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SCHUR–WEYL DUALITY FOR DELIGNE CATEGORIES II: THE LIMIT CASE

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This paper is a continuation of a previous paper by the author (*Int. Math. Res. Not.* 2015:18 (2015), 8959–9060), which gave an analogue to the classical Schur–Weyl duality in the setting of Deligne categories.

Given a finite-dimensional unital vector space V (a vector space V with a chosen nonzero vector $\mathbb{1}$), we constructed in that paper a complex tensor power of V: an Ind-object of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ which is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$, where $\overline{\mathfrak{P}}_{\mathbb{1}} \subset \mathrm{GL}(V)$ is the mirabolic subgroup preserving the vector $\mathbb{1}$.

This construction allowed us to obtain an exact contravariant functor $\widehat{\mathrm{SW}}_{\nu,V}$ from the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (the abelian envelope of the category $\underline{\mathrm{Rep}}(S_\nu)$) to a certain localization of the parabolic category O associated with the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$.

In this paper, we consider the case when $V = \mathbb{C}^\infty$. We define the appropriate version of the parabolic category O and its localization, and show that the latter is equivalent to a "restricted" inverse limit of categories $\hat{O}^{\mathfrak{p}}_{\nu,\mathbb{C}^N}$ with N tending to infinity. The Schur–Weyl functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N}$ then give an antiequivalence between this category and the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$.

This duality provides an unexpected tensor structure on the category $\hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$.

1. Introduction

1.1. The Karoubian rigid symmetric monoidal categories $\underline{\mathrm{Rep}}(S_\nu)$, $\nu \in \mathbb{C}$, were defined by P. Deligne [2007] as a polynomial family of categories interpolating the categories of finite-dimensional representations of the symmetric groups; namely, at points $n = \nu \in \mathbb{Z}_+$ the category $\underline{\mathrm{Rep}}(S_{\nu=n})$ allows an essentially surjective additive symmetric monoidal functor onto the standard category $\mathrm{Rep}(S_n)$. The categories $\underline{\mathrm{Rep}}(S_\nu)$ were subsequently studied by Deligne and others (e.g., J. Comes and V. Ostrik [2011; 2014]).

In [Entova Aizenbud 2015a], we gave an analogue to the classical Schur–Weyl duality in the setting of Deligne categories. To do that, we defined the "complex

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tensor power" of a finite-dimensional unital complex vector space (i.e., a vector space V with a distinguished nonzero vector $\mathbb{1}$). This complex tensor power of V, denoted by $V^{\underline{\otimes} \nu}$, is an Ind-object in the category $\underline{\mathrm{Rep}}(S_\nu)$, and comes with an action of $\mathfrak{gl}(V)$ on it; moreover, this Ind-object is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$, where $\overline{\mathfrak{P}}_{\mathbb{1}} \subset \mathrm{GL}(V)$ is the mirabolic subgroup preserving the vector $\mathbb{1}$.

The "ν-th tensor power" of V is defined for any $\nu \in \mathbb{C}$; for $n = \nu \in \mathbb{Z}_+$, the functor $\underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ takes this Ind-object of $\underline{\mathrm{Rep}}(S_{\nu=n})$ to the usual tensor power $V^{\otimes n}$ in $\mathrm{Rep}(S_n)$. Moreover, the action of $\mathfrak{gl}(V)$ on the former object corresponds to the action of $\mathfrak{gl}(V)$ on $V^{\otimes n}$.

This let us define an additive contravariant functor, called the Schur–Weyl functor:

$$\mathrm{SW}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to O_V^{\mathfrak{p}}, \quad \mathrm{SW}_{\nu,V} := \mathrm{Hom}_{\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)}(\,\cdot\,, V^{\underline{\otimes} \nu}).$$

Here $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ is the abelian envelope of the category $\underline{\mathrm{Rep}}(S_\nu)$ (this envelope was described in [Comes and Ostrik 2014; Deligne 2007, Chapter 8]). The category $O_V^{\mathfrak{p}}$ is a version of the parabolic category O for $\mathfrak{gl}(V)$ associated with the pair $(V, \mathbb{1})$, which is defined as follows.

We define $O_V^{\mathfrak{p}}$ to be the category of Harish-Chandra modules for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$ on which the group $\mathrm{GL}(V/\mathbb{C}\mathbb{1})$ acts by polynomial maps, and which satisfy some additional finiteness conditions (similar to the ones in the definition of the usual BGG category O).

