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# A COMBINATORIAL APPROACH TO VOICULESCU'S BI-FREE PARTIAL TRANSFORMS

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**We present a combinatorial approach to the 2-variable bi-free partial $S$- and $T$-transforms recently discovered by Voiculescu. This approach produces an alternate definition of said transforms using $(l, r)$-cumulants.**

## 1. Introduction

Voiculescu [2014] introduced the notion of bi-free pairs of faces as a means to simultaneously study left and right actions of algebras on reduced free product spaces. Substantial work has been performed since then in order to better understand bi-freeness and its applications [Charlesworth et al. 2015a; 2015b; Skoufranis 2015; Voiculescu 2016; Mastnak and Nica 2015; Gu et al. 2015]. Specifically, the results of [Voiculescu 1986] were generalized to the bi-free setting in [Voiculescu 2016] through the development of a 2-variable bi-free partial $R$-transform using analytic techniques. A combinatorial construction of the bi-free partial $R$-transform was given in [Skoufranis 2015] using results from [Charlesworth et al. 2015b].

Along similar lines, modifying his $S$-transform introduced in [Voiculescu 1987], Voiculescu [2015] associated to a pair $(a, b)$ of operators in a noncommutative probability space a 2-variable bi-free partial $S$-transform, denoted by $S_{a,b}(z, w)$. Using ideas from [Haagerup 1997], he demonstrated that if $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free then

$$S_{a_1a_2,b_1b_2}(z, w) = S_{a_1,b_1}(z, w)S_{a_2,b_2}(z, w). \tag{1}$$

He also constructed a 2-variable bi-free partial $T$-transform $T_{a,b}(z, w)$ to study the convolution product where additive convolution is used for the left variables and multiplicative convolution is used for the right variables. In particular, the defining characteristic of $T_{a,b}(z, w)$ is that if $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free then

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w). \tag{2}$$

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*Keywords:* bi-free probability, partial bi-free $S$-transform, partial bi-free $T$-transform, bi-free convolutions.

The goal of this paper is to provide a combinatorial proof of the results of [Voiculescu 2015]. The paper is structured as follows. Section 2 establishes all preliminary results, background, and notation necessary for the remainder of the paper. A reader would benefit greatly from knowledge of the combinatorial approach to the free $S$-transform from [Nica and Speicher 1997] and knowledge of the combinatorial approach to bi-freeness from [Charlesworth et al. 2015b] (or the summary in [Charlesworth et al. 2015a]). Section 3 provides an equivalent description of $T_{a,b}(z, w)$ using $(l, r)$-cumulants and provides a combinatorial proof of equation (2). Section 4 provides an equivalent description of $S_{a,b}(z, w)$ using $(l, r)$-cumulants and provides a combinatorial proof of equation (1).

An intriguing question arises in taking products of bi-free pairs of operators: is the "correct" multiplication to use on the right pair of algebras the usual one or its opposite? In other words, if $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free pairs of operators, which product should be used, $(a_1a_2, b_1b_2)$ or $(a_1a_2, b_2b_1)$? It is not difficult to see that the resulting distributions can be different; see [Charlesworth et al. 2015a]. Further, by Theorem 5.2.1 of [Charlesworth et al. 2015b] the $(l, r)$-cumulants of $(a_1a_2, b_2b_1)$ can be computed via a convolution product of the $(l, r)$-cumulants of $(a_1, b_1)$ and $(a_2, b_2)$ involving a bi-noncrossing Kreweras complement, just as in the free case. However, the product of Voiculescu's bi-free partial $S$-transforms of $(a_1, b_1)$ and $(a_2, b_2)$ is the bi-free partial $S$-transform of $(a_1a_2, b_1b_2)$. As we will see in Section 4, this is not just a matter of differences in notation and therefore one needs to carefully consider which product to use.

## 2. Background and preliminaries

In this section, we recall the necessary background required for this paper. We refer the reader to the summary in [Charlesworth et al. 2015a, Section 2] for more background on scalar-valued bi-free probability. This section also serves the purpose of setting notation for the remainder of the paper, which we endeavour to make consistent with [Voiculescu 2015]. We treat all series as formal power series, with commuting variables in the multivariate cases.

**2.1.** ***Free transforms.*** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space (that is, a unital algebra $\mathcal{A}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(I) = 1$) and let $a \in \mathcal{A}$. The Cauchy transform of $a$ is

$$G_a(z) := \varphi((zI - a)^{-1}) = \frac{1}{z} \sum_{n\geq 0} \varphi(a^n) z^{-n},$$

and the moment series of $a$ is

$$h_a(z) := \varphi((I - az)^{-1}) = \sum_{n\geq 0} \varphi(a^n) z^n = \frac{1}{z} G_a\left(\frac{1}{z}\right).$$

Recall one defines $K_a(z)$ to be the inverse of $G_a(z)$ in a neighbourhood of 0 so that $G_a(K_a(z)) = z$. Thus $R_a(z) := K_a(z) - \frac{1}{z}$ is the $R$-transform of $a$ and

$$h_a\Big(\frac{1}{K_a(z)}\Big) = K_a(z)G_a(K_a(z)) = zK_a(z). \tag{3}$$

Furthermore, if $\kappa_n(a)$ denotes the $n$-th free cumulant of $a$ and the cumulant series of $a$ is

$$c_a(z) := \sum_{n\geq 1} \kappa_n(a)z^n,$$

then one can verify that

$$1 + c_a(z) = zK_a(z). \tag{4}$$

To define the $S$-transform of $a$, we assume $\varphi(a) \neq 0$ and let $\psi_a(z) := h_a(z) - 1$. Since $\psi_a(0) = 0$ and $\psi'_a(z) = \varphi(a) \neq 0$, $\psi_a(z)$ has a formal power series inverse under composition, denoted $\psi_a^{\langle -1\rangle}(z)$. We define $\mathcal{X}_a(z) := \psi_a^{\langle -1\rangle}(z)$ so that

$$h_a(\mathcal{X}_a(z)) = 1 + \psi_a(\mathcal{X}_a(z)) = 1 + z. \tag{5}$$

The $S$-transform of $a$ is then defined to be

$$S_a(z) := \frac{1+z}{z}\mathcal{X}_a(z). \tag{6}$$

**2.2. *Free multiplicative functions and convolution.*** Let $\mathrm{NC}(n)$ denote the lattice of noncrossing partitions on $\{1, \dots, n\}$ with its usual refinement order, let $0_n$ denote the minimal element of $\mathrm{NC}(n)$, and let $1_n = \{1, 2, \dots, n\}$ denote the maximal element of $\mathrm{NC}(n)$. For $\pi, \sigma \in \mathrm{NC}(n)$ with $\pi \leq \sigma$, the interval between $\pi$ and $\sigma$, denoted $[\pi, \sigma]$, is the set

$$[\pi, \sigma] = \{\rho \in \mathrm{NC}(n) \mid \pi \leq \rho \leq \sigma\}.$$

A procedure is described in [Speicher 1994] which decomposes each interval of noncrossing partitions into a product of full partitions of the form

$$[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$$

where $k_j \geq 0$.

The incidence algebra of noncrossing partitions, denoted $\mathcal{I}(NC)$, is the algebra of all functions

$$f : \bigcup_{n\geq 1} \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$$

such that $f(\pi, \sigma) = 0$ unless $\pi \leq \sigma$, equipped with pointwise addition and a convolution product defined by

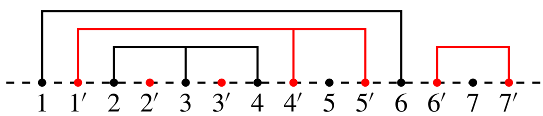
$$(f * g)(\pi, \sigma) := \sum_{\rho\in[\pi,\sigma]} f(\pi, \rho)g(\rho, \sigma).$$

Recall $f \in \mathcal{I}(\mathrm{NC})$ is called multiplicative if whenever $[\pi, \sigma]$ has a canonical decomposition $[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$, then

$$f(\pi, \sigma) = f(0_1, 1_1)^{k_1} f(0_2, 1_2)^{k_2} f(0_3, 1_3)^{k_3} \cdots .$$

Thus the value of a multiplicative function $f$ on any pair of noncrossing partitions is completely determined by the values of $f$ on full noncrossing partition lattices. We will denote the set of all multiplicative functions by $\mathcal{M}$ and the set all multiplicative functions $f$ with $f(0_1, 1_1) = 1$ by $\mathcal{M}_1$.

If $f, g \in \mathcal{M}$, one can verify that $f * g = g * f$. Furthermore, there is a nicer expression for convolution of multiplicative functions. Given a noncrossing partition $\pi \in \mathrm{NC}(n)$, the Kreweras complement of $\pi$, denoted $K(\pi)$, is the noncrossing partition on $\{1, \dots, n\}$ with noncrossing diagram obtained by drawing $\pi$ via the standard noncrossing diagram on $\{1, \dots, n\}$, placing nodes $1', 2', \dots, n'$ with $k'$ directly to the right of $k$, and drawing the largest noncrossing partition on $1', 2', \dots, n'$ that does not intersect $\pi$, which is then $K(\pi)$. The diagram below exhibits that if $\pi = \{\{1, 6\}, \{2, 3, 4\}, \{5\}, \{7\}\}$, then $K(\pi) = \{\{1, 4, 5\}, \{2\}, \{3\}, \{6, 7\}\}$.

