THE NEF CONES OF AND MINIMAL-DEGREE CURVES IN THE HILBERT SCHEMES OF POINTS ON CERTAIN SURFACES

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We determine the nef cones of the Hilbert schemes of points on certain surfaces \( X \) with \( h^1(X, \mathcal{O}_X) = 0 \). Then we apply the results to Hirzebruch surfaces, and study the minimal-degree curves in the Hilbert schemes of points on Hirzebruch surfaces. Our results generalize those in Li, Qin, and Zhang (2003).

1. Introduction

Hilbert schemes are classical objects in algebraic geometry, and have been studied extensively since their constructions by Grothendieck. Hilbert schemes of points on smooth surfaces are known to be smooth and irreducible, and have deep connections with combinatorics, representation theory and string theory. Ample divisors on these Hilbert schemes were considered in [Beltrametti and Sommese 1991; 1993; Catanese and Göttzsche 1990]. The nef cones of the Hilbert schemes of points on smooth surfaces were first investigated in [Li et al. 2003] when the surface is the projective plane. Recently, these nef cones were further understood in [Arcara et al. 2013; Bertram and Coskun 2013; Bolognese et al. 2015] via Bridgeland stability.

In this paper, we generalize the methods and results in [Li et al. 2003], and prove a structure theorem for the nef cones of the Hilbert schemes of points on certain surfaces. To state our result, let \( X \) be a smooth projective complex surface. The nef cone of \( X \) is the span of the nef divisors on \( X \). We use \( \text{NE}(X) \) to denote the cone spanned by all the effective curves in \( X \). It is well-known that \( \text{NE}(X) \) is dual to the nef cone of \( X \). Let \( X^{[n]} \) be the Hilbert scheme of points in \( X \). By [Fogarty 1968; Iarrobino 1977], \( X^{[n]} \) is a smooth irreducible variety of dimension \( 2n \).

**Theorem 1.1.** Let \( n \geq 2 \), and let the surface \( X \) satisfy \( h^1(X, \mathcal{O}_X) = 0 \). Assume that the nef cone of \( X \) is the span of the divisors \( F_1, \ldots, F_t \), and the cone \( \text{NE}(X) \) is the

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span of the curves $C_1, \ldots, C_t$ with $F_i \cdot C_j = \delta_{i,j}$ for all $i$ and $j$. Assume further that
\[ O_X \left( (n - 1) \sum_{i=1}^{t} F_i \right) \]
is $(n - 1)$-very ample. Then

(i) the nef cone of the Hilbert scheme $X^{[n]}$ is spanned by

\[ D_{F_1}, \ldots, D_{F_t}, (n - 1) \sum_{i=1}^{t} D_{F_i} - B_n/2; \]

(ii) the cone $\text{NE}(X^{[n]})$ is spanned by the classes

\[ \beta_{C_1} - (n - 1)\beta_n, \ldots, \beta_{C_t} - (n - 1)\beta_n, \beta_n. \]

In the above theorem, $B_n$ denotes the boundary divisor of the Hilbert scheme $X^{[n]}$ consisting of the elements $\xi \in X^{[n]}$ which are not smooth as subschemes of $X^{[n]}$, and $\beta_n$ is the minimal curve class contracted by the Hilbert–Chow morphism $X^{[n]} \to X^{(n)}$ sending an element $\xi \in X^{[n]}$ to its support (with multiplicities) in the $n$-th symmetric product $X^{(n)}$ of $X$. We refer to (2-4), (2-3) and Definition 2.1 for the definitions of $D_F$, $\beta_C$ and $(n - 1)$-very ampleness, respectively. Theorem 1.1 is proved in Section 2. Our main idea in the proof of Theorem 1.1 is to construct curves in $X^{[n]}$ which provide us with information about the nef divisors in $X^{[n]}$.

In Section 3, we apply Theorem 1.1 to the case when $X$ is a Hirzebruch surface, and recover a result in [Bertram and Coskun 2013]. Moreover, when $X$ is a Hirzebruch surface, we classify all the curves in the Hilbert scheme $X^{[n]}$ whose homology classes are contained in the list (1-2). These curves have minimal degree in the sense that their intersection numbers with certain very ample divisors in $X^{[n]}$ are all equal to 1. We compute the normal bundles of these curves, and prove that their moduli spaces are unobstructed, i.e., are smooth with the expected dimensions.

Conventions. Let $0 \leq k \leq n$ and $V$ be an $n$-dimensional vector space. We use the Grassmannian $\mathbb{G}(V, k)$ to denote the set of all $k$-dimensional quotients of $V$, or equivalently, the set of all $(n - k)$-dimensional subspaces of $V$. Also, we take $\mathbb{P}(V) = \mathbb{G}(V, 1)$. So the set of lines in $\mathbb{P}(V)$ is the Grassmannian $\mathbb{G}(V, 2)$.

2. The nef cones of the Hilbert schemes of points on certain surfaces

In this section, we study the nef cones of the Hilbert schemes of points on certain surfaces with $h^1(X, \mathcal{O}_X) = 0$. Our goal is to prove Theorem 1.1.

Let $X$ be a smooth projective complex surface, and $X^{[n]}$ be the Hilbert scheme of points in $X$. An element in $X^{[n]}$ is represented by a length-$n$ 0-dimensional closed
subscheme $\xi$ of $X$. For $\xi \in X^{[n]}$, let $I_{\xi}$ be the corresponding sheaf of ideals and $O_{\xi}$ the structure sheaf. The subset

$$(2-1) \quad B_n = \{ \xi \in X^{[n]} \mid |\text{Supp}(\xi)| < n \}$$

is defined to be the boundary of $X^{[n]}$. Let $C$ be a real surface in $X$, and fix distinct points $x_1, \ldots, x_{n-1} \in X$ which are not contained in $C$. Define

$$(2-2) \quad \beta_n = \{ \xi + x_2 + \cdots + x_{n-1} \in X^{[n]} \mid \text{Supp}(\xi) = \{x_1\} \},$$

$$(2-3) \quad \beta_C = \{ x + x_1 + \cdots + x_{n-1} \in X^{[n]} \mid x \in C \},$$

$$(2-4) \quad D_C = \{ \xi \in X^{[n]} \mid \text{Supp}(\xi) \cap C \neq \emptyset \}.$$

Note that $\beta_C$ is a curve, and $D_C$ a divisor, in $X^{[n]}$ when $C$ is a smooth algebraic curve in $X$. We extend the notions $\beta_C$ and $D_C$ to all the divisors $C$ in $X$ by linearity.

