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IWAHORI–HECKE ALGEBRAS FOR KAC–MOODY GROUPS OVER LOCAL FIELDS

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We define the Iwahori–Hecke algebra ${}^I\mathcal{H}$ for an almost split Kac–Moody group G over a local nonarchimedean field. We use the hovel $\mathscr{I}$ associated to this situation, which is the analogue of the Bruhat–Tits building for a reductive group. The fixer K_I of some chamber in the standard apartment plays the role of the Iwahori subgroup. We can define ${}^I\mathcal{H}$ as the algebra of some K_I-bi-invariant functions on G with support consisting of a finite union of double classes. As two chambers in the hovel are not always in a same apartment, this support has to be in some large subsemigroup G^+ of G. In the split case, we prove that the structure constants of ${}^I\mathcal{H}$ are polynomials in the cardinality of the residue field, with integer coefficients depending on the geometry of the standard apartment. We give a presentation of this algebra , similar to the Bernstein–Lusztig presentation in the reductive case, and embed it in a greater algebra ${}^{BL}\mathcal{H}$, algebraically defined by the Bernstein–Lusztig presentation. In the affine case, this algebra ${}^{BL}\mathcal{H}$ contains the Cherednik's double affine Hecke algebra. Actually, our results apply to abstract "locally finite" hovels, so that we can define the Iwahori–Hecke algebra with unequal parameters.

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Introduction

A bit of history. Iwahori–Hecke algebras were first introduced in number theory by Erich Hecke [1937]. He defined an algebra, now called the Hecke algebra, generated by some operators on modular forms. Then, based on an idea of André Weil, Goro Shimura [1959] defined an algebra attached to a group containing a subgroup (under some hypotheses) as the algebra spanned by some double cosets and recovered Hecke's algebra. Nagayoshi Iwahori [1964] showed that, in the case of a Chevalley group over a finite field containing a Borel subgroup, Shimura's algebra can be defined in terms of bi-invariant functions on the group. He further gave a presentation by generators and relations of this algebra. Examples of such groups containing a suitable subgroup are given by BN-pairs and the theory of buildings. Nagayoshi Iwahori and Hideya Matsumoto [1965] found a famous instance in a Chevalley group over a p-adic field corresponding to the Bruhat–Tits building associated to the situation. In fact, it is possible to define these algebras only in terms of building theory; see, e.g., [Parkinson 2006] for a contemporary treatment.

In a previous article, Gaussent and Rousseau [2008] introduced the analogue of the Bruhat–Tits building in Kac–Moody theory, and called it, a hovel. Guy Rousseau [2011] developed the notion further and gave an axiomatic definition, applicable in a broader context.

In this paper, we first define, in terms of the hovel, the Iwahori–Hecke algebra associated to a Kac–Moody group over a local field containing the equivalent of the Iwahori subgroup. Then, we study the properties of this algebra, like the structure constants of the product, some presentations by generators and relations, and an interesting example where we recover the double affine Hecke algebras.

For the rest of the introduction, we give a more detailed account of our work.

The case of simple algebraic groups. To begin, we recall the situation in the finite dimensional case. Let $\mathcal{K}$ be a local nonarchimedean field, with residue field $\mathbb{F}_q$. Suppose G is a split, simple and simply connected algebraic group over $\mathcal{K}$ and K an open compact subgroup. The space $\mathcal{H}_K$ of complex functions on G, bi-invariant by K and with compact support, is an algebra for the natural convolution product. Ichiro Satake [1963] studied such algebras to define the spherical functions and proved, in particular, that $\mathcal{H}_K$ is commutative for a good choice K_s of K, maximal compact. The corresponding convolution algebra $\mathcal{H}_{K_s} = {}^s\mathcal{H}(G)$ is now called the spherical Hecke algebra. From [Iwahori and Matsumoto 1965], we know that there exists an interesting open subgroup K_I, so called the Iwahori subgroup, of K_s with a Bruhat decomposition $G = K_I . W . K_I$, where W is an infinite Coxeter group. The corresponding convolution algebra $\mathcal{H}_{K_I} = {}^{\mathrm{I}}\mathcal{H}(G)$, called the Iwahori–Hecke algebra, may be described as the abstract Hecke algebra associated to this Coxeter group and the parameter q. There is another presentation of this Hecke algebra, stated by Joseph Bernstein and proved in the most general case by George Lusztig [1989]. This presentation emphasizes the role of the translations in W and uses new relations, now often called the Bernstein–Lusztig relations. In the building-like definition of these algebras, the group K_s (resp., K_I) is the fixer of a special vertex (resp., a chamber) for the action of G on the Bruhat–Tits building $\mathscr{I}$ [Bruhat and Tits 1972].

The Kac–Moody setting. Kac–Moody groups are interesting generalizations of semisimple groups, hence it is natural to define the Iwahori–Hecke algebras also in the Kac–Moody setting.

So, from now on, let G be a Kac–Moody group over $\mathcal{K}$, assumed minimal or "algebraic", i.e., as studied by Jacques Tits [1987] in the split case and by Bertrand Rémy [2002] in the almost split case. Unfortunately there is, up to now, no good topology on G and no good compact subgroup, so the "convolution product" has to be defined by other means. Alexander Braverman and David Kazhdan [2011] succeeded in defining geometrically such a spherical Hecke algebra, when G is split and untwisted affine; see also the survey [Braverman and Kazhdan 2013]. We were able, in [Gaussent and Rousseau 2014], to generalize their construction to any Kac–Moody group over $\mathcal{K}$. Using results of [Garland 1995; Braverman et al. 2014], Braverman, Kazhdan and Manish Patnaik [Braverman et al. 2016] constructed the spherical Hecke algebra and the Iwahori–Hecke algebra by algebraic computations in the Kac–Moody group, still assumed split and untwisted affine (and even simply laced for some statements). Those algebras are convolution algebras of functions on G bi-invariant under some analogue group K_s or K_I (contained in K_s), but there are two new features: the support has to be in a subsemigroup G^+ of G and the description of the Iwahori–Hecke algebra has to use Bernstein–Lusztig type relations since W is no longer a Coxeter group.

Iwahori–Hecke algebras in the Kac–Moody setting. Similar to [Gaussent and Rousseau 2014], our idea is to define the Iwahori–Hecke algebra using the hovel associated to the almost split Kac–Moody group G that we built in [Gaussent and Rousseau 2008; Rousseau 2011; 2010]. This hovel $\mathscr{I}$ is a set with an action of G and a covering by subsets called apartments. They are in one-to-one correspondence with the maximal split subtori, hence permuted transitively by G. Each apartment A is a finite dimensional real affine space. Its stabilizer N in G acts on A via a generalized affine Weyl group $W = W^v \ltimes Y$, where $Y \subset \vec{A}$ is a discrete subgroup of translations. The group W stabilizes a set $\mathcal{M}$ of affine hyperplanes called walls. So, $\mathscr{I}$ looks much like the Bruhat–Tits building of a reductive group. But as the root system Φ is infinite, the set of walls $\mathcal{M}$ is not locally finite. Further, two points in $\mathscr{I}$ are not always in a same apartment. This is why $\mathscr{I}$ is called a hovel. However, there exists on $\mathscr{I}$ a G-invariant preorder $\leq$ which induces on each apartment A the preorder given by the Tits cone $\mathcal{T} \subset \vec{A}$.

Now, we consider the fixer K_I in G of some (local) chamber C_0^+ in a chosen standard apartment $\mathbb{A}$; it is our Iwahori subgroup. Fix a ring R. The Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ will be defined as the space of some K_I-bi-invariant functions on G with values in R. In other words, it will be the space ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ of some G-invariant functions on $\mathcal{C}_0^+ \times \mathcal{C}_0^+$, where $\mathcal{C}_0^+ = G/K_I$ is the orbit of C_0^+ in the set $\mathcal{C}$ of chambers of $\mathscr{I}$. The convolution product is easy to guess from this point of view:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z \in \mathcal{C}_0^+} \varphi(C_x, C_z)\psi(C_z, C_y)$$

(if this sum means something). As for points, two chambers in $\mathscr{I}$ are not always in a same apartment, i.e., the Bruhat–Iwahori decomposition fails: $G \neq K_I . N . K_I$. So, we have to consider pairs of chambers $(C_x, C_y) \in \mathcal{C}_0^+ \times_\leq \mathcal{C}_0^+$, i.e., $C_x \in \mathcal{C}_0^+$ has x for vertex, $C_y \in \mathcal{C}_0^+$ has y for vertex, and $x \leq y$. This implies that C_x, C_y are in a same apartment. For ${}^{\mathrm{I}}\mathcal{H}_R$, this means that the support of $\varphi \in \mathcal{H}_R$ has to be in $K_I \backslash G^+ / K_I$ where $G^+ = \{g \in G \mid 0 \leq g . 0\}$ is a semigroup. We suppose moreover this support to be finite. In addition, $K_I \backslash G^+ / K_I$ is in bijective correspondence with the subsemigroup $W^+ = W^v \ltimes Y^+$ of W, where $Y^+ = Y \cap \mathcal{T}$.

With this definition we are able to prove that ${}^{\mathrm{I}}\mathcal{H}_R$ is really an algebra, which generalizes the known Iwahori–Hecke algebras in the semisimple case; see Section 2.

The structure constants. The structure constants of ${}^{\mathrm{I}}\mathcal{H}_R$ are the nonnegative integers $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$, for $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u} \in W^+$, such that

$$T_{\boldsymbol{w}} * T_{\boldsymbol{v}} = \sum_{\boldsymbol{u} \in W^+} a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}} T_{\boldsymbol{u}},$$

where $T_{\boldsymbol{w}}$ is the characteristic function of $K_I . \boldsymbol{w} . K_I$ and the sum is finite. Each chamber in $\mathscr{I}$ has only a finite number of adjacent chambers along a given panel. These numbers are called the parameters of $\mathscr{I}$ and form a finite set $\mathcal{Q}$. In the split case, there is only one parameter q: the number of elements of the residue field of $\mathcal{K}$. We conjecture that each $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$ is a polynomial in these parameters with integral coefficients depending only on the geometry of the model apartment $\mathbb{A}$ and on W. We prove this only partially: this is true if G is split or if we replace "polynomial" by "Laurent polynomial" (see Section 6.7); this is also true for $\boldsymbol{w}, \boldsymbol{v}$ "generic" (see Section 3.8). Actually in the generic case, we give, in Section 3, an explicit formula for $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$.

Generators and relations. If the parameters in $\mathcal{Q}$ are invertible in the ring R, we are able, in Section 4, to deduce from the geometry of $\mathscr{I}$ a set of generators and some relations in ${}^{\mathrm{I}}\mathcal{H}_R$. The family $(T_\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$ is an R-basis of ${}^{\mathrm{I}}\mathcal{H}_R$. And the subalgebra $\sum_{w \in W^v} R . T_w$ is the abstract Hecke algebra $\mathcal{H}_R(W^v)$ associated to the Coxeter group W^v, generated by the $T_i = T_{r_i}$, where the r_i are the fundamental reflections in W^v. So, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module. We get also some commuting relations between the T_λ and the T_w, including some relations of Bernstein–Lusztig type (see Theorem 4.8).

From all these relations, we deduce algebraically in Section 5 that there exists a new basis $(X^\lambda * T_w)_{\lambda\in Y^+, w\in W^v}$ of ${}^{\mathrm{I}}\mathcal{H}_R$, associated to some new elements $X^\lambda \in {}^{\mathrm{I}}\mathcal{H}_R$. These elements satisfy $X^\lambda = T_\lambda$ for $\lambda \in Y^{++} = Y \cap \overline{C^v_f}$, where C^v_f is the fundamental Weyl chamber, and $X^\lambda * X^\mu = X^{\lambda+\mu} = X^\mu * X^\lambda$ for $\lambda, \mu \in Y^+$. As, for any $\lambda \in Y^+$, there is $\mu \in Y^{++}$ with $\lambda + \mu \in Y^{++}$; these X^λ are some quotients of some elements T_μ. The Bernstein–Lusztig type relations may be translated to this new basis. When R contains sufficiently high roots of the parameters in $\mathcal{Q}$ (e.g., if $R \supset \mathbb{R}$), we may replace the T_w and X^λ by some $R^\times$-multiples H_w and Z^λ. We get a new basis $(Z^\lambda * H_w)_{\lambda\in Y^+l, w\in W^v}$ of ${}^{\mathrm{I}}\mathcal{H}_R$, satisfying a set of relations very close to the Bernstein–Lusztig presentation in the semisimple case; see Section 5.7.

In Section 6, we define the Bernstein–Lusztig–Hecke algebra ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ algebraically: it is the free module with basis written $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ over the algebra $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma_i'^{\pm 1})_{i\in I}]$, where σ_i, σ_i' are indeterminates (with some identifications). The product $*$ is given by the same relations as above for the $Z^\lambda * H_w$; one just extends $\lambda \in Y^+$ to $\lambda \in Y$ and replaces $\sqrt{q_i}, \sqrt{q_i'}$ by σ_i, σ_i'. We prove then that, up to a change of scalars, ${}^{\mathrm{I}}\mathcal{H}_R$ may be identified to a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$. This Bernstein–Lusztig algebra may be considered as a ring of quotients of the Iwahori–Hecke algebra.

Ordered affine hovel. Actually, this article is written in a more general framework, explained in Section 1: we work with $\mathscr{I}$ an abstract ordered affine hovel (as defined in [Rousseau 2011]), and we take G to be a strongly transitive group of (positive, "vectorially Weyl") automorphisms. In Section 7, we drop the assumption that G is vectorially Weyl to define extended versions ${}^{\mathrm{I}}\widetilde{\mathcal{H}}$ and ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}$ of ${}^{\mathrm{I}}\mathcal{H}$ and ${}^{\mathrm{BL}}\mathcal{H}$. In the affine case, we prove that they are graded algebras and that the summand of degree 0 of ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}$ is very close to Cherednik's double affine Hecke algebra.

1. General framework

1.1. ***Vectorial data.*** We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where V is a finite dimensional real vector space, W^v a subgroup of $\mathrm{GL}(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a family in V, and $(\alpha_i)_{i\in I}$ a family in the dual V^*. We suppose this family free, i.e., the set $\{\alpha_i \mid i \in I\}$ linearly independent and ask these data to satisfy the conditions of [Rousseau 2011, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V which is an element in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete, we consider the Kac–Moody case of [op. cit., 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac–Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v,\ i \in I\} \subset Q = \bigoplus_{i \in I} \mathbb{Z}.\alpha_i.$$

We set $\Phi^{\pm} = \Phi \cap Q^{\pm}$, where $Q^{\pm} = \pm\big(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i\big)$. Also,

$$Q^{\vee} = \Big(\bigoplus_{i \in I} \mathbb{Z}.\alpha_i^{\vee}\Big), \quad \text{and} \quad Q^{\vee}_{\pm} = \pm\Big(\bigoplus_{i \in I} (\mathbb{Z}_{\geq 0}).\alpha_i^{\vee}\Big).$$

We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$,

$$r_\alpha = w.r_i.w^{-1} \quad \text{and} \quad r_\alpha(v) = v - \alpha(v)\alpha^{\vee},$$

where the coroot $\alpha^{\vee} = w(\alpha_i^{\vee})$ depends only on α.

The set Φ is an (abstract, reduced) real root system in the sense of [Moody and Pianzola 1989; 1995; Bardy 1996]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta^+_{\text{im}} \cup \Delta^-_{\text{im}}$ of all roots (with $-\Delta^-_{\text{im}} = \Delta^+_{\text{im}} \subset Q^+$, W^v-stable) defined in [Kac 1990]. It is an (abstract, reduced) root system in the sense of [Bardy 1996].

The *fundamental positive chamber* is $C^v_f = \{v \in V \mid \alpha_i(v) > 0,\ \text{for all } i \in I\}$. Its closure $\overline{C^v_f}$ is the disjoint union of the vectorial faces

$$F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \text{ for all } i \in J,\ \text{and } \alpha_i(v) > 0 \text{ for all } i \in I \setminus J\}$$

for $J \subset I$. We set $V_0 = F^v(I)$. The positive and negative vectorial faces are the sets $w.F^v(J)$ and $-w.F^v(J)$, respectively, for $w \in W^v$ and $J \subset I$. The support of such a face is the vector space it generates. The set J or the face $w.F^v(J)$ or an element of this face is called *spherical* if the group $W^v(J)$ generated by $\{r_i \mid i \in J\}$ is finite. An element of a vectorial chamber $\pm w.C^v_f$ is called *regular*.

The *Tits cone* $\mathcal{T}$ is the (disjoint) union of the positive vectorial faces. Its interior $\mathcal{T}^\circ$ consists of those faces that are also spherical. It is a W^v-stable convex cone in V.

We say that $\mathbb{A}^v = (V, W^v)$ is a *vectorial apartment*. A *positive automorphism* of $\mathbb{A}^v$ is a linear bijection $\varphi : \mathbb{A}^v \to \mathbb{A}^v$ stabilizing $\mathcal{T}$ and permuting the roots and corresponding coroots; so it normalizes W^v and permutes the vectorial walls $M^v(\alpha) = \text{Ker}(\alpha)$. As W^v acts simply transitively on the positive (resp., negative) vectorial chambers, any subgroup $\widetilde{W}^v$ of the group $\text{Aut}^+(\mathbb{A}^v)$ (of positive automorphisms of $\mathbb{A}^v$) containing W^v may be written $\widetilde{W}^v = \Omega \ltimes W^v$, where Ω is the stabilizer in $\widetilde{W}^v$ of C^v_f (resp., $-C^v_f$). This group Ω induces a group of permutations of I (by $\omega(\alpha_i) = \alpha_{\omega(i)}$ and $\omega(\alpha_i^{\vee}) = \alpha^{\vee}_{\omega(i)}$); but it may be greater than the whole group of permutations in general, even infinite if $\bigcap \text{Ker}\,\alpha_i \neq \{0\}$.

1.2. ***The model apartment.*** As in [Rousseau 2011, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by $\text{Ker}(\alpha)$ for $\alpha \in \Phi$.

We ask this apartment to be *semidiscrete* and the origin 0 to be *special*. This means that these walls are the hyperplanes defined as

$$M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \quad \text{for } \alpha \in \Phi \text{ and } k \in \Lambda_\alpha,$$

with $\Lambda_\alpha = k_\alpha.\mathbb{Z}$ a nontrivial discrete subgroup of $\mathbb{R}$. Using [Gaussent and Rousseau 2014, Lemma 1.3] (i.e., by replacing Φ by another system Φ_1) we may (and shall) assume that $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$.

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$, and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed points the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; we assume that W^a stabilizes $\mathcal{M}$. We know that $W^a = W^v \ltimes Q^\vee$ and we write $W^a_{\mathbb{R}} = W^v \ltimes V$; here $Q^\vee$ and V have to be understood as groups of translations.

An automorphism of $\mathbb{A}$ is an affine bijection $\varphi : \mathbb{A} \to \mathbb{A}$ stabilizing the set of pairs $(M, \alpha^\vee)$ of a wall M and the coroot associated with $\alpha \in \Phi$ such that $M = M(\alpha, k)$, $k \in \mathbb{Z}$. We write $\overrightarrow{\varphi} : V \to V$ the linear application associated to φ. The group $\mathrm{Aut}(\mathbb{A})$ of these automorphisms contains W^a and normalizes it. We consider also the group $\mathrm{Aut}^W_{\mathbb{R}}(\mathbb{A}) = \{\varphi \in \mathrm{Aut}(\mathbb{A}) \mid \overrightarrow{\varphi} \in W^v\} = \mathrm{Aut}(\mathbb{A}) \cap W^a_{\mathbb{R}}$.

For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$ is an half-space; it is called an *half-apartment* if $k \in \mathbb{Z}$. We write $D(\alpha, \infty) = \mathbb{A}$.

The Tits cone $\mathcal{T}$ and its interior $\mathcal{T}^o$ are convex and W^v-stable cones; therefore, we can define two W^v-invariant preorder relations on $\mathbb{A}$:

$$x \leq y \;\Leftrightarrow\; y - x \in \mathcal{T} \quad \text{and} \quad x \overset{o}{<} y \;\Leftrightarrow\; y - x \in \mathcal{T}^o.$$

If W^v has no fixed point in $V \setminus \{0\}$ and no finite factor, then they are orders; but, in general, they are not.

1.3. ***Faces, sectors, and chimneys.*** The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments. As in [Bruhat and Tits 1972], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$. For the definition of that notion and its properties, see [loc. cit.] or [Gaussent and Rousseau 2008].

If F is a subset of $\mathbb{A}$ containing an element x in its closure, the germ of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of $\mathbb{A}$ which contain intersections of F and neighborhoods of x. In particular, if $x \neq y \in \mathbb{A}$, we denote the germ in x of the segment $[x, y]$ by $[x, y)$ and the germ in x of the segment $]x, y]$ by $]x, y)$.

Given F a filter of subsets of $\mathbb{A}$, its *enclosure* $\mathrm{cl}_{\mathbb{A}}(F)$ is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$, where $k_\alpha \in \mathbb{Z} \cup \{\infty\}$. Its *closure* $\overline{F}$ is the filter made of the subsets of $\mathbb{A}$ containing the closure $\overline{S}$ of some $S \in F$.

A *local face* F in the apartment $\mathbb{A}$ is associated to its vertex, a point $x \in \mathbb{A}$, and its direction, a vectorial face F^v in V. It is defined as $F = \mathrm{germ}_x(x + F^v)$ and we denote it by $F = F^\ell(x, F^v)$. Its closure is $\overline{F^\ell}(x, F^v) = \mathrm{germ}_x(x + \overline{F^v})$

There is an order on the local faces: in fact, the three assertions F is a face of F', F' covers F, and $F \le F'$ are by definition equivalent to $F \subset \overline{F'}$. The dimension of a local face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the support of F; if $F = F^\ell(x, F^v)$, then $\mathrm{supp}(F) = x + \mathrm{supp}(F^v)$.

A local face $F = F^\ell(x, F^v)$ is spherical if the direction of its support meets the open Tits cone (i.e., when F^v is spherical), then its pointwise stabilizer W_F in W^a is finite. We shall actually speak only of local faces here, and sometimes forget the word local.

Any point $x \in \mathbb{A}$ is contained in a unique face $F(x, V_0) \subset \mathrm{cl}_{\mathbb{A}}(\{x\})$ which is minimal of positive and negative direction (but seldom spherical). For any local face $F^\ell = F^\ell(x, F^v)$, there is a unique face F (as defined in [Rousseau 2011]) containing F^ℓ. Then $\overline{F^\ell} \subset \overline{F} = \mathrm{cl}_{\mathbb{A}}(F^\ell) = \mathrm{cl}_{\mathbb{A}}(F)$ is also the enclosure of any interval-germ $]x, y) = \mathrm{germ}_x(]x, y])$ included in F^ℓ.

A *local chamber* is a maximal local face, i.e., a local face $F^\ell(x, \pm w . C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local chamber* is $C_0^+ = \mathrm{germ}_0(C^v_f)$.

A *(local) panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension $n - 1$; its support is a wall.

A *sector* in $\mathbb{A}$ is a V-translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w . C^v_f$, with $w \in W^v$. The point x is its *base point* and C^v its *direction*. Two sectors have the same direction if and only if they are conjugate by V-translation, and if and only if their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a V-translate of $\mathfrak{s}$; it is well determined by the direction C^v. So, the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V, and the set of sector-germs in $\mathbb{A}$ are in canonical bijection. We denote the sector-germ associated to the negative fundamental vectorial chamber $-C^v_f$ by $\mathfrak{S}_{-\infty}$.

A *sector-face* in $\mathbb{A}$ is a V-translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm w . F^v(J)$. The sector-face-germ of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (i.e., $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

A *chimney* in $\mathbb{A}$ is associated to a face $F = F(x, F_0^v)$, called its basis, and to a vectorial face F^v, its direction; it is the filter

$$\mathfrak{r}(F, F^v) = \mathrm{cl}_{\mathbb{A}}(F + F^v).$$

A chimney $\mathfrak{r} = \mathfrak{r}(F, F^v)$ is *splayed* if F^v is spherical; it is *solid* if its support (as a filter, i.e., the smallest affine subspace containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid. The enclosure of a sector-face $\mathfrak{f} = x + F^v$ is a chimney.

A ray δ with origin in x and containing $y \neq x$ (or the interval $]x, y]$, the segment $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $x \overset{o}{<} y$ or $y \overset{o}{<} x$. With these new notions, a chimney can be defined as the enclosure of a preordered ray and a preordered segment-germ sharing the same origin. The chimney is splayed if and only if the ray is generic.

1.4. ***The hovel.*** In this section, we recall the definition and some properties of an ordered affine hovel given by Rousseau [2011].

1.4.1. An apartment of type $\mathbb{A}$ is a set A endowed with a set $\mathrm{Isom}^W(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that, if $f_0 \in \mathrm{Isom}^W(\mathbb{A}, A)$, then $f \in \mathrm{Isom}^W(\mathbb{A}, A)$ if and only if there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism (resp., a Weyl-isomorphism, a vectorially Weyl isomorphism) between two apartments $\varphi : A \to A'$ is a bijection such that, for any $f \in \mathrm{Isom}^W(\mathbb{A}, A)$, $f' \in \mathrm{Isom}^W(\mathbb{A}, A')$, $f'^{-1} \circ \varphi \circ f$ is contained in $\mathrm{Aut}(\mathbb{A})$ (resp., in W^a, in $\mathrm{Aut}^W_{\mathbb{R}}(\mathbb{A})$); the group of these isomorphisms is written $\mathrm{Isom}(A, A')$ (resp., $\mathrm{Isom}^W(A, A')$, $\mathrm{Isom}^W_{\mathbb{R}}(A, A')$). As the filters in $\mathbb{A}$ defined in Section 1.3 above (e.g., local faces, sectors, walls, etc.) are permuted by $\mathrm{Aut}(\mathbb{A})$, they are well defined in any apartment of type $\mathbb{A}$ and exchanged by any isomorphism.

Definition. An *ordered affine hovel of type* $\mathbb{A}$ (or, for short, a *masure of type* $\mathbb{A}$) is a set $\mathscr{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that:

(MA1) any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$;

(MA2) if F is a point, a germ of a preordered interval, a generic ray, or a solid chimney in an apartment A, and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$;

(MA3) if $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and F;

(MA4) if two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then their intersection contains $\mathrm{cl}_A(\mathfrak{R} \cup F)$ and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$;

(MA5) if x, y are two points contained in two apartments A and A', and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

We ask here that $\mathscr{I}$ be thick of *finite thickness*: the number of local chambers containing a given (local) panel has to be finite and at least 3. This number is the same for any panel in a given wall M [Rousseau 2011, 2.9]; we denote it by $1 + q_M$.

An automorphism (resp., a Weyl-automorphism, a vectorially Weyl automorphism) of $\mathscr{I}$ is a bijection $\varphi : \mathscr{I} \to \mathscr{I}$ such that $A \in \mathcal{A} \iff \varphi(A) \in \mathcal{A}$ and then $\varphi|_A : A \to \varphi(A)$ is an isomorphism (resp., a Weyl-isomorphism, a vectorially Weyl isomorphism).

1.4.2. For $x \in \mathscr{I}$, the set $\mathcal{T}_x^+\mathscr{I}$ (resp. $\mathcal{T}_x^-\mathscr{I}$) of segment-germs $[x, y)$ for $y > x$ (resp., $y < x$) may be considered as a building, the positive (resp., negative) tangent building. The corresponding faces are the local faces of positive (resp., negative) direction and of vertex x. The associated Weyl group is W^v. If the W-distance (calculated in $\mathcal{T}_x^\pm\mathscr{I}$) of two local chambers is $d^W(C_x, C'_x) = w \in W^v$, to any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$ corresponds a unique minimal gallery from C_x to C'_x of type $(i_1, \dots, i_n)$. We shall say, by abuse of notation, that this gallery is of type w.

The buildings $\mathcal{T}_x^+\mathscr{I}$ and $\mathcal{T}_x^-\mathscr{I}$ are actually twinned. The codistance $d^{*W}(C_x, D_x)$ of two opposite sign chambers C_x and D_x is the W-distance $d^W(C_x, \mathrm{op}\, D_x)$, where $\mathrm{op}\, D_x$ denotes the opposite chamber to D_x in an apartment containing C_x and D_x. Similarly two segment-germs $\eta \in \mathcal{T}_x^+\mathscr{I}$ and $\zeta \in \mathcal{T}_x^-\mathscr{I}$ are said opposite if they are in a same apartment A and opposite in this apartment (i.e., in the same line, with opposite directions).