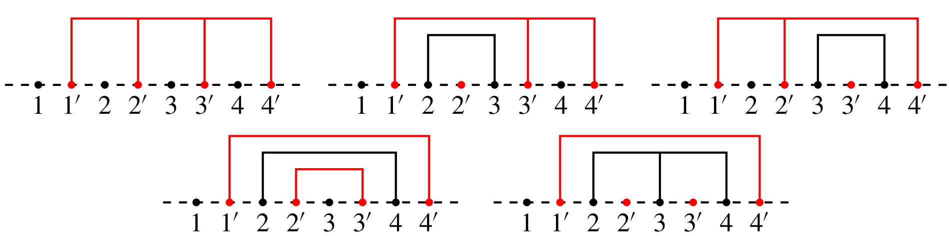



For $f, g \in \mathcal{M}$, convolution may be written as

$$(f * g)(0_n, 1_n) = \sum_{\pi \in \mathrm{NC}(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

Note that [Nica and Speicher 1997] demonstrated that if $a, b \in \mathcal{A}$ are free and if $f$ (respectively $g$) is the multiplicative function associated to the cumulants of $a$ (respectively $b$) defined by $f(0_n, 1_n) = \kappa_n(a)$ (respectively $g(0_n, 1_n) = \kappa_n(b)$), then $\kappa_n(ab) = \kappa_n(ba) = (f * g)(0_n, 1_n)$. Furthermore, for $\pi \in \mathrm{NC}(n)$ with blocks $\{V_k\}_{k=1}^m$, we have $f(0_n, \pi) = \kappa_\pi(a) = \prod_{k=1}^m \kappa_{|V_k|}(a)$.

Another convolution product on $\mathcal{M}_1$ from [loc. cit.] is required. Let $\mathrm{NC}'(n)$ denote all noncrossing partitions $\pi$ on $\{1, \dots, n\}$ such that $\{1\}$ is a block in $\pi$. It is not difficult to construct a natural isomorphism between $\mathrm{NC}'(n)$ and $\mathrm{NC}(n-1)$. The following diagrams illustrate all elements $\mathrm{NC}'(4)$, together with their Kreweras complements.



We desire to make an observation, which may be proved by induction. Given two noncrossing partitions $\pi$ and $\sigma$, let $\pi \vee \sigma$ denote the smallest noncrossing partition larger than both $\pi$ and $\sigma$. Fix $\pi \in \mathrm{NC}'(n)$. If $\sigma$ is the noncrossing partition on $\{1, 1', 2, 2', \dots, n, n'\}$ (with the ordering being the order of listing) with blocks $\{k, k'\}$ for all $k$, then the only noncrossing partition $\tau$ on $\{1', \dots, n'\}$ such that $\pi \cup \tau$ is noncrossing (under the ordering $1, 1', 2, 2', \dots, n, n'$) and $(\pi \cup \tau) \vee \sigma = 1_{2n}$ is $\tau = K(\pi)$.

For $f, g \in \mathcal{M}_1$, the "pinched-convolution" of $f$ and $g$, denoted $f \check{*} g$, is the unique element of $\mathcal{M}_1$ such that

$$(f \check{*} g)[0_n, 1_n] := \sum_{\pi \in \mathrm{NC}'(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

The pinched-convolution product is not commutative on $\mathcal{M}_1$.

Given an element $f \in \mathcal{M}$, define the formal power series

$$\phi_f(z) := \sum_{n \geq 1} f(0_n, 1_n) z^n.$$

In particular, if $f$ is the multiplicative function associated to the cumulants of $a$ defined by $f(0_n, 1_n) = \kappa_n(a)$, then $\phi_f(z) = c_a(z)$. Several formulae involving $\phi_f(z)$ are developed in [Nica and Speicher 1997]. In particular, [loc. cit., Proposition 2.3] demonstrates that if $f, g \in \mathcal{M}_1$ then $\phi_f(\phi_{f \check{*} g}(z)) = \phi_{f * g}(z)$ and thus

$$\phi_{f \check{*} g}\big(\phi_{f * g}^{\langle -1 \rangle}(z)\big) = \phi_f^{\langle -1 \rangle}(z). \tag{7}$$

Furthermore, [loc. cit., Theorem 1.6] demonstrates that

$$z \cdot \phi_{f \check{*} g}^{\langle -1 \rangle}(z) = \phi_f^{\langle -1 \rangle}(z) \phi_g^{\langle -1 \rangle}(z). \tag{8}$$

An immediate consequence of equation (8) is that if $\varphi(a) = 1$ then

$$S_a(z) = \frac{1}{z} c_a^{\langle -1 \rangle}(z). \tag{9}$$

**2.3. *Bi-freeness.*** For a map $\chi : \{1, \dots, n\} \to \{l, r\}$, the set of bi-noncrossing partitions on $\{1, \dots, n\}$ associated to $\chi$ is denoted by $\mathrm{BNC}(\chi)$. Note $\mathrm{BNC}(\chi)$ becomes a lattice where $\pi \leq \sigma$ provided every block of $\pi$ is contained in a single block of $\sigma$. The largest partition in $\mathrm{BNC}(\chi)$, which is $\{\{1, \dots, n\}\}$, is denoted by $1_\chi$. The work in [Charlesworth et al. 2015b] demonstrates that $\mathrm{BNC}(\chi)$ is naturally isomorphic to $\mathrm{NC}(n)$ via a permutation of $\{1, \dots, n\}$ induced by $\chi$.

The $(l, r)$-cumulant associated to a map $\chi : \{1, \dots, n\} \to \{l, r\}$, given elements $\{a_n\}_{n=1}^n \subseteq \mathcal{A}$, was defined in [Mastnak and Nica 2015] and is denoted by $\kappa_\chi(a_1, \dots, a_n)$. Note $\kappa_\chi$ is linear in each entry. The main result of [Charlesworth et al. 2015b] is that if $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free two-faced pairs in $(\mathcal{A}, \varphi)$, $\chi : \{1, \dots, n\} \to \{l, r\}$, $\epsilon : \{1, \dots, n\} \to \{l, r\}$, $c_{l,k} = a_k$, and $c_{r,k} = b_k$, then

$$\kappa_\chi(c_{\chi(1),\epsilon(1)}, \dots, c_{\chi(n),\epsilon(n)}) = 0$$

whenever $\epsilon$ is not constant.

Given a $\pi \in \mathrm{BNC}(\chi)$, each block $B$ of $\pi$ corresponds to the bi-noncrossing partition $1_{\chi_B}$ for some $\chi_B : B \to \{l, r\}$ (where the ordering on $B$ is induced from $\{1, \dots, n\}$). We write

$$\kappa_\pi(a_1, \dots, a_n) = \prod_{B \text{ a block of } \pi} \kappa_{1_{\chi_B}}((a_1, \dots, a_n)|_B),$$

where $(a_1, \dots, a_n)|_B$ denotes the $|B|$-tuple with indices not in $B$ removed. Similarly, if $V$ is a union of blocks of $\pi$, we denote by $\pi|_V$ the bi-noncrossing partition obtained by restricting $\pi$ to $V$.

For $n, m \geq 0$, we often consider the maps $\chi_{n,m} : \{1, \dots, n+m\} \to \{l, r\}$ such that $\chi(k) = l$ if $k \leq n$ and $\chi(k) = r$ if $k > n$. For notational purposes, it is useful to think of $\chi_{n,m}$ as a map on $\{1_l, 2_l, \dots, n_l, 1_r, 2_r, \dots, m_r\}$ under the identification $k \mapsto k_l$ if $k \leq n$ and $k \mapsto (k-n)_r$ if $k > n$. Furthermore, we write $\mathrm{BNC}(n, m)$ for $\mathrm{BNC}(\chi_{n,m})$, $1_{n,m}$ for $1_{\chi_{n,m}}$, and, for $n, m \geq 1$, $\kappa_{n,m}(a_1, \dots, a_n, b_1, \dots, b_m)$ for $\kappa_{1_{n,m}}(a_1, \dots, a_n, b_1, \dots, b_m)$. Finally, for $n, m \geq 1$, we set $\kappa_{n,m}(a, b) = \kappa_{1_{n,m}}(a, b)$, $\kappa_{n,0}(a, b) = \kappa_n(a)$, and $\kappa_{0,m}(a, b) = \kappa_n(b)$.