Nakajima [1997] and Grojnowski [1996] geometrically constructed a Heisenberg algebra action on the cohomology of the Hilbert schemes $X^{[n]}$. Let $H^*(X)$ be the total cohomology of $X$ with $\mathbb{C}$-coefficients. Denote the Heisenberg operators of Nakajima and Grojnowski by $a_m(\alpha)$ where $m \in \mathbb{Z}$ and $\alpha \in H^*(X)$. Set

$$H_X = \bigoplus_{n=0}^{+\infty} H^*(X^{[n]}).$$

Then the space $H_X$ is an irreducible representation of the Heisenberg algebra generated by the operators $a_m(\alpha)$ with the highest weight vector being

$$|0\rangle = 1 \in H^*(X^{[0]}) = \mathbb{C}.$$

It follows that the $n$-th component $H^*(X^{[n]})$ in the Fock space $H_X$ is linearly spanned by the Heisenberg monomial classes

$$a_{-n_1}(\alpha_1) \cdots a_{-n_k}(\alpha_k)|0\rangle,$$

where $k \geq 0$, $n_1, \ldots, n_k > 0$ and $n_1 + \cdots + n_k = n$. We have

$$(2-5) \quad \beta_n = a_{-2}(x)a_{-1}(x)^{n-2}|0\rangle,$$

$$(2-6) \quad \beta_C = a_{-1}(C)a_{-1}(x)^{n-1}|0\rangle,$$

$$(2-7) \quad B_n = \frac{1}{(n-2)!} a_{-1}(1_X)^{n-2}a_{-2}(1_X)|0\rangle,$$

$$(2-8) \quad D_C = \frac{1}{(n-1)!} a_{-1}(1_X)^{n-1}a_{-1}(C)|0\rangle,$$

where $x$ and $1_X$ denote the cohomology classes corresponding to a point $x \in X$ and the surface $X$, respectively. Abusing notation, we also use $C$ to denote the cohomology class corresponding to the real surface $C$. 
The following important definition is from [Beltrametti and Sommese 1991].

**Definition 2.1.** Let \( n \geq 1 \). A line bundle \( L \) on the surface \( X \) is \((n - 1)\)-very ample if the restriction \( H^0(X, L) \to H^0(X, \mathcal{O}_\xi \otimes L) \) is surjective for every \( \xi \in X^{[n]} \).

The concept of \((n - 1)\)-very ampleness relates \( X^{[n]} \) to a Grassmannian as follows. The surjective map in **Definition 2.1** represents an element in \( \mathcal{G}(H^0(X, L), n) \). So if \( L \) is \((n - 1)\)-very ample, then we obtain a morphism

\[
\phi_n(L) : X^{[n]} \to \mathcal{G}(H^0(X, L), n).
\]

Let \( h = h^0(X, L) \), and let \( \mathcal{P} : \mathcal{G}(\mathbb{C}^h, n) \to \mathbb{P}((\wedge^{h-n} \mathbb{C}^h)^*) \) be the Plücker embedding. Then we see from the Appendix of [Beltrametti and Sommese 1991] that

\[
(2-10) \quad (\mathcal{P} \circ \phi_n(L))^* \mathcal{H} = \mathcal{O}_{X^{[n]}}(D_{c_1(L)} - B_n/2),
\]

where \( \mathcal{H} \) is the hyperplane line bundle over the projective space \( \mathbb{P}((\wedge^{h-n} \mathbb{C}^h)^*) \).

**Lemma 2.2.** If \( L \) is \((n - 1)\)-very ample, then the divisor \( D_{c_1(L)} - B_n/2 \) is nef. If \( L \) is \( n \)-very ample, then the divisor \( D_{c_1(L)} - B_n/2 \) is very ample. \( \square \)

The first statement in **Lemma 2.2** follows immediately from (2-10), and the second statement was proved in [Catanese and Göttsche 1990].

In \( X^{[n]} \times X \), we have the universal codimension-2 subscheme

\[
(2-11) \quad Z_n = \{ (\xi, x) \in X^{[n]} \times X \mid x \in \text{Supp}(\xi) \} \subset X^{[n]} \times X.
\]

Define the incidence variety \( X^{[n-1,n]} = \{ (\xi, \eta) \in X^{[n-1]} \times X^{[n]} \mid \xi \subset \eta \} \). It is well-known [Cheah 1998; Tikhomirov 1994] that \( X^{[n-1,n]} \) is smooth and of dimension \( 2n \). Define

- \( f_n : X^{[n-1,n]} \to X^{[n-1]} \) with \( f_n(\xi, \eta) = \xi \),
- \( g_n : X^{[n-1,n]} \to X^{[n]} \) with \( g_n(\xi, \eta) = \eta \),
- \( \rho : X^{[n-1,n]} \to X \) with \( \rho(\xi, \eta) = \text{Supp}(I_{\xi}/I_{\eta}) \).

Set \( \phi_n = (f_n, \rho) : X^{[n-1,n]} \to X^{[n-1]} \times X \). By Proposition 2.2 in [Ellingsrud and Strømme 1998], \( \phi_n \) is the blowing-up morphism of \( X^{[n-1]} \times X \) along \( Z_{n-1} \).