We now consider the localization of $O_V^{\mathfrak{p}}$ obtained by taking the full subcategory of $O_V^{\mathfrak{p}}$ consisting of modules of degree ν (i.e., modules on which $\mathrm{Id}_V \in \mathrm{End}(V)$ acts by the scalar ν), and localizing by the Serre subcategory of $\mathfrak{gl}(V)$-polynomial modules. This quotient is denoted by $\widehat{O}^{\mathfrak{p}}_{\nu,V}$. It turns out that for any unital finite-dimensional space $(V, \mathbb{1})$ and any $\nu \in \mathbb{C}$, the contravariant functor $\widehat{\mathrm{SW}}_{\nu,V}$ makes $\widehat{O}^{\mathfrak{p}}_{\nu,V}$ a Serre quotient of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$.

In this paper, we will consider the categories $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ for $N \in \mathbb{Z}_+$ and for $N = \infty$. Defining appropriate restriction functors

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

allows us to consider the inverse limit of the system $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 0}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 1})$. Inside this inverse limit we consider a full subcategory which is equivalent to $\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$; this subcategory is the "restricted inverse limit" of $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 0}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 1})$ and will be denoted by $\varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. This category has an intrinsic description, which we give in this paper (intuitively, this is the inverse limit among finite-length categories).

Similarly to [Entova Aizenbud 2015a], we define the complex tensor power of the unital vector space $(\mathbb{C}^\infty, \mathbb{1} := e_1)$, and the corresponding Schur–Weyl contravariant functor $\mathrm{SW}_{\nu,\mathbb{C}^\infty}$. As in the finite-dimensional case, this functor induces an exact contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$, and we have the following commutative diagram:

$$\begin{tikzcd} \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[r, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] \arrow[dr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ & \hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u] \end{tikzcd}$$

The contravariant functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ and $\widehat{\mathrm{SW}}_{\nu,\lim}$ turn out to be antiequivalences induced by the Schur–Weyl functors $\mathrm{SW}_{\nu,\mathbb{C}^n}$. The antiequivalences $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ and $\widehat{\mathrm{SW}}_{\nu,\lim}$ induce an unexpected rigid symmetric monoidal category structure on

$$\hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

We obtain an interesting corollary: the duality in this category given by the tensor structure will coincide with the one arising from the usual notion of duality in BGG category O.

1.2. ***Notation.*** The base field throughout the paper will be $\mathbb{C}$. The notation and definitions used in this paper can be found in [Entova Aizenbud 2015a, Section 2]. In particular, lowercase Greek letters will denote Young diagrams, and $\ell(\lambda)$ will denote the number of rows in λ, while $|\lambda|$ will denote the number of boxes in λ.

We will use the definition of a finite-length abelian category given below.

Definition 1.2.1. Let $\mathcal{C}$ be an abelian category, and C be an object of $\mathcal{C}$. A *Jordan–Hölder filtration* for C is a finite sequence of subobjects of C

$$0 = C_0 \subset C_1 \subset \cdots \subset C_n = C$$

such that each subquotient C_{i+1}/C_i is simple.

The Jordan–Hölder filtration might not be unique, but the simple factors C_{i+1}/C_i are unique (up to reordering and isomorphisms). Consider the multiset of the simple factors: each simple factor is considered as an isomorphism class of simple objects, and its multiplicity is the multiplicity of its isomorphism class in the Jordan–Hölder filtration of C. This multiset is denoted by $\mathrm{JH}(C)$, and its elements are called the *Jordan–Hölder components* of C. The *length* of the object C, denoted by $\ell_{\mathcal{C}}(C)$, is defined to be the size of the finite multiset $\mathrm{JH}(C)$.

Definition 1.2.2. An abelian category $\mathcal{C}$ is called a *finite-length category* if every object admits a Jordan–Hölder filtration.

1.3. ***Structure of the paper.*** Sections 2 and 3 contain preliminaries on the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$, the categories of polynomial representations of $\mathfrak{gl}_N$ (where $N \in \mathbb{Z}_+ \cup \{\infty\}$) and the parabolic category O for $\mathfrak{gl}_N$. These sections are based on [Entova Aizenbud 2015a; 2015b; Sam and Snowden 2015].

In Section 4, we define the version of the parabolic category O for $\mathfrak{gl}_N$ which we will consider (including the case when $N = \infty$; see Section 4.2), and recall the necessary information about this category.

In Section 5, we give a description of the parabolic category O for $\mathfrak{gl}_\infty$ as a restricted inverse limit of the parabolic categories O for $\mathfrak{gl}_n$ as n tends to infinity.

