ is a fundamental domain of Γ in $\mathbb{B}$ and η_2 and η_1^2 are suitably normalized volume forms related to the Γ-invariant Bergman metric on $\mathbb{B}$.

On the other hand, on the smooth compactification $\widetilde{Y}(\Gamma)$ of $Y(\Gamma)$ there are also the usual Chern numbers. Before we relate these two kinds of Chern numbers, we note that $\widetilde{Y}(\Gamma)$ arises from $Y(\Gamma)$ by replacing a cusp $\kappa \in \partial_\Gamma \mathbb{B}$ by an elliptic curve E_κ. Hence, the difference between $\widetilde{Y}(\Gamma)$ and $Y(\Gamma)$ is encoded in the *compactification divisor* $T_\Gamma = \sum_{\kappa \in \partial_\Gamma \mathbb{B}} E_\kappa$.

Proposition 3.2 [Holzapfel 1998a, Proposition 4.3.6]. *Let Γ be a neat Picard modular group, $Y(\Gamma) = \Gamma \backslash \mathbb{B}$ and $\widetilde{Y}(\Gamma)$ be the smooth compactification of $Y(\Gamma)$. Also, let T_Γ denote the compactification divisor and $(T_\Gamma . T_\Gamma)$ the self-intersection number of T_Γ. Then*:

(1) $c_2(\widetilde{Y}(\Gamma)) = c_2(Y(\Gamma))$,

(2) $c_1^2(\widetilde{Y}(\Gamma)) = 3c_2(Y(\Gamma)) + (T_\Gamma . T_\Gamma)$.

Chern numbers of quotients by principal congruence subgroups. By the above Proposition 3.2, two magnitudes are essential for the computation of Chern invariants of the smooth compactification of a Picard modular surface $Y(\Gamma) = \Gamma \backslash \mathbb{B}$, namely the volume $c_2(Y(\Gamma))$ and the self-intersection number $(T_\Gamma . T_\Gamma)$. In the case of a ball quotient by a principal congruence subgroup, these two numbers can be given in terms of a number theoretic expression. Let us for brevity write $Y(\mathfrak{a})$ for the ball quotient $\Gamma_K(\mathfrak{a}) \backslash \mathbb{B}$ by the principal congruence subgroup $\Gamma_K(\mathfrak{a})$. In this context we also write $Y(1)$ for the quotient $\Gamma_K \backslash \mathbb{B}$ by a full Picard modular group.

Proposition 3.3 (see [Holzapfel 1998a, Theorem 5A.4.7]). *Let $K = \mathbb{Q}(\sqrt{-d})$ and let Γ_K be the full Picard modular group corresponding to K. Let D be the discriminant of K and $\chi_D(\,) = \left(\frac{D}{\cdot}\right)$ the Dirichlet character associated with K.*

Then

$$c_2(Y(1)) = c_2(\Gamma_K\backslash\mathbb{B}) = \delta_K \frac{|D|^{5/2}}{32\pi^3} L(3, \chi_D),$$

where $L(s, \chi_D)$ denotes the Dirichlet L-function associated with χ_D, and δ_K is the order of the center of Γ_K, that is, $\delta_K = \frac{1}{3}$ if $d = 3$, and $\delta_K = 1$ otherwise.

Let $\mathbb{P}\Gamma$ be the quotient of Γ by its center. We already know $c_2(Y(\mathfrak{a})) = c_2(\widetilde{Y}(\mathfrak{a}))$. As $\widetilde{Y}(\mathfrak{a})$ is a smooth Galois covering of $\widetilde{Y}(1)$ and $[\mathbb{P}\Gamma_K : \mathbb{P}\Gamma_K(\mathfrak{a})] = [\Gamma_K : \Gamma_K(\mathfrak{a})]/\delta_K$ is the degree of this covering, Proposition 3.3 implies the following result:

Corollary 3.4. *For each integral ideal $\mathfrak{a}$ of K and principal congruence subgroup $\Gamma_K(\mathfrak{a})$, we have*

$$c_2(Y(\mathfrak{a})) = [\Gamma_K : \Gamma_K(\mathfrak{a})] \frac{|D|^{5/2}}{32\pi^3} L(3, \chi_D).$$

Remark 3.5. The above volume formula can be seen as a special case of a general formula developed by G. Prasad, expressing the volume of a quotient of a Lie group by an arithmetic subgroup, by values of L-functions. The reader can consult [Prasad and Yeung 2007] for the case of the unitary group corresponding to a hermitian form of signature $(2, 1)$.

A similar arithmetic expression for the self-intersection number of the compactification divisor can be deduced using [Holzapfel 1980, Lemma 4.7, Lemma 4.8]:

Proposition 3.6. *Let h_K denote the ideal class number of K and let η_K and $\vartheta_{\mathfrak{a}}$ be defined as*

$$\vartheta_{\mathfrak{a}} = \min\{n \in \mathbb{N} \mid \mathfrak{a} \mid n\sqrt{-d}\},$$

$$\eta_K = \begin{cases} 1 & \text{if } D = -4, \\ \frac{1}{6} & \text{if } D = -3, \\ 2 & \text{if } D \neq -4,\ D \equiv 0 \bmod 4, \\ \frac{1}{2} & \text{if } D \neq -3,\ D \equiv 1 \bmod 4. \end{cases}$$

Then the following formula for the self-intersection number $(T_{\Gamma_K(\mathfrak{a})}.T_{\Gamma_K(\mathfrak{a})})$ holds:

$$(T_{\Gamma_K(\mathfrak{a})}.T_{\Gamma_K(\mathfrak{a})}) = -\frac{h_K \eta_K}{\vartheta_{\mathfrak{a}}^2} [\Gamma_K : \Gamma_K(\mathfrak{a})].$$

Values of Dirichlet L-functions at integers. In order to get concrete values of Chern numbers of Picard modular surfaces, we need to know the value $L(3, \chi_D)$ explicitly. Let us briefly discuss the well-known method for the computation of this number.

We first recall the functional equation for Dirichlet L-functions (see, for instance, [Washington 1982], Chapter 4, in particular Theorem 4.2). This equation implies

$$L(3,\chi_D) = -\frac{2\pi^3}{|D|^{5/2}} L(-2,\chi_D) = \frac{2\pi^3}{3|D|^{5/2}} B_{3,\chi_D}.$$

There, B_{n,χ_D} denotes the n-th generalized Bernoulli number associated with χ_D (see [Washington 1982, Chapter 4] for the definition). The generalized Bernoulli numbers can be computed easily by the following formula, which, for instance, can be found in [Washington 1982, Proposition 4.2]:

$$B_{n,\chi_D} = |D|^{n-1} \sum_{k=1}^{|D|} \chi_D(k) B_n\Big(\frac{k}{|D|}\Big). \tag{3-1}$$

In the above formula, $B_n(X)$ denotes the n-th Bernoulli polynomial, which can be defined as

$$B_n(X) = \sum_{k=0}^{n} \binom{n}{k} B_k X^{n-k}$$

with B_k denoting the usual k-th Bernoulli number. It is now easy to provide a list of values $L(3,\chi_D)$, knowing $B_3(X) = \frac{1}{2}X - \frac{3}{2}X^2 + X^3$. Here we give the first values of B_{3,χ_D}:

$\lvert D\rvert$	3	4	7	8	11	15	19	20	23	24	31	35
B_{3,χ_D}	$2/3$	$3/2$	$48/7$	9	18	48	66	90	144	138	288	108

4. Applications

In this section we discuss some applications of the explicit formulas for the Chern invariants of compactified ball quotients by principal congruence subgroups.

Classification.

Theorem 4.1. *Let $\mathfrak{a}$ be an integral ideal of $K = \mathbb{Q}(\sqrt{-d})$ such that $N(\mathfrak{a}) > 3$ and $\mathfrak{a} \cap \mathbb{Z}$ and the prime 2 are coprime. Then $\widetilde{Y}(\mathfrak{a})$, the smooth compactification of $\Gamma_K(\mathfrak{a})\backslash\mathbb{B}$, is a surface of general type, with the (additional) possible exception for $K = \mathbb{Q}(\sqrt{-7})$ and $\mathfrak{a} \cap \mathbb{Z}$ ramified in K.*

Proof. First we note that $\widetilde{Y}(\mathfrak{a})$ is a smooth and compact algebraic surface, since $\Gamma_K(\mathfrak{a})$ is neat by Lemma 2.6 (see also Theorem 3.1). In order to show that $\widetilde{Y}(\mathfrak{a})$ is of general type it is sufficient to check that $c_1^2(\widetilde{Y}(\mathfrak{a})) > 9$. This follows from the classification theory of algebraic surfaces. It is also sufficient to check the inequality only for the surfaces $\widetilde{Y}(\mathfrak{p})$, where $\mathfrak{p}$ is a prime ideal, since the Kodaira dimension can not decrease by the passage to a finite covering. Hence, if $\widetilde{Y}(\mathfrak{p})$ is of general type, then so is $\widetilde{Y}(\mathfrak{a})$ for any ideal $\mathfrak{a}$ such that $\mathfrak{p}|\mathfrak{a}$.

From Proposition 3.2, Corollary 3.4, and Proposition 3.6, we obtain the following formula for $c_1^2(\widetilde{Y}(\mathfrak{p}))$:

$$c_1^2(\widetilde{Y}(\mathfrak{p})) = [\Gamma_K : \Gamma_K(\mathfrak{a})]\left(\frac{3|D|^{5/2}}{32\pi^3}L(3,\chi_D) - \frac{h_K\eta_K}{\vartheta_{\mathfrak{p}}^2}\right). \tag{4-1}$$

Looking at the index formula in Theorem 2.5, we see that $[\Gamma_K : \Gamma_K(\mathfrak{p})] > 9$ if $N(\mathfrak{p}) > 3$ and $\mathfrak{p} \cap \mathbb{Z} \neq (2)$. Thus, it suffices to show that

$$\frac{3|D|^{5/2}}{32\pi^3}L(3,\chi_D) - \frac{h_K\eta_K}{\vartheta_{\mathfrak{p}}^2} \geq 1. \tag{4-2}$$

First we show that the inequality (4-2) holds if $|D| > 35$.

Recall first the analytic class number formula

$$h_K = \frac{\mu_K\sqrt{-D}}{2\pi}L(1,\chi_D) = -(\mu_K/4)B_{1,\chi_D},$$

where μ_K denotes the number of roots of unity contained in K. Using the facts on generalized Bernoulli numbers summarized on page 334, and in particular the formula (3-1), we get a trivial bound $h_K \leq |D|/4$ which holds for $|D| \neq 3, 4$. On the other hand, comparing the Euler factors, the value $L(3,\chi_D)$ is estimated below by a known value of the Riemann zeta function, namely, $L(3,\chi_D) > 1/\sqrt{\zeta_{\mathbb{Q}}(2)} = \sqrt{6}/\pi$. More precisely, one shows that $(1-p^{-2})(1-\chi_D(p)p^{-3})^2 < 1$ for all primes, from which the lower bound follows. Noting additionally that always $\eta_K \leq 2$ and $\vartheta_{\mathfrak{p}} \geq 1$, the inequality (4-2) will follow from the inequality

$$|D|^{5/2} > \frac{16\pi^4}{3\sqrt{6}}(2+|D|). \tag{4-3}$$

This inequality holds for $|D| > 35$.

For discriminants D with $|D| \leq 35$ we have to check the inequality $c_1^2(\widetilde{Y}(\mathfrak{p})) > 9$ case by case. For this, one first uses the functional equation to reformulate (4-1) as

$$c_1^2(\widetilde{Y}(\mathfrak{p})) = [\Gamma_K : \Gamma_K(\mathfrak{a})]\left(\frac{3}{16}B_{3,\chi_D} - \frac{h_K\eta_K}{\vartheta_{\mathfrak{p}}^2}\right)$$

and then uses the exact values B_{3,χ_D} from the table on page 335. For instance, let us consider the case $D = -35$. With $h_K = 2$, $\eta_K = \frac{1}{2}$ and $B_{3,\chi_{-35}} = 108$ we have $c_1^2 = [\Gamma_K : \Gamma_K(\mathfrak{p})]\left(\frac{81}{4} - 1/\vartheta_{\mathfrak{p}}^2\right) \geq \frac{81}{4}[\Gamma_K : \Gamma_K(\mathfrak{p})] > 9$. The other cases are treated in the same manner. The cases $|D| = -3, -4, -7, -8$ need to be considered more carefully, since in these situations c_1^2 can be negative.

$D = -8$. Let us, for brevity, write $I_{\mathfrak{p}}$ instead of $[\Gamma_K : \Gamma_K(\mathfrak{p})]$. For the self-intersection we have $c_1^2 = I_{\mathfrak{p}}\left\{\frac{27}{48} - 2/\vartheta_{\mathfrak{p}}^2\right\}$. Now, it is easy to see that $c_1^2 < 0$ if and only if

$\vartheta_{\mathfrak{p}} = 1$, which is exactly the case when $p = \mathfrak{p} \cap \mathbb{Z}$ is ramified in $K = \mathbb{Q}(\sqrt{-2})$. From this it follows that $p = 2$, and this case is excluded. Otherwise, $c_1^2 > 9$ if $I_{\mathfrak{p}} > 144$. This is the case for $N(\mathfrak{p}) > 3$ and p unramified, as the index formula shows.

$D = -7$. Here we have $c_1^2 = I_{\mathfrak{p}}\{\frac{3}{7} - \frac{1}{2}\vartheta_{\mathfrak{p}}^2\}$. This expression is negative if and only if $\vartheta_{\mathfrak{p}} = 1$, which means that $p = \mathfrak{p} \cap \mathbb{Z}$ is ramified. This is the exception we have to exclude. Indeed, for $\mathfrak{p} = (\sqrt{-7})$, we have $c_2 = 48$ and $c_1^2 = -24$. In other cases $c_1^2 > 9$.

$D = -4$. Here we get a negative c_1^2 for $\vartheta_{\mathfrak{p}} = 1$ or 2. But $\vartheta_{\mathfrak{p}} = 1$ is only possible for $\mathfrak{p} = (1)$ and $\vartheta_{\mathfrak{p}} = 2$ is only possible for those $\mathfrak{p}$ which contain the prime 2.

$D = -3$. In this case $c_1^2 < 0$ is only possible for a prime ideal $\mathfrak{p}$ such that p is ramified in K. But then $N(\mathfrak{p}) = 3$, a case which is excluded. □

Remark 4.2. Let us remark that the above proof works for any neat Picard modular group. Let, for instance, K be the imaginary quadratic field of discriminant $D = -24$ and Γ be the congruence subgroup $\Gamma_K(2\sqrt{D})$. This group is neat (see [Dimitrov and Ramakrishnan 2015, Proposition 2.6]). The group Γ is a subgroup of $\Gamma_K(\mathfrak{p}_3)$, where $\mathfrak{p}_3$ is the prime ideal over the rational prime $p = 3$. By Lemma 2.4, $[\Gamma_K : \Gamma] > 24$. The arguments of the proof of Theorem 4.1 then lead to $c_1^2(Y(2\sqrt{D})) > 9$ and we conclude that $Y(2\sqrt{D})$ is of general type.

The invariants of $\widetilde{Y}(\mathfrak{a})$ are in general very large due to the fact that the index, which grows very fast with the norm of the ideal, dominates the expressions. However, the surfaces $\widetilde{Y}(\mathfrak{a})$ are interesting from the point of view of surface geography, since the pairs $(c_1^2(\widetilde{Y}(\mathfrak{a})), c_2(\widetilde{Y}(\mathfrak{a})))$ are located in an interesting region of the (c_1^2, c_2)-plane, namely the ratio c_1^2/c_2 is close to 3 (see [Holzapfel 1980], and the first part of [Hunt 1989], for a detailed discussion on surface geography). More precisely, the ratio $c_1^2(\widetilde{Y}(\mathfrak{a}))/c_2(\widetilde{Y}(\mathfrak{a}))$ tends to 3 when either the norm $N(\mathfrak{a})$ or the discriminant $|D|$ tends to infinity (see also [Holzapfel 1980]). This is easily seen since

$$\frac{c_1^2(\widetilde{Y}(\mathfrak{a}))}{c_2(\widetilde{Y}(\mathfrak{a}))} = 3 - \frac{h_K \eta_K / \vartheta_{\mathfrak{a}}^2}{\frac{|D|^{5/2}}{32\pi^3} L(3, \chi_D)}. \tag{4-4}$$

Now, as $\vartheta_{\mathfrak{a}}^2 \to \infty$ for $N(\mathfrak{a}) \to \infty$, we see that for a fixed D the ratio tends to 3. Let us, on the other hand, fix $\mathfrak{a}$ but let D vary. We first observe, as $L(3, \chi_D)$ is bounded by the value of the Riemann zeta function at $s = 3$, namely $\zeta(3)^{-1} < L(3, \chi_D) < \zeta(3)$, that there are constants C and C' not depending on D such that

$$C' \frac{h_K}{|D|^{5/2}} \geq \frac{\frac{h_K \eta_K}{\vartheta_{\mathfrak{a}}^2}}{\frac{|D|^{5/2}}{32\pi^3} L(3, \chi_D)} \geq C \frac{h_K}{|D|^{5/2}}. \tag{4-5}$$

There are well-known bounds for the class number of an imaginary quadratic field, namely, for any $\epsilon > 0$, there are constants $C_1 = C_1(\epsilon)$ and $C_2 = C_2(\epsilon)$ such that

$$C_1|D|^{1/2+\epsilon} > h_K > C_2|D|^{1/2-\epsilon}.$$

Bringing these bounds into (4-5), we see that the expression $\frac{h_K\eta_K}{\vartheta_{\mathfrak{a}}^2} / \frac{|D|^{5/2}}{32\pi^3} L(3,\chi_D)$ tends to zero as $|D| \to \infty$. Hence

$$\lim_{|D|\to\infty} \frac{c_1^2(\widetilde{Y}(\mathfrak{a}))}{c_2(\widetilde{Y}(\mathfrak{a}))} = 3.$$

Already for small discriminants and norms, the Chern number ratio is very close to 3 (see [Holzapfel 1980], Table 1).

Dimension formulas. Another application of the above explicit formulas for the Chern numbers of Picard modular surfaces is an explicit formula for the dimensions of spaces of cusp forms relative to a Picard modular group Γ.

A holomorphic function f on the ball is called a *cusp form of weight k with respect to* Γ if it satisfies the following conditions:

- For all $\gamma \in \Gamma$ and all $z \in \mathbb{B}$, $f(\gamma z) = j(\gamma, z)^k f(z)$, where $j(\gamma, z)$ denotes the determinant of the Jacobian matrix corresponding to the holomorphic map $\gamma : \mathbb{B} \to \mathbb{B}$ at the point z.
- The function $f(z)$ vanishes at cusps of Γ, i.e., it vanishes on $\partial_\Gamma\mathbb{B} = \Gamma\backslash\partial_{\mathbb{Q}}\mathbb{B}$.

Let $\Gamma \subset \Gamma_K$ be a neat Picard modular group. In the following, $S_k(\Gamma)$ will denote the space of cusp forms of weight k with respect to Γ. By a result of Hemperly [1972], the cusp forms of weight k can be interpreted as the sections of a certain line bundle $\mathbb{L}^{(k)}$ on the smooth compactification $\widetilde{Y}(\Gamma)$. Thus, $\dim S_k(\Gamma) = \dim H^0(\widetilde{Y}(\Gamma), \mathbb{L}^{(k)})$. As an application of the Riemann–Roch theorem and the Kodaira vanishing theorem, one gets the following formula for $\dim S_k(\Gamma)$:

Theorem 4.3 (see [Holzapfel 1998b]). *Let Γ be a neat Picard modular group and $k \geq 2$. Then*

$$\dim S_k(\Gamma) = [\Gamma_K : \Gamma]\big(\tfrac{1}{6}(9k(k-1)+2)c_2(Y(\Gamma)) + \tfrac{1}{12}(T_\Gamma.T_\Gamma)\big).$$

Now it is clear that we have an explicit formula for the dimensions of spaces of cusp forms of weight $k \geq 2$ which are automorphic with respect to a neat principal congruence subgroup $\Gamma_K(\mathfrak{a})$, namely,

$$S_k(\Gamma_K(\mathfrak{a})) = \frac{[\Gamma_K : \Gamma_K(\mathfrak{a})]}{6}\left((9k(k-1)+2)\frac{|D|^{5/2}}{32\pi^3}L(3,\chi_D) - \frac{h_K\eta_K}{2\vartheta_{\mathfrak{a}}^2}\right).$$

These dimensions are very large in general, even in the case of small discriminants and small norms.

References

[Baeza 1973] R. Baeza, "Eine Zerlegung der unitären Gruppe über lokalen Ringen", *Arch. Math. (Basel)* **24** (1973), 144–157. MR Zbl

[Bass 1964] H. Bass, "K-theory and stable algebra", *Inst. Hautes Études Sci. Publ. Math.* 22 (1964), 5–60. MR

[Borel and Harish-Chandra 1962] A. Borel and Harish-Chandra, "Arithmetic subgroups of algebraic groups", *Ann. of Math.* (2) **75** (1962), 485–535. MR Zbl

[Dimitrov and Ramakrishnan 2015] M. Dimitrov and D. Ramakrishnan, "Arithmetic quotients of the complex ball and a conjecture of Lang", *Doc. Math.* **20** (2015), 1185–1205. MR Zbl

[Hemperly 1972] J. C. Hemperly, "The parabolic contribution to the number of linearly independent automorphic forms on a certain bounded domain", *Amer. J. Math.* **94** (1972), 1078–1100. MR Zbl

[Holzapfel 1980] R.-P. Holzapfel, "A class of minimal surfaces in the unknown region of surface geography", *Math. Nachr.* **98** (1980), 211–232. MR Zbl

[Holzapfel 1998a] R.-P. Holzapfel, *Ball and surface arithmetics*, Aspects of Mathematics **E29**, Friedr. Vieweg & Sohn, Braunschweig, 1998. MR Zbl

[Holzapfel 1998b] R.-P. Holzapfel, "Zeta dimension formula for Picard modular cusp forms of neat natural congruence subgroups", *Abh. Math. Sem. Univ. Hamburg* **68** (1998), 169–192. MR Zbl

[Hunt 1989] B. Hunt, "Complex manifold geography in dimension 2 and 3", *J. Differential Geom.* **30**:1 (1989), 51–153. MR Zbl

[Lang 1970] S. Lang, *Algebraic number theory*, Addison-Wesley, Reading, MA, 1970. MR Zbl

[Prasad and Yeung 2007] G. Prasad and S.-K. Yeung, "Fake projective planes", *Invent. Math.* **168**:2 (2007), 321–370. MR Zbl

[Washington 1982] L. C. Washington, *Introduction to cyclotomic fields*, Graduate Texts in Mathematics **83**, Springer, New York, 1982. MR Zbl

Received November 17, 2015. Revised January 19, 2016.

AMIR DŽAMBIĆ
MATHEMATISCHES SEMINAR
CHRISTIAN-ALBRECHTS-UNIVERSITÄT ZU KIEL
LUDEWIG-MEYN-STRASSE 4
24098 KIEL
GERMANY
dzambic@math.uni-kiel.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.341

RADIAL LIMITS OF BOUNDED NONPARAMETRIC PRESCRIBED MEAN CURVATURE SURFACES

MOZHGAN (NORA) ENTEKHABI AND KIRK E. LANCASTER

Dedicated to the memory of Alan Ross Elcrat

Consider a solution $f \in C^2(\Omega)$ of a prescribed mean curvature equation

$$\operatorname{div} \frac{\nabla f}{\sqrt{1+|\nabla f|^2}} = 2H(x, f) \quad \text{in } \Omega,$$

where $\Omega \subset \mathbb{R}^2$ is a domain whose boundary has a corner at $\mathcal{O} = (0, 0) \in \partial\Omega$. If $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite and Ω has a reentrant corner at $\mathcal{O}$, then the (nontangential) radial limits of f at $\mathcal{O}$,

$$Rf(\theta) := \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta),$$

are shown to exist, independent of the boundary behavior of f on $\partial\Omega$, and to have a specific type of behavior. If $\sup_{x\in\Omega} |f(x)|$ and $\sup_{x\in\Omega} |H(x, f(x))|$ are both finite and the trace of f on one side has a limit at $\mathcal{O}$, then the (nontangential) radial limits of f at $\mathcal{O}$ exist, the tangential radial limit of f at $\mathcal{O}$ from one side exists and the radial limits have a specific type of behavior.

1. Introduction and statement of main theorems

Consider the prescribed mean curvature equation

$$Nf = 2H(\,\cdot\,, f) \quad \text{in } \Omega, \tag{1}$$

where Ω is a domain in $\mathbb{R}^2$ whose boundary has a corner at $\mathcal{O} \in \partial\Omega$, $Nf = \nabla\cdot Tf = \operatorname{div}(Tf)$, $Tf = (\nabla f)/\sqrt{1+|\nabla f|^2}$, $H : \Omega\times\mathbb{R}\to\mathbb{R}$ and H satisfies one of the conditions which guarantees that "cusp solutions" (e.g., [Lancaster and Siegel 1996a, §5; 1996b]) do not exist; for example, $H(\boldsymbol{x}, t)$ is strictly increasing in t for each $\boldsymbol{x}$ or is real-analytic (e.g., constant). We will assume $\mathcal{O} = (0, 0)$. Let $\Omega^* = \Omega \cap B_{\delta^*}(\mathcal{O})$, where $B_{\delta^*}(\mathcal{O})$ is the ball in $\mathbb{R}^2$ of radius δ^* about $\mathcal{O}$. Polar coordinates relative to $\mathcal{O}$ will be denoted by r and θ. We assume that $\partial\Omega$ is piecewise smooth and there exists $\alpha \in (0, \pi)$ such that $\partial\Omega \cap B_{\delta^*}(\mathcal{O})$ consists of

MSC2010: 35B40, 53A10.
Keywords: prescribed mean curvature, radial limits.

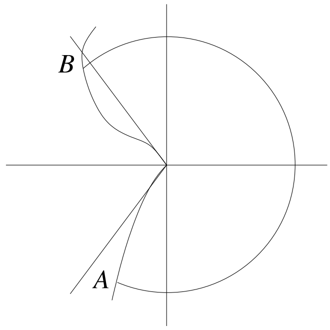


Figure 1. The domain Ω^*.

two arcs $\partial^+\Omega^*$ and $\partial^-\Omega^*$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached (see Figure 1 of [Lancaster and Siegel 1997] or Figure 1).

Suppose

$$\sup_{x\in\Omega} |f(x)| < \infty \quad \text{and} \quad \sup_{x\in\Omega} |H(x, f(x))| < \infty. \tag{2}$$

We shall prove

Theorem 1. *Let $f \in C^2(\Omega)$ satisfy* (1) *and suppose* (2) *holds and $\alpha \in (\pi/2, \pi)$. Then for each $\theta \in (-\alpha, \alpha)$,*

$$Rf(\theta) := \lim_{r\downarrow 0} f(r\cos\theta, r\sin\theta)$$

exists and $Rf(\cdot)$ is a continuous function on $(-\alpha, \alpha)$ which behaves in one of the following ways:

(i) *$Rf : (-\alpha, \alpha) \to \mathbb{R}$ is a constant function* (*so f has a nontangential limit at $\mathcal{O}$*).

(ii) *There exist α_1 and α_2 so that $-\alpha \le \alpha_1 < \alpha_2 \le \alpha$ and Rf is constant on $(-\alpha, \alpha_1]$ and $[\alpha_2, \alpha)$ and strictly increasing or strictly decreasing on (α_1, α_2).*

(iii) *There exist $\alpha_1, \alpha_L, \alpha_R, \alpha_2$ so that $-\alpha \le \alpha_1 < \alpha_L < \alpha_R < \alpha_2 \le \alpha, \alpha_R = \alpha_L + \pi$, and Rf is constant on $(-\alpha, \alpha_1]$, $[\alpha_L, \alpha_R]$, and $[\alpha_2, \alpha)$ and either strictly increasing on $(\alpha_1, \alpha_L]$ and strictly decreasing on $[\alpha_R, \alpha_2)$ or strictly decreasing on $(\alpha_1, \alpha_L]$ and strictly increasing on $[\alpha_R, \alpha_2)$.*

At a convex corner (i.e., $\alpha \in (0, \pi/2]$), Theorem 1 is not applicable. The additional assumption that the trace of f on one side (e.g., $\partial^-\Omega^*$) has a limit at $\mathcal{O}$ implies the radial limits of f exist.

Theorem 2. *Let $f \in C^2(\Omega) \cap C^0(\Omega \cup \partial^-\Omega^* \setminus \{\mathcal{O}\})$ satisfy* (1). *Suppose* (2) *holds and $m = \lim_{\partial^-\Omega^* \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x})$ exists. Then for each $\theta \in (-\alpha, \alpha)$, $Rf(\theta)$ exists and $Rf(\cdot)$ is a continuous function on $[-\alpha, \alpha)$, where $Rf(-\alpha) := m$. If $\alpha \in (0, \pi/2]$,*

Rf can behave as in (i) or (ii) in Theorem 1. If $\alpha \in (\pi/2, \pi)$, Rf can behave as in (i), (ii) or (iii) in Theorem 1.

The conclusions of these theorems were first obtained in [Lancaster 1985] for minimal surfaces satisfying Dirichlet boundary conditions and then for nonparametric prescribed mean curvature surfaces satisfying Dirichlet [Elcrat and Lancaster 1986; Lancaster 1988] or contact angle [Lancaster and Siegel 1996a] boundary conditions; see also [Jin and Lancaster 1997; Lancaster 1991]. Notice that Theorem 1 applies to a solution of a capillary surface problem whose domain has a reentrant corner even when the contact angle equals 0 and/or π on some (or all) of $\partial\Omega^*$.

Remark. The assumption that Ω has a reentrant corner at $\mathcal{O} \in \partial\Omega$ or that the trace of f from one side of $\partial\Omega$ is continuous at $\mathcal{O}$ is critical here; the nonexistence of radial limits at $(1, 0)$ when $\Omega = B_1(\mathcal{O})$ and the boundary data is symmetric with respect to the horizontal axis is demonistrated in [Lancaster 1989] and in Theorem 3 of [Lancaster and Siegel 1996a]. In [Lancaster 1987], it was conjectured that the existence of radial limits at corners for bounded solutions of Dirichlet problems for the minimal surface equation in $\mathbb{R}^2$, independent of boundary conditions. Although [Lancaster 1989] proved this conjecture false, Theorems 1 and 2 show it is true in many cases.

2. Proof of Theorem 1

Since $f \in C^2(\Omega)$ (and so in $C^0(\Omega)$), we may assume that f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| > \delta\}$ for each $\delta \in (0, \delta^*)$; if this is not true, we may replace Ω with U, $U \subset \Omega$, such that $\partial\Omega \cap \partial U = \{\mathcal{O}\}$ and $\partial U \cap B_{\delta^*}(\mathcal{O})$ consists of two arcs $\partial^+ U$ and $\partial^- U$, whose tangent lines approach the lines $L^+ : \theta = \alpha$ and $L^- : \theta = -\alpha$, respectively, as the point $\mathcal{O}$ is approached. Set

$$S_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \Omega^*\} \quad \text{and} \quad \Gamma_0^* = \{(\boldsymbol{x}, f(\boldsymbol{x})) : \boldsymbol{x} \in \partial\Omega^* \setminus \{\mathcal{O}\}\};$$

the points where $\partial B_{\delta^*}(\mathcal{O})$ intersect $\partial\Omega$ are labeled $A \in \partial^-\Omega^*$ and $B \in \partial^+\Omega^*$. From the calculation on page 170 of [Lancaster and Siegel 1996a], we see that the area of S_0^* is finite; let M_0 denote this area. For $\delta \in (0, 1)$, set

$$p(\delta) = \sqrt{\frac{8\pi M_0}{\ln(1/\delta)}}.$$

Let $E = \{(u, v) : u^2 + v^2 < 1\}$. As in [Elcrat and Lancaster 1986; Lancaster and Siegel 1996a], there is a parametric description of the surface S_0^*,

$$Y(u, v) = \big(a(u, v), b(u, v), c(u, v)\big) \in C^2(E : \mathbb{R}^3), \tag{3}$$

which has the following properties:

(a_1) Y is a diffeomorphism of E onto S_0^*.

(a_2) Set $G(u, v) = \big(a(u, v), b(u, v)\big)$, $(u, v) \in E$. Then $G \in C^0(\overline{E} : \mathbb{R}^2)$.

(a_3) Let $\sigma = G^{-1}(\partial\Omega^* \setminus \{\mathcal{O}\})$; then σ is a connected arc of ∂E and Y maps σ strictly monotonically onto Γ_0^*. We may assume the endpoints of σ are $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ and there exist points $\boldsymbol{a}, \boldsymbol{b} \in \sigma$ such that $G(\boldsymbol{a}) = A$, $G(\boldsymbol{b}) = B$, G maps the (open) arc $\boldsymbol{o}_1\boldsymbol{b}$ onto $\partial^+\Omega$, and G maps the (open) arc $\boldsymbol{o}_2\boldsymbol{a}$ onto $\partial^-\Omega$. (Note that $\boldsymbol{o}_1$ and $\boldsymbol{o}_2$ are not assumed to be distinct at this point; Figures 4a and 4b of [Lancaster and Siegel 1997] illustrate this situation.)

(a_4) Y is conformal on E: $Y_u \cdot Y_v = 0$, $Y_u \cdot Y_u = Y_v \cdot Y_v$ on E.

(a_5) $\triangle Y := Y_{uu} + Y_{vv} = H(Y) Y_u \times Y_v$ on E.

Here by the (open) arcs $\boldsymbol{o}_1\boldsymbol{b}$ and $\boldsymbol{o}_2\boldsymbol{a}$ we mean the component of $\partial E \setminus \{\boldsymbol{o}_1, \boldsymbol{b}\}$ which does not contain $\boldsymbol{a}$ and the component of $\partial E \setminus \{\boldsymbol{o}_2, \boldsymbol{a}\}$ which does not contain $\boldsymbol{b}$ respectively. Let $\sigma_0 = \partial E \setminus \sigma$.

There are two cases we wish to consider:

(A) $\boldsymbol{o}_1 = \boldsymbol{o}_2$.

(B) $\boldsymbol{o}_1 \neq \boldsymbol{o}_2$.

These correspond to Cases 5 and 3 respectively in Step 1 of the proof of Theorem 1 in [Lancaster and Siegel 1996a]. Let us first assume that (A) holds and set $\boldsymbol{o} = \boldsymbol{o}_1 = \boldsymbol{o}_2$. Let h denote a function on the annulus $\mathcal{A} = \{\boldsymbol{x} : r_1 \le |\boldsymbol{x}| \le r_2\}$ which vanishes on the circle $|\boldsymbol{x}| = r_2$ and whose graph is an unduloid surface with constant mean curvature $-H_0$ which becomes vertical at $|\boldsymbol{x}| = r_1$ and at $|\boldsymbol{x}| = r_2$ (see Figure 2) for suitable $r_1 < r_2$ (e.g., [Lancaster and Siegel 1996a, pp. 170–171]). Let q denote the modulus of continuity of h (i.e., $|h(\boldsymbol{x}_1) - h(\boldsymbol{x}_2)| \le q(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$).

For each $\boldsymbol{p} \in \mathbb{R}^2$ with $|\boldsymbol{p}| = r_1$, set $\mathcal{A}(\boldsymbol{p}) = \{\boldsymbol{x} : r_1 \le |\boldsymbol{x} - \boldsymbol{p}| \le r_2\}$ and define $h_{\boldsymbol{p}} : \mathcal{A}(\boldsymbol{p}) \to \mathbb{R}$ by $h_{\boldsymbol{p}}(\boldsymbol{x}) = h(\boldsymbol{x} - \boldsymbol{p})$.

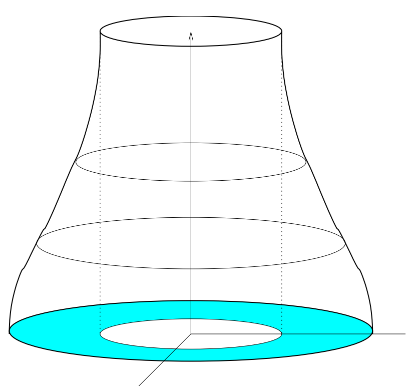


Figure 2. The graph of h over $\mathcal{A}$.

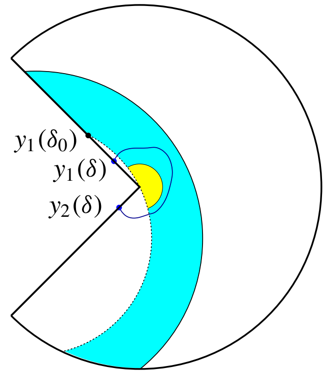


Figure 3. $\Omega^* \cap \mathcal{A}(\boldsymbol{p}_1)$, $C'_{\rho(\delta)}$ (blue curve), $B_{\eta(\delta)}(\mathcal{O}) \cap \mathcal{A}(\boldsymbol{p}_1)$ (yellow).

For $r > 0$, set $B_r = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}| < r\}$, $C_r = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}| = r\}$ and let l_r be the length of the image curve $Y(C_r)$; also let $C'_r = G(C_r)$ and $B'_r = G(B_r)$. From the Courant–Lebesgue lemma (e.g., [Courant 1950, Lemma 3.1]), we see that for each $\delta \in (0, 1)$, there exists a $\rho = \rho(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength l_ρ of $Y(C_\rho)$ is less than $p(\delta)$. For $\delta > 0$, let $k(\delta) = \inf_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \inf_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$ and $m(\delta) = \sup_{\boldsymbol{u} \in C_{\rho(\delta)}} c(\boldsymbol{u}) = \sup_{\boldsymbol{x} \in C'_{\rho(\delta)}} f(\boldsymbol{x})$; notice that $m(\delta) - k(\delta) \leq l_\rho < p(\delta)$.

For each $\delta \in (0, 1)$ with $\sqrt{\delta} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$, there are two points in $C_{\rho(\delta)} \cap \partial E$; we denote these points as $\boldsymbol{e}_1(\delta) \in \boldsymbol{ob}$ and $\boldsymbol{e}_2(\delta) \in \boldsymbol{oa}$ and set $\boldsymbol{y}_1(\delta) = G(\boldsymbol{e}_1(\delta))$ and $\boldsymbol{y}_2(\delta) = G(\boldsymbol{e}_2(\delta))$. Notice that $C'_{\rho(\delta)}$ is a curve in $\overline{\Omega}$ which joins $\boldsymbol{y}_1 \in \partial^+\Omega^*$ and $\boldsymbol{y}_2 \in \partial^-\Omega^*$ and $\partial\Omega \cap C'_{\rho(\delta)} \setminus \{\boldsymbol{y}_1, \boldsymbol{y}_2\} = \varnothing$; therefore there exists $\eta = \eta(\delta) > 0$ such that $B_{\eta(\delta)}(\mathcal{O}) = \{\boldsymbol{x} \in \Omega : |\boldsymbol{x}| < \eta(\delta)\} \subset B'_{\rho(\delta)}$ (see Figure 3).

Fix $\delta_0 \in (0, \delta^*)$ with $\sqrt{\delta_0} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$. Let $\boldsymbol{p}_1 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_1| = r_1$ and $|\boldsymbol{p}_1 - \boldsymbol{y}_1(\delta_0)| = r_1$ such that $\boldsymbol{p}_1$ lies below (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_1(\delta_0)$. Let $\boldsymbol{p}_2 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_2| = r_1$ and $|\boldsymbol{p}_2 - \boldsymbol{y}_2(\delta_0)| = r_1$ such that $\boldsymbol{p}_2$ lies above (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_2(\delta_0)$. Set $\Omega_0 = \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_1| > r_1\} \cup \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_2| > r_1\}$ (see Figure 4).

Claim. *f is uniformly continuous on Ω_0.*

Proof. Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) < \frac{1}{4}\epsilon$ and $p(\delta) < r_2 - r_1$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b_j^\pm : \mathcal{A}(\boldsymbol{p}_j) \to \mathbb{R}$ by

$$b_j^\pm(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h_{\boldsymbol{p}_j}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{A}(\boldsymbol{p}_j)$$

for $j \in \{1, 2\}$. Notice that

$$b_j^-(\boldsymbol{x}) < f(\boldsymbol{x}) < b_j^+(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_j), \quad j \in \{1, 2\}.$$

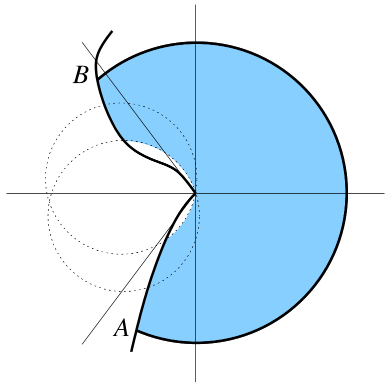


Figure 4. Ω_0.

If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfy $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$, then there exist $\boldsymbol{x}_3 \in \mathcal{A}(\boldsymbol{p}_1) \cap \mathcal{A}(\boldsymbol{p}_2)$ with $|\boldsymbol{x}_3| < \eta(\delta)$ such that $|\boldsymbol{x}_1 - \boldsymbol{x}_3| < \eta(\delta)$ and $|\boldsymbol{x}_2 - \boldsymbol{x}_3| < \eta(\delta)$ and so

$$(4)\quad |f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| \le |f(\boldsymbol{x}_1) - f(\boldsymbol{x}_3)| + |f(\boldsymbol{x}_1) - f(\boldsymbol{x}_3)| < 4p(\delta) + 4q(p(\delta)) < \epsilon.$$

Since f is uniformly continuous on $\{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x}| \ge \frac{1}{2}\eta(\delta)\}$, there exists a $\lambda > 0$ such that if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega^*$ satisfy $|\boldsymbol{x}_1 - \boldsymbol{x}_2| \ge \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < \lambda$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Now set $d = d(\epsilon) = \min\{\lambda, \frac{1}{2}\eta(\delta)\}$. If $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \le \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_1| < \frac{1}{2}\eta(\delta)$, then $|\boldsymbol{x}_1| < \eta(\delta)$ and $|\boldsymbol{x}_2| < \eta(\delta)$; hence $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$ by (4). Next, if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$, $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon) \le \lambda$, $|\boldsymbol{x}_1| \ge \frac{1}{2}\eta(\delta)$ and $|\boldsymbol{x}_2| \ge \frac{1}{2}\eta(\delta)$, then $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. Therefore, for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ with $|\boldsymbol{x}_1 - \boldsymbol{x}_2| < d(\epsilon)$, we have $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < \epsilon$. □

If $\{(r\cos(\theta^-(\delta_0)), r\sin(\theta^-(\delta_0))) : r \ge 0\}$ is the tangent ray to $\partial\mathcal{A}(\boldsymbol{p}_2)$ at $\mathcal{O}$, $\{(r\cos(\theta^+(\delta_0)), r\sin(\theta^+(\delta_0))) : r \ge 0\}$ is the tangent ray to $\partial\mathcal{A}(\boldsymbol{p}_1)$ at $\mathcal{O}$ and $\theta^-(\delta_0), \theta^+(\delta_0) \in (-\alpha, \alpha)$, then it follows from the claim that $f \in C^0(\overline{\Omega}_0)$, the radial limits $Rf(\theta)$ of f at $\mathcal{O}$ exist for $\theta \in [\theta^-(\delta_0), \theta^+(\delta_0)]$ and the radial limits are identical (i.e., $Rf(\theta) = f(\mathcal{O})$ for all $\theta \in [\theta^-(\delta_0), \theta^+(\delta_0)]$.) Since

$$(5)\qquad \lim_{\delta_0 \downarrow 0} \theta^-(\delta_0) = -\alpha \quad \text{and} \quad \lim_{\delta_0 \downarrow 0} \theta^+(\delta_0) = \alpha,$$

Theorem 1 is proven in this case.

Next assume that (B) holds. For $r > 0$ and $j \in \{1, 2\}$, set $B_r^j = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}_j| < r\}$, $C_r^j = \{\boldsymbol{u} \in \overline{E} : |\boldsymbol{u} - \boldsymbol{o}_j| = r\}$, and let l_r^j be the length of the image curve $Y(C_r^j)$; also let $C_r^{j,\prime} = G(C_r^j)$ and $B_r^{j,\prime} = G(B_r^j)$. From the Courant–Lebesgue lemma, we see that for each $\delta \in (0, 1)$ and $j \in \{1, 2\}$, there exists a $\rho_j = \rho_j(\delta) \in (\delta, \sqrt{\delta})$ such that the arclength $l_{j,\rho}$ of $Y(C_{\rho_j}^j)$ is less than $p(\delta)$.

We will only consider $\delta \le \delta_0$, where δ_0 is small enough that the endpoints of $C_{\rho_j(\delta)}^j$ lie on $\sigma_0 \cup \sigma_N^j$ for $j \in \{1, 2\}$ and $C_{\sqrt{\delta_0}}^1 \cap C_{\sqrt{\delta_0}}^2 = \varnothing$, where $\sigma_N^1 = \boldsymbol{o}_1\boldsymbol{b}$ and

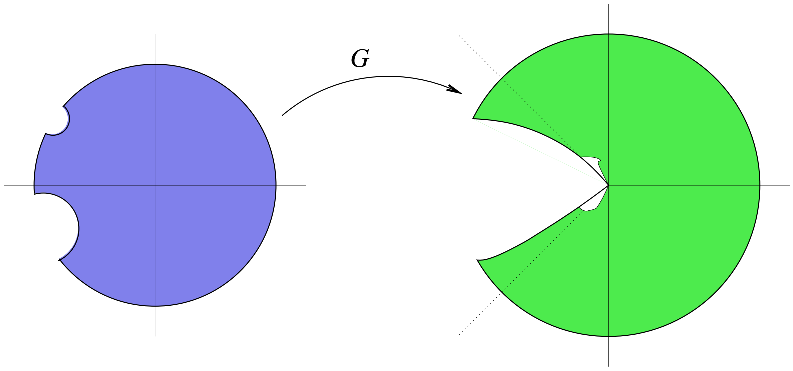


Figure 5. $E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}})$ and Ω_1.

$\sigma_N^2 = \boldsymbol{o}_2\boldsymbol{a}$. For each $\delta \in (0, \delta_0)$, the fact that $l_{j,\ \rho_j(\delta)}$ is finite for $j \in \{1, 2\}$ implies that

$$\lim_{C^{j,\prime}_{\rho_j(\delta)} \ni \boldsymbol{x} \to \mathcal{O}} f(\boldsymbol{x}) \quad \text{exists for } j \in \{1, 2\}.$$

If we set $\Omega_1 = G(E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}}))$ and define $\phi : \partial\Omega_1 \to \mathbb{R}$ by $\phi = f$, then ϕ has (at worst) a jump discontinuity at $\mathcal{O}$. If we consider ϕ to be the Dirichlet data for the boundary value problem

$$\text{div}(Th) = 2H(\cdot\,, f) \quad \text{in } \Omega_1, \tag{6}$$

$$h = \phi \quad \text{on } \partial\Omega_1 \setminus \{\mathcal{O}\}, \tag{7}$$

then $h = f$ is the unique solution of this boundary value problem and so we may parametrize the graph of f over Ω_1 in isothermal coordinates as above and the arguments in [Elcrat and Lancaster 1986; Lancaster 1988; Lancaster and Siegel 1996a] can be used to show that c is uniformly continuous on Ω_1 and so extends to be continuous on $\overline{\Omega}_1$. That is, let $k : E \setminus (\overline{B^1_{\rho_1(\delta)}} \cup \overline{B^2_{\rho_2(\delta)}}) \to E$ be a conformal map. From the works just cited we see that $c \circ k^{-1} \in C^0(\overline{E})$ and so $c \in C^0(\overline{E \setminus (B^1_{\rho_1(\delta)} \cup B^2_{\rho_2(\delta)})})$. Since

$$\bigcup_{\delta \in (0,1)} (E \setminus (B^1_{\rho_1(\delta)} \cup B^2_{\rho_2(\delta)})) = E,$$

we see $c \in C^0(\overline{E} \setminus \{\boldsymbol{o}_1, \boldsymbol{o}_2\})$.

As at the end of Step 1 of the proof of Theorem 1 of [Lancaster and Siegel 1996a], we define $X : B \to \mathbb{R}^3$ by $X = Y \circ g$ and $K : B \to \mathbb{R}^2$ by $K = G \circ g$, where $B = \{(u, v) \in \mathbb{R}^2 : u^2 + v^2 < 1,\ v > 0\}$ and $g : \overline{B} \to \overline{E}$ is an indirectly conformal (or anticonformal) map from $\overline{B}$ onto $\overline{E}$ such that $g(1, 0) = \boldsymbol{o}_1$, $g(-1, 0) = \boldsymbol{o}_2$ and $g(u, 0) \in \boldsymbol{o}_1\boldsymbol{o}_2$ for each $u \in [-1, 1]$. Notice that $K(u, 0) = \mathcal{O}$ for $u \in [-1, 1]$ (see Figure 6). Set

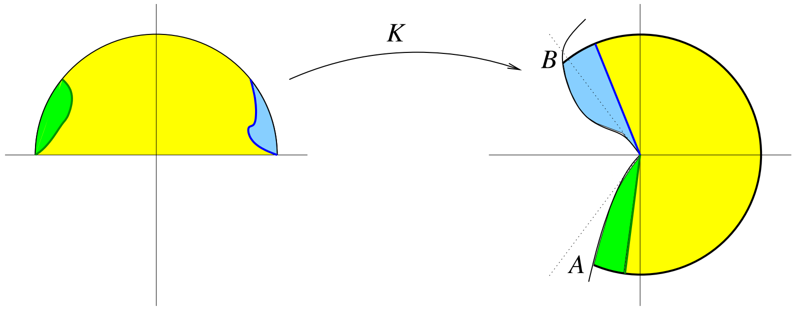


Figure 6. $L(\alpha_2)$, $K^{-1}(L(\alpha_2))$ (blue curves); $L(\alpha_1)$, $K^{-1}(L(\alpha_1))$ (green curves).

$x = a \circ g$, $y = b \circ g$ and $z = c \circ g$, so that $X(u, v) = (x(u, v), y(u, v), z(u, v))$ for $(u, v) \in B$. Now, from Step 2 of the proof of Theorem 1 of [Lancaster and Siegel 1996a],

$$X \in C^0\big(\overline{B} \setminus \{(\pm 1, 0)\} : \mathbb{R}^3\big) \cap C^{1,\iota}\big(B \cup \{(u, 0) : -1 < u < 1\} : \mathbb{R}^3\big)$$

for some $\iota \in (0, 1)$ and $X(u, 0) = (0, 0, z(u, 0))$ cannot be constant on any non-degenerate interval in $(-1, 1)$. Define $\Theta(u) = \arg(x_v(u, 0) + i y_v(u, 0))$. From equation (12) of [Lancaster and Siegel 1996a], we see that

$$\alpha_1 = \lim_{u \downarrow -1} \Theta(u) \quad \text{and} \quad \alpha_2 = \lim_{u \uparrow 1} \Theta(u);$$

here $\alpha_1 < \alpha_2$. As in Steps 2–5 of the proof of Theorem 1 of [Lancaster and Siegel 1996a], we see that $Rf(\theta)$ exists when $\theta \in (\alpha_1, \alpha_2)$,

$$\overline{G^{-1}(L(\alpha_2))} \cap \partial E = \{\boldsymbol{o}_1\} \quad \big(\text{and } \overline{K^{-1}(L(\alpha_2))} \cap \partial B = \{(1, 0)\}\big) \quad \text{if } \alpha_2 < \alpha,$$
$$\overline{G^{-1}(L(\alpha_1))} \cap \partial E = \{\boldsymbol{o}_2\} \quad \big(\text{and } \overline{K^{-1}(L(\alpha_1))} \cap \partial B = \{(-1, 0)\}\big) \quad \text{if } \alpha_1 > -\alpha,$$

where $L(\theta) = \big\{(r \cos\theta, r \sin\theta) \in \Omega : 0 < r < \delta^*\big\}$, and one of the following cases holds:

(a) Rf is strictly increasing or strictly decreasing on (α_1, α_2).

(b) There exist α_L, α_R so that $\alpha_1 < \alpha_L < \alpha_R < \alpha_2$, $\alpha_R = \alpha_L + \pi$, and Rf is constant on $[\alpha_L, \alpha_R]$ and either increasing on $(\alpha_1, \alpha_L]$ and decreasing on $[\alpha_R, \alpha_2)$ or decreasing on $(\alpha_1, \alpha_L]$ and increasing on $[\alpha_R, \alpha_2)$.

If $\alpha_2 = \alpha$ and $\alpha_1 = -\alpha$, then Theorem 1 is proven. Otherwise, suppose $\alpha_2 < \alpha$ and fix $\delta_0 \in (0, \delta^*)$ and Ω_0 (see Figure 4) as before in case (i).

Claim. *Suppose $\alpha_2 < \alpha$. Then f is uniformly continuous on Ω_0^+, where*

$$\Omega_0^+ := \big\{(r \cos\theta, r \sin\theta) \in \Omega_0 : 0 < r < \delta^*,\ \alpha_2 < \theta < \pi\big\}.$$

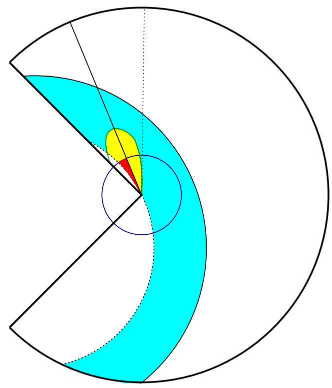


Figure 7. $\Omega^* \cap \mathcal{A}(\boldsymbol{p}_1)$ (blue, yellow and red regions), $\partial B_{\eta(\delta)}(\mathcal{O})$ (blue circle).

Proof. Suppose $\alpha - \alpha_2 < \pi$ (see the blue region in Figure 6). Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) < \frac{1}{4}\epsilon$ and $p(\delta) < r_2 - r_1$. Let $C_r = \{(u, v) \in \overline{B} : |(u, v) - (1, 0)| = r\}$ and let l_r be the arclength of the image curve $X(C_r)$. The Courant–Lebesgue lemma implies that for each $\delta \in (0, 1)$, there exists a $\rho(\delta) \in (\delta, \sqrt{\delta})$ such that $l_{\rho(\delta)} < p(\delta)$. Denote the endpoints of $C_{\rho(\delta)}$ as $(u_1(\delta), v_1(\delta))$ and $(u_2(\delta), 0)$, where $(u_1(\delta))^2 + (v_1(\delta))^2 = 1$, $v_1(\delta) > 0$ and $u_2(\delta) \in (-1, 1)$. Notice $\Theta(u_2(\delta)) < \alpha_2$; let us assume that δ is small enough that $\alpha - \Theta(u_2(\delta)) < \pi$.

Now $X(C_{\rho(\delta)})$ is a curve whose tangent ray at $\mathcal{O}$ exists and has direction $\theta = \Theta(u_2(\delta))$ and $\partial\Omega \cap X\big(C_{\rho(\delta)} \setminus \{(u_1(\delta), v_1(\delta)), (u_2(\delta), 0)\}\big) = \varnothing$; hence there exists $\eta = \eta(\delta) > 0$ such that $\{\boldsymbol{x} \in \Omega_0^+ : |\boldsymbol{x}| < \eta(\delta)\}$ (the red region in Figure 7) is a subset of $\Omega_0 \cap X\big(\{(u, v) \in \overline{B} : |(u, v) - (1, 0)| < \rho(\delta)\}\big)$ (the yellow region plus the red region in Figure 7). From (4) and the arguments in the proof of the claim in case (i), we see that f is uniformly continuous on Ω_0^+.

If $\alpha - \alpha_2 \geq \pi$, we argue as in the proof of the claim in case (i) and see that f is uniformly continuous on Ω_0^+. □

Thus $f \in C^0(\overline{\Omega}_0^+)$; hence (5) implies that $Rf(\theta) = \lim_{\tau \uparrow \alpha_2} Rf(\tau)$ for all $\theta \in [\alpha_2, \alpha)$. Suppose $\alpha_1 > -\alpha$. Then, as above, f is uniformly continuous on

$$\Omega_0^- := \{(r\cos\theta, r\sin\theta) \in \Omega_0 : 0 < r < \delta^*, -\pi < \theta < \alpha_1\}$$

and $f \in C^0(\overline{\Omega}_0^-)$; hence (5) implies

$$Rf(\theta) = \lim_{\tau \downarrow \alpha_1} Rf(\tau) \quad \text{for all } \theta \in (-\alpha, \alpha_1].$$

Thus Theorem 1 is proven. □

3. Proof of Theorem 2

The parametric representation (3) with properties $(a_1) - (a_5)$ continues to be valid and either case (A) or case (B) holds true.

Suppose case (A) holds. Let q_1 denote the modulus of continuity of the trace of f on the (closed) set $\partial^-\Omega^*$ (i.e., $|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| \le q_1(|\boldsymbol{x}_1 - \boldsymbol{x}_2|)$ if $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \partial^-\Omega^*$). Fix $\delta_0 \in (0, \delta^*)$ with $\sqrt{\delta_0} < \min\{|\boldsymbol{o} - \boldsymbol{a}|, |\boldsymbol{o} - \boldsymbol{b}|\}$. Let $\boldsymbol{p}_1 \in \mathbb{R}^2$ satisfy $|\boldsymbol{p}_1| = r_1$ and $|\boldsymbol{p}_1 - \boldsymbol{y}_1(\delta_0)| = r_1$ such that $\boldsymbol{p}_1$ lies above (and to the left of) the line through $\mathcal{O}$ and $\boldsymbol{y}_1(\delta_0)$. Set $\Omega_0 = \{\boldsymbol{x} \in \Omega^* : |\boldsymbol{x} - \boldsymbol{p}_1| > r_1\}$.

Claim. *f is uniformly continuous on Ω_0.*

Proof. Let $\epsilon > 0$. Choose $\delta \in (0, \delta_0)$ such that $p(\delta) + q(p(\delta)) + q_1(p(\delta)) < \frac{1}{2}\epsilon$ and $p(\delta) < r_2 - r_1$. Pick a point $\boldsymbol{w} \in C'_{\rho(\delta)}$ and define $b_1^{\pm} : \mathcal{A}(\boldsymbol{p}_1) \to \mathbb{R}$ by

$$b_1^{\pm}(\boldsymbol{x}) = f(\boldsymbol{w}) \pm p(\delta) \pm h_{\boldsymbol{p}_1}(\boldsymbol{x}), \quad \boldsymbol{x} \in \mathcal{A}(\boldsymbol{p}_1).$$

Notice that

$$b_1^-(\boldsymbol{x}) < f(\boldsymbol{x}) < b_1^+(\boldsymbol{x}) \quad \text{for } \boldsymbol{x} \in B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_1).$$

Now there exists $\eta = \eta(\delta) > 0$ such that $\{\boldsymbol{x} \in \Omega_0 : |\boldsymbol{x}| < \eta(\delta)\}$ (the red regions in Figure 8) is a subset of $B'_{\rho(\delta)} \cap \mathcal{A}(\boldsymbol{p}_1)$ (the yellow regions plus the red regions in Figure 8). Thus, for $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \Omega_0$ satisfying $|\boldsymbol{x}_1| < \eta(\delta)$, $|\boldsymbol{x}_2| < \eta(\delta)$, we have

$$|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)| < 2p(\delta) + 2q(p(\delta)) + 2q_1(p(\delta)) < \epsilon.$$

The remainder of the proof of the claim follows as before.

The proof of Theorem 2 in this case now follows the proof of Theorem 1 in the same case.

If case (B) holds, then the proof of Theorem 2 is essentially the same as the proof of Theorem 1; the only significant difference is that $z \in C^0(\overline{B} \setminus \{(1, 0)\})$ (and $c \in C^0(\overline{E} \setminus \{\boldsymbol{o}_1\})$) and hence $Rf(\theta)$ exists for $\theta \in [-\alpha, \alpha)$. $\square$

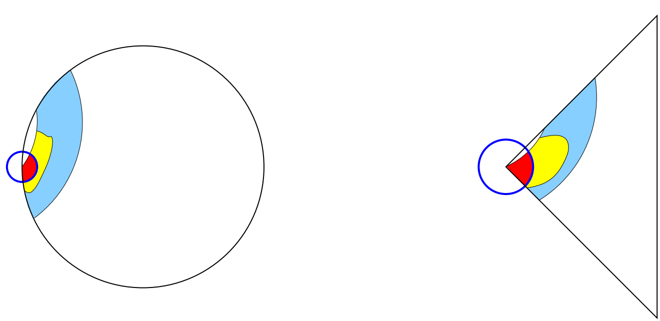


Figure 8. $\Omega_0 \cap \mathcal{A}(\boldsymbol{p}_1)$ (blue, yellow and red regions), $\partial B_{\eta(\delta)}(\mathcal{O})$ (blue circle).

References

[Courant 1950] R. Courant, *Dirichlet's principle, conformal mapping, and minimal murfaces*, Interscience, New York, 1950. MR 0036317

[Elcrat and Lancaster 1986] A. R. Elcrat and K. E. Lancaster, "Boundary behavior of a nonparametric surface of prescribed mean curvature near a reentrant corner", *Trans. Amer. Math. Soc.* **297**:2 (1986), 645–650. MR 854090

[Jin and Lancaster 1997] Z. Jin and K. Lancaster, "Behavior of solutions for some Dirichlet problems near reentrant corners", *Indiana Univ. Math. J.* **46**:3 (1997), 827–862. MR 1488339

[Lancaster 1985] K. E. Lancaster, "Boundary behavior of a nonparametric minimal surface in $\mathbf{R}^3$ at a nonconvex point", *Analysis* **5**:1-2 (1985), 61–69. MR 791492

[Lancaster 1987] K. E. Lancaster, "Boundary behavior of nonparametric minimal surfaces—some theorems and conjectures", pp. 37–41 in *Variational methods for free surface interfaces* (Menlo Park, CA, 1985), edited by P. Concus and R. Finn, Springer, New York, 1987. MR 872886

[Lancaster 1988] K. E. Lancaster, "Nonparametric minimal surfaces in $\mathbf{R}^3$ whose boundaries have a jump discontinuity", *Internat. J. Math. Math. Sci.* **11**:4 (1988), 651–656. MR 959444

[Lancaster 1989] K. E. Lancaster, "Existence and nonexistence of radial limits of minimal surfaces", *Proc. Amer. Math. Soc.* **106**:3 (1989), 757–762. MR 969523

[Lancaster 1991] K. E. Lancaster, "Boundary behavior near re-entrant corners for solutions of certain elliptic equations", *Rend. Circ. Mat. Palermo* (2) **40**:2 (1991), 189–214. MR 1151584

[Lancaster and Siegel 1996a] K. E. Lancaster and D. Siegel, "Existence and behavior of the radial limits of a bounded capillary surface at a corner", *Pacific J. Math.* **176**:1 (1996), 165–194. MR 1433987

[Lancaster and Siegel 1996b] K. E. Lancaster and D. Siegel, "Behavior of a bounded non-parametric H-surface near a reentrant corner", *Z. Anal. Anwendungen* **15**:4 (1996), 819–850. MR 1422643

[Lancaster and Siegel 1997] K. E. Lancaster and D. Siegel, "Correction to: "Existence and behavior of the radial limits of a bounded capillary surface at a corner"", *Pacific J. Math.* **179**:2 (1997), 397–402. MR 1452541

Received October 26, 2015. Revised February 13, 2016.

MOZHGAN (NORA) ENTEKHABI
DEPARTMENT OF MATHEMATICS, STATISTICS AND PHYSICS
WICHITA STATE UNIVERSITY
WICHITA, KS 67260-0033
UNITED STATES
mozhgan.entekhabi@gmail.com

KIRK E. LANCASTER
DEPARTMENT OF MATHEMATICS, STATISTICS AND PHYSICS
WICHITA STATE UNIVERSITY
WICHITA, KS 67260-0033
UNITED STATES
lancaster@math.wichita.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.353

A REMARK ON THE NOETHERIAN PROPERTY OF POWER SERIES RINGS

BYUNG GYUN KANG AND PHAN THANH TOAN

Let α be a (finite or infinite) cardinal number. An ideal of a ring R is called an α-generated ideal if it can be generated by a set with cardinality at most α. A ring R is called an α-generated ring if every ideal of R is an α-generated ideal. When α is finite, the class of α-generated rings has been studied in literature by scholars such as I. S. Cohen and R. Gilmer. In this paper, the class of α-generated rings when α is infinite (in particular, when $\alpha = \aleph_0$, the smallest infinite cardinal number) is considered. Surprisingly, it is proved that the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$. In other words, if every ideal of $R[\![X]\!]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[\![X]\!]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α, it is proved that R is an α-generated ring if and only if $R[X]$ is an α-generated ring, which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. Let $\mathbb{O}$ be the ring of algebraic integers. Under the continuum hypothesis, we show that $\mathbb{O}[\![X]\!]$ contains an $|\mathbb{O}[\![X]\!]|$-generated (and hence uncountably generated) ideal which is not a β-generated ideal for any cardinal number $\beta < |\mathbb{O}[\![X]\!]|$ and that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different for the power series ring $R[\![X]\!]$.

1. Introduction

In this paper, a ring means a commutative ring with identity. Let R be a ring and let n be a positive integer. An ideal I of R is called an n-generated ideal if I can be generated by a set with cardinality $\leq n$. We call R an n-generated ring if every ideal of R is an n-generated ideal. This class of n-generated rings was first introduced and studied by Cohen [1950]. Principal ideal rings are obviously 1-generated rings. It is well known that Dedekind domains are 2-generated rings. For each

Phan Thanh Toan is the corresponding author.

MSC2010: 03E10, 13A15, 13B25, 13E05, 13F25.

Keywords: countably generated ideal, n-generated ideal, Noetherian ring, polynomial ring, power series ring.

integer $n \geq 2$, Matson [2009] gave an example of an n-generated ring which is not an $(n-1)$-generated ring. Cohen [1950] proved that if D is an n-generated integral domain, then the Krull dimension of D is at most 1. An easy proof of this result was later given by Gilmer [1973]. He further showed [1972b] that the result is valid for n-generated rings (see also [Sally 1978, Theorem 1.2, p. 51]). By definition, every n-generated ring is a Noetherian ring. However, the converse does not hold [Cohen 1950]. Gilmer [1973] proved a nice result that if R is an n-generated ring, then the integral closure of R is a Noetherian Prüfer ring. He also showed that a Noetherian ring R is an n-generated ring (for some n) if and only if there exists a positive integer m such that R_M is an m-generated ring for each maximal ideal M of R [Gilmer 1972a; 1972b]. For more on n-generated ideals and rings, we refer the readers to [Ameziane Hassani et al. 1996; Cohen 1950; Gilmer 1972b; 1973; Heinzer and Lantz 1983; Matsuda 1979; McLean 1982; Okon et al. 1992; Okon and Vicknair 1992; 1993; Rush 1991; 1992; Sally 1978; Shalev 1986].

According to Cohen and Gilmer's results, the class of n-generated rings is rather small. It is a subclass of Noetherian rings with Krull dimension at most 1. We generalize the definition of n-generated rings in a natural way as follows. Let α be a (finite or infinite) cardinal number (e.g., $\alpha = 1, 2, \dots, \aleph_0, \aleph_1, \dots$). An ideal I of a ring R is called an α-generated ideal if I can be generated by a set with cardinality at most α. R is called an α-generated ring if every ideal of R is an α-generated ideal. In this paper, we mainly deal with the class of α-generated rings when $\alpha = \aleph_0$, the smallest infinite cardinal number. By definition, an $\aleph_0$-generated ring is a ring whose ideals are countably generated. Trivial examples of $\aleph_0$-generated rings are those that have only countably many elements (so that each ideal has itself as a countable generating set). Every Noetherian ring is obviously an $\aleph_0$-generated ring. However, the converse does not hold. Polynomial rings $R[X_1, X_2, \dots, X_n, \dots]$ in countably infinite indeterminates over countable rings R, the ring $\mathbb{O}$ of algebraic integers, the ring $\mathrm{Int}(\mathbb{Z})$ of integer-valued polynomials on $\mathbb{Z}$, and 1-dimensional nondiscrete valuation domains are good examples of $\aleph_0$-generated rings that are not Noetherian rings.

Even though the class of $\aleph_0$-generated rings is strictly larger than the class of Noetherian rings, we show that, when restricted to power series rings, they are actually the same. In other words, the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$ (Theorem 13). This means if every ideal of $R[\![X]\!]$ is countably generated, then each of them is in fact finitely generated. This shows a strange behavior of the power series ring $R[\![X]\!]$ compared to that of the polynomial ring $R[X]$. Indeed, for any infinite cardinal number α, we prove that R is an α-generated ring if and only if $R[X]$ is an α-generated ring (Theorem 22), which is an analogue of the Hilbert basis theorem stating that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We show under the continuum hypothesis that (1) $\mathbb{O}[\![X]\!]$ contains an $|\mathbb{O}[\![X]\!]|$-generated (and hence uncountably generated) ideal which is not a β-generated ideal for any cardinal number $\beta < |\mathbb{O}[\![X]\!]|$ (Corollary 16) and that (2) $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring that is not an $\aleph_0$-generated ring (Corollary 17). In fact, these two results hold if $\mathbb{O}$ is replaced by any non-Noetherian countable ring R. As a consequence, it is shown that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different (while the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same) for the power series ring $R[\![X]\!]$.

2. Some examples of α-generated rings

For each integer $n \geq 2$, Matson [2009] gave an example of an n-generated ring which is not an $(n-1)$-generated ring. For an infinite cardinal number α, we give an example of an α-generated ring that is not a β-generated ring for any cardinal number $\beta < \alpha$.

Proposition 1. *For an infinite cardinal number α, there exists an α-generated ring that is not a β-generated ring for any cardinal number $\beta < \alpha$.*

Proof. Let R be any ring with cardinality $< \alpha$ and let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a set of indeterminates over R, where Λ is a set of cardinality α. Then the polynomial ring $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ is clearly an α-generated ring since it has cardinality α. We now show that the ideal J of $R[\{X_\lambda\}_{\lambda\in\Lambda}]$ generated by $\{X_\lambda\}_{\lambda\in\Lambda}$ is not a β-generated ideal for any cardinal number $\beta < \alpha$. Let β be any cardinal number such that $\beta < \alpha$. If J is a β-generated ideal, then $J = (\{f_\mu\})$ for some $f_\mu \in J$ such that $|\{f_\mu\}| \leq \beta$. Since each f_μ involves only finitely many indeterminates, $(\{X_\lambda\}_{\lambda\in\Lambda}) = J = (\{f_\mu\}) \subseteq (\{X_\lambda\}_{\lambda\in\Gamma})$ for some subset Γ of Λ such that $|\Gamma| < |\Lambda| = \alpha$, a contradiction. □

In the next section, we are going to prove that the concepts "$\aleph_0$-generated ring" and "Noetherian ring" are the same for the power series ring $R[\![X]\!]$. We however note that these two concepts are different in general. We give here some examples of (finite-dimensional or infinite-dimensional) $\aleph_0$-generated rings that are not Noetherian rings.