Next, let \( C \) be an irreducible curve in \( X \). Let \( \xi = x_1 + \cdots + x_{n-1} \in X^{[n-1]} \), where \( x_1, \ldots, x_{n-1} \) are distinct smooth points on \( C \). Let \( (C + \xi) \) be the closure of \( (C - \text{Supp}(\xi)) + \xi \) in \( X^{[n]} \). Alternatively, consider

\[
(2-12) \quad \begin{array}{c}
\tilde{C}_\xi \\
\downarrow \quad \subset \quad \tilde{X}_\xi \\
\downarrow \quad \subset \quad X^{[n-1,n]} \quad \xrightarrow{g_n} \quad X^{[n]} \\
\{\xi\} \times C \\
\downarrow \quad \subset \quad \{\xi\} \times X \\
\downarrow \quad \phi_n \\
X^{[n-1,n]} \times X
\end{array}
\]

where \( \tilde{C}_\xi \) and \( \tilde{X}_\xi \) are the strict transforms of \( \{\xi\} \times C \) and \( \{\xi\} \times X \) in \( X^{[n-1,n]} \), respectively. Since \( \phi_n \) is the blowing-up morphism of \( X^{[n-1]} \times X \) along \( Z_{n-1} \), it
follows that \( \tilde{X}_\xi \) is isomorphic to the blowup of \( \{ \xi \} \times X \cong X \) at \( x_1, \ldots, x_{n-1} \). For \( 1 \leq i \leq (n-1) \), let \( E_i \) be the exceptional divisor in \( \tilde{X}_\xi \) over \( x_i \). Then we obtain

\[
(2-13) \quad (\phi_n|_{\tilde{X}_\xi})^*([\xi] \times C) = C_\xi + \sum_{i=1}^{n-1} E_i
\]

in the Chow group \( A_1(\tilde{X}_\xi) \). Notice that \( g_n(C_\xi) = (C + \xi) \) and

\[
g_n(E_i) = M_2(x_i) + x_1 + \cdots + x_{i-1} + x_{i+1} + \cdots + x_{n-1}.
\]

In fact, since \( g_n|_{\tilde{X}_\xi}: \tilde{X}_\xi \rightarrow g_n(\tilde{X}_\xi) \) is an isomorphism, we have

\[
(2-14) \quad (g_n|_{\tilde{X}_\xi})^*(C_\xi) = (C + \xi) \quad \text{and} \quad (g_n|_{\tilde{X}_\xi})^*(E_i) = \beta_n.
\]

**Lemma 2.3.** With the above notation, \( (C + \xi) = \beta_C - (n-1)\beta_n \) in \( A_1(X^{[n]}) \).

**Proof.** Choose two smooth curves \( C_1 \) and \( C_2 \) in \( X \) such that \( C = C_1 - C_2 \) in \( A_1(X) \) and \( \text{Supp}(\xi) \cap (C_1 \cup C_2) = \emptyset \). Then in \( A_1(X^{[n]}) \), we have

\[
(2-15) \quad (g_n|_{\tilde{X}_\xi})^*(\phi_n|_{\tilde{X}_\xi})^*([\xi] \times C) = (g_n|_{\tilde{X}_\xi})^*(\phi_n|_{\tilde{X}_\xi})^*([\xi] \times C_1) - (g_n|_{\tilde{X}_\xi})^*(\phi_n|_{\tilde{X}_\xi})^*([\xi] \times C_2) = (C_1 + \xi) - (C_2 + \xi) = \beta_{C_1} - \beta_{C_2} = \beta_C.
\]

On the other hand, applying (2-13) and (2-14), we conclude that

\[
(g_n|_{\tilde{X}_\xi})^*(\phi_n|_{\tilde{X}_\xi})^*([\xi] \times C) = (g_n|_{\tilde{X}_\xi})^*(C_\xi + \sum_{i=1}^{n-1} E_i) = (C + \xi) + (n-1)\beta_n.
\]

Combining this with (2-15), we see that \( (C + \xi) = \beta_C - (n-1)\beta_n \) in \( A_1(X^{[n]}) \). \( \square \)

**Lemma 2.4.** Let \( F \) be a divisor on \( X \). If \( D_F - d(B_n/2) \) is nef, then \( d \geq 0 \) and \( F \cdot C \geq d(n-1) \) for every irreducible curve \( C \subset X \). In particular, \( F \) is nef.

**Proof.** Note that \( D_F \cdot \beta_n = 0 \) and \( B_n \cdot \beta_n = -2 \). Thus, we have

\[
0 \leq (D_F - d(B_n/2)) \cdot \beta_n = d.
\]

Since \( D_F \cdot \beta_C = F \cdot C \) and \( B_n \cdot \beta_C = 0 \), we conclude from **Lemma 2.3** that

\[
0 \leq (D_F - d(B_n/2)) \cdot (\beta_C - (n-1)\beta_n) = F \cdot C - d(n-1).
\]

**Lemma 2.5.** Let \( F \) be a divisor in \( X \). Let \( C \) be a smooth rational curve in \( X \), and consider the \( n \)-th symmetric product \( C^{(n)} = \text{Hilb}^n(C) \subset X^{[n]} \). Then

(i) every line in \( C^{(n)} \cong \mathbb{P}^n \) is homologous to \( \beta_C - (n-1)\beta_n \);

(ii) \( \mathcal{O}_{X^{[n]}}(D_F)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(C \cdot F) \) and \( \mathcal{O}_{X^{[n]}}(B_n/2)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(n-1) \).
\textbf{Proof.} (i) Let $x_1, \ldots, x_{n-1} \in C$ be distinct, and put $\gamma = C + (x_1 + \cdots + x_{n-1})$. Then $\gamma$ is a line in the projective space $C^{(n)} \cong \mathbb{P}^n$. By \textbf{Lemma 2.3},

$$\gamma \sim \beta_C - (n-1)\beta_n,$$

where $\sim$ denotes homologous relation. So every line in $C^{(n)} \cong \mathbb{P}^n$ is homologous to the class $\beta_C - (n-1)\beta_n$.