In Sections 6 and 7, we recall the definition of the complex tensor power $(\mathbb{C}^N)^{\underline{\otimes}\nu}$, and define the functors $\mathrm{SW}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \to O^{\mathfrak{p}}_{\nu,V}$, $\widehat{\mathrm{SW}}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \to \widehat{O}^{\mathfrak{p}}_{\nu,V}$ for a unital vector space $(V, \mathbb{1})$ (finite- or infinite-dimensional). In Section 7.2, we recall the finite-dimensional case (studied in [Entova Aizenbud 2015a]).

Section 8 discusses the restricted inverse limit construction in the case of the classical Schur–Weyl duality, which motivates our construction for the Deligne categories. Sections 9 and 10 prove the main results of the paper. Section 11 discusses the relation between the rigidity (duality) in $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and the duality in the parabolic category O for $\mathfrak{gl}_\infty$.

2. Deligne category $\underline{\mathbf{Rep}}(S_\nu)$

A detailed description of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ and its abelian envelope can be found in [Comes and Ostrik 2011; 2014; Deligne 2007; Etingof 2014; Entova Aizenbud 2015a].

2.1. ***General description.*** For any $\nu \in \mathbb{C}$, the category $\underline{\mathrm{Rep}}(S_\nu)$ is generated, as a $\mathbb{C}$-linear Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of S_n, and any object in $\underline{\mathrm{Rep}}(S_\nu)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $\nu \notin \mathbb{Z}_+$, $\underline{\mathrm{Rep}}(S_\nu)$ is a semisimple abelian category.

For $\nu \in \mathbb{Z}_+$, the category $\underline{\mathrm{Rep}}(S_\nu)$ has a tensor ideal $\mathfrak{I}_\nu$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \to Y$ such that $\mathrm{tr}(fu) = 0$ for any morphism $u : Y \to X$). In that case, the classical category $\mathrm{Rep}(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $\underline{\mathrm{Rep}}(S_{\nu=n})/\mathfrak{I}_\nu$ (equivalent as Karoubian rigid symmetric monoidal categories). The full, essentially surjective functor $\underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ defining this equivalence will be denoted by $\mathcal{S}_n$. Note that $\mathcal{S}_n$ sends $\mathfrak{h}$ to the permutation representation of S_n.

The indecomposable objects of $\underline{\mathrm{Rep}}(S_\nu)$, regardless of the value of ν, are parametrized (up to isomorphism) by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\underline{\mathrm{Rep}}(S_\nu)$ corresponding to the Young diagram τ by X_τ.

For $\nu =: n \in \mathbb{Z}_+$, the partitions λ for which X_λ has a nonzero image in the quotient $\underline{\mathrm{Rep}}(S_{\nu=n})/\mathfrak{I}_{\nu=n} \cong \mathrm{Rep}(S_n)$ are exactly the λ for which $\lambda_1 + |\lambda| \leq n$. If $\lambda_1 + |\lambda| \leq n$, then the image of λ in $\mathrm{Rep}(S_n)$ is the irreducible representation of S_n

corresponding to the Young diagram $\tilde{\lambda}(n)$: the Young diagram obtained by adding a row of length $n-|\lambda|$ on top of λ.

For each ν, we define an equivalence relation $\stackrel{\nu}{\sim}$ on the set of all Young diagrams: we say that $\lambda \stackrel{\nu}{\sim} \lambda'$ if the sequence $(\nu-|\lambda|, \lambda_1-1, \lambda_2-2, \dots)$ can be obtained from the sequence $(\nu-|\lambda'|, \lambda'_1-1, \lambda'_2-2, \dots)$ by permuting a finite number of entries. The equivalence classes thus obtained are in one-to-one correspondence with the blocks of the category $\underline{\mathrm{Rep}}(S_\nu)$ (see [Comes and Ostrik 2011]).

We say that a block is *trivial* if the corresponding equivalence class is trivial, i.e., has only one element (in that case, the block is a semisimple category).

The nontrivial equivalence classes (respectively, blocks) are parametrized by all Young diagrams of size ν; in particular, this happens only if $\nu \in \mathbb{Z}_+$. These classes are always of the form $\{\lambda^{(i)}\}_i$, with

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots$$

(each $\lambda^{(i)}$ can be explicitly described based on the Young diagram of size ν corresponding to this class).