Example 2. Let $R_1 := R[X_1, X_2, \ldots, X_n, \ldots]$ be the polynomial ring in countably infinite indeterminates over a countable ring R. Then R_1 is an $\aleph_0$-generated ring since it is countable. It is easy to see that the ideal of R_1 generated by $X_1, X_2, \ldots, X_n, \ldots$ is not a finitely generated ideal and hence R_1 is not a Noetherian ring.

In Example 2, the (Krull) dimension of R_1 is infinite. We now give examples of finite-dimensional $\aleph_0$-generated rings that are not Noetherian rings.

Example 3. Let $\mathbb{O}$ be the ring of algebraic integers (an algebraic integer is a complex number that is integral over $\mathbb{Z}$). It is well known that $\mathbb{O}$ is a 1-dimensional non-Noetherian Bézout domain (for example, see p. 72 of [Kaplansky 1974]). However, since $\mathbb{O}$ is countable, it is an $\aleph_0$-generated ring.

Example 4. Let $\mathrm{Int}(\mathbb{Z})$ be the ring of integer-valued polynomials on $\mathbb{Z}$, i.e.,

$$\mathrm{Int}(\mathbb{Z}) := \{f \in \mathbb{Q}[X] \mid f(\mathbb{Z}) \subseteq \mathbb{Z}\}.$$

By [Cahen and Chabert 1997, Proposition V.2.7], $\mathrm{Int}(\mathbb{Z})$ is a 2-dimensional non-Noetherian domain. However, since $\mathrm{Int}(\mathbb{Z})$ is countable, it is an $\aleph_0$-generated ring.

So far, the given examples of $\aleph_0$-generated rings are all countable. We finally show that there do exist uncountable $\aleph_0$-generated rings that are not Noetherian rings (see Example 6).

Proposition 5. *If V is a 1-dimensional nondiscrete valuation domain, then V is an $\aleph_0$-generated ring that is not a Noetherian ring.*

Proof. Since V is a 1-dimensional valuation domain, its value group is (isomorphic to) a subgroup of $\mathbb{R}$, the (additive) group of real numbers. This implies that every ideal of V is countably generated, i.e., V is an $\aleph_0$-generated ring. Since V is nondiscrete, it is not a Noetherian ring. □

Example 6. Let K be a field and let V be the valuation ring of the field $K(X; \mathbb{R})$, which is the quotient field of the group ring $K[X; \mathbb{R}]$ of $\mathbb{R}$ over K, associated with the valuation v defined by

$$v\left(\sum_{i=0}^{n} a_{r_i} X^{r_i}\right) := \min\{r_i \mid i = 0, 1, \ldots, n\}.$$

Then V is a 1-dimensional nondiscrete valuation domain with value group $\mathbb{R}$. Hence, V is an $\aleph_0$-generated ring that is not a Noetherian ring by Proposition 5. Since $\mathbb{R}$ is uncountable, so is V.

3. Power series rings over α-generated rings

In this section, we prove that the power series ring $R[\![X]\!]$ is an $\aleph_0$-generated ring if and only if $R[\![X]\!]$ is a Noetherian ring (and hence) if and only if R is a Noetherian ring. In order to prove this result, we only need to show that if R is a non-Noetherian ring, then the power series ring $R[\![X]\!]$ is not an $\aleph_0$-generated ring since, if R is a Noetherian ring, then $R[\![X]\!]$ is also a Noetherian ring (for example, see [Kaplansky 1974, Theorem 71]) and hence an $\aleph_0$-generated ring.

Suppose that R is a non-Noetherian ring. Our task is to construct an ideal J of $R[\![X]\!]$ that cannot be generated by any countable subset of J. The desired uncountable generating set for J is indexed by the following special uncountable set, which is called a fathomless set.

3.1. ***Fathomless sets.*** Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of positive integers, and let $\mathcal{U}$ be the set of all subsets U of $\mathbb{N}$ such that $U = \{n, n+1, \dots\}$ for some $n \in \mathbb{N}$. For two strictly increasing sequences $s = \{s_n\}$ and $t = \{t_n\}$ of positive integers, we set $s \gg t$ (we also write $t \ll s$) if for each positive integer k there is a set $U \in \mathcal{U}$ (depending on k) such that $s_n > kt_n$ for each $n \in U$, i.e., $s_n > kt_n$ for all large n.

Let $\mathcal{S}$ be the collection of all $\mathcal{A}$ having the following properties:

(1) $\mathcal{A}$ is a nonempty collection of strictly increasing sequences $s = \{s_n\}$ of positive integers.

(2) If $s \in \mathcal{A}$ then $s \gg b$, where b is the sequence defined by $b_n := n$ for all n.

(3) If $s, t \in \mathcal{A}$ and $s \neq t$, then $s \gg t$ or $t \gg s$.

If u is the sequence defined by $u_n := b_n^2$ for each n, then it is easy to see that u is a strictly increasing sequence of positive integers and that $u \gg b$. It follows that the set $\mathcal{S}$ is nonempty. We order $\mathcal{S}$ by set-theoretic inclusion. By Zorn's lemma, there exists a maximal element in $\mathcal{S}$. Let $\mathcal{A}$ be a maximal element in $\mathcal{S}$. This choice of $\mathcal{A}$ will be fixed through the rest of the article. For $s, t \in \mathcal{A}$, we define $s \underline{\ll} t$ if and only if $s = t$ or $s \ll t$. Then $(\mathcal{A}, \underline{\ll})$ becomes a totally ordered set.

Definition 7. A totally ordered set $(\mathcal{Y}, \underline{\ll})$ is called a *fathomless set* if, for every nonempty countable subset $\mathcal{C}$ of $\mathcal{Y}$, there exists an element $y \in \mathcal{Y}$ such that $y \ll \mathcal{C}$, i.e., $y \ll c$ for all $c \in \mathcal{C}$.

The following theorem tells us that the set $(\mathcal{A}, \underline{\ll})$ is a fathomless set (for the proof, see [Kang et al. 2013, Theorem 5]).

Theorem 8. *The set $(\mathcal{A}, \underline{\ll})$ is a fathomless set.*

Remark 9. By definition, every fathomless set is an uncountable set. Hence, the set $(\mathcal{A}, \underline{\ll})$ is uncountable.

3.2. ***Power series rings over $\aleph_0$-generated rings.*** Using the above fathomless set $(\mathcal{A}, \underline{\ll})$, we construct generators for an ideal J of $R[\![X]\!]$ that is not countably generated. The following observation is useful:

Proposition 10. *For a ring R, the following are equivalent*:

(1) *R is a non-Noetherian ring.*

(2) *There exists a sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R such that*

$$a_m \notin (a_0, a_1, \dots, a_{m-1})$$

for each $m \geq 1$.

Proof. (1) $\Longrightarrow$ (2) Suppose that R is a non-Noetherian ring. Then there exists an ideal I of R such that I is not finitely generated. We will find the desired sequence $a_0, a_1, \dots, a_m, \dots$ of elements in R by using induction as follows. Choose an element $a_0 \in I$. Since $(a_0) \subsetneq I$, there exists an element $a_1 \in I \setminus (a_0)$. Suppose that there exist elements $a_0, a_1, \ldots, a_{m-1}$ in I $(m \geq 2)$ such that $a_i \notin (a_0, a_1, \ldots, a_{i-1})$ for each $1 \leq i \leq m-1$. Since $(a_0, a_1, \ldots, a_{m-1}) \subsetneq I$, there exists an element $a_m \in I \setminus (a_0, a_1, \ldots, a_{m-1})$.

(2) $\Longrightarrow$ (1) The ideal $(a_0, a_1, \ldots, a_m, \ldots)$ of R is not finitely generated. □

Let R be a non-Noetherian ring. By Proposition 10, there exists a sequence $a_0, a_1, \ldots, a_m, \ldots$ of elements in R such that

$$a_m \notin (a_0, a_1, \ldots, a_{m-1})$$

for each $m \geq 1$. For each integer $m \geq 0$, we let $I_m := (a_0, a_1, \ldots, a_m)$. Then $a_m \notin I_{m-1}$ for each $m \geq 1$. For each sequence $s = \{s_n\} \in \mathcal{A}$, we define

$$f_s := a_0 + a_1 X^{s_1} + a_2 X^{s_2} + \cdots + a_n X^{s_n} + \cdots \in R[\![X]\!].$$

We let J be the ideal of $R[\![X]\!]$ generated by all f_s with $s \in \mathcal{A}$.

Remark 11. The generators f_s of J are constructed by stretching out the coefficients of the power series $\sum_{n=0}^{\infty} a_n X^n$ so that its coefficient at X^{s_n} (s_n is much greater than n for all large n since $s \gg b$) is still a_n. This property will play a crucial role in showing that J is not a countably generated ideal.

An ideal I of a ring R is called an uncountably generated ideal if it is not a countably generated ideal. If R is a non-Noetherian ring, then so is the power series ring $R[\![X]\!]$. Hence, $R[\![X]\!]$ has some ideals that are not finitely generated. These ideals can be either countably generated or uncountably generated. However, we show that $R[\![X]\!]$ has at least one uncountably generated ideal if R is a non-Noetherian ring.

Theorem 12. *If R is a non-Noetherian ring, then the power series ring $R[\![X]\!]$ has at least one uncountably generated ideal.*

Proof. It suffices to show that the ideal J constructed above is not a countably generated ideal. Suppose on the contrary that J is countably generated. Then there exists a countable subset $\mathcal{B}$ of $\mathcal{A}$ such that J is generated by $\{f_s \mid s \in \mathcal{B}\}$. Since $\mathcal{A}$ is a fathomless set, there exists a sequence $v \in \mathcal{A}$ such that $v \ll \mathcal{B}$. Since $f_v \in J$, f_v is a finite sum of elements of the form $h(s) f_s$,

$$f_v = \sum_s h(s) f_s, \tag{1}$$

where $h(s) \in R[\![X]\!]$ and $s \in \mathcal{B}$. Since $v \ll \mathcal{B}$, by taking a finite intersection of members of $\mathcal{U}$, we can find a set $U \in \mathcal{U}$ such that $v_m < s_m$ for each $m \in U$ and for each s appearing in the finite sum (1). Choose any number $m \in U$. Since $v_m < s_m$, the coefficient of f_s at X^j belongs to I_{m-1} for all $j \leq v_m$. It follows that the coefficient of $h(s) f_s$ at X^{v_m} belongs to I_{m-1}. This holds for every s appearing in the finite sum (1). Therefore, the coefficient of $\sum_s h(s) f_s$ at X^{v_m} belongs to I_{m-1}. This is a contradiction since the coefficient of $f_v = \sum_s h(s) f_s$ at X^{v_m} is a_m and $a_m \notin I_{m-1}$. □

We can now obtain the main result of the paper.

Theorem 13. *For a ring R, the following are equivalent*:

(1) *$R[\![X]\!]$ is an $\aleph_0$-generated ring.*

(2) *$R[\![X]\!]$ is a Noetherian ring.*

(3) *R is a Noetherian ring.*

Proof. We only need to prove that (1) implies (3). However, this follows from Theorem 12. □

Since a ring R is a Noetherian ring if and only if every countably generated ideal is finitely generated, we have the following corollary.

Corollary 14. *For a ring R, the following are equivalent*:

(1) *Every ideal of $R[\![X]\!]$ is countably generated.*

(2) *Every countably generated ideal of $R[\![X]\!]$ is finitely generated.*

Corollary 15. *If R is an $\aleph_0$-generated ring, then the power series ring $R[\![X]\!]$ is not necessarily an $\aleph_0$-generated ring.*

Proof. Let R be any $\aleph_0$-generated ring which is not a Noetherian ring (see Proposition 5 and Examples 2, 3, and 4). Then $R[\![X]\!]$ is not an $\aleph_0$-generated ring by Theorem 13. □

Corollary 16. *Let $\mathbb{O}$ be the ring of algebraic integers and let $\alpha = |\mathbb{O}[\![X]\!]|$. Then, under the continuum hypothesis, $\mathbb{O}[\![X]\!]$ contains an α-generated ideal that is not a β-generated ideal for any cardinal number $\beta < \alpha$.*

Proof. We have $\alpha = |\mathbb{O}[\![X]\!]| = 2^{\aleph_0} = \aleph_1$ under the continuum hypothesis. By Theorem 12, $\mathbb{O}[\![X]\!]$ has an uncountably generated ideal J. Obviously, J is an α-generated ideal since $|J| \leq |\mathbb{O}[\![X]\!]| = \alpha$. But, since J is not countably generated, it is not a β-generated ideal for any cardinal number $\beta < \aleph_1 = \alpha$. □

The following corollary shows that the concepts "$\aleph_1$-generated ring" and "$\aleph_0$-generated ring" are different for the power series ring $R[\![X]\!]$.

Corollary 17. *Under the continuum hypothesis, $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring but not an $\aleph_0$-generated ring.*

Proof. Since $|\mathbb{O}[\![X]\!]| = \aleph_1$, $\mathbb{O}[\![X]\!]$ is an $\aleph_1$-generated ring. The fact that $\mathbb{O}[\![X]\!]$ is not an $\aleph_0$-generated ring follows from Corollary 16. □

Remark 18. Corollaries 16 and 17 hold for any non-Noetherian countable ring (see Examples 2, 3, and 4).

In the rest of this section, we consider power series rings over n-generated rings, where n is a positive integer. We first note that, for the power series ring $R[\![X]\!]$ to be an n-generated ring, it is necessary that the ring R be zero-dimensional.

Proposition 19. *If the power series ring $R[\![X]\!]$ is an n-generated ring for some positive integer n, then* $\dim R = 0$.

Proof. If $R[\![X]\!]$ is an n-generated ring, then $\dim R[\![X]\!] \leq 1$. It is easy to see that $\dim R + 1 \leq \dim R[\![X]\!]$. Thus, $\dim R + 1 \leq \dim R[\![X]\!] \leq 1$ and hence $\dim R = 0$. □

Proposition 20. *Suppose that D is an integral domain. Then the following are equivalent*:

(1) *$D[\![X]\!]$ is an n-generated ring for some positive integer n.*

(2) *$D[\![X]\!]$ is a* 1*-generated ring.*

(3) *D is a field.*

Proof. (3) $\Longrightarrow$ (2) $\Longrightarrow$ (1) These are obvious.

(1) $\Longrightarrow$ (3) By Proposition 19, $\dim D = 0$. Thus, D is a field by the assumption that D is an integral domain. □

Remark 21. By Proposition 20, if D is an (n-generated) integral domain which is not a field, then the power series ring $D[\![X]\!]$ is never an m-generated ring for any positive integer m. In particular, $D[\![X]\!]$ has an ideal that is not an $(n+1)$-generated ideal despite the fact that all prime ideals of $D[\![X]\!]$ are $(n+1)$-generated ideals (see [Kaplansky 1974, Theorem 70]).

4. Polynomial rings over α-generated rings

In this section, we prove that for any infinite cardinal number α, a ring R is an α-generated ring if and only if the polynomial ring $R[X]$ is an α-generated ring, which is an analogue of the Hilbert basis theorem, which states that R is a Noetherian ring if and only if $R[X]$ is a Noetherian ring. We also note that the result fails if α is a finite cardinal number.

Theorem 22. *For any infinite cardinal number α, a ring R is an α-generated ring if and only if the polynomial ring $R[X]$ is an α-generated ring.*

Proof. We follow the standard proof of the Hilbert basis theorem. Any homomorphic image of an α-generated ring is obviously an α-generated ring. Hence, if $R[X]$ is an α-generated ring, then so is R.

For the converse, suppose that R is an α-generated ring and let J be an ideal of $R[X]$. We show that J is an α-generated ideal. For each $n \geq 0$, let I_n be the set of $r \in R$ such that $r = 0$ or r is the leading coefficient of a polynomial in J of degree n. For each $n \geq 0$, since R is an α-generated ring, I_n is an α-generated ideal. Let

$$\{r_{n\lambda} \mid \lambda \in \Lambda_n\}$$

be a generating set of I_n with cardinality at most α. For each $r_{n\lambda}$, let $f_{n\lambda} \in J$ be a polynomial of degree n with leading coefficient $r_{n\lambda}$. We will show that the ideal J of $R[X]$ is generated by the set

$$F := \{f_{n\lambda} \mid n \geq 0, \lambda \in \Lambda_n\},$$

which also has cardinality at most α. Denote by (F) the ideal of $R[X]$ generated by F. Since $F \subseteq J$, we have $(F) \subseteq J$. Conversely, the polynomials of degree 0 in J are precisely the elements of I_0 and hence are contained in (F). Proceeding by induction, assume that (F) contains all polynomials of J of degree less than k and let $g \in J$ have degree k and leading coefficient r. Then

$$r = \sum_{i=1}^{m} s_i r_{k\lambda_i}$$

for some $s_i \in R$ and $\lambda_i \in \Lambda_k$. The polynomial $\sum_{i=1}^{m} s_i f_{k\lambda_i}$ also has degree k and leading coefficient r. Hence,

$$g - \sum_{i=1}^{m} s_i f_{k\lambda_i} \in J$$

has degree at most $k - 1$. By the induction hypothesis,

$$g - \sum_{i=1}^{m} s_i f_{k\lambda_i} \in (F)$$

and hence $g \in (F)$. Therefore, $J = (F)$ is an α-generated ideal. □

Proposition 23. *If the polynomial ring $R[X]$ is an n-generated ring for some positive integer n, then* $\dim R = 0$.

Proof. As in the proof of Proposition 19, we have $\dim R + 1 \leq \dim R[X] \leq 1$ and hence $\dim R = 0$. □

As in the power series ring case, we can prove the following:

Proposition 24. *Suppose that D is an integral domain. Then the following are equivalent*:

(1) *$D[X]$ is an n-generated ring for some positive integer n.*

(2) *$D[X]$ is a 1-generated ring.*

(3) *D is a field.*

Remark 25. By Proposition 24, if D is an (n-generated) integral domain which is not a field, then the polynomial ring $D[X]$ is never an m-generated ring for any positive integer m. In particular, Theorem 22 always fails for any finite cardinal number α.

Remarks 26. (1) For rings with zero-divisors, the polynomial ring $R[X]$ may not be a 2-generated ring even if the ring R is a 1-generated ring. For example, let $R = V/(a^3)$, where V is a 1-dimensional discrete valuation domain (or equivalently, a local principal ideal domain) with maximal ideal $M = (a)$. Then R is a 1-generated ring. However, if $\overline{M}$ denotes the maximal ideal $M/(a^3)$ of R, then $\overline{M}^2 \neq 0$. By [Matsuda 1979, (5.7)], $R[X]$ is not a 2-generated ring.

(2) More generally, for any integer $n \geq 2$, let $R = V/(a^{n+1})$, where V is the same as above. Then R is a 1-generated ring. However, by [Matsuda 1979, (5.13)], the polynomial ring $R[X]$ is not an n-generated ring.

Acknowledgements

The authors would like to thank the referee for his/her helpful comments and suggestions. This research was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) funded by the Ministry of Education (2013R1A1A2020783).

References

[Ameziane Hassani et al. 1996] S. Ameziane Hassani, M. Fontana, and S.-E. Kabbaj, "Group rings $R[G]$ with 3-generated ideals when R is Artinian", *Comm. Algebra* **24**:4 (1996), 1253–1280. MR 1380596

[Cahen and Chabert 1997] P.-J. Cahen and J.-L. Chabert, *Integer-valued polynomials*, Mathematical Surveys and Monographs **48**, American Mathematical Society, Providence, RI, 1997. MR 1421321

[Cohen 1950] I. S. Cohen, "Commutative rings with restricted minimum condition", *Duke Math. J.* **17** (1950), 27–42. MR 0033276

[Gilmer 1972a] R. Gilmer, *Multiplicative ideal theory*, Pure and Applied Mathematics **12**, Dekker, New York, 1972. MR 0427289

[Gilmer 1972b] R. Gilmer, "On commutative rings of finite rank", *Duke Math. J.* **39** (1972), 381–383. MR 0292810

[Gilmer 1973] R. Gilmer, "The n-generator property for commutative rings", *Proc. Amer. Math. Soc.* **38** (1973), 477–482. MR 0309922

[Heinzer and Lantz 1983] W. Heinzer and D. Lantz, "Commutative rings with acc on n-generated ideals", *J. Algebra* **80**:1 (1983), 261–278. MR 690717

[Kang et al. 2013] B. G. Kang, K. A. Loper, T. G. Lucas, M. H. Park, and P. T. Toan, "The Krull dimension of power series rings over non-SFT rings", *J. Pure Appl. Algebra* **217**:2 (2013), 254–258. MR 2969250

[Kaplansky 1974] I. Kaplansky, *Commutative rings*, revised ed., The University of Chicago Press, 1974. MR 0345945

[Matson 2009] A. Matson, "Rings of finite rank and finitely generated ideals", *J. Commut. Algebra* **1**:3 (2009), 537–546. MR 2524865

[Matsuda 1979] R. Matsuda, "Torsion-free abelian semigroup rings, V", *Bull. Fac. Sci. Ibaraki Univ. Ser. A* **11** (1979), 1–37. MR 536898

[McLean 1982] K. R. McLean, "Local rings with bounded ideals", *J. Algebra* **74**:2 (1982), 328–332. MR 647243

[Okon and Vicknair 1992] J. S. Okon and J. P. Vicknair, "Group rings with n-generated ideals", *Comm. Algebra* **20**:1 (1992), 189–217. MR 1145332

[Okon and Vicknair 1993] J. S. Okon and J. P. Vicknair, "One-dimensional monoid rings with n-generated ideals", *Canad. Math. Bull.* **36**:3 (1993), 344–350. MR 1245819

[Okon et al. 1992] J. S. Okon, D. E. Rush, and P. Vicknair, "Commutative semigroup rings with two-generated ideals", *J. London Math. Soc.* (2) **45**:3 (1992), 417–432. MR 1180253

[Rush 1991] D. E. Rush, "Rings with two-generated ideals", *J. Pure Appl. Algebra* **73**:3 (1991), 257–275. MR 1124788

[Rush 1992] D. E. Rush, "Artin rings with two-generated ideals", *Canad. Math. Bull.* **35**:1 (1992), 133–135. MR 1157475

[Sally 1978] J. D. Sally, *Numbers of generators of ideals in local rings*, Dekker, New York, 1978. MR 0485852

[Shalev 1986] A. Shalev, "On the number of generators of ideals in local rings", *Adv. in Math.* **59**:1 (1986), 82–94. MR 825089

Received November 29, 2015. Revised January 15, 2016.

BYUNG GYUN KANG
DEPARTMENT OF MATHEMATICS
POHANG UNIVERSITY OF SCIENCE AND TECHNOLOGY
POHANG 37673
SOUTH KOREA
bgkang@postech.ac.kr

PHAN THANH TOAN
DIVISION OF COMPUTATIONAL MATHEMATICS AND ENGINEERING
INSTITUTE FOR COMPUTATIONAL SCIENCE
TON DUC THANG UNIVERSITY
HO CHI MINH CITY
VIETNAM
and
FACULTY OF MATHEMATICS AND STATISTICS
TON DUC THANG UNIVERSITY
HO CHI MINH CITY
VIETNAM
phanthanhtoan@tdt.edu.vn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.365

CURVES WITH PRESCRIBED INTERSECTION WITH BOUNDARY DIVISORS IN MODULI SPACES OF CURVES

XIAO-LEI LIU

We construct curves in moduli spaces of curves with prescribed intersection with boundary divisors. As applications, we obtain families of curves with maximal slope as well as extremal test curves for the weakly positive cone of the moduli space.

1. Introduction

Let $\mathcal{M}_g$ be the moduli space of smooth curves of genus g over the field of complex numbers $\mathbb{C}$, and $\Delta_0, \Delta_1, \ldots, \Delta_{[g/2]}$ the boundary divisors of the Deligne–Mumford compactification $\overline{\mathcal{M}}_g$. Denote by $\mathcal{H}_g$ the moduli space of smooth hyperelliptic curves of genus g; then the restriction of Δ_0 to the closure $\overline{\mathcal{H}}_g$ breaks up into $\Xi_0, \Xi_1, \ldots, \Xi_{[(g-1)/2]}$. The restriction of Δ_i to $\overline{\mathcal{H}}_g$ is often denoted by Θ_i for $i > 0$; see [Harris and Morrison 1998].

A family of curves of genus g over a curve Y is a fibration $f : X \to Y$ whose general fibers are smooth curves of genus g, where X is a smooth projective surface. Let $\omega_{X/Y}$ be the relative dualizing sheaf. If f is nontrivial, the *slope* of f is $\omega^2_{X/Y}/\deg f_*\omega_{X/Y}$. Let $J_f : Y \to \overline{\mathcal{M}}_g$ denote the moduli map induced by f; see [Tan 2010].

Special curves in moduli spaces play an important rule in the study of birational geometry of moduli spaces: for example, the ample cone, the nef cone and the Mori cone of curves [Gibney 2009]. Before raising our problems, we firstly summarize some interesting properties of curves in moduli spaces with prescribed intersection with boundary divisors. In this paper, we always assume that curves in moduli spaces are complete irreducible and are not contained in boundary divisors.

If $C = J_f(Y) \subset \overline{\mathcal{H}}_g$ is a curve intersecting only Ξ_0 (resp. $\Delta_{[g/2]}$), then the semistable reduction of f has minimal (resp. maximal) slope; see [Liu 2016, Remark 3.9] and [Liu and Tan 2013, Theorem 3.1]. We refer to [Tan 2010] for related discussions. Moreover, if $C = J_f(Y) \subset \overline{\mathcal{M}}_g$ is disjoint from boundary divisors, then the semistable reduction of f is a Kodaira fibration; see [Kodaira 1967].

MSC2010: 14D22.
Keywords: intersection, boundary divisors, moduli spaces of curves.

On the other hand, curves intersecting exactly one boundary divisor of $\overline{\mathcal{H}}_g$ are the extremal test curves of the weakly positive cone in $\mathrm{Pic}(\overline{\mathcal{M}}_g) \otimes \mathbb{Q}$. Here the weakly positive cone consists of weakly positive $\mathbb{Q}$-Cartier divisors over $\mathcal{M}_g$; see Section 1 in [Moriwaki 1998] for the definition of weakly positive divisors, and Section 4 of that paper for this result.

Motivated by these properties, we want to study further properties of curves that intersect given boundary divisors. In this paper we discuss the existence of such curves.

Let C be a curve in $\overline{\mathcal{H}}_g$, and $\mathcal{B} = \{\Theta_1, \dots, \Theta_{[g/2]}, \Xi_0, \Xi_1, \dots, \Xi_{[(g-1)/2]}\}$ be the set of boundary divisors of $\overline{\mathcal{H}}_g$. Denote by $\mathcal{B}_C \subset \mathcal{B}$ the set of boundary divisors of $\overline{\mathcal{H}}_g$ that intersect C.

Problem 1.1. *For any nonempty subset $\mathcal{B}' \subseteq \mathcal{B}$, does there exist a curve C in $\overline{\mathcal{H}}_g$ such that the boundary divisors of $\overline{\mathcal{H}}_g$ that intersect C are those in $\mathcal{B}'$, i.e., $\mathcal{B}_C = \mathcal{B}'$?*

Let $\widetilde{\mathcal{M}}_{0,n}$ be the moduli space of stable unordered n-pointed rational curves. Let B_k be the boundary divisor of $\widetilde{\mathcal{M}}_{0,n}$ whose general point parametrizes the union of a k-pointed $\mathbb{P}^1$ and an $(n-k)$-pointed $\mathbb{P}^1$ for $2 \le k \le [n/2]$.

One can regard $\overline{\mathcal{H}}_g$ as the Hurwitz space parametrizing genus g admissible double covers of rational curves. Such a cover uniquely corresponds to a stable $(2g+2)$-pointed rational curve by marking the branch points of the cover. Thus $\overline{\mathcal{H}}_g$ can be further identified as $\widetilde{\mathcal{M}}_{0,2g+2}$. The natural isomorphism $\overline{\mathcal{H}}_g \cong \widetilde{\mathcal{M}}_{0,2g+2}$ induces the identifications $\Xi_i = B_{2i+2}$ and $\Theta_i = B_{2i+1}$. Also denote by $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$ the set of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$. Hence the existence of curves in $\overline{\mathcal{H}}_g$ in Problem 1.1 is the same as that in $\widetilde{\mathcal{M}}_{0,n}$ for $n = 2g+2$. For the sake of completeness, we consider the existence of curves in $\widetilde{\mathcal{M}}_{0,n}$. Precisely, we consider the following problem.

Problem 1.2. *For any nonempty subset $\mathcal{B}' \subseteq \mathcal{B}$, does there exist a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that the boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$ that intersect C are those in $\mathcal{B}'$, i.e., such that $\mathcal{B}_C = \mathcal{B}'$?*

The purpose of this paper is to answer these problems in a number of cases. We have the following uniform solution for small n.

Theorem 1.3. *Assume $n \le 17$. For any nonempty subset $\mathcal{B}'$ of $\mathcal{B}$, there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Unfortunately, our method is invalid for $n = 18$ (see Remark 6.1).

Corollary 1.4. *For $2 \le g \le 7$ and any nonempty $\mathcal{B}' \subset \mathcal{B}$, there exists a curve C in $\overline{\mathcal{H}}_g$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Corollary 1.5. *For $2 \le g \le 7$ and any nonempty $\mathcal{B}' \subset \Delta = \{\Delta_0, \Delta_1, \dots, \Delta_{[g/2]}\}$, there exists a curve C in $\overline{\mathcal{M}}_g$ with $\mathcal{B}_C = \mathcal{B}'$.*

In order to simplify the proof of the above theorem, we use the following two partial solutions of Problem 1.2.

Theorem 1.6. *Let $\mathcal{B}'$ be a subset of the set $\mathcal{B} = \{B_2, \dots, B_{[n/2]}\}$ of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$.*

(1) *For $|\mathcal{B}'| = 1$, if $\mathcal{B}' = \{B_i\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(2) *For $|\mathcal{B}'| = 2$, if $\mathcal{B}' = \{B_i, B_{i+1}\}$ or $\mathcal{B}' = \{B_i, B_{i+2}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(3) *For $|\mathcal{B}'| = 3$, if $\mathcal{B}'$ is one of $\{B_i, B_j, B_{i+j+1}\}$, $\{B_i, B_j, B_{i+j}\}$, $\{B_i, B_j, B_{i+j-1}\}$, or $\{B_i, B_j, B_{i+j-2}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

In particular, we recover the existence of a hyperelliptic family with maximal slope from [Liu and Tan 2013], where the authors construct explicit polynomials to define the family. As another application, we also give an alternative proof of the existence of extremal test curves for the weakly positive cone from Appendix A in [Moriwaki 1998], where the author uses the existence of a concrete linear system. Here we reduce the problem to the existence of rational functions and then give a unitive method. This new method greatly generalizes the former results.

Theorem 1.7. *Let $\mathcal{B}'$ be a subset of the set $\mathcal{B} = \{B_2, \dots, B_{[n/2]}\}$ of boundary divisors of $\widetilde{\mathcal{M}}_{0,n}$.*

(1) *If $\mathcal{B}' = \{B_2, B_{i_1}, \dots, B_{i_k}\}$ and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(2) *If $\mathcal{B}' = \{B_2, B_{k+1}, B_{k+2}, \dots, B_{[n/2]}\}$ and $2 \le k \le [n/2]$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

(3) *If $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 3$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Now we explain the main idea of the proofs. In order to construct curves in moduli spaces intersecting the given boundary divisors, we use the following three different methods:

(i) We regard a rational function ϕ of degree n as a 1-dimensional family of unordered n marked points in $\mathbb{P}^1$ in Section 3. Only the critical points of ϕ may correspond to points in boundary divisors; see the correspondence (3-1). Then we just need to construct rational functions with the desired critical points. The existence of such rational functions is from the main Theorem of [Scherbak 2002].

(ii) The graph G_ϕ of ϕ is a smooth rational curve in $\mathbb{P}^1 \times E$, where $E \cong \mathbb{P}^1$. Let p be the second projection and R the reducible curve consisting of G_ϕ and certain sections of p. Then the restriction $p|_R$ is a branched cover over E, and this cover induces different 1-dimensional families of marked points in $\mathbb{P}^1$. The correspondence (4-1) gives a relation between points in boundary divisors with ramification points of $p|_R$, i.e., between singular points of R and ramification points of $p|_{G_\phi}$. So we only need to construct curves R with suitable ramification points in Section 4. This method generalizes method (i), and both methods are effective for many cases.

On the other hand, if n is large, $\mathcal{B}'$ may have so many elements that there is no desired rational function, rendering the above two methods invalid; see Remark 6.1. For this difficulty, we have the following method for special $\mathcal{B}'$.

(iii) Taking the intersection of general very ample divisors of Hassett's weighted moduli spaces, we can obtain curves intersecting all the boundary divisors of these spaces. The proper transforms of these curves in $\widetilde{\mathcal{M}}_{0,n}$ by the reduction morphism are also the curves we need; see Section 5.

2. Existence of rational functions

Let $\phi(x) = f(x)/g(x)$ be a *rational function of degree n*; i.e., $f(x)$ and $g(x)$ are polynomials without common roots and $n = \max\{\deg f(x), \deg g(x)\}$. A point $z \in \mathbb{C}$ is a *critical point of multiplicity m* if z is a root of the Wronskian $W(x)$ of $f(x)$ and $g(x)$ of multiplicity m, where

$$W(x) = g^2(x)\phi'(x) = f'(x)g(x) - f(x)g'(x).$$

Let all the finite critical points of $\phi(x)$ be denoted $z_1, \dots, z_{l-1}$ with multiplicities $m_1, \dots, m_{l-1}$, respectively. According to the Riemann–Hurwitz formula, $2n-2 \geq m_1 + \cdots + m_{l-1}$. Let z_l be the point at infinity; then the difference

$$m_l = 2n - 2 - (m_1 + \cdots + m_{l-1}) \tag{2-1}$$

is *the multiplicity of $\phi(x)$ at infinity*. If $z_1, \dots, z_{l-1}$ are in general position and $1 \leq m_i \leq n-1$ for each $1 \leq i \leq l$, then we say that $\phi(x)$ is a rational function *of type* $(n; m_1, \dots, m_l)$.

Note that, up to the point at infinity, the types of rational functions here are the same as in [Scherbak 2002].

If $\phi_1(x)$ and $\phi_2(x)$ are two rational functions satisfying

$$\phi_2(x) = \frac{a\phi_1(x) + b}{c\phi_1(x) + d}, \quad ad - bc \neq 0,$$

then $\phi_1(x)$ and $\phi_2(x)$ have the same type, and we say that $\phi_1(x)$ and $\phi_2(x)$ are in the same class of rational functions.

Theorem 2.1 [Scherbak 2002]. *Let $l \geq 3$ and $n \geq 2$ be integers, and also let $1 \leq m_i \leq n-1$ for $1 \leq i \leq l$ be integers satisfying* (2-1). *Then the number $\#(n; m_1, \ldots, m_l)$ of classes of rational functions of type $(n; m_1, \ldots, m_l)$ is*

$$\#(n; m_1, \ldots, m_l) = \sum_{q=1}^{l-1} (-1)^{l-1-q} \sum_{1 \leq i_1 < \cdots < i_q \leq l-1} \binom{m_{i_1} + \cdots + m_{i_q} + q - n - 1}{l-3}, \tag{2-2}$$

and any nonempty class can be represented by the ratio of polynomials without multiple roots.

Proof. Let $\boldsymbol{m} = (m_1, \ldots, m_{l-1})$; then $\#(n; m_1, \ldots, m_l)$ is equal to $\#(n, l-1; \boldsymbol{m})$ in the main theorem in [Scherbak 2002], from which the result directly follows. □

As usual, we set $\binom{a}{b} = 0$ for $a < b$.

Corollary 2.2.

(1) *There exists a rational function $\phi(x)$ of type $(n; m_1, \ldots, m_{l-1}, m_l = n-1)$.*

(2) *There exists a rational function $\phi(x)$ of type $(n; m_1, m_2, m_3)$.*

Proof. (1) Since $m_l = n-1$, we have $m_1 + \cdots + m_{l-1} = n-1$, and the right side of (2-2) is

$$\binom{m_1 + \cdots + m_{l-1} + l - 1 - n - 1}{l-3} = \binom{l-3}{l-3} = 1.$$

So the desired rational function exists by Theorem 2.1.

(2) If $m_i = n-1$ for some i, then the existence is from (1). We may assume that $m_i \leq n-2$ for $i = 1, 2, 3$. Then the right side of (2-2) is

$$\binom{m_1 + m_2 + 2 - n - 1}{0} - \binom{m_1 - n}{0} - \binom{m_2 - n}{0} = \binom{m_1 + m_2 - n + 1}{0} = 1. \qquad \square$$

3. Curves from rational functions

Let $\phi(x)$ be a rational function of degree n; then it induces a degree n branched cover $D \cong \mathbb{P}^1 \to E \cong \mathbb{P}^1$. Varying a point $t \in E$, the union of the n preimage points also varies in D; hence it provides a 1-dimensional family E of unordered n points in $\mathbb{P}^1$. So we obtain a curve in $\widetilde{\mathcal{M}}_{0,n}$. When t hits a branch point, suppose that over t there is a ramification point z_i locally of type $y = \phi(x) = x^{m_i+1}$, i.e., a critical point of $\phi(x)$ with multiplicity m_i. Making a degree $m_i + 1$ base change, we then get an ordinary singularity of degree $m_i + 1$. Blowing up the singularity separates the $m_i + 1$ sheets, and thus t corresponds to a point in the boundary component B_{m_i+1} (or B_{n-1-m_i} if $m_i \geq [n/2]$, but not $m_i = n-1$ or $n-2$, which correspond to a point in the interior of $\widetilde{\mathcal{M}}_{0,n}$). Hence, if $\phi(x)$ is of type $(n; m_1, \ldots, m_l)$ and C is its corresponding curve in $\widetilde{\mathcal{M}}_{0,n}$, then

$$\mathcal{B}_C = \{B_{m_i+1} : 1 \leq m_i < [n/2]\} \cup \{B_{n-1-m_i} : [n/2] \leq m_i \leq n-3\}. \tag{3-1}$$

Theorem 3.1. *Let $\mathcal{B}' = \{B_{i_1}, \ldots, B_{i_k}\}$ be a subset of $\mathcal{B} = \{B_2, B_3, \ldots, B_{[n/2]}\}$ with $(i_1 - 1) + \cdots + (i_k - 1) = n - 1$; then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$. Moreover,*

(1) *if $\mathcal{B}' = \{B_2, B_{i_1}, \ldots, B_{i_k}\}$, and $(i_1 - 1) + \cdots + (i_k - 1) \le n - 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$;*

(2) *if $\mathcal{B}' = \{B_3, B_{i_1}, \ldots, B_{i_k}\}$, and $n - 1 - (i_1 - 1) - \cdots - (i_k - 1) = 2j \ge 2$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $(i_1 - 1) + \cdots + (i_k - 1) = n - 1$, set $i_{k+1} = n$; then there exists a rational function $\phi(x)$ of type $(n; i_1 - 1, \ldots, i_k - 1, i_{k+1} - 1)$ by Corollary 2.2(1). Thus the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi(x)$ satisfies $\mathcal{B}_C = \{B_{i_1}, \ldots, B_{i_k}\}$ by (3-1).

(1) Let $h = n - 1 - ((i_1 - 1) + \cdots + (i_k - 1)) \ge 1$ and $i_{k+1} = \cdots = i_{k+h} = 2$; then $(i_1 - 1) + \cdots + (i_{k+h} - 1) = n - 1$, and hence there exists a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \{B_2, B_{i_1}, \ldots, B_{i_k}\}$.

(2) If we take $i_{k+1} = \cdots = i_{k+j} = 3$, then $(i_1 - 1) + \cdots + (i_{k+j} - 1) = n - 1$, and there exists a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \{B_3, B_{i_1}, \ldots, B_{i_k}\}$. □

Theorem 3.2. *Let $\mathcal{B}'$ be a subset of $\mathcal{B} = \{B_2, B_3, \ldots, B_{[n/2]}\}$. If $\mathcal{B}'$ is one of $\{B_i, B_{i+1}\}$, $\{B_i, B_{i+2}\}$, or $\{B_i, B_j, B_{i+j+1}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $\mathcal{B}' = \{B_i, B_{i+1}\}$, then there exists a rational function $\phi_1(x)$ of type $(n; i, n-1-i, n-1)$ by Corollary 2.2(2). Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_1(x)$ satisfies $\mathcal{B}_C = \{B_i, B_{i+1}\}$ by (3-1).

By the same reasoning, for $\mathcal{B}' = \{B_i, B_{i+2}\}$, there exists a rational function $\phi_2(x)$ of type $(n; i+1, n-1-i, n-2)$, and the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_2(x)$ satisfies $\mathcal{B}_C = \{B_i, B_{i+2}\}$.

Similarly, for $\mathcal{B}' = \{B_i, B_j, B_{i+j+1}\}$, there exists a rational function $\phi_3(x)$ of type $(n; n-1-i, n-1-j, i+j)$, and the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ corresponding to $\phi_3(x)$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j+1}\}$. □

4. Curves from rational functions and sections

Let $\phi(x)$ be a rational function of type $(n - s; m_1, \ldots, m_l)$, where $s < n$ is a nonnegative integer. Let G_ϕ be its graph in $\mathbb{P}^1 \times E$, where $E \cong \mathbb{P}^1$; then G_ϕ is a smooth rational curve. We now consider the reducible curve

$$R_{\phi,\Gamma_1,\ldots,\Gamma_s} = G_\phi + \Gamma_1 + \cdots + \Gamma_s,$$

where the Γ_i are sections of the second projection

$$p : \mathbb{P}^1 \times E \to E, \quad p((x, t)) = t.$$

So the restriction of p to $R_{\phi,\Gamma_1,\dots,\Gamma_s}$

$$p : R_{\phi,\Gamma_1,\dots,\Gamma_s} \to E$$

is a cover of degree n. Similarly as in Section 3, varying a point $t \in E$, we obtain a 1-dimensional family of unordered n marked points in $\mathbb{P}^1$. Hence we get a curve C in $\widetilde{\mathcal{M}}_{0,n}$. Note that this construction is the same as that in Section 3 when $s = 0$.

Let $S_1 = (\Gamma_1 + \cdots + \Gamma_s) \cap G_\phi$, and let $S_2 = \{z_1, \dots, z_l\}$ be the set of all the critical points of $\phi(x)$, including the point at infinity. Here we identify the critical point z_i of ϕ with its image $(z_i, \phi(z_i))$ in G_ϕ.

If $z \in S_2 \cap S_1$, then the local equation of p at z is $x(x^{m+1} + t) = 0$, where m is the multiplicity of z. Making a degree $m + 1$ base change

$$t \mapsto u^{m+1},$$

we get an ordinary singularity of degree $m + 2$. Blowing it up, we then see that $p(z) \in E$ corresponds to a point in the boundary component B_{m+2} (or B_{n-2-m} if $[n/2] - 1 \le m \le n - 4$, but not if $m = n - 3$ or $n - 2$, corresponding to a point in the interior of $\widetilde{\mathcal{M}}_{0,n}$).

If $z \in S_2 - S_1$, then the point in $\widetilde{\mathcal{M}}_{0,n}$ corresponding to $p(z)$ has been considered in Section 3. Hence we have that

$$\begin{aligned}(4\text{-}1)\quad \mathcal{B}_C = \{B_2 : z \in S_1 \backslash S_2\} &\cup \left\{B_{m_i+1} : 1 \le m_i < \left[\tfrac{1}{2}n\right], z_i \in S_2 \backslash S_1\right\} \\ &\cup \left\{B_{n-1-m_i} : \left[\tfrac{1}{2}n\right] \le m_i \le n-3, z_i \in S_2 \backslash S_1\right\} \\ &\cup \left\{B_{m_i+2} : 1 \le m_i < \left[\tfrac{1}{2}n\right] - 1, z_i \in S_2 \cap S_1\right\} \\ &\cup \left\{B_{n-2-m_i} : \left[\tfrac{1}{2}n\right] - 1 \le m_i \le n-4, z_i \in S_2 \cap S_1\right\}.\end{aligned}$$

Since we have considered the case that $\mathcal{B}'$ contains B_2 in Theorem 3.1(1), we now consider the case where each section passes through a critical point of $\phi(x)$, that is, where $S_1 \subset S_2$.

Theorem 4.1. *Let $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ be a subset of $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$ with $n - 1 - ((i_1 - 1) + \cdots + (i_k - 1)) = 2j + 1 \ge 3$; then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. Set $i_{k+1} = \cdots = i_{k+j} = 3$, then

$$(i_1 - 1) + \cdots + (i_k - 1) + (3 - 1)j + ((n - 1) - 1) = 2((n - 1) - 1).$$

Hence there exists a rational function $\phi(x)$ of type $(n-1; i_1-1, \dots, i_{k+j}-1, n-2)$ by Corollary 2.2(1). Let Γ be the section passing through the critical point z_{k+j+1}, where the multiplicity of z_{k+j+1} of $\phi(x)$ is $n - 2$. Thus the reducible curve $R_{\phi,\Gamma} = G_\phi + \Gamma$ induces a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ with $\mathcal{B}_C = \mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$ by (4-1). □

Corollary 4.2. *If $\mathcal{B}' = \{B_3, B_{i_1}, \dots, B_{i_k}\}$, and $n-3-((i_1-1)+\cdots+(i_k-1)) \geq 0$, then there exists a curve C such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $n-3-((i_1-1)+\cdots+(i_k-1))$ is even, then it follows directly from Theorem 3.1(2); if $n-3-((i_1-1)+\cdots+(i_k-1))$ is odd, then it is from Theorem 4.1. □

Theorem 4.3. *Let $\mathcal{B}'$ be a subset of $\mathcal{B} = \{B_2, B_3, \dots, B_{[n/2]}\}$. If $\mathcal{B}'$ is one of $\{B_i\}$, $\{B_i, B_j, B_{i+j-2}\}$, $\{B_i, B_j, B_{i+j}\}$, or $\{B_i, B_j, B_{i+j-1}\}$, then there exists a curve C in $\widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. Let $\phi_1(x)$ be a rational function of type $(n-1; i-1, n-i-1, n-2)$ and Γ_1 the section passing through the critical point z_3 of $\phi_1(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by R_{ϕ_1,Γ_1} satisfies $\mathcal{B}_C = \{B_i\}$ by (4-1).

Let $\phi_2(x)$ be a rational function of type $(n-3; n-i-2, n-j-2, i+j-4)$ and Γ_{2i} the section passing through the critical point z_i $(i = 1, 2, 3)$ of $\phi_2(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by $R_{\phi_2,\Gamma_{21},\Gamma_{22},\Gamma_{23}}$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j-2}\}$ by (4-1).

Let $\phi_3(x)$ be a rational function of type $(n-2; n-2-i, n-2-j, i+j-2)$ and Γ_{3i} the section passing through the critical point z_i $(i = 1, 2)$ of $\phi_3(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by $R_{\phi_3,\Gamma_{31},\Gamma_{32}}$ satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j-1}\}$ by (4-1).

Let $\phi_4(x)$ be a rational function of type $(n-1; n-1-i, n-1-j, i+j-2)$ and Γ_4 the section passing through the critical point z_3 of $\phi_4(x)$. Then the curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ induced by R_{ϕ_4,Γ_4} satisfies $\mathcal{B}_C = \{B_i, B_j, B_{i+j}\}$ by (4-1). □

Proof of Theorem 1.6. It follows directly from Theorem 3.2 and Theorem 4.3. □

5. Curves from birational geometry of moduli spaces

Let $\overline{\mathcal{M}}_{0,n}$ be the moduli space of stable n-pointed rational curves. The space $\overline{\mathcal{M}}_{0,n}$ has a natural S_n action by reordering the marked points. Let $\rho : \overline{\mathcal{M}}_{0,n} \to \widetilde{\mathcal{M}}_{0,n}$ be the finite quotient morphism via S_n.

A *weight datum* $\mathcal{A} = (a_1, \dots, a_n)$ is a sequence of rational numbers such that $0 < a_i \leq 1$. Given a weight datum $\mathcal{A}$ satisfying $2g-2+\sum_{i=1}^n a_i > 0$, an n-pointed curve $(C; p_1, \dots, p_n)$ of genus g is *$\mathcal{A}$-stable* if

(1) C has, at worst, ordinary double points as singularities, and $p_1, \dots, p_n$ are smooth points of C;

(2) $\omega_C\left(\sum_{i=1}^n a_i p_i\right)$ is ample;

(3) $\operatorname{mult}_x\left(\sum_{i=1}^n a_i p_i\right) \leq 1$ for any $x \in C$.

For any weight datum $\mathcal{A}$ such that $2g-2+\sum_{i=1}^n a_i > 0$, there exists a projective coarse moduli space $\overline{\mathcal{M}}_{g,\mathcal{A}}$ [Hassett 2003, Theorem 2.1] of weighted n-pointed $\mathcal{A}$-stable curves. Note that $\overline{\mathcal{M}}_{g,\mathcal{A}} = \overline{\mathcal{M}}_{g,n}$ when $a_1 = \cdots = a_n = 1$.

Let $\mathcal{A}_1 = (a_1, \ldots, a_n)$ and $\mathcal{A}_2 = (b_1, \ldots, b_n)$ be two weight data and suppose that $a_i \geq b_i$ for all $i = 1, 2, \ldots, n$. Then there exists a *reduction morphism* [Hassett 2003, Theorem 4.1]

$$\varphi_{\mathcal{A}_1,\mathcal{A}_2} : \overline{\mathcal{M}}_{0,\mathcal{A}_1} \to \overline{\mathcal{M}}_{0,\mathcal{A}_2}.$$

If $(C, p_1, \ldots, p_n) \in \overline{\mathcal{M}}_{0,\mathcal{A}_1}$, then $\varphi_{\mathcal{A}_1,\mathcal{A}_2}(C, p_1, \ldots, p_n)$ is obtained by collapsing components of C on which $\omega_C + \sum b_i p_i$ fails to be ample.

Theorem 5.1. *Let $\mathcal{B}' = \{B_2, B_{k+1}, B_{k+2}, \ldots, B_{[n/2]}\}$ for $2 \leq k \leq [n/2]$; then there exists a curve $C \subset \widetilde{\mathcal{M}}_{0,n}$ such that $\mathcal{B}_C = \mathcal{B}'$.*

Proof. If $k = [n/2]$, then $\mathcal{B}' = \{B_2\}$, and existence was proved in Theorem 3.1(1).

If $k = 2$, then $\mathcal{B}' = \mathcal{B} = \{B_2, B_3, \ldots, B_{[n/2]}\}$. Taking the intersection of $n-4$ general very ample divisors of $\widetilde{\mathcal{M}}_{0,n}$, we obtain a curve C in $\widetilde{\mathcal{M}}_{0,n}$ that intersects all the boundary divisors; that is, $\mathcal{B}_C = \mathcal{B}$.

In the following, we assume that $3 \leq k \leq [n/2]-1$. Let $\mathcal{A}(k) = \{1/k, \ldots, 1/k\}$ be the symmetric weight datum that assigns $1/k$ to each marked point. Then $\sum_{i=1}^n a_i = n \cdot (1/k) > 2$. Let $\widetilde{\mathcal{M}}_{0,\mathcal{A}(k)}$ be the quotient of the weighted moduli space $\overline{\mathcal{M}}_{0,\mathcal{A}(k)}$ via the natural S_n action. Denote by $f_k : \widetilde{\mathcal{M}}_{0,n} \to \widetilde{\mathcal{M}}_{0,\mathcal{A}(k)}$ the corresponding reduction morphism. Since three points on $\mathbb{P}^1$ (the two marked points and the attaching node) have no nontrivial moduli, f_k does not contract B_2. The degree of the total weight on a rational tail with j $(3 \leq j \leq k)$ marked points is $(1/k) \cdot j + 1 \leq 2 = -\deg K_{\mathbb{P}^1}$, where the $+1$ comes from the attaching node. Hence it violates the stability condition. So f_k contracts the boundary divisors B_j for $3 \leq j \leq k$. Furthermore, we see that f_k contracts only these boundary divisors. Now taking the intersection of $n-4$ general very ample divisors, we obtain a curve C in $\widetilde{\mathcal{M}}_{0,\mathcal{A}(k)}$ that is disjoint with $f_k(B_j)$ for $3 \leq j \leq k$ and that intersects all the other (not contracted) boundary divisors $B_2, B_{k+1}, \ldots, B_{[n/2]}$ in the loci where f_k is an isomorphism. Then the proper transform C' of C in $\widetilde{\mathcal{M}}_{0,n}$ satisfies $\mathcal{B}_{C'} = \mathcal{B}' = \{B_2, B_{k+1}, \ldots, B_{[n/2]}\}$. □

Proof of Theorem 1.7. It follows from Theorems 3.1 and 5.1 and Corollary 4.2. □

6. Proof of Theorem 1.3

Let us first introduce some notation. Below, $\{i_1, \ldots, i_k\}$ stands for $\{B_{i_1}, \ldots, B_{i_k}\}$. If $\mathcal{B}' = \{i_1, \ldots, i_k\}$ is one of the sets in Theorem 1.6, we call $\mathcal{B}'$ *of type* $T_{i_1,\ldots,i_k}$. For example, if $\mathcal{B}' = \{2, 3, 5\}$, we call $\mathcal{B}'$ of type $T_{2,3,5}$. If $\mathcal{B}' = \{2, i_1, \ldots, i_k\}$ is one of the sets in Theorem 1.7(1)–(2), we call $\mathcal{B}'$ *of type* $T_{2,*}$. If $\mathcal{B}' = \{3, i_1, \ldots, i_k\}$ is one of the sets in Theorem 1.7(3), we call $\mathcal{B}'$ *of type* $T_{3,*}$.

Proof of Theorem 1.3. For $n \leq 11$, it is easy to check that the result follows directly from Theorem 1.6 and Theorem 1.7. In the following we always assume that $12 \leq n \leq 17$. Then we have the following three cases.

Case 1. If $\mathscr{B}'$ is one of the sets in Theorems 1.6 and 1.7, we are done.

Case 2. If $n = 14$ and $\mathscr{B}' = \{4, 7\}$, then there exists a rational function $\phi_1(x)$ of type (13; 3, 3, 6, 12) by Corollary 2.2(1). Denote by Γ_1 the section passing through the critical point z_4 of $\phi_1(x)$, where the multiplicity of z_4 is 12. Then the curve C_1 in $\widetilde{\mathscr{M}}_{0,14}$ corresponding to R_{ϕ_1,Γ_1} satisfies $\mathscr{B}_{C_1} = \{4, 7\}$ by (4-1).

If $n = 17$ and $\mathscr{B}' = \{4, 7\}$, then there exists a rational function $\phi_2(x)$ of type (16; 3, 6, 6, 15) by Corollary 2.2(1). Denote by $\Gamma_2(x)$ the section passing through the critical point z_4 of $\phi_2(x)$, where the multiplicity of z_4 is 15. Then the curve C_2 in $\widetilde{\mathscr{M}}_{0,17}$ corresponding to R_{ϕ_2,Γ_2} satisfies $\mathscr{B}_{C_2} = \{4, 7\}$ by (4-1).

Case 3. We discuss the remaining $\mathscr{B}'$ case by case using the method in Section 3.

Let $\mathscr{B}' = \{i_1, \dots, i_k\}$ be a set of boundary divisors of $\widetilde{\mathscr{M}}_{0,n}$. If there is a sequence $(m_1, \dots, m_l)$ satisfying

(i) $m_1 + \dots + m_l = 2n - 2$,

(ii) for any $1 \le t \le l$, $m_t \in \{i_1 - 1, \dots, i_k - 1, n-1-i_1, \dots, n-1-i_k, n-1, n-2\}$,

(iii) for any $1 \le j \le k$, there is $1 \le t_j \le l$ such that $i_j = m_{t_j} + 1$ or $n - 1 - m_{t_j}$,

(iv) the right side of (2-2) is positive,

then there exists a rational function of type $(n; m_1, \dots, m_l)$ by Theorem 2.1. Thus we get a curve C in $\widetilde{\mathscr{M}}_{0,n}$ such that $\mathscr{B}_C = \mathscr{B}'$ by (3-1). So we finish the proof of the theorem by giving a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) for each n and $\mathscr{B}'$; see Table 1–Table 10.

We now show the meaning of these tables. If $\mathscr{B}'$ is in Case 1, we give its type in the tables. If $\mathscr{B}'$ is in Case 3, we give a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) in the tables. For example, if $\mathscr{B}' = \{2, 3\}$, then $(m_1, \dots, m_l)$ is $T_{2,*}$ in Table 1; by

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{3, 4}	$T_{3,*}$	{3, 5}	$T_{3,*}$
{3, 6}	$T_{3,*}$	{4, 5}	$T_{4,5}$	{4, 6}	$T_{4,6}$
{5, 6}	$T_{5,6}$				
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 4, 5}	$T_{2,*}$	{2, 4, 6}	$T_{2,*}$	{2, 5, 6}	$T_{2,*}$
{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$	{3, 5, 6}	$T_{3,5,6}$
{4, 5, 6}	(4, 5, 6, 7)				
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 5, 6}	(2, 5, 6, 9)
{2, 4, 5, 6}	$T_{2,*}$	{3, 4, 5, 6}	(3, 5, 6, 8)		

Table 1. $\widetilde{\mathscr{M}}_{0,12}$.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
$\{2, 3\}$	$T_{2,*}$	$\{2, 4\}$	$T_{2,*}$	$\{2, 5\}$	$T_{2,*}$
$\{2, 6\}$	$T_{2,*}$	$\{3, 4\}$	$T_{3,*}$	$\{3, 5\}$	$T_{3,*}$
$\{3, 6\}$	$T_{3,*}$	$\{4, 5\}$	$T_{4,5}$	$\{4, 6\}$	$T_{4,6}$
$\{5, 6\}$	$T_{5,6}$				
$\{2, 3, 4\}$	$T_{2,*}$	$\{2, 3, 5\}$	$T_{2,*}$	$\{2, 3, 6\}$	$T_{2,*}$
$\{2, 4, 5\}$	$T_{2,*}$	$\{2, 4, 6\}$	$T_{2,*}$	$\{2, 5, 6\}$	$T_{2,*}$
$\{3, 4, 5\}$	$T_{3,4,5}$	$\{3, 4, 6\}$	$T_{3,4,6}$	$\{3, 5, 6\}$	$T_{3,5,6}$
$\{4, 5, 6\}$	$(3, 6, 7, 8)$				
$\{2, 3, 4, 5\}$	$T_{2,*}$	$\{2, 3, 4, 6\}$	$T_{2,*}$	$\{2, 3, 5, 6\}$	$T_{2,*}$
$\{2, 4, 5, 6\}$	$T_{2,*}$	$\{3, 4, 5, 6\}$	$(2, 3, 4, 6, 9)$		

Table 2. $\widetilde{\mathcal{M}}_{0,13}$.

this we mean that there is a curve C in $\widetilde{\mathcal{M}}_{0,12}$ of type $T_{2,*}$ with $\mathscr{B}_C = \{B_2, B_3\}$. If $\mathscr{B}' = \{4, 5, 6\}$, then $(m_1, \dots, m_l) = (4, 5, 6, 7)$ in Table 1; by this we mean that there exists a curve C in $\widetilde{\mathcal{M}}_{0,12}$ with $\mathscr{B}_C = \{4, 5, 6\}$, where C is induced by a rational function of type $(12; 4, 5, 6, 7)$.

Note that only the cases of $\mathscr{B}'$ with $|\mathscr{B}'| = 1$ and $|\mathscr{B}'| = [n/2] - 1$ are not contained in these tables, since the theorem for such $\mathscr{B}'$ holds true for any n; see Theorem 4.3(1) and Theorem 5.1 for $k = 2$. □

Remark 6.1. Assume that $n = 18$ and $\mathscr{B}' = \{B_3, B_4, \dots, B_9\}$. Suppose that there exists a sequence $(m_1, \dots, m_l)$ satisfying (i)–(iv) for $\mathscr{B}'$. From (ii) and (iii), we know that

$$m_1 + \cdots + m_l \geq \sum_{i=3}^{9} (i-1) = 35 > 2n - 2 = 34,$$

which contradicts (i). Hence the method in Section 3 is invalid for $\widetilde{\mathcal{M}}_{0,18}$. Similarly, we know that the method in Section 4 is also invalid.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
$\{2, 3\}$	$T_{2,*}$	$\{2, 4\}$	$T_{2,*}$	$\{2, 5\}$	$T_{2,*}$
$\{2, 6\}$	$T_{2,*}$	$\{2, 7\}$	$T_{2,*}$	$\{3, 4\}$	$T_{3,*}$
$\{3, 5\}$	$T_{3,*}$	$\{3, 6\}$	$T_{3,*}$	$\{3, 7\}$	$T_{3,*}$
$\{4, 5\}$	$T_{4,5}$	$\{4, 6\}$	$T_{4,6}$	$\{4, 7\}$	see Case 2
$\{5, 6\}$	$T_{5,6}$	$\{5, 7\}$	$T_{5,7}$	$\{6, 7\}$	$T_{6,7}$

Table 3. $\widetilde{\mathcal{M}}_{0,14}$ for $|\mathscr{B}'| = 2$.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
$\{2, 3, 4\}$	$T_{2,*}$	$\{2, 3, 5\}$	$T_{2,*}$	$\{2, 3, 6\}$	$T_{2,*}$
$\{2, 3, 7\}$	$T_{2,*}$	$\{2, 4, 5\}$	$T_{2,*}$	$\{2, 4, 6\}$	$T_{2,*}$
$\{2, 4, 7\}$	$T_{2,*}$	$\{2, 5, 6\}$	$T_{2,*}$	$\{2, 5, 7\}$	$T_{2,*}$
$\{2, 6, 7\}$	$T_{2,*}$	$\{3, 4, 5\}$	$T_{3,4,5}$	$\{3, 4, 6\}$	$T_{3,4,6}$
$\{3, 4, 7\}$	$T_{3,4,7}$	$\{3, 5, 6\}$	$T_{3,5,6}$	$\{3, 5, 7\}$	$T_{3,5,7}$
$\{3, 6, 7\}$	$T_{3,6,7}$	$\{4, 5, 6\}$	$(4, 5, 8, 9)$	$\{4, 5, 7\}$	$T_{4,5,7}$
$\{4, 6, 7\}$	$(3, 3, 5, 6, 9)$	$\{5, 6, 7\}$	$(5, 6, 7, 8)$		
$\{2, 3, 4, 5\}$	$T_{2,*}$	$\{2, 3, 4, 6\}$	$T_{2,*}$	$\{2, 3, 4, 7\}$	$T_{2,*}$
$\{2, 3, 5, 6\}$	$T_{2,*}$	$\{2, 3, 5, 7\}$	$T_{2,*}$	$\{2, 3, 6, 7\}$	$(1, 2, 6, 7, 10)$
$\{2, 4, 5, 6\}$	$T_{2,*}$	$\{2, 4, 5, 7\}$	$(1, 4, 6, 6, 9)$	$\{2, 4, 6, 7\}$	$(1, 5, 5, 6, 9)$
$\{2, 5, 6, 7\}$	$T_{2,*}$	$\{3, 4, 5, 6\}$	$(3, 5, 8, 10)$	$\{3, 4, 5, 7\}$	$(3, 3, 4, 6, 10)$
$\{3, 4, 6, 7\}$	$(3, 6, 7, 10)$	$\{3, 5, 6, 7\}$	$(2, 5, 5, 6, 8)$	$\{4, 5, 6, 7\}$	$(4, 6, 7, 9)$

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
$\{2, 3, 4, 5, 6\}$	$(1, 2, 3, 4, 5, 11)$	$\{2, 3, 4, 5, 7\}$	$(1, 2, 3, 4, 6, 10)$
$\{2, 3, 4, 6, 7\}$	$(1, 2, 3, 5, 6, 9)$	$\{2, 3, 5, 6, 7\}$	$(1, 2, 4, 5, 6, 8)$
$\{2, 4, 5, 6, 7\}$	$T_{2,*}$	$\{3, 4, 5, 6, 7\}$	$(2, 3, 4, 5, 6, 6)$

Table 4. $\widetilde{\mathcal{M}}_{0,14}$ for $3 \le |\mathscr{B}'| \le 5$.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
$\{2, 3\}$	$T_{2,*}$	$\{2, 4\}$	$T_{2,*}$	$\{2, 5\}$	$T_{2,*}$
$\{2, 6\}$	$T_{2,*}$	$\{2, 7\}$	$T_{2,*}$	$\{3, 4\}$	$T_{3,*}$
$\{3, 5\}$	$T_{3,*}$	$\{3, 6\}$	$T_{3,*}$	$\{3, 7\}$	$T_{3,*}$
$\{4, 5\}$	$T_{4,5}$	$\{4, 6\}$	$T_{4,6}$	$\{4, 7\}$	$(6, 6, 6, 10)$
$\{5, 6\}$	$T_{5,6}$	$\{5, 7\}$	$T_{5,7}$	$\{6, 7\}$	$T_{6,7}$
$\{2, 3, 4\}$	$T_{2,*}$	$\{2, 3, 5\}$	$T_{2,*}$	$\{2, 3, 6\}$	$T_{2,*}$
$\{2, 3, 7\}$	$T_{2,*}$	$\{2, 4, 5\}$	$T_{2,*}$	$\{2, 4, 6\}$	$T_{2,*}$
$\{2, 4, 7\}$	$T_{2,*}$	$\{2, 5, 6\}$	$T_{2,*}$	$\{2, 5, 7\}$	$T_{2,*}$
$\{2, 6, 7\}$	$T_{2,*}$	$\{3, 4, 5\}$	$T_{3,4,5}$	$\{3, 4, 6\}$	$T_{3,4,6}$
$\{3, 4, 7\}$	$T_{3,4,7}$	$\{3, 5, 6\}$	$T_{3,5,6}$	$\{3, 5, 7\}$	$T_{3,5,7}$
$\{3, 6, 7\}$	$T_{3,6,7}$	$\{4, 5, 6\}$	$(4, 5, 9, 10)$	$\{4, 5, 7\}$	$T_{4,5,7}$
$\{4, 6, 7\}$	$(3, 7, 8, 10)$	$\{5, 6, 7\}$	$(4, 7, 8, 9)$		
$\{2, 3, 4, 5\}$	$T_{2,*}$	$\{2, 3, 4, 6\}$	$T_{2,*}$	$\{2, 3, 4, 7\}$	$T_{2,*}$
$\{2, 3, 5, 6\}$	$T_{2,*}$	$\{2, 3, 5, 7\}$	$T_{2,*}$	$\{2, 3, 6, 7\}$	$T_{2,*}$
$\{2, 4, 5, 6\}$	$T_{2,*}$	$\{2, 4, 5, 7\}$	$T_{2,*}$	$\{2, 4, 6, 7\}$	$(1, 3, 5, 7, 12)$
$\{2, 5, 6, 7\}$	$T_{2,*}$	$\{3, 4, 5, 6\}$	$T_{3,*}$	$\{3, 4, 5, 7\}$	$(2, 7, 9, 10)$
$\{3, 4, 6, 7\}$	$(3, 6, 8, 11)$	$\{3, 5, 6, 7\}$	$(2, 4, 5, 6, 11)$	$\{4, 5, 6, 7\}$	$(3, 4, 5, 6, 10)$

Table 5. $\widetilde{\mathcal{M}}_{0,15}$ for $2 \le |\mathscr{B}'| \le 4$.

$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$
{2, 3, 4, 5, 6}	(1, 2, 3, 4, 8, 10)	{2, 3, 4, 5, 7}	(2, 3, 4, 7, 12)
{2, 3, 4, 6, 7}	(2, 3, 5, 6, 12)	{2, 3, 5, 6, 7}	(1, 4, 5, 7, 11)
{2, 4, 5, 6, 7}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 4, 5, 7, 10)

Table 6. $\widetilde{\mathcal{M}}_{0,15}$ for $|\mathcal{B}'| = 5$.