(ii) Since $\gamma \cdot D_F|_{C^{(n)}} = \gamma \cdot D_F = (\beta_C - (n-1)\beta_n) \cdot D_F = C \cdot F$, we get

$$\mathcal{O}_{X^{[n]}}(D_F)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(C \cdot F).$$

Using a similar method, we obtain $\mathcal{O}_{X^{[n]}}(B_n/2)|_{C^{(n)}} = \mathcal{O}_{C^{(n)}}(n-1)$. \hfill $\square$

In the rest of the paper, we assume that $h^1(X, \mathcal{O}_X) = 0$. Then

(2-16) \hspace{1cm} \text{Pic}(X^{[n]}) \cong \text{Pic}(X) \oplus \mathbb{Z} \cdot (B_n/2)

by [Fogarty 1973]. Under this isomorphism, the divisor $D_C \in \text{Pic}(X^{[n]})$ corresponds to $C \in \text{Pic}(X)$. Let $\{\alpha_1, \ldots, \alpha_s\}$ be a linear basis of $H^2(X)$. Then

(2-17) \hspace{1cm} \{D_{\alpha_1}, \ldots, D_{\alpha_s}, B_n\}

is a linear basis of $H^2(X^{[n]})$. Represent $\alpha_1, \ldots, \alpha_s$ by real surfaces $C_1, \ldots, C_s \subset X$, respectively. Then a linear basis of $H_2(X^{[n]})$ is given by

(2-18) \hspace{1cm} \{\beta_{C_1}, \ldots, \beta_{C_s}, \beta_n\}.

We are now ready to prove our main result in this paper.

\textbf{Theorem 2.6.} Let $n \geq 2$, and let the surface $X$ satisfy $h^1(X, \mathcal{O}_X) = 0$. Assume that the nef cone of $X$ is the span of the divisors $F_1, \ldots, F_t$, and the cone NE($X$) is the span of the curves $C_1, \ldots, C_t$ with $F_i \cdot C_j = \delta_{i,j}$ for all $i$ and $j$. Assume further that $\mathcal{O}_X((n-1) \sum_{i=1}^t F_i)$ is $(n-1)$-very ample. Then

(i) the nef cone of the Hilbert scheme $X^{[n]}$ is spanned by

(2-19) \hspace{1cm} D_{F_1}, \ldots, D_{F_t}, (n-1) \sum_{i=1}^t D_{F_i} - B_n/2;

(ii) the cone NE($X^{[n]}$) is spanned by the classes

(2-20) \hspace{1cm} \beta_{C_1} - (n-1)\beta_n, \ldots, \beta_{C_t} - (n-1)\beta_n, \beta_n.

\textbf{Proof.} (i) For $1 \leq j \leq n$, let $p_j : X^n \rightarrow X$ be the projection to the $j$-th factor. Let $X^{(n)}$ be the $n$-th symmetric product of $X$, and let $\nu_n : X^n \rightarrow X^{(n)}$ be the quotient map. Let $\rho_n : X^{[n]} \rightarrow X^{(n)}$ be the Hilbert–Chow morphism sending an element
\[ \xi \in X^{[n]} \] to its support (with multiplicities) in \( X^{(n)} \). For each \( F_i \), there exists a divisor \( H_i \) on \( X^{(n)} \) such that

\[
\rho_n^* H_i = D_{F_i}, \quad \nu_n^* H_i = \sum_{j=1}^n p_j^* F_i.
\]

It follows that since \( F_i \) is nef, the divisor \( D_{F_i} \) is nef as well. Since the line bundle \( O_X((n-1) \sum_{i=1}^t F_i) \) is \((n-1)\)-very ample, we conclude from Lemma 2.2 that \((n-1) \sum_{i=1}^t D_{F_i} - B_n/2 \) is a nef divisor. Thus, the cone \( C_1 \) spanned by the divisors in (2-19) is contained in the nef cone of \( X^{[n]} \).

Conversely, assume that \( D_F - d B_n/2 \) is a nef divisor on \( X^{[n]} \). Let \( F = \sum_{i=1}^t a_i F_i \).

By Lemma 2.4, \( d \geq 0 \) and \( F \cdot C \geq d(n-1) \) for every irreducible curve \( C \subset X \). So

\[
a_i = F \cdot C_i \geq d(n-1)
\]

for every \( i \). Now the nef divisor \( D_F - d B_n/2 \) can be written as

\[
\sum_{i=1}^t a_i D_{F_i} - d B_n/2 = \sum_{i=1}^t (a_i - d(n-1)) D_{F_i} + d \left( (n-1) \sum_{i=1}^t D_{F_i} - B_n/2 \right).
\]

Therefore, \( D_F - d B_n/2 \in C_1 \). It follows that \( C_1 \) is the nef cone of \( X^{[n]} \).

(ii) First of all, note that since the divisor \( F_i \) is nef and \( F_i \cdot C_i = 1 \), the curve \( C_i \) contains at least one reduced irreducible component. So the curve \( \beta_{C_i} - (n-1)\beta_n \) is well-defined, and the cone \( C_2 \) spanned by the curves in (2-20) is contained in the cone \( \text{NE}(X^{[n]}) \). Conversely, assume that \( \sum_{i=1}^t b_i \beta_{C_i} + c \beta_n \) is contained in \( \text{NE}(X^{[n]}) \). Then \( \left( \sum_{i=1}^t b_i \beta_{C_i} + c \beta_n \right) \cdot H \geq 0 \) for every nef divisor \( H \) in \( X^{[n]} \). Letting \( H = D_{F_i} \), we get \( b_i \geq 0 \) for every \( i \). Letting \( H = (n-1) \sum_{i=1}^t D_{F_i} - B_n/2 \), we obtain \( (n-1) \sum_{i=1}^t b_i + c \geq 0 \). Therefore, we have

\[
\sum_{i=1}^t b_i \beta_{C_i} + c \beta_n = \sum_{i=1}^t b_i (\beta_{C_i} - (n-1) \beta_n) + \left( (n-1) \sum_{i=1}^t b_i + c \right) \beta_n \in C_2.
\]

It follows that \( C_2 \) coincides with the cone \( \text{NE}(X^{[n]}) \).

**Corollary 2.7.** Under the same assumptions as in Theorem 2.6, if \( \gamma \) is an irreducible curve in the Hilbert scheme \( X^{[n]} \), then \( \gamma \) is homologous to

\[
\sum_{i=1}^t b_i (\beta_{C_i} - (n-1) \beta_n) + c \beta_n
\]

for some nonnegative integers \( b_1, \ldots, b_t, c \) not all equal to zero.