**2.4.** ***Bi-free transforms.*** Given two elements $a, b \in \mathcal{A}$, we define the ordered joint moment and cumulant series of the pair $(a, b)$ to be

$$H_{a,b}(z, w) := \sum_{n,m\geq 0} \varphi(a^n b^m) z^n w^m \quad \text{and} \quad C_{a,b}(z, w) := \sum_{n,m\geq 0} \kappa_{n,m}(a, b) z^n w^m,$$

respectively (where $\kappa_{0,0}(a, b) = 1$). Note [Skoufranis 2015, Theorem 7.2.4] demonstrates that

$$h_a(z) + h_b(w) = \frac{h_a(z) h_b(w)}{H_{a,b}(z, w)} + C_{a,b}(z h_a(z), w h_b(w)) \tag{10}$$

through combinatorial techniques. It is also demonstrated that (10) is equivalent to Voiculescu's [2016] 2-variable bi-free partial $R$-transform.

For computational purposes, it is helpful to consider the series

$$K_{a,b}(z, w) := \sum_{n,m\geq 1} \kappa_{n,m}(a, b) z^n w^m = C_{a,b}(z, w) - c_a(z) - c_b(w) - 1. \tag{11}$$

Also of use are the series

$$F_{a,b}(z,w) := \varphi((zI-a)^{-1}(1-wb)^{-1}) = \frac{1}{z}\sum_{n,m\geq 0}\varphi(a^n b^m) z^{-n} w^m = \frac{1}{z} H_{a,b}\Big(\frac{1}{z}, w\Big). \tag{12}$$

**2.5.** ***Bi-free cumulants of products.*** Of paramount importance to this paper is the ability to write $(l, r)$-cumulants of products as sums of $(l, r)$-cumulants. We recall a result from [Charlesworth et al. 2015a, Section 9].

Let $m, n \geq 1$ with $m < n$. Fix a sequence of integers

$$k(0) = 0 < k(1) < \cdots < k(m) = n.$$

For $\chi : \{1, \ldots, m\} \to \{l, r\}$, define $\hat{\chi} : \{1, \ldots, n\} \to \{l, r\}$ via

$$\hat{\chi}(q) = \chi(p_q),$$

where $p_q$ is the unique element of $\{1, \ldots, m\}$ such that $k(p_q - 1) < q \leq k(p_q)$.

There exists an embedding of $\mathrm{BNC}(\chi)$ into $\mathrm{BNC}(\hat{\chi})$ via $\pi \mapsto \hat{\pi}$ where the $p$-th node of $\pi$ is replaced by the block $\{k(p-1)+1, \ldots, k(p)\}$. It is easy to see that $\widehat{1_\chi} = 1_{\hat{\chi}}$ and $\widehat{0_\chi}$ is the partition with blocks $\{\{k(p-1)+1, \ldots, k(p)\}\}_{p=1}^{m}$. Given two partitions $\pi, \sigma \in \mathrm{BNC}(\chi)$, let $\pi \vee \sigma$ denote the smallest element of $\mathrm{BNC}(\chi)$ greater than $\pi$ and $\sigma$.

Using ideas from [Nica and Speicher 2006, Theorem 11.12], [Charlesworth et al. 2015a, Theorem 9.1.5] showed that if $\{a_k\}_{k=1}^{n} \subseteq \mathcal{A}$, then

$$\kappa_{1_\chi}(a_1 \cdots a_{k(1)}, a_{k(1)+1} \cdots a_{k(2)}, \ldots, a_{k(m-1)+1} \cdots a_{k(m)}) = \sum_{\substack{\sigma \in \mathrm{BNC}(\hat{\chi}) \\ \sigma \vee \widehat{0_\chi} = 1_{\hat{\chi}}}} \kappa_\sigma(a_1, \ldots, a_n). \tag{13}$$

## 3. Bi-free partial $T$-transform

We begin with Voiculescu's bi-free partial $T$-transform, as the combinatorics are slightly simpler than the bi-free partial $S$-transform.

**Definition 3.1** [Voiculescu 2015, Definition 3.1]. Let $(a, b)$ be a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b) \neq 0$. The 2-variable partial bi-free $T$-transform of $(a, b)$ is the holomorphic function on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$ defined by

$$T_{a,b}(z,w) = \frac{w+1}{w}\Big(1 - \frac{z}{F_{a,b}(K_a(z), \mathcal{X}_b(w))}\Big). \tag{14}$$

It is useful to note the following equivalent definition of the bi-free partial $T$-transform. To simplify the discussion, we show the equality in the case $\varphi(b) = 1$.

This does not hinder the proof of the desired result, namely Theorem 3.5 (see Remark 3.3).

**Proposition 3.2.** *If* $(a, b)$ *is a two-faced pair in a noncommutative probability space* $(\mathcal{A}, \varphi)$ *with* $\varphi(b) = 1$, *then, as formal power series,*

$$T_{a,b}(z, w) = 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big). \tag{15}$$

*Proof.* Using equations (3), (5), and (10), we obtain that

$$\frac{1}{H_{a,b}(1/K_a(z), \mathcal{X}_b(w))} = \frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)} \frac{1}{1+w} C_{a,b}(z, (1+w)\mathcal{X}_b(w)).$$

Therefore, using equations (6), (9), (11), (12), and (14), we obtain that

$$\begin{aligned}
&T_{a,b}(z, w)\\
&\quad = \frac{w+1}{w}\left(1 - \frac{z}{(1/K_a(z))H_{a,b}(1/K_a(z), \mathcal{X}_b(w))}\right)\\
&\quad = \frac{w+1}{w}\left(1 - zK_a(z)\left(\frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)}\frac{1}{1+w} C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\right)\right)\\
&\quad = \frac{1}{w}\big(-zK_a(z) + C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&\quad = \frac{1}{w}\big(-zK_a(z) + 1 + c_a(z) + c_b\big(c_b^{\langle -1\rangle}(w)\big) + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&\quad = \frac{1}{w}\big(w + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big)\\
&\quad = 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).
\end{aligned}$$

□

**Remark 3.3.** One might be concerned that we have restricted to the case $\varphi(b) = 1$. However, if we use (15) as the definition of the bi-free partial $T$-transform and if $\lambda \in \mathbb{C} \setminus \{0\}$, then $T_{a,b}(z, w) = T_{a,\lambda b}(z, w)$. Indeed, $c_{\lambda b}(w) = c_b(\lambda w)$, so we have $c_{\lambda b}^{\langle -1\rangle}(w) = \frac{1}{\lambda} c_b^{\langle -1\rangle}(w)$. Therefore, since $\kappa_{n,m}(a, \lambda b) = \lambda^m \kappa_{n,m}(a, b)$, we see that

$$K_{a,\lambda b}\big(z, c_{\lambda b}^{\langle -1\rangle}(w)\big) = K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).$$

Thus there is no loss in assuming $\varphi(b) = 1$.

**Remark 3.4.** Note that Proposition 3.2 immediately provides the $T$-transform portion of [Voiculescu 2015, Proposition 4.2]. Indeed if $a$ and $b$ are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b) \neq 0$ and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $T_{a,b}(z, w) = 1$.

We desire to prove the following theorem (which was one of two main results of [Voiculescu 2015]) using combinatorial techniques and Proposition 3.2.

**Theorem 3.5** [Voiculescu 2015, Theorem 3.1]. *Let $(a_1, b_1)$ and $(a_2, b_2)$ be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b_1) \neq 0$ and $\varphi(b_2) \neq 0$. Then*

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w)$$

*on* $(\mathbb{C} \setminus \{0\})^2$ *near* $(0, 0)$.

To simplify the proof of the result, we assume that $\varphi(b_1) = \varphi(b_2) = 1$. Note that $\varphi(b_1b_2) = 1$ by freeness of the right algebras in bi-free pairs. Furthermore, let $g_j$ denote the multiplicative function associated to the cumulants of $b_j$ defined by $g_j(0_n, 1_n) = \kappa_n(b_j)$. Recall that if $g$ is the multiplicative function associated to the cumulants of $b_1b_2$, then $g = g_1 * g_2$. Therefore $\phi_g^{\langle -1\rangle}(w) = c_{b_1b_2}^{\langle -1\rangle}(w)$ and $\phi_{g_j}^{\langle -1\rangle}(w) = c_{b_j}^{\langle -1\rangle}(w)$. Note that $g, g_1, g_2 \in \mathcal{M}_1$ by assumption.

By Proposition 3.2 it suffices to show that

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1}{w}\Theta_1(z, w)\Theta_2(z, w), \tag{16}$$

where

$$\Theta_j(z, w) = K_{a_j,b_j}\big(z, \phi_{g_j}^{\langle -1\rangle}(w)\big).$$

Recall

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1} \kappa_{n,m}(a_1 + a_2, b_1b_2)z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m)$ with blocks

$$\{\{k_l\}\}_{k=1}^n \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^m.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1{+}a_2, b_1b_2) = \sum_{\substack{\pi\in\mathrm{BNC}(n,2m)\\ \pi\vee\sigma_{n,m}=1_{n,2m}}} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}).$$

Notice that if $\pi \in \mathrm{BNC}(n, 2m)$ and $\pi \vee \sigma_{n,m} = 1_{n,2m}$, then any block of $\pi$ containing a $k_l$ must contain a $j_r$ for some $j$. Furthermore, if $1 \leq k < j \leq n$ are such that $k_l$ and $j_l$ are in the same block of $\pi$, then $q_l$ must be in the same block as $k_l$ for all $k \leq q \leq j$. Moreover, since $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free, we note that

$$\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) = 0$$

if $\pi$ contains a block containing a $(2k)_r$ and a $(2j-1)_r$ for some $k, j$.