Lemma [Rousseau 2011, 2.9]. *Let D be a half-apartment in $\mathscr{I}$ and $M = \partial D$ its wall* (*i.e.*, *its boundary*). *One considers a panel F in M and a local chamber C in $\mathscr{I}$ covering F. Then there is an apartment containing D and C.*

1.4.3. We assume that $\mathscr{I}$ has a strongly transitive group of automorphisms G, i.e., all isomorphisms involved in the above axioms are induced by elements of G; see [Rousseau 2012, 4.10; Ciobotaru and Rousseau 2015]. We choose in $\mathscr{I}$ a fundamental apartment which we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathscr{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ in G induces a group $W = \nu(N) \subset \mathrm{Aut}(\mathbb{A})$ of affine automorphisms of $\mathbb{A}$ which permutes the walls, local faces, sectors, sector-faces, etc., and contains the affine Weyl group $W^a = W^v \ltimes Q^\vee$ [Rousseau 2012, 4.13.1].

We denote the stabilizer of $0 \in \mathbb{A}$ in G by K and the pointwise stabilizer (or fixer) of C_0^+ by K_I; this group K_I is called the *Iwahori subgroup*.

1.4.4. We ask $W = \nu(N)$ to be *positive* and *vectorially Weyl* for its action on the vectorial faces. This means that the associated linear map $\vec{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $W = \nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that: $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$. An element $\boldsymbol{w} \in W$ will often be written $\boldsymbol{w} = \lambda.w$, with $\lambda \in Y$ and $w \in W^v$.

We ask Y to be *discrete* in V. This is clearly satisfied if Φ generates V^*, i.e., $(\alpha_i)_{i \in I}$ is a basis of V^*.

1.4.5. Note that there is only a finite number of constants q_M as in the definition of thickness. Indeed, we must have $q_{wM} = q_M$, $\forall w \in \nu(N)$ and $w.M(\alpha, k) = M(w(\alpha), k), \forall w \in W^v$. So now, fix $i \in I$, as $\alpha_i(\alpha_i^\vee) = 2$ the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i, k)$ (for $k \in \mathbb{Z}$) with two orbits. So, $Q^\vee \subset W^a$ has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$: one containing the $q_i = q_{M(\alpha_i,0)}$ and the other containing the $q'_i = q_{M(\alpha_i,\pm 1)}$. Hence, the number of (possibly) different q_M is at most $2.|I|$. We denote this set of parameters by $\mathcal{Q} = \{q_i, q'_i \mid i \in I\}$.

If $\alpha_i(\alpha_j^\vee)$ is odd for some $j \in I$, the translation by $\alpha_j^\vee$ exchanges the two walls $M(\alpha_i, 0)$ and $M(\alpha_i, -\alpha_i(\alpha_j^\vee))$; so $q_i = q'_i$. More generally, we see that $q_i = q'_i$ when $\alpha_i(Y) = \mathbb{Z}$, i.e., $\alpha_i(Y)$ contains an odd integer. If $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$, one knows that the element $r_i r_j r_i$ of $W^v(\{i, j\})$ exchanges α_i and $-\alpha_j$, so $q_i = q'_i = q_j = q'_j$.

Actually many of the following results (in sections 2, 3) are true without assuming the existence of G: we have only to assume that the parameters q_M satisfy the above conditions.

1.4.6. The main examples of all the above situation are provided by the hovels of almost split Kac–Moody groups over fields complete for a discrete valuation and with a finite residue field, see Section 7.2 below.

1.4.7. *Remarks.* (a) In the following, we sometimes use results of [Gaussent and Rousseau 2008] even though, in this paper we deal with split Kac–Moody groups and residue fields containing $\mathbb{C}$. But the cited results are easily generalizable to our present framework, using the above references.

(b) All isomorphisms in [Rousseau 2011] are Weyl-isomorphisms, and, when G is strongly transitive, all isomorphisms constructed in that reference are induced by an element of G.

1.5. ***Type* 0 *vertices.*** The elements of Y, through the identification $Y = N.0$, are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$ and $Y^{++} = Y \cap \overline{C^v_f}$. The type 0 vertices in $\mathcal{I}$ are the points on the orbit $\mathcal{I}_0$ of 0 by G. This set $\mathcal{I}_0$ is often called the affine Grassmannian as it is equal to G/K, where $K = \mathrm{Stab}_G(\{0\})$. But in general, G is not equal to $KYK = KNK$ [Gaussent and Rousseau 2008, 6.10], i.e., $\mathcal{I}_0 \neq K.Y$.

We know that $\mathcal{I}$ is endowed with a G-invariant preorder $\leq$ which induces the known one on $\mathbb{A}$. Moreover, if $x \leq y$, then x and y are in a same apartment [Rousseau 2011, 5.9]. We set $\mathcal{I}^+ = \{x \in \mathcal{I} \mid 0 \leq x\}$, $\mathcal{I}_0^+ = \mathcal{I}_0 \cap \mathcal{I}^+$, and $G^+ = \{g \in G \mid 0 \leq g.0\}$; so $\mathcal{I}_0^+ = G^+.0 = G^+/K$. As $\leq$ is a G-invariant preorder, G^+ is a semigroup.

If $x \in \mathcal{I}_0^+$ there is an apartment A containing 0 and x (by definition of $\leq$) and all apartments containing 0 are conjugated to $\mathbb{A}$ by K (see (MA2)); so $x \in K.Y^+$ as $\mathcal{I}_0^+ \cap \mathbb{A} = Y^+$. But $\nu(N \cap K) = W^v$ and $Y^+ = W^v.Y^{++}$, with uniqueness of the element in Y^{++}. So $\mathcal{I}_0^+ = K.Y^{++}$, more precisely $\mathcal{I}_0^+ = G^+/K$ is the union of the

KyK/K for $y \in Y^{++}$. This union is disjoint, for the above construction does not depend on the choice of A; see Section 1.9(a).

Hence, we have proved that the map $Y^{++} \to K\backslash G^+/K$ is one-to-one and onto.

1.6. ***Vectorial distance and $Q^\vee$-order.*** For x in the Tits cone $\mathcal{T}$, we denote by x^{++} the unique element in $\overline{C^v_f}$ conjugated by W^v to x.

Let $\mathcal{I} \times_\leq \mathcal{I} = \{(x, y) \in \mathcal{I} \times \mathcal{I} \mid x \leq y\}$ be the set of increasing pairs in $\mathcal{I}$. Such a pair (x, y) is always in a same apartment $g.\mathbb{A}$; so $(g^{-1}).y - (g^{-1}).x \in \mathcal{T}$ and we define the *vectorial distance* $d^v(x, y) \in \overline{C^v_f}$ by $d^v(x, y) = ((g^{-1}).y - (g^{-1}).x)^{++}$. It does not depend on the choices we made (by Section 1.9.a below).

For $(x, y) \in \mathcal{I}_0 \times_\leq \mathcal{I}_0 = \{(x, y) \in \mathcal{I}_0 \times \mathcal{I}_0 \mid x \leq y\}$, the vectorial distance $d^v(x, y)$ takes values in Y^{++}. Actually, as $\mathcal{I}_0 = G.0$, K is the stabilizer of 0 and $\mathcal{I}_0^+ = K.Y^{++}$ (with uniqueness of the element in Y^{++}), the map d^v induces a bijection between the set $\mathcal{I}_0 \times_\leq \mathcal{I}_0/G$ of G-orbits in $\mathcal{I}_0 \times_\leq \mathcal{I}_0$ and Y^{++}.

Further, d^v gives the inverse of the map $Y^{++} \to K\backslash G^+/K$, as any $g \in G^+$ is in $K.d^v(0, g.0).K$.

For $x, y \in \mathbb{A}$, we say that $x \leq_{Q^\vee} y$ when $y - x \in Q^\vee_+$, and $x \leq_{Q^\vee_\mathbb{R}} y$ when

$$y - x \in Q^\vee_{\mathbb{R}+} = \sum_{i \in I} \mathbb{R}_{\geq 0}.\alpha_i^\vee.$$

We get thus a preorder which is an order at least when $(\alpha_i^\vee)_{i \in I}$ is free or $\mathbb{R}_+$-free, i.e., $\sum a_i \alpha_i^\vee = 0$, $a_i \geq 0$ implies $a_i = 0$, for all i.

1.7. ***Paths.*** We consider piecewise linear continuous paths $\pi : [0, 1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v.\lambda$ for some $\lambda \in \overline{C^v_f}$. Such a path is called a λ-*path*; it is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$.

For any $t \neq 0$ (resp., $t \neq 1$), we let $\pi'_-(t)$ (resp., $\pi'_+(t)$) denote the derivative of π at t from the left (resp., from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda$-class such that $\pi'_\pm(t) = w_\pm(t).\lambda$, where $(W^v)_\lambda$ is the stabilizer in W^v of λ.

Hecke paths of shape λ (with respect to the sector germ $\mathfrak{S}_{-\infty} = \text{germ}_\infty(-C^v_f)$) are λ-paths satisfying some further precise conditions, see [Kapovich and Millson 2008, 3.27] or [Gaussent and Rousseau 2014, 1.8]. For us their interest will appear just below in Section 1.8.

But to give a formula for the structure constants of the forthcoming Iwahori–Hecke algebra, we will need slightly different Hecke paths whose definition is detailed in Section 3.3.

1.8. ***Retractions onto Y^+.*** For all $x \in \mathcal{I}^+$ there is an apartment containing x and $C_0^- = \text{germ}_0(-C^v_f)$ [Rousseau 2011, 5.1] and this apartment is conjugated to $\mathbb{A}$ by an element of K fixing C_0^-; see (MA2). So, by the usual arguments, as well as [op. cit., 5.5] (see below Proposition 1.10(a)), we can define the retraction $\rho_{C_0^-}$ of $\mathscr{I}^+$ into $\mathbb{A}$ with center C_0^-; its image is $\rho_{C_0^-}(\mathscr{I}^+) = \mathcal{T} = \mathscr{I}^+ \cap \mathbb{A}$ and $\rho_{C_0^-}(\mathscr{I}_0^+) = Y^+$.

Using axioms (MA3) and (MA4) [Gaussent and Rousseau 2008, 4.4], we may also define the retraction $\rho_{-\infty}$ of $\mathscr{I}$ onto $\mathbb{A}$ with center the sector-germ $\mathfrak{S}_{-\infty}$.

More generally, we may define the retraction ρ of $\mathscr{I}$ (resp., of the subset $\mathscr{I}_{\geq z} = \{y \in \mathscr{I} \mid y \geq z\}$, for a fixed z) onto an apartment A with center any sector germ (resp., any local chamber of negative direction with vertex z). For any such retraction ρ, the image of any segment $[x, y]$ with $(x, y) \in \mathscr{I} \times_{\leq} \mathscr{I}$ and $d^v(x, y) = \lambda \in \overline{C_f^v}$ (resp., and moreover $x, y \in \mathscr{I}_{\geq z}$) is a λ-path [Gaussent and Rousseau 2008, 4.4]. In particular, $\rho(x) \leq \rho(y)$.

Actually, the image by $\rho_{-\infty}$ of any segment $[x, y]$ with $(x, y) \in \mathscr{I} \times_{\leq} \mathscr{I}$ and $d^v(x, y) = \lambda \in Y^{++}$ is a Hecke path of shape λ with respect to $\mathfrak{S}_{-\infty}$ [Gaussent and Rousseau 2008, th. 6.2], and we have the following lemma.

Lemma. (a) *For $\lambda \in Y^{++}$ and $w \in W^v$, we have $w.\lambda \in \lambda - Q_+^\vee$, i.e., $w.\lambda \leq_{Q^\vee} \lambda$.*

(b) *Let π be a Hecke path of shape $\lambda \in Y^{++}$ with respect to $\mathfrak{S}_{-\infty}$, from $y_0 \in Y$ to $y_1 \in Y$. Then, for $0 \leq t < t' < 1$,*

$$\begin{aligned} &\lambda = \pi'_+(t)^{++} = \pi'_-(t')^{++};\\ &\pi'_+(t) \leq_{Q^\vee} \pi'_-(t') \leq_{Q^\vee} \pi'_+(t') \leq_{Q^\vee} \pi'_-(1);\\ &\pi'_+(0) \leq_{Q^\vee} \lambda;\\ &\pi'_+(0) \leq_{Q^\vee_\mathbb{R}} (y_1 - y_0) \leq_{Q^\vee_\mathbb{R}} \pi'_-(1) \leq_{Q^\vee} \lambda;\\ &y_1 - y_0 \leq_{Q^\vee} \lambda.\end{aligned}$$

Moreover $y_1 - y_0$ is in the convex hull $\operatorname{conv}(W^v.\lambda)$ *of all $w.\lambda$ for $w \in W^v$, more precisely in the convex hull* $\operatorname{conv}(W^v.\lambda, \geq \pi'_+(0))$ *of all $w'.\lambda$ for $w' \in W^v$, $w' \leq w$, where w is the element with minimal length such that $\pi'_+(0) = w.\lambda$.*

(c) *If, moreover, $(\alpha_i^\vee)_{i \in I}$ is free, we may replace above $\leq_{Q^\vee_\mathbb{R}}$ by $\leq_{Q^\vee}$.*

(d) *If $x \leq z \leq y$ in $\mathscr{I}_0$, then $d^v(x, y) \leq_{Q^\vee} d^v(x, z) + d^v(z, y)$.*

N.B. In the following, we always assume $(\alpha_i^\vee)_{i \in I}$ free.

Proof. Everything is proved in [Gaussent and Rousseau 2014, 2.4], except the second paragraph of (b). Actually we see in [loc. cit.] that $y_1 - y_0$ is the integral of the locally constant vector-valued function $\pi'_+(t) = w_+(t).\lambda$, where $w_+(t)$ is decreasing for the Bruhat order [op. cit., 5.4], hence the result. □

1.9. ***Chambers of type* 0.** Let $\mathscr{C}_0^\pm$ be the set of all local chambers with vertices of type 0 and positive or negative direction. A local chamber of vertex $x \in \mathscr{I}_0$ will often be written C_x and its direction C_x^v. We consider $\mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+ = \{(C_x, C_y) \in \mathscr{C}_0^+ \times \mathscr{C}_0^+ \mid x \leq y\}$. We sometimes write $C_x \leq C_y$ when $x \leq y$.

Proposition [Rousseau 2011, 5.4 and 5.1]. *Let $x, y \in \mathscr{I}$ with $x \leq y$. We consider two local faces F_x, F_y with respective vertices x, y.*

(a) *$\{x, y\}$ is included in an apartment and two such apartments A, A' are isomorphic by a Weyl-isomorphism in G, fixing* $\mathrm{cl}_A(\{x, y\}) = \mathrm{cl}_{A'}(\{x, y\}) \supset [x, y]$.

(b) *There is an apartment containing F_x and F_y, unless F_x and F_y are respectively of positive and negative direction. In this case we have to assume moreover $x \overset{o}{<} y$ or $x = y$ to get the same result.*

Consequences. (1) We define $W^+ = W^v \ltimes Y^+$ which is a subsemigroup of W.

If $C_x \in \mathscr{C}_0^+$, we know by (b) above, that there is an apartment A containing C_0^+ and C_x. But all apartments containing C_0^+ are conjugated to $\mathbb{A}$ by K_I (MA2), so there is $k \in K_I$ with $k^{-1}.C_x \subset \mathbb{A}$. Now the vertex $k^{-1}.x$ of $k^{-1}.C_x$ satisfies $k^{-1}.x \geq 0$, so there is $\boldsymbol{w} \in W^+$ such that $k^{-1}.C_x = \boldsymbol{w}.C_0^+$.

When $g \in G^+$, we have $g.C_0^+ \in \mathscr{C}_0^+$ and there are $k \in K_I$, $\boldsymbol{w} \in W^+$ satisfying $g.C_0^+ = k.\boldsymbol{w}.C_0^+$, i.e., $g \in K_I.W^+.K_I$. We have proved the *Bruhat decomposition* $G^+ = K_I.W^+.K_I$.

(2) Let $x \in \mathscr{I}_0$ and $C_y \in \mathscr{C}_0^+$ with $x \leq y$, $x \neq y$. We consider an apartment A containing x and C_y (by (b) above) and write $C_y = F(y, C_y^v)$ in A. For $y' \in y + C_y^v$ sufficiently close to y, $\alpha(y' - x) \neq 0$ for any root α. So $]x, y')$ is in a unique local chamber $\mathrm{pr}_x(C_y)$ of vertex x; this chamber satisfies $[x, y) \subset \overline{\mathrm{pr}_x(C_y)} \subset \mathrm{cl}_A(\{x, y'\})$ and does not depend on the choice of y'. Moreover, if A' is another apartment containing x and C_y, we may suppose $y' \in A \cap A'$ and $]x, y')$, $\mathrm{cl}_A(\{x, y'\})$, $\mathrm{pr}_x(C_y)$ are the same in A'. The local chamber $\mathrm{pr}_x(C_y)$ is well determined by x and C_y; it is the projection of C_y in $\mathcal{T}_x^+\mathscr{I}$.

The same things may be done changing accordingly $+$ to $-$ and $\leq$ to $\geq$. But, in the above situation, if $C_x \in \mathscr{C}_0^+$, we have to assume $x \overset{o}{<} y$ to define the analogous $\mathrm{pr}_y(C_x) \in \mathscr{C}_0^+$.

Proposition 1.10. *In the setting of Section 1.9,*

(a) *If $x \overset{o}{<} y$ or F_x and F_y are, respectively, of negative and positive direction, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G fixing the convex hull of F_x and F_y (in A or A').*

(b) *If $x = y$ and the directions of F_x and F_y have the same sign, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G, $\varphi : A \to A'$, fixing F_x and F_y. If moreover F_x is a local chamber, any minimal gallery from F_x to F_y is fixed by φ (and in $A \cap A'$).*

(c) *If F_x and F_y are of positive directions and F_y is spherical, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G fixing F_x and F_y.*

Remark. The conclusion in (c) above is less precise than in (a) or in Section 1.9(a). We may actually improve it when the hovel is associated to a very good family of parahorics, as defined in [Rousseau 2012] and already used in [Gaussent and Rousseau 2008]. Then, using the notion of half-good fixers, we may assume that the isomorphism in (c) above fixes some kind of enclosure of F_x and F_y (containing the convex hull). This particular case includes the case of an almost split Kac–Moody group over a local field.

Proof. The assertions (a) and (b) are Propositions 5.5 and 5.2 of [Rousseau 2011], respectively. To prove (c) we improve a little the proof of 5.5 in that reference and use the classical trick that says that it is enough to assume that either F_x or F_y is a local chamber. We assume now that $F_x = C_x$ is a local chamber; the other case is analogous.

We consider an element Ω_x (resp., Ω_y) of the filter C_x (resp., F_y) contained in $A \cap A'$. We have $x \in \overline{\Omega_x}$, $y \in \overline{\Omega_y}$, and one may suppose Ω_x is open and Ω_y is open in the support of F_y. There is an isomorphism $\varphi : A \to A'$ fixing Ω_x. Let $y' \in \Omega_y$; we want to prove that $\varphi(y') = y'$. As F_y is spherical, $x \le y \overset{o}{<} y'$; hence, $x \overset{o}{<} y'$. So $x' \le y'$ for any $x' \in \Omega_x$ (Ω_x sufficiently small). Moreover $[x', y'] \cap \Omega_x$ is an open neighborhood of x' in $[x', y']$. By the following lemma, we get $\varphi(y') = y'$. □

Lemma. *Let us consider two apartments A, A' in $\mathscr{I}$, a subset $\Omega \subset A \cap A'$, a point $z \in A \cap A'$ and an isomorphism $\varphi : A \to A'$ fixing (pointwise) Ω. We assume that there is $z' \in \Omega$ with $z' \le z$ or $z' \ge z$ and $[z', z] \cap \Omega$ open in $[z', z]$. Then $\varphi(z) = z$.*

N.B. This lemma asserts, in particular, that any isomorphism $\varphi : A \to A'$ fixing a local facet $F \subset A \cap A'$ fixes $\overline{F}$.

Proof. Note that $\varphi|_{[z',z]}$ is an affine bijection of $[z', z]$ onto its image in A', which is the identity in a neighborhood of z'. But Section 1.9(a) shows that $[z', z] \subset A \cap A'$ and the identity of $[z', z]$ is an affine bijection (for the affine structures induced by A and A'). Hence $\varphi|_{[z',z]}$ coincides with this affine bijection; in particular $\varphi(z) = z$. □

1.11. ***W-distance.*** Let $(C_x, C_y) \in \mathscr{C}_0^+ \times_{\le} \mathscr{C}_0^+$; there is an apartment A containing C_x and C_y. We identify $(\mathbb{A}, C_0^+)$ with (A, C_x), i.e., we consider the unique $f \in \mathrm{Isom}^W_{\mathbb{R}}(\mathbb{A}, A)$ such that $f(C_0^+) = C_x$. Then $f^{-1}(y) \ge 0$ and there is $\boldsymbol{w} \in W^+$ such that $f^{-1}(C_y) = \boldsymbol{w} . C_0^+$. By Proposition 1.10(c), $\boldsymbol{w}$ does not depend on the choice of A.

We define the *W-distance* between the two local chambers C_x and C_y to be this unique element: $d^W(C_x, C_y) = \boldsymbol{w} \in W^+ = Y^+ \rtimes W^v$. If $\boldsymbol{w} = \lambda . w$, with $\lambda \in Y^+$ and $w \in W^v$, we write also $d^W(C_x, y) = \lambda$. As $\le$ is G-invariant, the W-distance is also G-invariant. When $x = y$, this definition coincides with the one in Section 1.4.2.

If $C_x, C_y, C_z \in \mathscr{C}_0^+$, with $x \le y \le z$, are in a same apartment, we have the Chasles relation: $d^W(C_x, C_z) = d^W(C_x, C_y) . d^W(C_y, C_z)$.

When $C_x = C_0^+$ and $C_y = g\,.\,C_0^+$ (with $g \in G^+$), $d^W(C_x, C_y)$ is the only $\boldsymbol{w} \in W^+$ such that $g \in K_I\,.\,\boldsymbol{w}\,.\,K_I$. We have thus proved the uniqueness in the Bruhat decomposition: $G^+ = \coprod_{\boldsymbol{w}\in W^+} K_I\,.\,\boldsymbol{w}\,.\,K_I$.

The W-distance classifies the orbits of K_I on $\{C_y \in \mathscr{C}_0^+ \mid y \geq 0\}$, hence also the orbits of G on $\mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+$.

2. Iwahori–Hecke Algebras

Throughout this section, we assume that $(\alpha_i^\vee)_{i\in I}$ is free and we consider any commutative ring with unity R. To each $\boldsymbol{w} \in W^+$, we associate a function $T_{\boldsymbol{w}}$ from $\mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+$ to R defined by

$$T_{\boldsymbol{w}}(C, C') = \begin{cases} 1 & \text{if } d^W(C, C') = \boldsymbol{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the following free R-module

$${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}} = \left\{ \varphi = \sum_{\boldsymbol{w}\in W^+} a_{\boldsymbol{w}} T_{\boldsymbol{w}} \;\middle|\; a_{\boldsymbol{w}} \in R,\ a_{\boldsymbol{w}} = 0 \text{ except for a finite number of } \boldsymbol{w} \right\},$$

We endow this R-module with the convolution product:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z} \varphi(C_x, C_z)\,\psi(C_z, C_y).$$

where $C_z \in \mathscr{C}_0^+$ is such that $x \leq z \leq y$. It is clear that this product is associative and R-bilinear. We prove below that this product is well defined.

As in [Gaussent and Rousseau 2014, 2.1], we see easily that ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ can be identified with the natural convolution algebra of functions $G^+ \to R$, bi-invariant under K_I and with finite support.

Lemma 2.1. *Let $\mathfrak{S}^- \subset A$ be a sector-germ with negative direction in an apartment A, let $\rho_- : \mathscr{I} \to A$ be the corresponding retraction, and let $\boldsymbol{w} \in W^+$. Then the set*

$$P = \{d^W(\rho_-(C_x), \rho_-(C_y)) \in W^+ \mid \textit{for all } (C_x, C_y) \in \mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+,\ d^W(C_x, C_y) = \boldsymbol{w}\}$$

is finite and included in a finite subset P' of W^+ depending only on $\boldsymbol{w}$ and on the position of C_x with respect to $\mathfrak{S}^-$ (i.e., on the codistance $w_x \in W^v$ from C_x to the local chamber C_x^- in x of direction $\mathfrak{S}^-$).

Let us write $\boldsymbol{w} = \lambda\,.\,w$ for $\lambda \in Y^+$ and $w \in W^v$. If we assume C_x and $\mathfrak{S}^-$ are opposite (i.e., $w_x = 1$), then any $\boldsymbol{v} = \mu\,.\,v \in P'$ satisfies $\lambda \leq_{Q^\vee} \mu \leq_{Q^\vee} \lambda^{++}$ and μ is in $\mathrm{conv}(W^v.\lambda^{++})$*. More precisely μ is in the convex hull* $\mathrm{conv}(W^v.\lambda^{++}, \geq \lambda)$ *of all $w'.\lambda^{++}$ for $w' \in W^v$, $w' \leq w_\lambda$, where w_λ is the element with minimal length such that $\lambda = w_\lambda\,.\,\lambda^{++}$.*

If moreover $\lambda \in Y^{++}$*, then* $\mu = \lambda$ *and* $v \leq w$*. In particular, for* $\boldsymbol{w} = \lambda \in Y^{++}$*,* $P = \{\boldsymbol{w}\} = \{\lambda\}$*.*

Proof. We consider an apartment A_1 containing C_x and C_y. We set $C'_y = C_x + (y - x)$ in A_1. By identifying $(\mathbb{A}, C_0^+)$ with (A_1, C_x), we have $y = x + \lambda$, and by identifying $(\mathbb{A}, C_0^+)$ with (A_1, C'_y), we have $C_y = wC'_y$.

We have to prove that the possibilities for $\rho_-(C_y)$ vary in a finite set determined by $\rho_-(C_x)$, $\boldsymbol{w}$, and w_x. We shall prove this by successively showing the same kind of result for $\rho_-([x, y))$, $\rho_-(y)$, and $\rho_-(C'_y)$. Up to isomorphism, one may suppose that $C_x \subset A$.

(a) Fixing a reduced decomposition for w_λ gives a minimal gallery between C_x and $[x, y)$. By retraction, we get a gallery with the same type from $\rho_-(C_x)$ to $\rho_-([x, y))$. The possible foldings of this gallery determine the possibilities for $\rho_-([x, y))$. More precisely, $\rho_-([x, y)) = x + w'(\lambda_A^{++})[0, 1)$ for $w' \leq w_\lambda$ and λ_A^{++} the image in A of λ^{++} by the identification of $(\mathbb{A}, C_0^+)$ with (A, C_x).

(b) Now fix $\rho_-([x, y))$. By Section 1.8(b), $\rho_-([x, y])$ is a Hecke path π of shape λ^{++} (with respect to $\mathfrak{S}^-$). Its derivative $\pi'_+(0)$ is well determined by $\rho_-([x, y))$. We identify A with $\mathbb{A}$ in such a way that $\mathfrak{S}^-$ has direction $-C_f^v$. Then $\lambda_A^{++} = w_x(\lambda^{++})$ and $\pi'_+(0) = w'w_x(\lambda^{++})$, with w' as above. By Section 1.8(b),

$$\pi'_+(0) \leq_{Q^\vee} \rho_-(y) - \rho_-(x) \leq_{Q^\vee} \lambda^{++}.$$

So there are a finite number of possibilities for $\rho_-(y)$.

(c) Now fix $\rho_-([x, y))$ and $\rho_-(y)$, and investigate the possibilities for $\rho_-(C'_y)$. We shall use a segment $[x', y']$ in A_1 parallel to $[x, y]$ and prove successively that there are a finite number of possibilities for $\rho_-(x')$, $\rho_-([x', y'))$, $\rho_-(y')$, and $\rho_-(C'_y)$. So we choose $\xi \in Y^{++}$ and in the interior of the fundamental chamber C_f^v. In the apartment A_1, with (A_1, C_x) identified with $(\mathbb{A}, C_0^+)$, we consider $x' = x + \xi$ and $y' = y + \xi$ (hence, $y' = x' + \lambda$).

As in (a) and (b) above, we get that there are a finite number of possibilities for $\rho_-(x')$. So we fix $\rho_-(x')$.

(c1) On one side, we may also enlarge in A_1 the segment $[x, x']$ by considering the segment $[x', x'']$, where $x'' = x' + \varepsilon\xi = x + (1 + \varepsilon)\xi$, with $\varepsilon > 0$ small.