$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$	$\mathcal{B}'$	$(m_1, \ldots, m_l)$
{2, 3}	$T_{2,*}$	{2, 4}	$T_{2,*}$	{2, 5}	$T_{2,*}$
{2, 6}	$T_{2,*}$	{2, 7}	$T_{2,*}$	{2, 8}	$T_{2,*}$
{3, 4}	$T_{3,*}$	{3, 5}	$T_{3,*}$	{3, 6}	$T_{3,*}$
{3, 7}	$T_{3,*}$	{3, 8}	$T_{3,*}$	{4, 5}	$T_{4,5}$
{4, 6}	$T_{4,6}$	{4, 7}	(8, 11, 11)	{4, 8}	(3, 3, 3, 7, 7, 7)
{5, 6}	$T_{5,6}$	{5, 7}	$T_{5,7}$	{5, 8}	(4, 4, 4, 4, 7, 7)
{6, 7}	$T_{6,7}$	{6, 8}	$T_{6,8}$	{7, 8}	$T_{7,8}$
{2, 3, 4}	$T_{2,*}$	{2, 3, 5}	$T_{2,*}$	{2, 3, 6}	$T_{2,*}$
{2, 3, 7}	$T_{2,*}$	{2, 3, 8}	$T_{2,*}$	{2, 4, 5}	$T_{2,*}$
{2, 4, 6}	$T_{2,*}$	{2, 4, 7}	$T_{2,*}$	{2, 4, 8}	$T_{2,*}$
{2, 5, 6}	$T_{2,*}$	{2, 5, 7}	$T_{2,*}$	{2, 5, 8}	$T_{2,*}$
{2, 6, 7}	$T_{2,*}$	{2, 6, 8}	$T_{2,*}$	{2, 7, 8}	$T_{2,*}$
{3, 4, 5}	$T_{3,4,5}$	{3, 4, 6}	$T_{3,4,6}$	{3, 4, 7}	$T_{3,4,7}$
{3, 4, 8}	$T_{3,4,8}$	{3, 5, 6}	$T_{3,5,6}$	{3, 5, 7}	$T_{3,5,7}$
{3, 5, 8}	$T_{3,5,8}$	{3, 6, 7}	$T_{3,6,7}$	{3, 6, 8}	$T_{3,6,8}$
{3, 7, 8}	$T_{3,7,8}$	{4, 5, 6}	(9, 10, 11)	{4, 5, 7}	$T_{4,5,7}$
{4, 5, 8}	$T_{4,5,8}$	{4, 6, 7}	(5, 6, 8, 11)	{4, 6, 8}	$T_{4,6,8}$
{4, 7, 8}	(6, 6, 7, 11)	{5, 6, 7}	(5, 6, 9, 10)	{5, 6, 8}	(4, 7, 9, 10)
{5, 7, 8}	(6, 7, 7, 10)	{6, 7, 8}	(6, 7, 8, 9)		
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 4, 8}	$T_{2,*}$	{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$
{2, 3, 5, 8}	$T_{2,*}$	{2, 3, 6, 7}	$T_{2,*}$	{2, 3, 6, 8}	$T_{2,*}$
{2, 3, 7, 8}	(2, 7, 8, 13)	{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	$T_{2,*}$
{2, 4, 5, 8}	$T_{2,*}$	{2, 4, 6, 7}	$T_{2,*}$	{2, 4, 6, 8}	(1, 3, 5, 5, 7, 9)
{2, 4, 7, 8}	(1, 3, 6, 7, 13)	{2, 5, 6, 7}	(1, 4, 6, 9, 10)	{2, 5, 6, 8}	(1, 4, 5, 7, 13)
{2, 5, 7, 8}	(4, 6, 7, 13)	{2, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6}	(2, 3, 4, 9, 12)
{3, 4, 5, 7}	(2, 2, 3, 4, 8, 11)	{3, 4, 5, 8}	(2, 7, 10, 11)	{3, 4, 6, 7}	(2, 8, 9, 11)
{3, 4, 6, 8}	(3, 3, 5, 7, 12)	{3, 4, 7, 8}	(2, 2, 7, 8, 11)	{3, 5, 6, 7}	(2, 2, 4, 5, 8, 9)
{3, 5, 6, 8}	(2, 2, 7, 9, 10)	{3, 5, 7, 8}	(2, 2, 4, 7, 7, 8)	{3, 6, 7, 8}	(5, 6, 7, 12)
{4, 5, 6, 7}	(4, 6, 9, 11)	{4, 5, 6, 8}	(3, 4, 5, 7, 11)	{4, 5, 7, 8}	(4, 7, 8, 11)
{4, 6, 7, 8}	(3, 5, 6, 7, 9)	{5, 6, 7, 8}	(5, 7, 8, 10)		

Table 7. $\widetilde{\mathcal{M}}_{0,16}$ for $2 \le |\mathcal{B}'| \le 4$.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
$\{2,3,4,5,6\}$	$(1,2,3,5,9,10)$	$\{2,3,4,5,7\}$	$(1,2,3,6,8,10)$
$\{2,3,4,5,8\}$	$(1,2,3,4,7,13)$	$\{2,3,4,6,7\}$	$(1,2,3,5,6,13)$
$\{2,3,4,6,8\}$	$(2,3,5,7,13)$	$\{2,3,4,7,8\}$	$(1,1,3,6,7,12)$
$\{2,3,5,6,7\}$	$(1,2,4,5,6,12)$	$\{2,3,5,6,8\}$	$(1,1,4,5,7,12)$
$\{2,3,5,7,8\}$	$(1,4,6,7,12)$	$\{2,3,6,7,8\}$	$(1,2,5,6,7,9)$
$\{2,4,5,6,7\}$	$(1,3,4,5,6,11)$	$\{2,4,5,6,8\}$	$(1,3,4,5,7,10)$
$\{2,4,5,7,8\}$	$(1,3,4,6,7,9)$	$\{2,4,6,7,8\}$	$(1,3,5,6,7,8)$
$\{2,5,6,7,8\}$	$T_{2,*}$	$\{3,4,5,6,7\}$	$(2,3,4,5,6,10)$
$\{3,4,5,6,8\}$	$(2,3,4,5,7,9)$	$\{3,4,5,7,8\}$	$(2,2,4,5,6,11)$
$\{3,4,6,7,8\}$	$(2,3,5,6,7,7)$	$\{3,5,6,7,8\}$	$(2,5,6,7,10)$
$\{4,5,6,7,8\}$	$(3,4,5,5,6,7)$		
$\{2,3,4,5,6,7\}$	$(1,2,3,4,5,6,9)$	$\{2,3,4,5,6,8\}$	$(1,2,4,5,7,11)$
$\{2,3,4,5,7,8\}$	$(1,1,2,3,6,7,10)$	$\{2,3,4,6,7,8\}$	$(1,2,3,7,8,9)$
$\{2,3,5,6,7,8\}$	$(1,2,4,5,5,6,7)$	$\{2,4,5,6,7,8\}$	$T_{2,*}$
$\{3,4,5,6,7,8\}$	$(2,3,3,4,5,6,7)$		

Table 8. $\widetilde{\mathcal{M}}_{0,16}$ for $5 \le |\mathcal{B}'| \le 6$.

$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$	$\mathcal{B}'$	$(m_1, \dots, m_l)$
$\{2,3\}$	$T_{2,*}$	$\{2,4\}$	$T_{2,*}$	$\{2,5\}$	$T_{2,*}$
$\{2,6\}$	$T_{2,*}$	$\{2,7\}$	$T_{2,*}$	$\{2,8\}$	$T_{2,*}$
$\{3,4\}$	$T_{3,*}$	$\{3,5\}$	$T_{3,*}$	$\{3,6\}$	$T_{3,*}$
$\{3,7\}$	$T_{3,*}$	$\{3,8\}$	$T_{3,*}$	$\{4,5\}$	$T_{4,5}$
$\{4,6\}$	$T_{4,6}$	$\{4,7\}$	see Case 2	$\{4,8\}$	$(3,3,7,7,12)$
$\{5,6\}$	$T_{5,6}$	$\{5,7\}$	$T_{5,7}$	$\{5,8\}$	$(7,7,7,11)$
$\{6,7\}$	$T_{6,7}$	$\{6,8\}$	$T_{6,8}$	$\{7,8\}$	$T_{7,8}$
$\{2,3,4\}$	$T_{2,*}$	$\{2,3,5\}$	$T_{2,*}$	$\{2,3,6\}$	$T_{2,*}$
$\{2,3,7\}$	$T_{2,*}$	$\{2,3,8\}$	$T_{2,*}$	$\{2,4,5\}$	$T_{2,*}$
$\{2,4,6\}$	$T_{2,*}$	$\{2,4,7\}$	$T_{2,*}$	$\{2,4,8\}$	$T_{2,*}$
$\{2,5,6\}$	$T_{2,*}$	$\{2,5,7\}$	$T_{2,*}$	$\{2,5,8\}$	$T_{2,*}$
$\{2,6,7\}$	$T_{2,*}$	$\{2,6,8\}$	$T_{2,*}$	$\{2,7,8\}$	$T_{2,*}$
$\{3,4,5\}$	$T_{3,4,5}$	$\{3,4,6\}$	$T_{3,4,6}$	$\{3,4,7\}$	$T_{3,4,7}$
$\{3,4,8\}$	$T_{3,4,8}$	$\{3,5,6\}$	$T_{3,5,6}$	$\{3,5,7\}$	$T_{3,5,7}$
$\{3,5,8\}$	$T_{3,5,8}$	$\{3,6,7\}$	$T_{3,6,7}$	$\{3,6,8\}$	$T_{3,6,8}$
$\{3,7,8\}$	$T_{3,7,8}$	$\{4,5,6\}$	$(4,5,11,12)$	$\{4,5,7\}$	$T_{4,5,7}$
$\{4,5,8\}$	$T_{4,5,8}$	$\{4,6,7\}$	$(5,6,9,12)$	$\{4,6,8\}$	$T_{4,6,8}$
$\{4,7,8\}$	$(6,6,8,12)$	$\{5,6,7\}$	$(5,6,10,11)$	$\{5,6,8\}$	$(4,7,10,11)$
$\{5,7,8\}$	$(6,7,8,11)$	$\{6,7,8\}$	$(6,7,9,10)$		

Table 9. $\widetilde{\mathcal{M}}_{0,17}$ for $2 \le |\mathcal{B}'| \le 3$.

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4, 5}	$T_{2,*}$	{2, 3, 4, 6}	$T_{2,*}$	{2, 3, 4, 7}	$T_{2,*}$
{2, 3, 4, 8}	$T_{2,*}$	{2, 3, 5, 6}	$T_{2,*}$	{2, 3, 5, 7}	$T_{2,*}$
{2, 3, 5, 8}	$T_{2,*}$	{2, 3, 6, 7}	$T_{2,*}$	{2, 3, 6, 8}	$T_{2,*}$
{2, 3, 7, 8}	$T_{2,*}$	{2, 4, 5, 6}	$T_{2,*}$	{2, 4, 5, 7}	$T_{2,*}$
{2, 4, 5, 8}	$T_{2,*}$	{2, 4, 6, 7}	$T_{2,*}$	{2, 4, 6, 8}	$T_{2,*}$
{2, 4, 7, 8}	(1, 3, 7, 9, 12)	{2, 5, 6, 7}	$T_{2,*}$	{2, 5, 6, 8}	(1, 5, 7, 8, 11)
{2, 5, 7, 8}	(4, 6, 8, 14)	{2, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6}	$T_{3,*}$
{3, 4, 5, 7}	$T_{3,*}$	{3, 4, 5, 8}	$T_{3,*}$	{3, 4, 6, 7}	$T_{3,*}$
{3, 4, 6, 8}	(2, 8, 10, 12)	{3, 4, 7, 8}	(3, 7, 9, 13)	{3, 5, 6, 7}	(2, 9, 10, 11)
{3, 5, 6, 8}	(2, 4, 5, 8, 13)	{3, 5, 7, 8}	(2, 4, 6, 7, 13)	{3, 6, 7, 8}	(5, 6, 8, 13)
{4, 5, 6, 7}	(4, 6, 10, 12)	{4, 5, 6, 8}	(3, 4, 5, 8, 12)	{4, 5, 7, 8}	(4, 7, 9, 12)
{4, 6, 7, 8}	(3, 6, 6, 7, 10)	{5, 6, 7, 8}	(5, 7, 9, 11)		

$\mathscr{B}'$	$(m_1, \dots, m_l)$	$\mathscr{B}'$	$(m_1, \dots, m_l)$
{2, 3, 4, 5, 6}	$T_{2,*}$	{2, 3, 4, 5, 7}	$T_{2,*}$
{2, 3, 4, 5, 8}	(1, 2, 3, 4, 8, 14)	{2, 3, 4, 6, 7}	(2, 2, 3, 5, 6, 14)
{2, 3, 4, 6, 8}	(2, 3, 5, 8, 14)	{2, 3, 4, 7, 8}	(2, 3, 6, 7, 14)
{2, 3, 5, 6, 7}	(1, 4, 5, 9, 13)	{2, 3, 5, 6, 8}	(2, 4, 5, 7, 14)
{2, 3, 5, 7, 8}	(1, 4, 6, 8, 13)	{2, 3, 6, 7, 8}	(1, 5, 6, 7, 13)
{2, 4, 5, 6, 7}	(3, 4, 5, 6, 14)	{2, 4, 5, 6, 8}	(1, 3, 7, 10, 11)
{2, 4, 5, 7, 8}	(1, 3, 8, 9, 11)	{2, 4, 6, 7, 8}	(1, 5, 6, 8, 12)
{2, 5, 6, 7, 8}	$T_{2,*}$	{3, 4, 5, 6, 7}	(2, 4, 5, 9, 12)
{3, 4, 5, 6, 8}	(3, 4, 5, 7, 13)	{3, 4, 5, 7, 8}	(2, 3, 7, 9, 11)
{3, 4, 6, 7, 8}	(2, 5, 6, 7, 12)	{3, 5, 6, 7, 8}	(2, 5, 6, 8, 11)
{4, 5, 6, 7, 8}	(3, 5, 6, 7, 11)		
{2, 3, 4, 5, 6, 7}	(1, 3, 4, 5, 6, 13)	{2, 3, 4, 5, 6, 8}	(1, 2, 4, 5, 8, 12)
{2, 3, 4, 5, 7, 8}	(1, 2, 3, 4, 6, 7, 9)	{2, 3, 4, 6, 7, 8}	(1, 2, 3, 7, 9, 10)
{2, 3, 5, 6, 7, 8}	(1, 2, 5, 6, 7, 11)	{2, 4, 5, 6, 7, 8}	$T_{2,*}$
{3, 4, 5, 6, 7, 8}	(2, 3, 4, 6, 7, 10)		

Table 10. $\widetilde{\mathscr{M}}_{0,17}$ for $4 \le |\mathscr{B}'| \le 6$.

Acknowledgements

I thank Fei Ye and Zhenhua Qu for discussions on the cone of moduli spaces. I am very grateful to Prof. Shengli Tan for his encouragement and lots of helpful discussions. I would like to sincerely thank Hao Sun and Yifei Chen for reading the earlier version and for their valuable comments. The result of Theorem 5.1 is due to the referee, and I thank him very much for his help. Finally, I would like to express my appreciation to the referee for pointing out how to improve the paper and providing many valuable suggestions in rewriting the manuscript to make the paper more readable.

References

[Gibney 2009] A. Gibney, "Numerical criteria for divisors on $\overline{M}_g$ to be ample", *Compos. Math.* **145**:5 (2009), 1227–1248. MR Zbl

[Harris and Morrison 1998] J. Harris and I. Morrison, *Moduli of curves*, Graduate Texts in Mathematics **187**, Springer, New York, 1998. MR Zbl

[Hassett 2003] B. Hassett, "Moduli spaces of weighted pointed stable curves", *Adv. Math.* **173**:2 (2003), 316–352. MR Zbl

[Kodaira 1967] K. Kodaira, "A certain type of irregular algebraic surfaces", *J. Analyse Math.* **19** (1967), 207–215. MR Zbl

[Liu 2016] X.-L. Liu, "Modular invariants and singularity indices of hyperelliptic fibrations", preprint, 2016. To appear in *Chin. Ann. Math.* (*B*). arXiv

[Liu and Tan 2013] X.-L. Liu and S.-L. Tan, "Families of hyperelliptic curves with maximal slopes", *Sci. China Math.* **56**:9 (2013), 1743–1750. MR Zbl

[Moriwaki 1998] A. Moriwaki, "Relative Bogomolov's inequality and the cone of positive divisors on the moduli space of stable curves", *J. Amer. Math. Soc.* **11**:3 (1998), 569–600. MR Zbl

[Scherbak 2002] I. Scherbak, "Rational functions with prescribed critical points", *Geom. Funct. Anal.* **12**:6 (2002), 1365–1380. MR Zbl

[Tan 2010] S.-L. Tan, "Chern numbers of a singular fiber, modular invariants and isotrivial families of curves", *Acta Math. Vietnam.* **35**:1 (2010), 159–172. MR Zbl

Received December 1, 2014. Revised December 19, 2015.

XIAO-LEI LIU
ACADEMY OF MATHEMATICS AND SYSTEMS SCIENCE
CHINESE ACADEMY OF SCIENCE
55, EAST ZHONGGUANCUN ROAD
HAIDIAN
BEIJING, 100190
CHINA
xlliu1124@amss.ac.cn

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.381

VIRTUAL RATIONAL BETTI NUMBERS OF NILPOTENT-BY-ABELIAN GROUPS

BEHROOZ MIRZAII AND FATEMEH Y. MOKARI

We study the virtual rational Betti numbers of a nilpotent-by-abelian group G, where the abelianization N/N' of its nilpotent part N satisfies certain tameness property. More precisely, we prove that if N/N' is $2(c(n-1)-1)$-tame as a G/N-module, where c is the nilpotency class of N, then

$$\mathrm{vb}_j(G) := \sup_{M \in \mathcal{A}_G} \dim_{\mathbb{Q}} H_j(M, \mathbb{Q})$$

is finite for all $0 \leq j \leq n$, where $\mathcal{A}_G$ is the set of all finite-index subgroups of G.

Introduction

The virtual rational Betti numbers of a finitely generated group studies the growth of the Betti numbers of the group as one follows passage to subgroups of finite index. Following [Bridson and Kochloukova 2015; Kochloukova and Mokari 2015], we define the n-th virtual rational Betti number of a finitely generated group G as

$$\mathrm{vb}_n(G) := \sup_{M \in \mathcal{A}_G} \dim_{\mathbb{Q}} H_n(M, \mathbb{Q}),$$

where $\mathcal{A}_G$ is the set of all subgroups of finite index in G.

Bridson and Kochloukova [2015] introduced and studied the first virtual rational Betti number of a finitely generated group G and showed that if G is either a finitely presented nilpotent-by-abelian group or an abelian-by-polycyclic group of type FP_3, then $\mathrm{vb}_1(G)$ is finite. Moreover, they conjectured that this should be true for all finitely presented soluble groups. As they have shown the finiteness of the first virtual rational Betti numbers of a metabelian group G, with normal abelian subgroup A and abelian quotient Q is closely related to the 2-tameness of A as a Q-module, an invariant of metabelian groups introduced by Bieri and Strebel [1980].

Mokari is supported by a Capes/CNPq Ph.D. grant.
MSC2010: 20J05, 20J06.
Keywords: virtual Betti numbers, homology of groups, nilpotent-by-abelian groups, nilpotent action.

Kochloukova and Mokari [2015] extended these results to higher virtual rational Betti numbers of abelian-by-polycyclic groups, by replacing higher tameness with finitely generatedness of high tensor powers of abelian normal subgroups. More precisely, let A be a normal abelian subgroup of G such that the quotient group $Q := G/A$ is polycyclic. If Q is not abelian, we assume that G is of type FP_3. Then it is shown in [Kochloukova and Mokari 2015, Theorem A] that if $\bigotimes_{\mathbb{Q}}^{2n}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ is finitely generated as a $\mathbb{Q}Q$-module via the diagonal action, then $\mathrm{vb}_j(G)$ is finite for $0 \le j \le n$. Note that if G is metabelian, then finitely generatedness of $\bigotimes_{\mathbb{Q}}^{2n}(A \otimes_{\mathbb{Z}} \mathbb{Q})$ is equivalent to $2n$-tameness of A as a Q-module (see Theorem 4.1).

Finitely generated soluble groups occurring in applications are often nilpotent -by-abelian-by-finite, that is, any such group G contains subgroups $N \trianglelefteq H \trianglelefteq G$ such that N is nilpotent, H/N abelian and G/H finite. In this paper, we study the virtual rational Betti numbers of nilpotent-by-abelian-by-finite groups. Since $\mathrm{vb}_n(G) = \mathrm{vb}_n(H)$ (Lemma 5.5), it is sufficient to study virtual rational Betti numbers of nilpotent-by-abelian groups. Here is our main theorem.

Theorem 5.4 (see p. 396). *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups, where G is finitely generated, N is nilpotent of class c and Q is abelian. If N/N' is $2(c(n-1)+1)$-tame, then for any $0 \le j \le n$, $\mathrm{vb}_j(G)$ is finite.*

As a motivation for the study of virtual rational Betti numbers, one can mention a result of Lück which says that the L_2-Betti numbers can be computed as a limit involving the ordinary Betti numbers of subgroups of finite index. Here we show that for these groups there is no growth, i.e., the sequences remain bounded. This result therefore confirms Lück's formula by establishing a stronger property for this class of groups [Lück 1994].

To prove our main theorem we needed to study certain aspects of homology of nilpotent groups. Nilpotent groups have a great deal of commutativity built into their structure and they are groups that are "almost abelian". So it is natural to expect that some of the properties of homology of abelian groups, in some way, may be shared by nilpotent groups. In this article, we will study two such properties. For more similarity between homology of abelian and nilpotent groups we refer the interested reader to [Dwyer 1975; Robinson 1976; Hilton et al. 1975].

The n-th homology of an abelian group A with rational coefficients is isomorphic to $\bigwedge_{\mathbb{Q}}^n(A \otimes_{\mathbb{Z}} \mathbb{Q})$. We prove the analogue of this result for nilpotent groups. More precisely, if N is a nilpotent group of class c, then we show that there exists a natural filtration of $H_j(N, \mathbb{Q})$,

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Q}),$$

such that for any $0 \le k \le l$, E_k/E_{k-1} is a natural subquotient of a vector space from the set

$$\left\{ \bigotimes_{\mathbb{Q}}^{s} V \right\}_{0 \le s \le c(j-1)+1},$$

where $V := (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$. When our group is free nilpotent, we show that the above theorem is true even with integral coefficients. Although the existence of the above filtration is not a surprise and can be obtain by easy induction, but the bound $c(j-1)+1$ is new and important for our applications. Furthermore, for groups with small c we show that this bound is sharp. The proofs of these results occupy Sections 1 and 2.

Let N be a nilpotent normal subgroup of a group G. If G acts nilpotently on N/N', then Theorem 2.1 implies that G acts nilpotently on $H_k(N, \mathbb{Q})$. But with a direct method we can prove a more general result. Let T be an RG-module, where R is a commutative ring. In Section 3, we will show that if G acts nilpotently on both N/N' and T, then G acts nilpotently on each $H_k(N, T)$ and $H^k(N, T)$. As an application, we show that if moreover G/N is finite and l-torsion and $1/l \in R$, then the natural action of G/N on $H_k(N, T)$ and $H^k(N, T)$ is trivial and therefore the natural maps

$$\operatorname{corr}_N^G : H_k(N, T) \to H_k(G, T), \quad \operatorname{res}_N^G : H^k(G, T) \to H^k(N, T)$$

are isomorphisms.

Both of these results about the homology of nilpotent groups are used in the proof of our main theorem (Theorem 5.4).

1. Differentials of the Lyndon–Hochschild–Serre spectral sequence

Let G be a group, A an abelian normal subgroup of G and $Q := G/A$. Let

$${}_M\mathcal{E}^2_{p,q} = H_p(Q, H_q(A, M)) \Rightarrow H_{p+q}(G, M)$$

be the Lyndon–Hochschild–Serre spectral sequence associated to the exact sequence of groups

$$A \rightarrowtail G \twoheadrightarrow Q,$$

where here M is either $\mathbb{Z}$ or $\mathbb{Q}$ with the trivial action of G. In this section, we would like to give an explicit formula for the differentials

$$d^2_{2,q} : {}_{\mathbb{Q}}\mathcal{E}^2_{2,q} \to {}_{\mathbb{Q}}\mathcal{E}^2_{0,q+1},$$

for any $q \ge 0$, when A is central, i.e., $A \subseteq Z(G)$.

Let $\phi : A \otimes_{\mathbb{Z}} H_q(A, M) \to H_{q+1}(A, M)$ be the natural product map [Brown 1994, Chapter V, §5], say induced by the shuffle product on the bar resolution, and consider the following composition

$$H^2(Q,A)\otimes_{\mathbb{Z}} H_p(Q,H_q(A,M)) \xrightarrow{-\cap-} H_{p-2}(Q,A\otimes_{\mathbb{Z}} H_q(A,M)) \xrightarrow{H_{p-2}(\mathrm{id}_Q,\phi)} H_{p-2}(Q,H_{q+1}(A,M)), \tag{1-1}$$

where $-\cap-$ is the cap product [Brown 1994, Chapter V, §3].

Let ρ be the element of $H^2(Q,A)$ associated to $A \rightarrowtail G \twoheadrightarrow Q$ [Brown 1994, Chapter IV, Theorem 3.12] and set

$$\Delta(\rho) := H_{p-2}(\mathrm{id}_Q,\phi)\circ(\rho\cap -) : H_p(Q,H_q(A,M)) \to H_{p-2}(Q,H_{q+1}(A,M)).$$

Proposition 1.1 [André 1965, p. 2670]. *Let an exact sequence $A \rightarrowtail G \twoheadrightarrow Q$ be given as in above. Then*

$$d^2_{p,q} = d'^2_{p,q} + \Delta(\rho),$$

where $d'^2_{p,q}$ is the differential of the Lyndon–Hochschild–Serre spectral sequence associated to the semidirect product extension $A \rightarrowtail A \rtimes Q \twoheadrightarrow Q$.

Now let A be a central subgroup of G. Then the conjugate action of Q on A is trivial and thus $A \rtimes Q = A \times Q$. It is well-known and easy to prove that in this case, for any p and q, $d'^2_{p,q} = 0$ and therefore

$$d^2_{p,q} = \Delta(\rho). \tag{1-2}$$

Moreover, since A is central, the action of Q on $H_q(A,M)$ is trivial. Thus for $M = \mathbb{Q}$, the universal coefficient theorem implies that

$${}_{\mathbb{Q}}\mathcal{E}^2_{p,q} = H_p(Q,\mathbb{Z})\otimes_{\mathbb{Z}} H_q(A,\mathbb{Q}) \simeq H_p(Q,\mathbb{Z})\otimes_{\mathbb{Z}} \textstyle\bigwedge^q_{\mathbb{Q}}(A\otimes_{\mathbb{Z}}\mathbb{Q}).$$

If $p = 2$, then (1-1) finds the following form

$$H^2(Q,A)\otimes_{\mathbb{Z}} H_2(Q,\mathbb{Z})\otimes_{\mathbb{Z}} H_q(A,\mathbb{Q}) \xrightarrow{(-\cap-)\otimes\mathrm{id}} A\otimes_{\mathbb{Z}} H_q(A,\mathbb{Q}) \xrightarrow{\phi} H_{q+1}(A,\mathbb{Q}),$$

where

$$-\cap- : H^2(Q,A)\otimes_{\mathbb{Z}} H_2(Q,\mathbb{Z}) \to A$$

is the cap product. Therefore from formula (1-2), we obtain the following explicit formula

$$\begin{aligned} d^2_{2,q} : {}_{\mathbb{Q}}\mathcal{E}^2_{2,q} = H_2(Q,\mathbb{Z})\otimes_{\mathbb{Z}} \textstyle\bigwedge^q_{\mathbb{Q}}(A\otimes_{\mathbb{Z}}\mathbb{Q}) \to {}_{\mathbb{Q}}\mathcal{E}^2_{0,q+1} = \textstyle\bigwedge^{q+1}_{\mathbb{Q}}(A\otimes_{\mathbb{Z}}\mathbb{Q}),& \\ x\otimes(a_1\wedge\cdots\wedge a_q) \mapsto (\rho\cap x)\wedge a_1\wedge\cdots\wedge a_q.& \end{aligned}$$

Thus we have proved the following proposition.

Proposition 1.2. *Let G be a group, A a central subgroup of G and $Q := G/A$. Let*

$$_{\mathbb{Q}}\mathcal{E}^2_{p,q} = H_p(Q, H_q(A, \mathbb{Q})) \Rightarrow H_{p+q}(G, \mathbb{Q})$$

be the Lyndon–Hochschild–Serre spectral sequence associated to the extension $A \rightarrowtail G \twoheadrightarrow Q$. Then for any $q \geq 0$, the differential

$$d^2_{2,q} : {}_{\mathbb{Q}}\mathcal{E}^2_{2,q} = H_2(Q, \mathbb{Z}) \otimes_{\mathbb{Z}} \textstyle\bigwedge^q_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}) \to {}_{\mathbb{Q}}\mathcal{E}^2_{0,q+1} = \textstyle\bigwedge^{q+1}_{\mathbb{Q}}(A \otimes_{\mathbb{Z}} \mathbb{Q}),$$

is given by the formula $x \otimes (a_1 \wedge \cdots \wedge a_q) \mapsto (\rho \cap x) \wedge a_1 \wedge \cdots \wedge a_q$. Here ρ is the element of $H^2(G, A)$ associated to the above extension and the map $- \cap - : H^2(Q, A) \otimes_{\mathbb{Z}} H_2(Q, \mathbb{Z}) \to A$ is the cap product. If A is torsion free, then the same result is true for

$$d^2_{2,q} : {}_{\mathbb{Z}}\mathcal{E}^2_{2,q} \to {}_{\mathbb{Z}}\mathcal{E}^2_{0,q+1}.$$

The following corollary will be needed in the next section.

Corollary 1.3. *Let G, A, Q and ${}_{\mathbb{Q}}\mathcal{E}^2_{p,q}$ be as in Proposition 1.2. If $A \subseteq Z(G) \cap G'$, then*

$$d^2_{2,q} : {}_{\mathbb{Q}}\mathcal{E}^2_{2,q} \to {}_{\mathbb{Q}}\mathcal{E}^2_{0,q+1}$$

is surjective for any $q \geq 0$ and therefore

$${}_{\mathbb{Q}}\mathcal{E}^\infty_{0,q} = {}_{\mathbb{Q}}\mathcal{E}^3_{0,q} = 0.$$

Moreover, if A is torsion free, then the same results hold for

$$d^2_{2,q} : {}_{\mathbb{Z}}\mathcal{E}^2_{2,q} \to {}_{\mathbb{Z}}\mathcal{E}^2_{0,q+1}.$$

Proof. The spectral sequence ${}_M\mathcal{E}^2_{p,q}$, gives us the five term exact sequence

$$H_2(G, M) \to H_2(Q, M) \xrightarrow{d^2_{2,0}} H_1(A, M)_Q \to H_1(G, M) \to H_1(Q, M) \to 0,$$

[Brown 1994, Chapter VII, Corollary 6.4]. Clearly $H_1(G, \mathbb{Z}) \simeq H_1(Q, \mathbb{Z}) \simeq G/G'$. Since the action of Q on A is trivial, we have $H_1(A, \mathbb{Z})_Q \simeq H_1(A, \mathbb{Z}) = A$. Thus from the above exact sequence, we obtain the surjective map

$$d^2_{2,0} : H_2(Q, \mathbb{Z}) \twoheadrightarrow A.$$

However, from the above, we know that this map is given by the formula $x \mapsto \rho \cap x$. Now by Proposition 1.2, $d^2_{2,q}$ is surjective and this immediately implies that $\mathcal{E}^\infty_{0,q} = \mathcal{E}^3_{0,q} = 0$. $\square$

2. Homology of nilpotent groups

Let N be a nilpotent group of class c and consider its lower central series,

$$1 = \gamma_{c+1}(N) \subset \gamma_c(N) \subset \cdots \subset \gamma_2(N) \subset \gamma_1(N) = N.$$

From the exact sequence

$$\gamma_c(N) \rightarrowtail N \twoheadrightarrow N/\gamma_c(N),$$

we obtain the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(N/\gamma_c(N), H_q(\gamma_c(N), T)) \Rightarrow H_{p+q}(N, T), \tag{2-1}$$

where T is an N-module.

Since $\gamma_{c+1}(N) = [\gamma_c(N), N] = 1$, it follows that $\gamma_c(N) \subseteq Z(N)$. So the conjugate action of $N/\gamma_c(N)$ on $\gamma_c(N)$ is trivial. This also implies that the action of $N/\gamma_c(N)$ on $H_q(\gamma_c(N), T)$ is trivial, provided that the action of N on T is trivial.

Theorem 2.1. *Let N be a nilpotent group of class c. Then there exists a natural filtration of $H_j(N, \mathbb{Q})$,*

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Q}),$$

such that for any $0 \le k \le l$, E_k/E_{k-1} is a natural subquotient of a vector space from the set

$$\Big\{ \bigotimes_{\mathbb{Q}}^{s} V \Big\}_{0 \le s \le c(j-1)+1},$$

where $V := (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$.

Proof. We prove the claim by induction on c. All filtrations, homomorphisms and subquotients that will be considered in this proof are natural. If $c = 1$, then $N' = \gamma_2(N) = 1$. Thus N is abelian and by [Brown 1994, Theorem 6.4, Chapter V] we have

$$H_j(N, \mathbb{Q}) \simeq (\textstyle\bigwedge_{\mathbb{Z}}^j N) \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \textstyle\bigwedge_{\mathbb{Q}}^j V.$$

Clearly $\bigwedge_{\mathbb{Q}}^j V$ is of the form $\big(\bigotimes_{\mathbb{Q}}^j V\big)/T$, for some subspace T of $\bigotimes_{\mathbb{Q}}^j V$. Since $j = 1(j-1)+1 = c(j-1)+1$, our claim is valid for $c = 1$.

Now let $c \ge 2$ and assume that the claim of the theorem is true for all nilpotent groups of class d, $1 \le d \le c-1$. The spectral sequence (2-1) gives us

$$0 = F_{-1}H_j \subseteq F_0H_j \subseteq \cdots \subseteq F_{j-1}H_j \subseteq F_jH_j = H_j(N, \mathbb{Q}),$$

a filtration of $H_j(N, \mathbb{Q})$, such that $E^\infty_{i,j-i} \simeq F_iH_j/F_{i-1}H_j$, $0 \le i \le j$. By Corollary 1.3, $E^\infty_{0,j} = 0$, so

$$F_0H_j = F_0H_j/F_{-1}H_j \simeq E^\infty_{0,j} = 0.$$

We know that $E^{\infty}_{i,j-i}$ is a subquotient of

$$E^{2}_{i,j-i} \simeq H_i(N/\gamma_c(N), \mathbb{Q}) \otimes_{\mathbb{Q}} H_{j-i}(\gamma_c(N), \mathbb{Q}).$$

The group $\gamma_c(N)$ is abelian, so

$$H_{j-i}(\gamma_c(N), \mathbb{Q}) \simeq \textstyle\bigwedge_{\mathbb{Q}}^{j-i}(\gamma_c(N) \otimes_{\mathbb{Z}} \mathbb{Q}).$$

There is a natural surjective map $\bigotimes_{\mathbb{Z}}^{c}(N/N') \twoheadrightarrow \gamma_c(N)$, which induces a surjective map

$$\textstyle\bigwedge_{\mathbb{Q}}^{j-i}\Bigl(\bigotimes_{\mathbb{Q}}^{c} V\Bigr) \twoheadrightarrow \bigwedge_{\mathbb{Q}}^{j-i}(\gamma_c(N) \otimes_{\mathbb{Z}} \mathbb{Q}),$$

and clearly from this we obtain a surjective map

$$\bigotimes_{\mathbb{Q}}^{c(j-i)} V \twoheadrightarrow H_{j-i}(\gamma_c(N), \mathbb{Q}). \tag{2-2}$$

This implies that $F_iH_j/F_{i-1}H_j$ is a subquotient of

$$H_i(N/\gamma_c(N), \mathbb{Q}) \otimes_{\mathbb{Q}} \bigotimes_{\mathbb{Q}}^{c(j-i)} V. \tag{2-3}$$

On the other hand, since $N/\gamma_c(N)$ is nilpotent of class $c-1$, by the induction hypothesis, for any $1 \le i \le j$, we have a filtration of $H_i(N/\gamma_c(N), \mathbb{Q})$,

$$0 = G_{0,i} \subseteq G_{1,i} \subseteq \cdots \subseteq G_{k_i-1,i} \subseteq G_{k_i,i} = H_i(N/\gamma_c(N), \mathbb{Q}),$$

such that for any $0 \le t \le k_i$, $G_{t,i}/G_{t-1,i}$ is a subquotient of some $\bigotimes_{\mathbb{Q}}^{s_{t,i}} V$, where $0 \le s_{t,i} \le (c-1)(i-1)+1$. (Note that $(N/\gamma_c(N))/(N/\gamma_c(N))' = N/N'$). This together with (2-3) imply that $F_iH_j/F_{i-1}H_j$ is a subquotient of some $\bigotimes_{\mathbb{Q}}^{s_i} V$, where

$$0 \le s_i \le (c-1)(i-1)+1+c(j-i) = c(j-1)-i+2 \le c(j-1)+1.$$

This finishes the induction step and so the proof of the theorem. $\square$

With some restriction on N, one can obtain similar results for integral homology.

Proposition 2.2. *Let N be a free nilpotent group of class c. Then there exists a natural filtration of $H_j(N, \mathbb{Z})$,*

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Z}),$$

such that for any $0 \le k \le l$, E_k/E_{k-1} is a natural subquotient of a $\mathbb{Z}$-module from the set

$$\Bigl\{\bigotimes_{\mathbb{Q}}^{s} V\Bigr\}_{0 \le s \le c(j-1)+1}, \quad \textit{where } V := N/N'.$$

Proof. Since N is a free nilpotent group, $\gamma_c(N)$ is torsion free. Thus

$$H_n(\gamma_c(N), \mathbb{Z}) \simeq \textstyle\bigwedge_{\mathbb{Z}}^n \gamma_c(N)$$

(see [Brown 1994, Theorem 6.4, Chapter V]) and so it is torsion free. This implies that

$$E^2_{i,j-i} \simeq H_i(N/\gamma_c(N), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{j-i}(\gamma_c(N), \mathbb{Z}).$$

Now the proof is similar to the proof of Theorem 2.1. □

Remark 2.3. We believe that $c(j-1)+1$ is a sharp bound for the existence of a filtration with the above property for $H_j(N, \mathbb{Q})$. At least this is true for the extreme cases $c = 1$ (abelian N) or $j = 1$ (first homology group case). Also the above proof shows that $E_1 = F_1 H_j$ is a quotient of

$$\bigotimes_{\mathbb{Z}}^{c(j-1)+1} V.$$

This gives an evidence for the fact that the bound $c(j-1)+1$ in Theorem 2.1 is sharp.

Remark 2.4. If N is a nilpotent group of class c, then the above theorem also is true for $H_2(N, \mathbb{Z})$. By this we mean that there exist a natural filtration of $H_2(N, \mathbb{Z})$,

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_2(N, \mathbb{Z}),$$

such that for any $0 \le k \le l$, E_k/E_{k-1} is a natural subquotient of a $\mathbb{Z}$-module from the set

$$\Big\{\bigotimes_{\mathbb{Z}}^{s}(N/N')\Big\}_{0 \le s \le c+1}.$$

This follows from the above proof, using the facts that for an abelian group A, $H_2(A, \mathbb{Z}) \simeq A \wedge A$ and also for $0 \le i \le 2$,

$$E^2_{i,2-i} \simeq H_i(N/\gamma_c(N), \mathbb{Z}) \otimes_{\mathbb{Z}} H_{2-i}(\gamma_c(N), \mathbb{Z}).$$

If $c = 2$, the complete structure of $H_2(N, \mathbb{Z})$ is established in [Kochloukova 1997]. This description is simple if N is torsion-free. In this case $N/\gamma_2(N)$ is torsion-free and we obtain a filtration

$$0 \subseteq F_1 H_2 \subseteq F_2 H_2 = H_2(N, \mathbb{Z})$$

such that

$$F_1 H_2 \simeq \frac{(N/N') \otimes_{\mathbb{Z}} N'}{\langle xN' \otimes [y, z] + yN' \otimes [z, x] + zN' \otimes [x, y] \mid x, y, z \in N\rangle},$$

$$F_2 H_2 / F_1 H_2 \simeq \ker\big((N/N') \wedge (N/N') \longrightarrow N',\, xN' \wedge yN' \mapsto [x, y]\big).$$

Remark 2.5. Let N be a free nilpotent group of finite rank and of class $c = 2$. Then by [Kuz'min and Semenov 1998, p. 532], the differential

$$d^2_{p,q} : E^2_{p,q} = \textstyle\bigwedge^p_{\mathbb{Z}}(N/N') \otimes_{\mathbb{Z}} \bigwedge^q_{\mathbb{Z}} N' \to E^2_{p-2,q+1} = \bigwedge^{p-2}_{\mathbb{Z}}(N/N') \otimes_{\mathbb{Z}} \bigwedge^{q+1}_{\mathbb{Z}} N'$$

of the spectral sequence (2-1) is given by the formula

$$\begin{aligned} &d^2_{p,q}(a_1 N' \wedge \cdots \wedge a_p N' \otimes x_1 \wedge \cdots \wedge x_q) \\ &\quad = \sum_{k<l} (-1)^{k+l-1} a_1 N' \wedge \ldots \widehat{a_k N'} \ldots \widehat{a_l N'} \cdots \wedge a_p N' \otimes [a_k, a_l] \wedge x_1 \wedge \cdots \wedge x_q. \end{aligned}$$

Also in [Kuz'min and Semenov 1998, Theorem 4], it is shown that

$$H_j(N, \mathbb{Z}) \simeq \bigoplus_{i=1}^{j} E^3_{i,j-i}$$

(note that $E^3_{0,j} = 0$). This means that the filtration of $H_j(N, \mathbb{Z})$ induced by the spectral sequence,

$$0 = F_0 H_j \subseteq F_1 H_j \subseteq \cdots \subseteq F_{j-1} H_j \subseteq F_j H_j = H_j(N, \mathbb{Z}),$$

has the form

$$F_i H_j / F_{i-1} H_j \simeq E^3_{i,j-i} \subseteq \big(\textstyle\bigwedge^i_{\mathbb{Z}}(N/N') \otimes_{\mathbb{Z}} \bigwedge^{j-i+1}_{\mathbb{Z}} N'\big)/T_{i,j-i},$$

where $T_{i,j-i}$ is generated by the elements

$$\sum_{k<l} (-1)^{k+l-1} y_1 \wedge \cdots \wedge \widehat{y_k} \wedge \cdots \wedge \widehat{y_l} \wedge \cdots \wedge y_{i+2} \otimes [y_k, y_l] \wedge x_1 \wedge \cdots \wedge x_{j-i-1},$$

where $y_h \in N/N'$, $x_g \in N'$. This shows that $F_1 H_j \simeq E^3_{1,j-1}$ from the filtration is a quotient of $\bigotimes^{2j-1}_{\mathbb{Z}}(N/N')$ and is nontrivial. So the bound $2j - 1 = c(j-1) + 1$ in Theorem 2.1 is sharp.

Corollary 2.6. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups, where N is nilpotent of class c. Then there exist a natural filtration of $\mathbb{Q}Q$-submodules of $H_j(N, \mathbb{Q})$,*

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_j(N, \mathbb{Q}),$$

such that for any $0 \le k \le l$, E_k/E_{k-1} is a natural subquotient of a $\mathbb{Q}Q$-module from the set

$$\left\{ \bigotimes_{\mathbb{Q}}^{s} V \right\}_{0 \le s \le c(j-1)+1},$$

where $V := (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$, and $\bigotimes^s_{\mathbb{Q}} V$ is considered as a $\mathbb{Q}Q$-module via the diagonal action of Q.

Proof. We have a natural action of Q on $H_q(\gamma_c(N), \mathbb{Q})$ and $H_p(N/\gamma_c(N), \mathbb{Q})$. From these we obtain a natural action of Q on the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(N/\gamma_c(N), H_q(\gamma_c(N), \mathbb{Q})) \Rightarrow H_{p+q}(N, \mathbb{Q}).$$

This means that the groups $E^2_{p,q}$ are $\mathbb{Q}Q$-modules and the differentials $d^2_{p,q}$ are homomorphisms of $\mathbb{Q}Q$-modules. This implies that we have a filtration of $\mathbb{Q}Q$-submodules of $H_j(N, \mathbb{Q})$

$$0 = F_{-1}H_j \subseteq F_0H_j \subseteq \cdots \subseteq F_{j-1}H_j \subseteq F_jH_j = H_j(N, \mathbb{Q}),$$

such that each $E^\infty_{i,j-i} \simeq F_iH_j/F_{i-1}H_j$, $0 \le i \le j$, is an isomorphism of $\mathbb{Q}Q$-modules.

It is also easy to see that if $\bigotimes^c_{\mathbb{Z}}(N/N')$ is considered as $\mathbb{Z}Q$-module via the diagonal action of Q, then the natural map $\bigotimes^c_{\mathbb{Z}}(N/N') \to \gamma_c(N)$ is a homomorphism of $\mathbb{Z}Q$-modules. Now if we follow the proof of Theorem 2.1, we see that in all steps of the proof the $\mathbb{Q}Q$-structure is preserved. This means that all subquotients considered in the proof of Theorem 2.1 are $\mathbb{Q}Q$-subquotients (i.e., the subquotient structure commutes with the Q-action) and the maps are $\mathbb{Q}Q$-homomorphisms, etc. Therefore, as in the proof of Theorem 2.1, we obtain the desired filtration. □

3. Nilpotent action on the homology of nilpotent groups

We say that a group G acts nilpotently on a G-module T, if T has a filtration of G-submodules

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_{k-1} \subseteq T_k = T,$$

such that the action of G on each quotient T_i/T_{i-1} is trivial.

Corollary 2.6 shows that if $Q = G/N$ acts nilpotently on N/N', then it act nilpotently on $H_j(N, \mathbb{Q})$ for any $j \ge 0$. This fact can be generalized as follows.

Theorem 3.1. *Let G be a group, N a nilpotent normal subgroup of G and let T be a G-module. If G acts nilpotently on N/N' and T, then, for any $k \ge 0$, G acts nilpotently on $H_k(N, T)$ and $H^k(N, T)$.*

Proof. We prove the claim for the homology functor. The proof for the cohomology functor is similar. The proof is in three steps.

Step 1. *N is abelian and T is a trivial G-module*: Let

$$0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_n = N$$

be a filtration of N such that G acts trivially on each quotient N_i/N_{i-1}. We prove this step by induction on the length of the filtration of N, i.e., on n. If $n = 1$, then the action of G on $N = N_1$ is trivial. So the action of G on $H_k(N, T)$ also is trivial. From the exact sequence of groups

$$N_1 \rightarrowtail N \twoheadrightarrow N/N_1,$$

we obtain the Lyndon–Hochschild–Serre spectral sequence

$$E'^{2}_{p,q} = H_p(N/N_1, H_q(N_1, T)) \Rightarrow H_{p+q}(N, T).$$

By above, G acts trivially (and so nilpotently) on $H_q(N_1, T)$. Since G/N_1 acts nilpotently on N/N_1 and N/N_1 has a filtration of length $n-1$, by induction hypothesis G/N_1, and so G, acts nilpotently on each $E'^{2}_{p,q}$. Since $E'^{\infty}_{p,q}$ is a subquotient of $E'^{2}_{p,q}$, G acts nilpotently on it too. Moreover, G acts naturally on the above spectral sequence which means that each $E'^{2}_{p,q}$ is a G-module and the differentials $d'^{2}_{p,q}$ are homomorphisms of G-modules. This implies that we have a filtration of G-submodules

$$0 = F_{-1}H_k \subseteq F_0H_k \subseteq \cdots \subseteq F_{k-1}H_k \subseteq F_kH_k = H_k(N, T),$$

such that each isomorphism $E'^{\infty}_{i,k-i} \simeq F_iH_k/F_{i-1}H_k$ is an isomorphism of G-modules. Thus G acts nilpotently on each quotient $F_iH_k/F_{i-1}H_k$. This implies that G acts nilpotently on $H_k(N, T)$.

Step 2. *N is abelian and T is any G-module*: Let

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_l = T$$

be a filtration of T, such that G acts trivially on each quotient T_i/T_{i-1}. In this case we prove the theorem by induction on l, the length of the filtration of T. If $l = 1$, then the action of G on $T = T_1$ is trivial, so we arrive at Step 1. From the exact sequence

$$0 \to T_1 \to T \to T/T_1 \to 0,$$

we obtain the long exact sequence

$$\cdots \to H_k(N, T_1) \to H_k(N, T) \to H_k(N, T/T_1) \to \cdots .$$

We know that G acts nilpotently on $H_k(N, T_1)$ and by the induction hypothesis G acts nilpotently on $H_k(N, T/T_1)$. Now the above exact sequence implies that G acts nilpotently on $H_k(N, T)$.

Step 3. *The general case*: The proof of this step is by induction on the nilpotent class c of N. If $c = 1$, then N is abelian and this is done in Step 2. Now assume that the claim is true for all nilpotent groups of class d, $1 \leq d \leq c-1$. Consider the lower central series of N,

$$1 = \gamma_{c+1}(N) \subset \gamma_c(N) \subset \cdots \subset \gamma_2(N) \subset \gamma_1(N) = N.$$

Note that $\gamma_c(N) \subseteq Z(N)$. The exact sequence of groups

$$\gamma_c(N) \rightarrowtail N \twoheadrightarrow N/\gamma_c(N),$$

gives us the Lyndon–Hochschild–Serre spectral sequence

$$E^2_{p,q} = H_p(N/\gamma_c(N), H_q(\gamma_c(N), T)) \Rightarrow H_{p+q}(N, T).$$

We have a natural surjective map

$$\bigotimes_{\mathbb{Z}}^{c}(N/N') \twoheadrightarrow \gamma_c(N),$$

which is a map of G-modules if we consider $\bigotimes_{\mathbb{Z}}^{c}(N/N')$ as a G-module via the diagonal action [Lennox and Robinson 2004, 1.2.11]. Since G acts nilpotently on N/N', it also acts nilpotently on $\bigotimes_{\mathbb{Z}}^{c}(N/N')$. Thus through the above surjective map, G also acts nilpotently on $\gamma_c(N)$. By Step 2, G acts nilpotently on $H_q(\gamma_c(N), T)$. On the other hand, $N/\gamma_c(N)$ is of nilpotent class $c-1$ and G acts nilpotently on $(N/\gamma_c(N))/(N/\gamma_c(N))' \simeq N/N'$. So by the induction hypothesis, G acts nilpotently on each $E^2_{p,q}$. Thus G acts nilpotently on each $E^\infty_{p,q}$. Finally by the convergence of the spectral sequence, one can show, as in Step 1, that G acts nilpotently on $H_k(N, T)$. This completes the proof of the theorem. □

If A is an abelian normal subgroup of G, then one can show that G is nilpotent if and only if G/A is nilpotent and G acts nilpotently on A [Hilton et al. 1975, Proposition 4.1, Chapter I]. One side of this fact can be generalized as follows.

Corollary 3.2. *Let G be a nilpotent group, N a normal subgroup of G and let T be a G-module. If G acts nilpotently on T, then for any $k \geq 0$, G/N acts nilpotently on $H_k(N, T)$ and $H^k(N, T)$.*

Proof. Since G/N' is nilpotent and N/N' is abelian, G/N', and so G, acts nilpotently on N/N'. Now the claim follows from Theorem 3.1. □

Lemma 3.3. *Let G be a finite group, R a commutative ring and T an RG-module such that G acts nilpotently.*

(i) *If $1/|G| \in R$, then T is a trivial G-module.*

(ii) *If G is nilpotent, l-torsion and $1/l \in R$, then T is a trivial G-module.*

Proof. (i) We know that the functor $- \otimes_G \mathbb{Z} = (-)_G$ is right exact. First we show that this is in fact an exact functor if it is considered as a functor from the category of RG-modules to the category of R-modules. Consider the maps

$$\alpha_G : T^G \to T_G, \quad m \mapsto \overline{m},$$
$$\overline{N} : T_G \to T^G, \quad \overline{m} \mapsto Nm,$$

where $N := \sum_{g\in G} g \in RG$. Then clearly $\overline{N}\circ\alpha$ and $\alpha\circ\overline{N}$ coincide with multiplication by $|G|$. Since $1/|G| \in R$, α_G is an isomorphism. This implies that $(-)_G$ is exact, because $(-)^G$ is left exact. Next, let

$$0 = T_0 \subseteq T_1 \subseteq \cdots \subseteq T_k = T$$

be a filtration of T such that G acts trivially on each T_i/T_{i-1}. By applying the exact functor $(-)_G$ to the exact sequence $0 \to T_1 \to T_2 \to T_2/T_1 \to 0$ and using the fact that G acts trivially on T_1 and T_2/T_1, we see that

$$0 \to T_1 \to (T_2)_G \to T_2/T_1 \to 0$$

is exact. Therefore $T_2 \simeq (T_2)_G$ and so the action of G on T_2 is trivial. In a similar way and by induction on i, one can show that the action of G on each T_i is trivial. Thus the action of G on $T_k = T$ is trivial.

(ii) First we prove that $(-)_G$ is exact and we do this by induction on the size of G. We may assume that $G \neq 1$. Since G is nilpotent, $Z(G) \neq 1$. Let H be a nontrivial cyclic subgroup of $Z(G)$. Then the map α_G coincides with the following composition of maps

$$T^G \xrightarrow{\simeq} (T^H)^{G/H} \xrightarrow{\alpha_H} (T_H)^{G/H} \xrightarrow{\alpha_{G/H}} (T_H)_{G/H} \xrightarrow{\simeq} T_G.$$

Now the exactness of the functor $(-)_G$ follows from (i) and the induction step. Finally, as in (i) we can prove that G acts trivially on T. □

Corollary 3.4. *Let G be a nilpotent group and N a normal subgroup of G such that G/N is finite and l-torsion. Let R be a commutative ring such that $1/l \in R$ and let T be an RG-module. If G acts nilpotently on T, then, for any $k \geq 0$, the natural action of G/N on $H_k(N, T)$ and $H^k(N, T)$ is trivial and therefore the natural maps*

$$\mathrm{corr}_N^G : H_k(N, T) \to H_k(G, T), \quad \mathrm{res}_N^G : H^k(G, T) \to H^k(N, T)$$

are isomorphisms.

Proof. The claim follows from Corollary 3.2 and Lemma 3.3. □

Corollary 3.5. *Let G be a nilpotent group and N a subgroup of G such that G/N is finite and l-torsion. Let R be a commutative ring such that $1/l! \in R$ and let T be an RG-module. If G acts nilpotently on T, then, for any $k \geq 0$, the natural maps*

$$\mathrm{corr}_N^G : H_k(N, T) \to H_k(G, T), \quad \mathrm{res}_N^G : H^k(G, T) \to H^k(N, T)$$

are isomorphisms.

Proof. It is well-known that N has a subgroup L such that L is normal in G and $[G : L] \leq [G : N]!$. Now by Corollary 3.4, the maps

$$\mathrm{corr}_L^G : H_k(L, T) \to H_k(G, T) \quad \text{and} \quad \mathrm{corr}_L^N : H_k(L, T) \to H_k(N, T)$$

are isomorphisms. Therefore $\mathrm{corr}_N^G : H_k(N, T) \to H_k(G, T)$ is an isomorphism. The cohomology case can be treated in a similar way. □

Example 3.6. In general, in Corollary 3.4 the condition that $[G : N] < \infty$ and $1/l \in R$ can not be removed. In fact, if N is a noncentral abelian normal subgroup of a nilpotent group G, e.g., G a nilpotent group of class $c = 3$ and $N = G'$, then clearly G does not act trivially on $H_1(N, \mathbb{Z}) = N$.

4. Bieri–Strebel invariant

The main condition of our main Theorem 5.4, proved below, is closely related to an invariant, introduced by Bieri and Strebel [1980], which has played a prominent role in the study of soluble groups which are finitely presented.

Let Q be a multiplicative finitely generated abelian group. A homomorphism of groups

$$v : Q \to \mathbb{R}$$

is called a valuation on Q. If Q has rank n, then $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \simeq \mathbb{R}^n$, so $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R})$ can be regarded as a topological vector space. Two valuation v and v' on Q are called equivalent if $v' = av$ for some $a \in \mathbb{R}^{>0}$. We denote the equivalence class of v by $[v]$ and the set $S(Q)$ of all equivalence classes of elements of $\mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \setminus \{0\}$ is called the valuation sphere, which can be identified with the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$. Notice that $S(Q)$ is empty precisely when $n = 0$, that is, Q is finite. For any valuation v on Q define

$$Q_v := \{q \in Q | v(q) \geq 0\},$$

which is a submonoid of Q.

For a ring R, let RQ_v be the monoid ring, which clearly is a subring of RQ. For a finitely generated RQ-module A, define

$$\Sigma_A(Q) := \{[v] \in S(Q) \mid A \text{ is finitely generated over } RQ_v\}.$$

A finitely generated RQ-module A is called m-tame if for any m elements

$$v_1, \dots, v_m \in \mathrm{Hom}_{\mathbb{Z}}(Q, \mathbb{R}) \setminus \{0\}$$

with $v_1 + \cdots + v_m = 0$, there is $1 \leq i \leq m$ such that $[v_i] \in \Sigma_A(Q)$.

Theorem 4.1. *Let Q be a finitely generated abelian group, K a field, A a finitely generated KQ-module and $m \geq 2$ an integer. Then the following statements are equivalent*:

(i) *A is m-tame as KQ-module*;

(ii) *$\bigotimes_K^m A$ is finitely generated as KQ-module via the diagonal Q-action*;

(iii) $\bigotimes_K^i A$ *are finitely generated as* KQ*-modules via the diagonal* Q*-action for* $i = 2, \dots, m$;

(iv) $\bigwedge_K^i A$ *are finitely generated as* KQ*-modules via the diagonal* Q*-action for* $i = 2, 3, \dots, m$;

(v) $\bigwedge_K^m A$ *is finitely generated as* KQ*-module via the diagonal* Q*-action.*

Proof. See [Bieri and Groves 1982, Theorem C] and [Kochloukova 1999, Corollary B]. □

Theorem 4.2. *Let* $A \rightarrowtail G \twoheadrightarrow Q$ *be a short exact sequence of groups with both* A *and* Q *abelian and* G *finitely generated. If* G *is of type* FP_m*, then* $A \otimes_{\mathbb{Z}} K$ *is* m*-tame as a* KQ*-module for every field* K.

Proof. See [Bieri and Groves 1982, Theorem D]. □

5. Virtual rational Betti numbers of nilpotent-by-abelian groups

The following two theorems are taken from [Bridson and Kochloukova 2015] and [Kochloukova and Mokari 2015], respectively, which are very important for the study of virtual rational Betti numbers of abelian-by-polycyclic groups. In this section we will use them for the study of virtual rational Betti numbers of nilpotent-by-abelian groups.

Theorem 5.1 (Bridson–Kochloukova). *Let* Q *be a finitely generated abelian group and* B *a finitely generated* $\mathbb{Q}Q$*-module. If* $B \otimes_{\mathbb{Q}} B$ *is a finitely generated* $\mathbb{Q}Q$*-module via the diagonal action of* Q*, then*

$$\sup_{M \in \mathcal{A}_Q} \dim_{\mathbb{Q}}(B \otimes_{\mathbb{Q}M} \mathbb{Q}) < \infty.$$

Proof. See [Bridson and Kochloukova 2015, Theorem 3.1]. □

Theorem 5.2 (Kochloukova–Mokari). *Let* Q *be a finitely generated abelian group and* B *a finitely generated* $\mathbb{Q}Q$*-module. If* $\sup_{m\geq 1} \dim_{\mathbb{Q}}(B \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty$*, then for any* $i \geq 0$,

$$\sup_{m\geq 1} \dim_{\mathbb{Q}} H_i(Q^m, B) < \infty.$$

Proof. See [Kochloukova and Mokari 2015, Theorem 2.4]. □

Lemma 5.3. *Let* Q *be a finitely generated abelian group. Let* V *be a* $\mathbb{Q}Q$*-module such that* $\bigotimes_{\mathbb{Q}}^n V$ *is a finitely generated* $\mathbb{Q}Q$*-module via the diagonal action of* Q*. If* $\sup_{m\geq 1} \dim_{\mathbb{Q}}\big(\big(\bigotimes_{\mathbb{Q}}^n V\big) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}\big) < \infty$*, then for any* $\mathbb{Q}Q$*-subquotient* U *of* $\bigotimes_{\mathbb{Q}}^n V$*, we have*

$$\sup_{m\geq 1} \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty.$$

Proof. First let us assume that U is a quotient of $\bigotimes_{\mathbb{Q}}^n V$, i.e., $U = \big(\bigotimes_{\mathbb{Q}}^n V\big)/T$, for some $\mathbb{Q}Q$-submodule T of $\bigotimes_{\mathbb{Q}}^n V$. Then clearly

$$\dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) \leq \dim_{\mathbb{Q}}\left(\left(\bigotimes_{\mathbb{Q}}^{n} V\right) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}\right),$$

and thus

$$\sup_{m\geq 1} \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) \leq \sup_{m\geq 1} \dim_{\mathbb{Q}}\left(\left(\bigotimes_{\mathbb{Q}}^{n} V\right) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}\right) < \infty.$$

Next let U be a $\mathbb{Q}Q$-submodule of some $W := \big(\bigotimes_{\mathbb{Q}}^n V\big)/T$. Then W/U is of the form $\big(\bigotimes_{\mathbb{Q}}^n V\big)/T'$ for some $\mathbb{Q}Q$-submodule T' of $\bigotimes_{\mathbb{Q}}^n V$ and so

$$\sup_{m\geq 1} \dim_{\mathbb{Q}}(W \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty, \qquad \sup_{m\geq 1} \dim_{\mathbb{Q}}((W/U) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty.$$

Now from the exact sequence $0 \to U \to W \to W/U \to 0$, we obtain the long exact sequence

$$\cdots \to \mathrm{tor}_1^{\mathbb{Q}Q^m}(W/U, \mathbb{Q}) \to U \otimes_{\mathbb{Q}Q^m} \mathbb{Q} \to W \otimes_{\mathbb{Q}Q^m} \mathbb{Q} \to (W/U) \otimes_{\mathbb{Q}Q^m} \mathbb{Q} \to 0,$$

which implies that

$$\dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) \leq \dim_{\mathbb{Q}} \mathrm{tor}_1^{\mathbb{Q}Q^m}(W/U, \mathbb{Q}) + \dim_{\mathbb{Q}}(W \otimes_{\mathbb{Q}Q^m} \mathbb{Q}). \tag{5-1}$$

Since $\sup_{m\geq 1} \dim_{\mathbb{Q}}((W/U) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty$, by Theorem 5.2 we obtain

$$\sup_{m\geq 1} \dim_{\mathbb{Q}} H_i(Q^m, W/U) < \infty. \tag{5-2}$$

But $\mathrm{tor}_i^{\mathbb{Q}Q^m}(W/U, \mathbb{Q}) = H_i(Q^m, W/U)$, thus by (5-1) and (5-2) we have

$$\sup_{m\geq 1} \dim_{\mathbb{Q}}(U \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty. \qquad \square$$

The next theorem is the main result of this paper.

Theorem 5.4. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups, where G is finitely generated, N is nilpotent of class c and Q is abelian. If N/N' is $2(c(n-1)+1)$-tame, then for any $0 \leq j \leq n$, $\mathrm{vb}_j(G)$ is finite.*

Proof. Let G_1 be a subgroup of finite index in G. Let Q_1 be the image of G_1 in Q and $N_1 := N \cap G_1$. Then clearly $[Q : Q_1] < \infty$, and $[N : N_1] < \infty$. From the associated Lyndon–Hochschild–Serre spectral sequence

$$E_{p,q}^2 = H_p(Q_1, H_q(N_1, \mathbb{Q})) \Rightarrow H_{p+q}(G_1, \mathbb{Q})$$

of the extension $N_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$, we obtain

$$\dim_{\mathbb{Q}} H_j(G_1, \mathbb{Q}) = \sum_{p=0}^{j} \dim_{\mathbb{Q}} E^{\infty}_{p,j-p} \leq \sum_{p=0}^{j} \dim_{\mathbb{Q}} E^{2}_{p,j-p}.$$

Since $[N : N_1] < \infty$, by Corollary 3.4, for any $k \geq 0$, we have

$$H_k(N_1, \mathbb{Q}) \simeq H_k(N, \mathbb{Q}).$$

Thus $E^2_{p,q} \simeq H_p(Q_1, H_q(N, \mathbb{Q}))$. On the other hand, since $[Q : Q_1] < \infty$, there exists $m \in \mathbb{N}$ such that $(Q/Q_1)^m = 1$. Hence $Q^m \subseteq Q_1$. Since Q_1/Q^m is finite, we have

$$H_p(Q_1, H_{j-p}(N, \mathbb{Q})) \simeq H_p(Q^m, H_{j-p}(N, \mathbb{Q}))_{Q_1/Q^m},$$

and this implies that

$$\dim_{\mathbb{Q}} H_p(Q_1, H_{j-p}(N, \mathbb{Q})) \leq \dim_{\mathbb{Q}} H_p(Q^m, H_{j-p}(N, \mathbb{Q})).$$

So to prove the theorem it is sufficient to prove that

$$\sup_{m \geq 1} \dim_{\mathbb{Q}} H_p(Q^m, H_{j-p}(N, \mathbb{Q})) < \infty.$$

By Corollary 2.6, $H_{j-p}(N, \mathbb{Q})$ has a natural filtration of $\mathbb{Q}Q$-submodules

$$0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_{l-1} \subseteq E_l = H_{j-p}(N, \mathbb{Q}),$$

such that for any $0 \leq k \leq l$, E_k/E_{k-1} is a natural subquotient of a $\mathbb{Q}Q$-module from the set

$$\left\{ \bigotimes_{\mathbb{Q}}^{s} V \right\}_{0 \leq s \leq c(j-p-1)+1},$$

where $V := (N/N') \otimes_{\mathbb{Z}} \mathbb{Q}$ and $\bigotimes_{\mathbb{Q}}^{s} V$ is considered as a $\mathbb{Q}Q$-module via the diagonal action of Q. By Theorem 4.1, $\bigotimes_{\mathbb{Q}}^{s} V$ is a finitely generated $\mathbb{Q}Q$-module for $0 \leq s \leq 2c(j-p-1)+2$. Thus by Theorem 5.1,

$$\sup_{m \geq 1} \dim_{\mathbb{Q}} \left(\left(\bigotimes_{\mathbb{Q}}^{s} V \right) \otimes_{\mathbb{Q}Q^m} \mathbb{Q} \right) < \infty \quad \text{for } 0 \leq s \leq c(j-p-1)+1.$$

Next, Lemma 5.3 implies that

$$\sup_{m \geq 1} \dim_{\mathbb{Q}}((E_i/E_{i-1}) \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty,$$

and by induction on i, one can show that, for any $1 \leq i \leq j-p$

$$\sup_{m \geq 1} \dim_{\mathbb{Q}}(E_i \otimes_{\mathbb{Q}Q^m} \mathbb{Q}) < \infty.$$

Therefore

$$\sup_{m\geq 1} \dim_{\mathbb{Q}}(H_{j-p}(N,\mathbb{Q})\otimes_{\mathbb{Q}Q^m}\mathbb{Q}) = \sup_{m\geq 1}\dim_{\mathbb{Q}}(E_l\otimes_{\mathbb{Q}Q^m}\mathbb{Q}) < \infty.$$

Now by Theorem 5.2, for any $0 \leq p \leq j$,

$$\sup_{m\geq 1}\dim_{\mathbb{Q}} H_p(Q^m, H_{j-p}(N,\mathbb{Q})) < \infty.$$

This completes the proof of the theorem. □

Lemma 5.5. *Let G be a group and H a subgroup of finite index in G. Then* $\mathrm{vb}_n(G)$ *is finite if and only if* $\mathrm{vb}_n(H)$ *is finite. In fact, for any $n \geq 0$,* $\mathrm{vb}_n(G) = \mathrm{vb}_n(H)$.

Proof. If H_0 is a subgroup of finite index in H, then $[G:H_0]=[G:H][H:H_0]<\infty$. So $\dim_{\mathbb{Q}} H_n(H_0,\mathbb{Q}) \leq \mathrm{vb}_n(G)$ and hence

$$\mathrm{vb}_n(H) \leq \mathrm{vb}_n(G).$$

If G_0 is a subgroup of finite index in G, then $[G_0 : G_0\cap H] \leq [G : H]$. So there is a normal subgroup N of G_0 such that $N \subseteq G_0\cap H$ and $[G_0 : N] < \infty$. Since $H_n(G_0,\mathbb{Q}) \simeq H_n(N,\mathbb{Q})_{G_0/N}$, $\dim_{\mathbb{Q}} H_n(G_0,\mathbb{Q}) \leq \dim_{\mathbb{Q}} H_n(N,\mathbb{Q})$. Now from $[H : N] < \infty$, it follows that $\dim_{\mathbb{Q}} H_n(G_0,\mathbb{Q}) \leq \dim_{\mathbb{Q}} H_n(N,\mathbb{Q}) \leq \mathrm{vb}_n(H)$. Therefore

$$\mathrm{vb}_n(G) \leq \mathrm{vb}_n(H).$$ □

Corollary 5.6. *Let G be a nilpotent-by-abelian-by-finite group, i.e., we have a chain of subgroups $N \trianglelefteq H \trianglelefteq G$, where N is nilpotent, H/N is abelian and $[G : H] < \infty$. If N is of class c and H/N' is of type* $\mathrm{FP}_{2c(n-1)+2}$, *then* $\mathrm{vb}_j(G)$ *is finite for any $0 \leq j \leq n$.*

Proof. Since H/N' is metabelian of type $\mathrm{FP}_{2c(j-p-1)+2}$, by Theorem 4.2 the Q-module $(N/N')\otimes_{\mathbb{Z}}\mathbb{Q}$ is $2(c(j-p-1)+1)$-tame. Now the claim follows from Lemma 5.5 and Theorem 5.4. □

Remark 5.7. Theorem 5.4 and Corollary 5.6 generalize [Bridson and Kochloukova 2015, Theorem 5.3 and Corollary 5.4] to higher homology groups.

For the first virtual rational Betti number we can improve the above result a bit.