**Proof.** By Theorem 2.6 (ii), \( b_1, \ldots, b_t, c \) are nonnegative and not all equal to 0. Intersecting the curve \( \gamma \) with the divisors \( D_{F_1}, \ldots, D_{F_t} \) and \( (n-1) \sum_{i=1}^t D_{F_i} - B_n/2 \), respectively, we see that \( b_1, \ldots, b_t, c \) must be integers. \( \square \)
3. Application to Hirzebruch surfaces

In this section, we apply Theorem 2.6 to the Hirzebruch surfaces and recover a result in [Bertram and Coskun 2013]. Then we study the curves in the Hilbert schemes of points on the Hirzebruch surfaces, which have the minimal degree. We compute their normal bundles, and prove that their moduli spaces are unobstructed.

Let $X$ denote the Hirzebruch surface $\mathbb{F}_e$ with $e \geq 0$. Let $f$ be a fiber of the ruling $\pi : X \to \mathbb{P}^1$, and $\sigma \subset X$ be a section of $\pi$ such that $\sigma^2 = -e$. Then

$$\text{Pic}(X) = \mathbb{Z} \cdot \sigma \oplus \mathbb{Z} \cdot f.$$ 

It is well-known that $a\sigma + bf$ is nef if and only if $a \geq 0$ and $b \geq ae$. The following lemma was proved in [Beltrametti and Sommese 1993].

Lemma 3.1. $O_X(a\sigma + bf)$ is $n$-very ample if and only if $a \geq n$ and $b \geq n + ae$. □

Proposition 3.2. Let $n \geq 2$, and let $X$ be the Hirzebruch surface $\mathbb{F}_e$. Then

(i) the nef cone of the Hilbert scheme $X^{[n]}$ is spanned by

(3-1) $D_f, \quad D_\sigma + eD_f, \quad (n-1)D_\sigma + (n-1)(1+e)D_f - B_n/2$;

(ii) the cone $\text{NE}(X^{[n]})$ is spanned by the classes

(3-2) $\beta_\sigma - (n-1)\beta_n, \quad \beta_f - (n-1)\beta_n, \quad \beta_n$.

Proof. The nef cone of $X$ is the span of $F_1 = f$ and $F_2 = \sigma + ef$, and the cone $\text{NE}(X)$ is the span of $C_1 = \sigma$ and $C_2 = f$. Note that $F_i \cdot C_j = \delta_{i,j}$ for all $i$ and $j$. In addition, by Lemma 3.1, the line bundle $O_X((n-1)F_1 + (n-1)F_2)$ is $(n-1)$-very ample. Hence our proposition follows from Theorem 2.6. □

Proposition 3.2 has been proved in [Bertram and Coskun 2013]. We now study the curves in $X^{[n]}$ whose homology classes are contained in the list (3-2). Let

$$L_n = n\sigma + n(1+e)f.$$ 

By Lemma 3.1, the line bundle $O_X(L_n)$ is $n$-very ample. By Lemma 2.2, the divisor

$$D_{L_n} - B_n/2 = nD_\sigma + n(1+e)D_f - B_n/2$$

in $X^{[n]}$ is very ample. We next lemma characterizes the homology classes in (3-2).

Lemma 3.3. Let $\gamma$ be a curve in $X^{[n]}$ with $\gamma \cdot (nD_\sigma + n(1+e)D_f - B_n/2) = 1$. Then $\gamma$ is a smooth rational curve. Moreover, $\gamma \sim \beta_n, \beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$.

Proof. Since $nD_\sigma + n(1+e)D_f - B_n/2$ is very ample, $\gamma$ is a smooth rational curve. By Corollary 2.7, $\gamma \sim a(\beta_\sigma - (n-1)\beta_n) + b(\beta_f - (n-1)\beta_n) + c\beta_n$ for some nonnegative integers $a, b, c$. Since $\gamma \cdot (nD_\sigma + n(1+e)D_f - B_n/2) = 1$, we obtain

$$a + b + c = 1.$$ 

Therefore, we have $\gamma \sim \beta_n, \beta_f - (n-1)\beta_n$ or $\beta_\sigma - (n-1)\beta_n$. □
The curves in $X^{[n]}$ homologous to $\beta_n$ have been classified in [Li et al. 2003]. In the rest of this section, we study the curves $\gamma \subset X^{[n]}$ which are homologous to $\beta_f - (n - 1)\beta_n$ or $\beta_\sigma - (n - 1)\beta_n$. By Lemma 3.1, the line bundle $O_X(L_{n-1})$ is $(n - 1)$-very ample. So by (2-9), we have the morphism

$$
\varphi := \varphi_n(O_X(L_{n-1})) : X^{[n]} \to \mathbb{P}(H^0(X, O_X(L_{n-1})), n).
$$

**Lemma 3.4.** Let $\gamma$ be an irreducible curve in $X^{[n]}$ satisfying

$$
\gamma \cdot ((n - 1)D_\sigma + (n - 1)(1 + e)D_f - B_n/2) = 0.
$$

Then $\gamma \sim \beta_f - (n - 1)\beta_n$ or $\beta_\sigma - (n - 1)\beta_n$. Moreover, $\gamma$ is contracted by $\varphi$.

**Proof.** The first part of the lemma is proved by an argument similar to the proof of Lemma 3.3. For the second part, we notice from (2-10) that

$$
(\mathfrak{P} \circ \varphi)^* \mathcal{H} = O_{X^{[n]}}(D_{L_{n-1}} - B_n/2)
= O_{X^{[n]}}((n - 1)D_\sigma + (n - 1)(1 + e)D_f - B_n/2).
$$

Therefore, the curve $\gamma$ is contracted by the morphism $\mathfrak{P} \circ \varphi$. Since $\mathfrak{P}$ is an embedding, the curve $\gamma$ is contracted by the morphism $\varphi$. \(\square\)

In the following, we fix a curve $\gamma \subset X^{[n]}$ homologous to $\beta_f - (n - 1)\beta_n$ or $\beta_\sigma - (n - 1)\beta_n$. Let $X^{(n)}$ be the $n$-th symmetric product of $X$ and $\nu_n : X^n \to X^{(n)}$ the quotient map. Let $\rho_n : X^{[n]} \to X^{(n)}$ be the Hilbert–Chow morphism sending an element $\xi \in X^{[n]}$ to its support (with multiplicities) in $X^{(n)}$. Let $p_1$ be the projection from $X^n$ to the first factor.