On the other side, $[x, x']$ can be described as a path $\pi_1 : [0, 1] \to A_1$, defined by $\pi_1(t) = x + t\xi$. The retracted path $\pi = \rho_-(\pi_1)$ satisfies

$$\rho_-(x') - \rho_-(x) \leq_{Q^\vee} \pi'_+(1) \leq_{Q^\vee} \lambda^{++},$$

again by Section 1.8. So there are a finite number of possibilities for $\pi'_+(1)$, i.e., for $\rho_-([x', x''))$. But there exists (in A_1) a minimal gallery of the type of a reduced decomposition of w_λ from the unique local chamber $(C_x + \xi)$ containing $[x', x'')$

to $[x', y')$. Hence, there exists a gallery of the same type between (a local chamber containing) $\rho_-([x', x''))$ and $\rho_-([x', y'))$. Therefore, there is a finite number of possibilities for $\rho_-([x', y'))$.

As in (b), we deduce that there are a finite number of possibilities for $\rho_-(y')$.

(c2) The path $\rho_-([y, y'])$ is a Hecke path of shape ξ from $\rho_-(y)$ to $\rho_-(y')$. By [Gaussent and Rousseau 2008, Corollary 5.9], there exist a finite number of such paths. In particular, there are a finite number of possibilities for the segment-germ $\rho_-([y, y'))$ and for $\rho_-(C'_y)$.

(d) Next, we fix $\rho_-(C'_y)$. Fixing a reduced decomposition for w gives a minimal gallery between C'_y and C_y, hence a gallery of the same type between $\rho_-(C'_y)$ and $\rho_-(C_y)$. So, the number of possible $\rho_-(C_y)$ is finite and $d^W(\rho_-(C'_y), \rho_-(C_y)) \leq w$.

(e) Finally, let us consider the case $w_x = 1$; hence, $\lambda_A^{++} = \lambda^{++}$. So, in (b), we get $\pi'_+(0) = w'(\lambda^{++})$ with $w' \leq w_\lambda$; hence, $\pi'_+(0) \geq_{Q^\vee} w_\lambda(\lambda^{++}) = \lambda$ and $\lambda \leq_{Q^\vee} \pi'_+(0) \leq_{Q^\vee} \rho_-(y) - \rho_-(x) = \mu \leq_{Q^\vee} \lambda^{++}$. If, moreover, λ is in Y^{++}, then $\lambda = \lambda^{++}$ and $\mu = \lambda$. The Hecke path $\rho_-([x, y])$ is of shape λ and equal to the segment $[\rho_-(x), \rho_-(x)+\lambda]$. Its dual dimension is 0 [op. cit., 5.7]. By [op. cit., 6.3], there is one and only one segment in $\mathscr{I}$ with end y that retracts onto this Hecke path: any apartment containing y and $\mathfrak{S}^-$ contains $[x, y]$. But C_x is in the enclosure of x and $C'_y = C_y$ (computation in A_1). So, any apartment containing $\mathfrak{S}^-$ and C'_y contains C_x. Therefore, we have $\lambda = d^W(C_x, C'_y) = d^W(\rho_-(C_x), \rho_-(C'_y))$.

The end of the proof of the lemma follows then from (d) above. □

Proposition 2.2. *Let $C_x, C_y, C_z \in \mathscr{C}_0^+$ be such that $x \leq z \leq y$ and*

$$d^W(C_x, C_z) = \boldsymbol{w} \in W^+ \quad \textit{and} \quad d^W(C_z, C_y) = \boldsymbol{v} \in W^+.$$

Then $d^W(C_x, C_y)$ varies in a finite subset $P_{\boldsymbol{w},\boldsymbol{v}}$ of W^+, depending only on $\boldsymbol{w}$ and $\boldsymbol{v}$.

Let us write $\boldsymbol{w} = \lambda.w$ and $\boldsymbol{v} = \mu.v$ for $\lambda, \mu \in Y^+$ and $w, v \in W^v$. If we assume $\lambda = \lambda^{++}$ and $w = 1$, then any $\boldsymbol{w}' = \nu.u \in P_{\boldsymbol{w},\boldsymbol{v}}$ satisfies $\lambda + \mu \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda + \mu^{++}$ and $\nu - \lambda \in \mathrm{conv}(W^v.\mu^{++}, \geq \mu) \subset \mathrm{conv}(W^v.\mu^{++})$.

If, moreover, $\mu = \mu^{++} \in Y^{++}$, then $\nu = \lambda + \mu$ and $u \leq v$. In particular, for $\boldsymbol{w} = \lambda$ and $\boldsymbol{w}' = \mu$ in Y^{++}, we have $P_{\boldsymbol{w},\boldsymbol{v}} = \{\lambda + \mu\}$.

Proof. Consider any apartment A containing C_x, the sector-germ $\mathfrak{S}^-$ opposite C_x and the retraction ρ_- as in Lemma 2.1. Then $\rho_-(C_x) = C_x$ and $d^W(C_x, \rho_-(C_z))$ varies in a finite subset P_x of W^+ depending on $\boldsymbol{w}$, by Lemma 2.1. If

$$d^W(C_x, \rho_-(C_z)) = \lambda'.w',$$

then the relative position $w_z \in W^v$ of C_z and $\mathfrak{S}^-$ is equal to w'. Applying once more Lemma 2.1 to C_z and C_y, we get that $d^W(\rho_-(C_z), \rho_-(C_y))$ varies in a finite subset $P_{w'}$ of W^+ depending only on $\boldsymbol{v}$ and w'. Finally, $d^W(C_x, \rho_-(C_y))$ varies in the finite subset

$$P_{\boldsymbol{w},\boldsymbol{v}} = \{\boldsymbol{w}'.\boldsymbol{v}' \in W^+ \mid \boldsymbol{w}' = \lambda'.w' \in P_x \text{ and } \boldsymbol{v}' \in P_{w'}\}.$$

Taking now A containing C_x and C_y, we get $d^W(C_x, C_y) = d^W(C_x, \rho_-(C_y)) \in P_{\boldsymbol{w},\boldsymbol{v}}$.

To finish, suppose $\lambda = \lambda^{++}$ and $w = 1$. By Lemma 2.1, $P_1 = \{\lambda\}$; so, $w' = w_z = 1$. By Lemma 2.1 again, every $\boldsymbol{v}' = \mu'.v' \in P_{w'}$ satisfies $\mu \le_{Q^\vee} \mu' \le_{Q^\vee} \mu^{++}$. Therefore, any $\boldsymbol{w}'' = \nu.u$ in $P_{\boldsymbol{w},\boldsymbol{v}}$ is equal to $(\lambda + \mu').v'$ for $\mu'.v' \in P_{w'} = P_1$; hence

$$\lambda + \mu \le_{Q^\vee} \nu = \lambda + \mu' \le_{Q^\vee} \lambda + \mu^{++}.$$

If moreover $\mu \in Y^{++}$, then $\nu = \lambda + \mu$ and $u \le v$. The last particular case is now clear. □

Proposition 2.3. *Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \le y$ and $d^W(C_x, C_y) = \boldsymbol{u} \in W^+$. We consider $\boldsymbol{w}$ and $\boldsymbol{v}$ in W^+. Then the number $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ of $C_z \in C_0^+$ with $x \le z \le y$, $d^W(C_x, C_z) = \boldsymbol{w}$ and $d^W(C_z, C_y) = \boldsymbol{v}$ is finite, i.e., in $\mathbb{N}$.*

If we assume $\boldsymbol{w} = \lambda$, $\boldsymbol{v} = \mu$ and $\boldsymbol{u} = \nu$, then $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} = a^{\nu}_{\lambda,\mu} \ge 1$ (resp., $a^{\nu}_{\lambda,\mu} = 1$) when $\lambda \in Y^{++}$, $\mu \in Y^+$ (resp., $\lambda, \mu \in Y^{++}$) and $\nu = \lambda + \mu$.

N.B. From the above conditions, we get $d^v(x, z) = \lambda^{++}$ and $d^v(z, y) = \mu^{++}$. By [Gaussent and Rousseau 2014, 2.5], the number of points z satisfying these conditions is finite.

Proof. According to the above note, we may fix z and count now the possible C_z.

Let C'_z be the local chamber in z containing $[z, y)$ and $[z, y')$ for y' in a sufficiently small element of the filter C_y. By convexity, C'_z is well determined by z and C_y. But in an apartment containing C_y and C_z (hence also C'_z), we see that $d^W(C'_z, C_z)$ is well determined by $\boldsymbol{v}$. So there is a gallery (of a fixed type) from C'_z to C_z, thus the number of possible C_z is finite.

Assume now that $\boldsymbol{w} = \lambda \in Y^{++}$, $\boldsymbol{v} = \mu \in Y^+$, and $\boldsymbol{u} = \lambda + \mu$. Taking an apartment A_1 containing C_x and C_y, it is clear that the local chamber C_z in A_1 such that $d^W(C_x, C_z) = \lambda$ satisfies also $d^W(C_z, C_y) = \mu$ (as $d^W(C_x, C_y) = \lambda + \mu$). So $a^{\lambda+\mu}_{\lambda,\mu} \ge 1$. We consider now any C_z satisfying the conditions, with moreover $\mu \in Y^{++}$.

As in Proposition 2.2, we choose A containing C_x and $\mathfrak{S}^-$ opposite C_x. We saw in Lemma 2.1(e) that any apartment containing C_z and $\mathfrak{S}^-$ contains C_x and $d^W(C_x, \rho_-(C_z)) = \lambda$. With the same lemma applied to C_z and C_y, we see that any apartment containing C_z and $\mathfrak{S}^-$ contains C_y. In particular, there is an apartment A_1 containing C_x, C_z, C_y; so $d^W(C_x, C_z) = \lambda$, $d^W(C_z, C_y) = \mu$, and $d^W(C_x, C_y) = \lambda + \mu$. But $\lambda, \mu \in Y^{++}$, so C_z is in the enclosure of C_x and C_y. Therefore, C_z is unique: any other apartment A_2 containing C_x and C_y also contains x, y (with $x \le y$) and $x' = x + \xi$, $y' = y + \xi$ (with $x' \le y'$), for $\xi \in C^v_x = C^v_y$ small; by Section 1.9(a), A_2 contains $z \in \mathrm{cl}_{A_1}(\{x, y\})$ and $z' = z + \xi \in \mathrm{cl}_{A_1}(\{x', y'\})$, hence also $C_z \subset \mathrm{cl}_{A_1}(]z, z'))$. □

Theorem 2.4. *For any ring R, ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ is an algebra with identity* $\mathrm{Id} = T_1$ *such that*

$$T_{\boldsymbol{w}} * T_{\boldsymbol{v}} = \sum_{\boldsymbol{u} \in P_{\boldsymbol{w},\boldsymbol{v}}} a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}} T_{\boldsymbol{u}}$$

and $T_\lambda * T_\mu = T_{\lambda+\mu}$ for $\lambda, \mu \in Y^{++}$.

Proof. It follows from Propositions 2.2 and 2.3, as the map $T_{\boldsymbol{w}} * T_{\boldsymbol{v}} : \mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+ \to R$ is clearly G-invariant. □

Definition 2.5. The algebra ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ is the Iwahori–Hecke algebra associated to $\mathscr{I}$ with coefficients in R.

The structure constants $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$ are nonnegative integers. We conjecture that they are polynomials in the parameters q_i, q_i' with coefficients in $\mathbb{Z}$ and that these polynomials depend only on $\mathbb{A}$ and W. We prove this in the following section for $\boldsymbol{w}$, $\boldsymbol{v}$ generic, see the precise hypothesis just below. We get also this conjecture for some $\mathbb{A}$, W when all q_i, q_i' are equal; in the general case we get only that they are Laurent polynomials, see Section 6.7.

Geometrically, it is possible to get more information about $T_\lambda * T_\mu$ when $\lambda \in Y^{++}$ and $\mu \in Y^+$, but we shall obtain them algebraically; see Corollary 5.3.

3. Structure constants

In this section, we compute the structure constants $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$ of the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$, assuming that $\boldsymbol{v} = \mu . v$ is regular and $\boldsymbol{w} = \lambda . w$ is spherical, i.e., μ is regular and λ is spherical; see Section 1.1 for the definitions. We will adapt some results obtained in the spherical case in [Gaussent and Rousseau 2014] to our situation.

These structure constants depend on the shape of the standard apartment $\mathbb{A}$ and on the numbers q_M of Section 1.4. Recall that the number of (possibly) different parameters is at most $2|I|$. We denote by $\mathcal{Q} = \{q_1, \dots, q_l, q_1' = q_{l+1}, \dots, q_l' = q_{2l}\}$ this set of parameters.

3.1. ***Centrifugally folded galleries of chambers.*** Let z be a point in the standard apartment $\mathbb{A}$. We have twinned buildings $\mathcal{T}_z^+\mathscr{I}$ (resp., $\mathcal{T}_z^-\mathscr{I}$). We consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp., closed chambers) are the local chambers $C = \mathrm{germ}_z(z + C^v)$ (resp., local closed chambers $\overline{C} = \mathrm{germ}_z(z + \overline{C^v})$), where C^v is a vectorial chamber; see [Gaussent and Rousseau 2008, 4.5] or [Rousseau 2011, §5]. The distances (resp., codistances) between these chambers are written d^W (resp., d^{*W}). To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_z = (\mathbb{A}_z^-, \mathbb{A}_z^+)$.

Choose in $\mathbb{A}_z^-$ a negative (local) chamber C_z^- and denote by C_z^+ its opposite in $\mathbb{A}_z^+$. Consider the system of positive roots Φ^+ associated to C_z^+. Actually, $\Phi^+ = w . \Phi_f^+$ if Φ_f^+ is the system Φ^+ defined in Section 1.1 and $C_z^+ = \mathrm{germ}_z(z + w . C_f^v)$. Denote by $(\alpha_i)_{i\in I}$ the corresponding basis of Φ and by $(r_i)_{i\in I}$ the corresponding generators of W^v. Note that this change of notation is limited to Section 3.

Fix a reduced decomposition of an element $w \in W^v$, $w = r_{i_1} \dots r_{i_r}$, and let $\boldsymbol{i} = (i_1, \dots, i_r)$ be the type of the decomposition. Now consider galleries of (local) chambers $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r)$ in the apartment $\mathbb{A}_z^-$ starting at C_z^- and of type $\boldsymbol{i}$.

The set of all these galleries is in bijection with $\Gamma(\boldsymbol{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$ via the map

$$(c_1, \dots, c_r) \mapsto (C_z^-, c_1 C_z^-, \dots, c_1 \dots c_r C_z^-).$$

Let $\beta_j = -c_1 \cdots c_j(\alpha_{i_j})$; then β_j is the root corresponding to the common limit hyperplane $M_j = M(\beta_j, -\beta_j(z))$ of type i_j of

$$C_{j-1} = c_1 \cdots c_{j-1} C_z^- \quad \text{and} \quad C_j = c_1 \cdots c_j C_z^-$$

and satisfying $\beta_j(C_j) \geq \beta_j(z)$.

Definition. Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_z^+$. A gallery $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r) \in \Gamma(\boldsymbol{i})$ is said to be centrifugally folded with respect to $\mathfrak{Q}$ if $C_j = C_{j-1}$ implies that M_j is a wall and separates $\mathfrak{Q}$ from $C_j = C_{j-1}$. We denote this set of centrifugally folded galleries by $\Gamma_{\mathfrak{Q}}^+(\boldsymbol{i})$.

3.2. ***Liftings of galleries.*** Next, let $\rho_{\mathfrak{Q}} : \mathcal{T}_z\mathcal{I} \to \mathbb{A}_z$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r)$ in $\Gamma(\boldsymbol{i})$, one can associate the set of all galleries of type $\boldsymbol{i}$ starting at C_z^- in $\mathcal{T}_z^-\mathcal{I}$ that retract onto $\boldsymbol{c}$; we denote this set by $\mathcal{C}_{\mathfrak{Q}}(\boldsymbol{c})$. We denote the set of minimal galleries (i.e., $C_{j-1} \neq C_j$) in $\mathcal{C}_{\mathfrak{Q}}(\boldsymbol{c})$ by $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$. Recall from [Gaussent and Rousseau 2014, Proposition 4.4], that the set $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$ is nonempty if and only if the gallery $\boldsymbol{c}$ is centrifugally folded with respect to $\mathfrak{Q}$. Recall also from [op. cit., Corollary 4.5], that if $\boldsymbol{c} \in \Gamma_{\mathfrak{Q}}^+(\boldsymbol{i})$, then the number of elements in $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$ is

$$\sharp\, \mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c}) = \prod_{j\in J_1} (q_j - 1) \times \prod_{j\in J_2} q_j$$

where $q_j = q_{M_j} \in \mathcal{Q}$,

$$J_1 = \{j \in \{1, \dots, r\} \mid c_j = 1\}$$

and

$$J_2 = \{j \in \{1, \dots, r\} \mid c_j = r_{i_j} \text{ and } M_j \text{ is a wall separating } \mathfrak{Q} \text{ from } C_j\}.$$

3.3. ***Liftings of Hecke paths.*** The Hecke paths we consider here are slight modifications of those used in [Gaussent and Rousseau 2014]. Let us fix a local positive chamber $C_x \in \mathscr{C}_0^+ \cap \mathbb{A}$. Actually, a Hecke path of shape μ^{++} with respect to C_x in $\mathbb{A}$ is a μ^{++}-path in $\mathbb{A}$ that we denote by $\pi = [z' = z_0, z_1, \dots, z_{\ell_\pi}, y]$ and that satisfies the following assumptions.

For all $z = \pi(t)$, $z \neq z_0 = \pi(0)$, we ask that $x \overset{o}{<} z$. Then we choose the local negative chamber C_z^- as $C_z^- = \mathrm{pr}_z(C_x)$. This means that $\overline{C_z^-}$ contains $[z, x)$ and $[z, x')$ for x' in a sufficiently small element of the filter C_x. Then we assume moreover that for all $k \in \{1, \dots, \ell_\pi\}$, there exists a $(W^v_{z_k}, C^-_{z_k})$-chain from $\pi'_-(t_k)$ to $\pi'_+(t_k)$, where $z_k = \pi(t_k)$. More precisely, this means that, for all $k \in \{1, \dots, \ell_\pi\}$, there exist finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s)$ of real roots such that, for all $j = 1, \dots, s$:

(i) $r_{\beta_j}(\xi_{j-1}) = \xi_j$,

(ii) $\beta_j(\xi_{j-1}) < 0$,

(iii) $r_{\beta_j} \in W^v_{\pi(t_k)}$, i.e., $\beta_j(\pi(t_k)) \in \mathbb{Z}$,

(iv) each β_j is positive with respect to C_x, i.e., $\beta_j(z_k - C_x) > 0$.

The centrifugally folded galleries are related to the lifting of Hecke paths by the following lemma that we proved in [Gaussent and Rousseau 2014, Lemma 4.6].

Suppose that $z \in \mathbb{A}$ with $x \overset{o}{<} z$. Let ξ and η be two segment-germs in $\mathbb{A}_z^+$. Let $-\eta$ and $-\xi$ be opposite, respectively, of η and ξ in $\mathbb{A}_z^-$. Let $\boldsymbol{i}$ be the type of a minimal gallery between C_z^- and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $d^W(C_z^-, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}_z^+$ containing η. Suppose that ξ and η are conjugated by W_z^v.

Lemma. *The following conditions are equivalent*:

(i) *There exists an opposite ζ to η in $\mathcal{T}_z^- \mathcal{I}$ such that $\rho_{\mathbb{A}_z, C_z^-}(\zeta) = -\xi$.*

(ii) *There exists a gallery $\boldsymbol{c} \in \Gamma^+_{\mathfrak{Q}}(\boldsymbol{i})$ ending in $-\eta$.*

(iii) *There exists a (W_z^v, C_z^-)-chain from ξ to η.*

Moreover the possible ζ are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{\mathfrak{Q}}(\boldsymbol{c})$ for $\boldsymbol{c}$ in the set $\Gamma^+_{\mathfrak{Q}}(\boldsymbol{i}, -\eta)$ of galleries in $\Gamma^+_{\mathfrak{Q}}(\boldsymbol{i})$ ending in $-\eta$.

For a Hecke path as above and for $k \in \{1, \dots, \ell_\pi\}$, we define the segment-germs $\eta_k = \pi_+(t_k) = \pi(t_k) + \pi'_+(t_k).[0, 1)$ and $-\xi_k = \pi_-(t_k) = \pi(t_k) - \pi'_-(t_k).[0, 1)$. As above, $\boldsymbol{i}_k$ is the type of a minimal gallery between $C^-_{z_k}$ and $C_{-\xi_k}$, where $C_{-\xi_k}$ is the negative (local) chamber such that $-\xi_k \subset \overline{C_{-\xi_k}}$ and $d^W(C^-_{z_k}, C_{-\xi_k})$ is of minimal length. Let $\mathfrak{Q}_k$ be a fixed chamber in $\mathbb{A}^+_{z_k}$ containing η_k and $\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)$ be the set of all the galleries $(C^-_{z_k}, C_1, \dots, C_r)$ of type $\boldsymbol{i}_k$ in $\mathbb{A}^-_{z_k}$, centrifugally folded with respect to $\mathfrak{Q}_k$ and with $-\eta_k \in \overline{C_r}$.

Let us denote the retraction $\rho_{\mathbb{A}, C_x} : \mathcal{I}_{\geq x} \to \mathbb{A}$ simply by ρ and recall that $y = \pi(1)$. Let $S_{C_x}(\pi, y)$ be the set of all segments $[z, y]$ such that $\rho([z, y]) = \pi$, in particular, $\rho(z) = z'$. The following two theorems are proved in the same way as Theorems 4.8 and 4.12 of [Gaussent and Rousseau 2014]; in particular, we lift the path π step by step starting from the end of π.

Theorem 3.4. *The set $S_{C_x}(\pi, y)$ is nonempty if and only if π is a Hecke path with respect to C_x. Then, we have a bijection*

$$S_{C_x}(\pi, y) \simeq \prod_{k=1}^{\ell_\pi} \coprod_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k,-\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}).$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ depending only on $\mathbb{A}$.

Theorem 3.5. *Let $\lambda, \mu, \nu \in Y^{++}$ with λ spherical. Then, the number $m_{\lambda,\mu}(\nu)$ of points z in $\mathscr{I}$ with $d^v(0, z) = \lambda$ and $d^v(z, \nu) = \mu$ is equal to*

$$m_{\lambda,\mu}(\nu) = \sum_{w\in W^v/(W^v)_\lambda} \sum_{\pi} \prod_{k=1}^{\ell_\pi} \sum_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k,-\eta_k)} \sharp\mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}), \tag{1}$$

where π runs over the set of Hecke paths of shape μ with respect to C_x from $w\,.\,\lambda$ to ν and ℓ_π, $\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)$, and $\mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c})$ are defined as above for each such π.

Remark. In Theorems 3.4 and 3.5 above and in [Gaussent and Rousseau 2014], it is interesting to note that if $t_{\ell_\pi} = 1$, i.e., $z_{\ell_\pi} = y$, then, in the above formulas, $-\eta_{\ell_\pi}$ and $\mathfrak{Q}_{\ell_\pi}$ are not well defined: $\pi_+(1)$ does not exist. We have to understand that

$$\coprod_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_{\ell_\pi}}(\boldsymbol{i}_{\ell_\pi},-\eta_{\ell_\pi})} \mathcal{C}^m_{\mathfrak{Q}_{\ell_\pi}}(\boldsymbol{c})$$

is the set of all minimal galleries of type $\boldsymbol{i}_{\ell_\pi}$ starting from C_y^-, whose cardinality is $\prod_{j=1}^r q_{i_j}$ if $\boldsymbol{i}_{\ell_\pi} = (i_1, \dots, i_r)$.

3.6. ***The formula.*** Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \le y$ and $d^W(C_x, C_y) = \boldsymbol{u} \in W^+$. We consider $\boldsymbol{w}$ and $\boldsymbol{v}$ in W^+. Then we know that the number $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ of $C_z \in \mathscr{C}_0^+$ with $x \le z \le y$, $d^W(C_x, C_z) = \boldsymbol{w}$, and $d^W(C_z, C_y) = \boldsymbol{v}$ is finite; see Proposition 2.3. In order to obtain a formula for that number, we first use equivalent conditions on the W-distance between the chambers.

Lemma. (1) *Assume λ is spherical. Let $C_z^- = \mathrm{pr}_z(C_x)$ and let w_λ^+ be the longest element such that $w_\lambda^+.\lambda \in \overline{C^v_f}$. Then*

$$d^W(C_x, C_z) = \lambda.w \iff \begin{cases} d^W(C_x, z) = \lambda, \\ d^{*W}(C_z^-, C_z) = w_\lambda^+ w. \end{cases}$$

(2) *Assume μ is regular. Let $C_z^+ = \mathrm{pr}_z(C_y)$ and let w_μ be the unique element such that $\mu^{++} = w_\mu\,.\,\mu \in \overline{C^v_f}$. Then*

$$d^W(C_z, C_y) = \mu\,.\,v \iff \begin{cases} d^W(C_z, C_z^+) = w_\mu^{-1}, \\ d^W(C_z^+, C_y) = \mu^{++} w_\mu v. \end{cases}$$

As we assume μ regular, $C'_y = \mathrm{pr}_y(C_z)$ is the unique local chamber in y containing $[y, z)$, and $C_z^+ = \mathrm{pr}_z(C_y)$ is the unique local chamber in z containing $[z, y)$. Also,

$$d^W(C_z^+, C_y) = \mu^{++} w_\mu v \iff d^v(z, y) = \mu^{++} \text{ and } d^{*W}(C'_y, C_y) = w_\mu v.$$

Proof. (1) By convexity, C_z^- is in any apartment containing C_x and C_z. Let us fix such an apartment A and identify (A, C_x) with $(\mathbb{A}, \mathrm{germ}_0(C_f^v))$. By definition, we have $d^W(C_x, z) = d^W(C_x, z + C_x)$. Then, of course, $d^W(C_x, z) = \lambda$. Next as λ is supposed spherical, the stabilizer $(W^v)_\lambda$ is finite, so w_λ^+ is well defined and $x \overset{o}{<} z$, so C_z^- is well defined. Moreover, $d^W(\mathrm{op}_A C_z^-, z + C_x) = w_\lambda^+$ and $d^W(z + C_x, C_z) = w$. Therefore, by Chasles, we get $d^W(\mathrm{op}_A C_z^-, C_z) = w_\lambda^+ w$, but, by definition, $d^{*W}(C_z^-, C_z) = d^W(\mathrm{op}_A C_z^-, z + C_z)$.

(2) The first assertion is the Chasles' relation, as C_z, C_y, C_z^+, (and C'_y) are in a same apartment A'. The second comes from the fact that, if μ is regular, then $d^W(C_z^+, C_{zy}^+) = d^v(z, y) \in Y^{++}$, where C_{zy}^+ is opposite C'_y at y in A'. Moreover, $d^{*W}(C'_y, C_y) = d^W(C_{zy}^+, C_y) \in W^v$ by definition, so we conclude by Chasles. □

Theorem 3.7. *Assume μ is regular and λ is spherical. We choose the standard apartment $\mathbb{A}$ containing C_x and C_y. Then*

$$a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}} = \sum_{\pi, t_{\ell_\pi}=1} \left(\prod_{k=1}^{\ell_\pi - 1} \sum_{\boldsymbol{c} \in \Gamma_{\mathfrak{Q}_k}^+(\boldsymbol{i}_k, -\eta_k)} \sharp \mathcal{C}_{\mathfrak{Q}_k}^m(\boldsymbol{c}) \right) \left(\sum_{\boldsymbol{d} \in \Gamma_{C_y}^+(\boldsymbol{i}_\ell, \tilde{C}_y)} \sharp \mathcal{C}_{C_y}^m(\boldsymbol{d}) \right) \left(\sum_{\boldsymbol{e} \in \Gamma_{C_{z_0}^-}^+(\boldsymbol{i}, C'_{z_0})} \sharp \mathcal{C}_{C_{z_0}^-}^m(\boldsymbol{e}) \right)$$
$$+ \sum_{\pi, t_{\ell_\pi}<1} \left(\prod_{k=1}^{\ell_\pi} \sum_{\boldsymbol{c} \in \Gamma_{\mathfrak{Q}_k}^+(\boldsymbol{i}_k, -\eta_k)} \sharp \mathcal{C}_{\mathfrak{Q}_k}^m(\boldsymbol{c}) \right) \left(\sum_{\boldsymbol{e} \in \Gamma_{C_{z_0}^-}^+(\boldsymbol{i}, C'_{z_0})} \sharp \mathcal{C}_{C_{z_0}^-}^m(\boldsymbol{e}) \right),$$

where the π in the first sum runs over the set of all Hecke paths in $\mathbb{A}$ with respect to C_x of shape μ^{++} from $x + \lambda = z_0$ to $x + \nu = y$ such that $t_{\ell_\pi} = 1$; whereas, in the second sum, the paths have to satisfy $t_{\ell_\pi} < 1$ and $d^{*W}(C_y^-, C_y) = w_\mu v$, where $C_y^- = \mathrm{pr}_y(C_x)$ is the local chamber in y containing $[y, x)$ and $[y, x')$ for x' in a sufficiently small element of the filter C_x.