For $n, m \geq 1$, let $\mathrm{BNC}_T(n, m)$ denote all $\pi \in \mathrm{BNC}(n, 2m)$ such that

$$\pi \vee \sigma_{n,m} = 1_{n,2m}$$

and $\pi$ contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for some $k, j$. Consequently, we obtain

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in\mathrm{BNC}_T(n,m)} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_T(n, m)$. Let $\mathrm{BNC}_T(n, m)_e$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing $1_l$ also contains a $(2k)_r$ for some $k$, and let $\mathrm{BNC}_T(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing $1_l$ also contains a $(2k-1)_r$ for some $k$. Note that $\mathrm{BNC}_T(n, m)_e$ and $\mathrm{BNC}_T(n, m)_o$ are disjoint and

$$\mathrm{BNC}_T(n, m)_e \cup \mathrm{BNC}_T(n, m)_o = \mathrm{BNC}_T(n, m)$$

by previous discussions. Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z, w) := \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in\mathrm{BNC}_T(n,m)_d} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m,$$

then

$$K_{a_1+a_2,b_1b_2}(z, w) = \Psi_e(z, w) + \Psi_o(z, w).$$

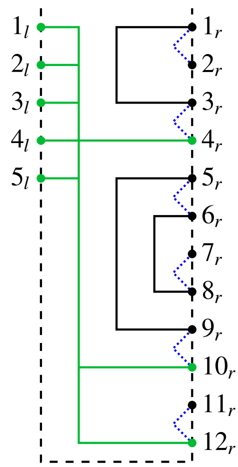
We derive expressions for $\Psi_e(z, w)$ and $\Psi_o(z, w)$ beginning with $\Psi_e(z, w)$.

**Lemma 3.6.** *Under the above notation and assumptions*,

$$\Psi_e(z, w) = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_e(z, w)$ by expanding $\kappa_\pi$ as a product of full $(l, r)$-cumulants and summing over all $\pi$ with the same block containing $1_l$.

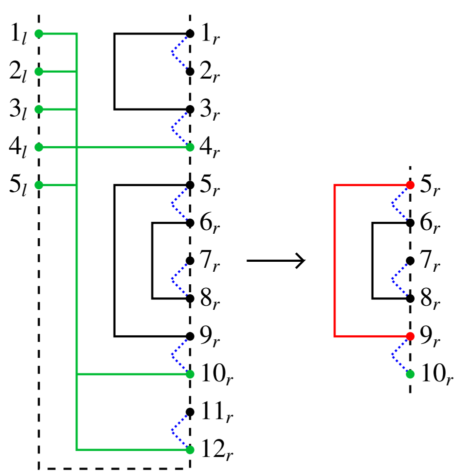
Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_T(n, m)_e$, then the block $V_\pi$ containing $1_l$ must also contain $(2k)_r$ for some $k$, and thus all of $(2m)_r, 1_l, 2_l, \ldots, n_l$ must be in $V_\pi$ in order for $\pi \vee \sigma_{n,m} = 1_{n,2m}$ to be satisfied. Below is an example of such a $\pi$. Two nodes are connected to each other with a solid line if and only if they lie in the same block of $\pi$ and two nodes are connected with a dotted line if and only if they are in the same block of $\sigma_{n,m}$. The condition $\pi \vee \sigma_{n,m} = 1_{n,2m}$ means one may travel from any one node to another using a combination of solid and dotted lines. Note we really should draw all of the left nodes above all of the right notes.




Let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^m$, let $s$ denote the number of elements of $E$ contained in $V_\pi$ (so $s \geq 1$), and let $1 \leq k_1 < k_2 < \cdots < k_s = m$ be such that $(2k_q)_r \in V_\pi$. Note $V_\pi$ divides the right nodes into $s$ disjoint regions. For each $1 \leq q \leq s$, let $j_q = k_q - k_{q-1}$, with $k_0 = 0$, and let $\pi_q$ denote the noncrossing partition obtained by restricting $\pi$ to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q - 1)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, if $\pi'_q$ is obtained from $\pi_q$ by adding the singleton block $\{(2k_q)_r\}$, then $\pi'_q|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_E)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{n,2m}$. The below diagram demonstrates an example of this restriction.



Consequently, by writing $\kappa_\pi$ as a product of cumulants, using linearity of $\kappa_\pi$, and using the fact that $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m$$
$$= \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)_e$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)_e \\ V_\rho = V_\pi}} \kappa_\rho(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m$$
$$= \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n, m \geq 1$ and all possible $V_\pi$ (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

$$\begin{aligned}\Psi_e(z, w) &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} \phi_{g_2 \check{*} g_1}(w) \\ &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n (\phi_{g_2 \check{*} g_1}(w))^s = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)),\end{aligned}$$

as desired. □

In order to discuss $\Psi_o(z, w)$, it is quite helpful to discuss a subcase. For $n, m \geq 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m+1)$ with blocks

$$\{\{k_l\}\}_{k=1}^{n} \cup \{1_r\} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Let $\mathrm{BNC}_T(n, m)'_o$ denote the set of all partitions $\pi \in \mathrm{BNC}(n, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$ and $\pi$ contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for any $k, j$.

**Lemma 3.7.** *Under the above notation and assumptions, if*

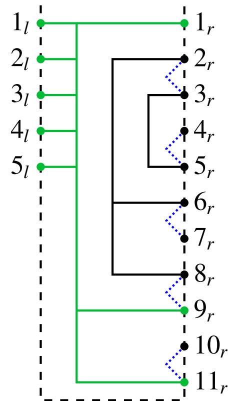
$$\Psi_{o'}(z, w)$$
$$:= \sum_{\substack{n \geq 1 \\ m \geq 0}} \Bigg( \sum_{\pi \in \mathrm{BNC}_T(n,m)'_o} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) \Bigg) z^n w^{m+1},$$

*then*

$$\Psi_{o'}(z, w) = \frac{w}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_{o'}(z, w)$ by expanding $\kappa_\pi$ as a product of full $(l, r)$-cumulants and summing over all $\pi$ with the same block containing $1_l$.

Fix $n \geq 1$ and $m \geq 0$. If $\pi \in \mathrm{BNC}_T(n, m)'_o$, then the block $V_\pi$ containing $1_l$ must contain $1_r$, $(2m+1)_r$, $1_l, 2_l, \ldots, n_l$ in order to have $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Below is an example of such a $\pi$.




Let $E = \{(2k)_r\}_{k=1}^{m}$, let $O = \{(2k-1)_r\}_{k=1}^{m+1}$, let $s$ denote the number of elements of $O$ contained in $V_\pi$ (so $s \geq 1$), and let $1 = k_1 < k_2 < \cdots < k_s = m+1$ be such that $(2k_q - 1)_r \in V_\pi$. Note $V_\pi$ divides the right nodes into $s-1$ disjoint regions. For each $1 \leq q \leq s-1$, let $j_q = k_{q+1} - k_q$ and let $\pi_q$ denote the noncrossing partition obtained by restricting $\pi$ to $\{(2k_q)_r, (2k_q+1)_r, \ldots, (2k_{q+1}-2)_r\}$. Note that $\sum_{q=1}^{s-1} j_q = m$. Furthermore, if $\pi'_q$ is obtained from $\pi_q$ by adding the singleton block $\{(2k_q - 1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ by $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Consequently, by writing $\kappa_\pi$ as a product of cumulants, using linearity of $\kappa_\pi$, and using the fact that $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^{m+1} = \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)'_o$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)'_o \\ V_\rho = V_\pi}} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^{m+1} = \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n \geq 1$, $m \geq 0$, and all possible $V_\pi$ (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

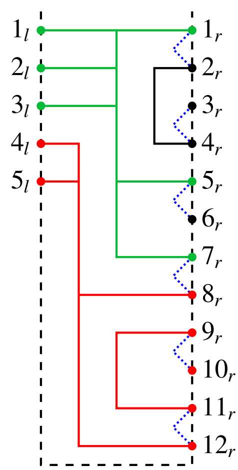
$$\begin{aligned}\Psi_{o'}(z, w) &= \sum_{n,s\geq 1} \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(w) \\ &= \frac{w}{\phi_{g_2 \check{*} g_1}(w)} \sum_{n,s\geq 1} \kappa_{n,s}(a_2, b_2) z^n (\phi_{g_2 \check{*} g_1}(w))^s \\ &= \frac{w}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)). \qquad \square\end{aligned}$$

**Lemma 3.8.** *Under the above notation and assumptions,*

$$\Psi_o(z, w) = \left(1 + \frac{1}{\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w)\right) K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).$$

*Proof.* For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_o(z, w)$ by expanding $\kappa_\pi$ as a product of full $(l, r)$-cumulants and summing over all $\pi$ with the same block containing $1_l$.