**Definition 3.5.** Define $C_\gamma$ to be the union of all the curves in $p_1(\nu_n^{-1}(\rho_n(\gamma)))$.

**Lemma 3.6.** Let $\gamma \sim \beta_f - (n - 1)\beta_n$ or $\beta_\sigma - (n - 1)\beta_n$. Then $C_\gamma \sim \sigma$ or $f$.

**Proof.** First of all, we claim that $C_\gamma \neq \emptyset$. Indeed, if $C_\gamma = \emptyset$, then $p_1(\nu_n^{-1}(\rho_n(\gamma)))$ is a finite set of points in $X$. Since the divisor $\sigma + (1 + e)f$ is very ample, we can choose a smooth curve $F \in |\sigma + (1 + e)f|$ such that $F \cap p_1(\nu_n^{-1}(\rho_n(\gamma))) = \emptyset$. Since the elements of $\gamma$ are supported in $p_1(\nu_n^{-1}(\rho_n(\gamma)))$, we must have $\gamma \cap D_F = \emptyset$. It follows that $\gamma \cdot D_F = 0$. However, this contradicts $\gamma \cdot D_F = 1$.

Next, assume that $C_\gamma \cdot (\sigma + (1 + e)f) \geq 2$. Take a point $\xi \in \gamma$ and a smooth point $x \in C_\gamma$ such that $x \notin \text{Supp}(\xi)$. Since $x \in C_\gamma \subset p_1(\nu_n^{-1}(\rho_n(\gamma)))$, there exists $\xi_\chi \in \gamma$ such that $\rho_n(\xi_\chi) = n_\chi x + \eta_\chi$, where $n_\chi \geq 1$, $\eta_\chi \in X^{(n-n_\chi)}$ and $x \notin \text{Supp}(\eta_\chi)$. Choose a smooth curve $F \in |\sigma + (1 + e)f|$ missing $\text{Supp}(\eta_\chi) \cup \text{Supp}(\xi)$, passing through $x$, and intersecting $C_\gamma$ transversally. Then $F \cap C_\gamma$ is a finite set. Since $C_\gamma \cdot F \geq 2$, $F \cap C_\gamma$ contains one more point $y \neq x$. Hence there exists $\xi_y \in \gamma$ with $y \in \text{Supp}(\xi_y)$. Thus $\xi_\chi, \xi_y \in \gamma \cap D_F$. Since $y \neq x$, $y \in F$ and $F$ misses $\text{Supp}(\eta_\chi)$, we get

$$
y \notin \{x\} \cup \text{Supp}(\eta_\chi) = \text{Supp}(\xi_\chi).
$$
So $\xi_x \neq \xi_y$. Since $\text{Supp}(\xi) \cap F = \emptyset$, we have $\xi \not\in D_F$. Since $\xi \in \gamma$ and $\gamma$ is a smooth rational curve, $\gamma$ is not contained in $D_F$. Therefore, $\gamma \cap D_F$ is a finite set of points. Since $\xi_x, \xi_y \in \gamma \cap D_F$ and $\xi_x \neq \xi_y$, we obtain $\gamma \cdot D_F \geq 2$, which contradicts $\gamma \cdot D_F = 1$.

It follows that $C_\gamma \cdot (\sigma + (1 + e)f) = 1$. Since the cone $\text{NE}(\mathcal{F}_e)$ is spanned by $\sigma$ and $f$, we conclude that $C_\gamma \sim \sigma$ or $f$. \hfill $\square$

By Lemma 3.6, $C_\gamma \sim \sigma$ or $f$. So $C_\gamma$ is a smooth rational curve, and

$$O_X(L_{n-1}|_{C_\gamma}) \cong O_{C_\gamma}(n - 1).$$

Let $V_{C_\gamma} \subset H^0(X, O_X(L_{n-1}))$ be the image of the injection

$$H^0(X, O_X(L_{n-1} - C_\gamma)) \to H^0(X, O_X(L_{n-1})),
$$

which is induced by the exact sequence

(3-3) \hspace{1cm} 0 \to O_X(L_{n-1} - C_\gamma) \to O_X(L_{n-1}) \to O_{C_\gamma}(n - 1) \to 0.

Similarly, for $\xi \in \gamma$, let $V_\xi \subset H^0(X, O_X(L_{n-1}))$ be the image of the injection

$$H^0(X, O_X(L_{n-1} \otimes I_\xi)) \to H^0(X, O_X(L_{n-1})).$$

Since $O_X(L_{n-1})$ is $(n - 1)$-very ample, we obtain

(3-4) \hspace{1cm} \dim V_\xi = h^0(X, O_X(L_{n-1})) - h^0(O_\xi) = h^0(X, O_X(L_{n-1})) - n.

Since the curve $\gamma$ is contracted to a point by the morphism $\varphi$, the subspaces $V_\xi$ of $H^0(X, O_X(L_{n-1}))$ are independent of $\xi \in \gamma$. Set $V_\gamma = V_\xi$ where $\xi \in \gamma$.