Moreover, $\boldsymbol{i}$ is a reduced decomposition of w_μ, C'_{z_0} is the local chamber at z_0 in $\mathbb{A}$ defined by $d^{*W}(C_{z_0}^-, C'_{z_0}) = w_\lambda^+ w$, $\boldsymbol{i}_\ell$ is the type of a minimal gallery from C_y^- to the local chamber C_y^* at y in $\mathbb{A}$ containing the segment-germ $\pi_-(y) = y - \pi'_-(1).[0, 1)$, and $\tilde{C}_y$ is the unique local chamber at y in $\mathbb{A}$ such that $d^{*W}(\tilde{C}_y, C_y) = w_\mu v$. The rest of the notation is as defined above.

Proof. Recall that in order to compute the structure constants, we use the retraction $\rho = \rho_{\mathbb{A}, C_x} : \mathscr{I} \to \mathbb{A}$, where C_x and C_y are fixed and in $\mathbb{A}$. We have $y = \rho(y) = x + \nu$, and the condition $d^W(C_x, z) = \lambda$ is equivalent to $\rho(z) = x + \lambda = z_0$. We want to prove a formula of the form

$$a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} = \sum_{\pi} \big(\text{number of liftings of } \pi\big) \times \big(\text{number of } C_z\big),$$

where π runs over some set of Hecke paths with respect to C_x of shape μ^{++} from $x+\lambda$ to $x+\nu$. It is possible to calculate like that for, in the case of a regular μ^{++}, $\rho(C_z^+)$ is well determined by π. Hence, the number of C_z only depends on π and not on the lifting of π.

The local chambers C_z satisfying $d^{*W}(C_z^-, C_z) = w_\lambda^+ w$ and $d^W(C_z, C_z^+) = w_\mu^{-1}$ are at the end of a minimal gallery starting at C_z^+ of type $\boldsymbol{i}$ and retracting by ρ_{A', C_z^-} onto the local chamber C'_z at z defined by $d^{*W}(C_z^-, C'_z) = w_\lambda^+ w$ in a fixed apartment A' containing C_x and C_z^+. So their number is given by the number of minimal galleries starting at C_z^+ of type $\boldsymbol{i}$ and retracting on a centrifugally folded gallery $\boldsymbol{e}$ of type $\boldsymbol{i}$ ending in C'_z. In other words, their number is given by the cardinality of the set $\mathcal{C}^m_{C_z^-}(\boldsymbol{e})$, for each $\boldsymbol{e} \in \Gamma^+_{C_z^-}(\boldsymbol{i}, C'_z)$. Using an isomorphism fixing C_x and sending A' to $\mathbb{A}$, we may replace in this formula z, C_z^-, C'_z, and C_z^+ by z_0, $C_{z_0}^-$, C'_{z_0}, and the unique local chamber $C_{z_0}^+$ in $\mathbb{A}$ containing the segment-germ $\pi_+(0) = z_0 + \pi'_+(0).[0, 1)$. Hence,

$$\text{number of } C_z = \sum_{\boldsymbol{e} \in \Gamma^+_{C_{z_0}^-}(\boldsymbol{i}, C'_{z_0})} \sharp\, \mathcal{C}^m_{C_{z_0}^-}(\boldsymbol{e}).$$

Now, we compute the number of liftings of a Hecke path π starting from the formula in Theorem 3.5 and according to the two conditions $d^W(C_x, z) = \lambda$ and $d^W(C_z^+, C_y) = \mu^{++} w_\mu v$. The first one fixes one element in the set $W^v/(W^v)_\lambda$, namely the coset of w_λ^+, i.e., $\pi(0) = x+\lambda$. The second one is equivalent to the fact that the segment $[z, y]$ is of type μ^{++} and $d^{*W}(C'_y, C_y) = w_\mu v$, as we have seen in the lemma above.

Further, we have that $t_{\ell_\pi} < 1$ if and only if $\pi_-(y) \in C_y^-$. If $\pi_-(y) \in C_y^-$ then $\rho(C'_y) = C'_y = C_y^-$, whence, $d^{*W}(C_y^-, C_y) = w_\mu v$. Since we lift the Hecke path into a segment backwards starting with its behavior at $y = \pi(1)$, there is nothing more to count.

If $t_{\ell_\pi} = 1$, then $\pi_-(y) \in C_y^* = \rho(C'_y) \neq C_y^-$. We want to lift the path but with the condition that $d^{*W}(C'_y, C_y) = w_\mu v$, which may be translated in $\rho'(C'_y) = \tilde{C}_y$, for $\rho' = \rho_{\mathbb{A}, C_y}$. Since μ^{++} is regular, to find $[y, z)$ it is enough to find C'_y, i.e., to lift $\tilde{C}_y$ with respect to ρ'. The liftings of $\tilde{C}_y$ are then given by the liftings of all the centrifugally folded galleries in $\mathbb{A}$ with respect to C_y of type $\boldsymbol{i}_\ell$ from C_y^- to $\tilde{C}_y$ to minimal galleries. Therefore, their number is given by the cardinality of the set $\mathcal{C}^m_{C_y}(\boldsymbol{d})$, for each $\boldsymbol{d} \in \Gamma^+_{C_y}(\boldsymbol{i}_\ell, \tilde{C}_y)$. The rest of the lifting procedure is the same as in the proof of Theorem 4.12 in [Gaussent and Rousseau 2014]. □

3.8. ***Consequence.*** The above explicit formula, together with the formula for $\sharp\mathcal{C}^m_{\mathfrak{Q}}(\boldsymbol{c})$ in Section 3.2, tell us that the structure constant $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is a polynomial in the parameters $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ and that this polynomial depends only on $\mathbb{A}$, W, $\boldsymbol{w}$, $\boldsymbol{v}$, and $\boldsymbol{u}$. So we have proved the conjecture following Definition 2.5 in this generic case: when λ is spherical and μ regular.

4. Relations

Here we study the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}^{\mathscr{I}}_R$ as a module over $\mathcal{H}_R(W^v)$ and we prove the first instance of the Bernstein–Lusztig relation. For short, we write ${}^{\mathrm{I}}\mathcal{H}_R = {}^{\mathrm{I}}\mathcal{H}^{\mathscr{I}}_R$ and $T_i = T_{r_i}$ (when $i \in I$).

Proposition 4.1. *Let $\lambda \in Y^+$, $w \in W^v$, and $i \in I$. Then*:

(1) $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$ *if and only if either* $(w(\alpha_i))(\lambda) < 0$ *or* $(w(\alpha_i))(\lambda) = 0$ *and* $\ell(wr_i) > \ell(w)$. *Otherwise* $T_{\lambda.w} * T_i = (q_i - 1)T_{\lambda.w} + q_i T_{\lambda.wr_i}$.

(2) $T_i * T_{\lambda.w} = T_{r_i(\lambda).r_i w}$ *if and only if either* $\alpha_i(\lambda) > 0$ *or* $\alpha_i(\lambda) = 0$ *and* $\ell(r_i w) > \ell(w)$. *Otherwise* $T_i * T_{\lambda.w} = (q_i - 1)T_{\lambda.w} + q_i T_{r_i(\lambda).r_i w}$.

Proof. We consider local chambers C_x, C_z, C_y with $x \le z \le y$ and $d^W(C_x, C_z) = \lambda.w$, $d^W(C_z, C_y) = r_i$. So there is an apartment A containing C_x, C_z (but perhaps not C_y) and, if we identify (A, C_x) to $(\mathbb{A}, C_0^+)$, we have $C_z = (\lambda.w)(C_x) = w(C_x) + \lambda$.

Moreover, $y = z$, $C_z \ne C_y$, and C_z, C_y share a panel F_i of type i. We write D for the half-apartment of A containing C_x and with wall ∂D containing F_i.

Actually the equation of ∂D in A is $(w(\alpha_i))(x') = (w(\alpha_i))(z)$. As $\alpha_i > 0$ on C_x, we have $(w(\alpha_i))(C_z) > (w(\alpha_i))(z)$. And so $(w(\alpha_i))(z) = (w(\alpha_i))(\lambda) < 0 = (w(\alpha_i))(x)$ (resp., $> 0 = (w(\alpha_i))(x)$) if and only if C_z is strictly on the same side (resp., the opposite side) of ∂D as x, hence as C_x; i.e., $C_z \subset D$ (resp., $C_z \not\subset D$). If now $(w(\alpha_i))(\lambda) = 0$, we may argue as if $\lambda = 0$, i.e., $C_z = w(C_x)$, then it is well known that $C_z \subset D$ if and only if $\ell(wr_i) > \ell(w)$. So,

$$C_z \subset D \iff \big((w(\alpha_i))(\lambda) < 0\big) \text{ or } \big((w(\alpha_i))(\lambda) = 0 \text{ and } \ell(wr_i) > \ell(w)\big).$$

Then, by Section 1.4.2, there exists an apartment A' containing C_y and D, hence also C_x, C_z, C_y. So $d^W(C_x, C_y) = \lambda.wr_i$. The panel $F_i = F^\ell(z, F_i^v) \subset A$ is a spherical local face, so, for any $p \in z + F_i^v \subset A$, we have $z \overset{o}{<} p$, hence $x \overset{o}{<} p$. By Proposition 1.10(a), any apartment A'' containing C_x and F_i contains C_z; moreover C_z is well determined by F_i and C_x. The number $a^{\lambda.wr_i}_{\lambda.w,r_i}$ of Proposition 2.3 is equal to 1 and we have proved that $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$.

If C_z is not in D, we denote by C'_z the local chamber in D with panel F_i. By the above argument, C'_z is well determined by F_i and C_x; moreover $d^W(C_x, C'_z) = \lambda.wr_i$. There are two cases: either $C_y = C'_z$ or not. If $C_y = C'_z$, then $d^W(C_x, C_y) = \lambda.wr_i$, and if C_x, C_y are given, there are q_i possibilities for C_z (all local chambers covering F_i

and different from C'_z): $a_{\lambda.w,r_i}^{\lambda.wr_i} = q_i$. If $C_y \neq C'_z$, then $d^W(C_x, C_y) = \lambda.w$ and, if C_x, C_y are given, there are $q_i - 1$ possibilities for C_z (all local chambers covering F_i and different from C'_z, C_y): $a_{\lambda.w,r_i}^{\lambda.w} = q_i - 1$.

We have proved (1) and we leave to the reader the similar proof of (2). □

4.2. ***The subalgebra $\mathcal{H}_R(W^v)$.*** We consider the R-submodule $\mathcal{H}_R(W^v)$ of ${}^{\mathrm{I}}\mathcal{H}_R$ with basis $(T_w)_{w\in W^v}$. As $d^W(C_x, C_y) \in W^v$ if and only if $x = y$, it is clearly a subalgebra of ${}^{\mathrm{I}}\mathcal{H}_R$. Actually $\mathcal{H}_R(W^v)$ is the Iwahori–Hecke algebra of the tangent building $\mathcal{T}_x^+\mathcal{I}$ for any $x \in \mathcal{I}$.

By Proposition 4.1,

$$\begin{cases} T_w * T_i = T_{wr_i} & \text{if } \ell(wr_i) > \ell(w), \\ T_w * T_i = (q_i - 1)T_w + q_i T_{wr_i} & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} T_i * T_w = T_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ T_i * T_w = (q_i - 1)T_w + q_i T_{r_i w} & \text{otherwise.} \end{cases}$$

In particular, $T_i^2 = (q_i - 1)T_i + q_i \,\mathrm{Id}$, and $T_w = T_{i_1} \cdots T_{i_n}$ for any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$.

Therefore, the algebra $\mathcal{H}_R(W^v)$ is the well known Hecke algebra associated to the Coxeter system $(W^v, \{r_i \mid i \in I\})$ with (in general unequal) parameters $(q_i)_{i\in I}$ and coefficients in the ring R. It is generated, as an R-algebra, by the T_i, for $i \in I$.

Suppose each q_i is invertible in R. Then, as is well known,

$$T_i^{-1} = q_i^{-1}\big(T_i - (q_i - 1)\,\mathrm{Id}\big) \in \mathcal{H}_R(W^v)$$

is the inverse of T_i. In particular any T_w is invertible: $T_w^{-1} = T_{i_n}^{-1} \cdots T_{i_1}^{-1}$ for any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$.

Remark. Assuming that q_i is invertible, it is easy to see from Proposition 4.1 that either $T_{\lambda.wr_i} = T_{\lambda.w} * T_i$ or $T_{\lambda.wr_i} = T_{\lambda.w} * T_i^{-1}$, and either $T_{r_i(\lambda).r_i w} = T_i * T_{\lambda.w}$ or $T_{r_i(\lambda).r_i w} = T_i^{-1} * T_{\lambda.w}$.

Corollary 4.3. *Suppose each q_i invertible in R and consider $\lambda \in Y^+$. We may write $\lambda = w.\lambda^{++}$, with $w \in W^v$. Then $T_\lambda = T_w * T_{\lambda^{++}} * T_w^{-1}$.*

Proof. Consider a reduced decomposition $w = r_{i_n} \cdots r_{i_1}$ and argue by induction on n. So, for $w' = r_{i_{n-1}} \cdots r_{i_1}$ and $\lambda' = w'.\lambda^{++}$, we have $T_{\lambda'} = T_{w'} * T_{\lambda^{++}} * T_{w'}^{-1}$. Now consider

$$T_w * T_{\lambda^{++}} * T_w^{-1} = T_{i_n} * T_{\lambda'} * T_{i_n}^{-1}.$$

But $\ell(r_{i_n} w') > \ell(w')$ and $\lambda^{++} \in Y^{++} \subset \overline{C_f^v}$, so $\alpha_{i_n}(w'.\lambda^{++}) \geq 0$, i.e., $\alpha_{i_n}(\lambda') \geq 0$. We get $T_{i_n} * T_{\lambda'} = T_{r_{i_n}(\lambda').r_{i_n}}$ by Proposition 4.1(2), and then $T_{i_n} * T_{\lambda'} * T_{i_n}^{-1} = T_{r_{i_n}(\lambda')} = T_\lambda$ by Proposition 4.1(1) (and the above remark). □

Corollary 4.4. *Let $\lambda \in Y^+$ and $w, w' \in W^v$. Then we may write*

$$T_{\lambda.w'} * T_w = \sum_{w'' \leq w} a_{\lambda.w',w}^{\lambda.w'w''} T_{\lambda.w'w''},$$

where each $a_{\lambda.w',w}^{\lambda.w'w''}$ is a polynomial in the q_i with coefficients in $\mathbb{Z}$, and, when $w' = 1$, $a_{\lambda,w}^{\lambda.w} > 0$ is a primitive monomial. This polynomial $a_{\lambda.w',w}^{\lambda.w'w''}$ depends only on $\mathbb{A}$ and on W.

Proof. Write $w = r_{i_1} \cdots r_{i_n}$ and argue by induction on n. The result is then clear from Proposition 4.1(1). Actually, $a_{\lambda,w}^{\lambda.w}$ is the product of certain q_{i_j}, $1 \leq j \leq n$. □

4.5. ***The Iwahori–Hecke algebra as a right $\mathcal{H}_R(W^v)$-module.*** We assume here that each q_i is invertible in R.

Given $\lambda \in Y^+$, we can conclude from Corollary 4.4 that $\{T_\lambda * T_w \mid w \in W^v\}$ and $\{T_{\lambda.w} \mid w \in W^v\}$ are two bases of the same R-module. The base-change matrix is triangular with respect to the Bruhat order on W^v and the coefficients are Laurent polynomials in the q_i, with coefficients in $\mathbb{Z}$ (primitive Laurent monomials on the diagonal). These polynomials depend only on $\mathbb{A}$ and W.

As $\{T_{\lambda.w} \mid \lambda \in Y^+, w \in W^v\}$ is an R-basis of ${}^{\mathrm{I}}\mathcal{H}_R$ and $\{T_w \mid w \in W^v\}$ is an R-basis of $\mathcal{H}_R(W^v)$, in particular, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module with basis $\{T_\lambda \mid \lambda \in Y^+\}$.

The R-algebra ${}^{\mathrm{I}}\mathcal{H}_R$ is generated by the T_i (for $i \in I$) and the T_λ (for $\lambda \in Y^+$) and even by the T_i (for $i \in I$) and the T_λ (for $\lambda \in Y^{++}$), as we see from Corollary 4.3.

Lemma 4.6. *Let $C_1, C_2 \in \mathcal{C}_0^+$ with vertices x_1, x_2 be such that $d^W(C_1, C_2) = \lambda \in Y^{++}$. We consider $i \in I$, F_1^i (resp., F_2^i) the panel of type i of C_1 (resp., C_2). In an apartment A_1 (resp., A_2) containing C_1 (resp., C_2), we consider the sector panel $\mathfrak{f}_1^-$ (resp., $\mathfrak{f}_2^+$) with base point x_1 (resp., x_2) and direction opposite the direction of F_1^i (resp., equal to the direction of F_2^i).*

Then there is an apartment A containing $\mathfrak{f}_1^-$, $\mathfrak{f}_2^+$, C_1, C_2 and, in this apartment A, the directions of $\mathfrak{f}_1^-$ and $\mathfrak{f}_2^+$, F_2^i and $\mathfrak{f}_1^-$ (resp., F_1^i and $\mathfrak{f}_2^+$) are opposite (resp., equal).

Proof. Choose $\lambda_i \in F^v(\{i\}) \cap Y \subset Y^{++}$, and write $\mathfrak{F}_j^\pm$ for the germ of $\mathfrak{f}_j^\pm$ and $F_j^{\pm v}$ for its direction in A_j. In A_1 (resp., A_2) we consider the splayed chimney $\mathfrak{r}_1^- = \mathfrak{r}(C_1, F_1^{-v})$ (resp., $\mathfrak{r}_2^+ = \mathfrak{r}(C_2, F_2^{+v})$) containing $\mathfrak{f}_1^-$ (resp., $\mathfrak{f}_2^+$) and, for $n \in \mathbb{N}$, the chamber of type 0: $C_1(-n) = C_1 - n\lambda_i \subset \mathfrak{r}_1^-$ (resp., $C_2(+n) = C_2 + n\lambda_i \subset \mathfrak{r}_2^+$); actually we identify $(\mathbb{A}, C_0^+)$ with (A_1, C_1) (resp., (A_2, C_2)) to consider λ_i in $\vec{A}_1$ (resp., $\vec{A}_2$).

Then $d^W(C_1(-n), C_1) = d^W(C_2, C_2(+n)) = n\lambda_i$ and $d^W(C_1, C_2) = \lambda$, both in Y^{++}. By (MA3) there is an apartment A containing the germs $\mathfrak{R}_1^-$ of $\mathfrak{r}_1^-$ and $\mathfrak{R}_2^+$ of $\mathfrak{r}_2^+$; hence, $C_1(-n)$ and $C_2(+n)$ for n great. By Proposition 2.2 and the last paragraph of the proof of Proposition 2.3, $d^W(C_1(-n), C_2(+n)) = \lambda + 2n\lambda_i \in Y^{++}$

and A contains C_1, C_2. By (MA4) A contains also $\mathfrak{f}_1^- \subset \mathfrak{r}_1^- \subset \mathrm{cl}_{A_1}(C_1, \mathfrak{R}_1^-)$ and $\mathfrak{f}_2^+ \subset \mathfrak{r}_2^+ \subset \mathrm{cl}_{A_2}(C_2, \mathfrak{R}_2^+)$. So all assertions of the lemma are satisfied. □

Proposition 4.7. *Let $C_1, C_2, C_3, C_4 \in \mathcal{C}_0^+$ be such that $d^W(C_1, C_2) = \lambda \in Y^{++}$, $d^W(C_2, C_3) = r_i$, and $d^W(C_3, C_4) = \mu \in Y^{++}$. Then there is a direction of wall* (*i.e., a parallel class of walls*) *M_i^∞* (*see* [Rousseau 2011, §4] *or* [Gaussent and Rousseau 2014, 5.5]), *chosen according to C_1, C_2* (*but independently from C_3, C_4*), *such that C_1, C_2, C_3, C_4 are in the extended tree $\mathscr{I}(M_i^\infty)$.*

Proof. We denote by $x_1, x_2 = x_3, x_4$ the three vertices of C_1, C_2, C_3, C_4 and by $F_1^i, F_2^i = F_3^i, F_4^i$ their panels of type i. We choose $\mathfrak{f}_1^-$ associated to C_1 and F_1^i in an apartment A_1 (resp., $\mathfrak{f}_4^+$ associated to C_4 and F_4^i in an apartment A_4), as in Lemma 4.6. By this lemma, using C_1 and C_2, the direction of $\mathfrak{f}_1^-$ opposites that of $F_2^i = F_3^i$ in some apartment A_2 and, using C_3 and C_4, the direction of $\mathfrak{f}_4^+$ is the same as that of $F_2^i = F_3^i$ in some apartment A_3. In A_3 (resp., A_2) we consider the sector face $\mathfrak{f}_3^+$ (resp., $\mathfrak{f}_2^-$) with base point $x_2 = x_3$ and same direction as $\mathfrak{f}_4^+$ or $F_2^i = F_3^i$ (resp., same direction as $\mathfrak{f}_1^-$ and opposite $F_2^i = F_3^i$).

We may use the lemma for $C_1, C_2, \mathfrak{f}_1^-, \mathfrak{f}_3^+$; so the directions of $\mathfrak{f}_1^-$ (or $\mathfrak{f}_2^-$) and $\mathfrak{f}_3^+$ (or $\mathfrak{f}_4^+$) are opposite and C_1, C_2 are in a same apartment A_5 of $\mathscr{I}(M_i^\infty)$, if we consider the direction of wall M_i^∞ associated to the directions of $\mathfrak{f}_1^-$ and $\mathfrak{f}_4^+$. Using now the lemma for $C_3, C_4, \mathfrak{f}_2^-, \mathfrak{f}_4^+$, we see that these filters are in a same apartment A_6 of $\mathscr{I}(M_i^\infty)$. □

Theorem 4.8. *Let $\lambda, \mu \in Y^{++}$ and $i \in I$, write $N = \inf(\alpha_i(\lambda), \alpha_i(\mu)) \in \mathbb{N}$, and, for $n \in \mathbb{N}$, $q_i^{*n} = q_i q_i' q_i q_i' \cdots$, with n terms in this product.*

(a) *If $N = \alpha_i(\mu) \le \alpha_i(\lambda)$, then $T_\lambda * T_i * T_\mu = T_{\lambda+\mu} * T_i$ for $N = 0$ and, for $N > 0$,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_{\lambda+\mu-N\alpha_i^\vee} * T_i + (q_i^{*N} - q_i^{*N-1}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}.$$

(b) *If $N = \alpha_i(\lambda) \le \alpha_i(\mu)$, then $T_\lambda * T_i * T_\mu = T_i * T_{\lambda+\mu}$ for $N = 0$ and, for $N > 0$,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_i * T_{\lambda+\mu-N\alpha_i^\vee} + (q_i^{*N} - q_i^{*N-1}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}.$$

Remarks. (1) The case (b) is less interesting for us, as we try to express any element in the basis of Section 4.5 for ${}^{\mathrm{I}}\mathcal{H}_R$ considered as a right $\mathcal{H}_R(W^v)$-module.

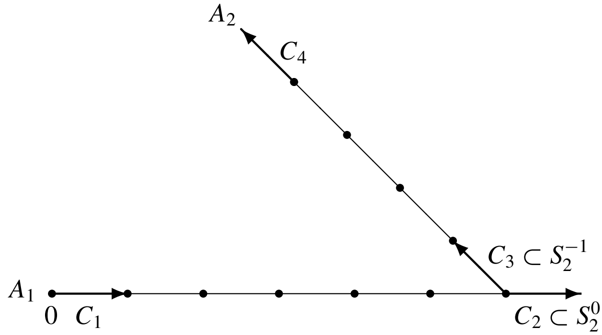
(2) In the case (a) we have $\mu - N\alpha_i^\vee = r_i(\mu)$ and $\lambda + \mu - N\alpha_i^\vee \in Y^{++}$, as $\alpha_i(\lambda + \mu - N\alpha_i^\vee) = \alpha_i(\lambda) - N$ and $\alpha_j(\lambda + \mu - N\alpha_i^\vee) \ge \alpha_j(\lambda) + \alpha_j(\mu)$ for $j \ne i$. So all ν such that T_ν appears on the right of the formula are in the $\alpha_i^\vee$-chain between $\lambda + \mu$ and $\lambda + r_i(\mu)$; in particular they are all in Y^{++}.

(3) We call relation (a) or relation (b) the Bernstein–Lusztig relation for the T_λ, (BLT) for short. We shall use it essentially when $\lambda = \mu$.

(4) When $\alpha_i(\lambda)$ or $\alpha_i(\mu)$ is odd, we know that $q_i' = q_i$; see Section 1.4.5.

Proof. We consider C_1, C_2, C_3, C_4, and M_i^∞ as in Proposition 4.7. When $N = 0$ the results come from Proposition 4.1. We concentrate on the case $0 < N = \alpha_i(\mu) \leq \alpha_i(\lambda)$; the other case is left to the reader. We have to evaluate $d^W(C_1, C_4)$ and, given C_1, C_4 satisfying $d^W(C_1, C_4) = \boldsymbol{u}$, to count the number of possible C_2, C_3. By Proposition 4.7 everything is in the extended tree $\mathscr{I}(M_i^\infty)$, which is semihomogeneous with thicknesses $1 + q_i$, $1 + q_i'$. By Proposition 4.1(2), C_3 is well determined by C_2, C_4 and lies in any apartment containing C_2, C_4; moreover $d^W(C_2, C_4) = r_i(\mu).r_i$.

We consider an apartment A_1 (resp., A_2) of $\mathscr{I}(M_i^\infty)$ containing C_1 and C_2 (resp., C_2 and C_4, hence also C_3), as illustrated in the figure:



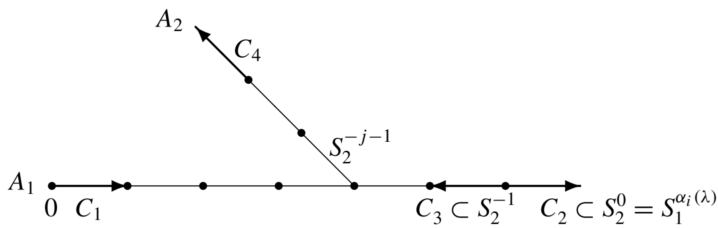
We identify (A_1, C_1) and (A_2, C_2) with $(\mathbb{A}, C_0^+)$; we consider the retraction ρ_1 (resp., ρ_2) of $\mathscr{I}(M_i^\infty)$ onto A_1 (resp., A_2) with center C_1 (resp., C_2). The closed chambers in an apartment of $\mathscr{I}(M_i^\infty)$ are stripes limited by walls of direction M_i^∞. In $A_1 = \mathbb{A}$, these walls are $M(\alpha_i, n)$, $n \in \mathbb{Z}$ and we write S_1^k the stripe $S_1^k = \{x \mid k \leq \alpha_i(x) \leq k+1\}$, in particular $C_1 \subset S_1^0$ and $C_2 \subset S_1^{\alpha_i(\lambda)}$. In $A_2 = \mathbb{A}$, we get also stripes $S_2^k = \{x \mid k \leq \alpha_i(x) \leq k+1\}$ such that $C_2 \subset S_2^0 = S_1^{\alpha_i(\lambda)}$, $C_3 \subset S_2^{-1}$ and $C_4 \subset S_2^{-N-1}$.

We have $C_2 = C_1 + \lambda$ in A_1 and $\rho_2(C_4) = C_3 + r_i(\mu)$ in A_2. To find $d^W(C_1, C_4)$ we have to determine the image of C_4 under ρ_1. It depends actually on the highest number j such that S_2^{-j} (hence also $S_2^0, \ldots, S_2^{-j+1}$) is in A_1. A classical result for affine buildings (clear for extended trees and generalized to hovels in [Rousseau 2011, 2.9.2]) tells, then, that there is an apartment containing the stripes $S_2^{-j-1}, \ldots, S_2^{-N-1}$ and the half-apartment $\bigcup_{k \leq \alpha_i(\lambda)-j-1} S_1^k$.