Fix $n, m \geq 1$, let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^m$, let $\pi \in \mathrm{BNC}_T(n, m)_o$, let $V_\pi$ denote the block of $\pi$ containing $1_l$, let $t$ (respectively $s$) denote the number of elements of $\{1_l, \ldots, n_l\}$ (respectively $O$) contained in $V_\pi$ (so $t, s \geq 1$). Since $\pi \vee \sigma_{n,m} = 1_{n,2m}$, $V_\pi$ must be of the form $\{k_l\}_{k=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s$ for some $1 = k_1 < k_2 < \cdots < k_s \leq m$. Below is an example of such a $\pi$.




Note that $V_\pi$ divides the right nodes into $s$ disjoint regions, where the bottom region is special as those nodes may connect to left nodes. For each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_s = m + 1$. Note that $\sum_{q=1}^s j_q = m$. For $q \neq s$, let $\pi_q$ denote the noncrossing partition obtained by restricting $\pi$ to

$$\{(2k_q)_r, (2k_q + 1)_r, \ldots, (2k_{q+1} - 2)_r\}.$$

As discussed in Lemma 3.6, if $\pi'_q$ is obtained from $\pi_q$ by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ since $\pi\vee\sigma_{n,m}=1_{n,2m}$.

Let $\pi'_s$ denote the bi-noncrossing partition obtained by restricting $\pi$ to

$$\{k_l\}_{k=t+1}^{n}\cup\{(2k_s)_r,\,(2k_s+1)_r,\,\ldots,\,(2m)_r\}$$

(which is shaded differently in the above diagram). Notice, since $\pi\vee\sigma_{n,m}=1_{2n,2m}$, that it must be the case that $\pi_s\in\mathrm{BNC}_T(n-t,\,j_s-1)'_o$.

By writing $\kappa_\pi$ as a product of cumulants, using linearity of $\kappa_\pi$, and using the fact that $(a_1,b_1)$ and $(a_2,b_2)$ are bi-free (and implicitly using $\varphi(b_1)=1$), we obtain

$$\begin{aligned}\kappa_\pi(&\underbrace{a_1+a_2,\ldots,a_1+a_2}_{n},\underbrace{b_1,b_2,b_1,b_2,\ldots,b_1,b_2}_{b_1\text{ occurs }m\text{ times}})z^nw^m\\ &=\kappa_{t,s}(a_1,b_1)z^t\left(\prod_{q=1}^{s-1}g_1(0_{j_q},\pi'_q)g_2(0_{j_q},K(\pi'_q))w^{j_q}\right)\\ &\qquad\cdot\kappa_{\pi_s}(\underbrace{a_1+a_2,\ldots,a_1+a_2}_{n-t},\underbrace{b_2,b_1,b_2,\ldots,b_1,b_2}_{b_2\text{ occurs }j_s\text{ times}})z^{n-t}w^{j_s}.\end{aligned}$$

Consequently, summing over all $\rho\in\mathrm{BNC}_T(n,m)_o$ with $V_\rho=V_\pi$, we obtain

$$\begin{aligned}\sum_{\substack{\rho\in\mathrm{BNC}_T(n,m)_o\\ V_\rho=V_\pi}}&\kappa_\rho(\underbrace{a_1+a_2,\ldots,a_1+a_2}_{n},\underbrace{b_1,b_2,b_1,b_2,\ldots,b_1,b_2}_{b_1\text{ occurs }m\text{ times}})z^nw^m\\ &=\kappa_{t,s}(a_1,b_1)z^t\left(\prod_{q=1}^{s-1}(g_1\check{*}g_2)(0_{j_q},1_{j_q})w^{j_q}\right)\\ &\quad\cdot\left(\sum_{\sigma\in\mathrm{BNC}_T(n-t,j_s-1)'_o}\kappa_\sigma(\underbrace{a_1+a_2,\ldots,a_1+a_2}_{n-t},\underbrace{b_2,b_1,b_2,\ldots,b_1,b_2}_{b_2\text{ occurs }j_s\text{ times}})z^{n-t}w^{j_s}\right)\end{aligned}$$

as all $\sigma\in\mathrm{BNC}_T(n-t,\,j_s-1)'_o$ occur.

We desire to sum over all $n,m\geq 1$ and all possible $V_\pi$. This produces all possible $t,s\geq 1$ and all $j_q\geq 1$. If we first sum those terms above with $t=n$, we see, using similar arguments to those used above, that

$$\sum_{\sigma\in\mathrm{BNC}_T(0,j_s-1)'_o}\kappa_\sigma(\underbrace{b_2,b_1,b_2,\ldots,b_1,b_2}_{b_2\text{ occurs }j_q\text{ times}})w^{j_s}=(g_1\check{*}g_2)(0_{j_s},1_{j_s})w^{j_s}.$$

Consequently, summing those terms with $t=n$ gives

$$\begin{aligned}\sum_{t,s\geq 1}\kappa_{t,s}(a_1,b_1)z^t\prod_{q=1}^{s}\phi_{g_1\check{*}g_2}(w)&=\sum_{t,s\geq 1}\kappa_{t,s}(a_1,b_1)z^t(\phi_{g_1\check{*}g_2}(w))^s\\ &=K_{a_1,b_1}(z,\phi_{g_1\check{*}g_2}(w)).\end{aligned}$$

Moreover, summing those terms with $t \neq n$ gives

$$\begin{aligned}\sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t \left( \prod_{q=1}^{s-1} \phi_{g_1 \check{*} g_2}(w) \right) \Psi_{o'}(z, w) \\ &= \sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t (\phi_{g_1 \check{*} g_2}(w))^{s-1} \Psi_{o'}(z, w) \\ &= \frac{1}{\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w) K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).\end{aligned}$$

Combining the above two sums completes the proof. ☐

*Proof of Theorem 3.5.* By Lemma 3.6 along with (7), we see that

$$\Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) = K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) = K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).$$

By Lemma 3.7 along with equations (7) and (8)), we see that

$$\begin{aligned}\Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big) &= \frac{\phi_g^{\langle -1\rangle}(w)}{\phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)} K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &= \frac{\frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w)\phi_{g_2}^{\langle -1\rangle}(w)}{\phi_{g_2}^{\langle -1\rangle}(w)} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big) \\ &= \frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

Furthermore, by Lemma 3.8 along with (7), we obtain

$$\begin{aligned}&\Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big) \\ &\quad= \left(1 + \frac{1}{\phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &\quad= \left(1 + \frac{1}{\phi_{g_1}^{\langle -1\rangle}(w)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &\quad= \left(1 + \frac{1}{w} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &\quad= K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) + \frac{1}{w} K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

As

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) + \Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big),$$

we have verified that equation (16) holds and thus the proof is complete. ☐

## 4. Bi-free partial $S$-transform

In this section, we study Voiculescu's bi-free partial $S$-transform through combinatorics. All notation in this section refers to the notation established in this section and not to the notation of Section 3.

**Definition 4.1** [Voiculescu 2015, Definition 2.1]. Let $(a, b)$ be a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$. The 2-variable partial bi-free $S$-transform of $(a, b)$ is the holomorphic function defined on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$ by

$$S_{a,b}(z, w) = \frac{z+1}{z}\frac{w+1}{w}\Big(1 - \frac{1+z+w}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))}\Big). \tag{17}$$

It is useful to note, in the following proposition, an equivalent definition of the bi-free partial $S$-transform. To simplify the discussion, we demonstrate the equality in the case $\varphi(a) = \varphi(b) = 1$. This does not hinder the proof of the desired result, namely Theorem 4.5 (see Remark 4.3).

**Proposition 4.2.** *If $(a, b)$ is a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) = \varphi(b) = 1$, then, as a formal power series,*

$$S_{a,b}(z, w) = 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big). \tag{18}$$

*Proof.* Using equations (5), (6), (9), and (10), we obtain that

$$\frac{1}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))} = \frac{1}{1+z} + \frac{1}{1+w} - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big).$$

Therefore, using equations (11) and (17), we obtain that

$$\begin{aligned} S_{a,b}(z, w) &= \frac{z+1}{z}\frac{w+1}{w}\Big(1 - (1+z+w)\Big(\frac{1}{1+z} + \frac{1}{1+w} \\ &\qquad - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\Big)\Big) \\ &= \frac{1}{zw}\big((1+z)(1+w) - (1+z+w)(2+z+w) \\ &\qquad + (1+z+w) C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big) \\ &= \frac{1}{zw}\Big(zw - (1+z+w)^2 \\ &\qquad + (1+z+w)\big(1+z+w + K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big)\Big) \\ &= 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big). \qquad \square \end{aligned}$$

**Remark 4.3.** Again, one might be concerned that we have restricted to the case $\varphi(a) = \varphi(b) = 1$. Using the same ideas as in Remark 3.3, if we use (18) as the definition of the $S$-transform and if $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, then $S_{a,b}(z, w) = S_{\lambda a, \mu b}(z, w)$. Hence there is no loss in assuming $\varphi(a) = \varphi(b) = 1$.

