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We present a combinatorial approach to the 2-variable bi-free partial S- and T-transforms recently discovered by Voiculescu. This approach produces an alternate definition of said transforms using (l, r)-cumulants.

1. Introduction

Voiculescu [2014] introduced the notion of bi-free pairs of faces as a means to simultaneously study left and right actions of algebras on reduced free product spaces. Substantial work has been performed since then in order to better understand bi-freeness and its applications [Charlesworth et al. 2015a; 2015b; Skoufranis 2015; Voiculescu 2016; Mastnak and Nica 2015; Gu et al. 2015]. Specifically, the results of [Voiculescu 1986] were generalized to the bi-free setting in [Voiculescu 2016] through the development of a 2-variable bi-free partial R-transform using analytic techniques. A combinatorial construction of the bi-free partial R-transform was given in [Skoufranis 2015] using results from [Charlesworth et al. 2015b].

Along similar lines, modifying his S-transform introduced in [Voiculescu 1987], Voiculescu [2015] associated to a pair (a, b) of operators in a noncommutative probability space a 2-variable bi-free partial S-transform, denoted by $S_{a,b}(z, w)$. Using ideas from [Haagerup 1997], he demonstrated that if (a_1, b_1) and (a_2, b_2) are bi-free then

$$S_{a_1a_2,b_1b_2}(z, w) = S_{a_1,b_1}(z, w)S_{a_2,b_2}(z, w). \tag{1}$$

He also constructed a 2-variable bi-free partial T-transform $T_{a,b}(z, w)$ to study the convolution product where additive convolution is used for the left variables and multiplicative convolution is used for the right variables. In particular, the defining characteristic of $T_{a,b}(z, w)$ is that if (a_1, b_1) and (a_2, b_2) are bi-free then

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w). \tag{2}$$

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The goal of this paper is to provide a combinatorial proof of the results of [Voiculescu 2015]. The paper is structured as follows. Section 2 establishes all preliminary results, background, and notation necessary for the remainder of the paper. A reader would benefit greatly from knowledge of the combinatorial approach to the free S-transform from [Nica and Speicher 1997] and knowledge of the combinatorial approach to bi-freeness from [Charlesworth et al. 2015b] (or the summary in [Charlesworth et al. 2015a]). Section 3 provides an equivalent description of $T_{a,b}(z, w)$ using (l, r)-cumulants and provides a combinatorial proof of equation (2). Section 4 provides an equivalent description of $S_{a,b}(z, w)$ using (l, r)-cumulants and provides a combinatorial proof of equation (1).

An intriguing question arises in taking products of bi-free pairs of operators: is the "correct" multiplication to use on the right pair of algebras the usual one or its opposite? In other words, if (a_1, b_1) and (a_2, b_2) are bi-free pairs of operators, which product should be used, (a_1a_2, b_1b_2) or (a_1a_2, b_2b_1)? It is not difficult to see that the resulting distributions can be different; see [Charlesworth et al. 2015a]. Further, by Theorem 5.2.1 of [Charlesworth et al. 2015b] the (l, r)-cumulants of (a_1a_2, b_2b_1) can be computed via a convolution product of the (l, r)-cumulants of (a_1, b_1) and (a_2, b_2) involving a bi-noncrossing Kreweras complement, just as in the free case. However, the product of Voiculescu's bi-free partial S-transforms of (a_1, b_1) and (a_2, b_2) is the bi-free partial S-transform of (a_1a_2, b_1b_2). As we will see in Section 4, this is not just a matter of differences in notation and therefore one needs to carefully consider which product to use.

2. Background and preliminaries

In this section, we recall the necessary background required for this paper. We refer the reader to the summary in [Charlesworth et al. 2015a, Section 2] for more background on scalar-valued bi-free probability. This section also serves the purpose of setting notation for the remainder of the paper, which we endeavour to make consistent with [Voiculescu 2015]. We treat all series as formal power series, with commuting variables in the multivariate cases.

2.1. ***Free transforms.*** Let $(\mathcal{A}, \varphi)$ be a noncommutative probability space (that is, a unital algebra $\mathcal{A}$ with a linear functional $\varphi : \mathcal{A} \to \mathbb{C}$ such that $\varphi(I) = 1$) and let $a \in \mathcal{A}$. The Cauchy transform of a is

$$G_a(z) := \varphi((zI - a)^{-1}) = \frac{1}{z} \sum_{n\geq 0} \varphi(a^n) z^{-n},$$

and the moment series of a is

$$h_a(z) := \varphi((I - az)^{-1}) = \sum_{n\geq 0} \varphi(a^n) z^n = \frac{1}{z} G_a\left(\frac{1}{z}\right).$$

Recall one defines $K_a(z)$ to be the inverse of $G_a(z)$ in a neighbourhood of 0 so that $G_a(K_a(z)) = z$. Thus $R_a(z) := K_a(z) - \frac{1}{z}$ is the R-transform of a and

$$h_a\Big(\frac{1}{K_a(z)}\Big) = K_a(z)G_a(K_a(z)) = zK_a(z). \tag{3}$$

Furthermore, if $\kappa_n(a)$ denotes the n-th free cumulant of a and the cumulant series of a is

$$c_a(z) := \sum_{n\geq 1} \kappa_n(a)z^n,$$

then one can verify that

$$1 + c_a(z) = zK_a(z). \tag{4}$$

To define the S-transform of a, we assume $\varphi(a) \neq 0$ and let $\psi_a(z) := h_a(z) - 1$. Since $\psi_a(0) = 0$ and $\psi_a'(z) = \varphi(a) \neq 0$, $\psi_a(z)$ has a formal power series inverse under composition, denoted $\psi_a^{\langle -1\rangle}(z)$. We define $\mathcal{X}_a(z) := \psi_a^{\langle -1\rangle}(z)$ so that

$$h_a(\mathcal{X}_a(z)) = 1 + \psi_a(\mathcal{X}_a(z)) = 1 + z. \tag{5}$$

The S-transform of a is then defined to be

$$S_a(z) := \frac{1+z}{z}\mathcal{X}_a(z). \tag{6}$$

2.2. ***Free multiplicative functions and convolution.*** Let $\mathrm{NC}(n)$ denote the lattice of noncrossing partitions on $\{1, \ldots, n\}$ with its usual refinement order, let 0_n denote the minimal element of $\mathrm{NC}(n)$, and let $1_n = \{1, 2, \ldots, n\}$ denote the maximal element of $\mathrm{NC}(n)$. For $\pi, \sigma \in \mathrm{NC}(n)$ with $\pi \leq \sigma$, the interval between π and σ, denoted $[\pi, \sigma]$, is the set

$$[\pi, \sigma] = \{\rho \in \mathrm{NC}(n) \mid \pi \leq \rho \leq \sigma\}.$$

A procedure is described in [Speicher 1994] which decomposes each interval of noncrossing partitions into a product of full partitions of the form

$$[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$$

where $k_j \geq 0$.

The incidence algebra of noncrossing partitions, denoted $\mathcal{I}(NC)$, is the algebra of all functions

$$f : \bigcup_{n\geq 1} \mathrm{NC}(n) \times \mathrm{NC}(n) \to \mathbb{C}$$

such that $f(\pi, \sigma) = 0$ unless $\pi \leq \sigma$, equipped with pointwise addition and a convolution product defined by

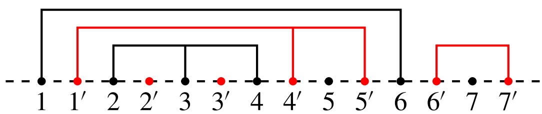
$$(f * g)(\pi, \sigma) := \sum_{\rho\in[\pi,\sigma]} f(\pi, \rho)g(\rho, \sigma).$$

Recall $f \in \mathcal{I}(\mathrm{NC})$ is called multiplicative if whenever $[\pi, \sigma]$ has a canonical decomposition $[0_1, 1_1]^{k_1} \times [0_2, 1_2]^{k_2} \times [0_3, 1_3]^{k_3} \times \cdots$, then

$$f(\pi, \sigma) = f(0_1, 1_1)^{k_1} f(0_2, 1_2)^{k_2} f(0_3, 1_3)^{k_3} \cdots .$$

Thus the value of a multiplicative function f on any pair of noncrossing partitions is completely determined by the values of f on full noncrossing partition lattices. We will denote the set of all multiplicative functions by $\mathcal{M}$ and the set all multiplicative functions f with $f(0_1, 1_1) = 1$ by $\mathcal{M}_1$.

If $f, g \in \mathcal{M}$, one can verify that $f * g = g * f$. Furthermore, there is a nicer expression for convolution of multiplicative functions. Given a noncrossing partition $\pi \in \mathrm{NC}(n)$, the Kreweras complement of π, denoted $K(\pi)$, is the noncrossing partition on $\{1, \ldots, n\}$ with noncrossing diagram obtained by drawing π via the standard noncrossing diagram on $\{1, \ldots, n\}$, placing nodes $1', 2', \ldots, n'$ with k' directly to the right of k, and drawing the largest noncrossing partition on $1', 2', \ldots, n'$ that does not intersect π, which is then $K(\pi)$. The diagram below exhibits that if $\pi = \{\{1, 6\}, \{2, 3, 4\}, \{5\}, \{7\}\}$, then $K(\pi) = \{\{1, 4, 5\}, \{2\}, \{3\}, \{6, 7\}\}$.