If $j = 0$, then S_2^{-1} or C_3 is not in A_1, so $\rho_1(C_3) = C_2$ and, more generally, $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda)+k-1}$, for $k \geq 1$. This is the case illustrated in the figure above. We get $\rho_1(C_4) = C_2 + \mu$ and $d^W(C_1, C_4) = \lambda + \mu$. When C_1 and C_4 are fixed with this W-distance, we have to count the number of possible C_2. But $C_3 \subset S_2^{-1}$ is in the enclosure of $C_1 \subset S_1^0$ and $C_4 \subset S_2^{-N-1}$: it is well determined by C_1 and C_4. Now

C_2 has to share its panel of type i with C_3 and to be neither in S_2^{-1} nor in $S_1^{\alpha_i(\lambda)-1}$; so there are $q_i - 1$ possibilities.

If $1 \le j \le N-1$, then A_1 contains $S_2^0 = S_1^{\alpha_i(\lambda)}$, $S_2^{-1} = S_1^{\alpha_i(\lambda)-1}, \dots, S_2^{-j} = S_1^{\alpha_i(\lambda)-j}$, but not $S_2^{-j-1}, \dots, S_2^{-N-1}$; this is the case illustrated below:



So $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda)-2j+k}$, for $k \ge j$. As in the proof of Proposition 4.7, we write $x_1, x_2 = x_3, x_4$ for the vertices of the local chambers C_1, C_2, C_3, C_4. The image of the line segment $[x_2, x_4] = [x_2, x_2 + \mu]$ under ρ_1 is

$$\rho_1([x_2, x_4]) = \left[x_2,\ x_2 + \frac{j}{N} r_i(\mu)\right] \cup \left[x_2 + \frac{j}{N} r_i(\mu),\ x_2 + \frac{j}{N} r_i(\mu) + \frac{N-j}{N}\mu\right].$$

As $N = \alpha_i(\mu)$ and $r_i(\mu) = \mu - N\alpha_i^\vee$, this means that $\rho_1(C_4) = C_2 + \mu - j\alpha_i^\vee$. When C_1 and C_4 are fixed with this W-distance, we have to count the number of possible C_2. As $S_1^0, \dots, S_1^{\alpha_i(\lambda)-j-1}, S_2^{-j-1}, \dots, S_2^{-N-1}$ are well determined by C_1, C_4, we have to count the possibilities for $(S_1^{\alpha_i(\lambda)-j}, \dots, S_1^{\alpha_i(\lambda)})$. As above, there are $q_i - 1$ possibilities for $S_1^{\alpha_i(\lambda)-j}$ (or $q_i' - 1$ if j is odd) and then q_i' (or q_i) possibilities for $S_1^{\alpha_i(\lambda)-j+1}$, etc. Finally the total number of possibilities is $(q_i - 1)q_i' q_i q_i' \cdots$ or $(q_i' - 1)q_i q_i' q_i \cdots$ (according to j being even or odd) with $j+1$ terms in the product. The last factor is necessarily q_i, so this total number is $(q_i^{*j+1} - q_i^{*j})$.

It is convenient to look at the cases $j = N$ or $j = N+1$ simultaneously. This means that $S_2^{-N} = S_1^{\alpha_i(\lambda)-N}$ is in A_1; in particular the panel F_4^i of type i of C_4 is in A_1, in the wall $\{x \mid \alpha_i(x) = \alpha_i(\lambda) - N\}$. More precisely F_4^i is the panel of type i of $C_4' = C_1 + \lambda + r_i(\mu) \subset A_1$. This means that $(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4) \ge 1$.

Conversely if C_1, C_4 are fixed satisfying this condition, we can find C_2, C_3 with the required W-distances. We have now to count the number of possibilities for C_2, C_3, i.e., for C_2 or for $(S_1^{\alpha_i(\lambda)-N}, \dots, S_1^{\alpha_i(\lambda)})$. The number of possibilities for $S_1^{\alpha_i(\lambda)-N}$ is exactly $(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$. Then the number of possibilities for $S_1^{\alpha_i(\lambda)-N+1}, \dots, S_1^{\alpha_i(\lambda)}$ is alternatively q_i or q_i'. Finally the total number of possibilities for C_2 is $q_i^{*N}(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$ (as, when N is odd, $q_i = q_i'$). □

5. New basis

In this section, we prove that left multiplication by T_μ, for $\mu \in Y^{++}$, is injective. That allows us to introduce a new basis of the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ in terms of $(T_w)_{w \in W^v}$ and $(X^\lambda)_{\lambda \in Y^+}$. From now on the main arguments are algebraic.

We suppose $\mathbb{Z} \subset R$ and each q_i, q'_i in $R^\times$, the set of invertible elements in R. As we saw in Section 4.5, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module with basis $\{T_\lambda \mid \lambda \in Y^+\}$.

For $\lambda \in Y^{++}$ and $H \in \mathcal{H}_R(W^v)$, we say that $T_\lambda * H$ is of degree λ.

For $i \in I$ and Ω a subset of the model apartment $\mathbb{A}$, we write $c(i)(\Omega)$ the convex hull of $\Omega \cup r_i(\Omega)$. For $(i_1, i_2, \dots, i_h) \in I^h$ and $(\lambda_0, \lambda_1, \dots, \lambda_h) \in (Y^{++})^{h+1}$, we define: $D(i_h)(\lambda_{h-1}, \lambda_h) = \lambda_{h-1} + c(i_h)(\lambda_h)$ and, by induction for k from $h-1$ to 1, $D(i_k, \dots, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_h) = \lambda_{k-1} + c(i_k)(D(i_{k+1}, \dots, i_h)(\lambda_k, \lambda_{k+1}, \dots, \lambda_h))$, and of course, $c(i_h)(\lambda_h) = c(i_h)(\{\lambda_h\})$.

Lemma 5.1. *With notation as above*:

(a) *If* $\lambda'_{h-1} \in D(i_h)(\lambda_{h-1}, \lambda_h)$, *then*

$$D(i_k, \dots, i_{h-2}, i_{h-1})(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-2}, \lambda'_{h-1}) \subset D(i_k, \dots, i_{h-1}, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-1}, \lambda_h).$$

(b) *If* $r_{i_1} r_{i_2} \cdots r_{i_h}$ *is a reduced word in* W^v *and* $\lambda \in D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h)$, *then* $\lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_h}(\lambda_h) \leq_{Q^\vee_{\mathbb{R}}} \lambda$.

Remark. If the expression $r_{i_1} r_{i_2} \cdots r_{i_h}$ is reduced, we get

$$D(i_1, \dots, i_h)(0, 0, \dots, 0, \lambda_h) = \mathrm{conv}(\{w(\lambda_h) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_h}\})$$

where $\leq_B$ denotes the Bruhat order.

Proof. The proof of (a) is easy.

(b) We have

$$D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h) \subset \lambda_0 + c(i_1)(\lambda_1) + c(i_1, i_2)(\lambda_2) + \cdots + c(i_1, i_2, \dots, i_h)(\lambda_h),$$

with

$$\begin{aligned} c(i_1, i_2, \dots, i_k)(\lambda_k) &= c(i_1)\big(c(i_2)\big(\cdots \big(c(i_k)(\lambda_k)\big) \cdots\big)\big) \\ &= \mathrm{conv}(\{w(\lambda_k) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}\}), \end{aligned}$$

where $0 \leq k \leq h$ and $\leq_B$ denotes the Bruhat order. For $w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}$, there is a sequence $w = w_0, w_1, \dots, w_r = r_{i_1} r_{i_2} \cdots r_{i_k}$ such that, for each $1 \leq i < r$, there is a reduced decomposition $w_{i+1} = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p} r_{j_{p+1}} \cdots r_{j_q}$ with $w_i = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_{p+1}} \cdots r_{j_q}$. Then

$$w_i(\lambda_k) = w_{i+1}(\lambda_k) + \alpha_{j_p}\big(r_{j_{p+1}} \cdots r_{j_q}(\lambda_k)\big) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha^\vee_{j_p})$$

and $Q^\vee_+$ contains the term $\big(r_{j_q} \cdots r_{j_{p+1}}(\alpha_{j_p})\big)(\lambda_k) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha^\vee_{j_p})$ by minimality of the expressions $r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p}$ and $r_{j_q} \cdots r_{j_{p+1}} r_{j_p}$. So by induction,

$$w(\lambda_k) \geq_{Q^\vee} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k) \quad \text{and} \quad w(\mu) \geq_{Q^\vee_{\mathbb{R}}} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$$

for any $\mu \in c(i_1, \dots, i_k)(\lambda_k)$. The expected result is now clear. □

Proposition 5.2. *For any expression* $H_k = T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ *with* $\lambda_i \in Y^{++}$, $H \in \mathcal{H}_{\mathbb{Z}}(W^v)$, *and any* $\mu \in Y^{++}$ *sufficiently great, the product* $T_\mu * H_k$ *may be written as an* R*-linear combination of elements* $T_\nu * H_\nu$ *with* $\nu \in \mu + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ *and* $H_\nu \in \mathcal{H}_R(W^v)$.

Moreover, if $r_{i_1} r_{i_2} \cdots r_{i_k}$ *is a reduced word and*

$$\nu_0 = \mu + \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k),$$

then $H_{\nu_0} \in R^\times T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H$ *and, more precisely, the constant in* $R^\times$ *is a primitive monomial in the* q_i, q_i'. *Further,* H_{ν_0} *is the only* H_ν *in*

$$(R \setminus \{0\}) . T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H.$$

N.B. So one may write $T_\mu * H_k = \sum_{\nu, w} a_{\nu,w} T_\nu * T_w$, with $a_{\nu,w} \in R$, ν running in $\mu + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$, and w in W^v. Moreover we get from the following proof, that each $a_{\nu,w}$ is a Laurent polynomial in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$; these polynomials depend only on the expression H_k, on $\mathbb{A}$, and on W.

Proof. The proof is easy in the following special case (I).

(I) We say that the expression of H_k is *normalizable of length k* when it satisfies the following properties:

(i) $\lambda_{k-1} - \lambda_k \in Y^{++}$,

(ii) For all h from k to 2, $\lambda_{h-2} - D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k) \subset \overline{C_f^v}$.

For such an expression, we write $D(H_k) = D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$.

We will then prove that $T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ is a $\mathbb{Z}[q_i, q_i']$-linear combination of normalizable elements H_{k-1}' of length $k-1$ such that $D(H_{k-1}') \subset D(H_k)$.

Using the fact $\lambda_{k-1} - \lambda_k \in Y^{++}$ and Theorem 4.8, or (BLT), for $T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k}$,

$$\text{(E)} \quad H_k = q_{i_k}^{*(\alpha_{i_k}(\lambda_k))} T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{i_{k-1}} * T_{\lambda_{k-1}^{(\alpha_{i_k}(\lambda_k))}} * (T_{i_k} * H) + \sum_{h=0}^{\alpha_{i_k}(\lambda_k)-1} (q_i^{*(h+1)} - q_i^{*(h)}) T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{i_{k-1}} * T_{\lambda_{k-1}^{(h)}} * H,$$

with $\lambda_{k-1}^{(h)} = \lambda_{k-1} + \lambda_k - h\alpha_{i_k}^\vee$, and in particular, $\lambda_{k-1}^{(\alpha_{i_k}(\lambda_k))} = \lambda_{k-1} + r_{i_k}(\lambda_k)$. Let us consider $\lambda_i' = \lambda_i$ for $i \le k-2$ and $\lambda_{k-1}' = \lambda_{k-1}^{(h)}$ for each $0 \le h \le \alpha_{i_k}(\lambda_k)$. Then $(\lambda_0', \dots, \lambda_{k-1}')$ satisfies $\lambda_{k-2}' - \lambda_{k-1}' \in Y^{++}$, by (ii) above for $h = k$ and $\lambda_{k-1}' \in D(i_k)(\lambda_{k-1}, \lambda_k)$, and, for all h from $k-1$ to 2,

$$\lambda_{h-2}' - D(i_h, \dots, i_{k-1})(\lambda_{h-1}', \dots, \lambda_{k-1}') \subset \overline{C_f^v}.$$

This last result comes from (ii) $\lambda'_{h-2} - D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k) \subset \overline{C^v_f}$ and the inclusion $D(i_h, \dots, i_{k-1})(\lambda'_{h-1}, \lambda'_h, \dots, \lambda'_{k-1}) \subset D(i_h, \dots, i_k)(\lambda_{h-1}, \lambda_h, \dots, \lambda_k)$, coming from Lemma 5.1(a). Since $T_{i_k} * H \in \mathcal{H}_R(W^v)$, every term of the right hand side of (E) is a normalizable element H'_{k-1} of length $k-1$ with $D(H'_{k-1}) \subset D(H_k)$.

By induction on each term, after k steps, we obtain H_k as a $\mathbb{Z}[q_i, q'_i]$-linear combination of $T_\nu * H_\nu$, with $\nu \in D(H_k)$ and $H_\nu \in \mathcal{H}_R(W^v)$.

Moreover, if the decomposition $r_{i_1} r_{i_2} \cdots r_{i_k}$ is reduced, we take

$$\nu_0 = \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$$

and look more carefully at the decomposition (E). For $0 \le h < \alpha_{i_k}(\lambda_k)$, we have $\nu_0 \notin D(T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{\lambda^{(h)}_{k-1}} * H) \subset D(H_k)$ by Lemma 5.1(b). Indeed, if

$$\lambda \in D(T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{\lambda^{(h)}_{k-1}} * H),$$

then, by minimality of $r_{i_1} r_{i_2} \cdots r_{i_k}$, we have $\nu_0 \le_{Q^\vee} \nu_0^{(h)} \le_{Q^\vee} \lambda$ with

$$\nu_0^{(h)} = \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_{k-1}}(\lambda^{(h)}_{i_{k-1}}) \neq \nu_0.$$

So the unique term of degree ν_0 of the final decomposition comes from the term of first kind (i.e., obtained like the first term of the right hand side of (E)) in every step of the reduction and is also the only term containing all the T_{i_j}. And so, we prove that, in front of the term $T_{\nu_0} * T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H$ obtained for ν_0, the constant is equal to the primitive monomial

$$C = q_{i_k}^{*(\alpha_{i_k}(\lambda_k))} q_{i_{k-1}}^{*(\alpha_{i_{k-1}}(\lambda_{k-1} + r_{i_k}(\lambda_k)))} \cdots q_{i_1}^{*(\alpha_{i_1}(\lambda_1 + r_{i_2}(\lambda_2) + \cdots + r_{i_2} \cdots r_{i_k}(\lambda_k)))}.$$

Let us consider now the general case but first prove the following result:

(II) If $H_k = T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$, with $\lambda_i \in Y^{++}$ and $H \in \mathcal{H}_R(W^v)$, we can choose $\mu_0 \in Y^{++}$ such that $T_{\mu_0} * H_k$ can be written as an R-linear combination of normalizable expressions H'_k of length at most k and with $D(H'_k) \subset \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$.

We prove this result for $H_{k-h} = T_{\lambda_h} * T_{i_{h+1}} * T_{\lambda_{h+1}} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ by decreasing induction on $0 \le h \le k-1$. For $h = k-1$, we have $H_1 = T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$. Choose $\mu_{k-1} = \lambda_k$; then, $T_{\mu_{k-1}} * H_1$ is normalizable of length 1 and

$$D(T_{\mu_{k-1}} * H_1) \subset \mu_{k-1} + D(i_k)(\lambda_{k-1}, \lambda_k).$$

Let $0 \le h \le k-2$ and suppose that we can choose $\mu_{h+1} \in Y^{++}$ such that $T_{\mu_{h+1}} * H_{k-(h+1)} = T_{\mu_{h+1}} * T_{\lambda_{h+1}} * T_{i_{h+2}} * \cdots * T_{i_k} * T_{\lambda_k} * H$ can be written as an R-linear combination of normalizable expressions $H'_{k-(h+1)}$ of length at most $k-(h+1)$ and with $D(H'_{k-(h+1)}) \subset \mu_{h+1} + D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k)$. Write these normalizable expressions $H'_{k-(h+1)} = T_{\lambda'_0} * T_{i'_1} * T_{\lambda'_1} * T_{i'_2} * \cdots * T_{i'_{k'}} * T_{\lambda'_{k'}} * H'$, where $k' \le k-(h+1)$ and $(\lambda'_0, \dots, \lambda'_{k'})$ satisfies (i) and (ii). Consider $\mu_h^{\min} \in Y^{++}$

such that $\mu_h^{\min} - D(i'_1, \dots, i'_{k'})(\lambda'_0, \lambda'_1, \dots, \lambda'_{k'}) \subset \overline{C_f^v}$ for all these expressions. We take $\mu_h = \mu_h^{\min} + 2\mu_{h+1} + r_{i_{h+1}}(\mu_{h+1})$. Then

$$\begin{aligned} T_{\mu_h} * H_{k-h} &= T_{\mu_h} * T_{\lambda_h} * T_{i_{h+1}} * H_{k-(h+1)} \\ &= T_{\mu_h^{\min}+\lambda_h+\mu_{h+1}} * T_{\mu_{h+1}+r_{i_{h+1}}(\mu_{h+1})} * T_{i_{h+1}} * H_{k-(h+1)}. \end{aligned}$$

By (BLT), we have:

$$\begin{aligned} \text{(E}'\text{)}\quad & q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h} \\ &= T_{\mu_h^{\min}+\lambda_h+2\mu_{h+1}} * T_{i_{h+1}} * T_{\mu_{h+1}} * H_{k-(h+1)} \\ &\quad - \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+\mu_h^{\min}+2\mu_{h+1}-j\alpha_{i_{h+1}}^\vee} * T_{\mu_{h+1}} * H_{k-(h+1)}. \end{aligned}$$

The choice of $\mu_h^{\min}$ and the hypothesis on $T_{\mu_{h+1}} * H_{k-(h+1)}$ allow us to say that we have written $T_{\mu_h} * H_{k-h}$ as an R-linear combination of normalizable expressions H'_{k-h} of length at most $k-h$ with

$$D(H'_{k-h}) \subset \mu_h^{\min} + 2\mu_{h+1} + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$$

for the first term and

$$D(H'_{k-h}) \subset \mu_h^{\min} + 2\mu_{h+1} - j\alpha_{i_{h+1}}^\vee + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$$

for the others. We need to be more precise to prove

$$D(H'_{k-h}) \subset \mu_h + D(i_{h+1}, \dots, i_k)(\lambda_h, \dots, \lambda_k).$$

By part (I) of this proof and the hypothesis on $T_{\mu_{h+1}} * H_{k-(h+1)}$, we know that this element can be written $\sum_\Lambda c_\Lambda T_\Lambda * H^\Lambda$ with $\Lambda = \mu_{h+1} + \Lambda'$, where $\Lambda' \in D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k) c_\Lambda \in R$ and $H^\Lambda \in \mathcal{H}_R(W^v)$. The first term of the right hand side of (E′) becomes

$$T_{\mu_h^{\min}+\lambda_h+2\mu_{h+1}} * T_{i_{h+1}} * \left(\sum_\Lambda c_\Lambda T_\Lambda * H^\Lambda\right) = T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda T_{\mu_h^{\min}} * T_{i_{h+1}} * T_\Lambda * H^\Lambda\right).$$

By the condition on $\mu_h^{\min}$ and (BLT), we write it

$$\begin{aligned} & T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda \left(q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\mu_h^{\min}+r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda\right)\right) \\ &\qquad + T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda \left(\sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda\right)\right). \end{aligned}$$

The first term of this sum will be

$$\sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda$$

and $\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}}(\Lambda)=\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}}(\mu_{h+1})+r_{i_{h+1}}(\Lambda')=\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')$ is an element of $\lambda_h+\mu_h+r_{i_{h+1}}(D(i_{h+2},\dots,i_k)(\lambda_{h+1},\dots,\lambda_k))$ which is included, as expected, in $\mu_h+D(i_{h+1},i_{h+2},\dots,i_k)(\lambda_h,\lambda_{h+1},\dots,\lambda_k)$.

The second term is

$$\sum_{\Lambda} c_\Lambda \Biggl(\sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda \Biggr).$$

And we see that in fact (E′) becomes (E″):

$$\text{(E}'') \quad q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h}$$
$$= \sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^\Lambda$$
$$+ \sum_{\Lambda} c_\Lambda \sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda$$
$$- \sum_{\Lambda} c_\Lambda \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+\mu_h^{\min}+2\mu_{h+1}-j\alpha_{i_{h+1}}^\vee} * T_\Lambda * H^\Lambda$$
$$= \sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^\Lambda$$
$$+ \sum_{\Lambda} c_\Lambda \varepsilon_\Lambda \sum_{j} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda,$$

where $\alpha_{i_{h+1}}(\mu_{h+1}) \le j \le \alpha_{i_{h+1}}(\Lambda)-1$ and $\varepsilon_\Lambda=+1$ if $\alpha_{i_{h+1}}(\mu_{h+1}) \le \alpha_{i_{h+1}}(\Lambda)$ (i.e., $\alpha_{i_{h+1}}(\Lambda') \ge 0$), and where $\alpha_{i_{h+1}}(\Lambda) \le j \le \alpha_{i_{h+1}}(\mu_{h+1})-1$ and $\varepsilon_\Lambda=-1$ if $\alpha_{i_{h+1}}(\mu_{h+1}) \ge \alpha_{i_{h+1}}(\Lambda)$ (i.e., $\alpha_{i_{h+1}}(\Lambda') \le 0$). For these values of j, by using $\Lambda - j\alpha_{i_{h+1}}^\vee = r_{i_{h+1}}(\mu_{h+1}) + j'\alpha_{i_{h+1}}^\vee + \Lambda'$ with $j'=\alpha_{i_{h+1}}(\mu_{h+1})-j$, we have

$$\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee = \lambda_h+\mu_h+j'\alpha_{i_{h+1}}^\vee+\Lambda'.$$

If $\alpha_{i_{h+1}}(\mu_{h+1}) \le \alpha_{i_{h+1}}(\Lambda)$, then $\alpha_{i_{h+1}}(\mu_{h+1})-\alpha_{i_{h+1}}(\Lambda)+1 \le j' \le 0$; that is, $-\alpha_{i_{h+1}}(\Lambda')+1 \le j' \le 0$. On the other hand, if $\alpha_{i_{h+1}}(\mu_{h+1}) \ge \alpha_{i_{h+1}}(\Lambda)$, then $\alpha_{i_{h+1}}(\mu_{h+1})-\alpha_{i_{h+1}}(\Lambda) \ge j' \ge 1$; that is $-\alpha_{i_{h+1}}(\Lambda') \ge j' \ge 1$. In all cases, $j'\alpha_{i_{h+1}}^\vee+\Lambda'$ is between Λ' and $r_{i_{h+1}}(\Lambda')$ and so, as expected,

$$\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee \in \mu_h+D(i_{h+1},i_{h+2},\dots,i_k)(\lambda_h,\lambda_{h+1},\dots,\lambda_k).$$

So we have proved that $T_{\mu_0} * H_k$ can be written as an R-linear combination of normalizable expressions H'_k of length at most k and with

$$D(H'_k) \subset \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k).$$

By (I) of the proof, we can write it as an R-linear combination of elements $T_\nu * H_\nu$ with $\nu \in \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ and $H_\nu \in \mathcal{H}_R(W^v)$.

As in (I), if the decomposition $r_{i_1} r_{i_2} \cdots r_{i_k}$, moreover, is reduced, then only the term

$$\sum_\Lambda c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda$$

(which contains $T_{i_{h+1}}$) in (E″) can give us a term of lowest degree

$$\mu_h + \lambda_h + r_{i_{h+1}}(\lambda_{h+1}) + \dots + r_{i_{h+1}} \cdots r_{i_k}(\lambda_k).$$

More precisely, the term of lowest degree comes from the term with

$$\Lambda_0 = \mu_{h+1} + \lambda_{h+1} + r_{i_{h+2}}(\lambda_{h+2}) + \dots + r_{i_{h+2}} \cdots r_{i_k}(\lambda_k)$$

for which we have $\alpha_{i_{k+1}}(\Lambda_0) \geq \alpha_{i_{k+1}}(\mu_{h+1})$. So, it's easy to see by induction that the coefficient of that term is a primitive monomial in the q_i, q'_i. □

Corollary 5.3. (a) *For $\lambda \in Y^+$ and $\mu \in Y^{++}$ sufficiently great, we have $T_\mu * T_\lambda = \sum_{\lambda \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda^{++}} T_{\mu+\nu} * H^\nu$ with $H^\nu \in \mathcal{H}_R(W^v)$.*

(b) *More precisely, if $H^\nu \neq 0$ then $\mu + \nu \in Y^{++}$ and ν is in the convex hull* conv$(W^v.\lambda^{++})$ *of $W^v.\lambda^{++}$ or, better, in the convex hull* conv$(W^v.\lambda^{++}, \geq \lambda)$ *of all $w'.\lambda^{++}$ for $w' \leq_B w_\lambda$, with w_λ the smallest element of W^v such that $\lambda = w_\lambda.\lambda^{++}$.*

(c) *For $\nu = \lambda$, H^λ is a strictly positive integer a_λ which may be written as a primitive monomial in q_i, q'_i, $i \in I$ (depending only on $\mathbb{A}$).*

(d) *In (a) above, we may write $H^\nu = \sum_{w \in W^v} a_{\mu,\lambda}^{\nu,w} T_w$ and, then each $a_{\mu,\lambda}^{\nu,w}$ is a Laurent polynomial in the parameters q_i, q'_i with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W.*

Proof. Only the result (c) is new (see Propositions 2.2 and 2.3), and we already saw that the constant term in H^λ is in $\mathbb{Z}_{>0}$. We have to prove that $H^\lambda \in \mathcal{H}_R(W^v)$ is actually a constant (for μ sufficiently great). Write $\lambda = w_\lambda(\lambda^{++})$ (with w_λ minimal in W^v for this property), choose a minimal decomposition $w_\lambda = r_{i_1} r_{i_2} \cdots r_{i_k}$, by Corollary 4.3 we have

$$T_\lambda = T_{i_1} * T_{i_2} * \dots * T_{i_k} * T_{\lambda^{++}} * T_{i_k}^{-1} * \dots * T_{i_1}^{-1}.$$

Then, by Proposition 5.2, for μ great, $T_\mu * T_\lambda$ may be written as an R-linear combination of elements $T_{\mu+\nu} * (H_1^\nu * T_{i_k}^{-1} * \dots * T_{i_1}^{-1})$ with $\nu \in D(i_1, \dots, i_k)(0, \dots, 0, \lambda^{++})$

and $H_1^\nu \in \mathcal{H}_R(W^v)$ with term of lowest degree $\nu_0 = \lambda$. Moreover,

$$H^\lambda = H_1^\lambda * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$$

is a primitive monomial in the q_i, q_i'.

To prove (d), notice that $T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$ may be written $\sum_{w \in W^v} a_w T_w$ with $a_w \in \mathbb{Z}[(q_i^{\pm 1})_{i \in I}]$, and apply Proposition 5.2 with $H = T_w$. □

Corollary 5.4. *In ${}^{\mathrm{I}}\mathcal{H}_R$, for $\mu \in Y^{++}$ the left multiplication by T_μ is injective.*

Proof. As $T_{\mu_1+\mu_2} = T_{\mu_1} * T_{\mu_2}$ for $\mu_1, \mu_2 \in Y^{++}$, we may assume μ sufficiently great. Let $H \in {}^{\mathrm{I}}\mathcal{H}_R \setminus \{0\}$. We may write $H = \sum_{j \in J} T_{\lambda_j} * H^j$ with $\lambda_j \in Y^+$ and $0 \neq H^j \in \mathcal{H}_R(W^v)$. We choose λ_{j_0} minimal among the λ_j for $\leq_{Q^\vee}$. Then

$$T_\mu * H = \sum_{j \in J} \sum_{\mu+\lambda_j \leq_{Q^\vee} \nu_j} T_{\nu_j} * H^{\nu_j, j} * H^j.$$

Hence $\nu_{j_0} = \mu + \lambda_{j_0}$ is minimal for $\leq_{Q^\vee}$ and $H^{\nu_{j_0}, j_0}$ is a monomial in q_i, q_i'; so $H^{\nu_{j_0}, j_0} * H^{j_0} \neq 0$ and $T_\mu * H \neq 0$. □

Theorem 5.5. (1) *For any $\lambda \in Y^+$, there is a unique $X^\lambda \in {}^{\mathrm{I}}\mathcal{H}_R$ such that for all $\mu \in Y^{++}$ with $\lambda + \mu \in Y^{++}$, we have $T_\mu * X^\lambda = T_{\lambda+\mu}$.*

(2) *More precisely,*

$$X^\lambda = b_\lambda T_\lambda + \sum_\nu T_\nu * H'^\nu,$$

where $H'^\nu \in \mathcal{H}_R(W^v)$, $\nu \in \mathrm{conv}(W^v.\lambda^{++}, \geq \lambda) \setminus \{\lambda\}$ and b_λ is a primitive monomial in $q_i^{-1}, q_i'^{-1}$.