2.2. ***Abelian envelope.*** As was mentioned before, the category $\underline{\mathrm{Rep}}(S_\nu)$ is defined as a Karoubian category. For $\nu \notin \mathbb{Z}_+$, it is semisimple and thus abelian, but for $\nu \in \mathbb{Z}_+$, it is not abelian. Fortunately, it has been shown that $\underline{\mathrm{Rep}}(S_\nu)$ possesses an "abelian envelope", that is, it can be embedded (as a full monoidal subcategory) into an abelian rigid symmetric monoidal category, and this abelian envelope has a universal mapping property (see [Comes and Ostrik 2014, Theorem 1.2; Deligne 2007, Conjecture 8.21.2]). We will denote the abelian envelope of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ by $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (with $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) := \underline{\mathrm{Rep}}(S_\nu)$ for $\nu \notin \mathbb{Z}_+$).

An explicit construction of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_{\nu=n})$ is given in [Comes and Ostrik 2014], and a detailed description of its structure can be found in [Entova Aizenbud 2015a]. It turns out that the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ is a highest weight category (with infinitely many weights) corresponding to the partially ordered set ({Young diagrams}, $\geq$), where

$$\lambda \geq \mu \iff \lambda \stackrel{\nu}{\sim} \mu, \quad \lambda \subset \mu$$

(namely, in a nontrivial $\stackrel{\nu}{\sim}$-class, $\lambda^{(i)} \geq \lambda^{(j)}$ if $i \leq j$).

Thus the isomorphism classes of simple objects in $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ are parametrized by the set of Young diagrams of arbitrary sizes. We will denote the simple object corresponding to λ by $\boldsymbol{L}(\lambda)$.

We will also use the fact that blocks of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$, just like the blocks of $\underline{\mathrm{Rep}}(S_\nu)$, are parametrized by $\stackrel{\nu}{\sim}$-equivalence classes. For each $\stackrel{\nu}{\sim}$-equivalence class, the corresponding block of $\underline{\mathrm{Rep}}(S_\nu)$ is the full subcategory of tilting objects in the corresponding block of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (see [Comes and Ostrik 2014, Proposition 2.9, Section 4]).

3. $\mathfrak{gl}_\infty$ and the restricted inverse limit of representations of $\mathfrak{gl}_n$

In this section, we discuss the category of polynomial representations of the Lie algebra $\mathfrak{gl}_\infty$ and its relation to the categories of polynomial representations of $\mathfrak{gl}_n$ for $n \geq 0$. The representations of the Lie algebra $\mathfrak{gl}_\infty$ are discussed in detail in [Penkov and Styrkas 2011; Dan-Cohen et al. 2016; Sam and Snowden 2015, Section 3]. Most of the constructions and the proofs of the statements appearing in this section can be found in [Entova Aizenbud 2015b, Section 7].

3.1. ***The Lie algebra*** $\mathfrak{gl}_\infty$**.** Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \dots$. Consider the Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices $A = (a_{ij})_{i,j\geq 1}$ with finitely many nonzero entries. We have a natural action of $\mathfrak{gl}_\infty$ on $\mathbb{C}^\infty$ and on the restricted dual $\mathbb{C}^\infty_* = \mathrm{span}_{\mathbb{C}}(e_1^*, e_2^*, e_3^*, \dots)$ (here e_i^* is the linear functional dual to e_i: $e_i^*(e_j) = \delta_{ij}$).

Let $N \in \mathbb{Z}_+ \cup \{\infty\}$, and let $m \geq 1$. We will consider the Lie subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_N$ which consists of matrices $A = (a_{ij})_{1\leq i,j\leq N}$ for which $a_{ij} = 0$ whenever $i > m$ or $j > m$. We will also denote by $\mathfrak{gl}_m^\perp$ the Lie subalgebra of $\mathfrak{gl}_N$ consisting of matrices $A = (a_{ij})_{1\leq i,j\leq N}$ for which $a_{ij} = 0$ whenever $i \leq m$ or $j \leq m$.

Remark 3.1.1. Note that $\mathfrak{gl}_m^\perp \cong \mathfrak{gl}_{N-m}$ for any $m \leq N$.

3.2. ***Categories of polynomial representations of*** $\mathfrak{gl}_N$**.** In this subsection, we take $N \in \mathbb{Z}_+ \cup \{\infty\}$. The notation $\mathbb{C}^N_*$ will stand for $(\mathbb{C}^N)^*$ whenever $N \in \mathbb{Z}_+$, and for $\mathbb{C}^\infty_*$ when $N = \infty$.

Consider the category $\mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ of polynomial representations of $\mathfrak{gl}_N$: this is the category of the representations of $\mathfrak{gl}_N$ which can be obtained as summands of a direct sum of tensor powers of the tautological representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$.