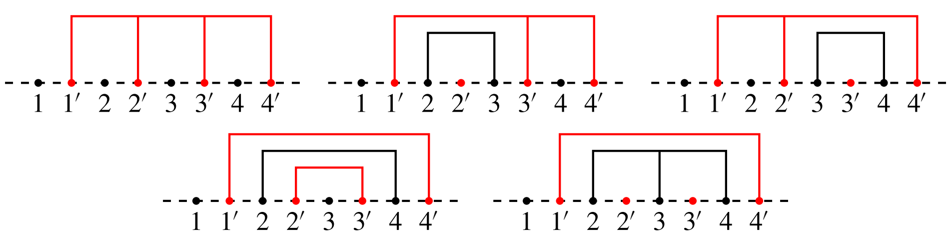


For $f, g \in \mathcal{M}$, convolution may be written as

$$(f * g)(0_n, 1_n) = \sum_{\pi \in \mathrm{NC}(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

Note that [Nica and Speicher 1997] demonstrated that if $a, b \in \mathcal{A}$ are free and if f (respectively g) is the multiplicative function associated to the cumulants of a (respectively b) defined by $f(0_n, 1_n) = \kappa_n(a)$ (respectively $g(0_n, 1_n) = \kappa_n(b)$), then $\kappa_n(ab) = \kappa_n(ba) = (f * g)(0_n, 1_n)$. Furthermore, for $\pi \in \mathrm{NC}(n)$ with blocks $\{V_k\}_{k=1}^m$, we have $f(0_n, \pi) = \kappa_\pi(a) = \prod_{k=1}^m \kappa_{|V_k|}(a)$.

Another convolution product on $\mathcal{M}_1$ from [loc. cit.] is required. Let $\mathrm{NC}'(n)$ denote all noncrossing partitions π on $\{1, \ldots, n\}$ such that $\{1\}$ is a block in π. It is not difficult to construct a natural isomorphism between $\mathrm{NC}'(n)$ and $\mathrm{NC}(n-1)$. The following diagrams illustrate all elements $\mathrm{NC}'(4)$, together with their Kreweras complements.



We desire to make an observation, which may be proved by induction. Given two noncrossing partitions π and σ, let $\pi \vee \sigma$ denote the smallest noncrossing partition larger than both π and σ. Fix $\pi \in \mathrm{NC}'(n)$. If σ is the noncrossing partition on $\{1, 1', 2, 2', \dots, n, n'\}$ (with the ordering being the order of listing) with blocks $\{k, k'\}$ for all k, then the only noncrossing partition τ on $\{1', \dots, n'\}$ such that $\pi \cup \tau$ is noncrossing (under the ordering $1, 1', 2, 2', \dots, n, n'$) and $(\pi \cup \tau) \vee \sigma = 1_{2n}$ is $\tau = K(\pi)$.

For $f, g \in \mathcal{M}_1$, the "pinched-convolution" of f and g, denoted $f \check{*} g$, is the unique element of $\mathcal{M}_1$ such that

$$(f \check{*} g)[0_n, 1_n] := \sum_{\pi \in \mathrm{NC}'(n)} f(0_n, \pi) g(0_n, K(\pi)).$$

The pinched-convolution product is not commutative on $\mathcal{M}_1$.

Given an element $f \in \mathcal{M}$, define the formal power series

$$\phi_f(z) := \sum_{n \geq 1} f(0_n, 1_n) z^n.$$

In particular, if f is the multiplicative function associated to the cumulants of a defined by $f(0_n, 1_n) = \kappa_n(a)$, then $\phi_f(z) = c_a(z)$. Several formulae involving $\phi_f(z)$ are developed in [Nica and Speicher 1997]. In particular, [loc. cit., Proposition 2.3] demonstrates that if $f, g \in \mathcal{M}_1$ then $\phi_f(\phi_{f \check{*} g}(z)) = \phi_{f * g}(z)$ and thus

$$\phi_{f \check{*} g}\big(\phi_{f * g}^{\langle -1 \rangle}(z)\big) = \phi_f^{\langle -1 \rangle}(z). \tag{7}$$

Furthermore, [loc. cit., Theorem 1.6] demonstrates that

$$z \cdot \phi_{f \check{*} g}^{\langle -1 \rangle}(z) = \phi_f^{\langle -1 \rangle}(z) \phi_g^{\langle -1 \rangle}(z). \tag{8}$$

An immediate consequence of equation (8) is that if $\varphi(a) = 1$ then

$$S_a(z) = \frac{1}{z} c_a^{\langle -1 \rangle}(z). \tag{9}$$

2.3. ***Bi-freeness.*** For a map $\chi : \{1, \dots, n\} \to \{l, r\}$, the set of bi-noncrossing partitions on $\{1, \dots, n\}$ associated to χ is denoted by $\mathrm{BNC}(\chi)$. Note $\mathrm{BNC}(\chi)$ becomes a lattice where $\pi \leq \sigma$ provided every block of π is contained in a single block of σ. The largest partition in $\mathrm{BNC}(\chi)$, which is $\{\{1, \dots, n\}\}$, is denoted by 1_χ. The work in [Charlesworth et al. 2015b] demonstrates that $\mathrm{BNC}(\chi)$ is naturally isomorphic to $\mathrm{NC}(n)$ via a permutation of $\{1, \dots, n\}$ induced by χ.

The (l, r)-cumulant associated to a map $\chi : \{1, \dots, n\} \to \{l, r\}$, given elements $\{a_n\}_{n=1}^n \subseteq \mathcal{A}$, was defined in [Mastnak and Nica 2015] and is denoted by $\kappa_\chi(a_1, \dots, a_n)$. Note κ_χ is linear in each entry. The main result of [Charlesworth et al. 2015b] is that if (a_1, b_1) and (a_2, b_2) are bi-free two-faced pairs in $(\mathcal{A}, \varphi)$, $\chi : \{1, \dots, n\} \to \{l, r\}$, $\epsilon : \{1, \dots, n\} \to \{l, r\}$, $c_{l,k} = a_k$, and $c_{r,k} = b_k$, then

$$\kappa_\chi(c_{\chi(1),\epsilon(1)}, \dots, c_{\chi(n),\epsilon(n)}) = 0$$

whenever ϵ is not constant.

Given a $\pi \in \mathrm{BNC}(\chi)$, each block B of π corresponds to the bi-noncrossing partition 1_{χ_B} for some $\chi_B : B \to \{l, r\}$ (where the ordering on B is induced from $\{1, \dots, n\}$). We write

$$\kappa_\pi(a_1, \dots, a_n) = \prod_{B \text{ a block of } \pi} \kappa_{1_{\chi_B}}((a_1, \dots, a_n)|_B),$$

where $(a_1, \dots, a_n)|_B$ denotes the $|B|$-tuple with indices not in B removed. Similarly, if V is a union of blocks of π, we denote by $\pi|_V$ the bi-noncrossing partition obtained by restricting π to V.

For $n, m \geq 0$, we often consider the maps $\chi_{n,m} : \{1, \dots, n+m\} \to \{l, r\}$ such that $\chi(k) = l$ if $k \leq n$ and $\chi(k) = r$ if $k > n$. For notational purposes, it is useful to think of $\chi_{n,m}$ as a map on $\{1_l, 2_l, \dots, n_l, 1_r, 2_r, \dots, m_r\}$ under the identification $k \mapsto k_l$ if $k \leq n$ and $k \mapsto (k-n)_r$ if $k > n$. Furthermore, we write $\mathrm{BNC}(n, m)$ for $\mathrm{BNC}(\chi_{n,m})$, $1_{n,m}$ for $1_{\chi_{n,m}}$, and, for $n, m \geq 1$, $\kappa_{n,m}(a_1, \dots, a_n, b_1, \dots, b_m)$ for $\kappa_{1_{n,m}}(a_1, \dots, a_n, b_1, \dots, b_m)$. Finally, for $n, m \geq 1$, we set $\kappa_{n,m}(a, b) = \kappa_{1_{n,m}}(a, b)$, $\kappa_{n,0}(a, b) = \kappa_n(a)$, and $\kappa_{0,m}(a, b) = \kappa_n(b)$.

2.4. ***Bi-free transforms.*** Given two elements $a, b \in \mathcal{A}$, we define the ordered joint moment and cumulant series of the pair (a, b) to be

$$H_{a,b}(z, w) := \sum_{n,m \geq 0} \varphi(a^n b^m) z^n w^m \quad \text{and} \quad C_{a,b}(z, w) := \sum_{n,m \geq 0} \kappa_{n,m}(a, b) z^n w^m,$$

respectively (where $\kappa_{0,0}(a, b) = 1$). Note [Skoufranis 2015, Theorem 7.2.4] demonstrates that

$$h_a(z) + h_b(w) = \frac{h_a(z) h_b(w)}{H_{a,b}(z, w)} + C_{a,b}(z h_a(z), w h_b(w)) \tag{10}$$

through combinatorial techniques. It is also demonstrated that (10) is equivalent to Voiculescu's [2016] 2-variable bi-free partial R-transform.

For computational purposes, it is helpful to consider the series

$$K_{a,b}(z, w) := \sum_{n,m \geq 1} \kappa_{n,m}(a, b) z^n w^m = C_{a,b}(z, w) - c_a(z) - c_b(w) - 1. \tag{11}$$

Also of use are the series

$$F_{a,b}(z,w) := \varphi((zI-a)^{-1}(1-wb)^{-1}) = \frac{1}{z}\sum_{n,m\geq 0}\varphi(a^n b^m)z^{-n}w^m = \frac{1}{z}H_{a,b}\Big(\frac{1}{z}, w\Big). \tag{12}$$

2.5. ***Bi-free cumulants of products.*** Of paramount importance to this paper is the ability to write (l,r)-cumulants of products as sums of (l,r)-cumulants. We recall a result from [Charlesworth et al. 2015a, Section 9].

Let $m, n \geq 1$ with $m < n$. Fix a sequence of integers

$$k(0) = 0 < k(1) < \cdots < k(m) = n.$$

For $\chi : \{1,\dots,m\} \to \{l,r\}$, define $\hat{\chi} : \{1,\dots,n\} \to \{l,r\}$ via

$$\hat{\chi}(q) = \chi(p_q),$$

where p_q is the unique element of $\{1,\dots,m\}$ such that $k(p_q - 1) < q \leq k(p_q)$.

There exists an embedding of $\mathrm{BNC}(\chi)$ into $\mathrm{BNC}(\hat{\chi})$ via $\pi \mapsto \hat{\pi}$ where the p-th node of π is replaced by the block $\{k(p-1)+1,\dots,k(p)\}$. It is easy to see that $\widehat{1_\chi} = 1_{\hat{\chi}}$ and $\widehat{0_\chi}$ is the partition with blocks $\{\{k(p-1)+1,\dots,k(p)\}\}_{p=1}^m$. Given two partitions $\pi, \sigma \in \mathrm{BNC}(\chi)$, let $\pi \vee \sigma$ denote the smallest element of $\mathrm{BNC}(\chi)$ greater than π and σ.