**Lemma 3.7.** If $n \geq e + 1$, then $V_{C_\gamma} = V_\gamma$.

**Proof.** Since $K_X = -2\sigma - (2 + e)f$, the divisor $L_{n-1} - C_\gamma - K_X$ is ample if $C_\gamma = \sigma$. Similarly, since $n \geq e + 1$, $L_{n-1} - C_\gamma - K_X$ is ample if $C_\gamma = f$. By the Kodaira vanishing theorem, $H^1(X, O_X(L_{n-1} - C_\gamma)) = 0$. So we see from (3-3) that

$$\dim V_{C_\gamma} = h^0(X, O_X(L_{n-1})) - h^0(C_\gamma, O_{C_\gamma}(n - 1)) = h^0(X, O_X(L_{n-1})) - n.$$

In view of (3-4), we conclude that

$$\dim V_{C_\gamma} = \dim V_\xi = \dim V_\gamma.$$

Thus, to prove our lemma, it remains to prove that $V_\gamma \subset V_{C_\gamma}$. Indeed, let $f \in V_\gamma$ be a section. Let $x \in C_\gamma$. Since $C_\gamma \subset p_1(v_n^{-1}(\rho_n(\gamma)))$, there exists $\xi \in \gamma$ such that $x \in \text{Supp}(\xi)$. Since $V_\gamma = V_\xi$, $f$ vanishes at every point in $\text{Supp}(\xi)$. In particular, $f$ vanishes at $x$. Hence, $f$ vanishes along the smooth curve $C_\gamma$. Therefore, $f \in V_{C_\gamma}$. It follows that $V_\gamma \subset V_{C_\gamma}$. \hfill $\square$
Proposition 3.8. Let \( n \geq \max(2, e + 1) \). Then a curve \( \gamma \subset X^{[n]} \) is homologous to \( \beta_f - (n - 1)\beta_n \) or \( \beta_\sigma - (n - 1)\beta_n \) if and only if there is a curve \( C \subset X \) such that \( C \sim \sigma \) or \( f \), and that \( \gamma \) is a line in \( \text{Hilb}^n(C) \subset X^{[n]} \). Moreover, the curve \( C \) is uniquely determined by the curve \( \gamma \).

Proof. The “if” part of the proposition follows from Lemma 2.5 (i). To prove the “only if” part, let \( \gamma \sim \beta_f - (n - 1)\beta_n \) or \( \beta_\sigma - (n - 1)\beta_n \). By Lemma 3.6, \( C := C_\gamma \sim \sigma \) or \( f \). Fix a section \( f_0 \in H^0(X, \mathcal{O}_X(C)) \) whose zero locus is \( C \). Let \( \xi \in \gamma \). Since \( \sigma + ef \) is basepoint-free, so is the divisor \( L_{n-1} - C \). Thus there exists \( f_1 \in H^0(X, \mathcal{O}_X(L_{n-1} - C)) \) such that \( f_1 \) does not vanish at any point in \( \text{Supp}(\xi) \). Now, \( f_0 \otimes f_1 \in V_\gamma \). By Lemma 3.7, \( f_0 \otimes f_1 \in V_\gamma = V_\xi \). Since \( f_1 \) does not vanish at any point in \( \text{Supp}(\xi) \), \( f_0 \) vanishes at \( \xi \). Hence, \( \xi \) is a closed subscheme of \( C \). It follows that \( \gamma \subset \text{Hilb}^n(C) \).

To show that \( \gamma \) is a line in \( \text{Hilb}^n(C) \subset X^{[n]} \), let \( F = \sigma + (e + 1)f \). By Lemma 2.5 (ii), \( \mathcal{O}_{X^{[n]}}(D_F)|_{\text{Hilb}^n(C)} = \mathcal{O}_{\text{Hilb}^n(C)}(1) \). So viewing \( \gamma \) as a curve in \( \text{Hilb}^n(C) \), we obtain

\[
\gamma \cdot c_1(\mathcal{O}_{\text{Hilb}^n(C)}(1)) = \gamma \cdot D_F = 1.
\]

Therefore, \( \gamma \) is a line in \( \text{Hilb}^n(C) \subset X^{[n]} \).

Finally, the uniqueness of \( C \) follows from the observation that if \( \xi \in X^{[n]} \) and \( n \geq 2 \), then \( \xi \) is contained in at most one curve \( C \subset X \) with \( C \sim \sigma \) or \( f \). \( \square \)

Next, we determine the normal bundle of a curve \( \gamma \) in \( X^{[n]} \) homologous to \( \beta_f - (n - 1)\beta_n \). By Proposition 3.8, there exists a unique fiber \( f_\gamma \) in \( X \) such that \( \gamma \) is a line in the \( n \)-th symmetric product \( f_\gamma^{(n)} = \text{Hilb}^n(f_\gamma) \subset X^{[n]} \). In particular,

\[
N_{\gamma \subset f_\gamma^{(n)}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)}.
\]

So we have the following exact sequence of normal bundles:

\[
0 \to \mathcal{O}_\gamma(1)^{\oplus(n-1)} \to N_{\gamma \subset X^{[n]}} \to N_{f_\gamma^{(n)} \subset X^{[n]}}|_\gamma \to 0.
\]

Lemma 3.9. Let \( n \geq \max(2, e + 1) \). Let \( \gamma \subset X^{[n]} \) be a curve homologous to \( \beta_f - (n - 1)\beta_n \), and \( f_\gamma \) be the unique fiber in \( X \) such that \( \gamma \) is a line in \( f_\gamma^{(n)} \). Then

(i) \( N_{f_\gamma^{(n)} \subset X^{[n]}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)} \oplus \mathcal{O}_\gamma(-1)^{\oplus(n-1)} \);

(ii) \( N_{\gamma \subset X^{[n]}} \cong \mathcal{O}_\gamma(1)^{\oplus(n-1)} \oplus \mathcal{O}_\gamma(1)^{\oplus(n-1)} \).