Proposition 5.8. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups, where N is nilpotent and Q is polycyclic. Let G/N' be of type* FP_3 *and let $\bigotimes_{\mathbb{Z}}^2 N/N'$ be finitely generated as $\mathbb{Z}Q$-module via the diagonal action. Then* $\mathrm{vb}_1(G)$ *is finite.*

Proof. Let G_1 be a normal subgroup of finite index in G. Let Q_1 be the image of the G_1 in Q and $N_1 = N \cap G_1$. The associated Lyndon–Hochschild–Serre spectral sequence of $N_1 \rightarrowtail G_1 \twoheadrightarrow Q_1$, i.e.,

$$E^2_{p,q} = H_p(Q_1, H_q(N_1, \mathbb{Q})) \Rightarrow H_{p+q}(G_1, \mathbb{Q}),$$

implies that

$$\begin{aligned}\dim_{\mathbb{Q}} H_1(G_1, \mathbb{Q}) &\le \dim_{\mathbb{Q}} E^2_{0,1} + \dim_{\mathbb{Q}} E^2_{1,0} \\ &= \dim_{\mathbb{Q}} H_0(Q_1, H_1(N_1, \mathbb{Q})) + \dim_{\mathbb{Q}} H_1(Q_1, \mathbb{Q}).\end{aligned}$$

Since any subgroup of a polycyclic group is polycyclic, by [Kochloukova and Mokari 2015, Lemma 3.2] we have $\dim_{\mathbb{Q}} H_1(Q_1, \mathbb{Q}) \le h(Q)$, where $h(Q)$ is the Hirsch length of Q. Since $[N : N_1] < \infty$, by Corollary 3.5 we have $H_1(N_1, \mathbb{Q}) \simeq H_1(N, \mathbb{Q})$. So to prove the claim it is sufficient to prove that

$$\sup_{[Q:Q_1]<\infty} \dim_{\mathbb{Q}}(N/N' \otimes_{Q_1} \mathbb{Q}) < \infty.$$

Let $A = N/N'$ and $H = G/N'$ and consider the exact sequence $A \rightarrowtail H \twoheadrightarrow Q$. If we put $A_0 = [A, H]$ and $Q_0 = H/A_0$ and if we follow the proof of Theorem A in [Kochloukova and Mokari 2015], we obtain

$$\sup_{[Q_0:Q_2]<\infty} \dim_{\mathbb{Q}}(A_0 \otimes_{Q_2} \mathbb{Q}) < \infty.$$

From the exact sequence $A_0 \rightarrowtail A \twoheadrightarrow A/A_0$, we obtain the exact sequence

$$A_0 \otimes_{Q_2} \mathbb{Q} \to A \otimes_{Q_2} \mathbb{Q} \to (A/A_0) \otimes_{Q_2} \mathbb{Q} \to 0,$$

which implies that

$$\dim_{\mathbb{Q}}(A \otimes_{Q_2} \mathbb{Q}) \le \dim_{\mathbb{Q}}(A_0 \otimes_{Q_2} \mathbb{Q}) + \dim_{\mathbb{Q}}((A/A_0) \otimes_{Q_2} \mathbb{Q}).$$

Now consider the exact sequence $A/A_0 \rightarrowtail Q_0 \overset{\beta}{\twoheadrightarrow} Q$ and let $Q_1 = \beta(Q_2)$. Since the action of A/A_0 over A is trivial, we have $A \otimes_{Q_1} \mathbb{Q} \simeq A \otimes_{Q_2} \mathbb{Q}$. Since A/A_0 is a finitely generated abelian group,

$$\sup_{[Q_0:Q_2]<\infty} \dim_{\mathbb{Q}}((A/A_0) \otimes_{Q_2} \mathbb{Q}) < \infty.$$

Therefore from the above relations we have

$$\sup_{[Q:Q_1]<\infty} \dim_{\mathbb{Q}}(A \otimes_{Q_1} \mathbb{Q}) < \infty.$$

This completes the proof of the theorem. □

Corollary 5.9. *Let $N \rightarrowtail G \twoheadrightarrow Q$ be an exact sequence of groups, where N is nilpotent and Q is nilpotent of class $c \le 2$. If G/N' is of type* FP_3, *then* $\mathrm{vb}_1(G)$ *is finite.*

Proof. By Lemma 3.5 in the proof of Corollary B in [Kochloukova and Mokari 2015], $\bigotimes^2_{\mathbb{Q}}(A_0 \otimes_{\mathbb{Z}} \mathbb{Q})$ is finitely generated as $\mathbb{Q}Q$-module via the diagonal action, where A_0 is as in the proof of Proposition 5.8. Now we can proceed as in the proof of Proposition 5.8. □

6. Some examples

6A. ***S-arithmetic groups.*** Unfortunately there is no classification of the nilpotent-by-abelian groups of type FP_n even in the case of $n = 2$, though the metabelian case was solved in [Bieri and Strebel 1980]. In this case type FP_2 turns out to be equivalent to finite presentability. Still in the case of soluble S-arithmetic groups there is a complete classification of finite presentability [Abels 1987, Theorem 7.5.2, Remark 4, Chapter VII]. They are finitely presented if and only if are of type FP_2. Note that soluble S-arithmetic groups are nilpotent-by-abelian-by-finite.

By a theorem of Borel–Serre [Abels 1987, Theorem 0.4.4], any S-arithmetic subgroup of a reductive group is of type FP_∞ and thus for such soluble subgroups the result of Corollary 5.6 is true for any $j \ge 0$. But such a result can be proved for other type of S-arithmetic groups.

The following example was considered in [Abels and Brown 1987]: Let p be a prime and

$$\Gamma_n \le \mathrm{GL}_{n+1}(\mathbb{Z}[1/p]),$$

where Γ_n is the group of upper triangular matrices A with $A_{1,1} = 1 = A_{n+1,n+1}$.

Theorem 6.1. *The group Γ_n is of type* FP_{n-1}, *but not of type* FP_n.

Proof. See [Abels and Brown 1987, Theorem A]. □

Let N_n be the subgroup of Γ_n containing all elements of Γ_n, where the main diagonal contains only entries 1. Then N_n is nilpotent and

$$Q_n = \Gamma_n/N_n \simeq \mathbb{Z}^{n-1}.$$

In this case the abelianization $V_n = N_n/[N_n, N_n]$ is isomorphic to $\mathbb{Z}[1/p]^n$, so $V_n \otimes_{\mathbb{Z}} \mathbb{Q} \simeq \mathbb{Q}^n$ is finite dimensional over $\mathbb{Q}$. Hence all tensor and exterior powers of V_n are finitely generated over $\mathbb{Q}Q_n$. Thus Theorem 4.1 implies that $V_n \otimes_{\mathbb{Z}} \mathbb{Q}$ is m-tame for any $m \ge 2$. Now by Theorem 5.4 we obtain the following result.

Proposition 6.2. *For any $j \ge 0$,* $\mathrm{vb}_j(\Gamma_n)$ *is finite.*

6B. ***Groups of finite torsion-free rank.*** It is a well-known theorem of Mal'cev that polycyclic groups are nilpotent-by-abelian-by-finite [Lennox and Robinson 2004, 3.1.14]. On the other hand, for a polycyclic group G, the group ring $\mathbb{Z}G$ is (right) noetherian [Lennox and Robinson 2004, 4.2.3] and thus G is of type FP_∞. Now by Corollary 5.6, all virtual rational Betti numbers of G are finite. A direct and much easier proof of this fact is given in [Kochloukova and Mokari 2015, Lemma 3.2]

A polycyclic group is a special case of constructible groups. A soluble group is called constructible if and only if it can be built from the trivial group in finitely many steps by taking descending HNN-extensions and finite extensions. It is well-known that the class of constructible soluble groups is closed with respect to taking homomorphic images and subgroups of finite index [Baumslag and Bieri 1976, Proposition 2, Theorem 4]. Moreover, they have finite Prüfer rank [Baumslag and Bieri 1976, Section 3.3, Remark 2] and thus are nilpotent-by-abelian-by-finite. The last part follows from the proof of [Robinson 1972, Theorem 10.38]. Furthermore, constructible soluble groups are finitely presented and are of type FP_∞ [Baumslag and Bieri 1976, Proposition 1]. Thus by Corollary 5.6 all virtual rational Betti numbers of these groups are finite.

Kochloukova and the second author gave a good bound for virtual rational Betti numbers of a polycyclic group [Kochloukova and Mokari 2015, Lemma 3.2]. Their proof work even for the larger class of groups of finite torsion-free rank. Polycyclic and constructible groups are of finite Prüfer rank and thus they are of finite torsion-free rank.

A group G, not necessarily soluble, is said to be of finite torsion-free rank if it has a series of subgroups

$$1 = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G,$$

such that each nontorsion factor G_i/G_{i-1} is infinite cyclic. One can show that the number of infinite cyclic factors is independent of the chosen series (see the proof of [Lennox and Robinson 2004, 1.3.3]) which it is called either the torsion-free rank or the Hirsch number of G and we denote it by $h(G)$.

Proposition 6.3. *Let G be a group of finite torsion-free rank. Then for any integer* $j \geq 0$, $\dim_{\mathbb{Q}} H_j(G, \mathbb{Q}) \leq \binom{h(G)}{j}$. *In particular*,

$$\mathrm{vb}_j(G) \leq \binom{h(G)}{j}.$$

Proof. The proof is similar to that of the case of polycyclic groups given in [Kochloukova and Mokari 2015, Lemma 3.2]. □

Acknowledgments

We would like to thank Professor D. H. Kochloukova for introducing the problem to us and for her constructive suggestion during the preparation of this paper. Section 6A was suggested by her. Also, J. R. Groves' proof — different from ours — of the special case of Theorem 2.1 for $c = 2$ was made available to us by Kochloukova. We would like to thank both for their help and suggestions.

References

[Abels 1987] H. Abels, *Finite presentability of S-arithmetic groups. Compact presentability of solvable groups*, Lecture Notes in Mathematics **1261**, Springer, Berlin, 1987. MR Zbl

[Abels and Brown 1987] H. Abels and K. S. Brown, "Finiteness properties of solvable S-arithmetic groups: an example", *J. Pure Appl. Algebra* **44**:1-3 (1987), 77–83. MR Zbl

[André 1965] M. André, "Le d_2 de la suite spectrale en cohomologie des groupes", *C. R. Acad. Sci. Paris* **260** (1965), 2669–2671. MR Zbl

[Baumslag and Bieri 1976] G. Baumslag and R. Bieri, "Constructable solvable groups", *Math. Z.* **151**:3 (1976), 249–257. MR Zbl

[Bieri and Groves 1982] R. Bieri and J. R. J. Groves, "Metabelian groups of type $(\mathrm{FP})_\infty$ are virtually of type (FP)", *Proc. London Math. Soc.* (3) **45**:2 (1982), 365–384. MR Zbl

[Bieri and Strebel 1980] R. Bieri and R. Strebel, "Valuations and finitely presented metabelian groups", *Proc. London Math. Soc.* (3) **41**:3 (1980), 439–464. MR Zbl

[Bridson and Kochloukova 2015] M. R. Bridson and D. H. Kochloukova, "The virtual first Betti number of soluble groups", *Pacific J. Math.* **274**:2 (2015), 497–510. MR Zbl

[Brown 1994] K. S. Brown, *Cohomology of groups*, Graduate Texts in Mathematics **87**, Springer, New York, NY, 1994. MR Zbl

[Dwyer 1975] W. G. Dwyer, "Vanishing homology over nilpotent groups", *Proc. Amer. Math. Soc.* **49** (1975), 8–12. MR Zbl

[Hilton et al. 1975] P. J. Hilton, G. Mislin, and J. Roitberg, *Localization of nilpotent groups and spaces*, North-Holland Mathematics Studies **15**, North-Holland, Amsterdam, 1975. MR Zbl

[Kochloukova 1997] D. H. Kochloukova, *The* FP_m*-conjecture for a class of metabelian groups and related topics*, Ph.D. thesis, Cambridge University, 1997.

[Kochloukova 1999] D. H. Kochloukova, "Finite generation of exterior and symmetric powers", *Math. Proc. Cambridge Philos. Soc.* **125**:1 (1999), 21–29. MR Zbl

[Kochloukova and Mokari 2015] D. H. Kochloukova and F. Y. Mokari, "Virtual rational Betti numbers of abelian-by-polycyclic groups", *J. Algebra* **443** (2015), 75–98. MR Zbl

[Kuz'min and Semenov 1998] Y. V. Kuz'min and Y. S. Semenov, "О гомологиях свободной нильпотентной группы класса 2", *Mat. Sb.* **189**:4 (1998), 49–82. Translated as "On the homology of a free nilpotent group of class 2" in *Sb. Math.* **189**:4 (1998), 527–560. MR Zbl

[Lennox and Robinson 2004] J. C. Lennox and D. J. S. Robinson, *The theory of infinite soluble groups*, Oxford University Press, 2004. MR Zbl

[Lück 1994] W. Lück, "Approximating L^2-invariants by their finite-dimensional analogues", *Geom. Funct. Anal.* **4**:4 (1994), 455–481. MR Zbl

[Robinson 1972] D. J. S. Robinson, *Finiteness conditions and generalized soluble groups, part 2*, Ergebnisse der Mathematik und ihrer Grenzgebiete **63**, Springer, Berlin, 1972. MR Zbl

[Robinson 1976] D. J. S. Robinson, "The vanishing of certain homology and cohomology groups", *J. Pure Appl. Algebra* **7**:2 (1976), 145–167. MR Zbl

Received August 30, 2015.

BEHROOZ MIRZAII
INSTITUTO DE CIÊNCIAS MATEMÁTICAS E DE COMPUTAÇÃO (ICMC)
UNIVERSIDADE DE SÃO PAULO
AVENIDA TRABALHADOR SÃO-CARLENSE, 400–CENTRO
13566-590 SÃO CARLOS-SP
BRAZIL
bmirzaii@icmc.usp.br

FATEMEH Y. MOKARI
INSTITUTO DE MATEMÁTICA, ESTATÍSTICA E COMPUTAÇÃO CIENTÍFICA (IMECC)
UNIVERSIDADE ESTADUAL DE CAMPINAS
RUA SÉRGIO BUARQUE DE HOLANDA, 651
13083-859 CAMPINAS–SP
BRAZIL
f.mokari61@gmail.com

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.405

A PLANAR SOBOLEV EXTENSION THEOREM FOR PIECEWISE LINEAR HOMEOMORPHISMS

EMANUELA RADICI

We prove that a planar piecewise affine homeomorphism φ defined on the boundary of the square can be extended to a piecewise affine homeomorphism h of the whole square, in such a way that $\|h\|_{W^{1,p}}$ is bounded from above by $\|\varphi\|_{W^{1,p}}$ for every $p \geq 1$.

1. Introduction

Let $\mathcal{Q}$ be the unit square in $\mathbb{R}^2$ and φ be a piecewise affine homeomorphism with finitely many affine components that maps $\partial\mathcal{Q}$ to a closed curve in $\mathbb{R}^2$. We call a piecewise affine map with finitely many affine components a finitely piecewise affine map. In this work, we provide a general recipe for extending φ to a function h of the whole square which maintains the finitely piecewise affine structure and whose Sobolev $W^{1,p}$-norm is controlled from above by $\|\varphi\|_{W^{1,p}}$. That such an extension exists is well known, and its construction is not difficult, but showing the existence of an extension with good control on its norm is a substantial problem. In fact, we will establish a bound

$$\|Dh\|_{L^p(\mathcal{Q})} \leq K\|D\varphi\|_{L^p(\partial\mathcal{Q})}$$

for a suitable geometric constant K which depends only on p. It is appropriate to explain briefly the context of our work and its utility. The problem of finding approximations of a planar homeomorphism $f : \Omega \to \mathbb{R}^2$ has a long history in the literature and recently it was realized to be relevant to the study of the regularity of minimizers for standard energies in the area of nonlinear elasticity. Many important results are already available on this topic. See, for instance, [Mora-Corral 2009; Bellido and Mora-Corral 2011; Iwaniec et al. 2011; Daneri and Pratelli 2015; Hencl and Pratelli 2015] for an overview of what is known. Let us recall the approach introduced in the last two of these references, where the authors create the approximation step by step, starting from an explicit subdivision of the domain Ω that depends on the Lebesgue points of Df. Although their settings and the regularity of their approximations are very different, in both papers the strategy

MSC2010: 46E35.
Keywords: piecewise affine approximation, Sobolev homeomorphisms.

is to split the domain into a countable disjoint union of simple polygons (in general triangles or squares) by introducing a locally finite one-dimensional grid, which consists of the union of the boundaries of such polygons. A first piecewise linear approximation of f is defined on the one-dimensional grid and, only in a second step, the approximation is extended in the interior of each simple polygon, being careful to get the regularity claimed. In this work we focus on the second step of this strategy, namely we already assume the existence of a piecewise linear function f_ε defined on a locally finite grid of squares, let us call it Δ, and we give all the ingredients needed for extending f_ε to a piecewise affine function of the whole domain Ω with suitably small $W^{1,p}$-norm. In fact, the extension of f_ε into a single square $\mathcal{Q}$ involves only the values on the boundary $\partial\mathcal{Q}$ of $\mathcal{Q}$, and it is useful to have an estimate like

$$\|f_\varepsilon\|_{W^{1,p}(\mathcal{Q})} < K\|f_\varepsilon\|_{W^{1,p}(\partial\mathcal{Q})},$$

for a suitable constant K. Let us already say that our proof does not depend on the precise value of p, thus it holds true for every $p \in [1, \infty)$. An analogous result was already proved in the cases $p = \infty$ in [Daneri and Pratelli 2015] and $p = 1$ in [Hencl and Pratelli 2015], while in this work we generalize to any $p > 1$ a technique introduced in [Hencl and Pratelli 2015]. Furthermore, an extension of this result seems to be true also in the Orlicz–Sobolev spaces (see [Campbell 2015]). For us, $\mathcal{Q}$ will be the rotated square centered in the origin and with corners in $(\pm 1, 0)$, $(0, \pm 1)$. Our result is the following.

Theorem 1.1. *Let $\varphi : \partial\mathcal{Q} \to \mathbb{R}^2$ be a one-to-one piecewise affine function. Then there exists a finitely piecewise affine homeomorphism $h : \mathcal{Q} \to \mathbb{R}^2$ that satisfies $h \equiv \varphi$ on $\partial\mathcal{Q}$ and, for any $p \geq 1$, there is a constant K depending only on p such that the estimate*

$$\int_{\mathcal{Q}} |Dh(x)|^p\, dx \leq K \int_{\partial\mathcal{Q}} |D\varphi(t)|^p\, d\mathcal{H}^1(t) \tag{1-1}$$

holds.

The plan of the paper is very simple: in the following section we make a short remark about the notation, the second section is devoted to the proof of Theorem 1.1 and in the last remark we explain the case of a generic square in the plane.

Notation. Let us briefly introduce the notation we use throughout the paper. We call $\mathcal{Q} = \{(x, y) \in \mathbb{R}^2 : |x| + |y| < 1\}$ the rotated square centered in the origin and $\boldsymbol{\mathcal{Q}}$ the image through h of $\mathcal{Q}$. With the capital letters A, B we always refer to points lying on the boundary of $\mathcal{Q}$, while P and R denote points in the interior of $\mathcal{Q}$. The points in $\partial\boldsymbol{\mathcal{Q}}$ and in the interior of $\boldsymbol{\mathcal{Q}}$ will be denoted similarly in bold style: $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{P}$, $\boldsymbol{R}$. When we use the same letter in normal and bold style, for example A and $\boldsymbol{A}$, this always means that $\boldsymbol{A}$ is the image of A through the mapping that we are considering in that moment. By AB and ABC we mean, respectively, the segment between A and B and the triangle of corners A, B and C (the same also for $\boldsymbol{AB}$, $\boldsymbol{ABC}$). The modulus of the horizontal and vertical derivatives of a function $f = (f_1, f_2) : \mathbb{R}^2 \to \mathbb{R}^2$ is denoted as

$$|D_1 f| = \sqrt{\left(\frac{\partial f_1}{\partial x}\right)^2 + \left(\frac{\partial f_2}{\partial x}\right)^2}, \qquad |D_2 f| = \sqrt{\left(\frac{\partial f_1}{\partial y}\right)^2 + \left(\frac{\partial f_2}{\partial y}\right)^2},$$

and with the symbol $\mathcal{H}^1$ we indicate the standard 1-dimensional Hausdorff measure. Finally, the letter K will always indicate a purely geometric constant that depends only on p. Since it is the existence, not the size, of K that matters, we do not calculate the explicit value of K but we show at each step that it remains finite and it stays independent from all the parameters but p.

2. Proof of Theorem 1.1

We generalize to every $1 \le p < \infty$ a strategy introduced in [Hencl and Pratelli 2015] for $p = 1$. To keep this work self contained, we present also the parts of the proof stated in [Hencl and Pratelli 2015] that do not depend on the exponent p.

Proof of Theorem 1.1. Since the proof of Theorem 1.1 is long, we split it into several steps for sake of clarity.

Step I. Choice of corners. It is useful to know that $|D\varphi|^p$ does not critically accumulate around the two opposite corners of $\partial\mathcal{Q}$, which we denote as $V_1 \equiv (0, -1)$ and $V_2 \equiv (0, 1)$. More precisely, the estimate we would like to have is the following:

$$\int_{B(V_i,r)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \le \widetilde{K} r \int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \quad \text{for all } r \in (0, \sqrt{2}),\ i \in \{1, 2\}. \tag{2-1}$$

This is not true for a generic φ, but in this step we show that there always exists a pair of opposite points P_1, P_2 on $\partial\mathcal{Q}$ that satisfies (2-1) in place of V_1, V_2 with $\widetilde{K} = 6$. Notice that it is always possible to apply a bi-Lipschitz transformation of $\partial\mathcal{Q}$ to itself which moves such P_1, P_2 to V_1, V_2 respectively. Since the bi-Lipschitz constant does not depend on the particular values of P_1 and P_2, then estimate (2-1) follows straightforwardly for a suitable constant $\widetilde{K}$. Thanks to this observation, the step is concluded once we prove the existence of such P_1, P_2. Two generic opposite points are a good choice unless at least one of them is in the set

$$\mathcal{A} = \left\{ A \in \partial\mathcal{Q} : \int_{B(A,r)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 > 6r \int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1 \text{ for some } r \in (0, \sqrt{2}) \right\}.$$

Thus, the existence of a good pair P_1, P_2 is established if $\mathcal{H}^1(\mathcal{A})$ is not too big. By a Vitali covering argument, it is always possible to cover $\mathcal{A}$ with a countable

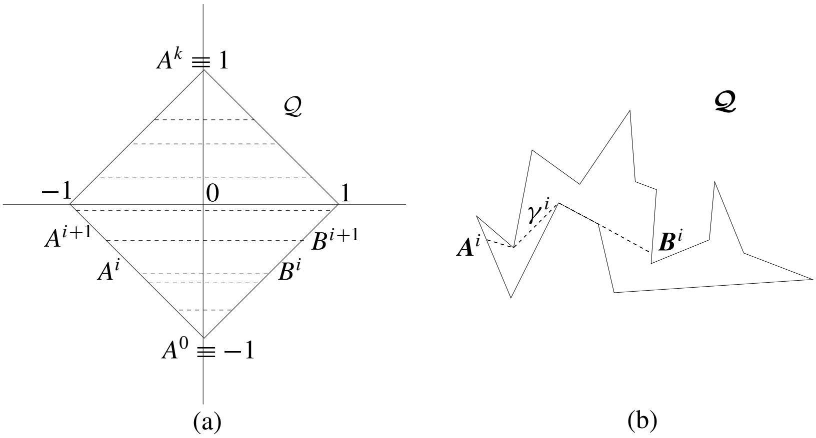


Figure 1. (a) A grid on $\mathcal{Q}$ and (b) a path γ^i in $\boldsymbol{\mathcal{Q}}$.

union of balls $B(A_i, 3r_i)$ such that $A_i \in \mathcal{A}$ and the sets $B(A_i, r_i) \cap \partial\mathcal{Q}$ are pairwise disjoint. In particular, one has

$$\mathcal{H}^1(\mathcal{A}) \leq \sum_i 6r_i \leq \sum_i \frac{\int_{B(A,r_i)\cap\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1}{\int_{\partial\mathcal{Q}} |D\varphi|^p \, d\mathcal{H}^1} \leq 1\,,$$

thus implying the existence of a pair of opposite points, P_1 and P_2 in $\partial\mathcal{Q} \setminus \mathcal{A}$, satisfying (2-1).

Step II. Construction of one-dimensional grids in $\mathcal{Q}$ and in $\boldsymbol{\mathcal{Q}}$. Let us denote with $\boldsymbol{\mathcal{Q}}$ the bounded component of $\mathbb{R}^2 \setminus \varphi(\partial\mathcal{Q})$, which is a polygon because φ is piecewise affine. Notice that the problem of finding a piecewise affine homeomorphism h which maps $\mathcal{Q}$ into $\boldsymbol{\mathcal{Q}}$ makes sense. Since we want $h \equiv \varphi$ on $\partial\mathcal{Q}$, our approach is the following: we start extending the function φ on a suitable one-dimensional grid on $\mathcal{Q}$, we then "complete" this grid to obtain a triangulation of $\mathcal{Q}$ and, at the end, we define h inside each triangle of the triangulation as the affine function extending the values on the boundary. In this step, we introduce the one-dimensional grid in $\mathcal{Q}$ and we construct a related grid in $\boldsymbol{\mathcal{Q}}$, which will be the image through h of the grid on $\mathcal{Q}$. For the construction, we make use of several horizontal segments A^iB^i whose endpoints are in $\partial\mathcal{Q}$. We call $A^i = (A^i_1, A^i_2)$ the endpoint that has negative first component and we choose the indexes i so that A^i_2 increases with respect to i from -1 to 1 (see Figure 1(a)). It is convenient to include also V_1, V_2 in the grid, therefore we denote them consistently as $A^0 \equiv B^0 \equiv V_1 \equiv (0, -1)$ and $A^k \equiv B^k \equiv V_2 \equiv (0, 1)$. We consider many horizontal segments A^iB^i such that for every i the restriction of φ to A^iA^{i+1} or B^iB^{i+1} is linear. Notice that this property is still valid even if we take more horizontal segments A^iB^i, therefore, we are allowed to add points A^i and B^i during the construction (of course, being careful in adding only finitely many). Once the grid on $\mathcal{Q}$ is fixed, we define a second one, this time on $\boldsymbol{\mathcal{Q}}$, that is the union of the geodesics γ^i inside $\overline{\boldsymbol{\mathcal{Q}}}$ connecting $\varphi(A^i)$ and $\varphi(B^i)$. Since $\boldsymbol{\mathcal{Q}}$ is a polygon, γ^i is piecewise affine and, moreover, the junction between any two consecutive affine pieces of γ^i lies in $\partial\boldsymbol{\mathcal{Q}}$ (see Figure 1(b)). In order to simplify the notation, we write the points of $\boldsymbol{\mathcal{Q}}$ in bold style: $\boldsymbol{A}^i \equiv \varphi(A^i)$ and $\boldsymbol{B}^i \equiv \varphi(B^i)$. Up to adding points between A^0 and A^1 (we do the same also for A^{k-1} and A^k), we can always assume that γ^1 (resp. γ^{k-1}) is either a segment $\boldsymbol{A}^1\boldsymbol{B}^1$ (resp. $\boldsymbol{A}^{k-1}\boldsymbol{B}^{k-1}$), or it is the union of the two segments $\boldsymbol{A}^1\boldsymbol{A}^0$ and $\boldsymbol{A}^0\boldsymbol{B}^1$ (resp. $\boldsymbol{A}^{k-1}\boldsymbol{A}^k$ and $\boldsymbol{A}^k\boldsymbol{B}^{k-1}$) lying entirely in $\partial\boldsymbol{\mathcal{Q}}$. It is well known in the literature that the shortest path that connects two generic points inside a simply connected set is unique. In particular, this result ensures that the paths γ^i are unique and implies that the grid on $\boldsymbol{\mathcal{Q}}$ is well defined.

Step III. Relevant properties of the paths γ^i. In this step we present some properties of the paths γ^i. The first property is a consequence of the uniqueness: whenever two paths γ^{i+1} and γ^i intersect each other they coincide from the first to the last point of intersection. In particular, γ^{i+1} and γ^i cannot cross each other, thus allowing us to distinguish three different parts on each path

$$\gamma^i = \gamma_1^i \cup \gamma_2^i \cup \gamma_3^i\,, \qquad \gamma^{i+1} = \gamma_1^{i+1} \cup \gamma_2^{i+1} \cup \gamma_3^{i+1}\,.$$

In detail, if $\boldsymbol{A}$ and $\boldsymbol{B}$ are the first and last points, respectively, of the common part between γ^i and γ^{i+1}, we call γ_1^i the path from $\boldsymbol{A}^i$ to $\boldsymbol{A}$ (analogous for γ_1^{i+1}), γ_2^i the path from $\boldsymbol{A}$ to $\boldsymbol{B}$ ($\gamma_2^i \equiv \gamma_2^{i+1}$) and γ_3^i the last part of the path from $\boldsymbol{B}$ to $\boldsymbol{B}^i$ (analogous for γ_3^{i+1}). When γ^i and γ^{i+1} have no intersection, we directly set $\gamma^i \equiv \gamma_1^i$ and $\gamma^{i+1} \equiv \gamma_1^{i+1}$, while γ_2^i, γ_3^i, γ_2^{i+1}, γ_3^{i+1} are empty paths. Let us observe that such subdivision of γ^i is related to the curve γ^{i+1} and there is no reason why it should match with the one related to γ^{i-1}. The last property of paths γ^i that we recall is fundamental for showing estimate (1-1) (for a formal proof see Step 5 of Theorem 2.1 in [Hencl and Pratelli 2015]). Let $\boldsymbol{P}$ be the last point of the curve γ_1^{i+1}; no matter whether $\boldsymbol{P}$ coincides with $\boldsymbol{A}$ or $\boldsymbol{B}^{i+1}$, the polygon having boundary $\gamma_1^{i+1} \cup \boldsymbol{A}^{i+1}\boldsymbol{P}$ is always convex (see Figure 2).

Step IV. Triangular grid on $\boldsymbol{\mathcal{Q}}$ and estimate on "vertical sides". In this step we are going to select finitely many points on the paths γ^i in order to get a triangular grid on $\boldsymbol{\mathcal{Q}}$. For all $i = 1, \dots, k-2$, we call $\boldsymbol{\mathcal{D}}_i$ the closure of the polygon having boundary $\boldsymbol{A}^i\boldsymbol{A}^{i+1} \cup \gamma^i \cup \gamma^{i+1} \cup \boldsymbol{B}^i\boldsymbol{B}^{i+1}$ (which lies inside the closure of $\boldsymbol{\mathcal{Q}}$), then we argue separately for each single $\boldsymbol{\mathcal{D}}_i$. For every "strip" $\boldsymbol{\mathcal{D}}_i$ we select some points on γ^i, γ^{i+1} depending on the relation between γ_1^i, γ_2^i, γ_3^i and γ_1^{i+1}, γ_2^{i+1}, γ_3^{i+1}. We argue separately for the cases $\gamma^i \cap \gamma^{i+1} \neq \varnothing$ and $\gamma^i \cap \gamma^{i+1} = \varnothing$. Both situations are depicted in Figure 3.

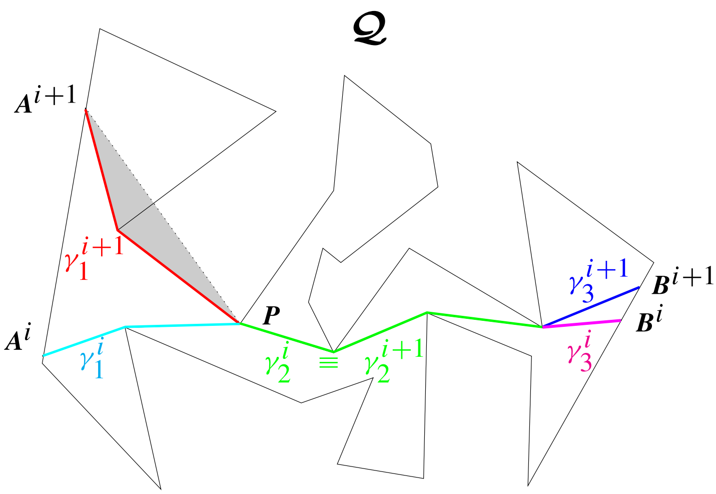


Figure 2. Convexity of the polygon delimited by $\gamma_1^{i+1} \cup A^{i+1}P$.

Let us start from the nonempty case (Figure 3(a)). For any endpoint of a linear piece of γ_1^{i+1} or γ_1^i we consider the corresponding point on the other path so that the segment connecting the two points is parallel to A^iA^{i+1}. We denote with P_j the points taken on γ_1^{i+1} and with R_j the corresponding point on γ_1^i. Notice that the convexity result introduced in Step III ensures that the segment P_jR_j is always well defined and, moreover, it satisfies $\mathcal{H}^1(P_jR_j) \le \mathcal{H}^1(A^iA^{i+1})$, thus, in particular,

$$\mathcal{H}^1(P_jR_j) \le \max\{\mathcal{H}^1(A^iA^{i+1}), \mathcal{H}^1(B^iB^{i+1})\}. \tag{2-2}$$

With a symmetric strategy we select other points P_j and R_j on γ_3^{i+1} and γ_3^i, respectively, by taking this time P_jR_j parallel to B^iB^{i+1}. Let us recall that (2-2) still holds true in this case, since now $\mathcal{H}^1(P_jR_j) \le \mathcal{H}^1(B^iB^{i+1})$. Finally, in the

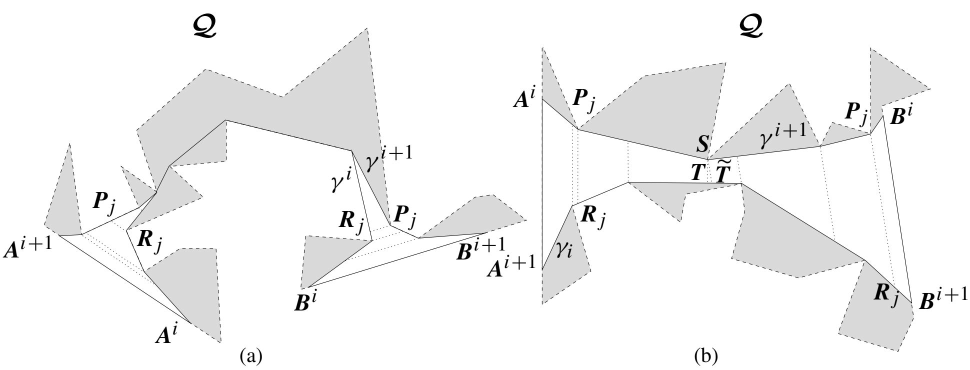


Figure 3. Selection of points on γ^i and γ^{i+1}.

common part $\gamma_2^i \equiv \gamma_2^{i+1}$, we take all the endpoints of the linear pieces of the path and by construction $\boldsymbol{P}_j \equiv \boldsymbol{R}_j$. Of course, (2-2) is trivially true because in this case $\mathcal{H}^1(\boldsymbol{P}_j\boldsymbol{R}_j)=0$. If $\gamma^i\cap\gamma^{i+1}=\varnothing$, the strategy is a little bit different (see Figure 3(b)). In the specific case in which $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ are parallel to each other, we can argue exactly as in the previous case: therefore all the points $\boldsymbol{P}_j$ selected on γ^{i+1} can be either endpoints of linear pieces of γ^{i+1} or the corresponding point of $\boldsymbol{R}_j$, where $\boldsymbol{R}_j$ is an endpoint for γ^i. Moreover, by construction, $\boldsymbol{P}_j\boldsymbol{R}_j$ is always parallel to both $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ and (2-2) is still satisfied thanks to Step III. For generic $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ we argue as follows. By symmetry, it is not restrictive to assume that $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ is vertical and the two lines with directions $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$ intersect in a point that is closer to $\boldsymbol{A}^{i+1}$ and $\boldsymbol{B}^{i+1}$ than $\boldsymbol{A}^i$ and $\boldsymbol{B}^i$ (as shown in Figure 3(b)). Let $\boldsymbol{S}$, $\boldsymbol{T}$, $\widetilde{\boldsymbol{T}}$ be three points such that $\boldsymbol{S}$ is on γ^{i+1} and $\boldsymbol{T}$, $\widetilde{\boldsymbol{T}}$ are on γ^i and $\boldsymbol{ST}$ is the shortest segment inside $\boldsymbol{\mathcal{D}}_i$ which is parallel to $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$ (notice that $\boldsymbol{S}$ can even happen to be $\boldsymbol{A}^{i+1}$ or $\boldsymbol{B}^{i+1}$) while $\widetilde{\boldsymbol{T}}$ is the intersection between γ^i and the half line starting from $\boldsymbol{S}$ with direction $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$. On one side, with the usual strategy, we select points $\boldsymbol{P}_j$, $\boldsymbol{R}_j$, with $\boldsymbol{P}_j$ between $\boldsymbol{A}^{i+1}$ and $\boldsymbol{S}$ and $\boldsymbol{R}_j$ between $\boldsymbol{A}^i$ and $\boldsymbol{T}$, so that $\boldsymbol{P}_j\boldsymbol{R}_j$ is parallel to $\boldsymbol{A}^i\boldsymbol{A}^{i+1}$. On the other side, we take $\boldsymbol{P}_j$, $\boldsymbol{R}_j$ with $\boldsymbol{P}_j$ between $\boldsymbol{S}$ and $\boldsymbol{B}^{i+1}$, $\boldsymbol{R}_j$ between $\widetilde{\boldsymbol{T}}$ and $\boldsymbol{B}^i$, and $\boldsymbol{P}_j\boldsymbol{R}_j$ parallel to $\boldsymbol{B}^i\boldsymbol{B}^{i+1}$. Finally, to any endpoint $\boldsymbol{R}_j$ of a linear piece of γ^i that happens to be between $\boldsymbol{T}$ and $\widetilde{\boldsymbol{T}}$, we associate $\boldsymbol{P}_j \equiv \boldsymbol{S}$. Notice that, by construction, estimate (2-2) is satisfied. We can then introduce the triangular grid on $\boldsymbol{\mathcal{D}}_i$ as the union of the boundaries of the triangles $\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{R}_j$ and $\boldsymbol{P}_{j+1}\boldsymbol{R}_j\boldsymbol{R}_{j+1}$ that are not degenerate. Moreover, recalling Step II, the polygons delimited by $\gamma^1\cup\boldsymbol{A}^1\boldsymbol{A}^0\cup\boldsymbol{A}^0\boldsymbol{B}^1$ and $\gamma^{k-1}\cup\boldsymbol{A}^{k-1}\boldsymbol{A}^k\cup\boldsymbol{A}^k\boldsymbol{B}^{k-1}$ are either triangles themselves or the union of two segments lying in $\partial\boldsymbol{\mathcal{Q}}$, thus we actually defined a triangular grid on the whole $\boldsymbol{\mathcal{Q}}$ and this concludes Step IV.

Step V. Triangular grid on $\mathcal{Q}$ and definition of $\tilde{h}$. We recall that our aim is to define a piecewise affine homeomorphism h mapping $\mathcal{Q}$ to $\boldsymbol{\mathcal{Q}}$ that matches with φ on $\partial\mathcal{Q}$ and satisfies estimate (1-1). In this step, we construct a function $\tilde{h}:\mathcal{Q}\to\boldsymbol{\mathcal{Q}}$ which is piecewise affine and coincides with φ on the boundary of $\mathcal{Q}$ and in Steps VI, VII we will prove that estimate (1-1) is satisfied by $\tilde{h}$. Let us already remark that in general $\tilde{h}$ will not be a one-to-one function, therefore we will have to suitably modify it later to obtain the homeomorphism h. We split the construction into three steps: first, we define the function $\tilde{h}$ only on $\partial\mathcal{Q}$ and on the segments A^iB^i, we then make use of the triangular grid of $\boldsymbol{\mathcal{Q}}$ defined in Step IV to find a triangular grid on $\mathcal{Q}$ and, finally, we use these triangular grids to extend $\tilde{h}$ on the whole $\mathcal{Q}$. We start by taking $\tilde{h}\equiv\varphi$ on $\partial\mathcal{Q}$ such that it maps all the horizontal segments A^iB^i to the respective path γ^i parametrized at constant speed. Notice that, in this way, $\tilde{h}$ is continuous and piecewise linear. Recalling the notation introduced in the previous step, we focus then on the polygon $\boldsymbol{\mathcal{D}}_i$: we associate to any $\boldsymbol{P}_j$ on γ^{i+1} the point P_j on $A^{i+1}B^{i+1}$ such that $\boldsymbol{P}_j=\tilde{h}(P_j)$, and to any $\boldsymbol{R}_j$ on γ^i the point R_j on A^iB^i such that $\boldsymbol{R}_j = \tilde{h}(R_j)$. For the sake of clarity we denote the endpoints $A^iA^{i+1}B^{i+1}B^i$ consistently with the notation, namely we call $P_0 \equiv A^{i+1}$, $P_N \equiv B^{i+1}$, $R_0 \equiv A^i$ and $R_N \equiv B^i$ for a suitable N. We can now define $\tilde{h}$ on the strip $\mathcal{D}_i := A^iA^{i+1} \cup A^{i+1}B^{i+1} \cup B^iB^{i+1} \cup A^iB^i$ as the function which is affine on each of the triangles $P_jP_{j+1}R_j$ or $P_{j+1}R_jR_{j+1}$ (notice that clearly $\tilde{h}$ can be degenerate on some triangles). In more detail, we define $\tilde{h}$ on $P_jP_{j+1}R_j$ as the affine function which maps $P_jP_{j+1}R_j$ onto $\boldsymbol{P}_j\boldsymbol{P}_{j+1}\boldsymbol{R}_j$ extending the values on the boundary (the very same definition is used for triangles of the form $P_{j+1}R_jR_{j+1}$). It still remains to define $\tilde{h}$ on the top and bottom triangles of $\mathcal{Q}$. Let us consider the bottom triangle $A^0A^1B^1$ (the definition is symmetric for $A^kA^{k-1}B^{k-1}$); then we know from Step II that γ^1 can be either a segment $\boldsymbol{A}^1\boldsymbol{B}^1$ or the union of two segments $\boldsymbol{A}^0\boldsymbol{A}^1$, $\boldsymbol{A}^0\boldsymbol{B}^1$ laying on $\partial\boldsymbol{\mathcal{Q}}$. In the first case we again define $\tilde{h}$ as the affine function that extends the values on the boundary. In the other case we will subdivide $A^0A^1B^1$ into two triangles A^0A^1P and A^0PB^1, where P is the point on A^1B^1 such that $\tilde{h}(P) = \boldsymbol{A}^0$, then we define $\tilde{h}$ to be constantly equal to $\boldsymbol{A}^0$ on the segment PA^0 and the degenerate affine function extending the values on the boundary on A^0A^1P and A^0PB^1. By construction, the function $\tilde{h}$ is piecewise affine, coincides with φ on $\partial\mathcal{Q}$ and it is also continuous.

Step VI. Estimate for $\int_{A^0A^1B^1}|D\tilde{h}|^p$. As mentioned above, this step and the following one are devoted to showing that the function $\tilde{h}$ satisfies the estimate (1-1). Here, in particular, we focus on the bottom triangle $\mathcal{T} := A^0A^1B^1$ (the very same argument holds also for the top triangle $A^kA^{k-1}B^{k-1}$), and we prove that

$$\int_{\mathcal{T}}|D\tilde{h}|^p \le K_1\int_{\partial\mathcal{Q}}|D\varphi|^p\, d\mathcal{H}^1\,, \tag{2-3}$$

where K_1 denotes a purely geometric constant. Recall that, by definition, $\tilde{h}(\mathcal{T})$ is either the nondegenerate triangle $\boldsymbol{A}^0\boldsymbol{A}^1\boldsymbol{B}^1$ or the union of the two segments $\boldsymbol{A}^1\boldsymbol{A}^0 \cup \boldsymbol{A}^0\boldsymbol{B}^1$. In the nondegenerate case, $D\tilde{h}$ is constant on $\mathcal{T}$, therefore we denote its constant value with $D^b\tilde{h}= (D_1^b\tilde{h},\, D_2^b\tilde{h})$. Moreover, by construction, $\tilde{h}(A^0A^1) = \boldsymbol{A}^0\boldsymbol{A}^1$ and $\tilde{h}(A^0B^1) = \boldsymbol{A}^0\boldsymbol{B}^1$, thus we can write the following relations:

$$\tfrac{\sqrt{2}}{2}|D_1^b\tilde{h} + D_2^b\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^0B^1)}\,, \qquad \tfrac{\sqrt{2}}{2}|-D_1^b\tilde{h} + D_2^b\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\mathcal{H}^1(A^0A^1)}\,,$$

which, in turn, imply that

$$|D^b\tilde{h}|^p \le \frac{2^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p + 2^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p}{\mathcal{H}^1(A^0A^1)^p}\,, \tag{2-4}$$

where we used the fact that $\mathcal{H}^1(A^0A^1) = \mathcal{H}^1(A^0B^1)$. On the other hand, we have that $|D\varphi|$ is constant on each of the segments A^0A^1, A^0B^1 and this gives us an estimate on $\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p$ and $\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p$, namely

$$\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p = \mathcal{H}^1(A^0A^1)^{p-1}\int_{A^0A^1}|D\varphi|^p\,d\mathcal{H}^1\,,$$
$$\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p = \mathcal{H}^1(A^0A^1)^{p-1}\int_{A^0B^1}|D\varphi|^p\,d\mathcal{H}^1\,.$$

By inserting both of these equations into (2-4) and using (2-1) with $r=\mathcal{H}^1(A^0A^1)$, one gets

$$|D^b\tilde{h}|^p \le \frac{2^p}{\mathcal{H}^1(A^0A^1)}\int_{B(A^0,\mathcal{H}^1(A^0A^1))\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1 \le 2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,.$$

Summarizing, in the nondegenerate case, we can finally deduce (2-3) from

$$\begin{aligned}\int_{\mathcal{T}}|D\tilde{h}|^p &= \frac{\mathcal{H}^1(A^0A^1)^2}{2}|D\tilde{h}|^p\\ &\le \frac{\mathcal{H}^1(A^0A^1)^2}{2}2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p d\mathcal{H}^1 \le 2^p\widetilde{K}\int_{\partial Q}|D\varphi|^p d\mathcal{H}^1.\end{aligned}$$

Now, let $\tilde{h}$ be a degenerate affine function on each of the two parts A^1PA^0 and A^0PB^1, where P satisfies $\tilde{h}(P)=\boldsymbol{A}^0$, as in Step V. Let us call $|D^l\tilde{h}|$ and $|D^r\tilde{h}|$ the constant values of $|D\tilde{h}|$ on the two parts. In this case the following relations hold:

$$|D_1^r\tilde{h}| = |D_1^l\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^1B^1)} = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\sqrt{2}\mathcal{H}^1(A^1A^0)}\,,$$
$$\tfrac{\sqrt{2}}{2}|D_2^l\tilde{h}-D_1^l\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\mathcal{H}^1(A^0A^1)}\,,$$
$$\tfrac{\sqrt{2}}{2}|D_1^r\tilde{h}+D_2^r\tilde{h}| = \frac{\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)}{\mathcal{H}^1(A^0B^1)}\,.$$

Therefore

$$|D_2^r\tilde{h}| \le \frac{3\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)+\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)}{\sqrt{2}\mathcal{H}^1(A^1A^0)}\,,$$

and in particular

$$\begin{aligned}|D^r\tilde{h}|^p &\le \frac{1}{(\sqrt{2}\mathcal{H}^1(A^1A^0))^p}\big(4^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{A}^1)^p+8^p\mathcal{H}^1(\boldsymbol{A}^0\boldsymbol{B}^1)^p\big)\\ &\le 2\frac{8^p}{\sqrt{2}^p}\frac{1}{\mathcal{H}^1(A^1A^0)}\int_{B(A^0,\mathcal{H}^1(A^1A^0))\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1\\ &\le 2\frac{8^p}{\sqrt{2}^p}\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,,\end{aligned}\tag{2-5}$$

where the last inequality is a consequence of Step I. An estimate for $|D^l\tilde{h}|^p$ analogous to (2-5) holds by a symmetric argument, then, as a consequence, one has

$$\begin{aligned}\int_{A^0A^1B^1}|D\tilde{h}|^p &= \int_{A^0PA^1}|D^l\tilde{h}|^p + \int_{A^0PB^1}|D^r\tilde{h}|^p \\ &\le \frac{\mathcal{H}^1(A^1A^0)^2}{2}(|D^r\tilde{h}|^p + |D^l\tilde{h}|^p) \le 2^{\frac{5}{2}p+1}\widetilde{K}\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 .\end{aligned}$$

Therefore (2-3) holds true for the degenerate case and also in general as soon as $K_1 \ge 2^{\frac{5}{2}p+1}\widetilde{K}$.

Step VII. Estimate for $\int_{\mathcal{Q}^-}|D\tilde{h}|^p$. In this step, we show that $\tilde{h}$ satisfies (1-1) also in $\mathcal{Q}^-$, which is the square $\mathcal{Q}$ without the top and the bottom triangles. Namely, we prove that

$$\int_{\mathcal{Q}^-}|D\tilde{h}|^p \le K_2\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 .$$

To do so, we need at first a similar estimate on a generic triangle $\mathcal{T}$ of the triangulation of $\mathcal{Q}$ which is inside $\mathcal{Q}^-$. To this end, let i be the index so that $\mathcal{T}$ is included in the polygon $\mathcal{D}_i := A_iB_i \cup A_{i+1}B_{i+1} \cup A_iA_{i+1} \cup B_iB_{i+1}$. We aim to show

$$\int_{\mathcal{T}}|D\tilde{h}|^p \le K'|\mathcal{T}|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + K''\frac{|\mathcal{T}|}{|A_2^{i+1}-A_2^i|}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1. \tag{2-6}$$

Let $\mathcal{T}$ be of the form $P_jP_{j+1}R_j$ (of course for the other triangles $P_{j+1}R_jR_{j+1}$ the very same argument can be applied) and, by symmetry, let us also assume that $\mathcal{D}_i$ lays below the x-axis. To simplify the notation, we denote r the distance between A^0 and the horizontal segment $A^{i+1}B^{i+1}$, and σ the distance between $A^{i+1}B^{i+1}$ and A^iB^i (which is equal to $|A_2^{i+1}-A_2^i|$ and to the height of $\mathcal{D}_i$). Since $\tilde{h}$ is affine on $\mathcal{T}$, we also denote by $|D^{\mathcal{T}}\tilde{h}|$ the constant value of $|D\tilde{h}|$ on $\mathcal{T}$. Arguing similarly to Step VI, we would like to estimate both the components $|D_1^{\mathcal{T}}\tilde{h}|$ and $|D_2^{\mathcal{T}}\tilde{h}|$. It follows by construction that

$$|D_1^{\mathcal{T}}\tilde{h}|^p = \frac{\mathcal{H}^1(\gamma^{i+1})^p}{(2r)^p} .$$

Since γ^{i+1} is defined to be the shortest path in $\overline{\boldsymbol{\mathcal{Q}}}$ connecting $\boldsymbol{A}^{i+1}$ to $\boldsymbol{B}^{i+1}$, then in particular it is shorter than the image through φ of the curve connecting A^{i+1} to B^{i+1} on $\partial\mathcal{Q}$ passing through A^0. Therefore, it satisfies the inequality

$$\mathcal{H}^1(\gamma^{i+1}) \le \int_{B(A^0,r\sqrt{2})\cap\partial\mathcal{Q}}|D\varphi|\,d\mathcal{H}^1 .$$

By Hölder's inequality and (2-1) we obtain

$$\mathcal{H}^1(\gamma^{i+1})^p \le (2r\sqrt{2})^{p/p'}\int_{B(A^0,r\sqrt{2})\cap\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 ,$$

and

$$(2\text{-}7)\quad |D_1^{\mathcal{T}}\tilde{h}|^p \le \frac{\sqrt{2}^{p/p'}}{2}\frac{1}{r}\int_{B(A^0,r\sqrt{2})\cap\partial Q}|D\varphi|^p\,d\mathcal{H}^1 \le \frac{\sqrt{2}^{p/p'}}{2}\widetilde{K}\int_{\partial Q}|D\varphi|^p\,d\mathcal{H}^1\,,$$

where p' is such that $\frac{1}{p}+\frac{1}{p'}=1$. Estimating $D_2^{\mathcal{T}}\tilde{h}$ is less straightforward. Let us call

$$d := \mathcal{H}^1(A^{i+1}P_j),\quad d' := \mathcal{H}^1(A^iQ_j)\,,\quad \ell := \max\{\mathcal{H}^1(\boldsymbol{A}^i\boldsymbol{A}^{i+1}),\mathcal{H}^1(\boldsymbol{B}^i\boldsymbol{B}^{i+1})\}\,.$$

Then we can write

$$(2\text{-}8)\qquad |(d'+\sigma-d)D_1^{\mathcal{T}}\tilde{h}+\sigma D_2^{\mathcal{T}}\tilde{h}| = \mathcal{H}^1(\boldsymbol{P}_j\boldsymbol{Q}_j)\le \ell\,,$$

and some geometrical considerations lead to an estimate of the term $|d-d'|\,|D_1^{\mathcal{T}}\tilde{h}|$. Indeed, the path γ^{i+1} is shorter than $\boldsymbol{A}^i\boldsymbol{A}^{i+1}\cup\gamma^i|_{A^i\boldsymbol{Q}_j}\cup\boldsymbol{Q}_j\boldsymbol{P}_j\cup\gamma^{i+1}|_{\boldsymbol{P}_j\boldsymbol{B}^{i+1}}$, providing that

$$\mathcal{H}^1(\gamma^{i+1})\le 2\ell+\mathcal{H}^1(\gamma^i)\frac{d'}{\mathcal{H}^1(A^i\boldsymbol{B}^i)}+\mathcal{H}^1(\gamma^{i+1})\left(1-\frac{d}{\mathcal{H}^1(A^{i+1}\boldsymbol{B}^{i+1})}\right),$$

which gives in particular

$$d\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^1(A^{i+1}\boldsymbol{B}^{i+1})}-d'\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^1(A^i\boldsymbol{B}^i)}\le 2\ell\,.$$

Since the symmetric argument involving γ^i gives the opposite inequality (this time we use that γ^i is shorter than $\boldsymbol{A}^i\boldsymbol{A}^{i+1}\cup\gamma^{i+1}|_{A^{i+1}\boldsymbol{P}_j}\cup\boldsymbol{P}_j\boldsymbol{Q}_j\cup\gamma^i|_{\boldsymbol{Q}_j\boldsymbol{B}^i}$), we get

$$\left|d\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^1(A^{i+1}\boldsymbol{B}^{i+1})}-d'\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^1(A^i\boldsymbol{B}^i)}\right|\le 2\ell\,.$$

Moreover, recalling that γ^{i+1} is parametrized at constant speed, it follows directly that

$$\begin{aligned}|d-d'|\,|D_1^{\mathcal{T}}\tilde{h}| &\le 2\ell+d'\left|\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^(A^{i+1}\boldsymbol{B}^{i+1})}-\frac{\mathcal{H}^1(\gamma^i)}{\mathcal{H}^(A^i\boldsymbol{B}^i)}\right|\\ &\le 2\ell+\left|\frac{\mathcal{H}^(A^i\boldsymbol{B}^i)}{\mathcal{H}^(A^{i+1}\boldsymbol{B}^{i+1})}\mathcal{H}^1(\gamma^{i+1})-\mathcal{H}^1(\gamma^i)\right|\\ &\le 2\ell+\left|\mathcal{H}^1(\gamma^{i+1})-\mathcal{H}^1(\gamma^i)\right|+2\sigma\frac{\mathcal{H}^1(\gamma^{i+1})}{\mathcal{H}^(A^{i+1}\boldsymbol{B}^{i+1})}\le 4\ell+2\sigma|D_1^{\mathcal{T}}\tilde{h}|.\end{aligned}$$

Inserting, then, the above estimate into (2-8), we get

$$|D_2^{\mathcal{T}}\tilde{h}|\le\frac{5}{\sigma}\ell+3|D_1^{\mathcal{T}}\tilde{h}|\,,$$

which in turn implies

$$|D_2^{\mathcal{T}}\tilde{h}|^p \leq \left(\frac{10l}{\sigma}\right)^p + 6^p|D_1^{\mathcal{T}}\tilde{h}|^p\,. \tag{2-9}$$

Let us notice that we can easily bound ℓ^p from above since φ is linear on A^iA^{i+1} and B^iB^{i+1}. Indeed, let ℓ be for instance equal to $\mathcal{H}^1(\boldsymbol{A}^i\boldsymbol{A}^{i+1})$; then

$$\ell^p = (\sigma\sqrt{2})^{p-1}\int_{A^iA^{i+1}}|D\varphi|^p\,d\mathcal{H}^1 \leq (\sigma\sqrt{2})^{p-1}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1\,, \tag{2-10}$$

where we used that $\mathcal{H}^1(A^iA^{i+1}) = \sigma\sqrt{2}$. Thus, by inserting (2-10) and (2-7) into (2-9) one gets

$$|D_2^{\mathcal{T}}\tilde{h}|^p \leq 6^p\frac{\sqrt{2}^{p/p'}}{2}\widetilde{K}\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + \frac{(10\sqrt{2})^p}{2}\frac{1}{\sigma}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1\,,$$

which, together with (2-7), gives (2-6) with $K' = 6^p\frac{1}{2}\sqrt{2}^{p/p'}\widetilde{K}$ and $K'' = \frac{1}{2}(10\sqrt{2})^p$. Moreover, by summing up among all the triangles $\mathcal{T}$ in $\mathcal{D}_i$ and observing that $|\mathcal{D}_i| \leq 2\sigma$ by construction, we have

$$\begin{aligned}\int_{\mathcal{D}_i}|D\tilde{h}|^p &\leq K'|\mathcal{D}_i|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + K''\frac{|\mathcal{D}_i|}{\sigma}\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1\\ &\leq K'|\mathcal{D}_i|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + 2K''\int_{A^iA^{i+1}\cup B^iB^{i+1}}|D\varphi|^p\,d\mathcal{H}^1\,.\end{aligned}$$

Finally, on the whole $\mathcal{Q}^-$ one gets

$$\begin{aligned}\int_{\mathcal{Q}^-}|D\tilde{h}|^p &\leq K'|\mathcal{Q}^-|\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1 + 2K''\int_{\partial\mathcal{Q}^-\cap\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1\\ &\leq K_2\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1\,,\end{aligned} \tag{2-11}$$

for a suitable $K_2 \geq 2\max\{K', K''\}$.

Step VIII. *Definition of h and conclusion.* We can now observe that, whenever $\tilde{h}$ is a homeomorphism, Theorem 1.1 follows directly. Indeed, $\tilde{h}$ coincides with φ on $\partial\mathcal{Q}$, it is finitely piecewise affine and, moreover, the estimates (2-3) and (2-11) provide that (1-1) is satisfied by $\tilde{h}$ and $K \geq \max\{K_1, K_2\}$. Unfortunately, in our construction the function $\tilde{h}$ happens to be one-to-one only when all the paths γ^i lie in the interior of $\boldsymbol{\mathcal{Q}}$ without intersecting each other. Of course in general this is not the case, but it is always possible to slightly modify the paths γ^i in order to get the one-to-one property. A possible configuration can be seen Figure 4. More precisely, it is always possible to separate a curve γ^{i+1} from either $\partial\boldsymbol{\mathcal{Q}}$ and γ^i so that the minimal distance between them is much smaller than the lengths of all the linear pieces of the paths γ and $\partial\boldsymbol{\mathcal{Q}}$. Let us notice that the minimal distance is

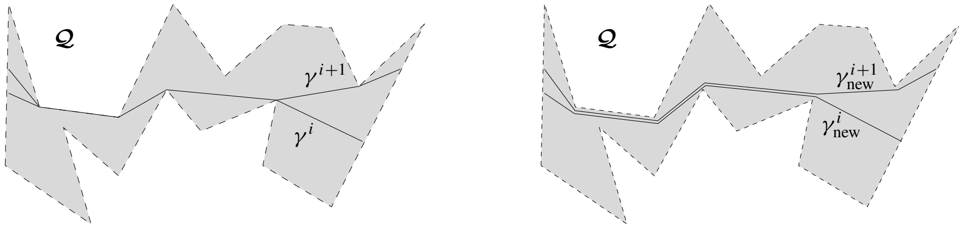


Figure 4. Modification of γ^i and γ^{i+1}.

strictly positive because there is only a finite number of paths and each of them is finitely piecewise linear. We finally define the function h in the very same way as we defined $\tilde{h}$ in Step V, but this time using the separated paths. Therefore, the function h is a homeomorphism, it is still finitely piecewise affine, it satisfies the boundary condition on $\partial\mathcal{Q}$ and, furthermore, estimate (1-1) is still valid up to increasing the geometric constant K by a quantity which is as small as we wish. This implies the validity of Theorem 1.1 and concludes the proof. $\square$

Remark 2.1. Let $\mathcal{Q}\subset\mathbb{R}^2$ be a generic square of length side r, $p\geq 1$ and $\varphi:\partial\mathcal{Q}\to\mathbb{R}^2$ piecewise linear. Then there exists a piecewise affine function $h:\mathcal{Q}\to\mathbb{R}^2$ that coincides with φ on the boundary $\partial\mathcal{Q}$ and a geometric constant K depending only on p such that

$$\int_{\mathcal{Q}}|Dh|^p \leq Kr\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1\,. \tag{2-12}$$

Indeed, there always exists an affine function ρ mapping the unit square $\mathcal{Q}_1$ onto $\mathcal{Q}$. Let us call, with a slight abuse of notation, ρ and its restriction to the boundary $\partial\mathcal{Q}_1$ with the same name. Then, by applying Theorem 1.1 to the function $\varphi\circ\rho$ and recalling that $|D\rho|=r$, it is possible to find a constant K and a piecewise affine function $\tilde{h}:\mathcal{Q}_1\to\mathbb{R}^2$ satisfying

$$\int_{\mathcal{Q}_1}|D\tilde{h}(x)|^p\,dx \leq K\int_{\partial\mathcal{Q}_1}r^p|D\varphi(\rho(t))|^p\,d\mathcal{H}^1(t)\,.$$

Finally, by defining $h:=\tilde{h}\circ\rho^{-1}$ and changing the variables, one gets

$$r^{p-2}\int_{\mathcal{Q}}|Dh|^p \leq r^{p-1}K\int_{\partial\mathcal{Q}}|D\varphi|^p\,d\mathcal{H}^1$$

and (2-12) follows.

References

[Bellido and Mora-Corral 2011] J. C. Bellido and C. Mora-Corral, "Approximation of Hölder continuous homeomorphisms by piecewise affine homeomorphisms", *Houston J. Math.* **37**:2 (2011), 449–500. MR 2794559 Zbl 1228.57009

[Campbell 2015] D. Campbell, "Diffeomorphic approximation of planar Sobolev homeomorphisms in Orlicz–Sobolev spaces", preprint, 2015, available at http://tinyurl.com/grptrrc.

[Daneri and Pratelli 2015] S. Daneri and A. Pratelli, "A planar bi-Lipschitz extension theorem", *Adv. Calc. Var.* **8**:3 (2015), 221–266. MR 3365742 Zbl 1331.26020

[Hencl and Pratelli 2015] S. Hencl and A. Pratelli, "Diffeomorphic approximation of $W^{1,1}$ planar Sobolev homeomorphisms", preprint, 2015, available at http://arxiv.org/abs/1502.07253.

[Iwaniec et al. 2011] T. Iwaniec, L. V. Kovalev, and J. Onninen, "Diffeomorphic approximation of Sobolev homeomorphisms", *Arch. Ration. Mech. Anal.* **201**:3 (2011), 1047–1067. MR 2824471 Zbl 1260.46023

[Mora-Corral 2009] C. Mora-Corral, "Approximation by piecewise affine homeomorphisms of Sobolev homeomorphisms that are smooth outside a point", *Houston J. Math.* **35**:2 (2009), 515–539. MR 2519545 Zbl 1182.57019

Received August 26, 2015. Revised November 25, 2015.

EMANUELA RADICI
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ERLANGEN
CAUERSTRASSE 11
D-90158 ERLANGEN
GERMANY
radici@math.fau.de

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.419

A COMBINATORIAL APPROACH TO VOICULESCU'S BI-FREE PARTIAL TRANSFORMS

PAUL SKOUFRANIS

We present a combinatorial approach to the 2-variable bi-free partial S- and T-transforms recently discovered by Voiculescu. This approach produces an alternate definition of said transforms using (l, r)-cumulants.

1. Introduction

Voiculescu [2014] introduced the notion of bi-free pairs of faces as a means to simultaneously study left and right actions of algebras on reduced free product spaces. Substantial work has been performed since then in order to better understand bi-freeness and its applications [Charlesworth et al. 2015a; 2015b; Skoufranis 2015; Voiculescu 2016; Mastnak and Nica 2015; Gu et al. 2015]. Specifically, the results of [Voiculescu 1986] were generalized to the bi-free setting in [Voiculescu 2016] through the development of a 2-variable bi-free partial R-transform using analytic techniques. A combinatorial construction of the bi-free partial R-transform was given in [Skoufranis 2015] using results from [Charlesworth et al. 2015b].

Along similar lines, modifying his S-transform introduced in [Voiculescu 1987], Voiculescu [2015] associated to a pair (a, b) of operators in a noncommutative probability space a 2-variable bi-free partial S-transform, denoted by $S_{a,b}(z, w)$. Using ideas from [Haagerup 1997], he demonstrated that if (a_1, b_1) and (a_2, b_2) are bi-free then

$$S_{a_1a_2,b_1b_2}(z, w) = S_{a_1,b_1}(z, w)S_{a_2,b_2}(z, w). \tag{1}$$

He also constructed a 2-variable bi-free partial T-transform $T_{a,b}(z, w)$ to study the convolution product where additive convolution is used for the left variables and multiplicative convolution is used for the right variables. In particular, the defining characteristic of $T_{a,b}(z, w)$ is that if (a_1, b_1) and (a_2, b_2) are bi-free then

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w). \tag{2}$$

MSC2010: 46L53, 46L54.

Keywords: bi-free probability, partial bi-free S-transform, partial bi-free T-transform, bi-free convolutions.

The goal of this paper is to provide a combinatorial proof of the results of [Voiculescu 2015]. The paper is structured as follows. Section 2 establishes all preliminary results, background, and notation necessary for the remainder of the paper. A reader would benefit greatly from knowledge of the combinatorial approach to the free S-transform from [Nica and Speicher 1997] and knowledge of the combinatorial approach to bi-freeness from [Charlesworth et al. 2015b] (or the summary in [Charlesworth et al. 2015a]). Section 3 provides an equivalent description of $T_{a,b}(z, w)$ using (l, r)-cumulants and provides a combinatorial proof of equation (2). Section 4 provides an equivalent description of $S_{a,b}(z, w)$ using (l, r)-cumulants and provides a combinatorial proof of equation (1).

An intriguing question arises in taking products of bi-free pairs of operators: is the "correct" multiplication to use on the right pair of algebras the usual one or its opposite? In other words, if (a_1, b_1) and (a_2, b_2) are bi-free pairs of operators, which product should be used, (a_1a_2, b_1b_2) or (a_1a_2, b_2b_1)? It is not difficult to see that the resulting distributions can be different; see [Charlesworth et al. 2015a]. Further, by Theorem 5.2.1 of [Charlesworth et al. 2015b] the (l, r)-cumulants of (a_1a_2, b_2b_1) can be computed via a convolution product of the (l, r)-cumulants of (a_1, b_1) and (a_2, b_2) involving a bi-noncrossing Kreweras complement, just as in the free case. However, the product of Voiculescu's bi-free partial S-transforms of (a_1, b_1) and (a_2, b_2) is the bi-free partial S-transform of (a_1a_2, b_1b_2). As we will see in Section 4, this is not just a matter of differences in notation and therefore one needs to carefully consider which product to use.

2. Background and preliminaries

In this section, we recall the necessary background required for this paper. We refer the reader to the summary in [Charlesworth et al. 2015a, Section 2] for more background on scalar-valued bi-free probability. This section also serves the purpose of setting notation for the remainder of the paper, which we endeavour to make consistent with [Voiculescu 2015]. We treat all series as formal power series, with commuting variables in the multivariate cases.

2.1. ***Free transforms.*** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space (that is, a unital algebra $\mathcal{A}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(I) = 1$) and let $a \in \mathcal{A}$. The Cauchy transform of a is

$$G_a(z) := \varphi((zI - a)^{-1}) = \frac{1}{z} \sum_{n\geq 0} \varphi(a^n) z^{-n},$$

and the moment series of a is

$$h_a(z) := \varphi((I - az)^{-1}) = \sum_{n\geq 0} \varphi(a^n) z^n = \frac{1}{z} G_a\Big(\frac{1}{z}\Big).$$

Recall one defines $K_a(z)$ to be the inverse of $G_a(z)$ in a neighbourhood of 0 so that $G_a(K_a(z)) = z$. Thus $R_a(z) := K_a(z) - \frac{1}{z}$ is the R-transform of a and

$$h_a\left(\frac{1}{K_a(z)}\right) = K_a(z)G_a(K_a(z)) = zK_a(z). \tag{3}$$

Furthermore, if $\kappa_n(a)$ denotes the n-th free cumulant of a and the cumulant series of a is

$$c_a(z) := \sum_{n\geq 1} \kappa_n(a)z^n,$$

then one can verify that

$$1 + c_a(z) = zK_a(z). \tag{4}$$

To define the S-transform of a, we assume $\varphi(a) \neq 0$ and let $\psi_a(z) := h_a(z) - 1$. Since $\psi_a(0) = 0$ and $\psi_a'(z) = \varphi(a) \neq 0$, $\psi_a(z)$ has a formal power series inverse under composition, denoted $\psi_a^{\langle -1\rangle}(z)$. We define $\mathcal{X}_a(z) := \psi_a^{\langle -1\rangle}(z)$ so that

$$h_a(\mathcal{X}_a(z)) = 1 + \psi_a(\mathcal{X}_a(z)) = 1 + z. \tag{5}$$

The S-transform of a is then defined to be

$$S_a(z) := \frac{1+z}{z}\mathcal{X}_a(z). \tag{6}$$

2.2. ***Free multiplicative functions and convolution.*** Let $\mathrm{NC}(n)$ denote the lattice of noncrossing partitions on $\{1, \dots, n\}$ with its usual refinement order, let 0_n denote the minimal element of $\mathrm{NC}(n)$, and let $1_n = \{1, 2, \dots, n\}$ denote the maximal element of $\mathrm{NC}(n)$. For $\pi, \sigma \in \mathrm{NC}(n)$ with $\pi \leq \sigma$, the interval between π and σ, denoted $[\pi, \sigma]$, is the set

$$[\pi, \sigma] = \{\rho \in \mathrm{NC}(n) \mid \pi \leq \rho \leq \sigma\}.$$

A procedure is described in [Speicher 1994] which decomposes each interval of noncrossing partitions into a product of full partitions of the form

$$[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$$

where $k_j \geq 0$.