Using ideas from [Nica and Speicher 2006, Theorem 11.12], [Charlesworth et al. 2015a, Theorem 9.1.5] showed that if $\{a_k\}_{k=1}^n \subseteq \mathcal{A}$, then

$$\kappa_{1_\chi}(a_1\cdots a_{k(1)}, a_{k(1)+1}\cdots a_{k(2)}, \dots, a_{k(m-1)+1}\cdots a_{k(m)}) = \sum_{\substack{\sigma\in \mathrm{BNC}(\widehat{\chi}) \\ \sigma\vee\widehat{0_\chi} = 1_{\hat{\chi}}}} \kappa_\sigma(a_1,\dots,a_n). \tag{13}$$

3. Bi-free partial T-transform

We begin with Voiculescu's bi-free partial T-transform, as the combinatorics are slightly simpler than the bi-free partial S-transform.

Definition 3.1 [Voiculescu 2015, Definition 3.1]. Let (a, b) be a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b) \neq 0$. The 2-variable partial bi-free T-transform of (a, b) is the holomorphic function on $(\mathbb{C}\setminus\{0\})^2$ near $(0, 0)$ defined by

$$T_{a,b}(z,w) = \frac{w+1}{w}\Big(1 - \frac{z}{F_{a,b}(K_a(z), \mathcal{X}_b(w))}\Big). \tag{14}$$

It is useful to note the following equivalent definition of the bi-free partial T-transform. To simplify the discussion, we show the equality in the case $\varphi(b) = 1$.

This does not hinder the proof of the desired result, namely Theorem 3.5 (see Remark 3.3).

Proposition 3.2. *If* (a, b) *is a two-faced pair in a noncommutative probability space* $(\mathcal{A}, \varphi)$ *with* $\varphi(b) = 1$, *then, as formal power series,*

$$T_{a,b}(z, w) = 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big). \tag{15}$$

Proof. Using equations (3), (5), and (10), we obtain that

$$\frac{1}{H_{a,b}(1/K_a(z), \mathcal{X}_b(w))} = \frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)} \frac{1}{1+w} C_{a,b}(z, (1+w)\mathcal{X}_b(w)).$$

Therefore, using equations (6), (9), (11), (12), and (14), we obtain that

$$\begin{aligned}
T_{a,b}&(z, w) \\
&= \frac{w+1}{w}\left(1 - \frac{z}{(1/K_a(z))H_{a,b}(1/K_a(z), \mathcal{X}_b(w))}\right) \\
&= \frac{w+1}{w}\left(1 - zK_a(z)\left(\frac{1}{zK_a(z)} + \frac{1}{1+w} - \frac{1}{zK_a(z)}\frac{1}{1+w} C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\right)\right) \\
&= \frac{1}{w}\big(-zK_a(z) + C_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big) \\
&= \frac{1}{w}\big(-zK_a(z) + 1 + c_a(z) + c_b\big(c_b^{\langle -1\rangle}(w)\big) + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big) \\
&= \frac{1}{w}\big(w + K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big)\big) \\
&= 1 + \frac{1}{w} K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).
\end{aligned}$$ □

Remark 3.3. One might be concerned that we have restricted to the case $\varphi(b) = 1$. However, if we use (15) as the definition of the bi-free partial T-transform and if $\lambda \in \mathbb{C} \setminus \{0\}$, then $T_{a,b}(z, w) = T_{a,\lambda b}(z, w)$. Indeed, $c_{\lambda b}(w) = c_b(\lambda w)$, so we have $c_{\lambda b}^{\langle -1\rangle}(w) = \frac{1}{\lambda} c_b^{\langle -1\rangle}(w)$. Therefore, since $\kappa_{n,m}(a, \lambda b) = \lambda^m \kappa_{n,m}(a, b)$, we see that

$$K_{a,\lambda b}\big(z, c_{\lambda b}^{\langle -1\rangle}(w)\big) = K_{a,b}\big(z, c_b^{\langle -1\rangle}(w)\big).$$

Thus there is no loss in assuming $\varphi(b) = 1$.

Remark 3.4. Note that Proposition 3.2 immediately provides the T-transform portion of [Voiculescu 2015, Proposition 4.2]. Indeed if a and b are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b) \neq 0$ and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $T_{a,b}(z, w) = 1$.

We desire to prove the following theorem (which was one of two main results of [Voiculescu 2015]) using combinatorial techniques and Proposition 3.2.

Theorem 3.5 [Voiculescu 2015, Theorem 3.1]. *Let (a_1, b_1) and (a_2, b_2) be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(b_1) \neq 0$ and $\varphi(b_2) \neq 0$. Then*

$$T_{a_1+a_2,b_1b_2}(z, w) = T_{a_1,b_1}(z, w)T_{a_2,b_2}(z, w)$$

on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.

To simplify the proof of the result, we assume that $\varphi(b_1) = \varphi(b_2) = 1$. Note that $\varphi(b_1b_2) = 1$ by freeness of the right algebras in bi-free pairs. Furthermore, let g_j denote the multiplicative function associated to the cumulants of b_j defined by $g_j(0_n, 1_n) = \kappa_n(b_j)$. Recall that if g is the multiplicative function associated to the cumulants of b_1b_2, then $g = g_1 * g_2$. Therefore $\phi_g^{\langle -1\rangle}(w) = c_{b_1b_2}^{\langle -1\rangle}(w)$ and $\phi_{g_j}^{\langle -1\rangle}(w) = c_{b_j}^{\langle -1\rangle}(w)$. Note that $g, g_1, g_2 \in \mathcal{M}_1$ by assumption.

By Proposition 3.2 it suffices to show that

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1}{w}\Theta_1(z, w)\Theta_2(z, w), \tag{16}$$

where

$$\Theta_j(z, w) = K_{a_j,b_j}\big(z, \phi_{g_j}^{\langle -1\rangle}(w)\big).$$

Recall

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1} \kappa_{n,m}(a_1 + a_2, b_1b_2)z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m)$ with blocks

$$\{\{k_l\}\}_{k=1}^n \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^m.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1+a_2, b_1b_2) = \sum_{\substack{\pi\in\mathrm{BNC}(n,2m)\\ \pi\vee\sigma_{n,m}=1_{n,2m}}} \kappa_\pi\big(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}\big).$$

Notice that if $\pi \in \mathrm{BNC}(n, 2m)$ and $\pi \vee \sigma_{n,m} = 1_{n,2m}$, then any block of π containing a k_l must contain a j_r for some j. Furthermore, if $1 \leq k < j \leq n$ are such that k_l and j_l are in the same block of π, then q_l must be in the same block as k_l for all $k \leq q \leq j$. Moreover, since (a_1, b_1) and (a_2, b_2) are bi-free, we note that

$$\kappa_\pi\big(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}\big) = 0$$

if π contains a block containing a $(2k)_r$ and a $(2j-1)_r$ for some k, j.

For $n, m \geq 1$, let $\mathrm{BNC}_T(n, m)$ denote all $\pi \in \mathrm{BNC}(n, 2m)$ such that

$$\pi \vee \sigma_{n,m} = 1_{n,2m}$$

and π contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for some k, j. Consequently, we obtain

$$K_{a_1+a_2,b_1b_2}(z, w) = \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in\mathrm{BNC}_T(n,m)} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_T(n, m)$. Let $\mathrm{BNC}_T(n, m)_e$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing 1_l also contains a $(2k)_r$ for some k, and let $\mathrm{BNC}_T(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that the block containing 1_l also contains a $(2k-1)_r$ for some k. Note that $\mathrm{BNC}_T(n, m)_e$ and $\mathrm{BNC}_T(n, m)_o$ are disjoint and

$$\mathrm{BNC}_T(n, m)_e \cup \mathrm{BNC}_T(n, m)_o = \mathrm{BNC}_T(n, m)$$

by previous discussions. Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z, w) := \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in\mathrm{BNC}_T(n,m)_d} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m,$$

then

$$K_{a_1+a_2,b_1b_2}(z, w) = \Psi_e(z, w) + \Psi_o(z, w).$$

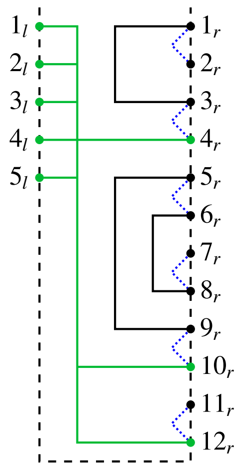
We derive expressions for $\Psi_e(z, w)$ and $\Psi_o(z, w)$ beginning with $\Psi_e(z, w)$.

Lemma 3.6. *Under the above notation and assumptions*,

$$\Psi_e(z, w) = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_e(z, w)$ by expanding κ_π as a product of full (l, r)-cumulants and summing over all π with the same block containing 1_l.

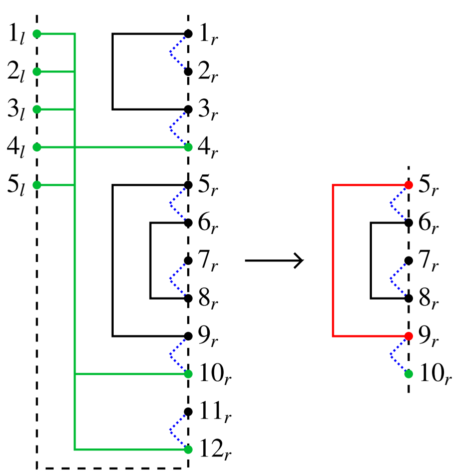
Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_T(n, m)_e$, then the block V_π containing 1_l must also contain $(2k)_r$ for some k, and thus all of $(2m)_r, 1_l, 2_l, \dots, n_l$ must be in V_π in order for $\pi \vee \sigma_{n,m} = 1_{n,2m}$ to be satisfied. Below is an example of such a π. Two nodes are connected to each other with a solid line if and only if they lie in the same block of π and two nodes are connected with a dotted line if and only if they are in the same block of $\sigma_{n,m}$. The condition $\pi \vee \sigma_{n,m} = 1_{n,2m}$ means one may travel from any one node to another using a combination of solid and dotted lines. Note we really should draw all of the left nodes above all of the right notes.