(3) *For $\lambda \in Y^{++}$, we have $X^\lambda = T_\lambda$, and for $\lambda, \lambda' \in Y^+$,*

$$X^\lambda * X^{\lambda'} = X^{\lambda+\lambda'} = X^{\lambda'} * X^\lambda.$$

Remarks. (a) We have two bases for the free right $\mathcal{H}_R(W^v)$-module ${}^{\mathrm{I}}\mathcal{H}_R$,

$$\{T_\lambda \mid \lambda \in Y^+\} \quad \text{and} \quad \{X^\lambda \mid \lambda \in Y^+\}.$$

The change of basis matrix is triangular (for the order $\geq_{Q^\vee}$) with diagonal coefficients primitive monomials in $q_i^{-1}, q_i'^{-1}$. From Corollary 5.3(d), we get that all coefficients of this matrix are Laurent polynomials in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and on W.

(b) By (1) above and Corollary 5.4, it is clear that the left multiplication by X^λ is injective, for any $\lambda \in Y^+$.

Proof. By Corollary 5.4, the uniqueness is clear and (3) follows from the relation $T_\lambda * T_\mu = T_{\lambda+\mu}$ of the Theorem 2.4. We have just to prove (1) and (2) for a $\mu \in Y^{++}$ (chosen sufficiently great).

We argue by induction on the height $\mathrm{ht}(\lambda^{++}-\lambda)$ of $\lambda^{++}-\lambda$ with respect to the free family $(\alpha_i^\vee)$ in $Q^\vee$. When the height is 0, $\lambda=\lambda^{++}$ and $X^\lambda=T_\lambda$. By Corollary 5.3, we write

$$T_\mu * T_\lambda = a_\lambda T_{\mu+\lambda} + \sum_{\substack{\lambda\leq_{Q^\vee}\nu\leq_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} T_{\mu+\nu} * H^\nu$$

with $H^\nu\in\mathcal{H}_R(W^v)$ and $\nu\in\mathrm{conv}(W^v.\lambda^{++})$; hence, $\nu^{++}\in\mathrm{conv}(W^v.\lambda^{++})$ (in particular, $\nu^{++}\leq_{Q^\vee}\lambda^{++}$); see Section 1.8(a).

So $\mathrm{ht}(\nu^{++}-\nu)<\mathrm{ht}(\lambda^{++}-\lambda)$. By induction and for μ sufficiently great, we can consider the element X^ν such that $T_{\mu+\nu}=T_\mu * X^\nu$; we can write it

$$X^\nu = \sum_{\nu\leq_{Q^\vee}\nu'\leq_{Q^\vee}\nu^{++}} T_{\nu'} * H^{\nu',\nu}$$

and we may take

$$\begin{aligned} X^\lambda &= a_\lambda^{-1}T_\lambda - \Big(\sum_{\substack{\lambda\leq_{Q^\vee}\nu\leq_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} X^\nu * H^\nu\Big)\\ &= a_\lambda^{-1}T_\lambda - \Big(\sum_{\substack{\lambda\leq_{Q^\vee}\nu\leq_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} \Big(\sum_{\nu\leq_{Q^\vee}\nu'\leq_{Q^\vee}\nu^{++}} T_{\nu'} * H^{\nu',\nu}\Big) * H^\nu\Big). \qquad \square \end{aligned}$$

Proposition 5.6. *For $\lambda\in Y^+$ and $i\in I$ we have the following relations*:

(a) *If $\alpha_i(\lambda)\geq 0$, then*

$$T_i * X^\lambda = q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1}\big(q_i^{*(h+1)}-q_i^{*(h)}\big)X^{\lambda-h\alpha_i^\vee}.$$

(b) *If $\alpha_i(\lambda)<0$, then*

$$T_i * X^\lambda = \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1}\big(q_i^{*(-\alpha_i(\lambda)+h+1)}-q_i^{*(-\alpha_i(\lambda)+h)}\big)X^{\lambda-h\alpha_i^\vee}.$$

N.B. These are the Bernstein–Lusztig relations for the X^λ, (BLX) for short.

Proof. If $\lambda\in Y^{++}$, by Theorem 4.8(a), we know that $X^\lambda * T_i * X^\lambda = X^{\lambda+\lambda} * T_i$ when $\alpha_i(\lambda)=0$ and, when $\alpha_i(\lambda)>0$,

$$X^\lambda * T_i * X^\lambda = q_i^{*\alpha_i(\lambda)}X^{\lambda+r_i(\lambda)} * T_i + (q_i^{*(\alpha_i(\lambda))}-q_i^{*(\alpha_i(\lambda)-1)})X^{\lambda+\lambda-(\alpha_i(\lambda)-1)\alpha_i^\vee} + \cdots + (q_i^{*2}-q_i)X^{\lambda+\lambda-\alpha_i^\vee} + (q_i-1)X^{\lambda+\lambda},$$

so we have the result.

In the general case, $\lambda \in Y^+$, we write $\lambda = \mu - \nu$ with μ, ν chosen in Y^{++}. By Theorem 5.5, $X^\nu * X^\lambda = X^\mu$. From (BLX) for X^μ and X^ν,

$$T_i * X^\mu = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i + \sum_{h=0}^{\alpha_i(\lambda+\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}$$

which can also be written

$$\begin{aligned} T_i * X^{\nu+\lambda} &= (T_i * X^\nu) * X^\lambda \\ &= \left(q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i + \sum_{h=0}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu-h\alpha_i^\vee}\right) * X^\lambda \\ &= q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda + \sum_{h=0}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}. \end{aligned}$$

If $\alpha_i(\lambda) \geq 0$, then

$$q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\mu)} * T_i + \sum_{h=\alpha_i(\nu)}^{\alpha_i(\lambda+\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}.$$

We take $h' = h - \alpha_i(\nu)$, then

$$X^{\nu+\lambda-h\alpha_i^\vee} = X^{\nu-\alpha_i(\nu)\alpha_i^\vee+\lambda-h'\alpha_i^\vee} = X^{r_i(\nu)+\lambda-h'\alpha_i^\vee}$$

and $q_i^{*(\alpha_i(\nu)+h')} = q_i^{*\alpha_i(\nu)} q_i^{*h'}$ (by $q_i = q_i'$ if $\alpha_i(\nu)$ is odd, and by an easy calculation if $\alpha_i(\nu)$ is even). So,

$$\begin{aligned} &q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda \\ &\qquad = q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * \left(q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h'=0}^{\alpha_i(\lambda)-1} \big(q_i^{*(h'+1)} - q_i^{*(h')}\big) X^{\lambda-h'\alpha_i^\vee}\right). \end{aligned}$$

And we are done, thanks to the injectivity of left multiplication by $X^{r_i(\nu)}$.

If $\alpha_i(\lambda) < 0$, we obtain

$$\begin{aligned} &q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda \\ &\qquad = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i - \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}. \end{aligned}$$

We have $q_i^{*(\alpha_i(\nu))} = q_i^{*(-\alpha_i(\lambda))} q_i^{*(\alpha_i(\lambda+\nu))}$ by an easy calculation if $\alpha_i(\nu)$ and $\alpha_i(\lambda)$ are even and because $q_i = q_i'$ whenever $\alpha_i(\nu)$ or $\alpha_i(\lambda)$ is odd. So,

$$X^{r_i(\nu)} * T_i * X^\lambda = \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda+\nu)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}$$

and we have (because of the injectivity of the left multiplication by $X^{r_i(\nu)}$)

$$\begin{aligned} T_i * X^\lambda &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\lambda+(\alpha_i(\nu)-h)\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))} q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big(q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h+1)} - q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h)}\big) X^{\lambda-h\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big(q_i^{*(-\alpha_i(\lambda)+h+1)} - q_i^{*(-\alpha_i(\lambda)+h)}\big) X^{\lambda-h\alpha_i^\vee}. \qquad \square \end{aligned}$$

5.7. ***The classical Bernstein–Lusztig relation.*** The *module* $\delta : Q^\vee \to R$ is defined by

$$\delta\Big(\sum_{i\in I} a_i \alpha_i^\vee\Big) = \prod_{i\in I} (q_i q_i')^{a_i}$$

[Gaussent and Rousseau 2014, 5.3.2]. After replacing eventually R by a bigger ring R' containing some square roots $\sqrt{q_i}, \sqrt{q_i'}$ of q_i, q_i' (with $\sqrt{q_i} = \sqrt{q_i'}$, if $q_i = q_i'$), we assume moreover that there exists a homomorphism $\delta^{1/2} : Y \to R^\times$, such that $\delta(\lambda) = (\delta^{1/2}(\lambda))^2$ for any $\lambda \in Q^\vee$ and $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i}\,.\sqrt{q_i'}$. In particular $\sqrt{q_i}^{\pm1}$ and $\sqrt{q_i'}^{\pm1}$ are well defined in $R^\times$. In the common example where $R = \mathbb{R}$ or $R = \mathbb{C}$, these expressions are chosen to be the classical ones: $\delta^{1/2}(Y) \subset \mathbb{R}_+^*$.

We define $H_i = \big(\sqrt{q_i}\big)^{-1} T_i$ and $Z^\lambda = \delta^{-1/2}(\lambda) X^\lambda$ for $\lambda \in Y^+$. When $w = r_{i_1} \cdots r_{i_n}$ is a reduced decomposition, we set $H_w = H_{i_1} * \cdots * H_{i_n}$; this does not depend on the chosen decomposition of w.

We may translate the relations (BLX) for these elements.

Proposition. *For $\lambda \in Y^{++}$, we have the relation*

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{k=0}^{\lfloor(\alpha_i(\lambda)-1)/2\rfloor} \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big) Z^{\lambda-(2k)\alpha_i^\vee} + \sum_{k=0}^{\lfloor\alpha_i(\lambda)/2\rfloor-1} \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big) Z^{\lambda-(2k+1)\alpha_i^\vee}.$$

Remarks. (1) This is the Bernstein–Lusztig relation for the Z^λ, (BLZ) for short.

(2) In the following section, we shall consider an algebra containing ${}^{\mathrm{I}}\mathcal{H}_R$ and, for any $i \in I$, an element $Z^{-\alpha_i{}^\vee}$ satisfying $Z^{\lambda-h\alpha_i{}^\vee} = Z^\lambda * (Z^{-\alpha_i{}^\vee})^h$ for $h \in \mathbb{N}$, $\lambda, \lambda - h\alpha_i^\vee \in Y^+$. In such an algebra the relation (BLZ) may be rewritten (using that $\sqrt{q_i} = \sqrt{q_i'}$ if $\alpha_i(\lambda)$ is odd) as the classical Bernstein–Lusztig relation (BL):

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)\frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)\frac{Z^{\lambda-\alpha_i^\vee} - Z^{r_i(\lambda)-\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}},$$

i.e., $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b\big(\sqrt{q_i}, \sqrt{q_i'}; Z^{-\alpha_i{}^\vee}\big)(Z^\lambda - Z^{r_i(\lambda)})$, where

$$b(t, u; z) = \frac{t - t^{-1} + (u - u^{-1})z}{1 - z^2}.$$

This is the same relation as in [Macdonald 2003, 4.2], up to the order; see (3).

(3) Actually this relation (BLZ) is still true when $\lambda \in Y^+$ and $\alpha_i(\lambda) \geq 0$ (same proof as below). If $\alpha_i(\lambda) < 0$, we leave to the reader the proof of the relation

$$T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i - \Bigg(\sum_{\substack{2 \leq h \leq -\alpha_i(\lambda) \\ h \text{ even}}} (q_i - 1) Z^{\lambda+h\alpha_i^\vee} + \sum_{\substack{1 \leq h \leq -\alpha_i(\lambda) \\ h \text{ odd}}} \Big(\sqrt{q_i \cdot q_i'} - \frac{\sqrt{q_i \cdot q_i'}}{q_i'}\Big) Z^{\lambda+h\alpha_i^\vee}\Bigg).$$

In the situation of (2) above, it may be rewritten

$$\begin{aligned} H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i &= \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)\frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)\frac{Z^{\lambda-\alpha_i^\vee} - Z^{r_i(\lambda)-\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}} \\ &= b\big(\sqrt{q_i}, \sqrt{q_i'}; Z^{-\alpha_i{}^\vee}\big)(Z^\lambda - Z^{r_i(\lambda)}). \end{aligned}$$

It is the same relation (BLZ) as above. Moreover, it's easy to see in the first equality that $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = Z^\lambda * H_i - H_i * Z^{r_i(\lambda)}$. Actually we shall see in Section 6 that this same relation is true for any $\lambda \in Y$ in a greater algebra containing elements Z^λ for $\lambda \in Y$.

Proof. From $Z^\lambda = \delta^{-1/2}(\lambda)X^\lambda$ and $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i \,.\, q_i'}$, we get

$$\begin{aligned} Z^{\lambda - h\alpha_i^\vee} &= \delta^{-1/2}(\lambda - h\alpha_i^\vee)X^{\lambda - h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda)(\delta^{1/2}(\alpha_i^\vee))^h X^{\lambda - h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda)\big(\sqrt{q_i \,.\, q_i'}\big)^h X^{\lambda - h\alpha_i^\vee}. \end{aligned}$$

By $\alpha_i(\lambda) \geq 0$ and (BLX),

$$\begin{aligned} &T_i * Z^\lambda \\ &\quad = q_i^{*(\alpha_i(\lambda))}\big(\sqrt{q_i \,.\, q_i'}\big)^{-\alpha_i(\lambda)} Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i^{*(h+1)} - q_i^{*h})\big(\sqrt{q_i \,.\, q_i'}\big)^{(-h)} Z^{\lambda - h\alpha_i^\vee}. \end{aligned}$$

Moreover, $q_i^{*h} = q_i q_i' q_i \cdots$ with h terms in the product, so $q_i^{*h} = \big(\sqrt{q_i \,.\, q_i'}\big)^h$ if h is even and $q_i^{*h} = q_i\big(\sqrt{q_i \,.\, q_i'}\big)^{(h-1)}$ if h is odd. So, if $\alpha_i(\lambda)$ is even, then

$$\begin{aligned} &T_i * Z^\lambda \\ &\quad = Z^{r_i(\lambda)} * T_i + \sum_{k=0}^{(\alpha_i(\lambda)-2)/2} (q_i - 1)Z^{\lambda-(2k)\alpha_i^\vee} + \sum_{k=0}^{(\alpha_i(\lambda)-2)/2} (q_i q_i' - q_i)\big(\sqrt{q_i q_i'}\big)^{-1} Z^{\lambda-(2k+1)\alpha_i^\vee}. \end{aligned}$$

If $\alpha_i(\lambda)$ is odd, then $q_i = q_i'$ and

$$T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i - 1)Z^{\lambda - h\alpha_i^\vee}.$$

In both cases, by $H_i = (\sqrt{q_i})^{-1}T_i$,

$$\begin{aligned} H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + &\sum_{k=0}^{\lfloor(\alpha_i(\lambda)-1)/2\rfloor} \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)Z^{\lambda-(2k)\alpha_i^\vee} \\ &+ \sum_{k=0}^{\lfloor\alpha_i(\lambda)/2\rfloor - 1} \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)Z^{\lambda-(2k+1)\alpha_i^\vee}. \quad \square \end{aligned}$$

6. Bernstein–Lusztig–Hecke Algebras

The aim of this section is to define, in a formal way, an associative algebra ${}^{\mathrm{BL}}\mathcal{H}_R$, called the Bernstein–Lusztig–Hecke algebra. This construction by generators and relations is motivated by the results obtained in the previous section (in particular Proposition 5.6) and we will be able next to identify ${}^{\mathrm{I}}\mathcal{H}_R$ and a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}_R$ (up to some hypotheses on R).

We use the same notation as before, even if the objects are somewhat different. This choice will be justified by the identification obtained at the end of this section.

We consider $\mathbb{A}$ as in Section 1.2 and $\mathrm{Aut}(\mathbb{A}) \supset W = W^v \ltimes Y \supset W^a$, with Y a discrete group of translations.

6.1. *The module $^{\mathrm{BL}}\mathcal{H}_{R_1}$.* We consider now the ring $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma_i'^{\pm 1})_{i\in I}]$ where the indeterminates σ_i, σ_i' satisfy the following relations (as q_i and q_i' in Section 1.4.5 because in the further identification, σ_i, σ_i' will play the role of $\sqrt{q_i}$ and $\sqrt{q_i'}$).

If $\alpha_i(Y) = \mathbb{Z}$, then $\sigma_i = \sigma_i'$.

If r_i and r_j are conjugated (i.e., if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$), then $\sigma_i = \sigma_j = \sigma_i' = \sigma_j'$.

We denote by $^{\mathrm{BL}}\mathcal{H}_{R_1}$ the free R_1-module with basis $(Z^\lambda H_w)_{\lambda\in Y,\, w\in W^v}$. For short, we write $H_i = H_{r_i}$, $H_w = Z^0 H_w$ and $Z^\lambda = Z^\lambda H_e$, where e is the unit element in W^v (and $H_e = Z^0$ will be the multiplicative unit element in $^{\mathrm{BL}}\mathcal{H}_{R_1}$).

Theorem 6.2. *There exists a unique multiplication $*$ on $^{\mathrm{BL}}\mathcal{H}_{R_1}$ which makes it an associative unitary R_1-algebra with unity H_e and satisfies the following conditions*:

(1) $Z^\lambda * H_w = Z^\lambda H_w$ *for all* $\lambda \in Y,\ w \in W^v$,

(2) $H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i - \sigma_i^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) < \ell(w), \end{cases}$ *for all* $i \in I,\ w \in W^v$,

(3) $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$ *for all* $\lambda \in Y,\ \mu \in Y$,

(4) $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)})$ *for all* $\lambda \in Y,\ i \in I$, *where*

$$b(t, u; z) = \frac{(t - t^{-1}) + (u - u^{-1})z}{1 - z^2}.$$

Remarks 6.3. (1) It is already known (see, e.g., [Humphreys 1990, Theorem 7.1] or [Bourbaki 1968, IV §2, exercise 23]) that the free submodule with basis $(H_w)_{w\in W^v}$ can be equipped, in a unique way, with a multiplication $*$ that satisfies (2) and gives it a structure of an associative unitary algebra called the "Hecke algebra of the group W^v over R_1" and denoted by $\mathcal{H}_{R_1}(W^v)$.

(2) The submodule $\mathcal{H}_{R_1}(Y)$ with basis $(Z^\lambda)_{\lambda\in Y}$ will be a commutative subalgebra.

(3) When all σ_i, σ_i' are equal, the existence of this algebra $^{\mathrm{BL}}\mathcal{H}$ is stated in [Garland and Grojnowski 1995] and justified by an action on some Grothendieck group.

(4) This R_1-algebra depends only on $\mathbb{A}$ and Y (i.e., $\mathbb{A}$ and W). We call it the Bernstein–Lusztig–Hecke algebra over R_1 (associated to $\mathbb{A}$ and W).

6.4. *Proof of Theorem 6.2.*

6.4.1. The uniqueness of the multiplication $*$ is clear: by associativity and distributivity, we have only to identify $H_w * Z^\mu$. If $w = r_{i_1} r_{i_2} \cdots r_{i_n}$ is a reduced decomposition, then, by (2), (4), and Remark 6.3(1),

$$H_w * Z^\mu = H_{i_1} * (H_{i_2} * (\cdots * (H_{i_n} * Z^\mu) \cdots))$$

has to be a well-defined linear combination of terms $Z^\nu H_u$: $H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$ with $a_k \in R_1$, $\nu_k \in Y$, and $u_k \in W^v$.

6.4.2. Construction of $*$. We define $H_w * Z^\mu$ as above and we have to prove that it does not depend on the reduced decomposition $w = r_{i_1} r_{i_2} \cdots r_{i_n}$.

(a) We define $L_i \in \mathrm{End}_{R_1}({}^{\mathrm{BL}}\mathcal{H}_{R_1})$ by

$$L_i(Z^\mu H_w) = H_i * (Z^\mu H_w) = Z^{r_i(\mu)}(H_i * H_w) + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\mu - Z^{r_i(\mu)}) * H_w,$$

where

$$H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i - \sigma_i{}^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) < \ell(w). \end{cases}$$

By Matsumoto's theorem [Bourbaki 1968, IV §1.5, Proposition 5], the expected independence will be a consequence of the braid relations, i.e.,

$$(*) \qquad L_i(L_j(L_i(\cdots(Z^\lambda H_w)\cdots))) = L_j(L_i(L_j(\cdots(Z^\lambda H_w)\cdots)))$$

(with $m_{i,j}$ factors L on each side), whenever the order $m_{i,j}$ of $r_i r_j$ is finite.

As $\mathcal{H}_{R_1}(W^v)$ is known to be an algebra, it is enough to prove $(*)$ for $w = 1$. We may also suppose $\alpha_j(\alpha_i^\vee) \neq 0$ as otherwise L_i and L_j commute clearly.

We choose $i, j \in I$ with $m_{i,j}$ finite; then $\pm\alpha_i$, $\pm\alpha_j$ generate a finite root system $\Phi_{i,j}$ of rank 2 (or 1 if $i = j$). Moreover, $Y' = \ker(\alpha_i) \cap \ker(\alpha_j) \cap Y$ is cotorsion free in Y. Let Y'' be a supplementary module containing $\alpha_i^\vee$ and $\alpha_j^\vee$; Y'' is a lattice (of rank 2 or 1) between the lattices $Q_{i,j}^\vee$ of coroots and $P_{i,j}^\vee$ of coweights, associated to $\Phi_{i,j}$.

Any $\lambda \in Y$ may be written $\lambda = \lambda' + \lambda''$ with $\lambda' \in Y'$ and $\lambda'' \in Y''$. By (4), $L_i(Z^{\lambda'}) = Z^{\lambda'} H_i$ and $L_j(Z^{\lambda'}) = Z^{\lambda'} H_j$. So we have to prove $(*)$ for $\lambda = \lambda'' \in Y''$. We shall do it by comparing with some Macdonald's results.

(b) Macdonald [2003] builds affine Hecke algebras $\mathcal{H}(W(R, L'))$ over $\mathbb{R}$, associated to any finite irreducible root system R and any lattice L' between the lattices of coroots and coweights; more precisely this algebra is associated to the extended affine Weyl group $W(R, L') = W(R) \ltimes L'$. It is defined by generators and relations, but it is proven that it is endowed with a basis $(Y^\lambda T(w))_{\lambda \in L', w \in W(R)}$ [op. cit., 4.2.7] and satisfies relations analogous to (1)–(4) as above. There are parameters $(\tau_i)_{i \in I}$ and τ_0 which are reals (but may be algebraically independent over $\mathbb{Q}$, so may be considered as indeterminates) and satisfy $\tau_i = \tau_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$. The relation (4) is satisfied with $\sigma_i = \tau_i$ and $\sigma_i' = \tau_i$ when $\alpha_i(L') = \mathbb{Z}$, $\sigma_i' = \tau_0$ when $\alpha_i(L') = 2\mathbb{Z}$.

(c) In the case $R = \Phi_{i,j}$, irreducible, $L' = Y''$, we may choose τ_i, τ_j, and τ_0 such that the relations (4) are the same, for us and Macdonald: either $\alpha_i(\alpha_j^\vee) = -1$ or $\alpha_j(\alpha_i^\vee) = -1$, so $\tau_0 = \sigma_i'$ or $\tau_0 = \sigma_j'$. In particular R_1 may be identified with a subring of $\mathbb{R}$. The operators L_i and L_j of both theories coincide on the elements

$Z^\lambda H_v$ (identified with $Y^\lambda T(v)$ in Macdonald's work) for $\lambda \in L' = Y''$ and $v \in \langle r_i, r_j \rangle$. So $(*)$ is satisfied as $\mathcal{H}(W(R, L'))$ is an associative algebra.

(d) So, if $H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$, with $a_k \in R_1$, $\nu_k \in Y$, $u_k \in W^v$, we define the product of $Z^\lambda H_w$ and $Z^\mu H_v$ by

$$(Z^\lambda H_w) * (Z^\mu H_v) = \sum_k a_k Z^{\lambda+\nu_k} * (H_{u_k} * H_v).$$

We get a distributive multiplication on ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ with unit H_e.

6.4.3. *Associativity.* (a) Using the associativity in $\mathcal{H}_{R_1}(Y)$ and $\mathcal{H}_{R_1}(W^v)$ and the formula in 6.4.2(d) above, it is clear that, for any $\lambda \in Y$, $w \in W^v$, $E_1, E_2 \in {}^{\mathrm{BL}}\mathcal{H}_{R_1}$,

(R1) $Z^\lambda * (E_1 * E_2) = (Z^\lambda * E_1) * E_2$,

(R2) $E_1 * (E_2 * H_w) = (E_1 * E_2) * H_w$.

We need also to prove (for $\lambda_1, \lambda_2 \in Y$, $w, w_1, w_2 \in W^v$, $E \in {}^{\mathrm{BL}}\mathcal{H}_{R_1}$),

(A) $H_w * (Z^{\lambda_1} * Z^{\lambda_2}) = (H_w * Z^{\lambda_1}) * Z^{\lambda_2}$,

(B) $H_{w_1} * (H_{w_2} * E) = (H_{w_1} * H_{w_2}) * E$.

Then the general associativity will follow: using (R1), (R2), (A), (B), and the formula in 6.4.2(d) for the product, it is not too difficult (and left to the reader) to prove that

$$\begin{aligned}(Z^{\lambda_1} H_{w_1}) * \big((Z^{\lambda_2} H_{w_2}) * (Z^{\lambda_3} H_{w_3})\big) &= Z^{\lambda_1} * (H_{w_1} * \big((Z^{\lambda_2} H_{w_2}) * Z^{\lambda_3})\big) * H_{w_3}\\ &= Z^{\lambda_1} * \big((H_{w_1} * Z^{\lambda_2}) * (H_{w_2} * Z^{\lambda_3})\big) * H_{w_3}\\ &= Z^{\lambda_1} * \big((H_{w_1} * (Z^{\lambda_2} H_{w_2})) * Z^{\lambda_3}\big) * H_{w_3}\\ &= \big((Z^{\lambda_1} H_{w_1}) * (Z^{\lambda_2} H_{w_2})\big) * (Z^{\lambda_3} H_{w_3}).\end{aligned}$$

(b) Proof of (B). This condition is equivalent to the fact that left multiplication by $\mathcal{H}_{R_1}(W^v)$ on ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ is an action. But the associative algebra $\mathcal{H}_{R_1}(W^v)$ is generated by the H_i with relations consisting of the braid relations and $H_i^2 = (\sigma_i - \sigma_i^{-1}) H_i + H_e$. As L_i is left multiplication by H_i, we have (B) if and only if these L_i satisfy the relation $(*)$ and

$$(**) \qquad L_i(L_i(Z^\lambda H_v)) = (\sigma_i - \sigma_i^{-1}) L_i(Z^\lambda H_v) + Z^\lambda H_v.$$

As in 6.4.2(b), we reduce the verification of $(**)$ to the case $v = 1$ and $\lambda \in Y''$ (associated to $i = j$), i.e., $\lambda \in Y'' = \mathbb{Q}\alpha_i^\vee \cap Y$. Then we look at Macdonald's construction of $\mathcal{H}(W(\{\pm\alpha_i\}, Y''))$ with $\tau_i = \sigma_i$, $\tau_0 = \sigma_i'$. We conclude, as in 6.4.2(c) that $(**)$ is satisfied.