The incidence algebra of noncrossing partitions, denoted $\mathcal{I}(NC)$, is the algebra of all functions

$$f : \bigcup_{n\geq 1} \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$$

such that $f(\pi, \sigma) = 0$ unless $\pi \leq \sigma$, equipped with pointwise addition and a convolution product defined by

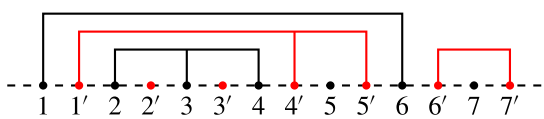
$$(f * g)(\pi, \sigma) := \sum_{\rho\in[\pi,\sigma]} f(\pi, \rho)g(\rho, \sigma).$$

Recall $f \in \mathcal{I}(\mathrm{NC})$ is called multiplicative if whenever $[\pi, \sigma]$ has a canonical decomposition $[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$, then

$$f(\pi, \sigma) = f(0_1, 1_1)^{k_1} f(0_2, 1_2)^{k_2} f(0_3, 1_3)^{k_3} \cdots .$$

Thus the value of a multiplicative function f on any pair of noncrossing partitions is completely determined by the values of f on full noncrossing partition lattices. We will denote the set of all multiplicative functions by $\mathcal{M}$ and the set all multiplicative functions f with $f(0_1, 1_1) = 1$ by $\mathcal{M}_1$.

If $f, g \in \mathcal{M}$, one can verify that $f * g = g * f$. Furthermore, there is a nicer expression for convolution of multiplicative functions. Given a noncrossing partition $\pi \in \mathrm{NC}(n)$, the Kreweras complement of π, denoted $K(\pi)$, is the noncrossing partition on $\{1, \dots, n\}$ with noncrossing diagram obtained by drawing π via the standard noncrossing diagram on $\{1, \dots, n\}$, placing nodes $1', 2', \dots, n'$ with k' directly to the right of k, and drawing the largest noncrossing partition on $1', 2', \dots, n'$ that does not intersect π, which is then $K(\pi)$. The diagram below exhibits that if $\pi = \{\{1, 6\}, \{2, 3, 4\}, \{5\}, \{7\}\}$, then $K(\pi) = \{\{1, 4, 5\}, \{2\}, \{3\}, \{6, 7\}\}$.

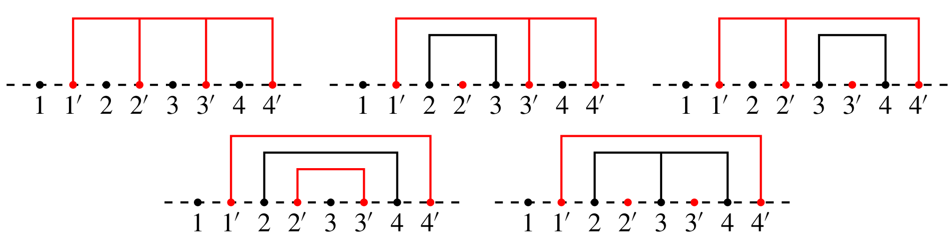


For $f, g \in \mathcal{M}$, convolution may be written as

$$(f * g)(0_n, 1_n) = \sum_{\pi \in \mathrm{NC}(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

Note that [Nica and Speicher 1997] demonstrated that if $a, b \in \mathcal{A}$ are free and if f (respectively g) is the multiplicative function associated to the cumulants of a (respectively b) defined by $f(0_n, 1_n) = \kappa_n(a)$ (respectively $g(0_n, 1_n) = \kappa_n(b)$), then $\kappa_n(ab) = \kappa_n(ba) = (f * g)(0_n, 1_n)$. Furthermore, for $\pi \in \mathrm{NC}(n)$ with blocks $\{V_k\}_{k=1}^m$, we have $f(0_n, \pi) = \kappa_\pi(a) = \prod_{k=1}^m \kappa_{|V_k|}(a)$.

Another convolution product on $\mathcal{M}_1$ from [loc. cit.] is required. Let $\mathrm{NC}'(n)$ denote all noncrossing partitions π on $\{1, \dots, n\}$ such that $\{1\}$ is a block in π. It is not difficult to construct a natural isomorphism between $\mathrm{NC}'(n)$ and $\mathrm{NC}(n-1)$. The following diagrams illustrate all elements $\mathrm{NC}'(4)$, together with their Kreweras complements.



We desire to make an observation, which may be proved by induction. Given two noncrossing partitions π and σ, let $\pi \vee \sigma$ denote the smallest noncrossing partition larger than both π and σ. Fix $\pi \in \mathrm{NC}'(n)$. If σ is the noncrossing partition on $\{1, 1', 2, 2', \dots, n, n'\}$ (with the ordering being the order of listing) with blocks $\{k, k'\}$ for all k, then the only noncrossing partition τ on $\{1', \dots, n'\}$ such that $\pi \cup \tau$ is noncrossing (under the ordering $1, 1', 2, 2', \dots, n, n'$) and $(\pi \cup \tau) \vee \sigma = 1_{2n}$ is $\tau = K(\pi)$.

For $f, g \in \mathcal{M}_1$, the "pinched-convolution" of f and g, denoted $f \check{*} g$, is the unique element of $\mathcal{M}_1$ such that

$$(f \check{*} g)[0_n, 1_n] := \sum_{\pi \in \mathrm{NC}'(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

The pinched-convolution product is not commutative on $\mathcal{M}_1$.

Given an element $f \in \mathcal{M}$, define the formal power series

$$\phi_f(z) := \sum_{n \geq 1} f(0_n, 1_n) z^n.$$

In particular, if f is the multiplicative function associated to the cumulants of a defined by $f(0_n, 1_n) = \kappa_n(a)$, then $\phi_f(z) = c_a(z)$. Several formulae involving $\phi_f(z)$ are developed in [Nica and Speicher 1997]. In particular, [loc. cit., Proposition 2.3] demonstrates that if $f, g \in \mathcal{M}_1$ then $\phi_f(\phi_{f \check{*} g}(z)) = \phi_{f * g}(z)$ and thus

$$\phi_{f \check{*} g}\big(\phi_{f * g}^{\langle -1 \rangle}(z)\big) = \phi_f^{\langle -1 \rangle}(z). \tag{7}$$

Furthermore, [loc. cit., Theorem 1.6] demonstrates that

$$z \cdot \phi_{f \check{*} g}^{\langle -1 \rangle}(z) = \phi_f^{\langle -1 \rangle}(z) \phi_g^{\langle -1 \rangle}(z). \tag{8}$$

An immediate consequence of equation (8) is that if $\varphi(a) = 1$ then

$$S_a(z) = \frac{1}{z} c_a^{\langle -1 \rangle}(z). \tag{9}$$

2.3. ***Bi-freeness.*** For a map $\chi : \{1, \dots, n\} \to \{l, r\}$, the set of bi-noncrossing partitions on $\{1, \dots, n\}$ associated to χ is denoted by $\mathrm{BNC}(\chi)$. Note $\mathrm{BNC}(\chi)$ becomes a lattice where $\pi \leq \sigma$ provided every block of π is contained in a single block of σ. The largest partition in $\mathrm{BNC}(\chi)$, which is $\{\{1, \dots, n\}\}$, is denoted by 1_χ. The work in [Charlesworth et al. 2015b] demonstrates that $\mathrm{BNC}(\chi)$ is naturally isomorphic to $\mathrm{NC}(n)$ via a permutation of $\{1, \dots, n\}$ induced by χ.

The (l, r)-cumulant associated to a map $\chi : \{1, \dots, n\} \to \{l, r\}$, given elements $\{a_n\}_{n=1}^n \subseteq \mathcal{A}$, was defined in [Mastnak and Nica 2015] and is denoted by $\kappa_\chi(a_1, \dots, a_n)$. Note κ_χ is linear in each entry. The main result of [Charlesworth et al. 2015b] is that if (a_1, b_1) and (a_2, b_2) are bi-free two-faced pairs in $(\mathcal{A}, \varphi)$, $\chi : \{1, \dots, n\} \to \{l, r\}$, $\epsilon : \{1, \dots, n\} \to \{l, r\}$, $c_{l,k} = a_k$, and $c_{r,k} = b_k$, then

$$\kappa_\chi(c_{\chi(1),\epsilon(1)}, \dots, c_{\chi(n),\epsilon(n)}) = 0$$

whenever ϵ is not constant.

Given a $\pi \in \mathrm{BNC}(\chi)$, each block B of π corresponds to the bi-noncrossing partition 1_{χ_B} for some $\chi_B : B \to \{l, r\}$ (where the ordering on B is induced from $\{1, \dots, n\}$). We write

$$\kappa_\pi(a_1, \dots, a_n) = \prod_{B \text{ a block of } \pi} \kappa_{1_{\chi_B}}((a_1, \dots, a_n)|_B),$$

where $(a_1, \dots, a_n)|_B$ denotes the $|B|$-tuple with indices not in B removed. Similarly, if V is a union of blocks of π, we denote by $\pi|_V$ the bi-noncrossing partition obtained by restricting π to V.

For $n, m \geq 0$, we often consider the maps $\chi_{n,m} : \{1, \dots, n+m\} \to \{l, r\}$ such that $\chi(k) = l$ if $k \leq n$ and $\chi(k) = r$ if $k > n$. For notational purposes, it is useful to think of $\chi_{n,m}$ as a map on $\{1_l, 2_l, \dots, n_l, 1_r, 2_r, \dots, m_r\}$ under the identification $k \mapsto k_l$ if $k \leq n$ and $k \mapsto (k-n)_r$ if $k > n$. Furthermore, we write $\mathrm{BNC}(n, m)$ for $\mathrm{BNC}(\chi_{n,m})$, $1_{n,m}$ for $1_{\chi_{n,m}}$, and, for $n, m \geq 1$, $\kappa_{n,m}(a_1, \dots, a_n, b_1, \dots, b_m)$ for $\kappa_{1_{n,m}}(a_1, \dots, a_n, b_1, \dots, b_m)$. Finally, for $n, m \geq 1$, we set $\kappa_{n,m}(a, b) = \kappa_{1_{n,m}}(a, b)$, $\kappa_{n,0}(a, b) = \kappa_n(a)$, and $\kappa_{0,m}(a, b) = \kappa_n(b)$.

2.4. ***Bi-free transforms.*** Given two elements $a, b \in \mathcal{A}$, we define the ordered joint moment and cumulant series of the pair (a, b) to be

$$H_{a,b}(z, w) := \sum_{n,m\geq 0} \varphi(a^n b^m) z^n w^m \quad \text{and} \quad C_{a,b}(z, w) := \sum_{n,m\geq 0} \kappa_{n,m}(a, b) z^n w^m,$$

respectively (where $\kappa_{0,0}(a, b) = 1$). Note [Skoufranis 2015, Theorem 7.2.4] demonstrates that

$$h_a(z) + h_b(w) = \frac{h_a(z) h_b(w)}{H_{a,b}(z, w)} + C_{a,b}(z h_a(z), w h_b(w)) \tag{10}$$

through combinatorial techniques. It is also demonstrated that (10) is equivalent to Voiculescu's [2016] 2-variable bi-free partial R-transform.

For computational purposes, it is helpful to consider the series

$$K_{a,b}(z, w) := \sum_{n,m\geq 1} \kappa_{n,m}(a, b) z^n w^m = C_{a,b}(z, w) - c_a(z) - c_b(w) - 1. \tag{11}$$

Also of use are the series

$$F_{a,b}(z,w) := \varphi((zI-a)^{-1}(1-wb)^{-1}) = \frac{1}{z}\sum_{n,m\geq 0}\varphi(a^n b^m)z^{-n}w^m = \frac{1}{z}H_{a,b}\Big(\frac{1}{z},w\Big). \tag{12}$$

2.5. ***Bi-free cumulants of products.*** Of paramount importance to this paper is the ability to write (l,r)-cumulants of products as sums of (l,r)-cumulants. We recall a result from [Charlesworth et al. 2015a, Section 9].

Let $m, n \geq 1$ with $m < n$. Fix a sequence of integers

$$k(0) = 0 < k(1) < \cdots < k(m) = n.$$

For $\chi : \{1,\ldots,m\} \to \{l,r\}$, define $\hat{\chi} : \{1,\ldots,n\} \to \{l,r\}$ via

$$\hat{\chi}(q) = \chi(p_q),$$

where p_q is the unique element of $\{1,\ldots,m\}$ such that $k(p_q-1) < q \leq k(p_q)$.

There exists an embedding of $\mathrm{BNC}(\chi)$ into $\mathrm{BNC}(\hat{\chi})$ via $\pi \mapsto \hat{\pi}$ where the p-th node of π is replaced by the block $\{k(p-1)+1,\ldots,k(p)\}$. It is easy to see that $\widehat{1_\chi} = 1_{\hat{\chi}}$ and $\widehat{0_\chi}$ is the partition with blocks $\{\{k(p-1)+1,\ldots,k(p)\}\}_{p=1}^m$. Given two partitions $\pi,\sigma \in \mathrm{BNC}(\chi)$, let $\pi\vee\sigma$ denote the smallest element of $\mathrm{BNC}(\chi)$ greater than π and σ.

Using ideas from [Nica and Speicher 2006, Theorem 11.12], [Charlesworth et al. 2015a, Theorem 9.1.5] showed that if $\{a_k\}_{k=1}^n \subseteq \mathcal{A}$, then

$$\kappa_{1_\chi}(a_1\cdots a_{k(1)}, a_{k(1)+1}\cdots a_{k(2)}, \ldots, a_{k(m-1)+1}\cdots a_{k(m)}) = \sum_{\substack{\sigma\in\mathrm{BNC}(\hat{\chi})\\ \sigma\vee\widehat{0_\chi}=1_{\hat{\chi}}}} \kappa_\sigma(a_1,\ldots,a_n). \tag{13}$$

3. Bi-free partial T-transform

We begin with Voiculescu's bi-free partial T-transform, as the combinatorics are slightly simpler than the bi-free partial S-transform.

Definition 3.1 [Voiculescu 2015, Definition 3.1]. Let (a,b) be a two-faced pair in a noncommutative probability space $(\mathcal{A},\varphi)$ with $\varphi(b)\neq 0$. The 2-variable partial bi-free T-transform of (a,b) is the holomorphic function on $(\mathbb{C}\setminus\{0\})^2$ near $(0,0)$ defined by

$$T_{a,b}(z,w) = \frac{w+1}{w}\Big(1 - \frac{z}{F_{a,b}(K_a(z),\mathcal{X}_b(w))}\Big). \tag{14}$$

It is useful to note the following equivalent definition of the bi-free partial T-transform. To simplify the discussion, we show the equality in the case $\varphi(b)=1$.

This does not hinder the proof of the desired result, namely Theorem 3.5 (see Remark 3.3).

Proposition 3.2. *If* (a, b) *is a two-faced pair in a noncommutative probability space* $(\mathcal{A}, \varphi)$ *with* $\varphi(b) = 1$, *then*, *as formal power series*,

$$T_{a,b}(z, w) = 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big). \tag{15}$$

Proof. Using equations (3), (5), and (10), we obtain that

$$\frac{1}{H_{a,b}(1/K_a(z), \mathcal{X}_b(w))} = \frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)} \frac{1}{1+w} C_{a,b}(z, (1+w)\mathcal{X}_b(w)).$$

Therefore, using equations (6), (9), (11), (12), and (14), we obtain that

$$\begin{aligned}
T_{a,b}&(z, w)\\
&= \frac{w+1}{w}\left(1 - \frac{z}{(1/K_a(z))H_{a,b}(1/K_a(z), \mathcal{X}_b(w))}\right)\\
&= \frac{w+1}{w}\left(1 - zK_a(z)\Big(\frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)}\frac{1}{1+w} C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\Big)\right)\\
&= \frac{1}{w}\big(-zK_a(z) + C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&= \frac{1}{w}\big(-zK_a(z) + 1 + c_a(z) + c_b\big(c_b^{\langle -1\rangle}(w)\big) + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&= \frac{1}{w}\big(w + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&= 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).
\end{aligned}$$

□

Remark 3.3. One might be concerned that we have restricted to the case $\varphi(b) = 1$. However, if we use (15) as the definition of the bi-free partial T-transform and if $\lambda \in \mathbb{C} \setminus \{0\}$, then $T_{a,b}(z, w) = T_{a,\lambda b}(z, w)$. Indeed, $c_{\lambda b}(w) = c_b(\lambda w)$, so we have $c_{\lambda b}^{\langle -1\rangle}(w) = \frac{1}{\lambda} c_b^{\langle -1\rangle}(w)$. Therefore, since $\kappa_{n,m}(a, \lambda b) = \lambda^m \kappa_{n,m}(a, b)$, we see that

$$K_{a,\lambda b}\big(z, c_{\lambda b}^{\langle -1\rangle}(w)\big) = K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).$$

Thus there is no loss in assuming $\varphi(b) = 1$.

Remark 3.4. Note that Proposition 3.2 immediately provides the T-transform portion of [Voiculescu 2015, Proposition 4.2]. Indeed if a and b are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b) \neq 0$ and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $T_{a,b}(z, w) = 1$.

We desire to prove the following theorem (which was one of two main results of [Voiculescu 2015]) using combinatorial techniques and Proposition 3.2.

Theorem 3.5 [Voiculescu 2015, Theorem 3.1]. *Let (a_1, b_1) and (a_2, b_2) be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b_1) \neq 0$ and $\varphi(b_2) \neq 0$. Then*

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w)$$

on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.

To simplify the proof of the result, we assume that $\varphi(b_1) = \varphi(b_2) = 1$. Note that $\varphi(b_1b_2) = 1$ by freeness of the right algebras in bi-free pairs. Furthermore, let g_j denote the multiplicative function associated to the cumulants of b_j defined by $g_j(0_n, 1_n) = \kappa_n(b_j)$. Recall that if g is the multiplicative function associated to the cumulants of b_1b_2, then $g = g_1 * g_2$. Therefore $\phi_g^{\langle -1\rangle}(w) = c_{b_1b_2}^{\langle -1\rangle}(w)$ and $\phi_{g_j}^{\langle -1\rangle}(w) = c_{b_j}^{\langle -1\rangle}(w)$. Note that $g, g_1, g_2 \in \mathcal{M}_1$ by assumption.

By Proposition 3.2 it suffices to show that

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1}{w}\Theta_1(z, w)\Theta_2(z, w), \tag{16}$$

where

$$\Theta_j(z, w) = K_{a_j,b_j}\big(z, \phi_{g_j}^{\langle -1\rangle}(w)\big).$$

Recall

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1} \kappa_{n,m}(a_1 + a_2, b_1b_2)z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m)$ with blocks

$$\{\{k_l\}\}_{k=1}^n \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^m.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1{+}a_2, b_1b_2) = \sum_{\substack{\pi\in\mathrm{BNC}(n,2m)\\ \pi\vee\sigma_{n,m}=1_{n,2m}}} \kappa_\pi\big(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}\big).$$

Notice that if $\pi \in \mathrm{BNC}(n, 2m)$ and $\pi \vee \sigma_{n,m} = 1_{n,2m}$, then any block of π containing a k_l must contain a j_r for some j. Furthermore, if $1 \leq k < j \leq n$ are such that k_l and j_l are in the same block of π, then q_l must be in the same block as k_l for all $k \leq q \leq j$. Moreover, since (a_1, b_1) and (a_2, b_2) are bi-free, we note that

$$\kappa_\pi\big(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}\big) = 0$$

if π contains a block containing a $(2k)_r$ and a $(2j-1)_r$ for some k, j.

For $n, m \geq 1$, let $\mathrm{BNC}_T(n, m)$ denote all $\pi \in \mathrm{BNC}(n, 2m)$ such that

$$\pi \vee \sigma_{n,m} = 1_{n,2m}$$

and π contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for some k, j. Consequently, we obtain

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in \mathrm{BNC}_T(n,m)} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_T(n, m)$. Let $\mathrm{BNC}_T(n, m)_e$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing 1_l also contains a $(2k)_r$ for some k, and let $\mathrm{BNC}_T(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing 1_l also contains a $(2k-1)_r$ for some k. Note that $\mathrm{BNC}_T(n, m)_e$ and $\mathrm{BNC}_T(n, m)_o$ are disjoint and

$$\mathrm{BNC}_T(n, m)_e \cup \mathrm{BNC}_T(n, m)_o = \mathrm{BNC}_T(n, m)$$

by previous discussions. Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z, w) := \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in \mathrm{BNC}_T(n,m)_d} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m,$$

then

$$K_{a_1+a_2,b_1b_2}(z, w) = \Psi_e(z, w) + \Psi_o(z, w).$$

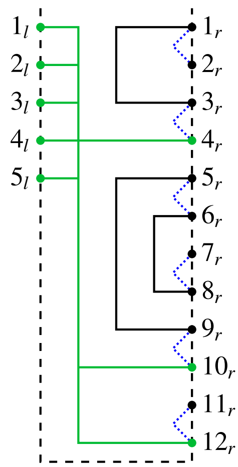
We derive expressions for $\Psi_e(z, w)$ and $\Psi_o(z, w)$ beginning with $\Psi_e(z, w)$.

Lemma 3.6. *Under the above notation and assumptions*,

$$\Psi_e(z, w) = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_e(z, w)$ by expanding κ_π as a product of full (l, r)-cumulants and summing over all π with the same block containing 1_l.

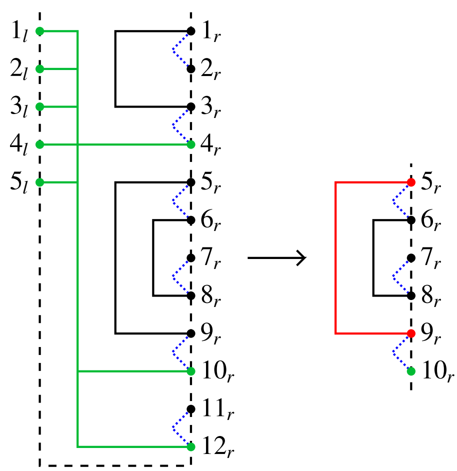
Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_T(n, m)_e$, then the block V_π containing 1_l must also contain $(2k)_r$ for some k, and thus all of $(2m)_r, 1_l, 2_l, \ldots, n_l$ must be in V_π in order for $\pi \vee \sigma_{n,m} = 1_{n,2m}$ to be satisfied. Below is an example of such a π. Two nodes are connected to each other with a solid line if and only if they lie in the same block of π and two nodes are connected with a dotted line if and only if they are in the same block of $\sigma_{n,m}$. The condition $\pi \vee \sigma_{n,m} = 1_{n,2m}$ means one may travel from any one node to another using a combination of solid and dotted lines. Note we really should draw all of the left nodes above all of the right notes.



Let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^m$, let s denote the number of elements of E contained in V_π (so $s \geq 1$), and let $1 \leq k_1 < k_2 < \cdots < k_s = m$ be such that $(2k_q)_r \in V_\pi$. Note V_π divides the right nodes into s disjoint regions. For each $1 \leq q \leq s$, let $j_q = k_q - k_{q-1}$, with $k_0 = 0$, and let π_q denote the noncrossing partition obtained by restricting π to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q - 1)_r\}.$$

Note that $\sum_{q=1}^s j_q = m$. Furthermore, if π'_q is obtained from π_q by adding the singleton block $\{(2k_q)_r\}$, then $\pi'_q|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_E)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{n,2m}$. The below diagram demonstrates an example of this restriction.



Consequently, by writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m \\ = \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)_e$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)_e \\ V_\rho = V_\pi}} \kappa_\rho(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m \\ = \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n, m \geq 1$ and all possible V_π (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

$$\begin{aligned} \Psi_e(z, w) &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} \phi_{g_2 \check{*} g_1}(w) \\ &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n (\phi_{g_2 \check{*} g_1}(w))^s = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)), \end{aligned}$$

as desired. □

In order to discuss $\Psi_o(z, w)$, it is quite helpful to discuss a subcase. For $n, m \geq 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m+1)$ with blocks

$$\{\{k_l\}\}_{k=1}^{n} \cup \{1_r\} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Let $\mathrm{BNC}_T(n, m)'_o$ denote the set of all partitions $\pi \in \mathrm{BNC}(n, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$ and π contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for any k, j.

Lemma 3.7. *Under the above notation and assumptions, if*

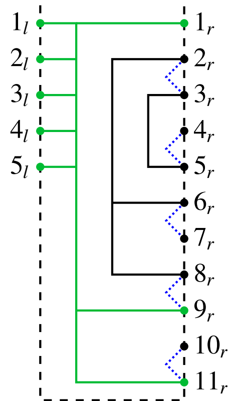
$$\Psi_{o'}(z, w) \\ := \sum_{\substack{n \geq 1 \\ m \geq 0}} \Bigg(\sum_{\pi \in \mathrm{BNC}_T(n,m)'_o} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) \Bigg) z^n w^{m+1},$$

then

$$\Psi_{o'}(z, w) = \frac{w}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_{o'}(z, w)$ by expanding κ_π as a product of full (l, r)-cumulants and summing over all π with the same block containing 1_l.

Fix $n \geq 1$ and $m \geq 0$. If $\pi \in \mathrm{BNC}_T(n, m)'_o$, then the block V_π containing 1_l must contain $1_r, (2m+1)_r, 1_l, 2_l, \dots, n_l$ in order to have $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Below is an example of such a π.



Let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^{m+1}$, let s denote the number of elements of O contained in V_π (so $s \geq 1$), and let $1 = k_1 < k_2 < \cdots < k_s = m+1$ be such that $(2k_q - 1)_r \in V_\pi$. Note V_π divides the right nodes into $s-1$ disjoint regions. For each $1 \leq q \leq s-1$, let $j_q = k_{q+1} - k_q$ and let π_q denote the noncrossing partition obtained by restricting π to $\{(2k_q)_r, (2k_q+1)_r, \dots, (2k_{q+1}-2)_r\}$. Note that $\sum_{q=1}^{s-1} j_q = m$. Furthermore, if π'_q is obtained from π_q by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ by $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Consequently, by writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^{m+1}$$
$$= \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)'_o$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)'_o \\ V_\rho = V_\pi}} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^{m+1}$$
$$= \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n \geq 1$, $m \geq 0$, and all possible V_π (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

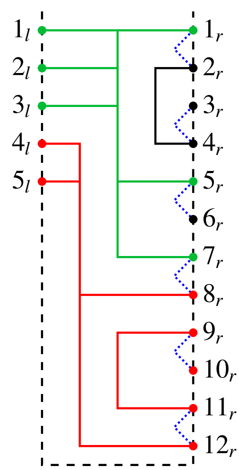
$$\begin{aligned}\Psi_{o'}(z, w) &= \sum_{n,s\geq 1} \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(w) \\ &= \frac{w}{\phi_{g_2 \check{*} g_1}(w)} \sum_{n,s\geq 1} \kappa_{n,s}(a_2, b_2) z^n (\phi_{g_2 \check{*} g_1}(w))^s \\ &= \frac{w}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)). \qquad \square\end{aligned}$$

Lemma 3.8. *Under the above notation and assumptions,*

$$\Psi_o(z, w) = \left(1 + \frac{1}{\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w)\right) K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_o(z, w)$ by expanding κ_π as a product of full (l, r)-cumulants and summing over all π with the same block containing 1_l.

Fix $n, m \geq 1$, let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^m$, let $\pi \in \mathrm{BNC}_T(n, m)_o$, let V_π denote the block of π containing 1_l, let t (respectively s) denote the number of elements of $\{1_l, \dots, n_l\}$ (respectively O) contained in V_π (so $t, s \geq 1$). Since $\pi \vee \sigma_{n,m} = 1_{n,2m}$, V_π must be of the form $\{k_l\}_{k=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s$ for some $1 = k_1 < k_2 < \cdots < k_s \leq m$. Below is an example of such a π.



Note that V_π divides the right nodes into s disjoint regions, where the bottom region is special as those nodes may connect to left nodes. For each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_s = m + 1$. Note that $\sum_{q=1}^s j_q = m$. For $q \neq s$, let π_q denote the noncrossing partition obtained by restricting π to

$$\{(2k_q)_r, (2k_q + 1)_r, \dots, (2k_{q+1} - 2)_r\}.$$

As discussed in Lemma 3.6, if π'_q is obtained from π_q by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ since $\pi \vee \sigma_{n,m} = 1_{n,2m}$.

Let π'_s denote the bi-noncrossing partition obtained by restricting π to

$$\{k_l\}_{k=t+1}^n \cup \{(2k_s)_r, (2k_s+1)_r, \dots, (2m)_r\}$$

(which is shaded differently in the above diagram). Notice, since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, that it must be the case that $\pi_s \in \mathrm{BNC}_T(n-t, j_s-1)'_o$.

By writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_1) = 1$), we obtain

$$\begin{aligned}
&\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m \\
&\qquad = \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} g_1(0_{j_q}, \pi'_q) g_2(0_{j_q}, K(\pi'_q)) w^{j_q} \right) \\
&\qquad\qquad \cdot \kappa_{\pi_s}(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n-t}, \underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_s \text{ times}})z^{n-t} w^{j_s}.
\end{aligned}$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)_o$ with $V_\rho = V_\pi$, we obtain

$$\begin{aligned}
&\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)_o \\ V_\rho = V_\pi}} \kappa_\rho(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m \\
&= \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} (g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q} \right) \\
&\quad \cdot \left(\sum_{\sigma \in \mathrm{BNC}_T(n-t, j_s-1)'_o} \kappa_\sigma(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n-t}, \underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_s \text{ times}})z^{n-t} w^{j_s} \right)
\end{aligned}$$

as all $\sigma \in \mathrm{BNC}_T(n-t, j_s-1)'_o$ occur.

We desire to sum over all $n, m \geq 1$ and all possible V_π. This produces all possible $t, s \geq 1$ and all $j_q \geq 1$. If we first sum those terms above with $t = n$, we see, using similar arguments to those used above, that

$$\sum_{\sigma \in \mathrm{BNC}_T(0, j_s-1)'_o} \kappa_\sigma(\underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_q \text{ times}}) w^{j_s} = (g_1 \check{*} g_2)(0_{j_s}, 1_{j_s}) w^{j_s}.$$

Consequently, summing those terms with $t = n$ gives

$$\begin{aligned}
\sum_{t,s \geq 1} \kappa_{t,s}(a_1, b_1) z^t \prod_{q=1}^{s} \phi_{g_1 \check{*} g_2}(w) &= \sum_{t,s \geq 1} \kappa_{t,s}(a_1, b_1) z^t (\phi_{g_1 \check{*} g_2}(w))^s \\
&= K_{a_1, b_1}(z, \phi_{g_1 \check{*} g_2}(w)).
\end{aligned}$$

Moreover, summing those terms with $t \neq n$ gives

$$\begin{aligned}\sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} \phi_{g_1 \check{*} g_2}(w)\right) \Psi_{o'}(z, w) \\ &= \sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t (\phi_{g_1 \check{*} g_2}(w))^{s-1} \Psi_{o'}(z, w) \\ &= \frac{1}{\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w) K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).\end{aligned}$$

Combining the above two sums completes the proof. $\square$

Proof of Theorem 3.5. By Lemma 3.6 along with (7), we see that

$$\Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) = K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) = K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).$$

By Lemma 3.7 along with equations (7) and (8)), we see that

$$\begin{aligned}\Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big) &= \frac{\phi_g^{\langle -1\rangle}(w)}{\phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)} K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &= \frac{\frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w)\phi_{g_2}^{\langle -1\rangle}(w)}{\phi_{g_2}^{\langle -1\rangle}(w)} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big) \\ &= \frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

Furthermore, by Lemma 3.8 along with (7), we obtain

$$\begin{aligned}\Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big) \\ &= \left(1 + \frac{1}{\phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &= \left(1 + \frac{1}{\phi_{g_1}^{\langle -1\rangle}(w)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &= \left(1 + \frac{1}{w} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &= K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) + \frac{1}{w} K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

As

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) + \Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big),$$

we have verified that equation (16) holds and thus the proof is complete. $\square$

4. Bi-free partial S-transform

In this section, we study Voiculescu's bi-free partial S-transform through combinatorics. All notation in this section refers to the notation established in this section and not to the notation of Section 3.

Definition 4.1 [Voiculescu 2015, Definition 2.1]. Let (a, b) be a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$. The 2-variable partial bi-free S-transform of (a, b) is the holomorphic function defined on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$ by

$$S_{a,b}(z, w) = \frac{z+1}{z}\frac{w+1}{w}\left(1 - \frac{1+z+w}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))}\right). \tag{17}$$

It is useful to note, in the following proposition, an equivalent definition of the bi-free partial S-transform. To simplify the discussion, we demonstrate the equality in the case $\varphi(a) = \varphi(b) = 1$. This does not hinder the proof of the desired result, namely Theorem 4.5 (see Remark 4.3).

Proposition 4.2. *If (a, b) is a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) = \varphi(b) = 1$, then, as a formal power series,*

$$S_{a,b}(z, w) = 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big). \tag{18}$$

Proof. Using equations (5), (6), (9), and (10), we obtain that

$$\frac{1}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))} = \frac{1}{1+z} + \frac{1}{1+w} - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big).$$

Therefore, using equations (11) and (17), we obtain that

$$\begin{aligned}
S_{a,b}(z, w) &= \frac{z+1}{z}\frac{w+1}{w}\bigg(1 - (1+z+w)\Big(\frac{1}{1+z} + \frac{1}{1+w} \\
&\qquad\qquad - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\Big)\bigg) \\
&= \frac{1}{zw}\big((1+z)(1+w) - (1+z+w)(2+z+w) \\
&\qquad\qquad + (1+z+w) C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big) \\
&= \frac{1}{zw}\Big(zw - (1+z+w)^2 \\
&\qquad\qquad + (1+z+w)\big(1+z+w+K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big)\Big) \\
&= 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big).
\end{aligned}$$ □

Remark 4.3. Again, one might be concerned that we have restricted to the case $\varphi(a) = \varphi(b) = 1$. Using the same ideas as in Remark 3.3, if we use (18) as the definition of the S-transform and if $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, then $S_{a,b}(z, w) = S_{\lambda a, \mu b}(z, w)$. Hence there is no loss in assuming $\varphi(a) = \varphi(b) = 1$.

Remark 4.4. Note Proposition 4.2 immediately provides the S-transform part of [Voiculescu 2015, Proposition 4.2]. Indeed if a and b are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$, $\varphi(b) \neq 0$, and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $S_{a,b}(z, w) = 1$.

We desire to prove the following, which is one of two main results of [Voiculescu 2015], using combinatorial techniques and Proposition 4.2.

Theorem 4.5 [Voiculescu 2015, Theorem 2.1]. *Let (a_1, b_1) and (a_2, b_2) be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a_j) \neq 0$ and $\varphi(b_j) \neq 0$. Then*

$$S_{a_1a_2, b_1b_2}(z, w) = S_{a_1,b_1}(z, w) S_{a_2,b_2}(z, w)$$

on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.

To simplify the proof of this result, we assume that $\varphi(a_j) = \varphi(b_j) = 1$. Note that $\varphi(a_1a_2) = \varphi(b_1b_2) = 1$ by freeness of the left algebras and of the right algebras in bi-free pairs. Furthermore, let f_j (respectively g_j) denote the multiplicative function associated to the cumulants of a_j (respectively b_j) defined by $f_j(0_n, 1_n) = \kappa_n(a_j)$ (respectively $g_j(0_n, 1_n) = \kappa_n(b_j)$). Recall that if f (respectively g) is the multiplicative function associated to the cumulants of a_1a_2 (respectively b_1b_2), then $f = f_1 * f_2$ (respectively $g = g_1 * g_2$). Thus

$$\phi_f^{\langle -1\rangle}(z) = c_{a_1a_2}^{\langle -1\rangle}(z), \qquad \phi_g^{\langle -1\rangle}(w) = c_{b_1b_2}^{\langle -1\rangle}(w),$$
$$\phi_{f_j}^{\langle -1\rangle}(z) = c_{a_j}^{\langle -1\rangle}(z), \qquad \phi_{g_j}^{\langle -1\rangle}(w) = c_{b_j}^{\langle -1\rangle}(w).$$

Note that $f, g, f_j, g_j \in \mathcal{M}_1$ by assumption.

By Proposition 4.2, it suffices to show that

$$K_{a_1a_2, b_1b_2}\big(\phi_f^{\langle -1\rangle}(w), \phi_g^{\langle -1\rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1+z+w}{zw}\Theta_1(z, w)\Theta_2(z, w) \tag{19}$$

where

$$\Theta_j(z, w) = K_{a_j,b_j}\big(\phi_{f_j}^{\langle -1\rangle}(w), \phi_{g_j}^{\langle -1\rangle}(w)\big).$$

Recall

$$K_{a_1a_2, b_1b_2}(z, w) = \sum_{n,m\geq 1} \kappa_{n,m}(a_1a_2, b_1b_2) z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(2n, 2m)$ with blocks

$$\{\{(2k-1)_l, (2k)_l\}\}_{k=1}^{n} \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^{m}.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1a_2, b_1b_2) = \sum_{\substack{\pi \in \mathrm{BNC}(2n,2m) \\ \pi \vee \sigma_{n,m} = 1_{2n,2m}}} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}).$$

Since (a_1, b_1) and (a_2, b_2) are bi-free, we note that

$$\kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) = 0$$

if π contains a block containing a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some k, j.

For $n, m \geq 1$, let $\mathrm{BNC}_S(n, m)$ be the set of all $\pi \in \mathrm{BNC}(2n, 2m)$ such that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$ and π contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some k, j. Consequently, we obtain

$$K_{a_1a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1} \left(\sum_{\pi \in \mathrm{BNC}_S(n,m)} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) \right) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_S(n, m)$. Notice that if $\pi \in \mathrm{BNC}_S(n, m)$, then π must contain a block with both a k_l and a j_r for some k, j, so that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$. If

$$V \subseteq \{1_l, \ldots, (2n)_l, 1_r, \ldots, (2m)_r\},$$

we define $\min(V)$ to be the integer k such that either $k_l \in V$ or $k_r \in V$ yet $j_l, j_r \notin V$ for all $j < k$.

Let $\mathrm{BNC}_S(n, m)_e$ denote all $\pi \in \mathrm{BNC}_S(n, m)$ such that $\min(V) \in 2\mathbb{Z}$ for the block V of π that has the smallest min-value over all blocks W of π such that there exist $k_l, j_r \in W$ for some k, j; that is, V is the first block, measured from the top, in the bi-noncrossing diagram of π that has both left and right nodes, and these nodes are of even index. Similarly, let $\mathrm{BNC}_S(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that $\min(V) \in 2\mathbb{Z}+1$ for the block V of π that has the smallest min-value over all blocks W of π such that there exist $k_l, j_r \in W$ for some k, j. Note $\mathrm{BNC}_S(n, m)_e$ and $\mathrm{BNC}_S(n, m)_o$ are disjoint and

$$\mathrm{BNC}_S(n, m)_e \cup \mathrm{BNC}_S(n, m)_o = \mathrm{BNC}_S(n, m).$$

Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z,w) := \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in \mathrm{BNC}_S(n,m)_d} \kappa_\pi(\underbrace{a_1,a_2,a_1,a_2,\ldots,a_1,a_2}_{a_1 \text{ occurs } n \text{ times}},\underbrace{b_1,b_2,b_1,b_2,\ldots,b_1,b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m,$$

then

$$K_{a_1a_2,b_1b_2}(z,w) = \Psi_e(z,w) + \Psi_o(z,w).$$

We derive expressions for $\Psi_e(z,w)$ and $\Psi_o(z,w)$ beginning with $\Psi_e(z,w)$. We do not use the same rigour as in Section 3, as most of the arguments are similar.

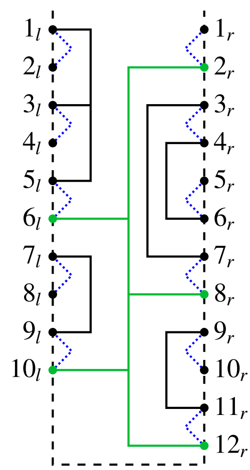
Lemma 4.6. *Under the above notation and assumptions,*

$$\Psi_e(z,w) = K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2\check{*} g_1}(w)).$$

Proof. Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n,m)_e$, let V_π denote the first (and, as it happens, only) block of π, as measured from the top of π's bi-noncrossing diagram, that has both left and right nodes. Since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, there exist $t, s \geq 1$, $1 \leq l_1 < l_2 < \cdots < l_t = n$, and $1 \leq k_1 < k_2 < \cdots < k_s = m$ such that

$$V_\pi = \{(2l_p)_l\}_{p=1}^t \cup \{(2k_q)_r\}_{q=1}^s.$$

Note V_π divides the remaining left nodes into t disjoint regions and the remaining right nodes into s disjoint regions. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



Let $E = \{(2k)_l\}_{k=1}^n \cup \{(2k)_r\}_{k=1}^m$ and $O = \{(2k-1)_l\}_{k=1}^n \cup \{(2k-1)_r\}_{k=1}^m$. For each $1 \leq p \leq t$, let $i_p = l_p - l_{p-1}$, where $l_0 = 0$, and let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting π to $\{(2l_{p-1}+1)_l, (2l_{p-1}+2)_l, \ldots, (2l_p - 1)_l\}$. Note that $\sum_{p=1}^t i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p)_l\}$, then $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \le q \le s$, let $j_q = k_q - k_{q-1}$, where $k_0 = 0$, and let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting π to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q - 1)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q)_r\}$, then $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n,m)_e$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_2, b_2)\Biggl(\prod_{p=1}^{t} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}\Biggr)\Biggl(\prod_{q=1}^{s} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}\Biggr).$$

Finally, if we sum over all possible $n, m \ge 1$ and all possible V_π (so, in the above equation, we get all possible $t, s \ge 1$ and all possible $i_p, j_q \ge 1$), we obtain that

$$\begin{aligned}
\Psi_e(z,w) &= \sum_{t,s \ge 1} \kappa_{t,s}(a_2, b_2)\Biggl(\prod_{p=1}^{t} \phi_{f_2 \check{*} f_1}(z)\Biggr)\Biggl(\prod_{q=1}^{s} \phi_{g_2 \check{*} g_1}(z)\Biggr) \\
&= \sum_{t,s \ge 1} \kappa_{t,s}(a_2, b_2)(\phi_{f_2 \check{*} f_1}(z))^t (\phi_{g_2 \check{*} g_1}(w))^s \\
&= K_{a_2, b_2}\bigl(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)\bigr). \qquad \square
\end{aligned}$$

In order to discuss $\Psi_o(z,w)$, it is quite helpful to discuss subcases. For $n, m \ge 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(2n+1, 2m+1)$ with blocks

$$\{\{1_l, 1_r\}\} \cup \{\{(2l)_l, (2l+1)_l\}\}_{l=1}^{n} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Define $\mathrm{BNC}_S(n,m)'_o$ to be the set of all $\pi \in \mathrm{BNC}(2n+1, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$ and π contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for any $\theta_1, \theta_2 \in \{l, r\}$ and any k, j. We wish to divide up $\mathrm{BNC}_S(n,m)'_o$ further. For $\pi \in \mathrm{BNC}_S(n,m)'_o$, let $V_{\pi,l}$ denote the block of π containing 1_l and $V_{\pi,r}$ the block of π containing 1_r. Then,

$$\begin{aligned}
&\mathrm{BNC}_S(n,m)_{o,0}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has no right nodes and } V_{\pi,r} \text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,r}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has no right nodes but } V_{\pi,r} \text{ has left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,l}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has right nodes but } V_{\pi,r} \text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,lr} = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} = V_{\pi,r}\}.
\end{aligned}$$

Due to the nature of bi-noncrossing partitions, the above sets are disjoint and have union $\mathrm{BNC}_S(n,m)'_o$.

For $d \in \{0, r, l, lr\}$, define

$$\Psi_{o,d}(z,w) := \sum_{n,m\geq 0}\Bigg(\sum_{\pi\in \mathrm{BNC}_S(n,m)_{o,d}} \kappa_\pi(\underbrace{a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^{n+1}w^{m+1}.$$

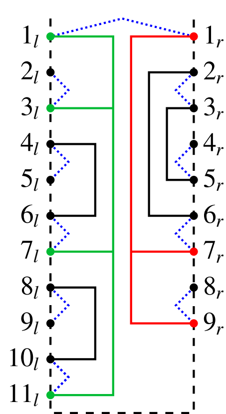
Lemma 4.7. *Under the above notation and assumptions,*

$$\Psi_{o,0}(z,w) = zw \cdot \frac{\phi_{f_2}(\phi_{f_2 \check{*} f_1}(z))\phi_{g_2}(\phi_{g_2\check{*} g_1}(w))}{\phi_{f_2\check{*} f_1}(z)\phi_{g_2\check{*} g_1}(w)}.$$

Proof. Fix $n, m \geq 0$. If $\pi \in \mathrm{BNC}_S(n,m)_{o,0}$, then, since $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$, there exist $t, s \geq 1$, $1 = l_1 < l_2 < \cdots < l_t = n+1$, and $1 = k_1 < k_2 < \cdots < k_s = m+1$ such that

$$V_{\pi,l} = \{(2l_p - 1)_l\}_{p=1}^t \quad \text{and} \quad V_{\pi,r} = \{(2k_q - 1)_r\}_{q=1}^s.$$

Note that $V_{\pi,l}$ divides the remaining left nodes into $t-1$ disjoint regions and $V_{\pi,r}$ divides the remaining right nodes into $s-1$ disjoint regions. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



If $i_p = l_{p+1} - l_p$ and $j_q = k_{q+1} - k_q$, then

$$\sum_{p=1}^{t-1} i_p = n \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^{n+1} w^{m+1}$$

for $\rho \in \mathrm{BNC}_S(n, m)_{o,0}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$zw \cdot \kappa_t(a_2)\kappa_s(b_2) \left(\prod_{p=1}^{t-1} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p} \right) \left(\prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q} \right).$$

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}
\Psi_e(z, w) &= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2) \left(\prod_{p=1}^{t-1} \phi_{f_2 \check{*} f_1}(z) \right) \left(\prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(z) \right) \\
&= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2) (\phi_{f_2 \check{*} f_1}(z))^{t-1} (\phi_{g_2 \check{*} g_1}(w))^{s-1} \\
&= zw \cdot \frac{\phi_{f_2}(\phi_{f_2 \check{*} f_1}(z)) \phi_{g_2}(\phi_{g_2 \check{*} g_1}(w))}{\phi_{f_2 \check{*} f_1}(z) \phi_{g_2 \check{*} g_1}(w)}.
\end{aligned}$$

□

Lemma 4.8. *Under the above notation and assumptions,*

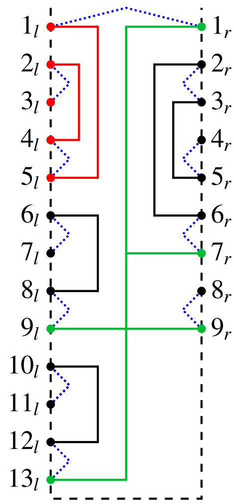
$$\Psi_{o,r}(z, w) = \frac{w \cdot \phi_{f_1 \check{*} f_2}(z)}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

Proof. Fix $n, m \geq 0$. Note $\mathrm{BNC}_S(0, m)_{o,r} = \varnothing$ by definition.

If $\pi \in \mathrm{BNC}_S(n, m)_{o,r}$, then, since $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$, there exist $t, s \geq 1$, $1 < l_1 < l_2 < \cdots < l_t = n + 1$, and $1 = k_1 < k_2 < \cdots < k_s = m + 1$ such that

$$V_{\pi,r} = \{(2l_p - 1)_l\}_{p=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s.$$

Note that $V_{\pi,r}$ divides the remaining right nodes into $s - 1$ disjoint regions and the remaining left nodes into t regions. However, the top region is special. If l_0 is the largest natural number such that $(2l_0 - 1)_l \in V_{\pi,l}$, then l_0 further divides the top region on the left into two regions. Note that each block of π can only contain nodes in one such region. The following is an example of such a π for which $l_0 = 3$, with one part of the special region $(1_l, \dots, 5_l)$ shaded differently.



Let $i_0 = l_0$, $i_p = l_p - l_{p-1}$ when $p \neq 0$, and $j_q = k_{q+1} - k_q$. Thus

$$\sum_{p=0}^{t} i_p = n+1 \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^{n+1} w^{m+1}$$

for $\rho \in \mathrm{BNC}_S(n, m)_{o,r}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$w \cdot \kappa_{t,s}(a_2, b_2) \left(\prod_{p=1}^{t} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p} \right) \\ \cdot \left(\prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q} \right) \left((f_1 \check{*} f_2)(0_{i_0}, 1_{i_0}) z^{i_0} \right).$$

Note for $p \geq 2$, each $(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}$ comes from the p-th region from the top on the left, whereas the top region on the left gives $(f_2 \check{*} f_1)(0_{i_1}, 1_{i_1}) z^{i_1}$ using the partitions below $(2l_0 - 1)_l$ and gives $(f_1 \check{*} f_2)(0_{i_0}, 1_{i_0}) z^{i_0}$ using the partitions above and including $(2l_0 - 1)_l$.

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}\Psi_e(z,w) &= w\sum_{t,s\geq 1}\kappa_{t,s}(a_2,b_2)\left(\prod_{p=1}^{t}\phi_{f_2\check{*}f_1}(z)\right)\left(\prod_{q=1}^{s-1}\phi_{g_2\check{*}g_1}(z)\right)\left(\phi_{f_1\check{*}f_2}(z)\right)\\ &= w\sum_{t,s\geq 1}\kappa_{t,s}(a_2,b_2)(\phi_{f_2\check{*}f_1}(z))^t(\phi_{g_2\check{*}g_1}(w))^{s-1}(\phi_{f_1\check{*}f_2}(z))\\ &= \frac{w\cdot\phi_{f_1\check{*}f_2}(z)}{\phi_{g_2\check{*}g_1}(w)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).\end{aligned}$$ □

Lemma 4.9. *Under the above notation and assumptions,*

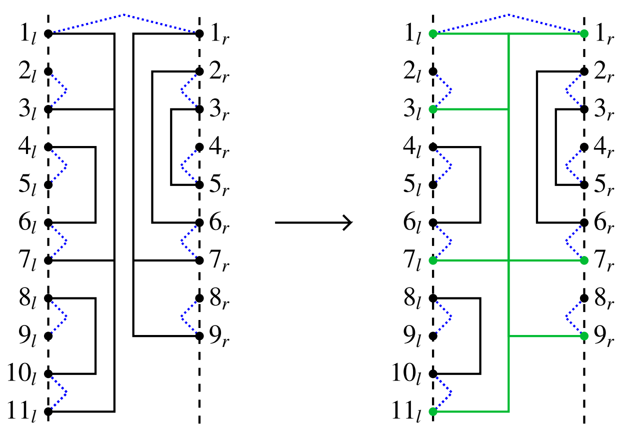
$$\Psi_{o,l}(z,w) = \frac{z\cdot\phi_{g_1\check{*}g_2}(w)}{\phi_{f_2\check{*}f_1}(z)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).$$

Proof. The proof can be obtained by applying a mirror to Lemma 4.8. □

Lemma 4.10. *Under the above notation and assumptions,*

$$\Psi_{o,lr}(z,w) = \frac{zw}{\phi_{f_2\check{*}f_1}(z)\phi_{g_2\check{*}g_1}(w)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).$$

Proof. The proof of this result follows from the proof of Lemma 4.7 by replacing each occurrence of $\kappa_t(a_2)\kappa_s(b_2)$ with $\kappa_{t,s}(a_2,b_2)$. Indeed there is a bijection from $\mathrm{BNC}_S(n,m)_{o,0}$ to $\mathrm{BNC}_S(n,m)_{o,lr}$ whereby, given $\pi\in\mathrm{BNC}_S(n,m)_{o,0}$, we produce $\pi'\in\mathrm{BNC}_S(n,m)_{o,lr}$ by joining $V_{\pi,l}$ and $V_{\pi,r}$ into a single block.



□

Lemma 4.11. *Under the above notation and assumptions,*

$$\Psi_o(z,w) = \frac{1}{\phi_{f_1\check{*}f_2}(z)\phi_{g_1\check{*}g_2}(w)}\Psi_{o'}(z,w)K_{a_1,b_1}(\phi_{f_1\check{*}f_2}(z),\phi_{g_1\check{*}g_2}(w)),$$

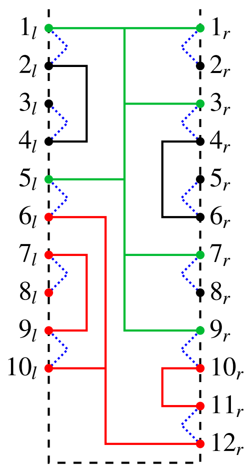
where

$$\Psi_{o'}(z,w) = \Psi_{o,0}(z,w)+\Psi_{o,r}(z,w)+\Psi_{o,l}(z,w)+\Psi_{o,lr}(z,w).$$

Proof. Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n,m)_o$, let V_π denote the first block of π, as measured from the top of π's bi-noncrossing diagram, that has both left and right nodes. Since $\pi \in \mathrm{BNC}_S(n,m)_o$, there exist $t, s \geq 1$, $1 = l_1 < l_2 < \cdots < l_t \leq n$, and $1 = k_1 < k_2 < \cdots < k_s \leq m$ such that

$$V_\pi = \{(2l_p - 1)_l\}_{p=1}^{t} \cup \{(2k_q - 1)_r\}_{q=1}^{s}.$$

Note V_π divides the remaining left nodes and right nodes into $t-1$ disjoint regions on the left, $s-1$ disjoint regions on the right, and one region on the bottom. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



Let

$$E = \{(2k)_l\}_{k=1}^{n} \cup \{(2k)_r\}_{k=1}^{m},$$
$$O = \{(2k-1)_l\}_{k=1}^{n} \cup \{(2k-1)_r\}_{k=1}^{m}.$$

For each $1 \leq p \leq t$, let $i_p = l_{p+1} - l_p$, where $l_{t+1} = n+1$, and, for $p \neq t$, let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting π to

$$\{(2l_p)_l, (2l_p + 1)_l, \ldots, (2l_{p+1} - 2)_l\}.$$

Note that $\sum_{p=1}^{t} i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p - 1)_l\}$, then $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_{s+1} = m+1$, and, for $q \neq s$, let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting π to

$$\{(2k_q)_r, (2k_q + 1)_r, \ldots, (2k_{q+1} - 2)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Finally, if π' is the bi-noncrossing partition obtained by restricting π to

$$\{(2l_t)_l, (2l_t+1)_l, \ldots, (2n)_l, (2k_s)_r, (2k_s+1)_r, \ldots, (2m)_r\}$$

(which is shaded differently in the above diagram), then $\pi' \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n, m)_o$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_1, b_1)\left(\prod_{p=1}^{t-1} (f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}\right)\left(\prod_{q=1}^{s-1} (g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}\right)$$
$$\cdot \left(\sum_{\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o} \kappa_\tau(\underbrace{a_2, a_1, a_2, a_1, \ldots, a_1, a_2}_{a_1 \text{ occurs } i_t-1 \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } j_s-1 \text{ times}}) z^{i_t} w^{j_s}\right).$$

Note that for $p \neq t$, each $(f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}$ comes from the p-th region from the top on the left, for $q \neq s$ each $(g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}$ comes from the q-th region from the top on the right, and all $\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$ are possible on the bottom.

Finally, if we sum over all possible $n, m \geq 1$ and all possible V_π (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}
\Psi_e(z, w) &= \sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1)\left(\prod_{p=1}^{t-1} \phi_{f_1 \check{*} f_2}(z)\right)\left(\prod_{q=1}^{s-1} \phi_{g_1 \check{*} g_2}(z)\right)\Psi_{o'}(z, w) \\
&= \sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1)(\phi_{f_1 \check{*} f_2}(z))^{t-1}(\phi_{g_1 \check{*} g_2}(w))^{s-1}\Psi_{o'}(z, w) \\
&= \frac{1}{\phi_{f_1 \check{*} f_2}(z)\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w) K_{a_1,b_1}(\phi_{f_1 \check{*} f_2}(z), \phi_{g_1 \check{*} g_2}(w)). \qquad \square
\end{aligned}$$

Proof of Theorem 4.5. Using (7) and (8), we see (via Lemmata 4.6–4.10) that

$$\begin{aligned}
\Psi_e\big(\phi^{\langle -1\rangle}_{f_1 * f_2}(z), \phi^{\langle -1\rangle}_{g_1 * g_2}(w)\big) &= K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z), \phi^{\langle -1\rangle}_{g_2}(w)\big), \\
\Psi_{o,0}\Big(\phi^{\langle -1\rangle}_{f_1 * f_2}(z), \phi^{\langle -1\rangle}_{g_1 * g_2}(w)\Big) &= \phi^{\langle -1\rangle}_{f_1 * f_2}(z)\phi^{\langle -1\rangle}_{g_1 * g_2}(w) \cdot \frac{zw}{\phi^{\langle -1\rangle}_{f_2}(z)\phi^{\langle -1\rangle}_{g_2}(w)} \\
&= \phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w),
\end{aligned}$$

$$\Psi_{o,r}\big(\phi_{f_1*f_2}^{\langle -1\rangle}(z), \phi_{g_1*g_2}^{\langle -1\rangle}(w)\big) = \frac{\phi_{f_1}^{\langle -1\rangle}(z)\phi_{g_1}^{\langle -1\rangle}(w)}{w} K_{a_2,b_2}\big(\phi_{f_2}^{\langle -1\rangle}(z), \phi_{g_2}^{\langle -1\rangle}(w)\big),$$
$$\Psi_{o,l}\big(\phi_{f_1*f_2}^{\langle -1\rangle}(z), \phi_{g_1*g_2}^{\langle -1\rangle}(w)\big) = \frac{\phi_{f_1}^{\langle -1\rangle}(z)\phi_{g_1}^{\langle -1\rangle}(w)}{z} K_{a_2,b_2}\big(\phi_{f_2}^{\langle -1\rangle}(z), \phi_{g_2}^{\langle -1\rangle}(w)\big),$$
$$\Psi_{o,lr}\big(\phi_{f_1*f_2}^{\langle -1\rangle}(z), \phi_{g_1*g_2}^{\langle -1\rangle}(w)\big) = \frac{\phi_{f_1}^{\langle -1\rangle}(z)\phi_{g_1}^{\langle -1\rangle}(w)}{zw} K_{a_2,b_2}\big(\phi_{f_2}^{\langle -1\rangle}(z), \phi_{g_2}^{\langle -1\rangle}(w)\big).$$

Since

$$\Phi_0\big(\phi_{f_1*f_2}^{\langle -1\rangle}(z), \phi_{g_1*g_2}^{\langle -1\rangle}(w)\big) = \frac{1}{\phi_{f_1}^{\langle -1\rangle}(z)\phi_{g_1}^{\langle -1\rangle}(w)} \Psi_{o'}\big(\phi_{f_1*f_2}^{\langle -1\rangle}(z), \phi_{g_1*g_2}^{\langle -1\rangle}(w)\big) K_{a_1,b_1}\big(\phi_{f_1}^{\langle -1\rangle}(z), \phi_{g_1}^{\langle -1\rangle}(w)\big)$$

by (7) and Lemma 4.11, and since

$$\frac{1}{z} + \frac{1}{w} + \frac{1}{zw} = \frac{1+z+w}{zw} \quad \text{and} \quad K_{a_1a_2,b_1b_2}(z, w) = \Psi_e(z, w) + \Psi_0(z, w),$$

we have verified that (19) holds and thus the proof is complete. □

References

[Charlesworth et al. 2015a] I. Charlesworth, B. Nelson, and P. Skoufranis, "Combinatorics of bi-freeness with amalgamation", *Comm. Math. Phys.* **338**:2 (2015), 801–847. MR Zbl

[Charlesworth et al. 2015b] I. Charlesworth, B. Nelson, and P. Skoufranis, "On two-faced families of non-commutative random variables", *Canad. J. Math.* **67**:6 (2015), 1290–1325. MR Zbl

[Gu et al. 2015] Y. Gu, H.-W. Huang, and J. Mingo, "An analogue of the Lévy–Hinčin formula for bi-free infinitely divisible distributions", *Indiana Univ. Math. J.* (Online publication August 2015).

[Haagerup 1997] U. Haagerup, "On Voiculescu's R- and S-transforms for free non-commuting random variables", pp. 127–148 in *Free probability theory* (Waterloo, ON, 1995), edited by D.-V. Voiculescu, Fields Institute Communications **12**, American Mathematical Society, Providence, RI, 1997. MR Zbl

[Mastnak and Nica 2015] M. Mastnak and A. Nica, "Double-ended queues and joint moments of left-right canonical operators on full Fock space", *Int. J. Math.* **26**:2 (2015), Article ID #1550016. MR Zbl

[Nica and Speicher 1997] A. Nica and R. Speicher, "A 'Fourier transform' for multiplicative functions on non-crossing partitions", *J. Algebraic Combin.* **6**:2 (1997), 141–160. MR Zbl

[Nica and Speicher 2006] A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series **335**, Cambridge University Press, 2006. MR Zbl

[Skoufranis 2015] P. Skoufranis, "Independences and partial R-transforms in bi-free probability", *Ann. Inst. Henri Poincaré Probab. Stat.* (Online publication May 2015).

[Speicher 1994] R. Speicher, "Multiplicative functions on the lattice of noncrossing partitions and free convolution", *Math. Ann.* **298**:4 (1994), 611–628. MR Zbl

[Voiculescu 1986] D.-V. Voiculescu, "Addition of certain noncommuting random variables", *J. Funct. Anal.* **66**:3 (1986), 323–346. MR Zbl

[Voiculescu 1987] D.-V. Voiculescu, "Multiplication of certain noncommuting random variables", *J. Operator Theory* **18**:2 (1987), 223–235. MR Zbl

[Voiculescu 2014] D.-V. Voiculescu, "Free probability for pairs of faces, I", *Comm. Math. Phys.* **332**:3 (2014), 955–980. MR Zbl

[Voiculescu 2015] D.-V. Voiculescu, "Free probability for pairs of faces, III: 2-Variables bi-free partial S- and T-transforms", preprint, 2015. arXiv

[Voiculescu 2016] D.-V. Voiculescu, "Free probability for pairs of faces, II: 2-Variables bi-free partial R-transform and systems with rank ≤ 1 commutation", *Ann. Inst. Henri Poincaré Probab. Stat.* **52**:1 (2016), 1–15. MR Zbl

Received June 12, 2015.

PAUL SKOUFRANIS
DEPARTMENT OF MATHEMATICS
TEXAS A&M UNIVERSITY
COLLEGE STATION, TX 77843
UNITED STATES

pskoufra@math.tamu.edu

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.449

VECTOR BUNDLE VALUED DIFFERENTIAL FORMS ON $\mathbb{N}Q$-MANIFOLDS

LUCA VITAGLIANO

Geometric structures on $\mathbb{N}Q$-manifolds, i.e., nonnegatively graded manifolds with a homological vector field, encode nongraded geometric data on Lie algebroids and their higher analogues. A particularly relevant class of structures consists of vector bundle valued differential forms. Symplectic forms, contact structures and, more generally, distributions are in this class. We describe vector bundle valued differential forms on nonnegatively graded manifolds in terms of nongraded geometric data. Moreover, we use this description to present, in a unified way, novel proofs of known results, and new results about degree-one $\mathbb{N}Q$-manifolds equipped with certain geometric structures, namely symplectic structures, contact structures, involutive distributions (already present in literature), locally conformal symplectic structures, and generic vector bundle valued higher order forms, in particular presymplectic and multisymplectic structures (not yet present in literature).

1. Introduction

Graded geometry encodes (nongraded) geometric structures in an efficient way. For instance, a vector bundle is the same as a degree-one nonnegatively graded manifold. In this respect $\mathbb{N}Q$-manifolds, i.e., nonnegatively graded manifolds equipped with a homological vector field, are of a special interest. Namely, they

MSC2010: 53D17, 58A50.
Keywords: graded manifolds, $\mathbb{N}Q$-manifolds, vector bundle valued differential forms, Lie algebroids, Spencer operators.

encode Lie algebroids in degree one, and higher versions of Lie algebroids (including homotopy Lie algebroids) in higher degrees [Voronov 2010] (see also [Bruce 2011; Kjeseth 2001b; Sati et al. 2012; Sheng and Zhu 2011; Bonavolontà and Poncin 2013; Vitagliano 2015b], and [Kjeseth 2001a; Vitagliano 2014] for applications of homotopy Lie algebroids). Accordingly, geometric data on Lie algebroids, or higher versions of them, that are compatible with the algebroid structure, can be encoded by suitable geometric structures on an $\mathbb{N}Q$-manifold that are preserved by the homological vector field. This is a general rule with various examples scattered in the literature. For instance, degree-one *symplectic* $\mathbb{N}Q$*-manifolds* are equivalent to Poisson manifolds (which can be understood as Lie algebroids of a special kind) [Roytenberg 2002]. Similarly, degree-one *contact* $\mathbb{N}Q$*-manifolds* [Grabowski 2013; Mehta 2013] are equivalent to Jacobi manifolds, and degree-one $\mathbb{N}Q$-manifolds equipped with a compatible involutive distribution are equivalent to Lie algebroids equipped with an *IM foliation* (see [Jotz and Ortiz 2011] for a definition) [Zambon and Zhu 2012]. More examples can be presented in degree two. For instance, degree-two symplectic $\mathbb{N}Q$-manifolds are equivalent to Courant algebroids [Roytenberg 2002], and degree-two contact $\mathbb{N}Q$-manifolds encode a contact version of Courant algebroids: *Grabowski's contact-Courant algebroids* [Grabowski 2013].

In all examples above the geometric structure on the $\mathbb{N}Q$-manifold is, or can be understood as, a *differential form with values in a vector bundle*. This motivates the study of vector bundle valued differential forms (vector valued forms, in the following) on graded manifolds, and, in particular, $\mathbb{N}Q$-manifolds. In this paper, we describe vector valued forms on nonnegatively graded manifolds in terms of nongraded geometric data (Theorem 10). Later we apply this description to the study of degree-one $\mathbb{N}Q$-manifolds equipped with a compatible vector valued form. In this way, we get a unified formalism for the description of degree-one contact $\mathbb{N}Q$-manifolds, symplectic $\mathbb{N}Q$-manifolds, and $\mathbb{N}Q$-manifolds equipped with a compatible involutive distribution. In particular, we manage to present alternative proofs of results (in degree one) of Roytenberg [2002], Grabowski [2013], Mehta [2013], and Zambon and Zhu [2012]. We also discuss three new examples. Namely, we show that degree-one presymplectic $\mathbb{N}Q$-manifolds (with an additional nondegeneracy condition) are basically equivalent to Dirac manifolds (Corollary 31). We also show that degree-one locally conformal symplectic $\mathbb{N}Q$-manifolds are equivalent to locally conformal Poisson manifolds (Theorem 35), and, more generally, degree-one $\mathbb{N}Q$-manifolds equipped with a compatible, higher degree, vector valued form are equivalent to Lie algebroids equipped with *Spencer operators* (Theorem 36). The latter have been recently introduced in [Crainic et al. 2015] (see also [Salazar 2013]) as infinitesimal counterparts of multiplicative vector valued forms on Lie groupoids. In particular, degree-one multisymplectic $\mathbb{N}Q$-manifolds are equivalent to Lie algebroids equipped with an *IM multisymplectic* *structure* [Bursztyn et al. 2015] (Theorem 39). We stress that we do only consider differential forms with values in vector bundles generated in one single degree (which, up to a shift, are actually generated in degree zero). This hypothesis simplifies the discussion a lot. We hope to discuss the general case, as well as higher-degree cases, elsewhere.

The paper is divided into three main sections and two appendixes. In Section 2, after a short review of vector valued Cartan calculus on graded manifolds, we present our description of vector valued forms on $\mathbb{N}$-manifolds in terms of nongraded geometric data (Theorem 10). As already remarked, this description allows one to present in a unified way various results scattered in the literature about the correspondence between geometric structures on degree-one $\mathbb{N}Q$-manifolds and (nongraded) geometric structures on Lie algebroids. In Section 3, we discuss 1-forms on degree-one $\mathbb{N}Q$-manifolds. Surjective 1-forms are the same as distributions and we discuss in some detail the contact and involutive cases. The results of this section (Theorem 23 and Theorem 25) are already present in literature, but they are presented here in a new and unified way that allows a straightforward generalization to (possibly degenerate) differential forms of higher order. In Section 4, we discuss 2-forms (on degree-one $\mathbb{N}Q$-manifolds). In particular, we present a novel proof of the remark of Roytenberg [2002] that degree-one symplectic $\mathbb{N}Q$-manifolds are equivalent to Poisson manifolds (Theorem 27). We also generalize Roytenberg result in two different directions, namely to presymplectic forms on one side (Theorem 30 and Corollary 31) and to locally conformal symplectic structures on the other side (Theorem 35). In Section 5 we discuss the general case of a differential form of arbitrarily high order. In particular, we relate compatible vector valued forms on $\mathbb{N}Q$-manifold and the Spencer operators of Crainic–Salazar–Struchiner [Crainic et al. 2015; Salazar 2013] (Theorem 36). Finally, we discuss degree-one multisymplectic $\mathbb{N}Q$-manifolds (Theorem 39). The paper is complemented by two appendixes. In Appendix A, we revisit briefly the concept of locally conformal symplectic manifolds [Vaisman 1985], and give a slightly more intrinsic definition of them. We also briefly review the relation between locally conformal symplectic manifolds and locally conformal Poisson manifolds [Vaisman 2007]. In Appendix B, we review the definition of Lie algebroids and their representations. As already remarked they play a key role in the paper.

1.1. ***Notation and conventions.*** Let $V = \bigoplus_i V_i$ be a graded vector space. We denote by $|v|$ the degree of a homogeneous element, i.e., $|v| = i$ whenever $v \in V_i$, unless otherwise stated.

Let $\mathcal{M}$ be a (graded) manifold, and $\mathcal{E} \to \mathcal{M}$ a (graded) vector bundle on it. We denote by M the support of $\mathcal{M}$. In the case when $\mathcal{M}$ is nonnegatively graded, M is also the degree-zero shadow of $\mathcal{M}$. Moreover, we denote by $C_i^\infty(\mathcal{M})$, (resp.

$\mathfrak{X}_i(\mathcal{M})$, $\Gamma_i(\mathcal{E})$) the vector space of degree-i smooth functions on $\mathcal{M}$ (resp. vector fields on $\mathcal{M}$, sections of $\mathcal{E}$). We also denote by $\mathfrak{X}_-(\mathcal{M})$ the graded vector space of negatively graded vector fields on $\mathcal{M}$. Sometimes, if there is no risk of confusion, we denote by $\mathcal{E}$ the (graded) $C^\infty(\mathcal{M})$-module of sections of $\mathcal{E}$. Similarly, we often identify (graded) vector bundle morphisms and (graded) homomorphisms between modules of sections.

We adopt the Einstein summation convention.