Let $E=\{(2k)_r\}_{k=1}^m$, let $O=\{(2k-1)_r\}_{k=1}^m$, let s denote the number of elements of E contained in V_π (so $s \geq 1$), and let $1 \leq k_1 < k_2 < \cdots < k_s = m$ be such that $(2k_q)_r \in V_\pi$. Note V_π divides the right nodes into s disjoint regions. For each $1 \leq q \leq s$, let $j_q = k_q - k_{q-1}$, with $k_0 = 0$, and let π_q denote the noncrossing partition obtained by restricting π to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q-1)_r\}.$$

Note that $\sum_{q=1}^s j_q = m$. Furthermore, if π_q' is obtained from π_q by adding the singleton block $\{(2k_q)_r\}$, then $\pi_q'|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi_q'|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi_q'|_E)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{n,2m}$. The below diagram demonstrates an example of this restriction.



Consequently, by writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m \\ = \kappa_{n,s}(a_2, b_2)z^n \prod_{q=1}^{s} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)_e$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)_e \\ V_\rho = V_\pi}} \kappa_\rho(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m \\ = \kappa_{n,s}(a_2, b_2)z^n \prod_{q=1}^{s} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n, m \geq 1$ and all possible V_π (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

$$\begin{aligned} \Psi_e(z, w) &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n \prod_{q=1}^{s} \phi_{g_2 \check{*} g_1}(w) \\ &= \sum_{n,s \geq 1} \kappa_{n,s}(a_2, b_2) z^n (\phi_{g_2 \check{*} g_1}(w))^s = K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)), \end{aligned}$$

as desired. □

In order to discuss $\Psi_o(z, w)$, it is quite helpful to discuss a subcase. For $n, m \geq 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(n, 2m+1)$ with blocks

$$\{\{k_l\}\}_{k=1}^{n} \cup \{1_r\} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Let $\mathrm{BNC}_T(n, m)'_o$ denote the set of all partitions $\pi \in \mathrm{BNC}(n, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$ and π contains no blocks containing both a $(2k)_r$ and a $(2j-1)_r$ for any k, j.

Lemma 3.7. *Under the above notation and assumptions, if*

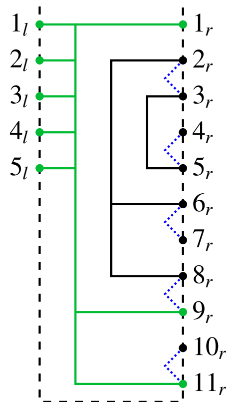
$$\Psi_{o'}(z, w) \\ := \sum_{\substack{n \geq 1 \\ m \geq 0}} \Bigg(\sum_{\pi \in \mathrm{BNC}_T(n,m)'_o} \kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \textit{ occurs } m \textit{ times}}) \Bigg) z^n w^{m+1},$$

then

$$\Psi_{o'}(z, w) = \frac{w}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2 \check{*} g_1}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_{o'}(z, w)$ by expanding κ_π as a product of full (l, r)-cumulants and summing over all π with the same block containing 1_l.

Fix $n \geq 1$ and $m \geq 0$. If $\pi \in \mathrm{BNC}_T(n, m)'_o$, then the block V_π containing 1_l must contain $1_r, (2m+1)_r, 1_l, 2_l, \ldots, n_l$ in order to have $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Below is an example of such a π.



Let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^{m+1}$, let s denote the number of elements of O contained in V_π (so $s \geq 1$), and let $1 = k_1 < k_2 < \cdots < k_s = m+1$ be such that $(2k_q - 1)_r \in V_\pi$. Note V_π divides the right nodes into $s-1$ disjoint regions. For each $1 \leq q \leq s-1$, let $j_q = k_{q+1} - k_q$ and let π_q denote the noncrossing partition obtained by restricting π to $\{(2k_q)_r, (2k_q+1)_r, \ldots, (2k_{q+1}-2)_r\}$. Note that $\sum_{q=1}^{s-1} j_q = m$. Furthermore, if π'_q is obtained from π_q by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ by $\pi \vee \sigma'_{n,m} = 1_{n,2m+1}$. Consequently, by writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_2) = 1$), we obtain

$$\kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^{m+1} = \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} g_2(0_{j_q}, \pi'_q) g_1(0_{j_q}, K(\pi'_q)) w^{j_q}.$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n, m)'_o$ with $V_\rho = V_\pi$, we obtain

$$\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)'_o \\ V_\rho = V_\pi}} \kappa_\pi(\underbrace{a_1+a_2, \ldots, a_1+a_2}_{n}, \underbrace{b_2, b_1, b_2, b_1, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^{m+1} = \kappa_{n,s}(a_2, b_2) z^n w \prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}.$$

Finally, if we sum over all possible $n \geq 1$, $m \geq 0$, and all possible V_π (so, in the above equation, we get all possible $s \geq 1$ and all possible $j_q \geq 1$), we obtain that

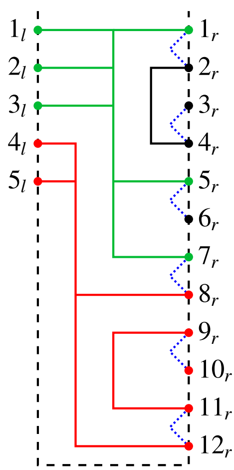
$$\begin{aligned}\Psi_{o'}(z,w) &= \sum_{n,s\geq 1} \kappa_{n,s}(a_2,b_2) z^n w \prod_{q=1}^{s-1} \phi_{g_2 \check{*} g_1}(w) \\ &= \frac{w}{\phi_{g_2\check{*}g_1}(w)} \sum_{n,s\geq 1} \kappa_{n,s}(a_2,b_2) z^n (\phi_{g_2\check{*}g_1}(w))^s \\ &= \frac{w}{\phi_{g_2\check{*}g_1}(w)} K_{a_2,b_2}(z, \phi_{g_2\check{*}g_1}(w)). \qquad \square\end{aligned}$$

Lemma 3.8. *Under the above notation and assumptions,*

$$\Psi_o(z,w) = \left(1 + \frac{1}{\phi_{g_1\check{*}g_2}(w)} \Psi_{o'}(z,w)\right) K_{a_1,b_1}(z, \phi_{g_1\check{*}g_2}(w)).$$

Proof. For each $n, m \geq 1$, we desire to rearrange the sum in $\Psi_o(z,w)$ by expanding κ_π as a product of full (l,r)-cumulants and summing over all π with the same block containing 1_l.

Fix $n, m \geq 1$, let $E = \{(2k)_r\}_{k=1}^m$, let $O = \{(2k-1)_r\}_{k=1}^m$, let $\pi \in \mathrm{BNC}_T(n,m)_o$, let V_π denote the block of π containing 1_l, let t (respectively s) denote the number of elements of $\{1_l, \ldots, n_l\}$ (respectively O) contained in V_π (so $t, s \geq 1$). Since $\pi \vee \sigma_{n,m} = 1_{n,2m}$, V_π must be of the form $\{k_l\}_{k=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s$ for some $1 = k_1 < k_2 < \cdots < k_s \leq m$. Below is an example of such a π.



Note that V_π divides the right nodes into s disjoint regions, where the bottom region is special as those nodes may connect to left nodes. For each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_s = m+1$. Note that $\sum_{q=1}^s j_q = m$. For $q \neq s$, let π_q denote the noncrossing partition obtained by restricting π to

$$\{(2k_q)_r, (2k_q+1)_r, \ldots, (2k_{q+1}-2)_r\}.$$

As discussed in Lemma 3.6, if π'_q is obtained from π_q by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_q|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_q|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_q|_O)$ since $\pi \vee \sigma_{n,m} = 1_{n,2m}$.

Let π'_s denote the bi-noncrossing partition obtained by restricting π to

$$\{k_l\}_{k=t+1}^{n} \cup \{(2k_s)_r, (2k_s+1)_r, \dots, (2m)_r\}$$

(which is shaded differently in the above diagram). Notice, since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, that it must be the case that $\pi_s \in \mathrm{BNC}_T(n-t, j_s-1)'_o$.

By writing κ_π as a product of cumulants, using linearity of κ_π, and using the fact that (a_1, b_1) and (a_2, b_2) are bi-free (and implicitly using $\varphi(b_1) = 1$), we obtain

$$\begin{aligned}
&\kappa_\pi(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m \\
&\qquad = \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} g_1(0_{j_q}, \pi'_q) g_2(0_{j_q}, K(\pi'_q)) w^{j_q} \right) \\
&\qquad\qquad \cdot \kappa_{\pi_s}(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n-t}, \underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_s \text{ times}}) z^{n-t} w^{j_s}.
\end{aligned}$$

Consequently, summing over all $\rho \in \mathrm{BNC}_T(n,m)_o$ with $V_\rho = V_\pi$, we obtain

$$\begin{aligned}
&\sum_{\substack{\rho \in \mathrm{BNC}_T(n,m)_o \\ V_\rho = V_\pi}} \kappa_\rho(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n}, \underbrace{b_1, b_2, b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m \\
&= \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} (g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q} \right) \\
&\qquad \cdot \left(\sum_{\sigma \in \mathrm{BNC}_T(n-t, j_s-1)'_o} \kappa_\sigma(\underbrace{a_1+a_2, \dots, a_1+a_2}_{n-t}, \underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_s \text{ times}}) z^{n-t} w^{j_s} \right)
\end{aligned}$$

as all $\sigma \in \mathrm{BNC}_T(n-t, j_s-1)'_o$ occur.