Proof. (i) First of all, let \( C \subset X \) be a smooth irreducible curve. Let \( \pi_n \) and \( q_n \) be the projections of \( X^{[n]} \times X \) to \( X^{[n]} \) and \( X \) respectively. Recall the universal codimension-2 subscheme \( \mathcal{Z}_n \subset X^{[n]} \times X \) from (2-11). By the results in [Altman et al. 1977], we have the isomorphism

\[
N_{C^{(n)} \subset X^{[n]}} \cong \pi_{n*}(q_n^*\mathcal{O}_X(C)|_{\mathcal{Z}_n})|_{C^{(n)}}.
\]
Let \( \tilde{Z}_n \) be the universal subscheme in \( C^{(n)} \times C \). Then we obtain
\[
\pi_{n*}(q_n^* O_X(C)|Z_n)|C^{(n)} \cong \tilde{\pi}_{n*}(\tilde{q}_n^* (O_X(C)|C)|\tilde{Z}_n),
\]
where \( \tilde{\pi}_n \) and \( \tilde{q}_n \) are the projections from \( C^{(n)} \times C \) to \( C^{(n)} \) and \( C \), respectively. So
\[
(3-6) \quad N_{C^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}(\tilde{q}_n^* (O_X(C)|C)|\tilde{Z}_n).
\]
Replacing \( C \) in (3-6) by \( f_y \), we get \( N_{f_y^{(n)} \subset X^{[n]}} \cong \tilde{\pi}_{n*}O_{\tilde{Z}_n} \). It is known that \( \tilde{Z}_n \subset f_y^{(n)} \times f_y \cong \mathbb{P}^n \times \mathbb{P}^1 \) is defined by the equation
\[
a_0 U^n + a_1 U^{n-1} V + \ldots + a_n V^n = 0,
\]
where \( a_0, a_1, \ldots, a_n \) and \( U, V \) are the homogeneous coordinates on \( \mathbb{P}^n \) and \( \mathbb{P}^1 \), respectively. So the line bundle \( O_{f_y^{(n)} \times f_y} (\tilde{Z}_n) \cong O_{\mathbb{P}^n \times \mathbb{P}^1} (\tilde{Z}_n) \) is of type \((1, n)\) in \( \text{Pic}(f_y^{(n)} \times f_y) \cong \text{Pic}(\mathbb{P}^n \times \mathbb{P}^1) \cong \mathbb{Z} \oplus \mathbb{Z} \).

Applying \( \tilde{\pi}_{n*} \) to the exact sequence
\[
0 \to O_{f_y^{(n)} \times f_y} (-\tilde{Z}_n) \to O_{f_y^{(n)} \times f_y} \to O_{\tilde{Z}_n} \to 0,
\]
we obtain the exact sequence
\[
0 \to O_{f_y^{(n)}} \to \tilde{\pi}_{n*}O_{\tilde{Z}_n} \to O_{f_y^{(n)}}(-1)^{\oplus(n-1)} \to 0.
\]
This exact sequence splits. Thus, \( \tilde{\pi}_{n*}O_{\tilde{Z}_n} \cong O_{f_y^{(n)}} \oplus O_{f_y^{(n)}}(-1)^{\oplus(n-1)} \). Hence
\[
N_{f_y^{(n)} \subset X^{[n]}} \cong \pi_{n*}O_{\tilde{Z}_n} \cong O_{f_y^{(n)}} \oplus O_{f_y^{(n)}}(-1)^{\oplus(n-1)}.
\]
(ii) By (i) and (3-5), we obtain the exact sequence
\[
0 \to O_{y^{(1)}}^{(n-1)} \to N_{\gamma \subset X^{[n]}} \to O_{\gamma} \oplus O_{\gamma}(-1)^{\oplus(n-1)} \to 0.
\]
Since this exact sequence splits, the proof of (ii) is complete. \( \square \)

Now we determine the normal bundle of a curve \( \gamma \) in \( X^{[n]} \) homologous to \( \beta_\sigma - (n-1)\beta_n \). Recall that the Hirzebruch surfaces \( \mathbb{F}_e \) are deformation equivalent to either \( \mathbb{F}_0 \) or \( \mathbb{F}_1 \). If \( X = \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \), then \( \sigma \) is a fiber of one of the two rulings on \( X \), so the normal bundle of a curve \( \gamma \) in \( X^{[n]} \) homologous to \( \beta_\sigma - (n-1)\beta_n \) has been computed by Lemma 3.9(ii). In the following, we concentrate on \( X = \mathbb{F}_1 \), which is the blowup of the projective plane at a point.

**Lemma 3.10.** Let \( n \geq 2 \) and \( X = \mathbb{F}_1 \). Let \( \gamma \subset X^{[n]} \) be a curve homologous to \( \beta_\sigma - (n-1)\beta_n \). Then, \( N_{\sigma^{(n)} \subset X^{[n]}} \cong O_{\sigma^{(n)}}(-1)^{\oplus n} \) and
\[
(3-7) \quad N_{\gamma \subset X^{[n]}} \cong O_{\gamma}^{(n-1)} \oplus O_{\gamma}(-1)^{\oplus n}.
\]
Applying (3-7) that a closed subvariety of homologous to \( \beta \). By Lemma 3.3, a curve \( N \) irreducible with dimension 2 or \( \beta \) the ruling on \( X \), we obtain \( \pi^*O(\beta) \) homologous to \( \beta \). By Proposition 3.8, let \( f \) be a fiber of \( X \), and (3-8), we see that \( \pi^*O(\beta) \) is irreducible and unobstructed, i.e., is smooth with the expected dimension.

By (2-10), the composition \( \varphi_n(L_n) : X^{[n]} \to \mathbb{P}^N \) (for a suitable positive integer \( N \)) is the embedding associated to the very ample divisor

\[
D_{L_n} - B_n/2 = nD_\sigma + n(1+e)D_f - B_n/2.
\]

By Lemma 3.3, a curve \( \gamma \subset X^{[n]} \) is mapped to a line in \( \mathbb{P}^N \) if and only if \( \gamma \) is homologous to \( \beta_n \), \( \beta_f - (n-1)\beta_n \) or \( \beta_\sigma - (n-1)\beta_n \). Therefore, regarding \( X^{[n]} \) as a closed subvariety of \( \mathbb{P}^N \), then the Hilbert scheme of lines in \( X^{[n]} \) is the disjoint union of \( \mathcal{M}(\beta_n) \), \( \mathcal{M}(\beta_f - (n-1)\beta_n) \) and \( \mathcal{M}(\beta_\sigma - (n-1)\beta_n) \).
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