2. Vector valued forms on graded manifolds

2.1. ***$\mathbb{N}Q$-manifolds and vector $\mathbb{N}Q$-bundles.*** We refer to [Roytenberg 2002; Mehta 2006; Cattaneo and Schätz 2011] for details about graded manifolds, and, in particular, $\mathbb{N}$-manifolds. In the following, we just recall some basic facts which will be often used below. We will work with the simplest possible notion of a *graded manifold*. Namely, any graded manifold $\mathcal{M}$ in this paper is equipped with one single $\mathbb{Z}$-grading in its algebra $C^\infty(\mathcal{M})$ of smooth functions (unless otherwise stated). Moreover, $C^\infty(\mathcal{M})$ is graded commutative with respect to the grading. We will call *degree* the grading. We will focus on $\mathbb{N}$-manifolds, i.e., *nonnegatively graded* manifolds. Recall that the *degree of an* $\mathbb{N}$*-manifold* is the highest degree of its coordinates. Similarly, the degree of a vector $\mathbb{N}$-bundle, i.e., a nonnegatively graded vector bundle over an $\mathbb{N}$-manifold, is the highest degree of its fiber coordinates.

Example 1. Every degree-one $\mathbb{N}$-manifold $\mathcal{M}$ is of the form $A[1]$ for some non-graded vector bundle $A \to M$, and one has $C^\infty(\mathcal{M}) = \Gamma(\wedge^\bullet A^*)$. In particular, degree-zero functions on $\mathcal{M}$ identify with functions on M, and degree-one functions on $\mathcal{M}$ identify with sections of A^*. Accordingly, vector fields of degree -1 on $\mathcal{M}$ identify with sections of A. In the following, we will tacitly understand the identifications $C_0^\infty(\mathcal{M}) \simeq C^\infty(M)$, $C_1^\infty(\mathcal{M}) \simeq \Gamma(A^*)$, and $\mathfrak{X}_{-1}(\mathcal{M}) \simeq \Gamma(A)$. The action of a vector field $X \in \Gamma(A)$ of degree -1 on a degree-one function $f \in \Gamma(A^*)$ is given by the *duality pairing*: $X(f) = \langle X, f\rangle$.

Example 2. Recall that every $\mathbb{N}$-manifold $\mathcal{M}$ is fibered over its degree-zero shadow M. Every degree-zero vector $\mathbb{N}$-bundle $\mathcal{E}$ over $\mathcal{M}$ is of the form $\mathcal{M} \times_M E$ for some nongraded vector bundle $E \to M$, and one has $\Gamma(\mathcal{E}) = C^\infty(\mathcal{M}) \otimes \Gamma(E)$ (where the tensor product is over $C^\infty(M)$). In particular, degree-zero sections of $\mathcal{E}$ identify with sections of E. In the following, we will tacitly understand the identification $\Gamma_0(\mathcal{E}) \simeq \Gamma(E)$.

A *Q-manifold* is a graded manifold $\mathcal{M}$ equipped with a homological vector field Q, i.e., a degree-one vector field Q such that $[Q, Q] = 0$. An $\mathbb{N}Q$-manifold is a nonnegatively graded Q-manifold.

Example 3. Every degree-one $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ is of the form $(A[1], d_A)$ for some nongraded Lie algebroid $A \to M$ (see Appendix B for a definition of Lie algebroid). Here d_A is the homological derivation induced in $\Gamma(\wedge^\bullet A^*) = C^\infty(\mathcal{M})$. The Lie bracket $[\![-,-]\!]$ in $\Gamma(A)$ and the anchor $\rho : \Gamma(A) \to \mathfrak{X}(M)$ can be recovered from Q via formulas

$$[\![X, Y]\!] = [[Q, X], Y], \quad \rho(X)(f) = [Q, X](f),$$

where $X, Y \in \Gamma(A)$ are also interpreted as vector fields of degree -1 on $\mathcal{M}$ (so that $[[Q, X], Y]$ also has degree -1), and $f \in C^\infty(M)$.

Similarly, we call a *Q-vector bundle* (resp. $\mathbb{N}Q$-vector bundle) a graded vector bundle $\mathcal{E} \to \mathcal{M}$ (resp. a vector $\mathbb{N}$-bundle) equipped with a homological derivation. In this respect, recall that a (graded) derivation of $\mathcal{E}$ is a graded, $\mathbb{R}$-linear map $\mathbb{X} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$ such that

$$\mathbb{X}(fe) = X(f)e + (-)^{|f||\mathbb{X}|} f\mathbb{X}(e), \quad f \in C^\infty(\mathcal{M}), \quad e \in \Gamma(\mathcal{E}),$$

for a (necessarily unique) vector field $X \in \mathfrak{X}(\mathcal{M})$ called the *symbol of* $\mathbb{X}$. Clearly, a derivation of $\mathcal{E}$ is completely determined by its symbol and its action on generators of $\Gamma(\mathcal{E})$.

Example 4. Denote by Δ the grading vector field on $\mathcal{M}$, i.e., $\Delta(f) = |f|f$, for all homogeneous functions f on $\mathcal{M}$. The grading $\Delta_{\mathcal{E}} : \Gamma(\mathcal{E}) \to \Gamma(\mathcal{E})$, $e \mapsto |e|e$, is a distinguished degree-zero derivation. Obviously, the symbol of $\Delta_{\mathcal{E}}$ is Δ.

Example 5. Let $\mathcal{M}$ be an $\mathbb{N}$-manifold, and let $\mathcal{E} = \mathcal{M} \times_M E$ be a degree-zero vector $\mathbb{N}$-bundle on it. Since $\Gamma(\mathcal{E})$ is generated in degree zero, then a negatively graded derivation $\mathbb{X}$ of $\mathcal{E}$ is completely determined by its symbol X and, therefore, it is the same as a negatively graded vector field on $\mathcal{M}$. Specifically, for a section of $\mathcal{E}$ of the form $f \otimes e$, $f \in C^\infty(\mathcal{M})$, $e \in \Gamma(E)$, one has $\mathbb{X}(f \otimes e) = X(f) \otimes e$. In the following, we will tacitly identify negatively graded derivations of $\mathcal{E}$ and negatively graded vector fields on $\mathcal{M}$.

Derivations of $\Gamma(\mathcal{E})$ are sections of a (graded) Lie algebroid $D\mathcal{E}$ over $\mathcal{M}$ with bracket given by the (graded) commutator, and anchor given by the symbol. A *homological derivation of* $\mathcal{E}$ is a degree-one derivation $\mathbb{Q}$, with symbol Q, such that $[\mathbb{Q}, \mathbb{Q}] = 0$ (in particular, Q is a homological vector field).

Example 6. Any degree-zero $\mathbb{N}Q$-vector bundle $(\mathcal{E}, \mathbb{Q})$ over a degree-one $\mathbb{N}$-manifold $\mathcal{M}$ is of the form $(A[1] \times_M E, d_E)$ for some nongraded Lie algebroid $A \to M$ equipped with a representation $E \to M$. Here d_E is the homological derivation induced on $\Gamma(\wedge^\bullet A^* \otimes E) = \Gamma(\mathcal{E})$. The algebroid structure on A corresponds to the symbol Q of $\mathbb{Q}$, while the (flat) A-connection ∇^E in E can be recovered from $\mathbb{Q}$ via formula

$$\nabla^E_X e = [\mathbb{Q}, X](e) = \mathbb{Q}(X(e)),$$

where $X \in \Gamma(A)$ is also interpreted as a derivation of $\mathcal{E}$ of degree -1 (see Example 5), and $e \in \Gamma(E)$.

2.2. ***Vector valued Cartan calculus on graded manifolds.*** Let $\mathcal{M}$ be a graded manifold and let $\mathcal{E}$ be a graded vector bundle over $\mathcal{M}$. Differential forms on $\mathcal{M}$ are functions on $T[1]\mathcal{M}$ which are polynomial on fibers of $T[1]\mathcal{M} \to \mathcal{M}$. In particular, the algebra $\Omega(\mathcal{M})$ of differential forms on $\mathcal{M}$ is equipped with two gradings: the "form" degree and the "internal, manifold" degree, which is usually referred simply as the *degree* (or, sometimes, the *weight*). The "total" degree is the sum of the form degree and the degree. Notice that the algebra $\Omega(\mathcal{M})$ is graded commutative with respect to the total degree. Similarly, $\mathcal{E}$-valued differential forms on $\mathcal{M}$ are sections of the vector bundle $T[1]\mathcal{M} \times_{\mathcal{M}} \mathcal{E} \to T[1]\mathcal{M}$ which are polynomial on fibers of $T[1]\mathcal{M} \to \mathcal{M}$. The $\Omega(\mathcal{M})$-module $\Omega(\mathcal{M}, \mathcal{E}) \simeq \Omega(\mathcal{M}) \otimes \Gamma(\mathcal{E})$ of $\mathcal{E}$-valued forms is equipped with two gradings. The "internal" degree will be referred to simply as the *degree*. We will denote by $|\omega|$ the degree of a homogeneous (with respect to the internal degree) $\mathcal{E}$-valued form ω.

Now, we briefly review the $\mathcal{E}$-valued version of Cartan calculus. Let $\mathbb{X}$ be a derivation of $\mathcal{E}$. There are unique derivations $i_{\mathbb{X}}$ and $L_{\mathbb{X}}$ of the vector bundle $T[1]\mathcal{M} \times_{\mathcal{M}} \mathcal{E} \to T[1]\mathcal{M}$ such that

(1) the symbol of $i_{\mathbb{X}}$ is the insertion i_X of the symbol X of $\mathbb{X}$,
(2) $i_{\mathbb{X}}$ vanishes on $\Gamma(\mathcal{E})$,
(3) the symbol of $L_{\mathbb{X}}$ is the Lie derivative L_X along the symbol X of $\mathbb{X}$,
(4) $L_{\mathbb{X}}$ agrees with $\mathbb{X}$ on $\Gamma(\mathcal{E})$.

Notice that, actually, $i_{\mathbb{X}}$ does only depend on the symbol of $\mathbb{X}$. For this reason, we will sometimes write i_X for $i_{\mathbb{X}}$.

Example 7. For any homogenous $\mathcal{E}$-valued form ω, $L_{\Delta_{\mathcal{E}}}\omega = |\omega|\omega$.

The following $\mathcal{E}$-valued Cartan identities hold:

$$[i_{\mathbb{X}}, i_{\mathbb{X}'}] = 0, \quad [L_{\mathbb{X}}, i_{\mathbb{X}'}] = i_{[\mathbb{X},\mathbb{X}']}, \quad [L_{\mathbb{X}}, L_{\mathbb{X}'}] = L_{[\mathbb{X},\mathbb{X}']}, \tag{1}$$

for all $\mathbb{X}, \mathbb{X}' \in \Gamma(D\mathcal{E})$, where the bracket $[-,-]$ denotes the graded commutator. Moreover,

$$i_{f\mathbb{X}} = f\, i_{\mathbb{X}}, \quad L_{f\mathbb{X}} = f L_{\mathbb{X}} + (-)^{|f|+|\mathbb{X}|} df\, i_{\mathbb{X}}, \tag{2}$$

for all $f \in C^\infty(\mathcal{M})$.

Now suppose that $\mathcal{E}$ is equipped with a flat connection ∇. Recall that a *connection* in $\mathcal{E}$ is a graded, homogeneous, $C^\infty(\mathcal{M})$-linear map $\nabla : \mathfrak{X}(\mathcal{M}) \to \Gamma(D\mathcal{E})$, denoted $X \mapsto \nabla_X$, such that the symbol of ∇_X is precisely X. In particular $|\nabla| = 0$. Derivation ∇_X is called the *covariant derivative along X*. A connection ∇ is *flat* if it is a morphism of (graded) Lie algebras, i.e., $[\nabla_X, \nabla_Y] = \nabla_{[X,Y]}$, for all $X, Y \in \mathfrak{X}(\mathcal{M})$. A connection ∇ in $\mathcal{E}$ determines a unique degree-one derivation d_∇ of the vector bundle $T[1]\mathcal{M} \times_{\mathcal{M}} \mathcal{E} \to T[1]\mathcal{M}$ such that

(1) the symbol of d_∇ is the de Rham differential $d \in \mathfrak{X}(T[1]\mathcal{M})$,

(2) $i_X d_\nabla e = \nabla_X e$ for all $e \in \Gamma(\mathcal{E})$ and $X \in \mathfrak{X}(\mathcal{M})$.

The derivation d_∇ is the *de Rham differential of* ∇. It is a homological derivation if and only if ∇ is flat.

Let ∇ be a flat connection in $\mathcal{E}$. The following identities hold:

$$[i_{\nabla_X}, d_\nabla] = L_{\nabla_X}, \quad [L_{\nabla_X}, d_\nabla] = 0, \quad [d_\nabla, d_\nabla] = 0, \tag{3}$$

for all $X \in \mathfrak{X}(\mathcal{M})$.

Remark 8. Specialize to the case when $\mathcal{M}$ is an ℕ-manifold and $\mathcal{E}$ is of degree zero. Then $\mathcal{E} = \mathcal{M} \times_M E$ for some vector bundle E over the degree-zero shadow M of $\mathcal{M}$. A connection ∇^0 in E induces a unique connection ∇ in $\mathcal{E}$ such that

$$\nabla_X e = \nabla^0_{\underline{X}} e,$$

for all $e \in \Gamma(E)$ and $X \in \mathfrak{X}_0(\mathcal{M})$, where $\underline{X} \in \mathfrak{X}(M)$ is the projection of X onto M. The connection ∇ is flat if and only if ∇^0 is flat. Moreover, every connection in $\mathcal{E}$ is of this kind. Notice that if ∇ is flat, whatever it is, the covariant derivative along the grading vector field Δ coincides with the grading derivation $\Delta_{\mathcal{E}}$. To see this it is enough to compare the action of ∇_Δ and $\Delta_{\mathcal{E}}$ on generators. Locally, $\Gamma(\mathcal{E})$ is generated by ∇^0-flat sections of E. Thus, let $e \in \Gamma(E)$ be ∇^0-flat. Then $\nabla_\Delta e = 0 = \Delta_{\mathcal{E}} e$. As an immediate consequence, every d_∇-closed E-valued differential form ω on $\mathcal{M}$ of positive degree n is also d_∇-exact, i.e., $\omega = d_\nabla \vartheta$, for a suitable ϑ. One can choose, for instance, $\vartheta = n^{-1} i_{\Delta_{\mathcal{E}}} \omega$. Indeed,

$$d_\nabla\Big(\frac{1}{n} i_{\Delta_{\mathcal{E}}} \omega\Big) = \frac{1}{n}[d_\nabla, i_{\Delta_{\mathcal{E}}}]\omega = \frac{1}{n} L_{\Delta_{\mathcal{E}}} \omega = \omega.$$

2.3. ***An alternative description of vector valued forms on ℕ-manifolds.*** In the following, we will only consider the case when $\mathcal{M}$ is an ℕ-manifold and $\mathcal{E}$ is generated in one single degree. Let M be the degree-zero shadow of $\mathcal{M}$. Then, up to an irrelevant shift, $\mathcal{E}$ is isomorphic to a pull-back $\mathcal{M} \times_M E$, where E is a nongraded vector bundle over M (see Example 2). Accordingly, we will often write $C^\infty(\mathcal{M}, E)$ for $\Gamma(\mathcal{E})$ and $\Omega(\mathcal{M}, E)$ for $\Omega(\mathcal{M}, \mathcal{E})$.

Remark 9. Despite the huge simplifications inherent to the hypothesis $\mathcal{E} \simeq \mathcal{M} \times_M E$, this case still captures many interesting situations. For instance, a degree-n symplectic ℕ-manifold [Roytenberg 2002] or contact ℕ-manifold [Grabowski 2013; Mehta 2013] can each be understood as an ℕ-manifold $\mathcal{M}$ equipped with a degree-n differential form with values in a vector bundle concentrated in just one degree (the trivial bundle $\mathcal{M} \times \mathbb{R}$ in the symplectic case, and a generically nontrivial line bundle concentrated in degree n in the contact case). We hope to discuss the case of a general vector bundle $\mathcal{E}$ elsewhere.

Theorem 10. *Let n be a positive integer. A degree-n differential k-form on $\mathcal{M}$ with values in E is equivalent to the following data*:

- *a degree-n (first order) differential operator $D : \mathfrak{X}_-(\mathcal{M}) \to \Omega^k(\mathcal{M}, E)$, and*
- *a degree-n $C^\infty(M)$-linear map $\ell : \mathfrak{X}_-(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M}, E)$,*

such that

$$D(fX) = f D(X) + (-)^{|X|} df\, \ell(X), \tag{4}$$

and, moreover,

$$L_X D(Y) - (-)^{|X||Y|} L_Y D(X) = D([X, Y]), \tag{5}$$

$$L_X \ell(Y) - (-)^{|X|(|Y|-1)} i_Y D(X) = \ell([X, Y]), \tag{6}$$

$$i_X \ell(Y) - (-)^{(|X|-1)(|Y|-1)} i_Y \ell(X) = 0. \tag{7}$$

for all $X, Y \in \mathfrak{X}_-(\mathcal{M})$, and $f \in C^\infty(M)$.

Remark 11. By induction on n, Theorem 10 provides a description of E-valued differential forms in terms of nongraded data. Indeed, D and ℓ take values in lower-degree forms and one can use degree-zero forms, namely E-valued forms on M as base of induction.

Proof. Let ω be a degree-n, E-valued differential k-form on $\mathcal{M}$. Define $D : \mathfrak{X}_-(\mathcal{M}) \to \Omega^k(\mathcal{M}, E)$ and $\ell : \mathfrak{X}_-(\mathcal{M}) \to \Omega^{k-1}(\mathcal{M}, E)$ by putting

$$D(X) := L_X \omega, \quad \text{and} \quad \ell(X) := i_X \omega, \quad X \in \mathfrak{X}_-(\mathcal{M}). \tag{8}$$

Properties (4), (5), (6), and (7) immediately follow from identities (1), and (2).

Conversely, let D and ℓ be as in the statement of the theorem, and prove that there exists a unique degree-n differential form $\omega \in \Omega^k(\mathcal{M}, E)$ fulfilling (8). We propose a local proof. One can pass to the global setting by partition of unity arguments. Let $\dots, z^a, \dots$ be positively graded coordinates on $\mathcal{M}$ and $\partial_a := \partial/\partial z^a$. In particular the grading derivation $\Delta_{\mathcal{E}}$ is locally given by

$$\Delta_{\mathcal{E}} = |z^a| z^a \partial_a.$$

Moreover, $\mathfrak{X}_-(\mathcal{M})$ is locally generated, as a $C^\infty(M)$-module, by vector fields

$$z^{b_1} \cdots z^{b_k} \partial_b, \quad |z^{b_1}| + \cdots + |z^{b_k}| - |z^b| < 0.$$

Put

$$\omega := \frac{|z^a|}{n} \big(z^a D(\partial_a) + dz^a \ell(\partial_a) \big)$$

and prove (8). First of all, for $X = X^a \partial_a$,

$$L_X\omega = \frac{|z^a|}{n} L_X\big(z^a D(\partial_a) + dz^a \ell(\partial_a)\big)$$
$$= \frac{|z^a|}{n}\big(X^a D(\partial_a) + (-)^{|z^a||X|} L_X D(\partial_a) + (-)^{|X|} dX^a \ell(\partial_a) + (-)^{|X|(|z^a|+1)} dz^a L_X \ell(\partial_a)\big).$$

In view of (5) and (6),

$$(-)^{|z^a||X|} L_X D(\partial_a) = L_{\partial_a} D(X) - D([\partial_a, X])$$

and

$$(-)^{|X|(|z^a|+1)} L_X \ell(\partial_a) = i_{\partial_a} D(X) - (-)^{|X|} \ell([\partial_a, X]),$$

so that

$$\begin{aligned}(9)\quad L_X\omega &= \frac{1}{n} L_{\Delta_\mathcal{E}} D(X) + \frac{|z^a|}{n}\big(X^a D(\partial_a) + (-)^{|X|} dX^a \ell(\partial_a) \\ &\qquad - z^a D([\partial_a, X]) - (-)^{|X|} dz^a \ell([\partial_a, X])\big) \\ &= \frac{n+|X|}{n} D(X) + \frac{|z^a|}{n}\big(X^a D(\partial_a) + (-)^{|X|} dX^a \ell(\partial_a) \\ &\qquad - z^a D([\partial_a, X]) - (-)^{|X|} dz^a \ell([\partial_a, X])\big).\end{aligned}$$

Similarly

$$(10)\qquad i_X\omega = \frac{n+|X|}{n}\ell(X) + \frac{|z^a|}{n}\big(X^a \ell(\partial_a) - z^a \ell([\partial_a, X])\big).$$

Using $X = \partial_b$ in (9) and (10) gives

$$(11)\qquad L_{\partial_b}\omega = D(\partial_b), \quad \text{and} \quad i_{\partial_b}\omega = \ell(\partial_b).$$

In view of identity (4), in order to prove (8), it is enough to restrict to vector fields X of the form $z^{b_1} \cdots z^{b_k} \partial_b$. This case can be treated by induction on k, using (11) as the base. Namely, use $X = z^{b_1} \cdots z^{b_k} \partial_b$ in (9), with $k > 0$. Since

$$[\partial_a, X] = \sum_i (-)^{(|z^{b_1}|+\cdots+|z^{b_{i-1}}|)|z^a|} \delta_a^{b_i} z^{b_1} \ldots \widehat{z^{b_i}} \cdots z^{b_k} \partial_b,$$

where a hat "$\widehat{-}$" denotes omission, by the induction hypothesis we have

$$D([\partial_a, X]) = L_{[\partial_a, X]}\omega, \quad \text{and} \quad \ell([\partial_a, X]) = i_{[\partial_a, X]}\omega.$$

A direct computation shows that the second summand in the right-hand side of (9) is equal to $-(|X|/n)L_X\omega$. Similarly, the second summand in the right-hand side of (10) is equal to $-(|X|/n)i_X\omega$. Notice that, since $k > 0$, we have $|X| > -n$ and can conclude that $L_X\omega = D(X)$, and similarly, $i_X\omega = \ell(X)$.

To prove uniqueness, it is enough to show that a degree-n differential form ω with values in E is completely determined by contraction with, and Lie derivative along, negatively graded derivations. Thus,

$$n\omega = L_{\Delta_{\mathcal{E}}}\omega = |z^a|(z^a L_{\partial_a}\omega + dz^a i_{\partial_a}\omega).$$

In particular ω is completely determined by $L_{\partial_a}\omega$ and $i_{\partial_a}\omega$. □

We will refer to the data (D, ℓ) corresponding to a vector valued form ω as the *Spencer data* of ω. Indeed, as we will show in Section 5, they are a vast generalization of the *Spencer operators* considered in [Crainic et al. 2015; Salazar 2013].

Example 12. Let $E \to M$ be a nongraded vector bundle equipped with a flat connection ∇, and let $\mathcal{M}$ be an $\mathbb{N}$-manifold. As discussed in Section 2.2, ∇ induces a flat connection in the graded vector bundle $\mathcal{M} \times_M E \to \mathcal{M}$ which we denote again by ∇. In its turn, the induced connection determines a homological derivation d_∇ of the vector bundle $T[1]\mathcal{M} \times_M E \to T[1]\mathcal{M}$ of E-valued forms on $\mathcal{M}$. Notice that d_∇ maps k-forms to $(k+1)$-forms. Now, let $\omega \in \Omega^k(\mathcal{M}, E)$ and let (D, ℓ) be the corresponding Spencer data. We want to describe the Spencer data (D', ℓ') of $d_\nabla\omega$. To do this, we first observe that a discussion similar to that in Remark 8 shows that, whatever ∇, the covariant derivative along a negatively graded vector field $X \in \mathfrak{X}_-(\mathcal{M})$ satisfies $\nabla_X = X$. Hence, from (3)

$$D'(X) = L_X d_\nabla\omega = L_{\nabla_X} d_\nabla\omega = d_\nabla L_X\omega = d_\nabla D(X)$$

and

$$\ell'(X) = i_X d_\nabla\omega = i_{\nabla_X} d_\nabla\omega = L_X\omega - (-)^{|X|} d_\nabla i_X\omega = D(X) - (-)^{|X|} d_\nabla \ell(X),$$

which completely describe (D', ℓ') in terms of (D, ℓ) and d_∇.

Example 13. Let $E \to M$ be a nongraded vector bundle. The first jet bundle $J^1E \to M$ fits in an exact sequence

$$0 \longrightarrow \Omega^1(M, E) \longrightarrow \Gamma(J^1E) \xrightarrow{p} \Gamma(E) \longrightarrow 0 \tag{12}$$

of $C^\infty(M)$-linear maps, where p is the canonical projection. Sequence (12) splits (beware, *over* $\mathbb{R}$ not over $C^\infty(M)$) via the universal first order differential operator $j^1 : \Gamma(E) \to \Gamma(J^1E)$. Accordingly, there is a first order differential operator $S : \Gamma(J^1E) \to \Omega^1(M, E)$ sometimes called the *Spencer operator*. The degree-n $\mathbb{N}$-manifold $\mathcal{M} = J^1E[n]$ comes equipped with an E-valued, degree-n *Cartan* 1*-form* θ. In order to define θ, recall that negatively graded vector fields on $\mathcal{M}$ are concentrated in degree $-n$, and $\mathfrak{X}_{-n}(\mathcal{M})$ identifies with $\Gamma(J^1E)$ as a $C^\infty(M)$-module. Now, θ is uniquely defined by the properties

$$i_{j^1e}\theta = e \quad \text{and} \quad L_{j^1e}\theta = 0, \tag{13}$$

for all $e \in \Gamma(E)$. It immediately follows from (13) that the Spencer data (D, ℓ) of θ identify with $(-)^n$ times the Spencer operator $S : \Gamma(J^1 E) \to \Omega^1(M, E)$ and the projection $\Gamma(J^1 E) \to \Gamma(E)$ respectively.

Example 14. Let M be a nongraded manifold. The degree-n $\mathbb{N}$-manifold $\mathcal{M} = T^*[n]M$ comes equipped with the obvious tautological, degree-n 1-form ϑ. Consider the degree-n 2-form $\omega = d\vartheta$. Negatively graded vector fields on $\mathcal{M}$ are concentrated in degree $-n$, and $\mathfrak{X}_{-n}(\mathcal{M})$ is naturally isomorphic to $\Omega^1(M)$ as a $C^\infty(M)$-module. It is easy to see that ω is uniquely defined by the properties

$$i_{df}\omega = df \quad \text{and} \quad L_{df}\omega = 0, \tag{14}$$

for all $f \in C^\infty(M)$. It immediately follows from (14) that the Spencer data (D, ℓ) of ω identify with $(-)^n$ times the exterior differential $d : \Omega^1(M) \to \Omega^2(M)$ and the identity $\mathrm{id} : \Omega^1(M) \to \Omega^1(M)$ respectively.

Example 15. Let $E \to M$ be a nongraded vector bundle equipped with a flat connection ∇. The degree-n $\mathbb{N}$-manifold $\mathcal{M} = T^*[n]M \otimes E$ is equipped with a tautological, degree-n E-valued 1-form ϑ. The flat connection ∇ induces a flat connection in the graded vector bundle $\mathcal{M} \times_M E \to \mathcal{M}$ which we denote again by ∇. Consider the homological derivation d_∇ as in Example 12. Notice that d_∇ agrees with the de Rham differential of ∇ on degree-zero forms, i.e., elements of $\Omega(M, E)$. Consider the degree-n 2-form $\omega = d_\nabla\vartheta$ with values in E. Negatively graded vector fields on $\mathcal{M}$ are concentrated in degree $-n$, and $\mathfrak{X}_{-n}(\mathcal{M})$ is isomorphic to $\Omega^1(M, E)$ as a $C^\infty(M)$-module. It is easy to see that ω is uniquely defined by the properties

$$i_{d_\nabla e}\omega = d_\nabla e \quad \text{and} \quad L_{d_\nabla e}\omega = 0, \tag{15}$$

for all $e \in \Gamma(E)$. It immediately follows from (15) that the Spencer data (D, ℓ) of ω identify with $(-)^n$ times the de Rham differential $d_\nabla : \Omega^1(M, E) \to \Omega^2(M, E)$ and the identity $\mathrm{id} : \Omega^1(M, E) \to \Omega^1(M, E)$ respectively.

In the three remaining sections we use Theorem 10 (and Proposition 17 below) to describe degree-one $\mathbb{N}Q$-manifolds equipped with a compatible vector valued differential form (see below) in terms of nongraded data. In particular, we manage to give alternative proofs of known results about compatible contact structures [Grabowski 2013; Mehta 2013], involutive distributions [Zambon and Zhu 2012], and symplectic forms [Roytenberg 2002] on degree-one $\mathbb{N}Q$-manifolds. We also manage to find new results about compatible, presymplectic and locally conformal symplectic structures, and, more generally, higher order vector valued forms on degree-one $\mathbb{N}Q$-manifolds. It turns out (Theorem 36) that a compatible degree-one differential k-form on a degree-one $\mathbb{N}Q$-manifold $(\mathcal{M}, Q)$ is equivalent to a Lie algebroid equipped with a structure recently identified in [Crainic et al. 2015] as the infinitesimal counterpart of a multiplicative vector valued form on a Lie groupoid (see also [Salazar 2013]), namely, a *k-th order Spencer operator*.

Let $\mathcal{M}$ be an $\mathbb{N}$-manifold, with degree-zero shadow M, and let $(\mathcal{E}, \mathbb{Q})$ be an $\mathbb{N}Q$-vector bundle over it. We denote by Q the symbol of $\mathbb{Q}$.

Definition 16. An $\mathcal{E}$-valued differential form on $\mathcal{M}$, ω, is *compatible with* $\mathbb{Q}$ if $L_{\mathbb{Q}}\omega = 0$.

Suppose $\mathcal{E}$ is of degree zero. Then $\mathcal{E} = \mathcal{M} \times_M E$ for a nongraded vector bundle $E \to M$. For later use, we conclude this section expressing the compatibility of an E-valued form ω on $\mathcal{M}$ with $\mathbb{Q}$ in terms of Spencer data.

Proposition 17. *Let $\omega \in \Omega^k(\mathcal{M}, E)$ be an E-valued k-form on $\mathcal{M}$ of degree $n > 0$, and let (D, ℓ) be its Spencer data. Then ω is compatible with $\mathbb{Q}$, i.e., $L_{\mathbb{Q}}\omega = 0$, if and only if*

$$\begin{aligned}(16)\quad A(X, Y) := D([[Q, X], Y]) &- L_{[\mathbb{Q},X]}D(Y)\\ &- (-)^{|X||Y|}\big(L_{[\mathbb{Q},Y]}D(X) - L_{\mathbb{Q}}L_Y D(X)\big) = 0,\end{aligned}$$

$$\begin{aligned}(17)\quad B(X, Y) := \ell([[Q, X], Y]) &- (-)^{|Y|}i_{[Q,X]}D(Y)\\ &- (-)^{|X||Y|}\big(L_{[\mathbb{Q},Y]}\ell(X) - L_{\mathbb{Q}}L_Y\ell(X)\big) = 0,\end{aligned}$$

$$(18)\quad C(X, Y) := i_{[Q,X]}\ell(Y) + (-)^{(|X|-1)(|Y|-1)}\big(i_{[Q,Y]}\ell(X) - L_{\mathbb{Q}}i_Y\ell(X)\big) = 0.$$

Remark 18. By induction on n, Proposition 17 provides a description of the compatibility condition between ω and $\mathbb{Q}$ in terms of nongraded data (see also Remark 19 below). Notice that, when $|\omega| = 1$, the last summand in (16), (17) and (18) vanishes by degree reasons.

Proof. First of all, notice that $[[\mathbb{Q}, X], Y]$ is negatively graded for all X, Y. Hence it identifies with $[[Q, X], Y]$. In particular, the left-hand side of (16), (17) and (18) are well-defined. Now, for any ω as in the statement, $L_{\mathbb{Q}}\omega$ is a form of degree $n+1$. Since every form of positive degree on $\mathcal{M}$ is completely determined by its Spencer data, $L_{\mathbb{Q}}\omega$ vanishes if and only if

$$L_Y L_X L_{\mathbb{Q}}\omega = i_Y L_X L_{\mathbb{Q}}\omega = L_Y i_X L_{\mathbb{Q}}\omega = i_Y i_X L_{\mathbb{Q}}\omega = 0,$$

for all $X, Y \in \mathfrak{X}_-(\mathcal{M})$. It immediately follows from the second Cartan identity (1) that condition $i_Y L_X L_{\mathbb{Q}}\omega = 0$ is actually redundant. It remains to compute $L_Y L_X L_{\mathbb{Q}}\omega$, $L_Y i_X L_{\mathbb{Q}}\omega$, and $i_Y i_X L_{\mathbb{Q}}\omega$. So

$$\begin{aligned}L_Y L_X L_{\mathbb{Q}}\omega &= L_Y L_{[X,\mathbb{Q}]}\omega - (-)^{|Y|}L_Y L_{\mathbb{Q}} L_X\omega\\ &= (-)^{|X|+|Y|(|X|+1)}\big(L_{[[\mathbb{Q},X],Y]}\omega - L_{[\mathbb{Q},X]}L_Y\omega\big)\\ &\quad - (-)^{|X|+|Y|}\big(L_{[\mathbb{Q},Y]}L_X\omega - L_{\mathbb{Q}}L_Y L_X\omega\big),\end{aligned}$$

which differs from $A(X,Y)$ in (16) by an overall sign $(-)^{|X|+|Y|(|X|+1)}$. Similarly,

$$\begin{aligned}L_Y i_X L_{\mathbb{Q}}\omega &= (-)^{|X|} L_Y i_{[\mathbb{Q},X]}\omega - (-)^{|X|} L_Y L_{\mathbb{Q}} i_X\omega \\ &= (-)^{(|X|+1)(|Y|+1)}\big(i_{[[\mathbb{Q},X],Y]}\omega - (-)^{|Y|} i_{[\mathbb{Q},X]} L_Y\omega\big) \\ &\quad + (-)^{|X|+|Y|}\big(L_{[\mathbb{Q},Y]} i_X\omega - L_{\mathbb{Q}} L_Y i_X\omega\big),\end{aligned}$$

which differs from $B(X,Y)$ in (17) by an overall sign $(-)^{(|X|+1)(|Y|+1)}$. Finally,

$$\begin{aligned}i_Y i_X L_{\mathbb{Q}}\omega &= (-)^{|X|} i_Y i_{[\mathbb{Q},X]}\omega - (-)^{|X|} i_Y L_{\mathbb{Q}} i_X\omega \\ &= (-)^{|X||Y|} i_{[\mathbb{Q},X]} i_Y\omega - (-)^{|X|+|Y|}\big(i_{[\mathbb{Q},Y]} i_X - L_{\mathbb{Q}} i_Y i_X\omega\big),\end{aligned}$$

which differs from $C(X,Y)$ in (18) by an overall sign $(-)^{|X||Y|}$. □

Remark 19. When $\mathcal{M}$ is the total space of a negatively graded vector bundle $V \to M$ (which is always the case up to a noncanonical isomorphism), a homological vector field on $\mathcal{M}$ is the same as an L_∞-algebroid structure on $\Gamma(V^*)$ (see, e.g., [Bonavolontà and Poncin 2013; Bruce 2011; Sati et al. 2012; Vitagliano 2015b]). We conjecture the existence of formulas expressing the compatibility between ω and $\mathbb{Q}$ in terms of the higher brackets (and the anchor) of this L_∞-algebroid, and the Spencer data of ω. Similarly, when no isomorphism $\mathcal{M} \simeq V$ is assigned, there should be formulas involving *Getzler higher derived brackets* on $\mathfrak{X}_-(\mathcal{M})$ [Getzler 2010]. Finding these formulas goes beyond the scopes of this paper and we postpone this task to a subsequent publication.

3. Vector valued 1-forms on $\mathbb{N}Q$-manifolds

3.1. ***Vector valued 1-forms and distributions.*** Let $\mathcal{M}$ be a degree-n $\mathbb{N}$-manifold, $n > 0$, and let $(\mathcal{E} = \mathcal{M} \times_M E, \mathbb{Q})$ be a degree-zero $\mathbb{N}Q$-vector bundle over it. According to Definition 16, a degree-n 1-form θ with values in E is *compatible* with $\mathbb{Q}$ if, by definition, $L_{\mathbb{Q}}\theta = 0$. Several interesting geometric structures are described by compatible 1-forms. For instance, compatible distributions on an $\mathbb{N}Q$-manifold are equivalent to surjective compatible 1-forms. Namely, Let $(\mathcal{M}, Q)$ be a degree-n $\mathbb{N}Q$-manifold, and let $\mathcal{D} \subset T\mathcal{M}$ be a distribution on $\mathcal{M}$. Consider the normal bundle $T\mathcal{M}/\mathcal{D}$. Projection $T\mathcal{M} \to T\mathcal{M}/\mathcal{D}$ can be interpreted as a degree-zero surjective 1-form with values in $T\mathcal{M}/\mathcal{D}$. We say that $\mathcal{D}$ is *cogenerated* in degree k if $T\mathcal{M}/\mathcal{D}$ is generated in degree $-k$. In this case, $T\mathcal{M}/\mathcal{D} = \mathcal{M} \times_M E[k]$ for a suitable nongraded vector bundle $E \to M$, and the projection $\theta_{\mathcal{D}} : T\mathcal{M} \to \mathcal{M} \times_M E$ can be interpreted as a degree-k, surjective, E-valued 1-form such that $\ker\theta_{\mathcal{D}} = \mathcal{D}$. Conversely, if $E \to M$ is a nongraded vector bundle and θ is a degree-k, surjective 1-form with values in E, then $\mathcal{D} := \ker\theta$ is a distribution such that $\theta_{\mathcal{D}} = \theta$.

Definition 20. A distribution $\mathcal{D}$ on $\mathcal{M}$, cogenerated in degree n, is *compatible* with Q if $[Q, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$.

Now, let $\mathcal{D}$ be a distribution cogenerated in degree-n and E be such that $T\mathcal{M}/\mathcal{D} = \mathcal{M} \times_M E[-n]$. If $\mathcal{D}$ is compatible with Q, then the commutator with Q restricts to a homological derivation of $\mathcal{D}$, hence it also descends to a homological derivation of the vector bundle $\mathcal{M} \times_M E$ which we denote by $\mathbb{Q}$. Moreover, $L_{\mathbb{Q}}\theta_{\mathcal{D}} = 0$. Indeed, for every vector field $X \in \mathfrak{X}(\mathcal{M})$,

$$i_X L_{\mathbb{Q}}\theta_{\mathcal{D}} = (-)^{|X|}\big(i_{[Q,X]}\theta_{\mathcal{D}} - \mathbb{Q}(i_X\theta_{\mathcal{D}})\big) = 0. \tag{19}$$

Conversely, if $(\mathcal{E}, \mathbb{Q})$ is a degree-zero $\mathbb{N}Q$-bundle and θ is a degree-n surjective 1-form with values in E, then it follows from (19) that $L_{\mathbb{Q}}\theta = 0$ if and only if 1) $\ker\theta$ is a distribution compatible with the symbol Q of $\mathbb{Q}$, and 2) $\mathbb{Q}$ is induced on $\Gamma(\mathcal{E})$ by the adjoint operator $[Q, -]$ on $\mathfrak{X}(\mathcal{M})$. One concludes that compatible distributions are the same as compatible surjective 1-forms.

Remark 21. The above discussion is actually independent of the degree of Q. Hence, it shows that an infinitesimal symmetry of $\mathcal{D}$, i.e., any vector field X such that $[X, \Gamma(\mathcal{D})] \subset \Gamma(\mathcal{D})$, determines a derivation $\mathbb{X}$ of $T\mathcal{M}/\mathcal{D}$ via

$$\mathbb{X}(Y \bmod \mathcal{D}) := [X, Y] \bmod \mathcal{D}.$$

The symbol of $\mathbb{X}$ is precisely X. Moreover, one can compute the Lie derivative $L_{\mathbb{X}}\theta_{\mathcal{D}}$, and find $L_{\mathbb{X}}\theta_{\mathcal{D}} = 0$.

Now recall that a distribution $\mathcal{D}$ on a (graded) manifold $\mathcal{M}$ comes equipped with a *curvature form*

$$\omega_{\mathcal{D}} : \Gamma(\mathcal{D}) \times \Gamma(\mathcal{D}) \to \Gamma(T\mathcal{M}/\mathcal{D}), \quad (X, Y) \mapsto [X, Y] \bmod \Gamma(\mathcal{D}).$$

The curvature form $\omega_{\mathcal{D}}$ measures how far is $\mathcal{D}$ from being involutive. The two limit cases $\omega_{\mathcal{D}}$ nondegenerate and $\omega_{\mathcal{D}} = 0$ are of special interest. The first one corresponds to *maximally nonintegrable distributions*, the second one to *involutive distributions*.

3.2. *Degree one contact $\mathbb{N}Q$-manifolds.* Let $\mathcal{M}$ be an $\mathbb{N}$-manifold and let C be a hyperplane distribution on it. Since $\mathcal{L} := T\mathcal{M}/C$ is a line bundle, it is generated in one single degree. A *degree-n contact $\mathbb{N}$-manifold* is an $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a *degree-n contact structure*, i.e., a hyperplane distribution C, such that the line bundle $\mathcal{L} := T\mathcal{M}/C$ is generated in degree n, and the curvature form ω_C is nondegenerate (see [Grabowski 2013] for an alternative definition exploiting the "symplectization trick").

Example 22. Let $L \to M$ be a nongraded line bundle. The kernel of the Cartan form θ on $J^1L[n]$ (see Example 13) is a degree-n contact structure.

It follows from the definition that, if $(\mathcal{M}, C)$ is a degree-n contact $\mathbb{N}$-manifold, then the degree of $\mathcal{M}$ is at most n. When $\mathcal{L}$ is a trivial line bundle, C is the kernel of a (nowhere vanishing) 1-form α which can be used to simplify the theory significantly (see [Mehta 2013]). In this case the contact structure is said to be *coorientable* and a choice of α provides a coorientation (i.e., an orientation of $\mathcal{L}$). In the general case, $\mathcal{L} := \mathcal{M} \times_M L[n]$ for a nongraded line bundle $L \to M$, and C is the kernel of a (degree-n) 1-form θ_C with values in a (generically nontrivial) line bundle L.

A degree-n contact structure on $\mathcal{M}$ determines a nondegenerate Jacobi bracket $\{-,-\}$ of degree $-n$ on $\Gamma(\mathcal{L})$, i.e., a Lie bracket of degree $-n$ which is a graded first order differential operator in each entry and such that the associated morphism $J^1\mathcal{L} \otimes J^1\mathcal{L} \to \mathcal{L}$ is nondegenerate (see also Appendix A). For the details about how to define the Jacobi bracket $\{-,-\}$ from C in the nongraded case see, for instance, [Crainic and Salazar 2015]. The generalization to the graded case can be carried out straightforwardly and the obvious details are left to the reader. A *degree-n contact $\mathbb{N}Q$-manifold* is a degree-n contact manifold $(\mathcal{M}, C)$ equipped with a homological vector field Q such that $[Q, \Gamma(C)] \subset \Gamma(C)$, in other words it is an $\mathbb{N}Q$-manifold equipped with a compatible degree-n contact structure. If $(\mathcal{M}, C, Q)$ is a contact $\mathbb{N}Q$-manifold, the homological vector field Q induces a homological derivation $\mathbb{Q}$ of $\mathcal{L}$ as discussed above. Thus, equivalently, a degree-n contact $\mathbb{N}Q$-manifold is a degree-n contact manifold $(\mathcal{M}, C)$ equipped with a homological derivation $\mathbb{Q}$ of $\mathcal{L}$ such that $L_{\mathbb{Q}}\theta_C = 0$.

Theorem 23 [Mehta 2013; Grabowski 2013]. *Every degree-one contact $\mathbb{N}$-manifold $(\mathcal{M}, C)$ is of the kind $(J^1L[1], \ker\theta)$, up to contactomorphisms, where $L \to M$ is a (nongraded) line bundle, and θ is the Cartan form on $J^1L[1]$. Moreover, there is a one-to-one correspondence between degree-one contact $\mathbb{N}Q$-manifolds and (nongraded) manifolds equipped with an abstract Jacobi structure (see the appendixes).*

Notice that Mehta does only discuss the case when C is coorientable, i.e., $T\mathcal{M}/C$ is globally trivial. Moreover, he selects a contact form, which amounts to selecting a global trivialization $T\mathcal{M}/C \simeq \mathcal{M} \times \mathbb{R}[1]$ (see [Mehta 2013] for details). On the other hand, in independent work Grabowski discusses the general case (he actually treats the degree-two case as well). His proof relies on the "symplectization trick" which consists in understanding a contact manifold as a homogeneous symplectic manifold (see [Grabowski 2013]) and then using already known results in the symplectic case. We propose an alternative proof avoiding the "symplectization trick" and focusing on the Spencer data of the structure 1-form of C. We refer to [Crainic and Salazar 2015] for details on abstract Jacobi structures.

Proof. Let $(\mathcal{M} = A[1], C)$ be a degree-one contact $\mathbb{N}$-manifold, and let $\mathcal{L} = T\mathcal{M}/C$ be the associated degree-one line-bundle. Then $\mathcal{L} = \mathcal{M} \times_M L[1]$ for a nongraded line bundle $L \to M$, and θ_C is a degree-one L-valued 1-form on $\mathcal{M}$. Denote by (D, ℓ)

the Spencer data of θ_C. The Jacobi bracket $\{-,-\}$ determines a isomorphism of graded vector bundles of degree -1 between $J^1\mathcal{L}$ and $D\mathcal{L}$. Since negatively graded derivations are completely determined by their symbol, this gives an isomorphism $\Gamma(J^1L) \simeq \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Gamma(A)$, hence a diffeomorphism $\mathcal{M} \simeq J^1L[1]$. The diagram

$$\begin{tikzcd}
\Gamma(L) \arrow[d, equal] & \Gamma(A) \arrow[l, "\ell"'] \arrow[r, "D"] \arrow[d, "\wr"] & \Omega^1(M, L) \arrow[d, equal] \\
\Gamma(L) & \Gamma(J^1L) \arrow[l, "p"'] \arrow[r, "-S"] & \Omega^1(M, L)
\end{tikzcd}$$

commutes, as can be easily seen. This shows that the diffeomorphism $\mathcal{M} \simeq J^1L[1]$ identifies θ_C with the Cartan form θ (see Example 13), thus proving the first part of the statement. In the following we identify $\mathcal{M}$ and $J^1L[1]$. For the second part of the statement, let $\mathbb{Q}$ be a homological derivation of $\mathcal{L}$ and let Q be its symbol. Moreover, let $(J^1L, \rho, [\![-,-]\!])$ and (L, ∇^L) be the Lie algebroid and the Lie algebroid representation associated to $\mathbb{Q}$. We use Proposition 17 to see when is $(\mathcal{M}, C, \mathbb{Q})$ a contact $\mathbb{N}Q$-manifold. Since θ is a 1-form, Equation (18) is automatically satisfied, and θ is compatible with $\mathbb{Q}$ if and only if $A(X, Y) = B(X, Y) = 0$, with $\omega = \theta$ and $X, Y \in \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Gamma(J^1L)$. In fact, one can even restrict to X, Y in the form $j^1\lambda, j^1\mu$, with $\lambda, \mu \in \Gamma(L)$. In this case, one gets

$$A(X, Y) = D([[Q, j^1\lambda], j^1\mu]) = -S[\![j^1\lambda, j^1\mu]\!],$$

and

$$B(X, Y) = \ell([[Q, j^1\lambda], j^1\mu]) + L_{[\mathbb{Q}, j^1\mu]}\lambda = p[\![j^1\lambda, j^1\mu]\!] + \nabla^L_{j^1\mu}\lambda,$$

where we used that $\ell(j^1\lambda) = i_{j^1\lambda}\theta = \lambda$, and $D(j^1\lambda) = L_{j^1\lambda}\theta = 0$ (see Example 13). Concluding, $(\mathcal{M}, C, \mathbb{Q})$ is a contact $\mathbb{N}Q$-manifold if and only if $p[\![j^1\lambda, j^1\mu]\!] = -\nabla^L_{j^1\mu}\lambda = \nabla^L_{j^1\lambda}\mu$ and $S[\![j^1\lambda, j^1\mu]\!] = 0$, i.e., if and only if $(J^1L, [\![-,-]\!], \rho)$ is the Lie algebroid associated to a Jacobi structure on $L \to M$, and ∇^L is its natural representation (see Appendix B). □

3.3. ***Involutive distributions on degree-one $\mathbb{N}Q$-manifolds.*** Compatible involutive distributions (cogenerated in degree one) on a degree-one $\mathbb{N}Q$-manifold are equivalent to infinitesimally multiplicative (IM) foliations of a special kind. Let $(A, [\![-,-]\!], \rho)$ be a Lie algebroid over a manifold M, and let $F \subset TM$ be an involutive distribution. An *IM foliation of A over F* [Jotz and Ortiz 2011] is a triple consisting of

- an involutive distribution F,
- a Lie subalgebroid $B \subset A$,
- a flat F-connection ∇ in the quotient bundle A/B,

such that

(1) sections X of A such that X mod B is ∇-flat form a Lie subalgebra in $\Gamma(A)$ with sections of B as a Lie ideal,

(2) ρ takes values in the stabilizer of F,

(3) $\rho|_B$ takes values in F.

As the terminology suggests, IM foliations are infinitesimal counterparts of involutive multiplicative distributions on Lie groupoids [Jotz and Ortiz 2011]. Zambon and Zhu [2012] proved that IM foliations can be also understood as degree-one ℕQ-manifolds equipped with an involutive distribution preserved by the homological vector field. In the following, we restrict to distributions cogenerated in degree one. In this particularly simple situation, we can provide an alternative proof of Zambon–Zhu result exploiting the description of vector valued forms in terms of their Spencer data.

Lemma 24. *Let* $(A, [\![-,-]\!], \rho)$ *be a Lie algebroid over a manifold* M. *If* (TM, B, ∇) *is an IM foliation of* A *over* TM, *then there is a flat* A*-connection* $\nabla^{A/B}$ *in* A/B *such that*

$$\nabla^{A/B}_X(Y \bmod B) = \nabla_{\rho(Y)}(X \bmod B) - [\![Y, X]\!] \bmod B, \tag{20}$$

and, moreover,

$$d_\nabla([\![X, Y]\!] \bmod B) = L_{\nabla^{A/B}_X} d_\nabla(Y \bmod B) - L_{\nabla^{A/B}_Y} d_\nabla(X \bmod B), \tag{21}$$

for all $X, Y \in \Gamma(A)$. *Conversely, if* $B \subset A$ *is a vector subbundle,* ∇ *is a flat connection in* A/B, *and* $\nabla^{A/B}$ *is a flat* A*-connection in* A/B *satisfying* (20) *and* (21), *then* (TM, B, ∇) *is an IM foliation of* A *over* TM.

Proof. For the first part of the statement, let (TM, B, ∇) be an IM foliation as in the statement. Denote by Γ_∇ the sheaf on M consisting of sections X of A such that X mod B is ∇-flat. Since $\Gamma(A/B)$ is locally generated by flat sections, $\Gamma(A)$ is locally generated by Γ_∇. Now, the left-hand side of (20) is clearly $C^\infty(M)$-linear in X. Moreover, it vanishes whenever $Y \in \Gamma(B)$. To see this, it is enough to compute on local generators $X \in \Gamma_\nabla$. In this case, the left-hand side of (20) reduces to $-[\![Y, X]\!]$ mod B which vanishes by property (1) of IM foliations whenever $Y \in \Gamma(B)$. One concludes that (20) defines a differential operator $\nabla^{A/B}_X$ in $\Gamma(A/B)$ for all $X \in \Gamma(A)$. It is easy to see that, besides being $C^\infty(M)$-linear in X, $\nabla^{A/B}_X$ is actually a derivation with symbol $\rho(X)$. Thus $\nabla^{A/B}$ is a well-defined A-connection in A/B. To see that it is flat, check that the curvature

$$R(X, Y)(Z \bmod B) := \big([\nabla^{A/B}_X, \nabla^{A/B}_Y] - \nabla^{A/B}_{[\![X,Y]\!]}\big)(Z \bmod B)$$

vanishes on all X, Y, Z. Since R is linear in the first two arguments, it is enough to check that it vanishes on $X, Y \in \Gamma_\nabla$. In this case $[\![X, Y]\!] \in \Gamma_\nabla$ as well and

$$R(X, Y)(Z \bmod B) = [\![[\![Z, Y]\!], X]\!] - [\![[\![Z, X]\!], Y]\!] + [\![Z, [\![X, Y]\!]]\!] = 0$$

by the Jacobi identity.

Finally, notice that Equation (21) is equivalent to

$$\begin{aligned}(22)\quad & \nabla_Z([\![X, Y]\!] \bmod B) \\ & = \big(\nabla^{A/B}_X \nabla_Z - \nabla_{[\rho(X), Z]}\big)(Y \bmod B) - \big(\nabla^{A/B}_Y \nabla_Z - \nabla_{[\rho(Y), Z]}\big)(X \bmod B),\end{aligned}$$

$X, Y \in \Gamma(A)$, and $Z \in \mathfrak{X}(M)$. Actually, (22) can be easily obtained from (21), by inserting Z in both sides, and using

$$[i_Z, L_{\nabla^{A/B}_X}] = i_{[Z, \rho(X)]}.$$

Thus it is enough to check that the expression

$$\begin{aligned}S(X, Y; Z) := {} & \nabla_Z([\![X, Y]\!] \bmod B) \\ & - \big(\nabla^{A/B}_X \nabla_Z - \nabla_{[\rho(X), Z]}\big)(Y \bmod B) + \big(\nabla^{A/B}_Y \nabla_Z - \nabla_{[\rho(Y), Z]}\big)(X \bmod B)\end{aligned}$$

vanishes for all X, Y, Z. A direct computation shows that S is $C^\infty(M)$-linear in its first two arguments. Therefore, it is enough to compute $S(X, Y; Z)$ for $X, Y \in \Gamma_\nabla$. In this case $[\![X, Y]\!] \in \Gamma_\nabla$ as well and $S(X, Y; Z)$ vanishes.

The second part of the statement immediately follows from (20) and (21). □

Theorem 25 [Zambon and Zhu 2012]. *There is a one-to-one correspondence between degree-one $\mathbb{N}Q$-manifolds equipped with a compatible involutive distribution, cogenerated in degree one, and Lie algebroids $A \to M$ equipped with an IM foliation over TM.*

Proof. Let $\mathcal{M} = A[1]$ be a degree-one $\mathbb{N}$-manifold, and let $\mathcal{D}$ be an involutive distribution on it, cogenerated in degree one. Denote by $\pi : \mathcal{M} \to M$ the projection of $\mathcal{M}$ onto its zero dimensional shadow. The quotient bundle $T\mathcal{M}/\mathcal{D}$ identifies with $\mathcal{M} \times_M E[1]$ for a nongraded vector bundle $E \to M$, and $\theta_\mathcal{D}$ is a degree-one E-valued 1-form on $\mathcal{M}$. Moreover, $\mathcal{D}$ projects surjectively onto TM, i.e., $\pi_* \mathcal{D} = TM$. In particular, for any vector field Z on M there is a (degree-zero) vector field $\tilde{Z} \in \Gamma(\mathcal{D})$ that is π-related to Z.

Denote by (D, ℓ) the Spencer data of $\theta_\mathcal{D}$. In particular, $\ell : \Gamma(A) \to \Gamma(E)$ is surjective. Let $B = \ker \ell$ so that E identifies with A/B and ℓ identifies with the projection $\Gamma(A) \to \Gamma(A/B)$. In the following we will understand this isomorphism. There is a unique first order differential operator $\delta : \Gamma(A/B) \to \Omega^1(M, A/B)$ such that diagram

$$\begin{tikzcd} \Gamma(A) \arrow[r, "D"] \arrow[d] & \Omega^1(M, A/B) \\ \Gamma(A/B) \arrow[ur, "\delta"'] & \end{tikzcd}$$

commutes. To see this it is enough to show that $\Gamma(B) \subset \ker D$. Since $\Gamma(B) = \Gamma_{-1}(\mathcal{D})$, and sections of $\mathcal{D}$ are infinitesimal symmetries by involutivity, $D(X) = L_X \theta_{\mathcal{D}} = 0$ for all $X \in \Gamma(B)$ (see Remark 21). It follows from (4) that $\delta(f\alpha) = f\delta\alpha - df \otimes \alpha$ for all $f \in C^\infty(M)$, and $\alpha \in \Gamma(A/B)$. Therefore, δ is minus the (first) de Rham differential of a unique connection ∇ in A/B. We claim that ∇ is a flat connection. Indeed, first of all, notice that for all $Z \in \mathfrak{X}(M)$ and $X \in \Gamma(A)$,

$$\nabla_Z(X \bmod B) = i_Z d_\nabla(X \bmod B) = -i_Z D(X) = -i_{\tilde{Z}} L_X \theta_{\mathcal{D}},$$

where $\tilde{Z}$ is any degree-zero vector field on $\mathcal{M}$ that is π-related to Z. We can choose $\tilde{Z} \in \Gamma(\mathcal{D})$ so that

$$\nabla_Z(X \bmod B) = -i_{\tilde{Z}} L_X \theta_{\mathcal{D}} = -i_{[X,\tilde{Z}]}\theta_{\mathcal{D}} = [\tilde{Z}, X] \bmod B.$$

Now, let Y, Z be vector fields on M, and let $\tilde{Y}, \tilde{Z}$ be vector fields in $\mathcal{D}$ that are π-related to them. Then, by involutivity, $[\tilde{Y}, \tilde{Z}]$ is in $\mathcal{D}$ and it is π-related to $[Y, Z]$. Thus

$$\begin{aligned} \nabla_{[Y,Z]}(X \bmod B) &= [[\tilde{Y}, \tilde{Z}], X] \bmod B \\ &= \big([\tilde{Y}, [\tilde{Z}, X]] - [\tilde{Z}, [\tilde{Y}, X]]\big) \bmod B \\ &= [\nabla_Y, \nabla_Z](X \bmod B). \end{aligned}$$

Conversely, let $B \subset A$ be a vector subbundle and let ∇ be a flat connection in A/B. Denote by $\ell : A \to A/B$ the projection. Then $(-d_\nabla \circ \ell, \ell)$ are Spencer data for an A/B-valued 1-form θ on $\mathcal{M}$. Put $\mathcal{D} = \ker\theta$. To see that $\mathcal{D}$ is involutive, notice that $\Gamma_{-1}(\mathcal{D}) = \ker \ell = \Gamma(B)$. Moreover, $\mathcal{D}$ projects surjectively on TM, therefore $\Gamma(\mathcal{D})$ is generated by 1) sections of B and 2) degree-zero vector fields in $\mathcal{D}$ that are projectable onto M. Commuting the latter with the former, one gets sections of B which are again in $\mathcal{D}$. It remains to show that the commutator of two projectable vector fields $\tilde{Z}, \tilde{Y}$ in $\mathcal{D}$ is again in $\mathcal{D}$, i.e., $i_{[\tilde{Y},\tilde{Z}]}\theta = 0$. Now $i_{[\tilde{Y},\tilde{Z}]}\theta = 0$ if and only if $L_X i_{[\tilde{Y},\tilde{Z}]}\theta = 0$ for all $X \in \mathfrak{X}_-(\mathcal{M}) = \Gamma(A)$. The same computation as above shows that

$$L_X i_{[\tilde{Y},\tilde{Z}]}\theta = \big([\nabla_Y, \nabla_Z] - \nabla_{[Y,Z]}\big)(X \bmod B) = 0,$$

where $Y = \pi_* \tilde{Y}$, and $Z = \pi_* \tilde{Z}$. We conclude that involutive distributions $\mathcal{D}$ on $\mathcal{M}$ cogenerated in degree one are equivalent to the following (nongraded) data: a vector subbundle $B \subset A$ and a flat connection in A/B.

Finally, let $\mathcal{D}$ be an involutive distribution on $\mathcal{M}$ cogenerated in degree one and let (B, ∇) be the corresponding nongraded data. Moreover, let $\mathbb{Q}$ be a homological derivation of $T\mathcal{M}/\mathcal{D} = \mathcal{M} \times_M A/B$, let Q be its symbol, and let $(A, [\![-,-]\!], \rho)$ and $(A/B, \nabla^{A/B})$ be the Lie algebroid and the Lie algebroid representation corresponding to $\mathbb{Q}$. The distribution $\mathcal{D}$ is compatible with Q, and $\mathbb{Q}$ is induced by $[Q, -]$, if and only if $\theta_{\mathcal{D}}$ is compatible with $\mathbb{Q}$. To see when this is the case, we use again Proposition 17. Identity (18) is automatically satisfied by $\omega = \theta_{\mathcal{D}}$. Additionally, for $\omega = \theta_{\mathcal{D}}$ and $X, Y \in \mathfrak{X}_-(\mathcal{M}) = \Gamma(A)$, one gets

$$\begin{aligned} A(X, Y) &= D([[Q, X], Y]) - L_{[\mathbb{Q},X]} D(Y) + L_{[\mathbb{Q},Y]} D(X) \\ &= -d_\nabla([\![X, Y]\!] \bmod B) + L_{\nabla^{A/B}_X} d_\nabla(Y \bmod B) - L_{\nabla^{A/B}_Y} d_\nabla(X \bmod B), \end{aligned}$$

and

$$\begin{aligned} B(X, Y) &= \ell([[Q, X], Y]) + i_{[Q,X]} D(Y) + L_{[\mathbb{Q},Y]} \ell(X) \\ &= [\![X, Y]\!] \bmod B - \nabla_{\rho(X)}(Y \bmod B) + \nabla^{A/B}_Y (X \bmod B), \end{aligned}$$

where we used that $D = -d_\nabla \circ \ell$. Proposition 17 and Lemma 24 then show that $\mathcal{D}$ is compatible with Q, and $\mathbb{Q}$ is induced by $[Q, -]$, if and only if (B, ∇, TM) is an IM foliation of A over TM, and $\nabla^{A/B}$ is the A-connection in the statement of the lemma. □

4. Vector Valued 2-forms on $\mathbb{N}Q$-manifolds

4.1. ***Degree one symplectic $\mathbb{N}Q$-manifolds.*** Recall that a degree-n symplectic $\mathbb{N}$-manifold is an $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a degree-n symplectic structure, i.e., a degree-n nondegenerate, closed 2-form ω.

It immediately follows from the definition that, if $(\mathcal{M}, \omega)$ is a degree-n symplectic $\mathbb{N}$-manifold, then the degree of $\mathcal{M}$ is at most n. If $n > 0$, then $\omega = d\vartheta$, with $\vartheta = n^{-1} i_\Delta \omega$.

Example 26. The degree-n 2-form ω on $T^*[n]M$ (see Example 14) is a degree-n symplectic structure.

A degree-n symplectic $\mathbb{N}Q$-manifold is a degree-n symplectic manifold $(\mathcal{M}, \omega)$ equipped with a homological vector field Q such that $L_Q \omega = 0$.

Theorem 27 [Roytenberg 2002]. *Every degree-one symplectic $\mathbb{N}$-manifold $(\mathcal{M}, \omega)$ is of the kind $(T^*[1]M, d\vartheta)$, up to symplectomorphisms, where ϑ is the tautological degree-one 1-form on $T^*[1]M$* (see Example 14). *Moreover, there is a one-to-one correspondence between degree-one symplectic $\mathbb{N}Q$-manifolds and (nongraded) Poisson manifolds.*

Roytenberg's proof exploits explicitly the Poisson bracket. We propose an alternative proof focusing on Spencer data. The advantage is that we can apply the same strategy to degenerate (Theorem 30 and Corollary 31) or higher-degree forms (Theorems 36 and 39) in a straightforward way.

Proof. Let $(\mathcal{M}, \omega)$ be a degree-one symplectic $\mathbb{N}$-manifold, and let (D, ℓ) be the Spencer data of ω. In particular, $\mathcal{M} = A[1]$ for some vector bundle $A \to M$. By nondegeneracy $\ell : \mathfrak{X}_{-1}(\mathcal{M}) \to \Omega^1(\mathcal{M})$ is an isomorphism $\Gamma(A) \simeq \Omega^1(M)$, i.e., $\mathcal{M} = A[1] \simeq T^*[1]M$. Moreover, since ω is closed, the diagram

$$\begin{tikzcd} \Gamma(A) \arrow[r, "D"] \arrow[d, "\ell"'] & \Omega^2(M) \\ \Omega^1(M) \arrow[ur, "-d"'] & \end{tikzcd} \tag{23}$$

commutes. This shows that diffeomorphism $\mathcal{M} \simeq T^*[1]M$ identifies ω with the canonical symplectic structure on $T^*[1]M$ (see Example 14), thus proving the first part of the statement. In the following we identify $\mathcal{M}$ and $T^*[1]M$. For the second part of the statement, let Q be a homological vector field on $\mathcal{M}$ and let $(T^*M, \rho, [\![-,-]\!])$ be the corresponding Lie algebroid. Similarly to the previous section, $(\mathcal{M}, \omega, Q)$ is a symplectic $\mathbb{N}Q$-manifold if and only if it satisfies (17), and (18), for all $X, Y \in \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Omega^1(M)$. Indeed, since ω is closed, condition (16) is actually a consequence of (17), and (18). It is easy to see that one can even restrict to X, Y in the form df, dg, with $f, g \in C^\infty(M)$. In this case, one gets

$$\begin{aligned} B(X, Y) &= \ell([[Q, df], dg]) + L_{[Q,df]}\ell(dg) = [\![df, dg]\!] + L_{\rho(df)}dg \\ &= [\![df, dg]\!] + d\rho(df)(g), \end{aligned}$$

and

$$C(X, Y) = i_{[Q,df]}\ell(dg) + i_{[Q,dg]}\ell(df) = \rho(df)(g) + \rho(dg)(f),$$

where we used that $\ell(df) = i_{df}\omega = df$, and $D(df) = L_{df}\omega = 0$ (see Example 14). Concluding, $(\mathcal{M}, \omega, Q)$ is a symplectic $\mathbb{N}Q$-manifold if and only if

$$\rho(df)(g) + \rho(dg)(f) = 0 \quad \text{and} \quad [\![dg, df]\!] = -d\rho(df)(g) = d\rho(dg)(f),$$

i.e., if and only if $(T^*M, \rho, [\![-,-]\!])$ is the Lie algebroid associated to a Poisson structure on M (see Appendix B). □

4.2. ***Degree one presymplectic $\mathbb{N}Q$-manifolds.*** In this subsection we relax the hypothesis about nondegeneracy of the 2-form in the previous subsection.

Definition 28. A *degree-n presymplectic $\mathbb{N}$-manifold* is a degree-n $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a degree-n presymplectic structure, i.e., a degree-n (possibly degenerate) closed 2-form ω. A *degree-n presymplectic $\mathbb{N}Q$-manifold* is a degree-n presymplectic manifold $(\mathcal{M}, \omega)$ equipped with a homological vector field Q such that $L_Q\omega = 0$.

Remark 29. Unlike the symplectic case, the existence of a presymplectic form on an $\mathbb{N}$-manifold $\mathcal{M}$ doesn't bound the degree of $\mathcal{M}$. This is the reason why we added a condition on the degree of $\mathcal{M}$ in the definition of a degree-n presymplectic $\mathbb{N}$-manifold above.

In what follows we show that degree-one presymplectic $\mathbb{N}Q$-manifolds (with an additional nondegeneracy condition) are basically equivalent to *Dirac manifolds*. Recall that a Dirac manifold is a manifold M equipped with a *Dirac structure*, i.e., a subbundle $\mathfrak{D} \subset TM \oplus T^*M$ such that 1) $\mathfrak{D}$ is maximally isotropic with respect to the canonical, split signature, symmetric form on $TM \oplus T^*M$

$$\langle\langle (X, \sigma), (X', \sigma') \rangle\rangle = i_X \sigma' + i_{X'} \sigma, \tag{24}$$

and 2) sections of $\mathfrak{D}$ are preserved by the Dorfman (equivalently, Courant) bracket

$$[(X, \sigma), (X', \sigma')]_{\mathrm{D}} := ([X, X'], L_X \sigma' - i_{X'} d\sigma), \tag{25}$$

$X, X' \in \mathfrak{X}(M)$, $\sigma, \sigma' \in \Omega^1(M)$. Any Dirac structure $\mathfrak{D}$ is a Lie algebroid, with anchor given by projection $TM \oplus T^*M \to TM$ and bracket given by the Dorfman bracket (25). Dirac manifolds encompass presymplectic and Poisson manifolds (see [Courant 1990; Bursztyn 2013] for more details). They are sometimes regarded as Lagrangian submanifolds in certain degree-two symplectic $\mathbb{N}Q$-manifolds. Corollary 31 below shows that they can be also regarded as suitable degree-one presymplectic $\mathbb{N}Q$-manifolds.

Let M be a manifold, denote by $\mathrm{pr}_T : TM \oplus T^*M \to TM$, and $\mathrm{pr}_{T^*} : TM \oplus T^*M \to T^*M$ the canonical projections.