We desire to sum over all $n, m \geq 1$ and all possible V_π. This produces all possible $t, s \geq 1$ and all $j_q \geq 1$. If we first sum those terms above with $t = n$, we see, using similar arguments to those used above, that

$$\sum_{\sigma \in \mathrm{BNC}_T(0, j_s-1)'_o} \kappa_\sigma(\underbrace{b_2, b_1, b_2, \dots, b_1, b_2}_{b_2 \text{ occurs } j_q \text{ times}}) w^{j_s} = (g_1 \check{*} g_2)(0_{j_s}, 1_{j_s}) w^{j_s}.$$

Consequently, summing those terms with $t = n$ gives

$$\begin{aligned}
\sum_{t,s \geq 1} \kappa_{t,s}(a_1, b_1) z^t \prod_{q=1}^{s} \phi_{g_1 \check{*} g_2}(w) &= \sum_{t,s \geq 1} \kappa_{t,s}(a_1, b_1) z^t (\phi_{g_1 \check{*} g_2}(w))^s \\
&= K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).
\end{aligned}$$

Moreover, summing those terms with $t \neq n$ gives

$$\sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t \left(\prod_{q=1}^{s-1} \phi_{g_1 \check{*} g_2}(w) \right) \Psi_{o'}(z, w) = \sum_{t,s\geq 1} \kappa_{t,s}(a_1, b_1) z^t (\phi_{g_1 \check{*} g_2}(w))^{s-1} \Psi_{o'}(z, w) = \frac{1}{\phi_{g_1 \check{*} g_2}(w)} \Psi_{o'}(z, w) K_{a_1,b_1}(z, \phi_{g_1 \check{*} g_2}(w)).$$

Combining the above two sums completes the proof. □

Proof of Theorem 3.5. By Lemma 3.6 along with (7), we see that

$$\Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) = K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) = K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).$$

By Lemma 3.7 along with equations (7) and (8)), we see that

$$\begin{aligned}\Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big) &= \frac{\phi_g^{\langle -1\rangle}(w)}{\phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)} K_{a_2,b_2}\big(z, \phi_{g_2 \check{*} g_1}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &= \frac{\frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w)\phi_{g_2}^{\langle -1\rangle}(w)}{\phi_{g_2}^{\langle -1\rangle}(w)} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big) \\ &= \frac{1}{w}\phi_{g_1}^{\langle -1\rangle}(w) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

Furthermore, by Lemma 3.8 along with (7), we obtain

$$\begin{aligned}&\Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big) \\ &= \left(1 + \frac{1}{\phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1 \check{*} g_2}\big(\phi_g^{\langle -1\rangle}(w)\big)\big) \\ &= \left(1 + \frac{1}{\phi_{g_1}^{\langle -1\rangle}(w)} \Psi_{o'}\big(z, \phi_g^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &= \left(1 + \frac{1}{w} K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big)\right) K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) \\ &= K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) + \frac{1}{w} K_{a_1,b_1}\big(z, \phi_{g_1}^{\langle -1\rangle}(w)\big) K_{a_2,b_2}\big(z, \phi_{g_2}^{\langle -1\rangle}(w)\big).\end{aligned}$$

As

$$K_{a_1+a_2,b_1b_2}\big(z, \phi_g^{\langle -1\rangle}(w)\big) = \Psi_e\big(z, \phi_g^{\langle -1\rangle}(w)\big) + \Psi_o\big(z, \phi_g^{\langle -1\rangle}(w)\big),$$

we have verified that equation (16) holds and thus the proof is complete. □

4. Bi-free partial S-transform

In this section, we study Voiculescu's bi-free partial S-transform through combinatorics. All notation in this section refers to the notation established in this section and not to the notation of Section 3.

Definition 4.1 [Voiculescu 2015, Definition 2.1]. Let (a, b) be a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$ and $\varphi(b) \neq 0$. The 2-variable partial bi-free S-transform of (a, b) is the holomorphic function defined on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$ by

$$S_{a,b}(z, w) = \frac{z+1}{z}\frac{w+1}{w}\Big(1 - \frac{1+z+w}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))}\Big). \tag{17}$$

It is useful to note, in the following proposition, an equivalent definition of the bi-free partial S-transform. To simplify the discussion, we demonstrate the equality in the case $\varphi(a) = \varphi(b) = 1$. This does not hinder the proof of the desired result, namely Theorem 4.5 (see Remark 4.3).

Proposition 4.2. *If (a, b) is a two-faced pair in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) = \varphi(b) = 1$, then, as a formal power series,*

$$S_{a,b}(z, w) = 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big). \tag{18}$$

Proof. Using equations (5), (6), (9), and (10), we obtain that

$$\frac{1}{H_{a,b}(\mathcal{X}_a(z), \mathcal{X}_b(w))} = \frac{1}{1+z} + \frac{1}{1+w} - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big).$$

Therefore, using equations (11) and (17), we obtain that

$$\begin{aligned} S_{a,b}(z, w) &= \frac{z+1}{z}\frac{w+1}{w}\Big(1 - (1+z+w)\Big(\frac{1}{1+z} + \frac{1}{1+w} \\ &\qquad\qquad - \frac{1}{1+z}\frac{1}{1+w} C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\Big)\Big) \\ &= \frac{1}{zw}\big((1+z)(1+w) - (1+z+w)(2+z+w) \\ &\qquad\qquad + (1+z+w) C_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big) \\ &= \frac{1}{zw}\Big(zw - (1+z+w)^2 \\ &\qquad\qquad + (1+z+w)\big(1+z+w + K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big)\big)\Big) \\ &= 1 + \frac{1+z+w}{zw} K_{a,b}\big(c_a^{\langle -1\rangle}(z), c_b^{\langle -1\rangle}(w)\big). \end{aligned}$$

□

Remark 4.3. Again, one might be concerned that we have restricted to the case $\varphi(a) = \varphi(b) = 1$. Using the same ideas as in Remark 3.3, if we use (18) as the definition of the S-transform and if $\lambda, \mu \in \mathbb{C} \setminus \{0\}$, then $S_{a,b}(z, w) = S_{\lambda a, \mu b}(z, w)$. Hence there is no loss in assuming $\varphi(a) = \varphi(b) = 1$.

Remark 4.4. Note Proposition 4.2 immediately provides the S-transform part of [Voiculescu 2015, Proposition 4.2]. Indeed if a and b are elements of a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a) \neq 0$, $\varphi(b) \neq 0$, and $\varphi(a^n b^m) = \varphi(a^n)\varphi(b^m)$ for all $n, m \geq 0$, then $\kappa_{n,m}(a, b) = 0$ for all $n, m \geq 1$ (see [Skoufranis 2015, Section 3.2]). Hence $K_{a,b}(z, w) = 0$, so $S_{a,b}(z, w) = 1$.

We desire to prove the following, which is one of two main results of [Voiculescu 2015], using combinatorial techniques and Proposition 4.2.

Theorem 4.5 [Voiculescu 2015, Theorem 2.1]. *Let (a_1, b_1) and (a_2, b_2) be bi-free two-faced pairs in a noncommutative probability space $(\mathcal{A}, \varphi)$ with $\varphi(a_j) \neq 0$ and $\varphi(b_j) \neq 0$. Then*

$$S_{a_1 a_2, b_1 b_2}(z, w) = S_{a_1,b_1}(z, w) S_{a_2,b_2}(z, w)$$

on $(\mathbb{C} \setminus \{0\})^2$ near $(0, 0)$.

To simplify the proof of this result, we assume that $\varphi(a_j) = \varphi(b_j) = 1$. Note that $\varphi(a_1 a_2) = \varphi(b_1 b_2) = 1$ by freeness of the left algebras and of the right algebras in bi-free pairs. Furthermore, let f_j (respectively g_j) denote the multiplicative function associated to the cumulants of a_j (respectively b_j) defined by $f_j(0_n, 1_n) = \kappa_n(a_j)$ (respectively $g_j(0_n, 1_n) = \kappa_n(b_j)$). Recall that if f (respectively g) is the multiplicative function associated to the cumulants of $a_1 a_2$ (respectively $b_1 b_2$), then $f = f_1 * f_2$ (respectively $g = g_1 * g_2$). Thus

$$\phi_f^{\langle -1 \rangle}(z) = c_{a_1 a_2}^{\langle -1 \rangle}(z), \qquad \phi_g^{\langle -1 \rangle}(w) = c_{b_1 b_2}^{\langle -1 \rangle}(w),$$
$$\phi_{f_j}^{\langle -1 \rangle}(z) = c_{a_j}^{\langle -1 \rangle}(z), \qquad \phi_{g_j}^{\langle -1 \rangle}(w) = c_{b_j}^{\langle -1 \rangle}(w).$$

Note that $f, g, f_j, g_j \in \mathcal{M}_1$ by assumption.

By Proposition 4.2, it suffices to show that

$$K_{a_1 a_2, b_1 b_2}\big(\phi_f^{\langle -1 \rangle}(w), \phi_g^{\langle -1 \rangle}(w)\big) = \Theta_1(z, w) + \Theta_2(z, w) + \frac{1+z+w}{zw} \Theta_1(z, w) \Theta_2(z, w) \tag{19}$$

where

$$\Theta_j(z, w) = K_{a_j, b_j}\big(\phi_{f_j}^{\langle -1 \rangle}(w), \phi_{g_j}^{\langle -1 \rangle}(w)\big).$$

Recall

$$K_{a_1 a_2, b_1 b_2}(z, w) = \sum_{n,m \geq 1} \kappa_{n,m}(a_1 a_2, b_1 b_2) z^n w^m.$$

For fixed $n, m \geq 1$, let $\sigma_{n,m}$ denote the element of $\mathrm{BNC}(2n, 2m)$ with blocks

$$\{\{(2k-1)_l, (2k)_l\}\}_{k=1}^{n} \cup \{\{(2k-1)_r, (2k)_r\}\}_{k=1}^{m}.$$

Thus (13) implies that

$$\kappa_{n,m}(a_1a_2, b_1b_2) = \sum_{\substack{\pi \in \mathrm{BNC}(2n,2m) \\ \pi \vee \sigma_{n,m} = 1_{2n,2m}}} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}).$$

Since (a_1, b_1) and (a_2, b_2) are bi-free, we note that

$$\kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) = 0$$

if π contains a block containing a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some k, j.