**Remark 4.4.** Note Proposition 4.2 immediately provides the $S$-transform part of [Voiculescu 2015, Proposition 4.2]. Indeed if $a$ and $b$ are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$, $\varphi(b) \neq 0$, and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $S_{a,b}(z, w) = 1$.

We desire to prove the following, which is one of two main results of [Voiculescu 2015], using combinatorial techniques and Proposition 4.2.

**Theorem 4.5** [Voiculescu 2015, Theorem 2.1]. *Let $(a_1, b_1)$ and $(a_2, b_2)$ be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a_j) \neq 0$ and $\varphi(b_j) \neq 0$. Then*

$$S_{a_1 a_2, b_1 b_2}(z, w) = S_{a_1, b_1}(z, w) S_{a_2, b_2}(z, w)$$

*on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.*

To simplify the proof of this result, we assume that $\varphi(a_j) = \varphi(b_j) = 1$. Note that $\varphi(a_1 a_2) = \varphi(b_1 b_2) = 1$ by freeness of the left algebras and of the right algebras in bi-free pairs. Furthermore, let $f_j$ (respectively $g_j$) denote the multiplicative function associated to the cumulants of $a_j$ (respectively $b_j$) defined by $f_j(0_n, 1_n) = \kappa_n(a_j)$ (respectively $g_j(0_n, 1_n) = \kappa_n(b_j)$). Recall that if $f$ (respectively $g$) is the multiplicative function associated to the cumulants of $a_1 a_2$ (respectively $b_1 b_2$), then $f = f_1 * f_2$ (respectively $g = g_1 * g_2$). Thus

$$\phi_f^{\langle -1 \rangle}(z) = c_{a_1 a_2}^{\langle -1 \rangle}(z), \qquad \phi_g^{\langle -1 \rangle}(w) = c_{b_1 b_2}^{\langle -1 \rangle}(w),$$
$$\phi_{f_j}^{\langle -1 \rangle}(z) = c_{a_j}^{\langle -1 \rangle}(z), \qquad \phi_{g_j}^{\langle -1 \rangle}(w) = c_{b_j}^{\langle -1 \rangle}(w).$$

Note that $f, g, f_j, g_j \in \mathcal{M}_1$ by assumption.

By Proposition 4.2, it suffices to show that

$$K_{a_1 a_2, b_1 b_2}\big(\phi_f^{\langle -1 \rangle}(w), \phi_g^{\langle -1 \rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1+z+w}{zw}\Theta_1(z, w)\Theta_2(z, w) \tag{19}$$

where

$$\Theta_j(z, w) = K_{a_j, b_j}\big(\phi_{f_j}^{\langle -1 \rangle}(w), \phi_{g_j}^{\langle -1 \rangle}(w)\big).$$

Recall

$$K_{a_1 a_2, b_1 b_2}(z, w) = \sum_{n,m \geq 1} \kappa_{n,m}(a_1 a_2, b_1 b_2) z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(2n, 2m)$ with blocks

$$\{\{(2k-1)_l, (2k)_l\}\}_{k=1}^{n} \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^{m}.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1a_2, b_1b_2) = \sum_{\substack{\pi\in\mathrm{BNC}(2n,2m)\\ \pi\vee\sigma_{n,m}=1_{2n,2m}}} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}).$$

Since $(a_1, b_1)$ and $(a_2, b_2)$ are bi-free, we note that

$$\kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) = 0$$

if $\pi$ contains a block containing a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some $k, j$.

For $n, m \geq 1$, let $\mathrm{BNC}_S(n, m)$ be the set of all $\pi \in \mathrm{BNC}(2n, 2m)$ such that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$ and $\pi$ contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some $k, j$. Consequently, we obtain

$$K_{a_1a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1}\left(\sum_{\pi\in\mathrm{BNC}_S(n,m)} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\right) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_S(n, m)$. Notice that if $\pi \in \mathrm{BNC}_S(n, m)$, then $\pi$ must contain a block with both a $k_l$ and a $j_r$ for some $k, j$, so that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$. If

$$V \subseteq \{1_l, \ldots, (2n)_l, 1_r, \ldots, (2m)_r\},$$

we define $\min(V)$ to be the integer $k$ such that either $k_l \in V$ or $k_r \in V$ yet $j_l, j_r \notin V$ for all $j < k$.

Let $\mathrm{BNC}_S(n, m)_e$ denote all $\pi \in \mathrm{BNC}_S(n, m)$ such that $\min(V) \in 2\mathbb{Z}$ for the block $V$ of $\pi$ that has the smallest min-value over all blocks $W$ of $\pi$ such that there exist $k_l, j_r \in W$ for some $k, j$; that is, $V$ is the first block, measured from the top, in the bi-noncrossing diagram of $\pi$ that has both left and right nodes, and these nodes are of even index. Similarly, let $\mathrm{BNC}_S(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that $\min(V) \in 2\mathbb{Z}+1$ for the block $V$ of $\pi$ that has the smallest min-value over all blocks $W$ of $\pi$ such that there exist $k_l, j_r \in W$ for some $k, j$. Note $\mathrm{BNC}_S(n, m)_e$ and $\mathrm{BNC}_S(n, m)_o$ are disjoint and

$$\mathrm{BNC}_S(n, m)_e \cup \mathrm{BNC}_S(n, m)_o = \mathrm{BNC}_S(n, m).$$

Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z, w) := \sum_{n,m \geq 1} \left( \sum_{\pi \in \mathrm{BNC}_S(n,m)_d} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) \right) z^n w^m,$$

then

$$K_{a_1 a_2, b_1 b_2}(z, w) = \Psi_e(z, w) + \Psi_o(z, w).$$

We derive expressions for $\Psi_e(z, w)$ and $\Psi_o(z, w)$ beginning with $\Psi_e(z, w)$. We do not use the same rigour as in Section 3, as most of the arguments are similar.

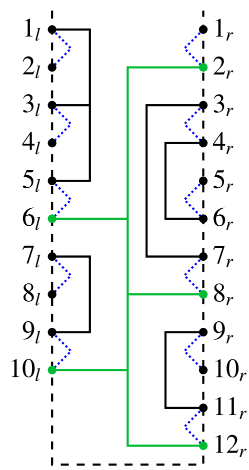
**Lemma 4.6.** *Under the above notation and assumptions,*

$$\Psi_e(z, w) = K_{a_2, b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n, m)_e$, let $V_\pi$ denote the first (and, as it happens, only) block of $\pi$, as measured from the top of $\pi$'s bi-noncrossing diagram, that has both left and right nodes. Since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, there exist $t, s \geq 1$, $1 \leq l_1 < l_2 < \cdots < l_t = n$, and $1 \leq k_1 < k_2 < \cdots < k_s = m$ such that

$$V_\pi = \{(2l_p)_l\}_{p=1}^t \cup \{(2k_q)_r\}_{q=1}^s.$$

Note $V_\pi$ divides the remaining left nodes into $t$ disjoint regions and the remaining right nodes into $s$ disjoint regions. Moreover, each block of $\pi$ can only contain nodes in one such region. Below is an example of such a $\pi$.




Let $E = \{(2k)_l\}_{k=1}^n \cup \{(2k)_r\}_{k=1}^m$ and $O = \{(2k-1)_l\}_{k=1}^n \cup \{(2k-1)_r\}_{k=1}^m$. For each $1 \leq p \leq t$, let $i_p = l_p - l_{p-1}$, where $l_0 = 0$, and let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting $\pi$ to $\{(2l_{p-1}+1)_l, (2l_{p-1}+2)_l, \ldots, (2l_p - 1)_l\}$. Note that $\sum_{p=1}^t i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p)_l\}$, then $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \le q \le s$, let $j_q = k_q - k_{q-1}$, where $k_0 = 0$, and let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting $\pi$ to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q - 1)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q)_r\}$, then $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n, m)_e$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_2, b_2)\left(\prod_{p=1}^{t} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}\right)\left(\prod_{q=1}^{s} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}\right).$$

Finally, if we sum over all possible $n, m \ge 1$ and all possible $V_\pi$ (so, in the above equation, we get all possible $t, s \ge 1$ and all possible $i_p, j_q \ge 1$), we obtain that

$$\begin{aligned}
\Psi_e(z, w) &= \sum_{t,s\ge 1} \kappa_{t,s}(a_2, b_2)\left(\prod_{p=1}^{t} \phi_{f_2 \check{*} f_1}(z)\right)\left(\prod_{q=1}^{s} \phi_{g_2 \check{*} g_1}(z)\right) \\
&= \sum_{t,s\ge 1} \kappa_{t,s}(a_2, b_2)(\phi_{f_2 \check{*} f_1}(z))^t (\phi_{g_2 \check{*} g_1}(w))^s \\
&= K_{a_2,b_2}\big(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)\big). \qquad \square
\end{aligned}$$