Theorem 30. *There is a one-to-one correspondence between degree-one presymplectic $\mathbb{N}Q$-manifolds and* (*nongraded*) *Lie algebroids $A \to M$ equipped with a vector bundle morphism $\Phi : A \to TM \oplus T^*M$ such that*

(1) *the anchor of A equals the composition* $\mathrm{pr}_T \circ \Phi$,

(2) *the image of Φ is an isotropic subbundle with respect to* (24), *and*

(3) Φ *intertwines the Lie bracket $[\![-,-]\!]$ on $\Gamma(A)$ and the Dorfman bracket* (25) *on $\Gamma(TM \oplus T^*M)$, i.e.,* $\Phi[\![X, Y]\!] = [\Phi(X), \Phi(Y)]_{\mathrm{D}}$ *for all $X, Y \in \Gamma(A)$.*

Proof. Let $(\mathcal{M}, \omega)$ be a degree-one presymplectic $\mathbb{N}$-manifold, and let (D, ℓ) be the corresponding Spencer data. Moreover, let Q be a homological vector field on $\mathcal{M}$, and let $(A, \rho, [\![-,-]\!])$ be the corresponding Lie algebroid. Since ω is closed, diagram (23) commutes and ω is completely determined by ℓ. Now, combine ℓ and $\rho : A \to TM$ in a vector bundle morphism $\Phi := (\rho, \ell) : A \to TM \oplus T^*M$. In particular, $\rho = \mathrm{pr}_T \circ \Phi$, i.e., Φ satisfies property (1) in the statement. Similarly as in the proof of Theorem 27, $(\mathcal{M}, \omega, Q)$ is a presymplectic $\mathbb{N}Q$-manifold if and only if (17) and (18) are satisfied for all $X, Y \in \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Omega^1(M)$. One gets

$$\begin{aligned} B(X, Y) &= \ell([[Q, X], Y]) + i_{[Q,X]}D(Y) + L_{[Q,Y]}\ell(X) \\ &= \ell([\![X, Y]\!]) - i_{\rho(X)}d\ell(Y) + L_{\rho(X)}\ell(X) \\ &= \mathrm{pr}_{T^*}(\Phi[\![X, Y]\!] - [\Phi(X), \Phi(Y)]_{\mathsf{D}}), \end{aligned}$$

and

$$C(X, Y) = i_{[Q,X]}\ell(Y) + i_{[Q,Y]}\ell(X) = i_{\rho(X)}\ell(Y) - i_{\rho(Y)}\ell(X) = \langle\!\langle \Phi(X), \Phi(Y)\rangle\!\rangle.$$

Since $\mathrm{pr}_T([\Phi(X), \Phi(Y)]_{\mathsf{D}}) = [\rho(X), \rho(Y)] = \rho[X, Y] = \mathrm{pr}_T\, \Phi[\![X, Y]\!]$, one concludes that $(\mathcal{M}, \omega, Q)$ is a presymplectic $\mathbb{N}Q$-manifold if and only if Φ, besides satisfying property (1) in the statement, does also satisfy properties (2) and (3).

Conversely, Let $\Phi : A \to TM \oplus T^*M$ be a vector bundle morphism. Put $\ell := \mathrm{pr}_{T^*} \circ \Phi$. It is easy to see that $(\ell, -d \circ \ell)$ is a pair of Spencer data corresponding to a degree-one presymplectic form ω on $\mathcal{M}$. If, additionally, Φ satisfies properties (1), (2), and (3) in the statement, then the same computations as above show that $(\mathcal{M}, \omega, Q)$ is a presymplectic $\mathbb{N}Q$-manifold. □

Corollary 31. *There is a one-to-one correspondence between degree-one presymplectic $\mathbb{N}Q$-manifolds $(\mathcal{M}, \omega, Q)$ such that*

(1) $\operatorname{rank} A = \dim M$, *and*

(2) $\ker \ell \cap \ker \rho = 0$,

where $(A \to M, \rho, [\![-,-]\!])$ is the Lie algebroid corresponding to $(\mathcal{M}, Q)$, and (ℓ, D) are Spencer data corresponding to ω, and (nongraded) Lie algebroids $A \to M$ equipped with a Lie algebroid isomorphism $\Phi : A \simeq \mathfrak{D}$ with values in a Dirac structure $\mathfrak{D} \subset TM \oplus T^*M$ over M.

Proof. Let $(\mathcal{M}, \omega, Q)$ be a degree-one presymplectic $\mathbb{N}Q$-manifold and let (A, Φ) be the corresponding nongraded data as in Theorem 30. The vector bundle morphism Φ is injective if and only if condition (2) in the statement is satisfied. In this case, Φ is an isomorphism onto its image $\mathfrak{D}$. Additionally, $\mathfrak{D}$ is maximally isotropic in $TM \oplus T^*M$, hence a Dirac structure, if and only if $\operatorname{rank} \mathfrak{D} = \operatorname{rank} A$ is precisely $\dim M$, i.e., condition (1) in the statement is satisfied. □

4.3. ***Degree one locally conformal symplectic $\mathbb{N}Q$-manifolds.*** The original definition of a locally conformal symplectic (lcs) structure is (equivalent to) the following [Vaisman 1985]: an *lcs structure* on a manifold M is a pair (ϕ, ω), where ϕ is a closed 1-form and ω is a nondegenerate 2-form on M such that $d\omega = \phi \wedge \omega$. Ordinary symplectic manifolds and lcs manifolds share some properties, but the latter are manifestly more general. Moreover, they are examples of Jacobi manifolds. In this paper we adopt an approach to lcs manifolds which is slightly more intrinsic than the traditional one (see Appendix A) in the same spirit as the *intrinsic approach* to contact and Jacobi geometry of [Crainic and Salazar 2015].

Definition 32. A degree-n *abstract lcs* $\mathbb{N}$-*manifold* is an $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a degree-n *abstract lcs structure*, i.e., a triple $(\mathcal{L}, \nabla, \omega)$ where $\mathcal{L} \to \mathcal{M}$ is a line $\mathbb{N}$-bundle, ∇ is a flat connection in $\mathcal{L}$, and ω is a degree-n, nondegenerate, d_∇-closed, $\mathcal{L}$-valued 2-form ω.

First of all, notice that $\mathcal{L}$, being a line bundle, is actually generated in one single degree $-k$. Up to a shift in degree in the above definition we may (and we actually will) assume $k = 0$. In particular, $\mathcal{L} = \mathcal{M} \times_M L$, for some (nongraded) vector bundle L on the degree-zero shadow M of $\mathcal{M}$, and ∇ is actually induced from a flat connection on L. Exactly as in the symplectic case [Roytenberg 2002] one shows that, if $\mathcal{M}$ possesses a degree-n abstract lcs structure, then, by nondegeneracy, the degree of $\mathcal{M}$ is at most n. If $n > 0$, then $\omega = d_\nabla \vartheta$, with $\vartheta = n^{-1} i_{\Delta_{\mathcal{L}}} \omega$.

Example 33. Consider the degree-n 2-form ω of Example 15. If $E = L$ is a line bundle then the triple $(T[1]\mathcal{M} \times_M L, \nabla, \omega)$ is a degree-n abstract lcs structure.

A degree-n abstract lcs symplectic $\mathbb{N}Q$-manifold is a degree-n abstract lcs manifold $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ equipped with a homological derivation $\mathbb{Q}$ of $\mathcal{L}$ such that $L_{\mathbb{Q}}\omega = 0$. The proposition below shows that, actually, $\mathbb{Q}$ is completely determined by its symbol.

Proposition 34. *Let* $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ *be an abstract lcs* $\mathbb{N}$-*manifold with homological derivation* $\mathbb{Q}$, *and let* Q *be the symbol of* $\mathbb{Q}$. *Then* $\mathbb{Q}$ *is the covariant derivative along* Q.

Proof. The derivations $\mathbb{Q}$ and ∇_Q share the same symbol Q and, therefore, their difference $\mathbb{Q} - \nabla_Q$ is a degree-one endomorphism of $\mathcal{L}$ which can only consist in multiplying sections by a degree-one function f on $\mathcal{M}$. Thus,

$$0 = L_{\mathbb{Q}}\omega = L_{\nabla_Q}\omega + f\omega = -d_\nabla i_{\mathbb{Q}}\omega + f\omega$$

So that $f\omega = d_\nabla i_{\mathbb{Q}}\omega$. It follows that

$$0 = d_\nabla(f\omega) = df \cdot \omega.$$

Hence, by nondegeneracy, $df = 0$. Since f is a function of positive degree, one concludes that $f = 0$. □

Theorem 35. *Every degree-one abstract lcs* $\mathbb{N}$-*manifold* $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ *is of the form* $(T^*[1]M \otimes L, (T^*[1]M \otimes L) \times_M L, \nabla, d_\nabla \vartheta)$, *up to isomorphisms* (*and a shift in the degree of* $\mathcal{L}$), *where* ϑ *is the tautological degree-one* L-*valued* 1-*form on* $T^*[1]M \otimes L$ *and* ∇ *is a flat connection in the line bundle* $L \to M$ (*see Example 15*). *Moreover*, *there is a one-to-one correspondence between degree-one abstract lcs*

$\mathbb{N}Q$-manifolds and (nongraded) abstract locally conformal Poisson manifolds (see Appendix A for a definition).

Proof. The proof is a suitable adaptation of both the proofs of Theorem 23 and Theorem 27. Let $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ be a degree-one abstract lcs $\mathbb{N}$-manifold, and let (D, ℓ) be the Spencer data of ω. In particular, $\mathcal{M} = A[1]$ for some vector bundle $A \to M$, and $\mathcal{L} = A[1] \times_M L$ for some line bundle $L \to M$ (up to a shift). Moreover ∇ is induced in $\mathcal{L}$ by a flat connection in L which, abusing the notation, we denote by ∇ again.

By nondegeneracy $\ell : \mathfrak{X}_{-1}(\mathcal{M}) \to \Omega^1(\mathcal{M}, L)$ is an isomorphism $\Gamma(A) \simeq \Omega^1(M, L)$, i.e., $\mathcal{M} = A[1] \simeq T^*[1]M \otimes L$. Moreover, since ω is d_∇-closed, the diagram

$$\begin{tikzcd} \Gamma(A) \arrow[r, "D"] \arrow[d, "\ell"'] & \Omega^2(M, L) \\ \Omega^1(M, L) \arrow[ur, "-d_\nabla"'] & \end{tikzcd}$$

commutes. This shows that the diffeomorphism $\mathcal{M} \simeq T^*[1]M \otimes L$ identifies ω with the canonical L-valued 2-form on $T^*[1]M \otimes L$ (see Example 15), thus proving the first part of the statement. In the following we identify $\mathcal{M}$ and $T^*[1]M \otimes L$. For the second part of the statement, let $\mathbb{Q}$ be a homological derivation of $\mathcal{L} = \mathcal{M} \times_M L$. The derivation $\mathbb{Q}$ is equivalent to a Lie algebroid $(T^*M \otimes L, \rho, [\![-,-]\!])$ equipped with a representation (L, ∇^L) (beware not to confuse the Lie algebroid connection ∇^L and the standard connection ∇). As above $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ is an abstract lcs $\mathbb{N}Q$-manifold with homological derivation $\mathbb{Q}$ if and only if (17), and (18) are satisfied for all $X, Y \in \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Omega^1(M, L)$. Similarly to the proof of Theorem 27, one can even restrict to X, Y in the form $d_\nabla\lambda, d_\nabla\mu$, with $\lambda, \mu \in \Gamma(L)$. In this case one gets

$$\begin{aligned} B(X, Y) &= \ell([[Q, d_\nabla\lambda], d_\nabla\mu]) + L_{[\mathbb{Q}, d_\nabla\lambda]}\ell(d_\nabla\mu) \\ &= [\![d_\nabla\lambda, d_\nabla\mu]\!] + \nabla^L_{\rho(d_\nabla\lambda)} d_\nabla\mu \\ &= [\![d_\nabla\lambda, d_\nabla\mu]\!] + d_\nabla\nabla^L_{\rho(d_\nabla\lambda)}\mu, \end{aligned}$$

and

$$C(X, Y) = i_{[Q, d_\nabla\lambda]}\ell(d_\nabla\mu) + i_{[Q, d_\nabla\mu]} d_\nabla\lambda = \nabla^L_{\rho(d_\nabla\lambda)}\mu + \nabla^L_{\rho(d_\nabla\mu)}\lambda,$$

where we used that $\ell(d_\nabla\lambda) = i_{d_\nabla\lambda}\omega = d_\nabla\lambda$, and $D(d_\nabla\lambda) = L_{d_\nabla\lambda}\omega = 0$ (see Example 14). Concluding, $(\mathcal{M}, \mathcal{L}, \nabla, \omega)$ is an abstract lcs $\mathbb{N}Q$-manifold with homological derivation $\mathbb{Q}$ if and only if $\nabla^L_{\rho(d_\nabla\lambda)}\mu + \nabla^L_{\rho(d_\nabla\mu)}\lambda = 0$ and $[\![d_\nabla\lambda, d_\nabla\mu]\!] = -d_\nabla\nabla^L_{\rho(d_\nabla\lambda)}\mu = d_\nabla\nabla^L_{\rho(d_\nabla\mu)}\lambda$, i.e., if and only if $(T^*M \otimes L, \rho, [\![-,-]\!])$ is the Lie algebroid associated to a locally conformal Poisson structure (L, ∇, P) on M, and (L, ∇^L) is its canonical representation. $\square$

5. Higher-degree forms on degree-one $\mathbb{N}Q$-manifolds

5.1. ***Vector valued forms on degree-one $\mathbb{N}Q$-manifolds and Spencer operators.*** In this section, we discuss general degree-one compatible vector valued forms on degree-one $\mathbb{N}Q$-manifolds. It turns out that they are equivalent to the recently introduced *Spencer operators on Lie algebroids* [Crainic et al. 2015]. Let $(A, [\![-,-]\!], \rho)$ be a Lie algebroid over a manifold M, (E, ∇^E) a representation of A, and let k be a nonnegative integer. An *E-valued k-Spencer operator* [Crainic et al. 2015] is a pair consisting of

- a (first order) differential operator $D : \Gamma(A) \to \Omega^k(M, E)$, and
- a $C^\infty(M)$-linear map $\ell : \Gamma(A) \to \Omega^{k-1}(M, E)$,

such that

$$D(fX) = fD(X) - df \wedge \ell(X)$$

and, moreover,

$$\begin{aligned} L_{\nabla^E_X} D(Y) - L_{\nabla^E_Y} D(X) &= D([\![X, Y]\!]), \\ L_{\nabla^E_X} \ell(Y) + i_{\rho(Y)} D(X) &= \ell([\![X, Y]\!]), \\ i_{\rho(X)} \ell(Y) + i_{\rho(Y)} \ell(X) &= 0, \end{aligned} \tag{26}$$

for all $X, Y \in \Gamma(A)$. There is a difference in signs between the above definition and the original one in [Crainic et al. 2015]. The original definition is recovered by replacing $D \to -D$. We chose the sign convention which makes formulas simpler in the present graded context.

Spencer operators are the infinitesimal counterparts of multiplicative vector valued forms on Lie groupoids. When the vector bundle is a trivial line bundle, they reduce to the IM forms of Bursztyn and Cabrera [2012] (see also [Bursztyn et al. 2009] for the 2-form case). Hence the result of this section is the expected generalization of the following (well) known facts:

- Jacobi manifolds can be understood either as infinitesimal counterparts of contact Lie groupoids [Crainic and Salazar 2015] or as degree-one contact $\mathbb{N}Q$-manifolds [Mehta 2013; Grabowski 2013].
- Poisson manifolds can be understood either as infinitesimal counterparts of symplectic Lie groupoids [Weinstein 1987] or as degree-one symplectic $\mathbb{N}Q$-manifolds [Roytenberg 2002].

Theorem 36. *There is a one-to-one correspondence between*

- *degree-one $\mathbb{N}$-manifolds equipped with a degree-zero $\mathbb{N}Q$-vector bundle $\mathcal{E}$ and a degree-one compatible $\mathcal{E}$-valued differential k-form, and*

- *Lie algebroids equipped with a representation* (E, ∇^E) *and an* E*-valued* k*-Spencer operator.*

Proof. Let $\mathcal{M}$ be a degree-one $\mathbb{N}$-manifold, and let $(\mathcal{E}, \mathbb{Q})$ be a degree-zero $\mathbb{N}Q$-vector bundle over it. In particular $\mathcal{E} = \mathcal{M} \times_M E$ for a nongraded vector bundle $E \to M$. Let $(T^*M, \rho, [\![-,-]\!])$ and (E, ∇^E) be the Lie algebroid and the Lie algebroid representation corresponding to $\mathbb{Q}$. Finally, let ω be a degree-one E-valued k-form on $\mathcal{M}$. Then, ω is compatible with $\mathbb{Q}$ if and only if (16), (17), and (18) are satisfied, for all $X, Y \in \mathfrak{X}_{-1}(\mathcal{M}) \simeq \Gamma(A)$. Denote by (D, ℓ) the Spencer data corresponding to ω. Then

$$\begin{aligned} A(X, Y) &= D([\![X, Y]\!]) - L_{\nabla^E_X} D(Y) + L_{\nabla^E_Y} D(X), \\ B(X, Y) &= \ell([\![X, Y]\!]) + i_{\rho(X)} D(Y) + L_{\nabla^E_Y} \ell(X), \\ C(X, Y) &= i_{\rho(X)} \ell(Y) + i_{\rho(Y)} \ell(X). \end{aligned}$$

Concluding, ω is compatible with $\mathbb{Q}$ if and only if (D, ℓ) is an E-valued k-Spencer operator on the Lie algebroid A. □

5.2. ***Degree one multisymplectic*** $\mathbb{N}$***Q-manifolds.*** We conclude this section specializing to degree-one multisymplectic $\mathbb{N}Q$-manifolds. Let k be a positive integer. Recall that a k-plectic manifold (see, for instance, [Rogers 2012], see also [Cantrijn et al. 1999] for more details on multisymplectic geometry) is a manifold N equipped with a k-plectic structure, i.e., a closed $(k+1)$-form ω which is nondegenerate in the sense that the vector bundle morphism $TN \to \wedge^k T^*N$, $X \mapsto i_X \omega$ is an embedding. As expected, degree-one multisymplectic $\mathbb{N}Q$-manifolds are equivalent to Lie algebroids equipped with an *IM multisymplectic structure*, also called a *higher Poisson structure* in [Bursztyn et al. 2015]. The latter are infinitesimal counterparts of multisymplectic groupoids. Specifically, an *IM k-plectic structure* on a Lie algebroid $(A, [\![-,-]\!], \rho)$ (see [Bursztyn et al. 2015]) is a $C^\infty(M)$-linear map $\ell : A \to \Omega^k(M)$ such that

$$i_{\rho(X)} \ell(Y) + i_{\rho(Y)} \ell(X) = 0, \tag{27}$$

$$L_{\rho(X)} \ell(Y) - i_{\rho(Y)} d\ell(X) = \ell([\![X, Y]\!]), \tag{28}$$

for all $X, Y \in \Gamma(A)$, and, moreover,

$$\ker \ell := \{a \in A : \ell(a) = 0\} = 0, \quad (\operatorname{im} \ell)^\circ := \{\zeta \in TM : i_\zeta \circ \ell = 0\} = 0.$$

Definition 37. A degree-n *k-plectic $\mathbb{N}$-manifold* is a degree-n $\mathbb{N}$-manifold $\mathcal{M}$ equipped with a degree-n *k-plectic structure*, i.e., a closed $(k+1)$-form which is *nondegenerate* in the sense that the degree-n vector bundle morphism $T\mathcal{M} \to S^k T^*[-1]\mathcal{M}$, $X \mapsto i_X \omega$ is an embedding. A *k-plectic $\mathbb{N}Q$-manifold* of degree-n is an $\mathbb{N}Q$-manifold equipped with a compatible k-plectic structure.

Example 38. Let M be an ordinary (nongraded) manifold. The degree-n $\mathbb{N}$-manifold $\mathcal{M} = (\wedge^k T^*)[n]M$ comes equipped with the obvious tautological, degree-n k-form ϑ. Consider the degree-n $(k+1)$-form $\omega = d\vartheta$. It is a degree-n k-plectic structure. Negatively graded vector fields on $\mathcal{M}$ identify with k-forms on M and it is easy to see, along similar lines as in Example 14, that the Spencer data (D, ℓ) of ω identify with $(-)^n$ times the exterior differential $d : \Omega^k(M) \to \Omega^{k+1}(M)$ and the identity $\mathrm{id} : \Omega^k(M) \to \Omega^k(M)$ respectively.

Theorem 39. *Degree one k-plectic $\mathbb{N}Q$-manifolds are in one-to-one correspondence with Lie algebroids equipped with an IM k-plectic structure.*

Proof. Let $\mathcal{M}$ be a degree-one $\mathbb{N}$-manifold, ω a degree-one $(k+1)$-form on it and let (D, ℓ) be the corresponding Spencer data. In particular, $\mathcal{M} = A[1]$ for some vector bundle $A \to M$. Moreover, ω is closed if and only if $i_X d\omega = 0$ for all negatively graded vector fields X on $\mathcal{M}$. Indeed, from $i_X d\omega = 0$ it also follows that $L_X d\omega = 0$. In other words, $d\omega = 0$ if and only if the diagram

$$\begin{tikzcd} \Gamma(A) \arrow[r, "D"] \arrow[d, "\ell"'] & \Omega^{k+1}(M) \\ \Omega^k(M) \arrow[ur, "-d"'] & \end{tikzcd}$$

commutes. Conversely, a $C^\infty(M)$-linear map $\ell : \Gamma(A) \to \Omega^k(M)$ uniquely determines a closed degree-one $(k+1)$-form on $\mathcal{M}$ whose Spencer data are $(-d \circ \ell, \ell)$. Concluding, degree-one $\mathbb{N}$-manifolds equipped with a closed $(k+1)$-form are equivalent to vector bundles $A \to M$ equipped with a linear map $\ell : \Gamma(A) \to \Omega^k(M)$.

Now, let Q be a homological vector field on $\mathcal{M}$ and let $(A, \rho, [\![-,-]\!])$ be the corresponding Lie algebroid. The $(k+1)$-form ω is compatible with Q if and only if $(-d \circ \ell, \ell)$ is a $(k+1)$-Spencer operator, i.e., ℓ fulfills (27) and (28) (Equation (26) then follows from $D = -d \circ \ell$).

Finally, we need to characterize nondegeneracy of the closed form ω in terms of ℓ. Recall that M can be understood as a submanifold in $\mathcal{M}$ via the "zero section" of $\mathcal{M} \to M$, and the vector bundle morphism $\Gamma : T\mathcal{M} \to S^k T^*[-1]\mathcal{M}$, $X \mapsto i_X\omega$, restricts to a vector bundle morphism $\Gamma|_M : T\mathcal{M}|_M \to S^k T^*[-1]\mathcal{M}|_M$. Now, there are canonical identifications $T\mathcal{M}|_M = TM \oplus A[1]$, and

$$S^k T^*[-1]\mathcal{M}|_M = \bigoplus_{i+j=k} \wedge^i T^*M \otimes S^j A^*[-1].$$

It follows from $|\omega| = 1$ that $\Gamma|_M$ does actually take values in $\wedge^{k-1} T^*M \otimes A^*[-1] \oplus \wedge^k T^*M$. More precisely, it identifies with the pair of vector bundle morphisms

$$A[1] \to \wedge^k T^*M, \quad X \mapsto \ell(X).$$

and

$$TM \to \wedge^{k-1} T^*M \otimes A^*[-1], \quad Z \mapsto i_Z \circ \ell.$$

Consequently, $\ker \ell$ and $(\operatorname{im} \ell)^\circ$ are trivial if and only if $\Gamma|_M$ is an embedding. It remains to show that ω is nondegenerate provided $\Gamma|_M$ is an embedding. This is easily seen, for instance, in local coordinates: let x^i be coordinates in M and z^a be (degree-one) fiber coordinates in $A[1] \to M$. Locally,

$$\omega = \omega_{a|i_1\cdots i_k} dz^a dx^{i_1} \cdots dx^{i_k} + \omega'_{a|i_1\cdots i_{k+1}} z^a dx^{i_1} \cdots dx^{i_{k+1}}.$$

In the basis $\{\, \partial/\partial z^a \mid \partial/\partial x^i \,\}$ of $\mathfrak{X}(\mathcal{M})$ and $\{\, dx^{i_1} \cdots dx^{i_k} \mid dz^a dx^{i_1} \cdots dx^{i_{k-1}} \mid \cdots \}$ of $\Omega^k(\mathcal{M})$, the vector bundle morphism Γ is represented by the matrix

$$\left(\begin{array}{c:c:c} \omega_{a|i_1\cdots i_k} & 0 & \cdots \\ \hdashline * & k\omega_{a|wei_1\cdots i_k} & \cdots \end{array}\right)$$

and $\Gamma|_M$ is represented by the same matrix with the lower-left block set to zero. This concludes the proof. □

Appendix A: Locally conformal symplectic manifolds revisited

We refer to [Vaisman 1985] for details about standard locally conformal symplectic (lcs) structures. Here, we present a slightly more intrinsic approach to them (A. M. Vinogradov, personal communication, 2014; see also [Vitagliano 2015a, Section 3]). Let M be a smooth manifold.

Definition 40. An *abstract lcs structure* on M is a triple (L, ∇, ω), where $L \to M$ is a line bundle, ∇ is a flat connection in L, and ω is a nondegenerate L-valued 2-form on M such that $d_\nabla \omega = 0$, where $d_\nabla : \Omega(M, L) \to \Omega(M, L)$ is the de Rham differential of ∇. A manifold equipped with an abstract lcs structure is an *abstract lcs manifold*.

Example 41. Let (L, ∇, ω) be an abstract lcs structure on M. If $L = M \times \mathbb{R}$ is the trivial line bundle, then ∇ is the same as a closed 1-form on M, specifically, the *connection* 1*-form* $\phi := -d_\nabla 1 \in \Omega^1(M)$. Moreover, ω is a standard (nondegenerate) 2-form on M and it is easy to see that (ϕ, ω) is a standard lcs structure, i.e., $d\omega = \phi \wedge \omega$. In particular, if $\phi = 0$, then ω is a symplectic structure.

The word "abstract" in Definition 40 refers to the fact that ω takes values in an "abstract" line-bundle L, as opposed to the concrete, trivial line bundle $M \times \mathbb{R}$. Similarly, one can define *"abstract" locally conformal Poisson manifolds* (see below) and, more generally, *"abstract" Jacobi manifolds*. An *abstract Jacobi structure* (called a Jacobi bundle in [Marle 1991]) on a manifold M is a line bundle L equipped with a Lie bracket $\{-,-\}$ on $\Gamma(L)$ which is a first order differential operator in each entry (see, e.g., [Crainic and Salazar 2015] for details). Abstract Jacobi manifolds where introduced by Kirillov [1976] under the name *local Lie algebras with one dimensional fibers*. An abstract lcs structure (L,∇,ω) on M determines an abstract Jacobi structure $(L,\{-,-\})$ as follows. First of all, by nondegeneracy, ω establishes an isomorphism $TM\to T^*M\otimes L,\ X\mapsto i_X\omega$. Denote by $\sharp : T^*M\otimes L\to TM$ the inverse isomorphism and, for $\lambda\in\Gamma(L)$, put $X_\lambda := \sharp(d_\nabla\lambda)\in\mathfrak{X}(M)$. Finally, put

$$\{\lambda,\mu\} := \omega(X_\lambda, X_\mu) = \nabla_{X_\lambda}\mu,$$

$\lambda,\mu\in\Gamma(L)$. Clearly, $\{-,-\}$ is a first order differential operator in each entry. Moreover, the Jacobi identity is equivalent to $d_\nabla\omega=0$. Thus, $(L,\{-,-\})$ is an abstract Jacobi structure on M. Notice that there exists a unique linear morphism $P:\wedge^2(T^*M\otimes L)\to L$ such that

$$\{\lambda,\mu\} = P(d_\nabla\lambda, d_\nabla\mu),\quad \text{for all } \lambda,\mu\in\Gamma(L).$$

Example 42. Let $L=M\times\mathbb{R}$ so that (L,∇,ω) is the same as a standard lcs structure (ϕ,ω). Then, for $f,g\in C^\infty(M)=\Gamma(L)$, X_f is implicitly defined by

$$i_{X_f}\omega = df - f\phi,$$

and

$$\{f,g\} := \omega(X_f, X_g) = X_f(g) - g\phi(X_f).$$

In particular, if $\phi=0$, then P is the Poisson bivector determined by the symplectic structure ω.

More generally, Let M be a smooth manifold, (L,∇) a line bundle over M equipped with a flat connection, and let $P:\wedge^2(T^*M\otimes L)\to L$ be a linear morphism. One can then define a bracket $\{-,-\}_P$ in $\Gamma(L)$ by putting

$$\{\lambda,\mu\}_P = P(d_\nabla\lambda, d_\nabla\mu),$$

$\lambda,\mu\in\Gamma(L)$.

Definition 43. An *abstract locally conformal Poisson structure* on M is a triple (L,∇,P), where $L\to M$ is a line bundle, ∇ is a flat connection in L, and P is a linear morphism $P:\wedge^2(T^*M\otimes L)\to L$ such that $\{-,-\}_P$ is a Lie bracket. A manifold equipped with an abstract locally conformal Poisson structure is an *abstract locally conformal Poisson manifold*.

Thus, abstract lcs manifolds are abstract locally conformal Poisson manifolds (much as standard symplectic manifolds are standard Poisson manifolds), but the latter are more general.

Example 44. Let (L, ∇, P) be an abstract locally conformal Poisson structure on M. If $L = M \times \mathbb{R}$ is the trivial line bundle, and $\phi := -d_\nabla 1 \in \Omega^1(M)$ is the connection 1-form, then P is a standard bivector on M and a lengthy but straightforward computation shows that (ϕ, P) is a *locally conformal Poisson structure* in the sense of [Vaisman 2007], i.e., $[P, P]_{\mathrm{ns}} = i_\phi P \wedge P$ (where $[-,-]_{\mathrm{ns}}$ is the Nijenhuis–Schouten bracket of multivectors). In particular, if $\phi = 0$, then P is a Poisson structure.

Finally, notice also that abstract locally conformal Poisson manifolds are abstract Jacobi manifolds (of a special kind).

Appendix B: Lie algebroids and their representations

Recall that a *Lie algebroid* over a manifold M is a vector bundle $A \to M$ equipped with 1) a $C^\infty(M)$-linear map $\rho : \Gamma(A) \to \mathfrak{X}(M)$ called the *anchor*, and 2) a Lie bracket $[\![-,-]\!]$ on $\Gamma(A)$ such that

$$[\![X, fY]\!] = \rho(X)(f)Y + f[\![X, Y]\!], \quad X, Y \in \Gamma(A), \quad f \in C^\infty(M).$$

Example 45. The tangent bundle TM is a Lie algebroid with Lie bracket given by the commutator of vector fields and anchor given by the identity.

Let $A \to M$ be a Lie algebroid. A *representation of* A is a vector bundle $E \to M$ equipped with a *flat A-connection* ∇^E, i.e., a $C^\infty(M)$-linear map

$$\nabla^E : \Gamma(A) \to \Gamma(DE), \quad X \mapsto \nabla^E_X,$$

such that the symbol of the derivation ∇^E_X is $\rho(X)$, and $[\nabla^E_X, \nabla^E_Y] = \nabla^E_{[\![X,Y]\!]}$, for all $X, Y \in \Gamma(A)$. Let (E, ∇^E) be a representation of A. The graded vector space $\Gamma(\wedge^\bullet A^* \otimes E)$ of alternating, $C^\infty(M)$-multilinear, $\Gamma(E)$-valued forms on $\Gamma(A)$ is naturally equipped with a homological operator d_E given by the following *Chevalley–Eilenberg formula*:

$$\begin{aligned}&(d_E\varphi)(X_1, \dots, X_{k+1})\\ &:= \sum_i (-)^{i+1}\nabla^E_{X_i}(\varphi(\dots, \widehat{X}_i, \dots)) + \sum_{i<j}(-)^{i+j}\varphi([\![X_i, X_j]\!], \dots, \widehat{X}_i, \dots, \widehat{X}_j, \dots),\end{aligned}$$

where $\varphi \in \Gamma(\wedge^k A^* \otimes E)$ is an alternating form with k-entries, $X_1, \dots, X_{k+1} \in \Gamma(A)$, and a hat $(\widehat{-})$ denotes omission.

Example 46. Let ∇ be a standard flat connection in a vector bundle E. Then (E, ∇) is a representation of the Lie algebroid TM and the de Rham operator d_∇ of ∇ is its associated homological operator.

Example 47. Let $A \to M$ be a Lie algebroid. Clearly $(M \times \mathbb{R}, \rho)$ is a canonical representation of A. In particular, $\Gamma(\wedge^* A)$ is equipped with a homological operator (in fact a derivation) which we denote by d_A.

Example 48. Let $(L, \{-,-\})$ be an abstract Jacobi structure on a manifold M. There is a unique Lie algebroid $(J^1 L, \rho, [\![-,-]\!])$ such that $[\![j^1\lambda, j^1\mu]\!] = j^1\{\lambda, \mu\}$, and $\rho(j^1\lambda)$ is the symbol of the first order differential operator (in fact a derivation) $\{\lambda, -\}$, where $\lambda, \mu \in \Gamma(L)$. Moreover, there is a unique representation (L, ∇^L) of $J^1 L$ such that $\nabla^L_{j^1\lambda}\mu = \{\lambda, \mu\}$. In particular,

$$[\![j^1\lambda, j^1\mu]\!] = j^1(\nabla^L_{j^1\lambda}\mu). \tag{29}$$

Conversely, let $(J^1 L, \rho, [\![-,-]\!])$ be a Lie algebroid equipped with a representation (L, ∇^L) such that (29) holds. For $\lambda, \mu \in \Gamma(L)$ put $\{\lambda, \mu\} := \nabla^L_{j^1\lambda}\mu$. Then $(L, \{-,-\})$ is an abstract Jacobi structure on M. This shows that abstract Jacobi structures $(L, \{-,-\})$ are equivalent to Lie algebroids $(J^1 L, \rho, [\![-,-]\!])$ equipped with a representation (L, ∇^L) such that (29) holds.

Example 49. Let $\{-,-\}$ be a Poisson structure on a manifold M. There is a unique Lie algebroid $(T^*M, \rho, [\![-,-]\!])$ such that $[\![df, dg]\!] = d\{f, g\}$, and $\rho(df)$ is the Hamiltonian vector field of f, where $f, g \in C^\infty(M)$. In particular,

$$[\![df, dg]\!] = d(\rho(df)(g)) \quad \text{and} \quad \rho(df)(g) + \rho(dg)(f) = 0. \tag{30}$$

Conversely, let $(T^*M, \rho, [\![-,-]\!])$ be a Lie algebroid such that (30) holds. For $f, g \in C^\infty(M)$ put $\{f, g\} := \rho(df)(g)$. Then $\{-,-\}$ is a Poisson structure on M. This shows that Poisson structures are equivalent to Lie algebroids $(T^*M, \rho, [\![-,-]\!])$ such that (30) holds.

Example 50. Let (L, ∇, ω) be an abstract locally conformal Poisson structure on a manifold M (see the previous appendix). There is a unique Lie algebroid $(T^*M \otimes L, \rho, [\![-,-]\!])$ such that $[\![d_\nabla\lambda, d_\nabla\mu]\!] = d_\nabla\{\lambda, \mu\}$, and $\rho(d_\nabla\lambda)$ is the symbol of the first order differential operator $\{\lambda, -\}$, where $\lambda, \mu \in \Gamma(L)$. Moreover, there is a unique representation (L, ∇^L) of $T^*M \otimes L$ such that $\nabla^L_{d_\nabla\lambda}\mu = \{\lambda, \mu\}$. In particular,

$$[\![d_\nabla\lambda, d_\nabla\mu]\!] = d_\nabla(\nabla^L_{d_\nabla\lambda}\mu) \quad \text{and} \quad \nabla^L_{d_\nabla\lambda}\mu + \nabla^L_{d_\nabla\mu}\lambda = 0. \tag{31}$$

Conversely, let $(T^*M \otimes L, \rho, [\![-,-]\!])$ be a Lie algebroid equipped with a representation (L, ∇^L) such that (31) holds. For $\lambda, \mu \in \Gamma(L)$ put $\{\lambda, \mu\} := \nabla^L_{d_\nabla\lambda}\mu$. Then $(L, \nabla, \{-,-\})$ is a locally conformal Poisson structure on M. Thus locally conformal Poisson structures are equivalent to Lie algebroids $(T^*M \otimes L, \rho, [\![-,-]\!])$ such that (31) holds.

Acknowledgement. I thank the anonymous referee for carefully reading the first manuscript and for her/his suggestions to improve the readability of the paper.

References

[Bonavolontà and Poncin 2013] G. Bonavolontà and N. Poncin, "On the category of Lie n-algebroids", *J. Geom. Phys.* **73** (2013), 70–90. MR Zbl

[Bruce 2011] A. J. Bruce, "From L_∞-algebroids to higher Schouten/Poisson structures", *Rep. Math. Phys.* **67**:2 (2011), 157–177. MR Zbl

[Bursztyn 2013] H. Bursztyn, "A brief introduction to Dirac manifolds", pp. 4–38 in *Geometric and topological methods for quantum field theory*, edited by A. Cardona et al., Cambridge Univ. Press, 2013. MR Zbl arXiv

[Bursztyn and Cabrera 2012] H. Bursztyn and A. Cabrera, "Multiplicative forms at the infinitesimal level", *Math. Ann.* **353**:3 (2012), 663–705. MR Zbl

[Bursztyn et al. 2009] H. Bursztyn, A. Cabrera, and C. Ortiz, "Linear and multiplicative 2-forms", *Lett. Math. Phys.* **90**:1-3 (2009), 59–83. MR Zbl

[Bursztyn et al. 2015] H. Bursztyn, A. Cabrera, and D. Iglesias, "Multisymplectic geometry and Lie groupoids", pp. 57–73 in *Geometry, mechanics, and dynamics*, edited by D. E. Chang et al., Fields Inst. Commun. **73**, Springer, New York, 2015. MR Zbl arXiv

[Cantrijn et al. 1999] F. Cantrijn, A. Ibort, and M. de León, "On the geometry of multisymplectic manifolds", *J. Austral. Math. Soc. Ser. A* **66**:3 (1999), 303–330. MR Zbl

[Cattaneo and Schätz 2011] A. S. Cattaneo and F. Schätz, "Introduction to supergeometry", *Rev. Math. Phys.* **23**:6 (2011), 669–690. MR Zbl

[Courant 1990] T. J. Courant, "Dirac manifolds", *Trans. Amer. Math. Soc.* **319**:2 (1990), 631–661. MR Zbl

[Crainic and Salazar 2015] M. Crainic and M. A. Salazar, "Jacobi structures and Spencer operators", *J. Math. Pures Appl.* (9) **103**:2 (2015), 504–521. MR Zbl

[Crainic et al. 2015] M. Crainic, M. A. Salazar, and I. Struchiner, "Multiplicative forms and Spencer operators", *Math. Z.* **279**:3-4 (2015), 939–979. MR Zbl

[Getzler 2010] E. Getzler, "Higher derived brackets", preprint, 2010. arXiv

[Grabowski 2013] J. Grabowski, "Graded contact manifolds and contact Courant algebroids", *J. Geom. Phys.* **68** (2013), 27–58. MR Zbl

[Jotz and Ortiz 2011] M. Jotz and C. Ortiz, "Foliated groupoids and their infinitesimal data", preprint, 2011. arXiv

[Kirillov 1976] A. A. Kirillov, "Local Lie algebras", *Uspehi Mat. Nauk* **31**:4(190) (1976), 57–76. In Russian; translated in *Russian Math. Surveys* **31**:4 (1976), 55–75. MR

[Kjeseth 2001a] L. Kjeseth, "A homotopy Lie–Rinehart resolution and classical BRST cohomology", *Homology Homotopy Appl.* **3**:1 (2001), 165–192. MR Zbl

[Kjeseth 2001b] L. Kjeseth, "Homotopy Rinehart cohomology of homotopy Lie–Rinehart pairs", *Homology Homotopy Appl.* **3**:1 (2001), 139–163. MR Zbl

[Marle 1991] C.-M. Marle, "On Jacobi manifolds and Jacobi bundles", pp. 227–246 in *Symplectic geometry, groupoids, and integrable systems* (Berkeley, 1989), edited by P. Dazord and A. Weinstein, Math. Sci. Res. Inst. Publ. **20**, Springer, New York, 1991. MR Zbl

[Mehta 2006] R. A. Mehta, *Supergroupoids, double structures, and equivariant cohomology*, Ph.D. thesis, University of California, Berkeley, 2006, http://search.proquest.com/docview/305363751. MR

[Mehta 2013] R. A. Mehta, "Differential graded contact geometry and Jacobi structures", *Lett. Math. Phys.* **103**:7 (2013), 729–741. MR Zbl

[Rogers 2012] C. L. Rogers, "L_∞-algebras from multisymplectic geometry", *Lett. Math. Phys.* **100**:1 (2012), 29–50. MR Zbl

[Roytenberg 2002] D. Roytenberg, "On the structure of graded symplectic supermanifolds and Courant algebroids", pp. 169–185 in *Quantization, Poisson brackets and beyond* (Manchester, 2001), edited by T. Voronov, Contemp. Math. **315**, Amer. Math. Soc., Providence, RI, 2002. MR Zbl

[Salazar 2013] M. A. Salazar, *Pfaffian groupoids*, Ph.D. thesis, Universiteit Utrecht, 2013, http://dspace.library.uu.nl/bitstream/handle/1874/275521/salazar-pinzon.pdf.

[Sati et al. 2012] H. Sati, U. Schreiber, and J. Stasheff, "Twisted differential string and fivebrane structures", *Comm. Math. Phys.* **315**:1 (2012), 169–213. MR Zbl

[Sheng and Zhu 2011] Y. Sheng and C. Zhu, "Higher Extensions of Lie Algebroids and Application to Courant Algebroids", preprint, 2011. arXiv

[Vaisman 1985] I. Vaisman, "Locally conformal symplectic manifolds", *Internat. J. Math. Math. Sci.* **8**:3 (1985), 521–536. MR Zbl

[Vaisman 2007] I. Vaisman, "Dirac structures and generalized complex structures on $TM \times \mathbb{R}^h$", *Adv. Geom.* **7**:3 (2007), 453–474. MR Zbl

[Vitagliano 2014] L. Vitagliano, "On the strong homotopy Lie–Rinehart algebra of a foliation", *Commun. Contemp. Math.* **16**:6 (2014), 1450007, 49. MR Zbl

[Vitagliano 2015a] L. Vitagliano, "Dirac–Jacobi Bundles", preprint, 2015. To appear in *J. Sympl. Geom.* arXiv

[Vitagliano 2015b] L. Vitagliano, "Representations of homotopy Lie–Rinehart algebras", *Math. Proc. Cambridge Philos. Soc.* **158**:1 (2015), 155–191. MR

[Voronov 2010] T. T. Voronov, "Q-manifolds and higher analogs of Lie algebroids", pp. 191–202 in *XXIX Workshop on Geometric Methods in Physics*, edited by P. Kielanowski et al., AIP Conf. Proc. **1307**, Amer. Inst. Phys., Melville, NY, 2010. MR Zbl arXiv

[Weinstein 1987] A. Weinstein, "Symplectic groupoids and Poisson manifolds", *Bull. Amer. Math. Soc.* (*N.S.*) **16**:1 (1987), 101–104. MR Zbl

[Zambon and Zhu 2012] M. Zambon and C. Zhu, "Distributions and quotients on degree 1 NQ-manifolds and Lie algebroids", *J. Geom. Mech.* **4**:4 (2012), 469–485. MR Zbl arXiv

Received April 14, 2015. Revised January 14, 2016.

LUCA VITAGLIANO
DEPARTMENT OF MATHEMATICS
UNIVERSITÀ DEGLI STUDI DI SALERNO
& ISTITUTO NAZIONALE DI FISICA NUCLEARE
GC SALERNO
VIA GIOVANNI PAOLO II N° 123
I-84084 FISCIANO (SA)
ITALY

lvitagliano@unisa.it

PACIFIC JOURNAL OF MATHEMATICS
Vol. 283, No. 2, 2016
dx.doi.org/10.2140/pjm.2016.283.483

DISCRIMINANTS AND THE MONOID OF QUADRATIC RINGS

JOHN VOIGHT

We consider the natural monoid structure on the set of quadratic rings over an arbitrary base scheme and characterize this monoid in terms of discriminants.

Quadratic field extensions K of $\mathbb{Q}$ are characterized by their discriminants. Indeed, there is a bijection

$$\left\{\begin{array}{c}\text{separable quadratic algebras}\\ \text{over } \mathbb{Q} \text{ up to isomorphism}\end{array}\right\} \xrightarrow{\sim} \mathbb{Q}^\times/\mathbb{Q}^{\times 2}, \quad \mathbb{Q}[\sqrt{d}]=\mathbb{Q}[x]/(x^2-d)\mapsto d\mathbb{Q}^{\times 2},$$

where a separable quadratic algebra over $\mathbb{Q}$ is either a quadratic field extension or the algebra $\mathbb{Q}[\sqrt{1}] \simeq \mathbb{Q}\times\mathbb{Q}$ of discriminant 1. In particular, the set of isomorphism classes of separable quadratic extensions of $\mathbb{Q}$ can be given the structure of an elementary abelian 2-group, with identity element the class of $\mathbb{Q}\times\mathbb{Q}$: we have, simply,

$$\mathbb{Q}[\sqrt{d_1}] * \mathbb{Q}[\sqrt{d_2}] = \mathbb{Q}[\sqrt{d_1 d_2}]$$

up to isomorphism. If d_1, d_2, $d_1 d_2 \in \mathbb{Q}^\times \setminus \mathbb{Q}^{\times 2}$ then $\mathbb{Q}(\sqrt{d_1 d_2})$ sits as the third quadratic subfield of the compositum $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$:

$$\begin{tikzcd}
 & \mathbb{Q}(\sqrt{d_1}, \sqrt{d_2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\mathbb{Q}(\sqrt{d_1}) \arrow[dr, dash] & \mathbb{Q}(\sqrt{d_1 d_2}) \arrow[d, dash] & \mathbb{Q}(\sqrt{d_2}) \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

Indeed, if σ_1 is the nontrivial element of $\operatorname{Gal}(\mathbb{Q}(\sqrt{d_1})/\mathbb{Q})$, then there is a unique extension of σ_1 to $\mathbb{Q}(\sqrt{d_1}, \sqrt{d_2})$ leaving $\mathbb{Q}(\sqrt{d_2})$ fixed, similarly with σ_2, and $\mathbb{Q}(\sqrt{d_1 d_2})$ is the fixed field of the composition $\sigma_1\sigma_2 = \sigma_2\sigma_1$.

This characterization of quadratic extensions works over any base field F with $\operatorname{char} F \neq 2$ and is summarized concisely in the Kummer theory isomorphism

$$H^1(\operatorname{Gal}(\overline{F}/F), \{\pm 1\}) = \operatorname{Hom}(\operatorname{Gal}(\overline{F}/F), \{\pm 1\}) \simeq F^\times/F^{\times 2}.$$

MSC2010: primary 11E99; secondary 11R11, 11R29.
Keywords: quadratic forms, quadratic rings, monoids, discriminants, discriminant algebras.

On the other hand, over a field F with $\operatorname{char} F = 2$, all separable quadratic extensions have trivial discriminant and instead they are classified by the (additive) Artin–Schreier group

$$F/\wp(F), \quad \text{where} \quad \wp(F) = \{r + r^2 : r \in F\},$$

with the class of $a \in F$ in correspondence with the isomorphism class of the extension $F[x]/(x^2 - x + a)$. By similar considerations as above, we again find a natural structure of an elementary abelian 2-group on the set of isomorphism classes of separable quadratic extensions of F.

One can extend this correspondence between quadratic extensions and discriminants integrally, as follows: Let R be a commutative ring. An *R-algebra* is a ring B equipped with an embedding $R \hookrightarrow B$ of rings (mapping $1 \in R$ to $1 \in B$) whose image lies in the center of B; we identify R with its image via this embedding. A *free quadratic* R-algebra (also called a *free quadratic ring* over R) is an R-algebra S (associative with 1) that is free of rank 2 as an R-module. Let S be a free quadratic R-algebra. Then $S/R \simeq \bigwedge^2 S \simeq R$ is projective, so there is an R-basis $1, x$ for S; we find that $x^2 = tx - n$ for some $t, n \in R$ and that S is commutative. The map $\sigma : S \to S$ induced by $x \mapsto t - x$ is the unique *standard involution* on S, an R-linear (anti)automorphism such that $y\sigma(y) \in R$ for all $y \in S$. The class in $R/R^{\times 2}$ of the *discriminant* of S,

$$d = d(S) = (x - \sigma(x))^2 = t^2 - 4n,$$

is independent of the choice of basis $1, x$. A discriminant d satisfies the congruence $d \equiv t^2 \pmod{4R}$, so for example if $R = \mathbb{Z}$ then $d \equiv 0, 1 \pmod 4$.

Now suppose that R is an integrally closed domain of characteristic not 2. Then there is a bijection

$$\left\{ \begin{matrix} \text{free quadratic rings over } R \\ \text{up to isomorphism} \end{matrix} \right\} \xrightarrow{\sim} \{d \in R : d \text{ is a square in } R/4R\}/R^{\times 2},$$
$$S \longmapsto d(S).$$

For example, over $R = \mathbb{Z}$, the free quadratic ring $S(d)$ over $\mathbb{Z}$ of discriminant $d \in \mathbb{Z} = \mathbb{Z}/\mathbb{Z}^{\times 2}$ with $d \equiv 0, 1 \pmod 4$ is given by

$$S(d) = \begin{cases} \mathbb{Z}[x]/(x^2) \hookrightarrow \mathbb{Q}[x]/(x^2) & \text{if } d = 0; \\ \mathbb{Z}[x]/(x^2 - \sqrt{d}x) \hookrightarrow \mathbb{Q} \times \mathbb{Q} & \text{if } d \neq 0 \text{ is a square;} \\ \mathbb{Z}[(d + \sqrt{d})/2] \hookrightarrow \mathbb{Q}(\sqrt{d}) & \text{otherwise.} \end{cases}$$

The set of discriminants under multiplication has the structure of a *commutative monoid*, a nonempty set equipped with a commutative binary operation and identity element. Hence, so does the set of isomorphism classes of free quadratic R-algebras, an operation we denote by $*$: the identity element is the class of

$R \times R \simeq R[x]/(x^2 - x)$. The class of the ring $R[x]/(x^2)$ with discriminant 0 is called an *absorbing element*.

More generally, a *quadratic* R-algebra or *quadratic ring over* R is an R-algebra S that is locally free of rank 2 as an R-module. By definition, a quadratic ring over R localized at any prime (or maximal) ideal of R is a free quadratic R-algebra. Being true locally, a quadratic R-algebra S is commutative and has a unique standard involution.

There is a natural description of quadratic R-algebras as a stack quotient, as follows: A free quadratic R-algebra equipped with a basis $1, x$ has multiplication table uniquely determined by $t, n \in R$ and has no automorphisms, so the functor that associates to a commutative ring R the set of free quadratic R-algebras with basis (up to isomorphism) is represented by two-dimensional affine space $\mathbb{A}^2$ (over $\mathbb{Z}$). The change of basis for a free quadratic R-algebra is of the form $x \mapsto u(x + r)$ with $u \in R^\times$ and $r \in R$, mapping

$$(t, n) \mapsto (u(t + 2r), u^2(n + tr + r^2)).$$

Therefore, we have a map from the set of free quadratic R-algebras with basis to the quotient of $\mathbb{A}^2(R)$ by $G(R) = (\mathbb{G}_m \rtimes \mathbb{G}_a)(R)$ with the above action. Working over $\operatorname{Spec} \mathbb{Z}$, the group scheme G is naturally a subgroup scheme of GL_2, but it does not act linearly on $\mathbb{A}^2$, and the Artin stack $[\mathbb{A}^2/G]$ has dimension zero over $\operatorname{Spec} \mathbb{Z}$! Nevertheless, the set $[\mathbb{A}^2/G](R)$ is in bijection with the set of quadratic R-algebras up to isomorphism.

Recall that a commutative R-algebra S is *separable* if S is (faithfully) projective as an $S \otimes_R S$-module via the map $x \otimes y \mapsto xy$. A free quadratic R-algebra S is separable if and only if $d(S) \in R^\times$; so, for example, the only separable (free) R-algebra over $\mathbb{Z}$ is the ring $\mathbb{Z} \times \mathbb{Z}$ of discriminant 1, an impoverishment indeed! A separable quadratic R-algebra S is étale over R, and R is equal to the fixed subring of the standard involution of S over R. (In many contexts, one then says that S is Galois over R with Galois group $\mathbb{Z}/2\mathbb{Z}$, though authors differ on precise terminology. See Lenstra [2008] for one approach to Galois theory for schemes.) If S and T are separable free quadratic R-algebras, where R is a Dedekind domain of characteristic not 2, having standard involutions σ and τ, respectively, then the monoid product $S * T$ defined above (by transporting the monoid structure on the set of discriminants) is the fixed subring of $S \otimes_R T$ by $\sigma \otimes \tau$, in analogy with the case of fields.

The characterization of free quadratic R-algebras by their discriminants is an example of the parametrization of algebraic structures, corresponding to the Lie group A_1 in the language of Bhargava [2002] (perhaps indexed by A_0 in the schema of Knus, Ojanguren, Parimala and Tignol [Knus et al. 1998, §15]).

Results in this direction go back to Gauss's composition law for binary quadratic forms and have been extended in recent years by Bhargava [2006], Wood [2011] and others. Indeed, several authors have considered the case of quadratic R-algebras, including Kanzaki [1973] and Small [1972]. In this article, we consider a very general instance of this monoidal correspondence between quadratic R-algebras and discriminants over an arbitrary base scheme.

Let X be a scheme. A *quadratic $\mathscr{O}_X$-algebra* is a coherent sheaf $\mathscr{S}$ of $\mathscr{O}_X$-algebras that is locally free of rank 2 as a sheaf of $\mathscr{O}_X$-modules. Equivalently, a quadratic $\mathscr{O}_X$-algebra is specified by a finite, locally free morphism of schemes $\phi : Y \to X$ of degree 2 (sometimes called a *double cover*): the sheaf $\phi_*\mathscr{O}_Y$ is a sheaf of $\mathscr{O}_X$-algebras that is locally free of rank 2. If $f : X \to Z$ is a morphism of schemes and $\mathscr{S}$ is a quadratic $\mathscr{O}_Z$-algebra, then the pullback $f^*\mathscr{S}$ is a quadratic $\mathscr{O}_X$-algebra. Let $\mathrm{Quad}(X)$ denote the set of isomorphism classes of quadratic $\mathscr{O}_X$-algebras and, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$, let $\mathrm{Quad}(X; \mathscr{L}) \subseteq \mathrm{Quad}(X)$ be the subset of those algebras $\mathscr{S}$ such that there exists an isomorphism $\bigwedge^2 \mathscr{S} \simeq \mathscr{L}$ of $\mathscr{O}_X$-modules.

Our first result provides an axiomatic description of the monoid structure on the set $\mathrm{Quad}(X)$ (see Theorem 3.27).

Theorem A. *There is a unique system of binary operations*

$$*_X : \mathrm{Quad}(X) \times \mathrm{Quad}(X) \to \mathrm{Quad}(X),$$

one for each scheme X, such that:

(i) $\mathrm{Quad}(X)$ *is a commutative monoid under $*_X$, with identity element the isomorphism class of $\mathscr{O}_X \times \mathscr{O}_X$.*

(ii) *The association $X \mapsto (\mathrm{Quad}(X), *_X)$ from schemes to commutative monoids is functorial in X: for each morphism $f : X \to Z$ of schemes, the diagram*

$$\begin{array}{ccc} \mathrm{Quad}(Z) \times \mathrm{Quad}(Z) & \xrightarrow{*_Z} & \mathrm{Quad}(Z) \\ \big\downarrow & & \big\downarrow{\scriptstyle f^*} \\ \mathrm{Quad}(X) \times \mathrm{Quad}(X) & \xrightarrow{*_X} & \mathrm{Quad}(X) \end{array}$$

is commutative.

(iii) *If $X = \operatorname{Spec} R$ and S and T are separable quadratic R-algebras with standard involutions σ and τ, then $S *_{\operatorname{Spec} R} T$ is the fixed subring of $S \otimes_R T$ under $\sigma \otimes \tau$.*

The binary operation is defined locally (Construction 3.14): if $X = \operatorname{Spec} R$, and $S = R \oplus Rx$ and $T = R \oplus Ry$ are free quadratic R-algebras with $x^2 = tx - n$ and $y^2 = sy - m$, then we define the free quadratic R-algebra

$$S * T = R \oplus Rw,$$

where

$$w^2 = (st)w - (mt^2 + ns^2 - 4nm).$$

This explicit description (in the free case over an affine base) is given by Hahn [1994, Exercises 14–20, pp. 42–43].

A general investigation of the monoid structure on quadratic algebras goes back at least to Loos [1996], who gives via a universal construction a tensor product on the larger category of unital quadratic forms (quadratic forms representing 1); this category is equivalent to the category of quadratic algebras for forms on a finitely generated, projective module of rank 2 [Loos 1996, Proposition 1.6] as long as one takes morphisms as isomorphisms in the category [Loos 2007, §1.4]. (See also §6.1 of Loos [2007] for further treatment.) The existence of the monoid structure was also established in an unpublished letter of Deligne to Rost and Bhargava (March 2, 2005) by a different method: he associates to every R-algebra its discriminant algebra (a quadratic algebra) and extends the natural operation of addition of $\mathbb{Z}/2\mathbb{Z}$-torsors from the étale case to the general case by geometric arguments. Our proof of Theorem A carries the same feel as these results, but it is accomplished in a more direct fashion and gives a characterization (in particular, uniqueness).

Recently, there has been renewed interest in the construction of discriminant algebras (sending an R-algebra A of rank n to a quadratic R-algebra) by Loos [2007], Rost [2002] and Biesel and Gioia [2015]. Indeed, Biesel and Gioia [2015, Section 8] describe the monoid operation in Theorem A over an affine base in the context of discriminant algebras. We hope that our theorem will have some applications in this context.

Our second result characterizes quadratic algebras in terms of discriminants. A *discriminant* (over X) is a pair $(d, \mathscr{L})$ such that $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module and $d \in (\mathscr{L}^\vee)^{\otimes 2}(X)$ is a global section that is a *square modulo* 4: there exists a global section $\bar{t} \in \mathscr{L}^\vee(X)/2\mathscr{L}^\vee(X) = \mathscr{L}^\vee \otimes \mathscr{O}_X/2\mathscr{O}_X$ such that

$$\bar{t} \otimes \bar{t} = \bar{d} \in (\mathscr{L}^\vee)^{\otimes 2}(X)/4(\mathscr{L}^\vee)^{\otimes 2}(X).$$

We can of course also think of $d \in (\mathscr{L}^\vee)^{\otimes 2}(X)$ as an $\mathscr{O}_X$-module homomorphism $d : \mathscr{L}^{\otimes 2} \to \mathscr{O}_X$; but such a global section is also equivalently given by a quadratic form $D : \mathscr{L} \to \mathscr{O}_X$ (see Section 2).

An isomorphism of discriminants $(d, \mathscr{L})$, $(d', \mathscr{L}')$ is an isomorphism $f : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $(f^\vee)^{\otimes 2}(d') = d$. For example, if $X = \operatorname{Spec} R$ for R a commutative ring and $\mathscr{L} = \mathscr{O}_X = \tilde{R}$, then, as above, a discriminant is specified by an element $d \in R$ such that $d \equiv t^2 \pmod{4R}$ for some $t \in R$ (noting that only $t \in R/2R$ matters), and two discriminants d, d' are isomorphic if and only if there exists $u \in R^\times$ such that $u^2 d' = d$. Thinking of a discriminant as a quadratic form $D : \mathscr{L} \to \mathscr{O}_X$, its image generates a locally principal ideal sheaf $\mathscr{I} \subseteq \mathscr{O}_X$, and the set of discriminants with given locally free image $\mathscr{I} \subseteq \mathscr{O}_X$, if nonempty, is a principal homogeneous space for the group $\mathscr{O}_X^\times/\mathscr{O}_X^{\times 2}$. Let $\operatorname{Disc}(X)$ denote the set of isomorphism classes of discriminants and $\mathrm{Disc}(X;\mathscr{L}) \subseteq \mathrm{Disc}(X)$ the subset with underlying line bundle $\mathscr{L}$. Then the tensor product

$$(d,\mathscr{L}) * (d',\mathscr{L}') = (d \otimes d', \mathscr{L} \otimes \mathscr{L}')$$

gives $\mathrm{Disc}(X)$ and $\mathrm{Disc}(X;\mathscr{O}_X)$ the structure of a commutative monoid with identity element the class of $(1,\mathscr{O}_X)$.

A quadratic $\mathscr{O}_X$-algebra $\mathscr{S}$ has a discriminant $\mathrm{disc}(\mathscr{S}) = \bigl(d(\mathscr{S}), \bigwedge^2\mathscr{S}\bigr)$, defined by

$$\begin{aligned} d(\mathscr{S}) &: \bigl(\textstyle\bigwedge^2\mathscr{S}\bigr)^{\otimes 2} \to \mathscr{O}_X, \\ (x \wedge y) &\otimes (z \wedge w) \mapsto (x\sigma(y) - \sigma(x)y)(z\sigma(w) - \sigma(z)w), \end{aligned}$$

where σ is the unique standard involution on $\mathscr{S}$. Although, a priori, the codomain of $d(\mathscr{S})$ is $\mathscr{S}$, in fact its image lies in $\mathscr{O}_X$: if $X = \operatorname{Spec} R$ and $\mathscr{S} = \operatorname{Spec} S$ and S is free with basis $1, x$, then $\bigl(\bigwedge^2 S\bigr)^{\otimes 2}$ is freely generated by $(1 \wedge x) \otimes (1 \wedge x)$ and

$$1 \wedge x \otimes 1 \wedge x \mapsto (x - \sigma(x))^2 \in R.$$

We have a natural forgetful map $\mathrm{Disc}(X) \to \mathrm{Pic}(X)$, where $(d,\mathscr{L})$ maps to the isomorphism class of $\mathscr{L}$.

We say a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ of commutative monoids is *exact* if f is injective, g is surjective, and for all $z, w \in B$ we have

$$g(z) = g(w) \iff xz = yw \text{ for some } x, y \in A;$$

equivalently, the sequence is exact if and only if f is injective and g induces an isomorphism of monoids $B/f(A) \simeq C$. (For a review of monoids, see Section 1.)

We now describe the monoid $\mathrm{Quad}(X)$. We begin with the statement that the forgetful map is compatible with the discriminant map, as follows.

Theorem B. *Let X be a scheme. Then the diagram of commutative monoids*

$$\begin{tikzcd} \mathrm{Quad}(X;\mathscr{O}_X) \arrow[r] \arrow[d, "\mathrm{disc}"] & \mathrm{Quad}(X) \arrow[r, "\wedge^2"] \arrow[d, "\mathrm{disc}"] & \mathrm{Pic}(X) \arrow[d, equal] \\ \mathrm{Disc}(X;\mathscr{O}_X) \arrow[r] & \mathrm{Disc}(X) \arrow[r] & \mathrm{Pic}(X) \end{tikzcd}$$

is functorial and commutative with exact rows and Zariski locally surjective columns.

By "Zariski locally surjective columns", we mean that there is an (affine) open cover of X where (under pullback) the columns are surjective. (Considering the corresponding sheaves over X, we also obtain a surjective map of sheaves; see Theorem 3.28.)

We now turn to describe the morphism $\mathrm{Quad}(X;\mathscr{O}_X) \to \mathrm{Disc}(X;\mathscr{O}_X)$. For this purpose, we work locally and assume $X = \operatorname{Spec} R$ for a commutative ring R; we abbreviate $\mathrm{Quad}(\mathrm{Spec}\, R) = \mathrm{Quad}(R)$ and $\mathrm{Quad}(\mathrm{Spec}\, R; \mathscr{O}_{\mathrm{Spec}\, R}) = \mathrm{Quad}(R; R)$, and similarly with discriminants.

We would like to able to fit the surjective map $\mathrm{Quad}(R) \xrightarrow{\mathrm{disc}} \mathrm{Disc}(R)$ of monoids into an exact sequence by identifying its kernel, but unfortunately the fibers of this map vary over the codomain. Instead, we will describe the action of a subgroup of $\mathrm{Quad}(R)$ on the fibers of the map disc; this is a natural generalization, as the fibers of a group homomorphism are principal homogeneous spaces for the kernel K and are noncanonically isomorphic as a K-set to K with the regular representation.

Recalling the case of quadratic extensions of a field F with $\mathrm{char}\, F = 2$, for a commutative ring R we define the *Artin–Schreier group* $\mathrm{AS}(R)$ to be the additive quotient

$$\mathrm{AS}(R) = \frac{R[4]}{\wp(R)[4]}, \quad \text{where} \quad \wp(R)[4] = \{n = r + r^2 \in R : r \in R\} \cap R[4]$$

and $R[4] = \{a \in R : 4a = 0\}$. We have a map $i : \mathrm{AS}(R) \to \mathrm{Quad}(R; R) \hookrightarrow \mathrm{Quad}(R)$ sending the class of $n \in \mathrm{AS}(R)$ to the isomorphism class of the algebra $S = R[x]/(x^2 - x + n)$. The group $\mathrm{AS}(R)$ is an elementary abelian 2-group since $2R[4] \subseteq \wp(R)$.

Our next main result is as follows (see Theorem 4.3).

Theorem C. *The fibers of the map* $\mathrm{disc} : \mathrm{Quad}(R) \to \mathrm{Disc}(R)$ *have a unique action of the group* $\mathrm{AS}(R)$ *compatible with the inclusion of monoids* $\mathrm{AS}(R) \hookrightarrow \mathrm{Quad}(R)$. *Moreover*, *the kernel of this action on the fiber* $\mathrm{disc}^{-1}(dR^{\times 2})$ *contains* $\mathrm{ann}_R(d)[4]$.

Roughly speaking, Theorems B and C together say that “a quadratic algebra is determined by its Steinitz class and its discriminant, locally up to an Artin–Schreier extension”. These theorems could be rephrased in terms of the Grothendieck group; however, due to the existence of an absorbing element, the group $K_0(\mathrm{Quad}(X))$ is trivial for all schemes X.

The article is organized as follows. In Section 1, we briefly review the relevant notions from monoid theory. In Section 2, we consider the monoid of discriminants; in Section 3 we define the monoid of quadratic R-algebras and prove Theorems A and B. In Section 4 we prove Theorem C.

1. Monoids

To begin, we review standard terminology for monoids. A reference for the material in this section is Bergman [2015, Chapter 3]; more generally, see Burris and Sankappanavar [2012] and McKenzie, McNulty and Taylor [McKenzie et al. 1987].

A *semigroup* is a nonempty set A equipped with an associative binary operation

$$* : A \times A \to A.$$

A *monoid* is a semigroup with identity element 1 for $*$ (necessarily unique). Any semigroup without 1 can be augmented to a monoid. Natural examples of monoids abound: the natural numbers $\mathbb{N} = \mathbb{Z}_{\geq 0}$ under addition, a ring R under its multiplication, and the set of endomorphisms of an algebraic object (such as a variety) under composition. A *group* is a monoid equipped with an inverse map ${}^{-1} : A \to A$.

Let A be a semigroup. We say A is *commutative* if $xy = yx$ for all $x, y \in A$. An *absorbing element* of A is an element $0 \in A$ such that $0x = x0 = 0$ for all $x \in A$; a monoid has at most one absorbing element. Multiplicative notation for A will in general be more natural for us; however, we will occasionally write A additively with operation $+$, in which case the identity element will be denoted 0 and, to avoid confusion, A will have no absorbing element. An element $x \in A$ is (left) *cancellative* if $xy = xz$ implies $y = z$ for all $y, z \in A$.

A *homomorphism* of semigroups is a map $f : A \to B$ such that $f(xy) = f(x)f(y)$ for all $x, y \in A$, and a homomorphism of monoids is a homomorphism of semigroups such that $f(1_A) = 1_B$.

Let $f : A \to B$ be a homomorphism of monoids. Unlike groups, the kernel $\ker f = \{x \in A : f(x) = 1\}$ of a monoid homomorphism does not determine the structure of the image of f; instead, we define the *kernel congruence* of f by

$$K_f = \{(x, y) : f(x) = f(y)\} \subseteq A \times A.$$

The set K_f defines a *congruence* on A, an equivalence relation compatible with the operation on A, i.e., if $(x, y), (z, w) \in K_f$ then $(xz, yw) \in K_f$. Conversely, given a congruence K on a monoid A, the set operation $[x] \cdot [y] = [x \cdot y]$ on equivalence classes $[x], [y] \in A/K$ is well-defined and the quotient map $A \to A/K$ via $x \mapsto [x]$ is a surjective homomorphism of monoids with kernel K; any homomorphism $f : A \to B$ with $K_f \supseteq K$ factors through $A \to A/K$.

The image $f(A) = \{f(x) : x \in A\}$ is a submonoid of B, but if A and B are noncommutative then not every submonoid is eligible to be the kernel of a homomorphism (just as not every subgroup is normal). As we will be interested only in commutative monoids, and this assumption simplifies the presentation, suppose from now on that A and B are commutative. Then the set

$$I_f = \{(z, w) : f(x)z = f(y)w \text{ for some } x, y \in A\} \subseteq B \times B$$

is a congruence called the *image congruence*. (Without the hypothesis of commutativity, I_f is a relation that is reflexive and symmetric, but not necessarily transitive nor a congruence; if A and B are possibly nonabelian groups, then I_f is transitive and is a congruence if and only if $f(A)$ is a normal subgroup of B.) Note that if $0 \in f(A)$ then $I_f = B \times B$. We write $B/f(A) = B/I_f$.

A sequence

$$A \xrightarrow{f} B \xrightarrow{g} C \tag{1.1}$$

is *exact* if f is injective, g is surjective, and $K_g = I_f$, i.e.,

$$g(z) = g(w) \iff xz = yw \text{ for some } x, y \in A;$$

equivalently, (1.1) is exact if f is injective and g induces an isomorphism $B/f(A) = B/I_f \xrightarrow{\sim} C$. A sequence of groups (1.1) is exact as a sequence of groups if and only if it is exact as a sequence of monoids.

Remark 1.2. We will not make use of long exact sequences of monoids here, nor write the customary 0 or 1 at the ends of our short exact sequences. Indeed, the straightforward extension of the notion from groups to monoids using the definitions above (kernel congruence equals image congruence) has a defect: the sequence

$$\mathbb{N} \xrightarrow{f} \mathbb{Z} \xrightarrow{j} 0$$

of monoids under addition has $I_f = \mathbb{Z} \times \mathbb{Z} = K_j$ even though f is not surjective. (The map f is, however, an epimorphism in the category of monoids.)

We will also make use of sheaves of monoids over a scheme X. A sequence $\mathscr{A} \xrightarrow{f} \mathscr{B} \xrightarrow{g} \mathscr{C}$ of sheaves of monoids is *exact* if the sheaf associated to the presheaf $U \mapsto \mathscr{B}(U)/f(\mathscr{A}(U))$ is isomorphic to $\mathscr{C}$, or equivalently if the induced sequence $\mathscr{A}_x \xrightarrow{f_x} \mathscr{B}_x \xrightarrow{g_x} \mathscr{C}_x$ of monoid stalks is exact for all $x \in X$.