For $n, m \geq 1$, let $\mathrm{BNC}_S(n, m)$ be the set of all $\pi \in \mathrm{BNC}(2n, 2m)$ such that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$ and π contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for some $\theta_1, \theta_2 \in \{l, r\}$ and for some k, j. Consequently, we obtain

$$K_{a_1a_2,b_1b_2}(z, w) = \sum_{n,m \geq 1} \left(\sum_{\pi \in \mathrm{BNC}_S(n,m)} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) \right) z^n w^m.$$

We desire to divide up this sum into two parts based on types of partitions in $\mathrm{BNC}_S(n, m)$. Notice that if $\pi \in \mathrm{BNC}_S(n, m)$, then π must contain a block with both a k_l and a j_r for some k, j, so that $\pi \vee \sigma_{n,m} = 1_{2n,2m}$. If

$$V \subseteq \{1_l, \ldots, (2n)_l, 1_r, \ldots, (2m)_r\},$$

we define $\min(V)$ to be the integer k such that either $k_l \in V$ or $k_r \in V$ yet $j_l, j_r \notin V$ for all $j < k$.

Let $\mathrm{BNC}_S(n, m)_e$ denote all $\pi \in \mathrm{BNC}_S(n, m)$ such that $\min(V) \in 2\mathbb{Z}$ for the block V of π that has the smallest min-value over all blocks W of π such that there exist $k_l, j_r \in W$ for some k, j; that is, V is the first block, measured from the top, in the bi-noncrossing diagram of π that has both left and right nodes, and these nodes are of even index. Similarly, let $\mathrm{BNC}_S(n, m)_o$ denote all $\pi \in \mathrm{BNC}_T(n, m)$ such that $\min(V) \in 2\mathbb{Z}+1$ for the block V of π that has the smallest min-value over all blocks W of π such that there exist $k_l, j_r \in W$ for some k, j. Note $\mathrm{BNC}_S(n, m)_e$ and $\mathrm{BNC}_S(n, m)_o$ are disjoint and

$$\mathrm{BNC}_S(n, m)_e \cup \mathrm{BNC}_S(n, m)_o = \mathrm{BNC}_S(n, m).$$

Therefore, if for $d \in \{o, e\}$ we define

$$\Psi_d(z, w) := \sum_{n,m\geq 1}\Bigg(\sum_{\pi\in \mathrm{BNC}_S(n,m)_d} \kappa_\pi(\underbrace{a_1, a_2, a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^n w^m,$$

then

$$K_{a_1a_2,b_1b_2}(z, w) = \Psi_e(z, w) + \Psi_o(z, w).$$

We derive expressions for $\Psi_e(z, w)$ and $\Psi_o(z, w)$ beginning with $\Psi_e(z, w)$. We do not use the same rigour as in Section 3, as most of the arguments are similar.

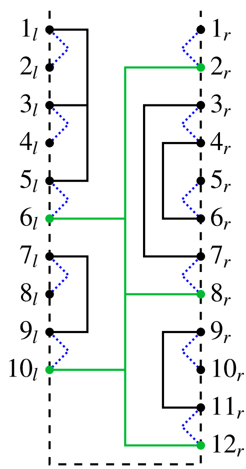
Lemma 4.6. *Under the above notation and assumptions,*

$$\Psi_e(z, w) = K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

Proof. Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n, m)_e$, let V_π denote the first (and, as it happens, only) block of π, as measured from the top of π's bi-noncrossing diagram, that has both left and right nodes. Since $\pi \vee \sigma_{n,m} = 1_{2n,2m}$, there exist $t, s \geq 1$, $1 \leq l_1 < l_2 < \cdots < l_t = n$, and $1 \leq k_1 < k_2 < \cdots < k_s = m$ such that

$$V_\pi = \{(2l_p)_l\}_{p=1}^t \cup \{(2k_q)_r\}_{q=1}^s.$$

Note V_π divides the remaining left nodes into t disjoint regions and the remaining right nodes into s disjoint regions. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



Let $E = \{(2k)_l\}_{k=1}^n \cup \{(2k)_r\}_{k=1}^m$ and $O = \{(2k-1)_l\}_{k=1}^n \cup \{(2k-1)_r\}_{k=1}^m$. For each $1 \leq p \leq t$, let $i_p = l_p - l_{p-1}$, where $l_0 = 0$, and let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting π to $\{(2l_{p-1}+1)_l, (2l_{p-1}+2)_l, \ldots, (2l_p - 1)_l\}$. Note that $\sum_{p=1}^t i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p)_l\}$, then $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \le q \le s$, let $j_q = k_q - k_{q-1}$, where $k_0 = 0$, and let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting π to

$$\{(2k_{q-1}+1)_r, (2k_{q-1}+2)_r, \ldots, (2k_q - 1)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q)_r\}$, then $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_E)$ in order to have $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \ldots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \ldots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}})z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n,m)_e$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_2, b_2)\Bigg(\prod_{p=1}^{t}(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p})z^{i_p}\Bigg)\Bigg(\prod_{q=1}^{s}(g_2 \check{*} g_1)(0_{j_q}, 1_{j_q})w^{j_q}\Bigg).$$

Finally, if we sum over all possible $n, m \ge 1$ and all possible V_π (so, in the above equation, we get all possible $t, s \ge 1$ and all possible $i_p, j_q \ge 1$), we obtain that

$$\begin{aligned}\Psi_e(z,w) &= \sum_{t,s\ge 1} \kappa_{t,s}(a_2,b_2)\Bigg(\prod_{p=1}^{t}\phi_{f_2\check{*}f_1}(z)\Bigg)\Bigg(\prod_{q=1}^{s}\phi_{g_2\check{*}g_1}(z)\Bigg)\\ &= \sum_{t,s\ge 1}\kappa_{t,s}(a_2,b_2)(\phi_{f_2\check{*}f_1}(z))^t(\phi_{g_2\check{*}g_1}(w))^s\\ &= K_{a_2,b_2}\big(\phi_{f_2\check{*}f_1}(z), \phi_{g_2\check{*}g_1}(w)\big). \qquad \square\end{aligned}$$

In order to discuss $\Psi_o(z,w)$, it is quite helpful to discuss subcases. For $n, m \ge 0$, let $\sigma'_{n,m}$ denote the element of $\mathrm{BNC}(2n+1, 2m+1)$ with blocks

$$\{\{1_l, 1_r\}\} \cup \{\{(2l)_l, (2l+1)_l\}\}_{l=1}^{n} \cup \{\{(2k)_r, (2k+1)_r\}\}_{k=1}^{m}.$$

Define $\mathrm{BNC}_S(n,m)'_o$ to be the set of all $\pi \in \mathrm{BNC}(2n+1, 2m+1)$ such that $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$ and π contains no blocks with both a $(2k)_{\theta_1}$ and a $(2j-1)_{\theta_2}$ for any $\theta_1, \theta_2 \in \{l, r\}$ and any k, j. We wish to divide up $\mathrm{BNC}_S(n,m)'_o$ further. For $\pi \in \mathrm{BNC}_S(n,m)'_o$, let $V_{\pi,l}$ denote the block of π containing 1_l and $V_{\pi,r}$ the block of π containing 1_r. Then,

$$\begin{aligned}
&\mathrm{BNC}_S(n,m)_{o,0}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has no right nodes and } V_{\pi,r} \text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,r}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has no right nodes but } V_{\pi,r} \text{ has left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,l}\\
&\qquad = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} \text{ has right nodes but } V_{\pi,r} \text{ has no left nodes}\},\\
&\mathrm{BNC}_S(n,m)_{o,lr} = \{\pi \in \mathrm{BNC}_S(n,m)'_o \mid V_{\pi,l} = V_{\pi,r}\}.
\end{aligned}$$

Due to the nature of bi-noncrossing partitions, the above sets are disjoint and have union $\mathrm{BNC}_S(n,m)'_o$.

For $d \in \{0, r, l, lr\}$, define

$$\Psi_{o,d}(z,w) := \sum_{n,m\geq 0}\Bigg(\sum_{\pi\in\mathrm{BNC}_S(n,m)_{o,d}} \kappa_\pi(\underbrace{a_2,a_1,a_2,\dots,a_1,a_2}_{a_1 \text{ occurs } n \text{ times}},\underbrace{b_2,b_1,b_2,\dots,b_1,b_2}_{b_1 \text{ occurs } m \text{ times}})\Bigg) z^{n+1}w^{m+1}.$$

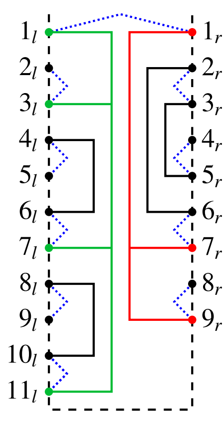
Lemma 4.7. *Under the above notation and assumptions,*

$$\Psi_{o,0}(z,w) = zw\cdot\frac{\phi_{f_2}(\phi_{f_2\check{*}f_1}(z))\phi_{g_2}(\phi_{g_2\check{*}g_1}(w))}{\phi_{f_2\check{*}f_1}(z)\phi_{g_2\check{*}g_1}(w)}.$$

Proof. Fix $n, m \geq 0$. If $\pi \in \mathrm{BNC}_S(n,m)_{o,0}$, then, since $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$, there exist $t, s \geq 1$, $1 = l_1 < l_2 < \cdots < l_t = n+1$, and $1 = k_1 < k_2 < \cdots < k_s = m+1$ such that

$$V_{\pi,l} = \{(2l_p - 1)_l\}_{p=1}^t \quad\text{and}\quad V_{\pi,r} = \{(2k_q - 1)_r\}_{q=1}^s.$$

Note that $V_{\pi,l}$ divides the remaining left nodes into $t-1$ disjoint regions and $V_{\pi,r}$ divides the remaining right nodes into $s-1$ disjoint regions. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



If $i_p = l_{p+1} - l_p$ and $j_q = k_{q+1} - k_q$, then

$$\sum_{p=1}^{t-1} i_p = n \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^{n+1} w^{m+1}$$

for $\rho \in \mathrm{BNC}_S(n, m)_{o,0}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$zw \cdot \kappa_t(a_2)\kappa_s(b_2)\left(\prod_{p=1}^{t-1}(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}\right)\left(\prod_{q=1}^{s-1}(g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q}\right).$$