In order to discuss $\Psi_o(z, w)$, it is quite helpful to discuss subcases. For $n, m \ge 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(2n+1, 2m+1)$ with blocks

$$\{\{1_l, 1_r\}\} \cup \{\{(2l)_l, (2l+1)_l\}\}_{l=1}^{n} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Define $\mathrm{BNC}_S(n, m)'_o$ to be the set of all $\pi \in \mathrm{BNC}(2n+1, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$ and $\pi$ contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for any $\theta_1, \theta_2 \in \{l, r\}$ and any $k, j$. We wish to divide up $\mathrm{BNC}_S(n, m)'_o$ further. For $\pi \in \mathrm{BNC}_S(n, m)'_o$, let $V_{\pi,l}$ denote the block of $\pi$ containing $1_l$ and $V_{\pi,r}$ the block of $\pi$ containing $1_r$. Then,

$$\begin{aligned}
&\mathrm{BNC}_S(n,m)_{o,0}\\
&\qquad=\{\pi\in\mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l}\text{ has no right nodes and }V_{\pi,r}\text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,r}\\
&\qquad=\{\pi\in\mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l}\text{ has no right nodes but }V_{\pi,r}\text{ has left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,l}\\
&\qquad=\{\pi\in\mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l}\text{ has right nodes but }V_{\pi,r}\text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,lr}=\{\pi\in\mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l}=V_{\pi,r}\}.
\end{aligned}$$

Due to the nature of bi-noncrossing partitions, the above sets are disjoint and have union $\mathrm{BNC}_S(n,m)'_o$.

For $d\in\{0,r,l,lr\}$, define

$$\Psi_{o,d}(z,w):=\sum_{n,m\geq 0}\Bigg(\sum_{\pi\in\mathrm{BNC}_S(n,m)_{o,d}}\kappa_\pi(\underbrace{a_2,a_1,a_2,\ldots,a_1,a_2}_{a_1\text{ occurs }n\text{ times}},\underbrace{b_2,b_1,b_2,\ldots,b_1,b_2}_{b_1\text{ occurs }m\text{ times}})\Bigg)z^{n+1}w^{m+1}.$$

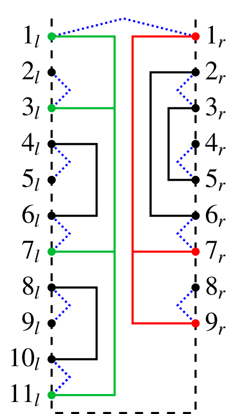
**Lemma 4.7.** *Under the above notation and assumptions,*

$$\Psi_{o,0}(z,w)=zw\cdot\frac{\phi_{f_2}(\phi_{f_2\check{*}f_1}(z))\phi_{g_2}(\phi_{g_2\check{*}g_1}(w))}{\phi_{f_2\check{*}f_1}(z)\phi_{g_2\check{*}g_1}(w)}.$$

*Proof.* Fix $n,m\geq 0$. If $\pi\in\mathrm{BNC}_S(n,m)_{o,0}$, then, since $\pi\vee\sigma'_{n,m}=1_{2n+1,2m+1}$, there exist $t,s\geq 1$, $1=l_1<l_2<\cdots<l_t=n+1$, and $1=k_1<k_2<\cdots<k_s=m+1$ such that

$$V_{\pi,l}=\{(2l_p-1)_l\}_{p=1}^t\quad\text{and}\quad V_{\pi,r}=\{(2k_q-1)_r\}_{q=1}^s.$$

Note that $V_{\pi,l}$ divides the remaining left nodes into $t-1$ disjoint regions and $V_{\pi,r}$ divides the remaining right nodes into $s-1$ disjoint regions. Moreover, each block of $\pi$ can only contain nodes in one such region. Below is an example of such a $\pi$.



If $i_p = l_{p+1} - l_p$ and $j_q = k_{q+1} - k_q$, then

$$\sum_{p=1}^{t-1} i_p = n \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^{n+1} w^{m+1}$$

for $\rho \in \mathrm{BNC}_S(n, m)_{o,0}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$zw \cdot \kappa_t(a_2)\kappa_s(b_2)\left(\prod_{p=1}^{t-1} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}\right)\left(\prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}\right).$$

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}\Psi_e(z, w) &= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2)\left(\prod_{p=1}^{t-1} \phi_{f_2 \check{*} f_1}(z)\right)\left(\prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(z)\right) \\ &= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2)(\phi_{f_2 \check{*} f_1}(z))^{t-1}(\phi_{g_2 \check{*} g_1}(w))^{s-1} \\ &= zw \cdot \frac{\phi_{f_2}(\phi_{f_2 \check{*} f_1}(z))\phi_{g_2}(\phi_{g_2 \check{*} g_1}(w))}{\phi_{f_2 \check{*} f_1}(z)\phi_{g_2 \check{*} g_1}(w)}. \qquad \square\end{aligned}$$

**Lemma 4.8.** *Under the above notation and assumptions*,

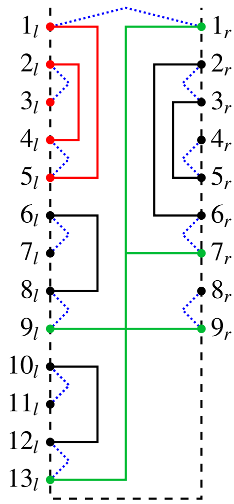
$$\Psi_{o,r}(z, w) = \frac{w \cdot \phi_{f_1 \check{*} f_2}(z)}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* Fix $n, m \geq 0$. Note $\mathrm{BNC}_S(0, m)_{o,r} = \varnothing$ by definition.

If $\pi \in \mathrm{BNC}_S(n, m)_{o,r}$, then, since $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$, there exist $t, s \geq 1$, $1 < l_1 < l_2 < \cdots < l_t = n + 1$, and $1 = k_1 < k_2 < \cdots < k_s = m + 1$ such that

$$V_{\pi,r} = \{(2l_p - 1)_l\}_{p=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s.$$

Note that $V_{\pi,r}$ divides the remaining right nodes into $s - 1$ disjoint regions and the remaining left nodes into $t$ regions. However, the top region is special. If $l_0$ is the largest natural number such that $(2l_0 - 1)_l \in V_{\pi,l}$, then $l_0$ further divides the top region on the left into two regions. Note that each block of $\pi$ can only contain nodes in one such region. The following is an example of such a $\pi$ for which $l_0 = 3$, with one part of the special region $(1_l, \dots, 5_l)$ shaded differently.




Let $i_0 = l_0$, $i_p = l_p - l_{p-1}$ when $p \neq 0$, and $j_q = k_{q+1} - k_q$. Thus

$$\sum_{p=0}^{t} i_p = n+1 \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^{n+1}w^{m+1}$$

for $\rho \in BNC_S(n, m)_{o,r}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$w \cdot \kappa_{t,s}(a_2, b_2)\left(\prod_{p=1}^{t}(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p})z^{i_p}\right) \cdot \left(\prod_{q=1}^{s-1}(g_2 \check{*} g_1)(0_{j_q}, 1_{j_q})w^{j_q}\right)\left((f_1 \check{*} f_2)(0_{i_0}, 1_{i_0})z^{i_0}\right).$$

Note for $p \geq 2$, each $(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p})z^{i_p}$ comes from the $p$-th region from the top on the left, whereas the top region on the left gives $(f_2 \check{*} f_1)(0_{i_1}, 1_{i_1})z^{i_1}$ using the partitions below $(2l_0 - 1)_l$ and gives $(f_1 \check{*} f_2)(0_{i_0}, 1_{i_0})z^{i_0}$ using the partitions above and including $(2l_0 - 1)_l$.