(c) The proof of (A) is by induction on $\ell(w)$. If $w = r_i$,

$$\begin{aligned}(H_i * Z^{\lambda_1}) * Z^{\lambda_2} &= (Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + \big(b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2} \\ &= Z^{r_i(\lambda_1)} * \big(Z^{r_i(\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_2} - Z^{r_i(\lambda_2)})\big) \\ &\qquad + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1)+\lambda_2}) \\ &= Z^{r_i(\lambda_1+\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{r_i(\lambda_1)+\lambda_2} - Z^{r_i(\lambda_1)+r_i(\lambda_2)}) \\ &\qquad + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1)+\lambda_2}) \\ &= Z^{r_i(\lambda_1+\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1+\lambda_2)}) \\ &= H_i * (Z^{\lambda_1} * Z^{\lambda_2})\end{aligned}$$

If the result is known when $\ell(w) = n$, let us consider $w = w' r_i$ with $\ell(w) = n+1$ and $\ell(w') = n$. Then

$$\begin{aligned}&H_w * (Z^{\lambda_1} * Z^{\lambda_2}) \\ &\qquad = H_{w'} * (H_i * Z^{\lambda_1+\lambda_2}) \\ &\qquad = H_{w'} * \big((H_i * Z^{\lambda_1}) * Z^{\lambda_2}\big) \\ &\qquad = H_{w'} * \Big((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + \big(b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2}\Big),\end{aligned}$$

where the first equality is because left multiplication by $\mathcal{H}_{R_1}(W^v)$ is an action, and the second equality is the case $\ell(w) = 1$. On the other hand,

$$\begin{aligned}&(H_w * Z^{\lambda_1}) * Z^{\lambda_2} \\ &\quad = (H_{w'} * (H_i * Z^{\lambda_1})) * Z^{\lambda_2} \\ &\quad = \big(H_{w'} * (Z^{r_i(\lambda_1)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2} \\ &\quad = (H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} + \big(H_{w'} * (b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)}))\big) * Z^{\lambda_2}.\end{aligned}$$

The second term of the right hand side is an R_1-linear combination of

$$(H_{w'} * Z^{\lambda_1 + k\alpha_i^\vee}) * Z^{\lambda_2}$$

and we see by induction that it is the same as

$$H_{w'} * \big((b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2}\big)$$

in $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$.

In the first term, $(H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} = ((H_{w'} * Z^{r_i(\lambda_1)}) * H_i)) * Z^{\lambda_2}$, we can write

$$H_{w'} * Z^{r_i(\lambda_1)} = \sum_k c_k Z^{\lambda_k} H_{w_k},$$

and we will use later in the same way

$$H_i * Z^{\lambda_2} = \sum_h a_h Z^{\mu_h} H_{v_h}$$

with $c_k, a_h \in R_1,\ \lambda_k, \mu_h \in Y$, and $w_k, v_h \in W^v$. So, we have

$$\begin{aligned}
&\left(\left(\sum_k c_k Z^{\lambda_k} H_{w_k}\right) * H_i\right) * Z^{\lambda_2} \\
&= \left(\sum_k c_k (Z^{\lambda_k} * (H_{w_k} * H_i))\right) * Z^{\lambda_2} && \text{(by (R2))} \\
&= \sum_k c_k Z^{\lambda_k} * ((H_{w_k} * H_i) * Z^{\lambda_2}) && \text{(by 6.4.2(d))} \\
&= \sum_k c_k Z^{\lambda_k} * (H_{w_k} * (H_i * Z^{\lambda_2})) && \text{(by (B))} \\
&= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * (H_i * Z^{\lambda_2}) && \text{(by (R1))} \\
&= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * \left(\sum_h a_h Z^{\mu_h} H_{v_h}\right) \\
&= \sum_{k,h} c_k a_h (Z^{\lambda_k} * H_{w_k}) * (Z^{\mu_h} * H_{v_h}) \\
&= \sum_{k,h} c_k a_h (((Z^{\lambda_k} * H_{w_k}) * Z^{\mu_h}) * H_{v_h}) && \text{(by (R2))} \\
&= \sum_h a_h (((H_{w'} * Z^{r_i(\lambda_1)}) * Z^{\mu_h}) * H_{v_h}) \\
&= \sum_h a_h ((H_{w'} * (Z^{r_i(\lambda_1)} * Z^{\mu_h})) * H_{v_h}) && \text{(by induction)} \\
&= \sum_h a_h H_{w'} * ((Z^{r_i(\lambda_1)} * Z^{\mu_h}) * H_{v_h}) && \text{(by (R2))} \\
&= H_{w'} * (Z^{r_i(\lambda_1)} * (H_i * Z^{\lambda_2})). && \text{(by (R1))}
\end{aligned}$$

This corresponds to the term $H_{w'} * ((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2})$ in $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$ so we obtain the equality when $\ell(w) = n + 1$.

6.5. *Change of scalars.*

6.5.1. Suppose that we are given a morphism φ from R_1 to a ring R. Then we are able to consider, by extension of scalars, ${}^{\mathrm{BL}}\mathcal{H}_R = R \otimes_{R_1} {}^{\mathrm{BL}}\mathcal{H}_{R_1}$ as an R-associative algebra. The family $(Z^\lambda H_w)_{\lambda \in Y,\, w \in W^v}$ is still a basis of the R-module ${}^{\mathrm{BL}}\mathcal{H}_R$.

6.5.2. In order to consider elements similar to the X^λ of Section 4, we are going to define a ring R_3 containing R_1 such that there exists a group homomorphism $\delta^{1/2} : Y \to R_3^\times$ with $\delta(\lambda) = \delta^{1/2}(\lambda)^2$ for any $\lambda \in Q^\vee$ and $\delta^{1/2}(\alpha_i^\vee) = \sigma_i . \sigma_i'$.

Since $Q^\vee$ is a submodule of the free $\mathbb{Z}$-module Y, by the elementary divisor theorem, if we denote by m the biggest elementary divisor, then $m\mu \in Q^\vee$ for any $\mu \in Y \cap (Q^\vee \otimes_{\mathbb{Z}} \mathbb{R})$. Let us consider the ring $R_3 = \mathbb{Z}[(\tau_i^{\pm 1}, \tau_i'^{\pm 1})_{i\in I}]$ (with τ_i, τ_i' satisfying conditions similar to those of Section 6.1) and the identification of R_1 as a subring of R_3 given by $\tau_i^m = \sigma_i$ and $\tau_i'^m = \sigma_i'$. Then, for $\lambda \in Y$ we have $m\lambda = \sum_{i\in I} a_i \alpha_i^\vee + \lambda_0$ with the $a_i \in \mathbb{Z}$ and $\lambda_0 \notin Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, and we can define

$$\delta^{1/2}(\lambda) = \prod_{i\in I} (\tau_i \tau_i')^{a_i}$$

and obtain a group homomorphism from Y to R_3, with the wanted properties.

In ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$, let us consider $X^\lambda = \delta^{1/2}(\lambda) Z^\lambda$ for $\lambda \in Y$ and $T_i = \sigma_i H_i = (\tau_i)^m H_i$. It's easy to see that $T_w = T_{i_1} * T_{i_2} * \cdots * T_{i_n}$ is independent of the choice of a reduced decomposition $r_{i_1} r_{i_2} \cdots r_{i_n}$ of w. It is clear that the family $(X^\lambda * T_w)_{\lambda\in Y,\, w\in W^v}$ is a new basis of the R_3-module ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$.

6.5.3. We can give new formulas to define $*$ in terms of these generators. The relation (4) of the definition of ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$ can be written as previously:

If $\alpha_i(\lambda) \geq 0$, then

$$\text{(BLZ+)} \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{\substack{0\leq k\leq \alpha_i(\lambda)-1 \\ k \text{ even}}} (\sigma_i - \sigma_i^{-1}) Z^{\lambda - k\alpha_i^\vee} + \sum_{\substack{0\leq k\leq \alpha_i(\lambda)-1 \\ k \text{ odd}}} (\sigma_i' - \sigma_i'^{-1}) Z^{\lambda - k\alpha_i^\vee}.$$

If $\alpha_i(\lambda) < 0$, then

$$\text{(BLZ}-\text{)} \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i - \sum_{\substack{2\leq k\leq -\alpha_i(\lambda) \\ k \text{ even}}} (\sigma_i - \sigma_i^{-1}) Z^{\lambda + k\alpha_i^\vee} - \sum_{\substack{1\leq k\leq -\alpha_i(\lambda) \\ k \text{ odd}}} (\sigma_i' - \sigma_i'^{-1}) Z^{\lambda + k\alpha_i^\vee}.$$

With the same arguments as in Section 5.7, these relations (after changing variables and writing $(\sigma_i^2)^{*n} = \sigma_i^2 \sigma_i'^2 \sigma_i^2 \sigma_i'^2 \cdots$ with n terms in this product) become:

If $\alpha_i(\lambda) \geq 0$, then

$$\text{(BLX+)} \quad T_i * X^\lambda = (\sigma_i^2)^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} \big((\sigma_i^2)^{*(h+1)} - (\sigma_i^2)^{*(h)} \big) X^{\lambda - h\alpha_i^\vee}.$$

If $\alpha_i(\lambda) < 0$, then

$$(\mathrm{BLX}-)\quad T_i * X^\lambda = \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big((\sigma_i^2)^{*(-\alpha_i(\lambda)+h+1)} - (\sigma_i^2)^{*(-\alpha_i(\lambda)+h)}\big) X^{\lambda - h\alpha_i^\vee}.$$

The other formulas easily give:

$$(2')\quad T_i * T_w = \begin{cases} T_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i^2 - 1) T_w + \sigma_i^2 T_{r_i w} & \text{if } \ell(r_i w) < \ell(w), \end{cases} \quad \text{for all } i \in I,\ w \in W^v,$$

$$(3')\quad X^\lambda * X^\mu = X^{\lambda+\mu} \quad \text{for all } \lambda \in Y,\ \mu \in Y.$$

In all these relations, we can see that the coefficients are in the subring $R_2 = \mathbb{Z}[(\sigma_i^{\pm 2}, \sigma_i'^{\pm 2})_{i \in I}]$ of R_1. So, if we consider ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$ the R_2-submodule with basis $(X^\lambda * T_w)_{\lambda \in Y,\, w \in W^v}$, the multiplication $*$ gives it a structure of associative unitary algebra over R_2.

6.6. ***The positive Bernstein–Lusztig–Hecke algebra.*** If we consider in ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$, the submodule with basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$, it is stable by multiplication $*$ (in (BLX+) and (BLX−) if $\lambda \in Y^+$ all the $\lambda \pm h\alpha_i^\vee$ written are also in Y^+). We denote by ${}^{\mathrm{BL}}\mathcal{H}_{R_2}^+$ this R_2-subalgebra of ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$. Actually, we can define such positive Hecke subalgebras inside all algebras in Section 6.5.

Like before, if we are given a morphism θ from R_2 to a ring R, we are able to consider, by extension of scalars, ${}^{\mathrm{BL}}\mathcal{H}_R^+ = R \otimes_{R_2} {}^{\mathrm{BL}}\mathcal{H}_{R_2}^+$. Let us consider the ring R of the Section 4 (such that $\mathbb{Z} \subset R$ and all q_i, q_i' are invertible in R); we can construct a morphism θ from R_2 to R by $\theta(\sigma_i^2) = q_i$ and $\theta(\sigma_i'^2) = q_i'$. So, we obtain an algebra ${}^{\mathrm{BL}}\mathcal{H}_R^+$ with basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$ and the same relations as in ${}^{\mathrm{I}}\mathcal{H}_R$. So:

Proposition. *Over R, the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ and the positive Bernstein–Lusztig–Hecke algebra ${}^{\mathrm{BL}}\mathcal{H}_R^+$ are isomorphic.*

Remark. ${}^{\mathrm{BLX}}\mathcal{H}_R$ is a ring of quotients of ${}^{\mathrm{BL}}\mathcal{H}_R^+ \simeq {}^{\mathrm{I}}\mathcal{H}_R$, as we added in it inverses of the $X^\lambda = T_\lambda$ for $\lambda \in Y^{++}$. Actually, from Proposition 5.2, Corollary 5.4, and similar results, one may prove that $S = \{T_\lambda \mid \lambda \in Y^{++}\}$ satisfies the right and left Ore condition and that the map from ${}^{\mathrm{BL}}\mathcal{H}_R^+$ to the corresponding quotient ring is injective; see, e.g., [McConnell and Robson 2001, 2.1.6 and 2.1.12].

6.7. ***Structure constants.*** Using Section 6.6, the structure constants of the convolution product $*$ of ${}^{\mathrm{I}}\mathcal{H}_R$, in the basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$, are Laurent polynomials in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W. By Theorem 5.5(a), we get the same result for the structure constants in the basis

$(T_\lambda * T_w)_{\lambda\in Y^+,\, w\in W^v}$ and then still the same result for the structure constants $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ in the basis $(T_{\boldsymbol{w}})_{\boldsymbol{w}\in W^+}$ (by Section 4.5).

This last result is not as precise as the one expected in the conjecture of Section 2. But there is at least one case where we can prove it:

Remark. Suppose $\mathscr{I}$ is the hovel associated to a split Kac–Moody group G over a local field $\mathcal{K}$; see [Gaussent and Rousseau 2014, §3]. Then all parameters q_i, q_i' are equal to the cardinality q of the residue field; moreover, we know that each $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is an integer and a Laurent polynomial in q, with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W. But, as G is split, the same thing is true (without changing $\mathbb{A}$ and W) for all unramified extensions of the field $\mathcal{K}$, hence for infinitely many q. So the Laurent polynomial $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is an integer for infinitely many integral values of the variable q; hence, it has to be a true polynomial. This result was also obtained independently by D. Muthiah [2015], when G is untwisted affine.

7. Extended affine cases and DAHAs

In this section, we define the extended Iwahori–Hecke algebras and explore their relationship with the double affine Hecke algebras introduced by Cherednik.

7.1. ***Extended groups of automorphisms.*** We may consider a group $\widetilde{G}$ containing the group G of Section 1.4 and an extension to $\widetilde{G}$ of the action of G on $\mathscr{I}$. We assume that $\widetilde{G}$ permutes the apartments and induces isomorphisms between them, hence $\widetilde{G}$ is equal to $G.\widetilde{N}$, where $\widetilde{N}\supset N$ is the stabilizer of $\mathbb{A}$ in $\widetilde{G}$. This group $\widetilde{N}$ has almost the same properties as the group N described in Section 1.4.4. But we assume now that $\widetilde{W}=\nu(\widetilde{N})\subset \mathrm{Aut}(\mathbb{A})$ is only positive for its action on the vectorial faces; this means that the associated linear map $\vec{w}$ of any $w\in\widetilde{W}$ is in $\mathrm{Aut}^+(\mathbb{A}^v)$. We assume moreover that $\widetilde{W}$ may be written $\widetilde{W}=\widetilde{W}^v\ltimes Y$, where $\widetilde{W}^v$ fixes the origin 0 of $\mathbb{A}$ and Y is the same group of translations as for G; see Section 1.4.4. In particular, $\widetilde{W}^v$ is isomorphic to the group $\{\vec{w}\mid w\in\widetilde{W}\}$ and may be written $\widetilde{W}^v=\Omega\ltimes W^v$ (see Section 1.1); moreover $\widetilde{W}=\Omega\ltimes W$, where Ω is the stabilizer of C^v_f in $\widetilde{W}$. Finally, we assume that G contains the fixer $\mathrm{Ker}\,\nu$ of $\mathbb{A}$ in $\widetilde{G}$ so that $G\lhd\widetilde{G}$ is the subgroup of all vectorially Weyl automorphisms in $\widetilde{G}$ and $\widetilde{G}/G\simeq\Omega$.

As $\widetilde{W}$ is positive, $\widetilde{G}$ preserves the preorder $\le$ on $\mathscr{I}$. So $\widetilde{G}^+=\{g\in\widetilde{G}\mid 0\le g.0\}$ is a semigroup with $\widetilde{G}^+\cap G=G^+$. And $\widetilde{W}^+=\Omega\ltimes W^+=\widetilde{W}^v\ltimes Y^+\subset\widetilde{W}$ is also a semigroup, with $\widetilde{W}^+\cap W=W^+$.

7.2. ***Examples: Kac–Moody and loop groups.***

7.2.1. One considers a field $\mathcal{K}$, complete for a normalized, discrete valuation with a finite residue field (of cardinality q). If $\mathfrak{G}$ is an almost split Kac–Moody group scheme over $\mathcal{K}$, then the Kac–Moody group $G=\mathfrak{G}(\mathcal{K})$ acts on an affine ordered hovel $\mathscr{I}$, with the properties described in Section 1.4. See [Rousseau 2010;

Gaussent and Rousseau 2014, §3] in the split case (where all q_i, q'_i are equal to q) and [Charignon 2009; 2010; Rousseau 2012] in general.

7.2.2. Let $\mathfrak{G}_0$ be a simply connected, almost simple, split, semisimple algebraic group of rank r over $\mathcal{K}$. Its fundamental maximal torus $\mathfrak{T}_0$ is $Q_0^\vee \otimes_{\mathbb{Z}} \mathfrak{Mult}$, where $Q_0^\vee$ and $P_0^\vee$ are the coroot lattice and coweight lattice, respectively, of the root system $\Phi_0 \subset V_0^*$ with Weyl group W_0^v.

Some central extension of (a subgroup of) the loop group $\mathfrak{G}_0(\mathcal{K}[t, t^{-1}]) \rtimes \mathcal{K}^\times$ by $\mathcal{K}^\times$ (where $x \in \mathcal{K}^\times$ acts on $\mathfrak{G}_0(\mathcal{K}[t, t^{-1}])$ via $t \mapsto xt$) is $G = \mathfrak{G}(\mathcal{K})$ for the most popular example $\mathfrak{G}$ of an untwisted, affine, split, Kac–Moody group scheme over $\mathcal{K}$. Its fundamental, maximal torus $\mathfrak{T}$ is $\mathfrak{Mult} \times \mathfrak{T}_0 \times \mathfrak{Mult} = Y \otimes_{\mathbb{Z}} \mathfrak{Mult}$, with cocharacter group $Y = \mathbb{Z}\mathfrak{c} \oplus Q_0^\vee \oplus \mathbb{Z}d$, where $\mathfrak{c}$ is the canonical central element and d is the scaling element.

The set Φ of real roots is $\{\alpha_0 + n\delta \mid \alpha_0 \in \Phi_0,\ n \in \mathbb{Z}\}$ in the dual V^* of

$$V = Y \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}\mathfrak{c} \oplus V_0 \oplus \mathbb{R}d,$$

where $\delta(a\mathfrak{c} + v_0 + bd) = b$ and $\alpha_0(a\mathfrak{c} + v_0 + bd) = \alpha_0(v_0)$. The corresponding Weyl group W^v is actually the affine Weyl group $W_0^a = W_0^v \ltimes Q_0^\vee$ acting linearly on V; its action on the hyperplane $d + V_0$ of $V/\mathbb{R}\mathfrak{c}$ is affine: W_0^v acts linearly on V_0 and $Q_0^\vee$ acts by translations. The group G is generated by $T = \mathfrak{T}(\mathcal{K})$ and root groups $U_\alpha \simeq \mathcal{K} = \mathfrak{Add}(\mathcal{K})$ for $\alpha \in \Phi$; if $\alpha = \alpha_0 + n\delta$, then $U_\alpha = \mathfrak{U}_{\alpha_0}(t^n.\mathcal{K})$.

The fundamental apartment $\mathbb{A}$ of the associated hovel is as described in Section 1.2 with $W = W^v \ltimes Y$ containing the affine Weyl group $W^a = W^v \ltimes Q^\vee$, where $Q^\vee = \mathbb{Z}\mathfrak{c} \oplus Q_0^\vee$.

This is the situation considered in [Braverman et al. 2016]. We saw in [Gaussent and Rousseau 2014, Remark 3.4] that our group K is the same as the K of [Braverman et al. 2016]. It is clear that the Iwahori group I of [op. cit.] is included in our group K_I. But from Section 1.4.2 and [op. cit., 3.1.2], we get two Bruhat decompositions $K = \bigsqcup_{w \in W^v} K_I . w . K_I = \bigsqcup_{w \in W^v} I . w . I$. So $K_I = I$ and, in this case, our results are the same as those of [op. cit.].

7.2.3. Let us consider a central schematic quotient $\mathfrak{G}_{00}$ of $\mathfrak{G}_0$. It is determined by the cocharacter group Y_{00} of its fundamental torus $\mathfrak{T}_{00}$: $Q_0^\vee \subset Y_{00} \subset P_0^\vee$ and $\mathfrak{T}_{00} = Y_{00} \otimes_{\mathbb{Z}} \mathfrak{Mult}$. The root system $\Phi_0 \subset V_0^*$ and the Weyl group $W_0^v \subset \mathrm{GL}(V_0)$ are the same.

We get a more general untwisted, affine, split Kac–Moody scheme $\mathfrak{G}_1$ by "amalgamating" $\mathfrak{G}$ and the $\mathcal{K}$-split torus $\mathfrak{T}_1 = Y_1 \otimes_{\mathbb{Z}} \mathfrak{Mult}$ (with $Y_1 = \mathbb{Z}\mathfrak{c} \oplus Y_{00} \oplus \mathbb{Z}d$) along $\mathfrak{T}$. A little more precisely, the Kac–Moody group $G_1 = \mathfrak{G}_1(\mathcal{K})$ is a quotient of the free product of G and $T_{00} = \mathfrak{T}_{00}(\mathcal{K}) = Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}^\times$ by some relations; essentially, T_{00} normalizes T and each U_α (hence also G) and one identifies both copies of T_0;

see [Rousseau 2010, 1.8]. The new fundamental torus is $\mathfrak{T}_1$. We keep the same V, Φ, W^v, $\mathbb{A}$, and $\mathscr{I}$, but now $W_1 = W^v \ltimes Y_1 \supset W \supset W^a$.

7.2.4. We may consider a central extension by $\mathcal{K}^\times$ of (a subgroup of) the loop group $\mathfrak{G}_{00}(\mathcal{K}[t, t^{-1}]) \rtimes \mathcal{K}^\times$. We get thus an extended Kac–Moody group $\widetilde{G}_2$ (not among the Kac–Moody groups of [Tits 1987] or [Rousseau 2010]) which may also be described by amalgamation: $\widetilde{G}$ is a quotient of the free product of G and $Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}[t, t^{-1}]^*$ by relations similar to those above; in particular the conjugation by $\lambda \otimes xt^n$ sends $U_{\alpha_0+p\delta}$ to $U_{\alpha_0+(p+n\alpha(\lambda))\delta}$. The group $\widetilde{G}_2$ contains G_1 as a normal subgroup; its fundamental torus is $T_1 = Y_1 \otimes_{\mathbb{Z}} \mathcal{K}^\times$, with normalizer $\widetilde{N}_2 = N_{\tilde{G}_2}(T_1)$ containing $Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}[t, t^{-1}]^* \supset Y_{00} \otimes_{\mathbb{Z}} t^{\mathbb{Z}} =: t^{Y_{00}}$.

The group $\widetilde{G}_2$ is generated by $t^{Y_{00}}$ and G_1 (which contains $N_1 = N_2 \cap G_1 \supset t^{Q_0^\vee}$); in particular $\widetilde{G}_2/G_1 \simeq Y_{00}/Q_0^\vee$. We keep the same V and Φ, but now the corresponding vectorial Weyl group is $\widetilde{W}_2^v = N_2/T_1 = W_0^v \ltimes Y_{00}$. As in Section 1.1, we may also write $\widetilde{W}_2^v = \Omega_2 \ltimes W^v$, where Ω_2 is the stabilizer in $\widetilde{W}_2^v$ of C_f^v. It is well known that Ω_2 is a finite group isomorphic to $Y_{00}/Q_0^\vee$; it is isomorphic to its image in the permutation group of the affine Dynkin diagram of $\mathfrak{G}_{00}$ or $\mathfrak{G}_0$ (indexed by I) and acts simply transitively on the special vertices of this diagram.

It is not too difficult to extend to $\widetilde{G}_2$ the action of G_1 on the hovel $\mathscr{I}$. The group $\widetilde{N}_2$ is the stabilizer of $\mathbb{A}$; it acts through $\widetilde{W}_2 = \widetilde{W}_2^v \ltimes Y_1 \supset W \supset W^a$. We are exactly in the situation of Section 7.1 with $(\widetilde{G}_2, G_1)$.

7.2.5. We may get new couples $(\widetilde{G}_j, G_j)$ satisfying Section 7.1 for the same hovel $\mathscr{I}$:

We may enlarge $\widetilde{G}_2$ and G_1 by amalgamating them with $T_3 = Y_3 \otimes_{\mathbb{Z}} \mathcal{K}^\times$ along T_1 (or with $T_{000} = Y_{000} \otimes_{\mathbb{Z}} \mathcal{K}^\times$ along T_{00}), where $Y_{00} \subset Y_{000} \subset P_0^\vee$ and

$$Y_3 = \mathbb{Z}.\tfrac{1}{m}.\mathfrak{c} \oplus Y_{000} \oplus \mathbb{Z}d,$$

with $m \in \mathbb{Z}_{>0}$. Then $\widetilde{W}_3^v = \widetilde{W}_2^v$, $\Omega_3 = \Omega_2$, $\widetilde{W}_3 = \widetilde{W}_2^v \ltimes Y_3$, and G_3 is still a Kac–Moody group with maximal torus T_3.

We may keep G_1 (or G_3) and take a semidirect product of $\widetilde{G}_2$ (or $\widetilde{G}_3$) by a group Γ of automorphisms of the Dynkin diagram of $\mathfrak{G}_0$, stabilizing Y_{00} (or Y_{00} and Y_{000}). Then $\widetilde{W}_4^v = \Gamma \ltimes \widetilde{W}_2^v$, $\Omega_4 = \Gamma \ltimes \Omega_2$, and $\widetilde{W}_4 = \widetilde{W}_4^v \ltimes Y_2$ (or $\widetilde{W}_4 = \widetilde{W}_4^v \ltimes Y_3$).

7.2.6. We may also add a split torus as direct factor to any of the preceding groups $\widetilde{G}_i$ or G_i, enlarge $\mathscr{I}$ by a trivial euclidean factor of the same dimension as the torus and add to $\widetilde{W}^v$ and Ω, as a direct factor, any automorphism group (possibly infinite) of this torus.

7.3. *Marked chambers.* We come back to the general situation of Section 7.1. We want a set of "geometric objects" in $\mathscr{I}$ on which $\widetilde{G}$ acts with the Iwahori subgroup K_I as one of the isotropy groups.

7.3.1. A *marked chamber* in the hovel $\mathscr{I}$ is the equivalence class of an isomorphism $\varphi : \mathbb{A} \to A \in \mathcal{A}$ sending the fundamental chamber C_0^+ to some local chamber C_x, modulo the equivalence

$$\varphi_1 \simeq \varphi_2 \iff \exists S \in C_0^+ \text{ such that } \varphi_1|_S = \varphi_2|_S.$$

It is simply written $\varphi : C_0^+ \to C_x$; this does not depend on A.

The group $\widetilde{G}$ permutes the marked chambers; for $g \in \widetilde{G}$ and φ as above, $g.\varphi = \varphi$ if and only if g fixes (pointwise) C_x. In particular, the isotropy group in $\widetilde{G}$ of $\widetilde{C}_0^+ = \mathrm{Id} : C_0^+ \to C_0^+ \subset \mathbb{A} \subset \mathscr{I}$ is $K_I \subset G$.

A local chamber of type 0, $C_x \in \mathscr{C}_0^+$ determines a unique marked chamber $\widetilde{C}_x^0 : C_0^+ \to C_x$ (called *normalized*) which is the restriction of some $\varphi \in \mathrm{Isom}_{\mathbb{R}}^W(\mathbb{A}, A)$; see Section 1.11. These normalized marked chambers are permuted transitively by G.

7.3.2. A marked chamber is said *of type* 0 if it is in the orbit under $\widetilde{G}$ of any of those $\widetilde{C}_x^0$. So the set $\widetilde{\mathscr{C}}_0^+$ of marked chambers of type 0 is $\widetilde{G}/K_I$.

By hypothesis $\widetilde{G}$ may be written $G.\widetilde{\Omega}$, where $\widetilde{\Omega} = \nu^{-1}(\Omega) \subset \widetilde{N}$ stabilizes C_0^+ (considered as in $\mathscr{I}$) and induces Ω on it. So $\widetilde{\mathscr{C}}_0^+ = \{\widetilde{C}_x = \widetilde{C}_x^0 \circ \omega^{-1} \mid C_x \in \mathscr{C}_0^+,\ \omega \in \Omega\}$.

7.4. $\widetilde{W}$-*distance.*

7.4.1. Let $\widetilde{C}_x : C_0^+ \to C_x$, $\widetilde{C}_y : C_0^+ \to C_y$ be in $\widetilde{\mathscr{C}}_0^+$ with $x \le y$. There is an apartment A containing C_x and C_y so $\widetilde{C}_x, \widetilde{C}_y$ may be extended to $\varphi, \psi \in \mathrm{Isom}(\mathbb{A}, A)$. We "identify" $(\mathbb{A}, C_0^+)$ with (A, C_x) via φ. Then $\varphi^{-1}(y) \ge 0$ and, as $\widetilde{C}_x, \widetilde{C}_y$ are in a same orbit of $\widetilde{G}$, there is $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$ such that $\psi = \varphi \circ \widetilde{\boldsymbol{w}}$. This $\widetilde{\boldsymbol{w}}$ does not depend on the choice of A by Proposition 1.10(c).