Like the formation of the integers from the natural numbers, one can construct the Grothendieck group $K_0(A)$ of a commutative monoid A, with the universal property that, for any monoid homomorphism $A \to G$ with G an abelian group, there exists a unique group homomorphism $K_0(A) \to G$ such that the diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & G \\ K_0(A) \arrow[ur, dashed, "\exists!"] & \end{tikzcd}$$

commutes. The group $K_0(A)$ is constructed as $A \times A$ under the equivalence relation $(x, x') \sim (y, y')$ if there exists $z \in A$ such that $xy'z = x'yz$. Note that if A has an absorbing element 0 then $K_0(A) = \{0\}$. The set of cancellative elements A_{canc} is the largest submonoid of A that can be embedded in a group, and the smallest such containing group is the Grothendieck group $K_0(A_{\text{canc}})$.

2. Discriminants

In this section, we define discriminants for quadratic rings over general schemes (Definition 2.11); for a discussion of discriminant modules overlapping the one presented here, see Knus [1991, §III.3] and Loos [1996, §1.2]. We also relate seminondegenerate quadratic forms on line bundles by their images (Lemma 2.8) and factor the monoid of discriminants over the Picard group (Proposition 2.18).

Let X be a scheme. A *quadratic form* over X is a pair $(\mathscr{M}, Q)$ where $\mathscr{M}$ is a locally free $\mathscr{O}_X$-module of finite rank and $Q : \mathscr{M} \to \mathscr{O}_X$ is a *quadratic map*, i.e., for all open sets $U \subseteq X$ we have:

(i) $Q(rx) = r^2 Q(x)$ for all $r \in \mathscr{O}_X(U)$ and $x \in \mathscr{M}(U)$.

(ii) The map $T : \mathscr{M}(U) \times \mathscr{M}(U) \to \mathscr{O}_X(U)$ defined by

$$T(x, y) = Q(x + y) - Q(x) - Q(y)$$

is $\mathscr{O}_X(U)$-bilinear; we call T the *associated bilinear form*.

An *isometry* between quadratic forms $Q : \mathscr{M} \to \mathscr{O}_X$ and $Q' : \mathscr{M}' \to \mathscr{O}_X$ is an $\mathscr{O}_X$-module isomorphism $f : \mathscr{M} \xrightarrow{\sim} \mathscr{M}'$ such that $Q' \circ f = Q$. A *similarity* between quadratic forms Q and Q' is a commutative square:

$$\begin{tikzcd} \mathscr{M} \arrow[r, "Q"] \arrow[d, "f", "\wr"'] & \mathscr{O}_X \arrow[d, "g"', "\wr"] \\ \mathscr{M}' \arrow[r, "Q'"] & \mathscr{O}_X \end{tikzcd}$$

and so an isometry is just a similarity with $g = \mathrm{id}$.

A quadratic form $(\mathscr{M}, Q)$ is also equivalently specified by $\mathscr{M}$ and an $\mathscr{O}_X$-module homomorphism $Q : \operatorname{Sym}_2 \mathscr{M} \to \mathscr{O}_X$, or a global section

$$Q \in \operatorname{Hom}(\operatorname{Sym}_2 \mathscr{M}, \mathscr{O}_X) = (\operatorname{Sym}_2 \mathscr{M})^\vee(X) \simeq \operatorname{Sym}^2(\mathscr{M}^\vee)(X).$$

(Here, $\operatorname{Sym}^2 \mathscr{M}$ denotes the second symmetric power of $\mathscr{M}$ and $\operatorname{Sym}_2 \mathscr{M}$ the submodule of symmetric second tensors of $\mathscr{M}$.)

A quadratic form $Q : \mathscr{M} \to \mathscr{O}_X$ with associated bilinear form $T : \mathscr{M} \times \mathscr{M} \to \mathscr{O}_X$ induces a homomorphism of $\mathscr{O}_X$-modules $\mathscr{M} \to \mathscr{M}^\vee$ defined by

$$\mathscr{M}(U) \to \mathscr{M}^\vee(U), \quad y \mapsto (x \mapsto T(x, y)).$$

Following Knus [1991, (I.3.2)], we say $Q : \mathscr{M} \to \mathscr{O}_X$ is *nondegenerate* if the associated map $\mathscr{M} \to \mathscr{M}^\vee$ is injective and *nonsingular* (or *regular*) if the associated map $\mathscr{M} \to \mathscr{M}^\vee$ is an isomorphism; these properties hold for Q if and only if they hold on an affine open cover. On an open set $U = \operatorname{Spec} R$ where $\mathscr{M}(U) = M$ is free of rank n, we define the *discriminant* of Q as $\operatorname{disc}(Q) = \det(T) \in R/R^{\times 2}$, the determinant of the bilinear form T with respect to a basis of $M \simeq R^n$; then Q is nondegenerate if and only if $\operatorname{disc}(Q)$ is a nonzerodivisor and nonsingular if and only if $\operatorname{disc}(Q)$ is a unit in R. When, further, n is odd, we define the *half-discriminant* (see, e.g., Knus [1991, (IV.3.1.3)]) by a universal formula, and we say that Q is *seminondegenerate* (resp. *seminonsingular* or *semiregular*) if the half-discriminant is a nonzerodivisor (resp. a unit), and we extend these notions globally to a quadratic form $Q : \mathscr{M} \to \mathscr{O}_X$ if they hold on an affine open cover.

Now let $\mathscr{L}$ be an invertible $\mathscr{O}_X$-module (i.e., locally free of rank 1). A quadratic form on $\mathscr{L}$, the case of our primary concern, is a quadratic map $Q : \mathscr{L} \to \mathscr{O}_X$, but it is given equivalently by a global section $q \in \operatorname{Sym}^2(\mathscr{L}^\vee)(X) \simeq (\mathscr{L}^\vee)^{\otimes 2}(X)$. Because will use this identification frequently, we make it explicit. The identification is defined locally on X, so suppose $X = \operatorname{Spec} R$ and L is a free module of rank 1 over R. Then $\operatorname{Sym}^2(L^\vee) \simeq (L^\vee)^{\otimes 2} \simeq (L^{\otimes 2})^\vee$. Suppose $q \in \operatorname{Sym}^2(L^\vee)$, so $q : L \otimes L \to R$ is an R-module homomorphism. Let $L = Re$ for some $e \in L$ and define the quadratic map $Q : L \to R$ by $Q(re) = r^2 q(e \otimes e)$; this definition is independent of the choice of e. Conversely, if $Q : L \to R$ is a quadratic map, then again letting $L = Re$ we define the R-module homomorphism $q : L \otimes L \to R$ by $q(e \otimes e) = Q(e)$.

Remark 2.1. One must remember the domain $\mathscr{L}$ in this identification. Indeed, if $i : \mathscr{T}^{\otimes 2} \simeq \mathscr{O}_X$ is an isomorphism of $\mathscr{O}_X$-modules, so that $\mathscr{T} \in \operatorname{Pic}(X)[2]$, then i defines a quadratic form $I : \mathscr{T} \to \mathscr{O}_X$, called a *neutral form*, giving rise to an isomorphism

$$(\mathscr{L}^\vee)^{\otimes 2} \xrightarrow{\sim} ((\mathscr{L} \otimes \mathscr{T})^\vee)^{\otimes 2}.$$

The notions of *(semi)nondegenerate* and *(semi)nonsingular* can be made quite explicit for quadratic forms of rank 1. These conditions are local, so let $Q : L \to R$ be a quadratic form with $L = Re$. Then Q is uniquely specified by the element $Q(e) = a \in R$ and the associated bilinear form is specified by $T(e, e) = 2a$, with $2a = \det(Q) \in R/R^{\times 2}$, because a different choice of basis $e' = ue$ gives $Q(e') = u^2 Q(e) = au^2$ with $u \in R^\times$. We find that Q is nondegenerate if and only if $2a$ is a nonzerodivisor and is regular if $2a \in R^\times$, and Q is seminondegenerate if a is a nonzerodivisor and seminonsingular if a is a unit.

Remark 2.2. The slightly unpleasant term *seminondegenerate* is not standard in the literature. It is common to use the term *nondegenerate* instead, but we do not do this here to avoid potential confusion.

Given two quadratic forms $Q : \mathscr{L} \to \mathscr{O}_X$ and $Q' : \mathscr{L}' \to \mathscr{O}_X$, corresponding to $q \in (\mathscr{L}^\vee)^{\otimes 2}(X)$ and $q' \in (\mathscr{L}'^\vee)^{\otimes 2}(X)$, from the element

$$q \otimes q' \in ((\mathscr{L}^\vee)^{\otimes 2} \otimes (\mathscr{L}'^\vee)^{\otimes 2})(X) \simeq ((\mathscr{L} \otimes \mathscr{L}')^\vee)^{\otimes 2}(X)$$

we define the corresponding tensor product $Q \otimes Q' : \mathscr{L} \otimes \mathscr{L}' \to \mathscr{O}_X$: following the identification above, over $X = \operatorname{Spec} R$, if $L = Re$ and $L' = Re'$ then $L \otimes L' = R(e \otimes e')$ and $(Q \otimes Q')(e \otimes e') = Q(e)Q(e')$. The tensor product gives the set of similarity classes of quadratic forms of rank 1 over X the structure of a commutative monoid.

Remark 2.3. The definition of the tensor product of quadratic forms is more subtle in general for forms of arbitrary rank; here we find the correct notion because we can think of rank 1 quadratic forms as rank 1 symmetric bilinear forms.

Definition 2.4. Let $Q : \mathscr{L} \to \mathscr{O}_X$ be a (rank 1) quadratic form. We say Q is *cancellative* if it is cancellative in the monoidal sense, as $q \in (\mathscr{L}^\vee)^{\otimes 2}(X)$: if $q' \in ((\mathscr{L}')^\vee)^{\otimes 2}$ and $q'' \in ((\mathscr{L}'')^\vee)^{\otimes 2}(X)$ have $q \otimes q'$ similar to $q \otimes q''$, then q' is similar to q''.

We say Q is *locally cancellative* if for all $x \in X$ there exists an affine open neighborhood $U \ni x$ such that $Q|_U$ is cancellative.

Proposition 2.5. *A* (*rank* 1) *quadratic form* $Q : \mathscr{L} \to \mathscr{O}_X$ *is locally cancellative if and only if* Q *is seminondegenerate.*

Moreover, *a locally cancellative rank* 1 *quadratic form* Q *over* X *is cancellative. If* X *is affine*, *then* Q *is cancellative if and only if it is locally cancellative.*

Proof. Both properties are local, so it suffices to check this over a ring $X = \operatorname{Spec} R$ such that the quadratic forms involved are free. To a quadratic form $Q : L = Re \to R$, we have $Q(e) = a \in R$. Similarly, if Q' and Q'' are other rank 1 quadratic forms with $Q(e') = a' \in R$ and $Q(e'') = a'' \in R$, then $(Q \otimes Q')(e \otimes e') = aa'$ and $(Q \otimes Q'')(e \otimes e'') = aa''$. We have $Q' \sim Q''$ if and only if there exists $u \in R^\times$ such that $a' = ua''$.

Thus, if Q is seminondegenerate then a is a nonzerodivisor, so $Q \otimes Q' \sim Q \otimes Q''$ implies $aa' = uaa''$, which implies $a' = ua''$, which implies $Q' \sim Q''$, so Q is locally cancellative. Conversely, if Q is locally cancellative and a is a zero divisor, with $aa' = 0$ and $a' \neq 0$, then taking $Q'(e') = a'$ and $Q''(e'') = 0$ we have $Q \otimes Q' \sim Q \otimes Q''$, so $Q' \sim Q''$ and thus there exists $u \in R^\times$ such that $a' = u(0) = 0$, a contradiction.

Now for the second statement. Let $Q : \mathscr{L} \to \mathscr{O}_X$ be locally cancellative. By the previous paragraph, Q is seminondegenerate. Suppose that $Q \otimes Q' \sim Q \otimes Q''$ for rank 1 quadratic forms Q' and Q''; we will show that $Q' \sim Q''$. Cancelling in $\operatorname{Pic}(X)$, we may assume without loss of generality that $\mathscr{L}' = \mathscr{L}''$. Let $U = \operatorname{Spec} R$ be an affine open subset of X such that $\mathscr{L}|_U = L = Re$ is free, as is $L' = Re' = L'' = Re''$. Let $a = Q(e)$ and, similarly, $a' = Q'(e')$ and $a'' = Q''(e'')$, so as in the previous paragraph we are given (a unique) $u \in R^\times$ such that $aa' = uaa''$. Since Q is locally cancellative, we have $a' = ua''$, which defines a similarity $Q' \sim Q''$. Repeating this on an open cover, the elements u glue to give an element $g \in \mathscr{O}_X(X)^\times$ and so, together with the identity map on $\mathscr{L}' = \mathscr{L}''$, we therefore have a similarity $Q' \sim Q''$.

The converse in the final statement follows immediately by taking $U = X$ if X is affine. $\square$

The following corollary is then immediate.

Corollary 2.6. *The subset of locally cancellative quadratic forms over* X *is a submonoid of the monoid of rank* 1 *quadratic forms over* X.

Remark 2.7. The global notion of cancellative is not as robust as one may like. Kleiman [1979] gives an example of a scheme X and a global section $t \in \mathscr{O}_X(X)$ that is a nonzerodivisor such that it becomes a zerodivisor in an affine open $t|_U \in \mathscr{O}_X(U)$.

The similarity class of a locally cancellative quadratic form is determined by its image ("effective Cartier divisors on a scheme are the same as invertible sheaves with a choice of regular global section" [Stacks 2015, Tag 01X0]), as follows.

Lemma 2.8. *There is a (functorial) isomorphism of commutative monoids*

$$\left\{\begin{array}{c}\text{similarity classes of rank 1,}\\ \text{locally cancellative quadratic forms}\\ Q:\mathscr{L}\to\mathscr{O}_X\\ \text{modulo neutral forms}\end{array}\right\} \xrightarrow{\sim} \left\{\begin{array}{c}\text{locally free ideal sheaves}\\ \mathscr{I}\subseteq\mathscr{O}_X\\ \text{such that } [\mathscr{I}]\in 2\operatorname{Pic}(X)\end{array}\right\},$$

where the similarity class of a quadratic form $Q:\mathscr{L}\to\mathscr{O}_X$ maps to the ideal $\mathscr{I}$ of $\mathscr{O}_X$ generated by the values $Q(\mathscr{L})$.

Ideal sheaves are a monoid under multiplication, so the "monoid" part of Lemma 2.8 says that the tensor product $Q\otimes Q'$ of two quadratic forms Q and Q' maps to the product $\mathscr{I}\mathscr{I}'$ of their associated ideal sheaves $\mathscr{I}$ and $\mathscr{I}'$.

Proof. First, we show the map is well-defined, which we may do locally. Let $Q:L\to R$ be a rank 1, locally cancellative quadratic form; then L is free, so we may write $L=Re$ and then $Q(L)=Q(e)R$. By Proposition 2.5, we know that $Q(e)$ is a nonzerodivisor, and this is independent of the choice of e (up to a unit of R). If $Q':L'\to R$ is similar to Q, then there exist R-linear isomorphisms $f:L\to L'$ and $g:R\to R$ such that $Q'(f(x))=g(Q(x))$ for all $x\in L$. Letting $e'=f(e)$ we have $L'=Re'$. The map g must be of the form $g(a)=ua$ for some $u\in R^\times$, so $Q'(L')=Q'(f(L))=uQ(L)=uQ(e)R=Q(e)R$, and thus the image is a well-defined principal ideal.

Now let $Q:\mathscr{L}\to\mathscr{O}_X$ be a locally cancellative quadratic form of rank 1, corresponding to the global section $q\in(\mathscr{L}^{\otimes 2})^\vee(X)$. We claim that $q:\mathscr{L}^{\otimes 2}\to\mathscr{O}_X$ is an isomorphism onto its image $\mathscr{I}=Q(\mathscr{L})=q(\mathscr{L}^{\otimes 2})$. If $U=\operatorname{Spec} R$ is an affine open set, where $\mathscr{L}|_U=Re$, then $q(L^{\otimes 2})=q(e\otimes e)R$; if, further, U is such that q is cancellative over U, we have that $q(e\otimes e)$ is a nonzerodivisor, so q is injective on U (and $q(L)$ is free). By hypothesis, such affine open sets U cover X, so we have $[\mathscr{I}]=[\mathscr{L}^{\otimes 2}]\in\operatorname{Pic}(X)$.

Next, let $q:\mathscr{L}\to\mathscr{O}_X$ and $q':\mathscr{L}'\to\mathscr{O}_X$ be locally cancellative quadratic forms such that $q(\mathscr{L})=q'(\mathscr{L}')=\mathscr{I}$. Since $[\mathscr{I}]=[\mathscr{L}^{\otimes 2}]=[(\mathscr{L}')^{\otimes 2}]$, tensoring q' by a neutral form we may assume that $f:\mathscr{L}\xrightarrow{\sim}\mathscr{L}'$. Then, on any affine open set $U=\operatorname{Spec} R\subseteq X$, where $\mathscr{L}|_U=Re$ and $\mathscr{L}'|_U=Re'$, we have $q(L)=q(e)R=q'(e')R$. Therefore, there exists $u\in R$ such that $q(e)=uq'(e')$, and $u'\in R$ such that $q'(e')=u'q(e)$. Thus $q(e)(1-uu')=0$. On an affine open set U where

q is cancellative, we have $uu' = 1$, so $u \in R^\times$. Moreover, the element u is unique, since if $q(e) = uq'(e') = vq'(e')$ then $(u - v)q'(e') = 0$ so, since q' is cancellative, we have $u = v$. Repeating this argument on an open cover where both $\mathscr{L}$ and $\mathscr{L}'$ are free, there exists (a unique) $u \in \mathscr{O}_X(X)^\times$ giving rise to an isomorphism $\mathscr{O}_X \xrightarrow{\sim} \mathscr{O}_X$ such that $q'f = uq$, so that q and q' are similar.

Finally, the map is surjective: we are given that there exists an invertible bundle $\mathscr{L}$ such that $\mathscr{L}^{\otimes 2} \simeq \mathscr{I}$. The embedding $\mathscr{L}^{\otimes 2} \simeq \mathscr{I} \hookrightarrow \mathscr{O}_X$ then defines a locally cancellative quadratic form $Q : \mathscr{L} \to \mathscr{O}_X$ with values $Q(\mathscr{L}) = \mathscr{I}$, as can readily be checked locally. □

Remark 2.9. A (locally) cancellative rank 1 quadratic form might not pull back to a (locally) cancellative form under an arbitrary morphism of schemes.

To work with discriminants, we will work modulo 2 and 4 as follows: The multiplication-by-4 map on $\mathscr{O}_X$ gives a closed immersion

$$X_{[4]} = X \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}/4\mathbb{Z} \hookrightarrow X$$

and the pullback $\mathscr{L}_{[4]} = \mathscr{L} \otimes \mathscr{O}_X / 4\mathscr{O}_X$ is an invertible $\mathscr{O}_{X_{[4]}}$-module, equipped with a map ${}_{[4]} : \mathscr{L} \to \mathscr{L}_{[4]}$. We can also, further, work modulo 2, obtaining $\mathscr{L}_{[2]}$.

Let R be a commutative ring. Then squaring gives a well-defined map of sets

$$\operatorname{sq} : R/2R \to R/4R, \quad \operatorname{sq}(t + 2R) = t^2 + 4R. \tag{2.10}$$

The map sq is functorial in R and canonically defined, so we can sheafify: if $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module, there is a unique map

$$\operatorname{sq} : \mathscr{L}_{[2]} \to (\mathscr{L}_{[4]})^{\otimes 2}$$

locally defined by (2.10). Explicitly, for an affine open set $U = \operatorname{Spec} R$ of X, where $\mathscr{L}(U) = L = Re$, we may write

$$\mathscr{L}_{[2]}(U) = (R/2R)e \quad \text{and} \quad \mathscr{L}_{[4]}^{\otimes 2}(U) = (R/4R)(e \otimes e),$$

and

$$\operatorname{sq}((t + 2R)e) = \operatorname{sq}(t + 2R)(e \otimes e)$$

is well-defined (independent of the choice of e).

Definition 2.11. A *discriminant* over X is a pair $(d, \mathscr{L})$ where $\mathscr{L}$ is an invertible $\mathscr{O}_X$-module and $d \in (\mathscr{L}^\vee)^{\otimes 2}(X)$ such that there exists $t \in \mathscr{L}_{[2]}^\vee(X)$ with

$$\operatorname{sq}(t) = d_{[4]} \in (\mathscr{L}_{[4]}^\vee)^{\otimes 2}(X). \tag{2.12}$$

If 2 is invertible in X, then $X_{[4]}$ is the empty scheme and the square condition (2.12) is vacuously satisfied.

Definition 2.13. An *isomorphism* between discriminants $(d, \mathscr{L})$ and $(d', \mathscr{L}')$ is an isomorphism $f : \mathscr{L} \xrightarrow{\sim} \mathscr{L}'$ such that $(f^\vee)^{\otimes 2} : (\mathscr{L}'^\vee)^{\otimes 2} \to (\mathscr{L}^\vee)^{\otimes 2}$ has $(f^\vee)^{\otimes 2}(d') = d$.

Equivalently, an isomorphism between discriminants is an isometry — not a similarity! — between the corresponding quadratic forms.

In what follows, we will often abbreviate d for a discriminant $(d, \mathscr{L})$ and refer to $\mathscr{L}$ as the underlying invertible sheaf.

Let $\operatorname{Disc}(X)$ denote the set of discriminants over X up to isomorphism. For an invertible sheaf $\mathscr{L}$ on X, let $\operatorname{Disc}(X; \mathscr{L}) \subseteq \operatorname{Disc}(X)$ denote the subset of isomorphism classes of discriminants d whose underlying invertible sheaf is (isomorphic to) $\mathscr{L}$. Define $\mathbf{Disc}(X)$ to be the sheaf associated to the presheaf $U \mapsto \operatorname{Disc}(U)$.

Lemma 2.14. $\operatorname{Disc}(X)$ *has the structure of commutative monoid under tensor product, with identity element represented by the class of* $(1, \mathscr{O}_X)$. *Moreover,* $\operatorname{Disc}(X; \mathscr{O}_X)$ *is a submonoid of* $\operatorname{Disc}(X)$ *with absorbing element* $(0, \mathscr{O}_X)$.

Proof. The binary operation of tensor product on quadratic forms restricts to a binary operation on discriminants: if $(d, \mathscr{L})$ and $(d', \mathscr{L}')$ are discriminants, with $t \in \mathscr{L}^\vee_{[2]}(X)$ satisfying $\operatorname{sq}(t) = d_{[4]} \in (\mathscr{L}^\vee_{[4]})^{\otimes 2}(X)$ and similarly for $(d', \mathscr{L}')$, then

$$\begin{aligned}\operatorname{sq}(t \otimes t') = d_{[4]} \otimes d'_{[4]} = (d \otimes d')_{[4]} \in ((\mathscr{L}^\vee_{[4]})^{\otimes 2} \otimes ((\mathscr{L}'_{[4]})^\vee)^{\otimes 2})(X)\\ \simeq ((\mathscr{L} \otimes \mathscr{L}')^\vee_{[4]})^{\otimes 2}(X).\end{aligned}$$

This definition is independent of the choice of a representative discriminant in an isomorphism class, so we obtain a binary operation on $\operatorname{Disc}(X)$. This operation is associative and commutative, and $(1, \mathscr{O}_X)$ is an identity by definition of the tensor product. The subset $\operatorname{Disc}(X; \mathscr{O}_X)$ is closed under tensor product, and $(0, \mathscr{O}_X)$ is visibly an absorbing element. $\square$

Lemma 2.15. *There is a functorial monoid isomorphism*

$$\{d \in \mathscr{O}_X(X) : d \text{ is a square modulo } 4\mathscr{O}_X(X)\}/\mathscr{O}_X(X)^{\times 2} \xrightarrow{\sim} \operatorname{Disc}(X; \mathscr{O}_X). \tag{2.16}$$

Proof. The explicit identification of the monoid homomorphism between elements

$$d \in \mathscr{O}_X(X) \simeq (\mathscr{O}_X^\vee)^{\otimes 2}(X)$$

and discriminants as quadratic forms is explained in the beginning of Section 2, with the discriminant condition passing through on both sides.

Suppose that f is an isomorphism between discriminants d and d'. Let $U = \operatorname{Spec} R \subseteq X$ be an affine open subset. Then $d|_U : R \to R$ is a quadratic map with $d(r) = r^2 d(1)$ for all $r \in R$. The restriction $f|_U : \mathscr{O}_X(U) = R \to R$ is an isomorphism and so is identified with a unique element $u \in R^\times$; thus, in R we have

$d'|_U(f|_U(1)) = d'|_U(u) = u^2 d'|_U(1) = d|_U(1)$; by gluing, there exists a (unique) global section $u \in \mathscr{O}_X(X)^\times$ such that $d = u^2 d'$. $\square$

Example 2.17. If $X = \operatorname{Spec} R$, where R is a PID or local ring, then, by Lemma 2.15, $\operatorname{Disc}(R) = \operatorname{Disc}(R;\, R)$ is canonically identified with

$$\{d \in R : d \text{ is a square in } R/4R\}/R^{\times 2}.$$

So, for $R = \mathbb{Z}$, since $\mathbb{Z}^{\times 2} = \{1\}$ we recover the usual set of discriminants as those integers $d \equiv 0, 1 \pmod 4$.

Given a discriminant $(d, \mathscr{L})$, we can forget the quadratic map d and consider only the isomorphism class of the $\mathscr{O}_X$-module $\mathscr{L}$; this gives a map

$$p : \operatorname{Disc}(X) \to \operatorname{Pic}(X).$$

Proposition 2.18. *The sequence*

$$\operatorname{Disc}(X;\, \mathscr{O}_X) \to \operatorname{Disc}(X) \xrightarrow{p} \operatorname{Pic}(X)$$

of commutative monoids is exact.

Proof. The map $p : \operatorname{Disc}(X) \to \operatorname{Pic}(X)$ is surjective because an invertible module $\mathscr{L}$ has the zero quadratic form $d = 0$, which is a discriminant taking $t = 0$. (One can hardly do better in general, since it may be the case that $\mathscr{L}(X) = \{0\}$ has no nonzero global sections.)

Let $i : \operatorname{Disc}(X;\, \mathscr{O}_X) \hookrightarrow \operatorname{Disc}(X)$ be the inclusion map. We show that $I_i = K_p$. The inclusion $I_i \subseteq K_p$ is easy, so we show the reverse inclusion. Let d and d' be discriminants and suppose $([d], [d']) \in K_p$; then the underlying invertible sheaves of d and d' are isomorphic, so we may assume without loss of generality that $d, d' \in (\mathscr{L}^\vee)^{\otimes 2}(X)$. To show $([d], [d']) \in I_i$, we need to show that there exist $\delta, \delta' \in \operatorname{Disc}(X;\, \mathscr{O}_X)$ such that $\delta \otimes d' = \delta' \otimes d$. For this purpose, we may take $\delta = \delta' = 0$. More generally, we could take any $\delta \in (d : d') = \{\delta \in \mathscr{O}_X(X) : \delta d' \in \langle d \rangle\} \subseteq \mathscr{O}_X(X)$. $\square$

3. Quadratic algebras

In this section, we give a monoid structure on the set of isomorphism classes of quadratic algebras. We begin by discussing the algebras over commutative rings, then work over a general base scheme. For more on quadratic rings and standard involutions, see Knus [1991, Chapter I, §1.3] and Voight [2011, §§1–2].

Let R be a commutative ring. An *R-algebra* is an associative ring B with 1 equipped with an embedding $R \hookrightarrow B$ of rings (mapping $1 \in R$ to $1 \in B$) whose image lies in the center of B; we identify R with this image in B. In particular, B is necessarily faithful as an R-module. A homomorphism of R-algebras is required to preserve 1.

Definition 3.1. A *quadratic* R-algebra (or *quadratic ring* over R) is an R-algebra S that is finite locally free of rank 2 as an R-module.

By *finite locally free* we mean that there is a cover of Spec R by standard open sets $D(f_i)$ with $i \in I$ such that the localization M_{f_i} is a free R_{f_i}-module for all $i \in I$. There are a number of other equivalent formulations of this condition [Stacks 2015, Tag 00NV], including that M is finitely presented and R-flat, that M is finite projective, and that M is finitely presented with the localization $M_{\mathfrak{p}}$ free for all primes $\mathfrak{p} \in \operatorname{Spec}(R)$.

Let S be a quadratic R-algebra. Then S is commutative, and there is a unique *standard involution* on S, an R-linear homomorphism $\sigma : S \to S^{\mathrm{op}} = S$ such that $\sigma(\sigma(x)) = x$ and $x\sigma(x) \in R$ for all $x \in S$ [Voight 2011, Lemma 2.9]. Consequently, we have a linear map $\operatorname{trd} : S \to R$ defined by $\operatorname{trd}(x) = x + \sigma(x)$ and a multiplicative map $\operatorname{nrd} : S \to R$ defined by $\operatorname{nrd}(x) = x\sigma(x) = \sigma(x)x$ with the property that $x^2 - \operatorname{trd}(x)x + \operatorname{nrd}(x) = 0$ for all $x \in S$.

Lemma 3.2. *If S is free as an R-module, then there is a basis* 1, x *for S as an R-module.*

Proof. Let x_1, x_2 be a basis for S; then there exist $a_1, a_2 \in R$ such that $1 = a_1x_1 + a_2x_2$. Let $x_1^2 = b_1x_1 + b_2x_2$ and $x_1x_2 = c_1x_1 + c_2x_2$ with $b_1, b_2, c_1, c_2 \in R$. Then

$$x_1 = x_1 \cdot 1 = a_1x_1^2 + a_2x_1x_2 = (a_1b_1 + a_2c_1)x_1 + (a_1b_2 + a_2c_2)x_2.$$

Thus $a_1b_1 + a_2c_1 = 1$. Let $x = -c_1x_1 + b_1x_2$. Then

$$\det\begin{pmatrix} a_1 & a_2 \\ -c_1 & b_1 \end{pmatrix} = a_1b_1 + a_2c_1 = 1 \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 \\ -c_1 & b_1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 1 \\ x \end{pmatrix}$$

so $1, x$ is a basis for S as an R-module. □

Remark 3.3. Lemma 3.2 does not generally extend to R-algebras of higher rank. On the one hand, if S is finite locally free we always have $S \simeq R \oplus S/R$ as R-modules [Voight 2011, Lemma 1.3]; on the other hand, if S is free this need not imply that S/R is free. (However, S/R is still locally free, so for the purposes of making local arguments you can refine an open cover to one over which S/R is in fact free to then find a basis containing 1.)

If S is free over R with basis $1, x$, then the multiplication table is determined by the multiplication $x^2 = tx - n$; consequently, we have a bijection

$$\left\{\begin{array}{c}\text{free quadratic } R\text{-algebras } S \text{ over } R, \\ \text{equipped with a basis } 1,\, x\end{array}\right\} \xrightarrow{\sim} R^2, \tag{3.4}$$

$$S \longmapsto (\operatorname{trd}(x), \operatorname{nrd}(x)) = (x + \bar{x}, x\bar{x}) = (t, n).$$

A change of basis for a free quadratic R-algebra is of the form $x \mapsto u(x+r)$ with $u \in R^\times$ and $a \in R$, mapping

$$(t,n) \mapsto (u(t+2r), u^2(n+tr+r^2)). \tag{3.5}$$

We have identified $R \subseteq S$ as a subring; the quotient S/R is locally free of rank 1. Therefore, we have a canonical identification

$$S/R \xrightarrow{\sim} \textstyle\bigwedge^2 S, \quad x+R \mapsto 1 \wedge x. \tag{3.6}$$

Lemma 3.7. *Let S be a quadratic R-algebra. Then the map*

$$\begin{aligned} d : \left(\textstyle\bigwedge^2 S\right)^{\otimes 2} &\to R, \\ (x \wedge y) \otimes (z \wedge w) &\mapsto (x\sigma(y) - \sigma(x)y)(z\sigma(w) - \sigma(z)w), \end{aligned} \tag{3.8}$$

is a discriminant.

We have

$$\begin{aligned} d((1 \wedge x)^{\otimes 2}) &= (x - \sigma(x))^2 = (2x - \operatorname{trd}(x))^2 \\ &= 4x^2 - 4x \operatorname{trd}(x) + \operatorname{trd}(x)^2 = \operatorname{trd}(x)^2 - 4\operatorname{nrd}(x) \end{aligned} \tag{3.9}$$

in the lemma, as one might expect. We accordingly call the quadratic map $d = d(S)$ in Lemma 3.7 the *discriminant* of S.

Proof. We define the map

$$t : \textstyle\bigwedge^2(S/2S) \to R/2R, \quad t(1 \wedge x) = \operatorname{trd}(x),$$

via the identification (3.6). The map t is well-defined since $\operatorname{trd}(x+r) = \operatorname{trd}(x) + 2r \equiv \operatorname{trd}(x) \pmod{2R}$. We then verify that

$$\operatorname{sq}(t)((1\wedge x)^{\otimes 2}) = t(1 \wedge x)^2 = \operatorname{trd}(x)^2 \equiv \operatorname{trd}(x)^2 - 4\operatorname{nrd}(x) = d((1\wedge x)^{\otimes 2}) \pmod{4R}$$

by (3.9). $\square$

Recall that a commutative R-algebra B is *separable* if B is projective as a $B \otimes_R B$-module via the map $x \otimes y \mapsto xy$. If $B \simeq R[x]/(f(x))$ with $f(x) \in R[x]$, then B is separable if and only if the ideal generated by $f(x)$ and its derivative $f'(x)$ is the unit ideal.

Lemma 3.10. *A quadratic R-algebra S is separable if and only if its discriminant d is an isomorphism.*

Proof. The map $d : \left(\bigwedge^2 S\right)^{\otimes 2} \to R$ is an isomorphism if and only if it is locally an isomorphism, so we reduce to the case where $S = R[x]/(x^2 - tx + n) = R[x]/(f(x))$. Then, by (3.9), d is an isomorphism if and only if $\operatorname{trd}(x)^2 - 4\operatorname{nrd}(x) \in R^\times$ if and only if S is separable [Knus 1991, Chapter I, (7.3.4)]. $\square$

Corollary 3.11. *If S is a separable quadratic R-algebra then $\bigwedge^2 S \in \mathrm{Pic}(R)[2]$.*

Now let X be a scheme.

Definition 3.12. A *quadratic $\mathscr{O}_X$-algebra* is a sheaf $\mathscr{S}$ of $\mathscr{O}_X$-algebras that is locally free of rank 2 as a sheaf of $\mathscr{O}_X$-modules: there is a basis of open sets U of X such that $\mathscr{S}(U)$ is free of rank 2 as an $\mathscr{O}_X(U)$-module.

Equivalently, a quadratic $\mathscr{O}_X$-algebra is given by a *double cover* $\phi : Y \to X$, a finite locally free morphism of schemes of degree 2: the sheaf $\phi_* \mathscr{O}_Y$ is a sheaf of $\mathscr{O}_X$-algebras that is locally free of rank 2. By uniqueness of the standard involution, we obtain a *standard involution* on $\mathscr{S}$, a standard involution on $\mathscr{S}(U)$ for all open sets U (covering each by affine open sets where $\mathscr{S}$ is free), and, in particular, maps trd and nrd on $\mathscr{S}$.

Analogous to Lemma 3.7, we have the following result.

Proposition 3.13. *Let $\mathscr{S}$ be a quadratic $\mathscr{O}_X$-algebra. Then there exists a unique discriminant $d : \left(\bigwedge^2 \mathscr{S}\right)^{\otimes 2} \to \mathscr{O}_X$ that coincides locally with the one defined by* (3.8).

Proof. Let $\mathscr{L} = \bigwedge^2 \mathscr{S}$. We must exhibit $t \in \mathscr{L}^{\vee}_{[2]}(X)$ such that

$$\mathrm{sq}(t) = d_{[4]} \in (\mathscr{L}^{\vee}_{[4]})^{\otimes 2}(X),$$

where [4] denotes working modulo 4, as in the previous section. We adapt the argument in Lemma 3.7 to a global setting. First working locally, let $U = \operatorname{Spec} R \subseteq X$ be an open set where $\mathscr{S}(U) = S$ and $\mathscr{L}(U) = L = \bigwedge^2 S$. Since $\mathrm{trd}(x+r) = \mathrm{trd}(x) + 2r$ for $r \in R$ and $x \in S$, the map

$$t : \textstyle\bigwedge^2(S/2S) \simeq L/2L \to R/2R, \quad x \wedge y \mapsto \mathrm{trd}(x+y),$$

is well-defined (since $t(x \wedge x) = \mathrm{trd}(2x) = 0$) and R-linear. This map does not depend on any choices, so repeating this on an open cover we obtain an element $t \in \mathscr{L}^{\vee}_{[2]}(X)$.

Now we verify that $\mathrm{sq}(t) = d_{[4]} \in (\mathscr{L}^{\vee}_{[4]})^{\otimes 2}(X)$. We may do so on an open cover, where $\mathscr{S}$ is free, so let $S = R \oplus Rx$ and $\bigwedge^2 S = L = R(1 \wedge x)^{\otimes 2}$. Then

$$d((1 \wedge x)^{\otimes 2}) = \mathrm{trd}(x)^2 - 4\,\mathrm{nrd}(x) \equiv \mathrm{trd}(x)^2 \pmod{4R}.$$

On the other hand, by definition we have

$$\mathrm{sq}(t)((1 \wedge x)^{\otimes 2}) \equiv t(1 \wedge x)^2 = \mathrm{trd}(1+x)^2 = (2 + \mathrm{trd}(x))^2 \equiv \mathrm{trd}(x)^2 \pmod{4R}.$$

The result follows. □

A quadratic $\mathscr{O}_X$-algebra $\mathscr{S}$ is separable if and only if d induces an isomorphism of $\mathscr{O}_X$-modules $\left(\bigwedge^2 \mathscr{S}\right)^{\otimes 2} \xrightarrow{\sim} \mathscr{O}_X$, as this is true on any affine open set.

Let $\mathrm{Quad}(X)$ denote the set of isomorphism classes of quadratic $\mathscr{O}_X$-algebras and, for an invertible $\mathscr{O}_X$-module $\mathscr{L}$, let $\mathrm{Quad}(X; \mathscr{L}) \subseteq \mathrm{Quad}(X)$ be the subset of those algebras $\mathscr{S}$ such that there exists an isomorphism $\bigwedge^2 \mathscr{S} \simeq \mathscr{L}$ of $\mathscr{O}_X$-modules. Similarly, define $\mathbf{Quad}(X)$ to be the sheaf associated to the presheaf $U \mapsto \mathrm{Quad}(U)$.

We now give $\mathrm{Quad}(X)$ the structure of a commutative monoid.

Construction 3.14. *Let $X = \operatorname{Spec} R$ and let $S = R \oplus Rx$ and $T = R \oplus Ry$ be free quadratic R-algebras with choice of basis. Let $x^2 = tx - n$ and $y^2 = sy - m$, so*

$$t = \operatorname{trd}(x), \quad n = \operatorname{nrd}(x), \quad s = \operatorname{trd}(y), \quad m = \operatorname{nrd}(y) \quad \textit{with } t, n, s, m \in R.$$

Then we define the free quadratic R-algebra

$$S * T = R \oplus Rw,$$

where

$$w^2 = (st)w - (mt^2 + ns^2 - 4nm). \tag{3.15}$$

Construction 3.14 has been known for some time, e.g., it is given by Hahn [1994, Exercises 14–20, pp. 42–43]. (See the introduction for further context and references.)

Lemma 3.16. *Construction 3.14 is functorial with respect to the base ring R. The operation $*$ gives the set of free quadratic R-algebras with basis the structure of commutative monoid with identity element $R \times R = R[x]/(x^2 - x)$ and absorbing element $R[x]/(x^2)$.*

Proof. Functoriality is clear, and $S * T = T * S$ for all free quadratic R-algebras S and T by the symmetry of the construction. It is routine to check associativity. To check that $E = R \times R$ is the identity element for $*$ we simply substitute $s = 1$ and $m = 0$ to obtain $S * E = S$; a similar check works for the absorbing element. □

Remark 3.17. Construction 3.14 generalizes the Kummer map, presented in the introduction. Indeed, suppose that R is a PID or local ring and $2 \in R^\times$. Then, by completing the square, any quadratic R-algebra S is of the form $S = R[x]/(x^2 - n) = R[\sqrt{n}]$, where $n = d(S)/4$. So, if $S = R[\sqrt{n}]$ and $T = R[\sqrt{m}]$, then

$$S * T = R[x]/(x^2 - 4nm) \simeq R[\sqrt{nm}].$$

At the same time, Construction 3.14 generalizes Artin–Schreier extensions of fields. Suppose that $R = k$ is a field of characteristic 2. Then every separable extension of k can be written in the form $k[x]/(x^2 - x + n)$, and

$$k[x]/(x^2 - x + n) * k[x]/(x^2 - x + m) = k[x]/(x^2 - x + (m + n)).$$

Since $4 = 0$, the discriminant of every such algebra has class 1 in $R/R^{\times 2}$.

In the above construction, if S and T are separable over R, so that they are (étale) Galois extensions of R [Lenstra 2008] (with the standard involutions σ and τ, respectively, as the nontrivial R-algebra automorphisms), then the algebra $S * T$

is the subalgebra of $S \otimes_R T$ fixed by the product of the involutions $\sigma \otimes \tau$ acting on $S \otimes_R T$ [Small 1972, Proposition 1].

In all cases, a direct calculation shows that (3.15) is satisfied by the element

$$w = x \otimes y + \sigma(x) \otimes \tau(y) \in S \otimes_R T;$$

this will figure in the proof of Theorem A. However, there is no reason why the R-algebra generated by w need be free of rank 2 over R; for example, if R has characteristic 2 with $\sigma(x) = x$ and $\tau(y) = y$, then $w = 0$. Thus, Construction 3.14 can be thought of as a formal way to create a fixed subalgebra of $S \otimes_R T$ under the involution given by the product of standard involutions.

Lemma 3.18. *Construction 3.14 is functorial with respect to isomorphisms: if*

$$\phi : S = R \oplus Rx \xrightarrow{\sim} S' = R \oplus Rx',$$
$$\psi : T = R \oplus Ry \xrightarrow{\sim} T' = R \oplus Ry',$$

are R-algebra isomorphisms of quadratic R-algebras, then there is a canonical isomorphism

$$\phi * \psi : S * T \xrightarrow{\sim} S' * T'.$$

Proof. There exist unique $u, v \in R^\times$ and $r, q \in R$ such that $\phi(x) = ux' + r$ and $\psi(y) = vy' + q$. Because ϕ is an R-algebra homomorphism, both $\phi(x)$ and x satisfy the same unique monic quadratic polynomial, and from

$$(ux' + r)^2 = t(ux' + r) - n$$

we conclude that

$$(x')^2 = u^{-1}(t - 2r)x' - u^{-2}(n - tr - r^2) = t'x - n',$$

so $t = ut' + 2r$ and $n = u^2n' + tr + r^2$. Similarly, we obtain $s = vs' + 2q$ and $m = v^2m' + sq + q^2$. We claim then that the map

$$\phi * \psi : S * T \xrightarrow{\sim} S' * T', \quad (\phi * \psi)(w) = (uv)w' + (qt + rs - 2qr),$$

is an isomorphism; for this we simply verify that

$$((uv)w' + (qt + rs - 2qr))^2 = st((uv)w' + (qt + rs - 2qr)) - (mt^2 + ns^2 - 4nm)$$

and the result follows. $\square$

Lemma 3.19. *Let $\mathscr{S}$ and $\mathscr{T}$ be quadratic $\mathscr{O}_X$-algebras. Then there is a unique quadratic $\mathscr{O}_X$-algebra $\mathscr{S} * \mathscr{T}$ up to $\mathscr{O}_X$-algebra isomorphism with the property that, on any affine open set $U \subseteq X$ such that $S = \mathscr{S}(U)$ and $T = \mathscr{T}(U)$ are free, we have*

$$(\mathscr{S} * \mathscr{T})(U) \simeq S * T,$$

as in Construction 3.14.

Proof. This lemma is a standard application of gluing; we give the argument for completeness. Let $\{U_i = \operatorname{Spec} R_i\}$ be an open affine cover of X on which

$$\mathscr{S}(U_i) = S_i = R_i \oplus R_i x_i \quad \text{and} \quad \mathscr{T}(U_i) = T_i = R_i \oplus R_i y_i$$

are free. We define $(\mathscr{S} * \mathscr{T})(U_i) = S_i * T_i = R_i \oplus R_i w_i$ according to Construction 3.14. We glue these according to the isomorphisms on $\mathscr{S}$ and $\mathscr{T}$ using Lemma 3.18, as follows: We have $U_i \cap U_j = U_j \cap U_i = \bigcup_k U_{ijk}$ covered by open sets $U_{ijk} = \operatorname{Spec} R_{ik} \simeq \operatorname{Spec} R_{jk}$ distinguished in U_i and U_j. Because $\mathscr{S}$ is a sheaf, we have compatible isomorphisms

$$\phi_{ijk} : R_{ik} \oplus R_{ik} x_i = \mathscr{S}(\operatorname{Spec} R_{ik}) \simeq \mathscr{S}(\operatorname{Spec} R_{jk}) = R_{jk} \oplus R_{jk} x_j$$

for each such open set. Similarly, we obtain compatible isomorphisms ψ_{ijk} for $\mathscr{T}$ over the same open cover. By Lemma 3.18, we obtain compatible isomorphisms

$$\phi_{ijk} * \psi_{ijk} : (\mathscr{S} * \mathscr{T})(\operatorname{Spec} R_{ik}) \simeq (\mathscr{S} * \mathscr{T})(\operatorname{Spec} R_{jk})$$

and can thereby glue on X to obtain a quadratic $\mathscr{O}_X$-algebra, unique up to $\mathscr{O}_X$-algebra isomorphism. $\square$

Corollary 3.20. *Construction 3.14 gives* $\operatorname{Quad}(X)$ *the structure of a commutative monoid, functorial in X, with identity element the isomorphism class of $\mathscr{O}_X \times \mathscr{O}_X$.*

Proof. Lemma 3.19 shows that Construction 3.14 extends to X and is well-defined on the set of isomorphism classes $\operatorname{Quad}(X)$ of quadratic $\mathscr{O}_X$-algebras. To check that we obtain a functorial commutative monoid, it is enough to show this when X is affine, and this follows from Lemmas 3.16 and 3.18. $\square$

Lemma 3.21. *If $\mathscr{S}$ is a separable quadratic $\mathscr{O}_X$-algebra, then $\mathscr{S} * \mathscr{S} \simeq \mathscr{O}_X \times \mathscr{O}_X$.*

Proof. By gluing, it is enough to show this on an affine cover. Suppose that $S = R[x]/(x^2 - tx + n)$ has discriminant $d = t^2 - 4n$. Then, by definition, we have

$$S * S = R[w]/(w^2 - t^2 w + 2n(t^2 - 2n));$$

with the substitution $w \leftarrow w - 2n$, we find that $S * S \simeq R[w]/(w^2 - dw)$. Since $d \in R^\times$, the replacement $w \leftarrow wd^{-1}$ yields an isomorphism $S * S \simeq R \times R$. $\square$

Remark 3.22. Given our description of the monoid product in the separable case, it follows that the submonoid of separable quadratic algebras is isomorphic to the group of isomorphism classes of étale quadratic covers $\check{\mathrm{H}}^1_{\text{ét}}(X, \mathbb{Z}/2\mathbb{Z})$, a group killed by 2; for more, see Knus [1991, §III.4].

Lemma 3.23. *If* $\mathscr{S}, \mathscr{T} \in \operatorname{Quad}(X)$ *then*

$$d(\mathscr{S} * \mathscr{T}) = d(\mathscr{S}) d(\mathscr{T}) \in \operatorname{Disc}(X)$$

and

$$\textstyle\bigwedge^2 (\mathscr{S} * \mathscr{T}) \simeq \bigwedge^2 \mathscr{S} \otimes \bigwedge^2 \mathscr{T}.$$

Proof. If $S = \mathscr{S}(U)$ and $T = \mathscr{T}(U)$ are as in Construction 3.14, then

$$\begin{aligned} d(S*T)((1\wedge(x\otimes y))^{\otimes 2}) &= (st)^2 - 4(mt^2+ns^2-4nm) \\ &= (t^2-4n)(s^2-4m) \\ &= d(S)((1\wedge x)^{\otimes 2})d(T)((1\wedge y)^{\otimes 2}). \end{aligned} \tag{3.24}$$

The first statement then follows. For the second, again on affine open sets we have the isomorphism

$$\textstyle\bigwedge^2 S\otimes_R \bigwedge^2 T \to \bigwedge^2(S*T), \quad (1\wedge x)\otimes(1\wedge y)\mapsto 1\wedge w, \tag{3.25}$$

which glues to give the desired isomorphism globally. □

Lemma 3.26. *The discriminant maps*

$$\mathrm{disc}:\mathbf{Quad}(X)\to\mathbf{Disc}(X) \quad \textit{and} \quad \mathrm{disc}:\mathbf{Quad}(X;\mathscr{O}_X)\to\mathbf{Disc}(X;\mathscr{O}_X)$$

are surjective homomorphisms of sheaves of commutative monoids.

Proof. The fact that these maps are homomorphisms of sheaves of monoids follows locally from Lemma 3.23. We show these maps are surjective locally, and for that we may assume $X = \operatorname{Spec} R$ and $L = Re$. We refer to Lemma 2.15 and Example 2.17: given any $d\in R$ such that $d = t^2-4n$ with $t, n\in R$, we have the quadratic ring $R[x]/(x^2-tx+n)$ of discriminant d. □

We are now ready to prove Theorem A.

Theorem 3.27. *Construction 3.14 is the unique system of binary operations*

$$* = *_X : \mathrm{Quad}(X)\times\mathrm{Quad}(X)\to\mathrm{Quad}(X),$$

one for each scheme X, such that:

(i) $\mathrm{Quad}(X)$ *is a commutative monoid under $*$, with identity element the class of $\mathscr{O}_X\times\mathscr{O}_X$.*

(ii) *For each morphism $f: X\to Y$ of schemes, the diagram*

$$\begin{array}{ccc} \mathrm{Quad}(Y)\times\mathrm{Quad}(Y) & \xrightarrow{*_Y} & \mathrm{Quad}(Y) \\ \downarrow & & \downarrow{\scriptstyle f^*} \\ \mathrm{Quad}(X)\times\mathrm{Quad}(X) & \xrightarrow{*_X} & \mathrm{Quad}(X) \end{array}$$

is commutative.

(iii) *If $X = \operatorname{Spec} R$, and S and T are separable quadratic R-algebras with standard involutions σ and τ, then $S*T$ is the fixed subring of $S\otimes_R T$ under $\sigma\otimes\tau$.*

Proof. By (3.4), the universal free quadratic algebra with basis is the algebra

$$S_{\text{univ}} = R_{\text{univ}}[x]/(x^2 - tx + n),$$

where $R_{\text{univ}} = \mathbb{Z}[t, n]$ is the polynomial ring in two variables over $\mathbb{Z}$; in other words, for any commutative ring R and free quadratic R-algebra S with basis, there is a unique map $f : R_{\text{univ}} \to R$ such that $S = f^* S_{\text{univ}} = S_{\text{univ}} \otimes_{f, R_{\text{univ}}} R$. By (ii), then, following this argument on an open affine cover, we see that the monoid structure on $\operatorname{Quad}(\operatorname{Spec} R_{\text{univ}})$ determines the monoid structure for all schemes X.

Dropping subscripts, consider $S = R[x]/(x^2 - tx + n)$ and $T = R[y]/(y^2 - sy + m)$, where $R = \mathbb{Z}[t, n, s, m]$; we show there is a unique way to define $S * T$.

To begin, we claim that $S * T$ is free over R. As R-modules, we can write $S * T = R \oplus Iz$, where $I \subseteq F = \operatorname{Frac}(R)$ is a projective R-submodule of F and the class $[I] \in \operatorname{Pic}(R)$ is well-defined. But $\operatorname{Pic}(\mathbb{Z}[t, n, s, m]) \simeq \operatorname{Pic}(\mathbb{Z}) = \{0\}$ ($\mathbb{Z}$ is *seminormal* [Gilmer and Heitmann 1980]), so $I \simeq R$.

Now let

$$D = (t^2 - 4n)(s^2 - 4m).$$

Then $S[1/D]$ and $T[1/D]$ are separable over $R[1/D]$, with involutions $\sigma(x) = t - x$ and $\tau(y) = s - y$. By (iii), the product $S[1/D] * T[1/D]$ is the subring of $S[1/D] \otimes_{R[1/D]} T[1/D]$ generated by

$$z = x \otimes y + \sigma(x) \otimes \tau(y) = 2(x \otimes y) - s(x \otimes 1) - t(1 \otimes y).$$

Then

$$\begin{aligned}
z^2 &= x^2 \otimes y^2 + 2nm + \sigma(x)^2 \otimes \tau(y)^2 \\
&= (tx - n) \otimes (sy - m) + 2nm + (t\sigma(x) - n) \otimes (s\tau(y) - m) \\
&= ts(x \otimes y + \sigma(x) \otimes \tau(y)) - mt\big((x + \sigma(x)) \otimes 1\big) - ns\big(1 \otimes (y + \tau(y))\big) + 4nm \\
&= (st)z - (mt^2 + ns^2 - 4nm).
\end{aligned}$$

In particular, $S[1/D] * T[1/D] \simeq R[1/D] \oplus R[1/D]z$.

By (ii), $(S * T)[1/D] \simeq S[1/D] * T[1/D]$, and we have $S * T \subseteq (S * T)[1/D]$. Since R is a UFD and $S * T$ is free over R, it is generated as an R-algebra by an element of the form $(az + b)/D^k$ for some $a, b \in R$ and $k \in \mathbb{Z}_{\geq 0}$. But, by (3.24), $d((S * T)[1/D]) = D$, so $d(S * T) = (a/D^k)^2 D \in R$; thus, $a/D^k \in R$. Since $\operatorname{trd}((az + b)/D^k) = (ast + 2b)/D^k \in R$, we conclude that $2b/D^k \in R$; since D is not divisible by 2, by Gauss's lemma we have $b/D^k \in R$, so without loss of generality we may take $b = 0$ and suppose $S * T$ is generated by az for some $a \in R$. Since $R^\times = \{\pm 1\}$ and $(S * T)[1/D]$ is generated by z, we must have $a = D^k$ for some $k \in \mathbb{Z}_{\geq 0}$. Finally, we consider

$$\frac{\mathbb{Z}[x]}{(x^2 - tx + n)} * \frac{\mathbb{Z}[y]}{(y^2 - y)} = \frac{\mathbb{Z}[z]}{(z^2 - D^k tz + D^{2k} n)}$$

over $\mathbb{Z}[t,n]$. The algebra on the right has discriminant $D^{2k}(t^2-4n)$, but, by (i), it must be isomorphic to the algebra on the left of discriminant (t^2-4n), so we must have $D^{2k}=1$, so $k=0$. Therefore $S*T=R\oplus Rz$. ☐

Having given the monoid structure, we conclude this section by proving Theorem B.

Theorem 3.28. *Let X be a scheme. Then the following diagram of commutative monoids is functorial and commutative with exact rows and surjective columns*:

$$\begin{tikzcd} \mathbf{Quad}(X;\mathscr{O}_X) \arrow[r] \arrow[d, "\mathrm{disc}"] & \mathbf{Quad}(X) \arrow[r, "\wedge^2"] \arrow[d, "\mathrm{disc}"] & \mathbf{Pic}(X) \arrow[d, equal] \\ \mathbf{Disc}(X;\mathscr{O}_X) \arrow[r] & \mathbf{Disc}(X) \arrow[r] & \mathrm{Pic}(X) \end{tikzcd}$$

Proof. The exactness of the bottom row follows from Proposition 2.18. The exactness of the top row and commutativity of the diagram follows by the same (trivial) argument. Surjectivity follows from Lemma 3.26. ☐

4. Proof of Theorem C

In this section, we prove Theorem C and conclude with some final discussion.

Let R be a commutative ring and let $R[4]=\{a\in R : 4a=0\}$. Let

$$\wp(R)=\{r+r^2 : r\in R\}$$

and let $\wp(R)[4]=\wp(R)\cap R[4]$. Note that $4(r+r^2)=0$ if and only if $(1+2r)^2=1$, so we have, equivalently,

$$\wp(R)[4]=\{r+r^2 : r\in R \text{ and } (1+2r)^2=1\}.$$

Lemma 4.1. *$\wp(R)[4]$ is a subgroup of $R[4]$ under addition.*

Proof. We have $0=0+0^2\in\wp(R)[4]$. If $n=r+r^2\in\wp(R)[4]$ and $m=s+s^2\in\wp(R)[4]$ then

$$(r+s+2rs)+(r+s+2rs)^2=(r+r^2)+(s+s^2)+4(r+r^2)(s+s^2)=n+m$$

and $4(n+m)=0$, so $n+m\in\wp(R)[4]$. Finally, if $n\in\wp(R)[4]$ then $-n=3n\in\wp(R)[4]$ by the preceding sentence. ☐

We define the *Artin–Schreier group* $\mathrm{AS}(R)$ to be the quotient

$$\mathrm{AS}(R)=\frac{R[4]}{\wp(R)[4]}.$$

Since $2R[4]\subseteq\wp(R)[4]$, the group $\mathrm{AS}(R)$ is an elementary abelian 2-group.

We define a map $i:\mathrm{AS}(R)\to\mathrm{Quad}(R;R)$ sending the class of $n\in\mathrm{AS}(R)$ to the isomorphism class of the algebra $S=R[x]/(x^2-x+n)$.

Proposition 4.2. *The map* $i : \mathrm{AS}(R) \to \mathrm{Quad}(R, R)$ *is a* (*well-defined*) *injective map of commutative monoids.*

Proof. Let $S = i(n) = R[x]/(x^2 - x + n)$ and $T = i(m) = R[y]/(y^2 - y + m)$ with $n, m \in \mathrm{AS}(R)$. Then $S \simeq T$ if and only if $y = u(x + r)$ for some $u \in R^\times$ and $r \in R$, which, by (3.5), holds if and only if $u(1 + 2r) = 1$ and $u^2(n + r + r^2) = m$; these are further equivalent to $1 + 2r \in R^\times$ and

$$n + r + r^2 = m(1 + 2r)^2 = (1 + 4r + 4r^2)m.$$

But $4m = 0$, so $n + r + r^2 = m$ and, since $4n = 0$, we have $4(r + r^2) = 0$. Thus $S \simeq T$ if and only if $(1 + 2r)^2 = 1$ and $n + r + r^2 = m$, as desired. It follows from Construction 3.14 that $S * T = R[w]/(w^2 - w + (n + m))$, since $4nm = 0$, and $i(0) = R[w]/(w^2 - w)$ is the identity, so i is a homomorphism of monoids. □

We now prove Theorem C, and recall $\mathrm{Quad}(R; R)$ is the set of isomorphism classes of free quadratic R-algebras.

Theorem 4.3. *Let R be a commutative ring and let $d \in R$ be a discriminant. Then the fiber* $\mathrm{disc}^{-1}(d)$ *of the map*

$$\mathrm{disc} : \mathrm{Quad}(R; R) \to \mathrm{Disc}(R; R)$$

above d has a unique action of the group $\mathrm{AS}(R)/\mathrm{ann}_R(d)[4]$ *compatible with the inclusion of monoids* $\mathrm{AS}(R) \hookrightarrow \mathrm{Quad}(R; R)$.

Proof. A (free) quadratic R-algebra with basis $1, x$ such that $(x - \sigma(x))^2 = d$ has the form $S = R[x]/(x^2 - tx + n)$ with $t^2 - 4n = d$. Let $m \in R[4]$. Then by Construction 3.14 we have

$$S * (R[y]/(y^2 - y + m)) = (R[y]/(y^2 - y + m)) * S = R[w]/(w^2 - tw + dm + n)$$

since $4m = 0$, so $dm = (t^2 - 4n)m = t^2 m$. Thus we have an action of $R[4]$ on the set of these quadratic R-algebras with basis and a free action of $R[4]/\mathrm{ann}_R(d)[4]$. Two quadratic R-algebras S and S' are in the same orbit if and only if $t' = t$ and $n' = dm + n$ for some $m \in R[4]$ if and only if $t' = t$ and $n' - n \in dR[4]$; therefore, the orbits are indexed noncanonically by the set

$$\{t \in R : t^2 \equiv d \pmod{4R}\} \times R[4]/dR[4].$$

We now descend to isomorphism classes: by Proposition 4.2, the monoid multiplication $*$ is well-defined on isomorphism classes, giving the unique action of $\mathrm{AS}(R)/\mathrm{ann}_R(d)[4]$ on the fiber over d. □

Under favorable hypotheses, the action of $\mathrm{AS}(R)$ is free, so we make the following definition.

Definition 4.4. An element $t \in R$ is *SEC* (*square even cancellative*) if

(i) t is a nonzerodivisor, and

(ii) $r^2, 2r \in tR$ implies $r \in tR$ for all $r \in R$.

A quadratic R-algebra S is *SEC* if $\operatorname{disc}(S)$ is a nonzerodivisor and there a basis $1, x$ for S such that $\operatorname{trd}(x)$ is SEC. A quadratic $\mathscr{O}_X$-algebra $\mathscr{S}$ is *SEC* if $\mathscr{S}$ is SEC on an open affine cover of X.

Proposition 4.5. *Let R be a commutative ring. Then the action of* $\operatorname{AS}(R)$ *on the set of SEC quadratic R-algebras of discriminant d is free.*

Proof. We continue as in the proof of Theorem 4.3. Let $m \in R[4]$ and let S be an SEC quadratic R-algebras with discriminant d. Let 1, x be a basis for $S = R[x]/(x^2 - tx + n)$ such that $t = \operatorname{trd}(x)$ is SEC. To show the action is free, suppose that

$$S * (R[y]/(y^2 - y + m)) = R[w]/(w^2 - tw + dm + n) \simeq S; \tag{4.6}$$

we show that $m \in \wp(R)$. Equation (4.6) holds if and only if there exist $u \in R^\times$ and $r \in R$ such that $t = u(t+2r)$ and $dm+n = u^2(n+tr+r^2)$. Consequently, $u^2 d = d$. Since S is SEC, d is a nonzerodivisor, so $u^2 - 1 = 0$ and thus $t^2 m = dm = tr + r^2$, so $r^2 = (tm - r)t$. We also have $2r = (1-u)t$, so since t is SEC and $r^2, 2r \in tR$, we conclude that $r \in tR$. Let $r = at$ with $a \in R$. Then, substituting, we have $t^2(a^2 + a - m) = r^2 + tr - t^2 m = 0$. Since t is a nonzerodivisor, we conclude that $a^2 + a - m = 0$, so $m \in \wp(R)$, as claimed. $\square$

Acknowledgements

The author would like to thank Asher Auel, Manjul Bhargava, James Borger, and Melanie Wood for helpful suggestions. The author is also indebted to the anonymous referee and Owen Biesel for very detailed and helpful comments and corrections. The author was supported by an NSF CAREER Award (DMS-1151047).

References

[Bergman 2015] G. M. Bergman, *An invitation to general algebra and universal constructions*, 2nd ed., Springer, Cham, 2015. MR 3309721 Zbl 1317.08001

[Bhargava 2002] M. Bhargava, “Gauss composition and generalizations”, pp. 1–8 in *Algorithmic number theory* (Sydney, 2002), edited by C. Fieker and D. R. Kohel, Lecture Notes in Computer Science **2369**, Springer, Berlin, 2002. MR 2041069 Zbl 1058.11030

[Bhargava 2006] M. Bhargava, “Higher composition laws and applications”, pp. 271–294 in *International Congress of Mathematicians* (Madrid, 2006), vol. 3, edited by M. Sanz-Solé et al., European Mathematical Society, Zürich, 2006. MR 2275598 Zbl 1129.11049

[Biesel and Gioia 2015] O. Biesel and A. Gioia, “A new discriminant algebra construction”, preprint, 2015. arXiv 1503.05318v1

[Burris and Sankappanavar 2012] S. N. Burris and H. P. Sankappanavar, "A course in universal algebra: the millennium edition", 2012, available at https://www.math.uwaterloo.ca/~snburris/htdocs/ualg.html.

[Gilmer and Heitmann 1980] R. Gilmer and R. C. Heitmann, "On Pic($R[X]$) for R seminormal", *J. Pure Appl. Algebra* **16**:3 (1980), 251–257. MR 558489 Zbl 0428.13003

[Hahn 1994] A. J. Hahn, *Quadratic algebras, Clifford algebras, and arithmetic Witt groups*, Springer, New York, 1994. MR 1249481 Zbl 0791.11021

[Kanzaki 1973] T. Kanzaki, "On the quadratic extensions and the extended Witt ring of a commutative ring", *Nagoya Math. J.* **49** (1973), 127–141. MR 0332778 Zbl 0238.13018

[Kleiman 1979] S. L. Kleiman, "Misconceptions about K_X", *Enseign. Math.* (2) **25**:3–4 (1979), 203–206. MR 570309 Zbl 0432.14013

[Knus 1991] M.-A. Knus, *Quadratic and Hermitian forms over rings*, Grundlehren der Math. Wissenschaften **294**, Springer, Berlin, 1991. MR 1096299 Zbl 0756.11008

[Knus et al. 1998] M.-A. Knus, A. Merkurjev, M. Rost, and J.-P. Tignol, *The book of involutions*, American Mathematical Society Colloquium Publications **44**, American Mathematical Society, Providence, RI, 1998. MR 1632779 Zbl 0955.16001

[Lenstra 2008] H. W. Lenstra, "Galois theory for schemes", Lecture notes, 3rd ed., Universiteit Leiden, 2008, available at http://websites.math.leidenuniv.nl/algebra/GSchemes.pdf.

[Loos 1996] O. Loos, "Tensor products and discriminants of unital quadratic forms over commutative rings", *Monatsh. Math.* **122**:1 (1996), 45–98. MR 1397923 Zbl 0872.11022

[Loos 2007] O. Loos, "Discriminant algebras of finite rank algebras and quadratic trace modules", *Math. Z.* **257**:3 (2007), 467–523. MR 2328810 Zbl 1136.11031

[McKenzie et al. 1987] R. N. McKenzie, G. F. McNulty, and W. F. Taylor, *Algebras, lattices, varieties*, vol. 1, Wadsworth and Brooks/Cole, Monterey, CA, 1987. MR 883644 Zbl 0611.08001

[Rost 2002] M. Rost, "The discriminant algebra of a cubic algebra", preprint, 2002, available at http://www.math.uni-bielefeld.de/~rost/data/cub-disc.pdf.

[Small 1972] C. Small, "The group of quadratic extensions", *J. Pure Appl. Algebra* **2**:2 (1972), 83–105. MR 0306170 Zbl 0242.13005

[Stacks 2015] P. Belmans, A. J. de Jong, et al., "The Stacks project", electronic reference, Columbia University, 2015, available at http://stacks.math.columbia.edu.

[Voight 2011] J. Voight, "Rings of low rank with a standard involution", *Illinois J. Math.* **55**:3 (2011), 1135–1154. MR 3069299 Zbl 06185698

[Wood 2011] M. M. Wood, "Gauss composition over an arbitrary base", *Adv. Math.* **226**:2 (2011), 1756–1771. MR 2737799 Zbl 1262.11049

Received May 8, 2015. Revised February 12, 2016.

JOHN VOIGHT
DEPARTMENT OF MATHEMATICS
DARTMOUTH COLLEGE
6188 KEMENY HALL
HANOVER, NH 03755
UNITED STATES
jvoight@gmail.com

CONTENTS

Volume 283, no. 1 and no. 2

Guidelines for Authors

Authors may submit articles at msp.org/pjm/about/journal/submissions.html and choose an editor at that time. Exceptionally, a paper may be submitted in hard copy to one of the editors; authors should keep a copy.

By submitting a manuscript you assert that it is original and is not under consideration for publication elsewhere. Instructions on manuscript preparation are provided below. For further information, visit the web address above or write to pacific@math.berkeley.edu or to Pacific Journal of Mathematics, University of California, Los Angeles, CA 90095–1555. Correspondence by email is requested for convenience and speed.

Manuscripts must be in English, French or German. A brief abstract of about 150 words or less in English must be included. The abstract should be self-contained and not make any reference to the bibliography. Also required are keywords and subject classification for the article, and, for each author, postal address, affiliation (if appropriate) and email address if available. A home-page URL is optional.

Authors are encouraged to use LaTeX, but papers in other varieties of TeX, and exceptionally in other formats, are acceptable. At submission time only a PDF file is required; follow the instructions at the web address above. Carefully preserve all relevant files, such as LaTeX sources and individual files for each figure; you will be asked to submit them upon acceptance of the paper.

Bibliographical references should be listed alphabetically at the end of the paper. All references in the bibliography should be cited in the text. Use of BibTeX is preferred but not required. Any bibliographical citation style may be used but tags will be converted to the house format (see a current issue for examples).

Figures, whether prepared electronically or hand-drawn, must be of publication quality. Figures prepared electronically should be submitted in Encapsulated PostScript (EPS) or in a form that can be converted to EPS, such as GnuPlot, Maple or Mathematica. Many drawing tools such as Adobe Illustrator and Aldus FreeHand can produce EPS output. Figures containing bitmaps should be generated at the highest possible resolution. If there is doubt whether a particular figure is in an acceptable format, the authors should check with production by sending an email to pacific@math.berkeley.edu.

Each figure should be captioned and numbered, so that it can float. Small figures occupying no more than three lines of vertical space can be kept in the text ("the curve looks like this:"). It is acceptable to submit a manuscript will all figures at the end, if their placement is specified in the text by means of comments such as "Place Figure 1 here". The same considerations apply to tables, which should be used sparingly.

Forced line breaks or page breaks should not be inserted in the document. There is no point in your trying to optimize line and page breaks in the original manuscript. The manuscript will be reformatted to use the journal's preferred fonts and layout.

Page proofs will be made available to authors (or to the designated corresponding author) at a website in PDF format. Failure to acknowledge the receipt of proofs or to return corrections within the requested deadline may cause publication to be postponed.

PACIFIC JOURNAL OF MATHEMATICS

Volume 283 No. 2 August 2016

0030-8730(201608)283:2;1-V