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}\Psi_e(z, w) &= w \sum_{t,s\geq 1} \kappa_{t,s}(a_2, b_2)\left(\prod_{p=1}^{t} \phi_{f_2 \check{*} f_1}(z)\right)\left(\prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(z)\right)\left(\phi_{f_1 \check{*} f_2}(z)\right) \\ &= w \sum_{t,s\geq 1} \kappa_{t,s}(a_2, b_2)(\phi_{f_2 \check{*} f_1}(z))^t (\phi_{g_2 \check{*} g_1}(w))^{s-1}(\phi_{f_1 \check{*} f_2}(z)) \\ &= \frac{w \cdot \phi_{f_1 \check{*} f_2}(z)}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).\end{aligned}$$ □

**Lemma 4.9.** *Under the above notation and assumptions,*

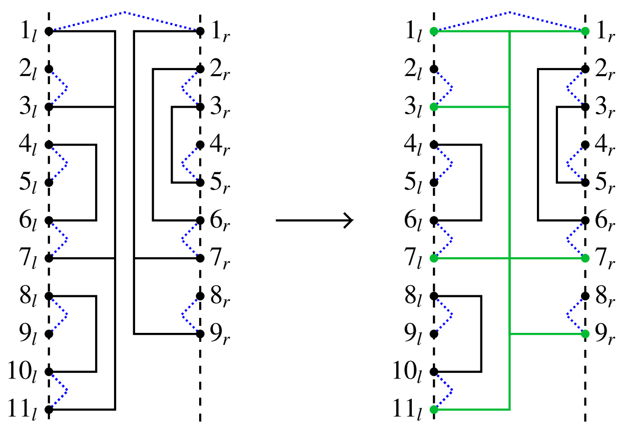
$$\Psi_{o,l}(z, w) = \frac{z \cdot \phi_{g_1 \check{*} g_2}(w)}{\phi_{f_2 \check{*} f_1}(z)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* The proof can be obtained by applying a mirror to Lemma 4.8. □

**Lemma 4.10.** *Under the above notation and assumptions,*

$$\Psi_{o,lr}(z, w) = \frac{zw}{\phi_{f_2 \check{*} f_1}(z)\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

*Proof.* The proof of this result follows from the proof of Lemma 4.7 by replacing each occurrence of $\kappa_t(a_2)\kappa_s(b_2)$ with $\kappa_{t,s}(a_2, b_2)$. Indeed there is a bijection from $\mathrm{BNC}_S(n, m)_{o,0}$ to $\mathrm{BNC}_S(n, m)_{o,lr}$ whereby, given $\pi \in \mathrm{BNC}_S(n, m)_{o,0}$, we produce $\pi' \in \mathrm{BNC}_S(n, m)_{o,lr}$ by joining $V_{\pi,l}$ and $V_{\pi,r}$ into a single block.




□

**Lemma 4.11.** *Under the above notation and assumptions,*

$$\Psi_o(z, w) = \frac{1}{\phi_{f_1 \check{*} f_2}(z)\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w) K_{a_1,b_1}(\phi_{f_1 \check{*} f_2}(z), \phi_{g_1 \check{*} g_2}(w)),$$

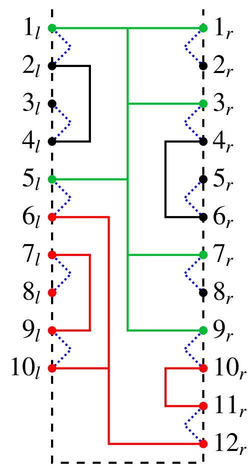
*where*

$$\Psi_{o'}(z, w) = \Psi_{o,0}(z, w) + \Psi_{o,r}(z, w) + \Psi_{o,l}(z, w) + \Psi_{o,lr}(z, w).$$

*Proof.* Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n, m)_o$, let $V_\pi$ denote the first block of $\pi$, as measured from the top of $\pi$'s bi-noncrossing diagram, that has both left and right nodes. Since $\pi \in \mathrm{BNC}_S(n, m)_o$, there exist $t, s \geq 1$, $1 = l_1 < l_2 < \cdots < l_t \leq n$, and $1 = k_1 < k_2 < \cdots < k_s \leq m$ such that

$$V_\pi = \{(2l_p - 1)_l\}_{p=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s.$$

Note $V_\pi$ divides the remaining left nodes and right nodes into $t-1$ disjoint regions on the left, $s-1$ disjoint regions on the right, and one region on the bottom. Moreover, each block of $\pi$ can only contain nodes in one such region. Below is an example of such a $\pi$.




Let

$$E = \{(2k)_l\}_{k=1}^n \cup \{(2k)_r\}_{k=1}^m,$$
$$O = \{(2k-1)_l\}_{k=1}^n \cup \{(2k-1)_r\}_{k=1}^m.$$

For each $1 \leq p \leq t$, let $i_p = l_{p+1} - l_p$, where $l_{t+1} = n+1$, and, for $p \neq t$, let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting $\pi$ to

$$\{(2l_p)_l, (2l_p + 1)_l, \ldots, (2l_{p+1} - 2)_l\}.$$

Note that $\sum_{p=1}^t i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p - 1)_l\}$, then $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_{s+1} = m+1$, and, for $q \neq s$, let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting $\pi$ to

$$\{(2k_q)_r, (2k_q + 1)_r, \ldots, (2k_{q+1} - 2)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Finally, if $\pi'$ is the bi-noncrossing partition obtained by restricting $\pi$ to

$$\{(2l_t)_l, (2l_t+1)_l, \dots, (2n)_l, (2k_s)_r, (2k_s+1)_r, \dots, (2m)_r\}$$

(which is shaded differently in the above diagram), then $\pi' \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n,m)_o$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_1,b_1)\biggl(\prod_{p=1}^{t-1}(f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}\biggr)\biggl(\prod_{q=1}^{s-1}(g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}\biggr)$$
$$\cdot \Biggl(\sum_{\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o} \kappa_\tau(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } i_t-1 \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } j_s-1 \text{ times}}) z^{i_t} w^{j_s}\Biggr).$$

Note that for $p \neq t$, each $(f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}$ comes from the $p$-th region from the top on the left, for $q \neq s$ each $(g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}$ comes from the $q$-th region from the top on the right, and all $\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$ are possible on the bottom.

Finally, if we sum over all possible $n, m \geq 1$ and all possible $V_\pi$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}
\Psi_e(z,w) &= \sum_{t,s \geq 1} \kappa_{t,s}(a_1,b_1)\biggl(\prod_{p=1}^{t-1} \phi_{f_1 \check{*} f_2}(z)\biggr)\biggl(\prod_{q=1}^{s-1} \phi_{g_1 \check{*} g_2}(z)\biggr)\Psi_{o'}(z,w) \\
&= \sum_{t,s \geq 1} \kappa_{t,s}(a_1,b_1)(\phi_{f_1 \check{*} f_2}(z))^{t-1}(\phi_{g_1 \check{*} g_2}(w))^{s-1}\Psi_{o'}(z,w) \\
&= \frac{1}{\phi_{f_1 \check{*} f_2}(z)\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z,w) K_{a_1,b_1}(\phi_{f_1 \check{*} f_2}(z), \phi_{g_1 \check{*} g_2}(w)). \qquad \square
\end{aligned}$$

*Proof of Theorem 4.5.* Using (7) and (8), we see (via Lemmata 4.6–4.10) that

$$\begin{aligned}
\Psi_e\bigl(\phi_{f_1 * f_2}^{\langle -1 \rangle}(z), \phi_{g_1 * g_2}^{\langle -1 \rangle}(w)\bigr) &= K_{a_2,b_2}\bigl(\phi_{f_2}^{\langle -1 \rangle}(z), \phi_{g_2}^{\langle -1 \rangle}(w)\bigr), \\
\Psi_{o,0}\Bigl(\phi_{f_1 * f_2}^{\langle -1 \rangle}(z), \phi_{g_1 * g_2}^{\langle -1 \rangle}(w)\Bigr) &= \phi_{f_1 * f_2}^{\langle -1 \rangle}(z)\phi_{g_1 * g_2}^{\langle -1 \rangle}(w) \cdot \frac{zw}{\phi_{f_2}^{\langle -1 \rangle}(z)\phi_{g_2}^{\langle -1 \rangle}(w)} \\
&= \phi_{f_1}^{\langle -1 \rangle}(z)\phi_{g_1}^{\langle -1 \rangle}(w),
\end{aligned}$$

$$\Psi_{o,r}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z),\phi^{\langle -1\rangle}_{g_1*g_2}(w)\big)=\frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{w}K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z),\phi^{\langle -1\rangle}_{g_2}(w)\big),$$
$$\Psi_{o,l}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z),\phi^{\langle -1\rangle}_{g_1*g_2}(w)\big)=\frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{z}K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z),\phi^{\langle -1\rangle}_{g_2}(w)\big),$$
$$\Psi_{o,lr}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z),\phi^{\langle -1\rangle}_{g_1*g_2}(w)\big)=\frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{zw}K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z),\phi^{\langle -1\rangle}_{g_2}(w)\big).$$

Since

$$\Phi_0\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z),\phi^{\langle -1\rangle}_{g_1*g_2}(w)\big)=\frac{1}{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}\Psi_{o'}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z),\phi^{\langle -1\rangle}_{g_1*g_2}(w)\big)K_{a_1,b_1}\big(\phi^{\langle -1\rangle}_{f_1}(z),\phi^{\langle -1\rangle}_{g_1}(w)\big)$$

by (7) and Lemma 4.11, and since

$$\frac{1}{z}+\frac{1}{w}+\frac{1}{zw}=\frac{1+z+w}{zw}\quad\text{and}\quad K_{a_1a_2,b_1b_2}(z,w)=\Psi_e(z,w)+\Psi_0(z,w),$$

we have verified that (19) holds and thus the proof is complete. □

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