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}
\Psi_e(z, w) &= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2)\left(\prod_{p=1}^{t-1}\phi_{f_2 \check{*} f_1}(z)\right)\left(\prod_{q=1}^{s-1}\phi_{g_2 \check{*} g_1}(z)\right) \\
&= zw \sum_{t,s \geq 1} \kappa_t(a_2)\kappa_s(b_2)(\phi_{f_2 \check{*} f_1}(z))^{t-1}(\phi_{g_2 \check{*} g_1}(w))^{s-1} \\
&= zw \cdot \frac{\phi_{f_2}(\phi_{f_2 \check{*} f_1}(z))\phi_{g_2}(\phi_{g_2 \check{*} g_1}(w))}{\phi_{f_2 \check{*} f_1}(z)\phi_{g_2 \check{*} g_1}(w)}.
\end{aligned}$$

□

Lemma 4.8. *Under the above notation and assumptions,*

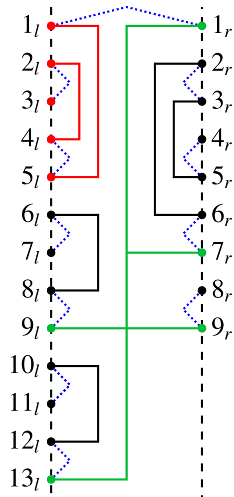
$$\Psi_{o,r}(z, w) = \frac{w \cdot \phi_{f_1 \check{*} f_2}(z)}{\phi_{g_2 \check{*} g_1}(w)} K_{a_2,b_2}(\phi_{f_2 \check{*} f_1}(z), \phi_{g_2 \check{*} g_1}(w)).$$

Proof. Fix $n, m \geq 0$. Note $\mathrm{BNC}_S(0, m)_{o,r} = \varnothing$ by definition.

If $\pi \in \mathrm{BNC}_S(n, m)_{o,r}$, then, since $\pi \vee \sigma'_{n,m} = 1_{2n+1,2m+1}$, there exist $t, s \geq 1$, $1 < l_1 < l_2 < \cdots < l_t = n+1$, and $1 = k_1 < k_2 < \cdots < k_s = m+1$ such that

$$V_{\pi,r} = \{(2l_p - 1)_l\}_{p=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s.$$

Note that $V_{\pi,r}$ divides the remaining right nodes into $s-1$ disjoint regions and the remaining left nodes into t regions. However, the top region is special. If l_0 is the largest natural number such that $(2l_0 - 1)_l \in V_{\pi,l}$, then l_0 further divides the top region on the left into two regions. Note that each block of π can only contain nodes in one such region. The following is an example of such a π for which $l_0 = 3$, with one part of the special region $(1_l, \dots, 5_l)$ shaded differently.



Let $i_0 = l_0$, $i_p = l_p - l_{p-1}$ when $p \neq 0$, and $j_q = k_{q+1} - k_q$. Thus

$$\sum_{p=0}^{t} i_p = n+1 \quad \text{and} \quad \sum_{q=1}^{s-1} j_q = m.$$

Using similar arguments to those in Lemma 4.6, expanding

$$\kappa_\rho(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^{n+1} w^{m+1}$$

for $\rho \in \mathrm{BNC}_S(n, m)_{o,r}$ and summing all terms with $V_{\rho,l} = V_{\pi,l}$ and $V_{\rho,r} = V_{\pi,r}$, we obtain

$$w \cdot \kappa_{t,s}(a_2, b_2) \left(\prod_{p=1}^{t} (f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p} \right) \\ \cdot \left(\prod_{q=1}^{s-1} (g_2 \check{*} g_1)(0_{j_q}, 1_{j_q}) w^{j_q} \right) \left((f_1 \check{*} f_2)(0_{i_0}, 1_{i_0}) z^{i_0} \right).$$

Note for $p \geq 2$, each $(f_2 \check{*} f_1)(0_{i_p}, 1_{i_p}) z^{i_p}$ comes from the p-th region from the top on the left, whereas the top region on the left gives $(f_2 \check{*} f_1)(0_{i_1}, 1_{i_1}) z^{i_1}$ using the partitions below $(2l_0 - 1)_l$ and gives $(f_1 \check{*} f_2)(0_{i_0}, 1_{i_0}) z^{i_0}$ using the partitions above and including $(2l_0 - 1)_l$.

Finally, if we sum over all possible $n, m \geq 0$ and all possible $V_{\pi,l}$ and $V_{\pi,r}$ (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}\Psi_e(z,w) &= w\sum_{t,s\geq 1}\kappa_{t,s}(a_2,b_2)\left(\prod_{p=1}^{t}\phi_{f_2\check{*}f_1}(z)\right)\left(\prod_{q=1}^{s-1}\phi_{g_2\check{*}g_1}(z)\right)\left(\phi_{f_1\check{*}f_2}(z)\right)\\ &= w\sum_{t,s\geq 1}\kappa_{t,s}(a_2,b_2)(\phi_{f_2\check{*}f_1}(z))^t(\phi_{g_2\check{*}g_1}(w))^{s-1}(\phi_{f_1\check{*}f_2}(z))\\ &= \frac{w\cdot\phi_{f_1\check{*}f_2}(z)}{\phi_{g_2\check{*}g_1}(w)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).\end{aligned}$$ □

Lemma 4.9. *Under the above notation and assumptions,*

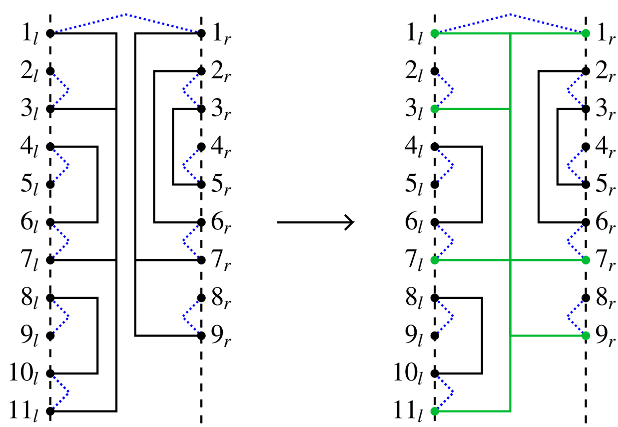
$$\Psi_{o,l}(z,w) = \frac{z\cdot\phi_{g_1\check{*}g_2}(w)}{\phi_{f_2\check{*}f_1}(z)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).$$

Proof. The proof can be obtained by applying a mirror to Lemma 4.8. □

Lemma 4.10. *Under the above notation and assumptions,*

$$\Psi_{o,lr}(z,w) = \frac{zw}{\phi_{f_2\check{*}f_1}(z)\phi_{g_2\check{*}g_1}(w)}K_{a_2,b_2}(\phi_{f_2\check{*}f_1}(z),\phi_{g_2\check{*}g_1}(w)).$$

Proof. The proof of this result follows from the proof of Lemma 4.7 by replacing each occurrence of $\kappa_t(a_2)\kappa_s(b_2)$ with $\kappa_{t,s}(a_2,b_2)$. Indeed there is a bijection from $\mathrm{BNC}_S(n,m)_{o,0}$ to $\mathrm{BNC}_S(n,m)_{o,lr}$ whereby, given $\pi\in\mathrm{BNC}_S(n,m)_{o,0}$, we produce $\pi'\in\mathrm{BNC}_S(n,m)_{o,lr}$ by joining $V_{\pi,l}$ and $V_{\pi,r}$ into a single block.



□

Lemma 4.11. *Under the above notation and assumptions,*

$$\Psi_o(z,w) = \frac{1}{\phi_{f_1\check{*}f_2}(z)\phi_{g_1\check{*}g_2}(w)}\Psi_{o'}(z,w)K_{a_1,b_1}(\phi_{f_1\check{*}f_2}(z),\phi_{g_1\check{*}g_2}(w)),$$

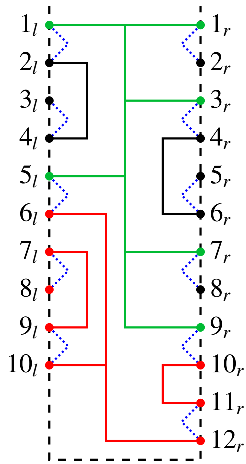
where

$$\Psi_{o'}(z,w) = \Psi_{o,0}(z,w)+\Psi_{o,r}(z,w)+\Psi_{o,l}(z,w)+\Psi_{o,lr}(z,w).$$

Proof. Fix $n, m \geq 1$. If $\pi \in \mathrm{BNC}_S(n, m)_o$, let V_π denote the first block of π, as measured from the top of π's bi-noncrossing diagram, that has both left and right nodes. Since $\pi \in \mathrm{BNC}_S(n, m)_o$, there exist $t, s \geq 1$, $1 = l_1 < l_2 < \cdots < l_t \leq n$, and $1 = k_1 < k_2 < \cdots < k_s \leq m$ such that

$$V_\pi = \{(2l_p - 1)_l\}_{p=1}^t \cup \{(2k_q - 1)_r\}_{q=1}^s.$$

Note V_π divides the remaining left nodes and right nodes into $t-1$ disjoint regions on the left, $s-1$ disjoint regions on the right, and one region on the bottom. Moreover, each block of π can only contain nodes in one such region. Below is an example of such a π.