We define the $\widetilde{W}$-*distance* between the marked chambers $\widetilde{C}_x$ and $\widetilde{C}_y$ as this unique element: $d^W(\widetilde{C}_x, \widetilde{C}_y) = \widetilde{\boldsymbol{w}} \in \widetilde{W}^+$. So we get a $\widetilde{G}$-invariant map

$$d^W : \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+ = \{(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times \widetilde{\mathscr{C}}_0^+ \mid x \le y\} \to \widetilde{W}^+.$$

7.4.2. For $(C_x, C_y) \in \mathscr{C}_0^+ \times_{\le} \mathscr{C}_0^+$, we have $d^W(\widetilde{C}_x^0, \widetilde{C}_y^0) = d^W(C_x, C_y)$ and, more generally, for $\omega_x, \omega_y \in \Omega$, we have $(\widetilde{C}_x^0 \circ \omega_x^{-1}, \widetilde{C}_y^0 \circ \omega_y^{-1}) \in \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+$ and

$$d^W(\widetilde{C}_x^0 \circ \omega_x^{-1}, \widetilde{C}_y^0 \circ \omega_y^{-1}) = \omega_x . d^W(C_x, C_y) . \omega_y^{-1} \in \widetilde{W}^+.$$

For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+$ and $\omega_x, \omega_y \in \Omega$, we have also

$$d^W(\widetilde{C}_x \circ \omega_x^{-1}, \widetilde{C}_y \circ \omega_y^{-1}) = \omega_x . d^W(\widetilde{C}_x, \widetilde{C}_y) . \omega_y^{-1} \in \widetilde{W}^+.$$

We deduce from this some interesting consequences:

7.4.3. If $\widetilde{C}_x, \widetilde{C}_y, \widetilde{C}_z$, with $x \le y \le z$, are in the same apartment, we have a Chasles relation:

$$d^W(\widetilde{C}_x, \widetilde{C}_z) = d^W(\widetilde{C}_x, \widetilde{C}_y) . d^W(\widetilde{C}_y, \widetilde{C}_z).$$

7.4.4. For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_\le \widetilde{\mathscr{C}}_0^+$, if $\widetilde{C}_x$ is normalized then $d^W(\widetilde{C}_x, \widetilde{C}_y) \in W^+$ if and only if $\widetilde{C}_y$ is normalized. The same is true with the roles of $\widetilde{C}_x$ and $\widetilde{C}_y$ reversed.

7.4.5. For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_\le \widetilde{\mathscr{C}}_0^+$,

$$d^W(\widetilde{C}_x, \widetilde{C}_y) = \omega \in \Omega \iff \widetilde{C}_y = \widetilde{C}_x \circ \omega.$$

In particular, $\widetilde{C}_y$ is uniquely determined by $\widetilde{C}_x$ and ω; moreover, $C_y = C_x$.

7.4.6. If $(C_x, C_y) \in \mathscr{C}_0^+ \times_\le \mathscr{C}_0^+$ and $d^W(C_x, C_y) = r_i \in W^v$ (resp., $\lambda \in Y^+$) and $\omega \in \Omega$, then $d^W(\widetilde{C}_x^0 \circ \omega^{-1}, \widetilde{C}_y^0 \circ \omega^{-1}) = \omega . r_i . \omega^{-1} = r_{\omega(i)}$ (resp., $\omega(\lambda) \in Y^+$), where we consider the action of Ω on I (resp., Y).

7.4.7. When $\widetilde{C}_x = \widetilde{C}_0^+$ and $\widetilde{C}_y = g . \widetilde{C}_0^+$ (with $g \in \widetilde{G}^+$), then $d^W(\widetilde{C}_x, \widetilde{C}_y)$ is the only $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$ such that $g \in K_I . \widetilde{\boldsymbol{w}} . K_I$. There is a Bruhat decomposition

$$\widetilde{G}^+ = \bigsqcup_{\widetilde{\boldsymbol{w}} \in \widetilde{W}^+} K_I . \widetilde{\boldsymbol{w}} . K_I .$$

The $\widetilde{W}$-distance classifies the orbits of K_I on $\{\widetilde{C}_y \in \widetilde{\mathscr{C}}_0^+ \mid y \ge 0\}$, hence also the orbits of $\widetilde{G}$ on $\widetilde{\mathscr{C}}_0^+ \times_\le \widetilde{\mathscr{C}}_0^+$.

7.5. *The extended Iwahori–Hecke algebra.*

7.5.1. We define this extended algebra for $\widetilde{G}$ as we did in Section 2 for G:

To each $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$, we associate a function $T_{\widetilde{\boldsymbol{w}}} : \widetilde{\mathscr{C}}_0^+ \times_\le \widetilde{\mathscr{C}}_0^+ \to R$, defined by

$$T_{\widetilde{\boldsymbol{w}}}(\widetilde{C}, \widetilde{C}') = \begin{cases} 1 & \text{if } d^W(\widetilde{C}, \widetilde{C}') = \widetilde{\boldsymbol{w}}, \\ 0 & \text{otherwise.} \end{cases}$$

And we consider the following free R-module of functions $\widetilde{\mathscr{C}}_0^+ \times_\le \widetilde{\mathscr{C}}_0^+ \to R$:

$$^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = \left\{ \varphi = \sum_{\widetilde{\boldsymbol{w}} \in \widetilde{W}^+} a_{\widetilde{\boldsymbol{w}}} T_{\widetilde{\boldsymbol{w}}} \;\middle|\; a_{\widetilde{\boldsymbol{w}}} \in R,\ a_{\widetilde{\boldsymbol{w}}} = 0 \text{ except for a finite number} \right\},$$

We endow this R-module with the convolution product given by

$$(\varphi * \psi)(\widetilde{C}_x, \widetilde{C}_y) = \sum_{\widetilde{C}_z} \varphi(\widetilde{C}_x, \widetilde{C}_z) \psi(\widetilde{C}_z, \widetilde{C}_y).$$

where $\widetilde{C}_z \in \widetilde{\mathscr{C}}_0^+$ is such that $x \le z \le y$. This product is associative and R-bilinear. We prove below that it is well defined.

As in Section 2, we see easily that $^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is the natural convolution algebra of the functions $\widetilde{G}^+ \to R$, bi-invariant under K_I and with finite support.

7.5.2. For $\omega \in \Omega$, $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$, the products $T_\omega * T_{\widetilde{\boldsymbol{w}}}$ and $T_{\widetilde{\boldsymbol{w}}} * T_\omega$ are well defined: actually $T_\omega * T_{\widetilde{\boldsymbol{w}}} = T_{\omega . \widetilde{\boldsymbol{w}}}$ and $T_{\widetilde{\boldsymbol{w}}} * T_\omega = T_{\widetilde{\boldsymbol{w}} . \omega}$; see Sections 7.4.3 and 7.4.5.

7.5.3. As the formula for $\varphi * \psi$ is clearly $\widetilde{G}$-invariant, we may fix $\widetilde{C}_x$ normalized to calculate $\varphi * \psi$. From Section 7.4.4, we deduce that, when $\boldsymbol{w}, \boldsymbol{v} \in W^+$, the product $T_{\boldsymbol{w}} * T_{\boldsymbol{v}}$ may be computed using only normalized marked chambers. So it is well defined and the same as in ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$.

From Section 7.5.2 we deduce now that the convolution product is well defined in ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$:

Proposition. *For any ring R, ${}^{I}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is an algebra; it contains ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ as a subalgebra.*

Definition. The algebra ${}^{I}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is the *extended Iwahori–Hecke algebra* associated to $\mathscr{I}$ and $\widetilde{G}$ with coefficients in R.

7.6. *Relations.*

7.6.1. From Section 7.5 we see that ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ contains the algebra

$$R[\Omega] = \bigoplus_{\omega \in \Omega} R \,.\, T_\omega$$

of the group Ω. Moreover, as an R-module, ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = R[\Omega] \otimes_R {}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$: we identify $T_{\omega.\boldsymbol{w}} = T_\omega * T_{\boldsymbol{w}}$ and $T_\omega \otimes T_{\boldsymbol{w}}$ for $\omega \in \Omega$ and $\boldsymbol{w} \in W^+$.

The multiplication in this tensor product is semidirect:

$$\begin{aligned}(T_\omega \otimes T_{\boldsymbol{w}}) \,.\, (T_{\omega'} \otimes T_{\boldsymbol{v}}) &= T_\omega * T_{\boldsymbol{w}} * T_{\omega'} * T_{\boldsymbol{v}} = T_{\omega.\boldsymbol{w}.\omega'} * T_{\boldsymbol{v}} \\ &= T_{\omega.\omega'.\boldsymbol{w}'} * T_{\boldsymbol{v}} = T_{\omega.\omega'} * T_{\boldsymbol{w}'} * T_{\boldsymbol{v}} = T_{\omega.\omega'} \otimes (T_{\boldsymbol{w}'} * T_{\boldsymbol{v}}),\end{aligned}$$

where $\boldsymbol{w}' = \omega'^{-1} \,.\, \boldsymbol{w} \,.\, \omega' =: \omega'^{-1}(\boldsymbol{w}) \in W^+$.

In particular, we get the following relations among some elements:

7.6.2. For $\omega \in \Omega$ and $\boldsymbol{w} \in W^+$,

$$T_\omega * T_{\boldsymbol{w}} * T_\omega^{-1} = T_{\omega(\boldsymbol{w})}.$$

If, moreover, $\boldsymbol{w} = r_i \in W^v$, then $\omega(r_i) = r_{\omega(i)}$ and

$$T_\omega * T_i * T_\omega^{-1} = T_{\omega(i)}.$$

If now $\boldsymbol{w} = \lambda \in Y^+$, then

$$T_\omega * T_\lambda * T_\omega^{-1} = T_{\omega(\lambda)},$$

with $\omega(\lambda) \in Y^+$.

7.6.3. From Theorem 5.5(1) and (2) above, it is clear that $T_\omega * X^\lambda * T_\omega^{-1} = X^{\omega(\lambda)}$ if $\omega \in \Omega$ and $\lambda \in Y^+$ (as Ω stabilizes $Y^{++} = Y \cap C_f^v$).

7.6.4. As the action of Ω on $\mathbb{A}$ is induced by automorphisms of $\mathscr{I}$, we have $q_i = q_{\omega(i)}$ and $q'_i = q'_{\omega(i)}$ for $\omega \in \Omega$ and $i \in I$. We may also choose the homomorphism $\delta^{1/2} : Y \to R^*$ of Section 5.7 invariant by Ω (for R great enough). So, for $\omega \in \Omega$, $w, r_i \in W^v$, and $\lambda \in Y$,

$$T_\omega * H_w * T_\omega^{-1} = H_{\omega(w)}, \quad T_\omega * H_i * T_\omega^{-1} = H_{\omega(i)}, \quad T_\omega * Z^\lambda * T_\omega^{-1} = Z^{\omega(\lambda)}.$$

7.7. ***The extended Bernstein–Lusztig–Hecke algebra.*** Notation from Section 7.1 is still in use. But we no longer assume the existence of a group $\widetilde{G}$ or G. The group $W = W^v \ltimes Y \lhd \widetilde{W}$ satisfies $\widetilde{W} = \Omega \ltimes W$ and the conditions of Section 6.

We consider the ring $\tilde{R} = \mathbb{Z}[(\tilde{\sigma}_i^{\pm 1}, (\tilde{\sigma}'_i)^{\pm 1})_{i\in I}]$, where the indeterminates $\tilde{\sigma}_i, \tilde{\sigma}'_i$ satisfy the same relations as σ_i, σ'_i in Section 6.1 and the additional relation (see Section 7.6.4 above)

$$\tilde{\sigma}_i = \tilde{\sigma}_j \text{ and } \tilde{\sigma}'_i = \tilde{\sigma}'_j \quad \text{if } \omega(i) = j \text{ for some } \omega \in \Omega.$$

We denote by ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ the free $\tilde{R}$-module with basis $(T_\omega Z^\lambda H_w)_{\omega\in\Omega,\, \lambda\in Y,\, w\in W^v}$ and write

$$H_w = T_1 Z^0 H_w, \quad H_i = T_1 Z^0 H_i, \quad Z^\lambda = T_1 Z^\lambda H_e, \text{ and } T_\omega = T_\omega Z^0 H_e.$$

Proposition. *There exists a unique multiplication $*$ on ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ which makes it an associative, unitary $\tilde{R}$-algebra with unity $H_e = T_1 = Z^0$ and satisfies the conditions* (1)–(4) *of Theorem 6.2 plus*

(5) $T_\omega * T_{\omega'} = T_{\omega.\omega'}$, $T_\omega * T_i * T_\omega^{-1} = T_{\omega(i)}$, *and* $T_\omega * T_\lambda * T_\omega^{-1} = T_{\omega(\lambda)}$ *for* $\omega, \omega' \in \Omega$, $i \in I$, *and* $\lambda \in Y$.

Proof. As $\tilde{R}$-modules, ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}} = \tilde{R}[\Omega] \otimes {}^{\mathrm{BL}}\mathcal{H}_{\tilde{R}}$, where the homomorphism $R_1 \to \tilde{R}$ is given by $\sigma_i \mapsto \tilde{\sigma}_i$, $\sigma'_i \mapsto \tilde{\sigma}'_i$. Now the multiplication is classical on $\tilde{R}[\Omega]$, given by Theorem 6.2 on ${}^{\mathrm{BL}}\mathcal{H}_{\tilde{R}}$, and semidirect for general elements. $\square$

Definition. This $\tilde{R}$-algebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ depends only on $\mathbb{A}$, Y and Ω (i.e., on $\mathbb{A}$ and $\widetilde{W}$). We call it the *extended Bernstein–Lusztig–Hecke algebra* associated to $\mathbb{A}$ and $\widetilde{W}$ with coefficients in $\tilde{R}$.

As in Section 6.6, we may identify, up to an extension of scalars, a subalgebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}^+_{\tilde{R}}$ of ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ with the extended Iwahori–Hecke algebra ${}^{\mathrm{I}}\widetilde{\mathcal{H}}^{\mathscr{I}}_{R}$.

7.8. ***The affine case.***

7.8.1. We suppose now that $(\mathbb{A}^v, W^v)$ is affine. So there is a smallest positive imaginary root $\delta = \sum a_i \alpha_i \in \Delta^+_{\mathrm{im}} \subset Q^+$ satisfying $\delta(\alpha_i^\vee) = 0$ for all $i \in I$, and a canonical central element $\mathfrak{c} = \sum a_i^\vee \alpha_i^\vee \in Q^\vee_+$ satisfying $\alpha_i(\mathfrak{c}) = 0$ for all $i \in I$. In particular, δ and $\mathfrak{c}$ are fixed by W^v and $\widetilde{W}^v$.

As $\delta \in Q^+$, it takes integral values on Y. For $n \in \mathbb{Z}$, we define

$$Y^n = \{\lambda \in Y \mid \delta(\lambda) = n\},$$

which is stable under W^v and $\widetilde{W}^v$. We have $Y = \bigsqcup_{n\in\mathbb{Z}} Y^n$ and $Y^+ = \big(\bigsqcup_{n>0} Y^n\big) \sqcup Y_c^0$, with $Y_c^0 = Y^0 \cap Y^+ = Y \cap \mathbb{Q}\mathfrak{c}$. We write $\lambda_c = \frac{1}{m}\mathfrak{c}$ a generator of Y_c^0 (with $m \in \mathbb{Z}_{>0}$). As $\delta(Q^\vee) = 0$, we have $\delta(\lambda) = \delta(\mu)$ whenever $\mu \le_{Q^\vee} \lambda$ or $\mu \le_{Q^\vee_{\mathbb{R}}} \lambda$ in Y.

7.8.2. Considering Proposition 2.2 and Theorem 5.5(2), the algebra is graded (for a suitable R) by

$${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}} = \bigoplus_{n\ge 0} {}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}n},$$

where ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}n}$ has for R-basis $\{T_\lambda * T_w \mid \lambda \in Y^n,\ w \in W^v\}$ if $n > 0$ and $\{T_\lambda * T_w \mid \lambda \in Y_c^0,\ w \in W^v\}$ if $n = 0$. For some rings R, we may replace each $T_\lambda * T_w$ by $X^\lambda * T_w$ or by $Z^\lambda * H_w$ to find new bases. Also,

$${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = \bigoplus_{n\ge 0} {}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}n},$$

where ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}n}$ has for R-basis $\{T_\omega * T_\lambda * T_w \mid \omega \in \Omega,\ \lambda \in Y^n,\ w \in W^v\}$ if $n > 0$ and $\{T_\omega * X^\lambda * T_w \mid \omega \in \Omega,\ \lambda \in Y_c^0,\ w \in W^v\}$ if $n = 0$. For some rings R, we may replace each $T_\omega * T_\lambda * T_w$ by $T_\omega * X^\lambda * T_w$ or by $T_\omega * Z^\lambda * H_w$ to find new bases. Furthermore,

$${}^{\mathrm{BL}}\mathcal{H}_{R_1} = \bigoplus_{n\in\mathbb{Z}} {}^{\mathrm{BL}}\mathcal{H}_{R_1}^n,$$

where ${}^{\mathrm{BL}}\mathcal{H}_{R_1}^n$ has for R_1-basis the $Z^\lambda H_w$ for $\lambda \in Y^n$ and $w \in W^v$, and

$${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}} = \bigoplus_{n\in\mathbb{Z}} {}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^n,$$

where ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^n$ has for $\tilde{R}$-basis the $T_\omega Z^\lambda H_w$ for $\omega \in \Omega,\ \lambda \in Y^n$, and $w \in W^v$.

These gradations are compatible with the identifications explained in Section 6.6 or Section 7.7.

7.8.3. For any $\widetilde{C}_x \in \widetilde{\mathscr{C}}_0^+$ and any $\lambda \in Y_c^0 = \mathbb{Z}\lambda_c$, there is a unique $\widetilde{C}_y \in \widetilde{\mathscr{C}}_0^+$ with $d^W(\widetilde{C}_x, \widetilde{C}_y) = \lambda$: the translation by λ in $\mathbb{A}$ stabilizes all enclosed sets and extends to $\mathscr{I}$ as a translation in any apartment. From this we see that

$$\begin{cases} T_\lambda * T_\mu = T_{\lambda+\mu} = T_\mu * T_\lambda & \text{for } \mu \in Y^+, \\ T_\lambda * X^\mu = X^{\lambda+\mu} = X^\mu * T_\lambda & \text{for } \mu \in Y, \\ T_\lambda * T_w = T_{\lambda.w} = T_{w.\lambda} = T_w * T_\lambda & \text{for } w \in W^v. \end{cases}$$

Such a T_λ is central and invertible in ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$, ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$, ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$, or ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$.

Actually ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}0}$ is the tensor product $R[Y_c^0]\otimes_R\mathcal{H}_R(W^v)$ with a direct multiplication (factor by factor) and ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}0} = R[Y_c^0] \otimes_R (R[\Omega] \otimes_R \mathcal{H}_R(W^v))$ with a semidirect multiplication.

7.9. ***The double affine Hecke algebra.*** The subalgebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^0$ is well known as Cherednik's double affine Hecke algebra (DAHA). More precisely, Cherednik [1992; 1995] considers an untwisted affine root system, as in [Kac 1990, Chapter 7]; but, as he works with roots instead of coroots, we write $\Phi^\vee$ for this system. He considers the case where $\widetilde{W}^v$ is the full extended Weyl group ($\widetilde{W}^v = W_0^v \ltimes P_0^\vee$ with the notation of Section 7.2), i.e., $\Omega \simeq P_0^\vee/Q_0^\vee$ acts on the extended Dynkin diagram, simply transitively on its "special" vertices. His choice for Y^0 is $Y^0 = \mathbb{Z}\cdot\frac{1}{m}.\mathfrak{c}\oplus P_0^\vee \subset P^\vee$ (and, e.g., $Y = Y^0 \oplus \mathbb{Z}d$), where $m \in \mathbb{Z}_{\geq 1}$ is suitably chosen. He then defines the DAHA as an algebra over a field of rational functions $\mathbb{C}(\underline{\delta}, (q_\nu)_{\nu\in\nu_R})$ with generators $(T_i)_{i\in I}$, $(X_\beta)_{\beta\in P_0^\vee}$ and some relations. It is easy to see that this DAHA is, up to scalar changes, a ring of quotients of our ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^0$ (for $\mathbb{A}$, $\widetilde{W}$ as described above): actually $\underline{\delta}$ stands for our Z^{λ_c}. Here is a partial dictionary to translate from [Cherednik 1992; Cherednik 1995] to our article: roots $\leftrightarrow$ coroots, $X_\beta \mapsto Z^\beta$, $T_i \mapsto H_i$, $q_i \mapsto \sigma_i$, $\Pi \mapsto \Omega$, $\pi_r \mapsto T_\omega$, $\underline{\delta} \mapsto T_{\lambda_c}$ and $\underline{\Delta} = \underline{\delta}^m \mapsto T_{\mathfrak{c}}$.

In [Cherednik 1992] there is another presentation of the same DAHA using the Bernstein presentation of $\mathcal{H}_R(W^v)$. This is also the point of view of [Macdonald 2003], where the framework is more general.

References

[Bardy 1996] N. Bardy, *Systèmes de racines infinis*, Mémoire Nouvelle Série **65**, Société Mathématique de France, Paris, 1996. MR 1484906

[Bourbaki 1968] N. Bourbaki, *Groupes et algèbres de Lie: Chapitres 4, 5, 6*, Hermann, Paris, 1968. MR 0240238

[Braverman and Kazhdan 2011] A. Braverman and D. Kazhdan, "The spherical Hecke algebra for affine Kac–Moody groups I", *Ann. of Math.* (2) **174**:3 (2011), 1603–1642. MR 2846488

[Braverman and Kazhdan 2013] A. Braverman and D. Kazhdan, "Representations of affine Kac–Moody groups over local and global fields: a survey of some recent results", pp. 91–117 in *Proceedings of the* 6*th European Congress of Mathematics* (Kraków, 2012), edited by A. Latała, Rafałand Ruciński et al., Eur. Math. Soc., Zürich, 2013. MR 3469117

[Braverman et al. 2014] A. Braverman, H. Garland, D. Kazhdan, and M. Patnaik, "An affine Gindikin–Karpelevich formula", pp. 43–64 in *Perspectives in representation theory* (New Haven, CT, 2012), edited by P. Etingof et al., Contemp. Math. **610**, Amer. Math. Soc., 2014. MR 3220625

[Braverman et al. 2016] A. Braverman, D. Kazhdan, and M. Patnaik, "Iwahori–Hecke algebras for p-adic loop groups", *Invent. Math.* **204**:2 (2016), 347–442. MR 3489701

[Bruhat and Tits 1972] F. Bruhat and J. Tits, "Groupes réductifs sur un corps local", *Inst. Hautes Études Sci. Publ. Math.* 41 (1972), 5–251. MR 0327923

[Charignon 2009] C. Charignon, "Structures immobilières pour un groupe de Kac–Moody sur un corps local", preprint, 2009. arXiv 0912.0442

[Charignon 2010] C. Charignon, *Immeubles affines et groupes de Kac–Moody, masures bordées*, Ph.D. Thesis, Université de Lorraine, Nancy, 2010, http://theses.fr/2010NAN10138.

[Cherednik 1992] I. Cherednik, "Double affine Hecke algebras, Knizhnik–Zamolodchikov equations, and Macdonald's operators", *Internat. Math. Res. Notices* 9 (1992), 171–180. MR 1185831

[Cherednik 1995] I. Cherednik, "Double affine Hecke algebras and Macdonald's conjectures", *Ann. of Math.* (2) **141**:1 (1995), 191–216. MR 1314036

[Ciobotaru and Rousseau 2015] C. Ciobotaru and G. Rousseau, "Strongly transitive actions on affine ordered hovels", preprint, 2015. arXiv 1504.00526

[Garland 1995] H. Garland, "A Cartan decomposition for p-adic loop groups", *Math. Ann.* **302**:1 (1995), 151–175. MR 1329451

[Garland and Grojnowski 1995] H. Garland and I. Grojnowski, "Affine Hecke algebras associated to Kac–Moody groups", preprint, 1995. arXiv q-alg/9508019

[Gaussent and Rousseau 2008] S. Gaussent and G. Rousseau, "Kac–Moody groups, hovels and Littelmann paths", *Ann. Inst. Fourier* (*Grenoble*) **58**:7 (2008), 2605–2657. MR 2498360

[Gaussent and Rousseau 2014] S. Gaussent and G. Rousseau, "Spherical Hecke algebras for Kac–Moody groups over local fields", *Ann. of Math.* (2) **180**:3 (2014), 1051–1087. MR 3245012

[Hecke 1937] E. Hecke, "Über Modulfunktionen und die Dirichletschen Reihen mit Eulerscher Produktentwicklung, I", *Math. Ann.* **114**:1 (1937), 1–28. MR 1513122 Zbl 0015.40202

[Humphreys 1990] J. E. Humphreys, *Reflection groups and Coxeter groups*, Cambridge Studies in Advanced Mathematics **29**, Cambridge University Press, 1990. MR 1066460

[Iwahori 1964] N. Iwahori, "On the structure of a Hecke ring of a Chevalley group over a finite field", *J. Fac. Sci. Univ. Tokyo Sect. I* **10** (1964), 215–236. MR 0165016

[Iwahori and Matsumoto 1965] N. Iwahori and H. Matsumoto, "On some Bruhat decomposition and the structure of the Hecke rings of $\mathfrak{p}$-adic Chevalley groups", *Inst. Hautes Études Sci. Publ. Math.* 25 (1965), 5–48. MR 0185016

[Kac 1990] V. G. Kac, *Infinite-dimensional Lie algebras*, 3rd ed., Cambridge University Press, 1990. MR 1104219

[Kapovich and Millson 2008] M. Kapovich and J. J. Millson, "A path model for geodesics in Euclidean buildings and its applications to representation theory", *Groups Geom. Dyn.* **2**:3 (2008), 405–480. MR 2415306

[Lusztig 1989] G. Lusztig, "Affine Hecke algebras and their graded version", *J. Amer. Math. Soc.* **2**:3 (1989), 599–635. MR 991016

[Macdonald 2003] I. G. Macdonald, *Affine Hecke algebras and orthogonal polynomials*, Cambridge Tracts in Mathematics **157**, Cambridge University Press, 2003. MR 1976581

[McConnell and Robson 2001] J. C. McConnell and J. C. Robson, *Noncommutative Noetherian rings*, Graduate Studies in Mathematics **30**, American Mathematical Society, Providence, 2001. MR 1811901

[Moody and Pianzola 1989] R. V. Moody and A. Pianzola, "On infinite root systems", *Trans. Amer. Math. Soc.* **315**:2 (1989), 661–696. MR 964901

[Moody and Pianzola 1995] R. V. Moody and A. Pianzola, *Lie algebras with triangular decompositions*, Wiley, New York, 1995. MR 1323858

[Muthiah 2015] D. Muthiah, "On Iwahori–Hecke algebras for p-adic loop groups: double coset basis and Bruhat order", preprint, 2015. arXiv 1502.00525

[Parkinson 2006] J. Parkinson, "Buildings and Hecke algebras", *J. Algebra* **297**:1 (2006), 1–49. MR 2206366

[Rémy 2002] B. Rémy, "Groupes de Kac–Moody déployés et presque déployés", pp. viii+348 Astérisque **277**, Société Mathématique de France, Paris, 2002. MR 1909671

[Rousseau 2010] G. Rousseau, "Groupes de Kac–Moody déployés sur un corps local, 2 masures ordonnées", preprint, 2010. To appear in *Bull. Soc. Math. France*. arXiv 1009.0138

[Rousseau 2011] G. Rousseau, "Masures affines", *Pure Appl. Math. Q.* **7**:3, Special Issue: In honor of Jacques Tits (2011), 859–921. MR 2848593

[Rousseau 2012] G. Rousseau, *Almost split Kac–Moody groups over ultrametric fields*, 2012. arXiv 1202.6232

[Satake 1963] I. Satake, "Theory of spherical functions on reductive algebraic groups over $\mathfrak{p}$-adic fields", *Inst. Hautes Études Sci. Publ. Math.* 18 (1963), 5–69. MR 0195863

[Shimura 1959] G. Shimura, "Sur les intégrales attachées aux formes automorphes", *J. Math. Soc. Japan* **11** (1959), 291–311. MR 0120372

[Tits 1987] J. Tits, "Uniqueness and presentation of Kac–Moody groups over fields", *J. Algebra* **105**:2 (1987), 542–573. MR 873684

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