Let

$$E = \{(2k)_l\}_{k=1}^n \cup \{(2k)_r\}_{k=1}^m,$$

$$O = \{(2k-1)_l\}_{k=1}^n \cup \{(2k-1)_r\}_{k=1}^m.$$

For each $1 \leq p \leq t$, let $i_p = l_{p+1} - l_p$, where $l_{t+1} = n+1$, and, for $p \neq t$, let $\pi_{l,p}$ denote the noncrossing partition obtained by restricting π to

$$\{(2l_p)_l, (2l_p + 1)_l, \ldots, (2l_{p+1} - 2)_l\}.$$

Note that $\sum_{p=1}^t i_p = n$. Furthermore, as explained in Lemma 3.6, if $\pi'_{l,p}$ is obtained from $\pi_{l,p}$ by adding the singleton block $\{(2l_p - 1)_l\}$, then $\pi'_{l,p}|_O$ is naturally an element of $\mathrm{NC}'(i_p)$ and $\pi'_{l,p}|_E$ is naturally an element of $\mathrm{NC}(i_p)$, which must be $K(\pi'_{l,p}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Similarly, for each $1 \leq q \leq s$, let $j_q = k_{q+1} - k_q$, where $k_{s+1} = m+1$, and, for $q \neq s$, let $\pi_{r,q}$ denote the noncrossing partition obtained by restricting π to

$$\{(2k_q)_r, (2k_q + 1)_r, \ldots, (2k_{q+1} - 2)_r\}.$$

Note that $\sum_{q=1}^{s} j_q = m$. Furthermore, as explained in Lemma 3.6, if $\pi'_{r,q}$ is obtained from $\pi_{r,q}$ by adding the singleton block $\{(2k_q-1)_r\}$, then $\pi'_{r,q}|_O$ is naturally an element of $\mathrm{NC}'(j_q)$ and $\pi'_{r,q}|_E$ is naturally an element of $\mathrm{NC}(j_q)$, which must be $K(\pi'_{r,q}|_O)$ in order to satisfy $\pi \vee \sigma_{n,m} = 1_{2n,2m}$.

Finally, if π' is the bi-noncrossing partition obtained by restricting π to

$$\{(2l_t)_l, (2l_t+1)_l, \dots, (2n)_l, (2k_s)_r, (2k_s+1)_r, \dots, (2m)_r\}$$

(which is shaded differently in the above diagram), then $\pi' \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$.

Expanding

$$\kappa_\rho(\underbrace{a_1, a_2, \dots, a_1, a_2}_{a_1 \text{ occurs } n \text{ times}}, \underbrace{b_1, b_2, \dots, b_1, b_2}_{b_1 \text{ occurs } m \text{ times}}) z^n w^m$$

for $\rho \in \mathrm{BNC}_S(n,m)_o$ and summing such terms with $V_\rho = V_\pi$, we obtain

$$\kappa_{t,s}(a_1,b_1)\left(\prod_{p=1}^{t-1}(f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}\right)\left(\prod_{q=1}^{s-1}(g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}\right)$$
$$\cdot\left(\sum_{\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o} \kappa_\tau(\underbrace{a_2, a_1, a_2, a_1, \dots, a_1, a_2}_{a_1 \text{ occurs } i_t-1 \text{ times}}, \underbrace{b_2, b_1, b_2, b_1, \dots, b_1, b_2}_{b_1 \text{ occurs } j_s-1 \text{ times}}) z^{i_t} w^{j_s}\right).$$

Note that for $p \neq t$, each $(f_1 \check{*} f_2)(0_{i_p}, 1_{i_p}) z^{i_p}$ comes from the p-th region from the top on the left, for $q \neq s$ each $(g_1 \check{*} g_2)(0_{j_q}, 1_{j_q}) w^{j_q}$ comes from the q-th region from the top on the right, and all $\tau \in \mathrm{BNC}_S(i_t-1, j_s-1)'_o$ are possible on the bottom.

Finally, if we sum over all possible $n, m \geq 1$ and all possible V_π (so, in the above equation, we get all possible $t, s \geq 1$ and all possible $i_p, j_q \geq 1$), we obtain that

$$\begin{aligned}
\Psi_e(z,w) &= \sum_{t,s\geq 1} \kappa_{t,s}(a_1,b_1)\left(\prod_{p=1}^{t-1}\phi_{f_1 \check{*} f_2}(z)\right)\left(\prod_{q=1}^{s-1}\phi_{g_1 \check{*} g_2}(z)\right)\Psi_{o'}(z,w) \\
&= \sum_{t,s\geq 1} \kappa_{t,s}(a_1,b_1)(\phi_{f_1 \check{*} f_2}(z))^{t-1}(\phi_{g_1 \check{*} g_2}(w))^{s-1}\Psi_{o'}(z,w) \\
&= \frac{1}{\phi_{f_1 \check{*} f_2}(z)\phi_{g_1 \check{*} g_2}(w)}\Psi_{o'}(z,w) K_{a_1,b_1}(\phi_{f_1 \check{*} f_2}(z), \phi_{g_1 \check{*} g_2}(w)). \qquad \square
\end{aligned}$$

Proof of Theorem 4.5. Using (7) and (8), we see (via Lemmata 4.6–4.10) that

$$\begin{aligned}
\Psi_e\big(\phi^{\langle -1\rangle}_{f_1 * f_2}(z), \phi^{\langle -1\rangle}_{g_1 * g_2}(w)\big) &= K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z), \phi^{\langle -1\rangle}_{g_2}(w)\big), \\
\Psi_{o,0}\Big(\phi^{\langle -1\rangle}_{f_1 * f_2}(z), \phi^{\langle -1\rangle}_{g_1 * g_2}(w)\Big) &= \phi^{\langle -1\rangle}_{f_1 * f_2}(z)\phi^{\langle -1\rangle}_{g_1 * g_2}(w) \cdot \frac{zw}{\phi^{\langle -1\rangle}_{f_2}(z)\phi^{\langle -1\rangle}_{g_2}(w)} \\
&= \phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w),
\end{aligned}$$

$$\Psi_{o,r}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z), \phi^{\langle -1\rangle}_{g_1*g_2}(w)\big) = \frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{w} K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z), \phi^{\langle -1\rangle}_{g_2}(w)\big),$$
$$\Psi_{o,l}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z), \phi^{\langle -1\rangle}_{g_1*g_2}(w)\big) = \frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{z} K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z), \phi^{\langle -1\rangle}_{g_2}(w)\big),$$
$$\Psi_{o,lr}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z), \phi^{\langle -1\rangle}_{g_1*g_2}(w)\big) = \frac{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)}{zw} K_{a_2,b_2}\big(\phi^{\langle -1\rangle}_{f_2}(z), \phi^{\langle -1\rangle}_{g_2}(w)\big).$$

Since

$$\Phi_0\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z), \phi^{\langle -1\rangle}_{g_1*g_2}(w)\big) = \frac{1}{\phi^{\langle -1\rangle}_{f_1}(z)\phi^{\langle -1\rangle}_{g_1}(w)} \Psi_{o'}\big(\phi^{\langle -1\rangle}_{f_1*f_2}(z), \phi^{\langle -1\rangle}_{g_1*g_2}(w)\big) K_{a_1,b_1}\big(\phi^{\langle -1\rangle}_{f_1}(z), \phi^{\langle -1\rangle}_{g_1}(w)\big)$$

by (7) and Lemma 4.11, and since

$$\frac{1}{z} + \frac{1}{w} + \frac{1}{zw} = \frac{1+z+w}{zw} \quad \text{and} \quad K_{a_1a_2,b_1b_2}(z,w) = \Psi_e(z,w) + \Psi_0(z,w),$$

we have verified that (19) holds and thus the proof is complete. □

References

[Charlesworth et al. 2015a] I. Charlesworth, B. Nelson, and P. Skoufranis, "Combinatorics of bi-freeness with amalgamation", *Comm. Math. Phys.* **338**:2 (2015), 801–847. MR Zbl

[Charlesworth et al. 2015b] I. Charlesworth, B. Nelson, and P. Skoufranis, "On two-faced families of non-commutative random variables", *Canad. J. Math.* **67**:6 (2015), 1290–1325. MR Zbl

[Gu et al. 2015] Y. Gu, H.-W. Huang, and J. Mingo, "An analogue of the Lévy–Hinčin formula for bi-free infinitely divisible distributions", *Indiana Univ. Math. J.* (Online publication August 2015).

[Haagerup 1997] U. Haagerup, "On Voiculescu's R- and S-transforms for free non-commuting random variables", pp. 127–148 in *Free probability theory* (Waterloo, ON, 1995), edited by D.-V. Voiculescu, Fields Institute Communications **12**, American Mathematical Society, Providence, RI, 1997. MR Zbl

[Mastnak and Nica 2015] M. Mastnak and A. Nica, "Double-ended queues and joint moments of left-right canonical operators on full Fock space", *Int. J. Math.* **26**:2 (2015), Article ID #1550016. MR Zbl

[Nica and Speicher 1997] A. Nica and R. Speicher, "A 'Fourier transform' for multiplicative functions on non-crossing partitions", *J. Algebraic Combin.* **6**:2 (1997), 141–160. MR Zbl

[Nica and Speicher 2006] A. Nica and R. Speicher, *Lectures on the combinatorics of free probability*, London Mathematical Society Lecture Note Series **335**, Cambridge University Press, 2006. MR Zbl

[Skoufranis 2015] P. Skoufranis, "Independences and partial R-transforms in bi-free probability", *Ann. Inst. Henri Poincaré Probab. Stat.* (Online publication May 2015).

[Speicher 1994] R. Speicher, "Multiplicative functions on the lattice of noncrossing partitions and free convolution", *Math. Ann.* **298**:4 (1994), 611–628. MR Zbl

[Voiculescu 1986] D.-V. Voiculescu, "Addition of certain noncommuting random variables", *J. Funct. Anal.* **66**:3 (1986), 323–346. MR Zbl

[Voiculescu 1987] D.-V. Voiculescu, "Multiplication of certain noncommuting random variables", *J. Operator Theory* **18**:2 (1987), 223–235. MR Zbl

[Voiculescu 2014] D.-V. Voiculescu, "Free probability for pairs of faces, I", *Comm. Math. Phys.* **332**:3 (2014), 955–980. MR Zbl

[Voiculescu 2015] D.-V. Voiculescu, "Free probability for pairs of faces, III: 2-Variables bi-free partial S- and T-transforms", preprint, 2015. arXiv

[Voiculescu 2016] D.-V. Voiculescu, "Free probability for pairs of faces, II: 2-Variables bi-free partial R-transform and systems with rank ≤ 1 commutation", *Ann. Inst. Henri Poincaré Probab. Stat.* **52**:1 (2016), 1–15. MR Zbl

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