It is easy to see that this is a semisimple abelian category, whose simple objects are parametrized (up to isomorphism) by all Young diagrams of arbitrary sizes whose length does not exceed N: the simple object corresponding to λ is $S^\lambda\mathbb{C}^N$.

Remark 3.2.1. Note that $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$ is the free abelian symmetric monoidal category generated by one object (see [Sam and Snowden 2015, Section 2.2.11]). It has an equivalent definition as the category of polynomial functors of bounded degree, which can be found in [Hong and Yacobi 2013; Sam and Snowden 2015].

Next, we define a natural $\mathbb{Z}_+$-grading on objects in $\text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ (cf. [Sam and Snowden 2015, Section 2.2.2]):

Definition 3.2.2. The objects in $\text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ have a natural $\mathbb{Z}_+$-grading. Given $M \in \text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$, we consider the decomposition $M = \bigoplus_\lambda S^\lambda\mathbb{C}^N \otimes \mathrm{mult}_\lambda$ (here

mult_λ is the multiplicity space of $S^\lambda\mathbb{C}^N$ in M), and we define

$$\mathrm{gr}_k(M) := \bigoplus_{\lambda:|\lambda|=k} S^\lambda\mathbb{C}^N \otimes \mathrm{mult}_\lambda.$$

Of course, the morphisms in Ind-$\mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ respect this grading.

3.3. ***Specialization and restriction functors.*** We now define specialization functors from the category of representations of $\mathfrak{gl}_\infty$ to the categories of representations of $\mathfrak{gl}_n$ (cf. [Sam and Snowden 2015, Section 3]):

Definition 3.3.1. We have

$$\Gamma_n : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}, \quad \Gamma_n := (\,\cdot\,)^{\mathfrak{gl}_n^\perp}.$$

One can easily check (cf. [Entova Aizenbud 2015b, Section 7]) that the functor Γ_n is well defined.

Lemma 3.3.2 [Penkov and Styrkas 2011; Sam and Snowden 2015, Section 3]. *The functors Γ_n are additive symmetric monoidal functors between semisimple symmetric monoidal categories. Their effect on the simple objects is described as follows: for any Young diagram λ, we have $\Gamma_n(S^\lambda\mathbb{C}^\infty) \cong S^\lambda\mathbb{C}^n$.*

Definition 3.3.3. Let $n \geq 1$. We define the functors

$$\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}_{n-1}^\perp}.$$

Again, one can easily show that these functors are well defined.

Remark 3.3.4. There is an alternative definition of the functors $\mathfrak{Res}_{n-1,n}$. One can think of the functor $\mathfrak{Res}_{n-1,n}$ acting on a $\mathfrak{gl}_n$-module M as taking the restriction of M to $\mathfrak{gl}_{n-1}$ and then considering only the vectors corresponding to "appropriate" central characters.

More specifically, we say that a $\mathfrak{gl}_n$-module M is of *degree d* if $\mathrm{Id}_{\mathbb{C}^n} \in \mathfrak{gl}_n$ acts by $d\,\mathrm{Id}_M$ on M. Also, given any $\mathfrak{gl}_n$-module M, we may consider the maximal submodule of M of degree d, and denote it by $\deg_d(M)$. This defines an endofunctor $\deg_d$ of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$. Note that a simple module $S^\lambda\mathbb{C}^n$ is of degree $|\lambda|$.

The notion of degree gives a decomposition

$$\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \cong \bigoplus_{d\in\mathbb{Z}_+} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},d},$$

where $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},d}$ is the full subcategory of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ consisting of all polynomial $\mathfrak{gl}_n$-modules of degree d. Then

$$\mathfrak{Res}_{n-1,n} = \bigoplus_{d\in\mathbb{Z}_+} \mathfrak{Res}_{d,n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}},$$

with

$$\mathfrak{Res}_{d,n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},d} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly},d}, \quad \mathfrak{Res}_{d,n-1,n} := \deg_d \circ \mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}},$$

where $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ is the usual restriction functor for the pair $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$.

Once again, the functors $\mathfrak{Res}_{n-1,n}$ are additive functors between semisimple categories.

Lemma 3.3.5. $\mathfrak{Res}_{n-1,n}(S^\lambda \mathbb{C}^n) \cong S^\lambda \mathbb{C}^{n-1}$ *for any Young diagram* λ (*recall that* $S^\lambda \mathbb{C}^{n-1} = 0$ *if* $\ell(\lambda) > n-1$).

Moreover, these functors are compatible with the functors Γ_n defined before.

Lemma 3.3.6. *For any* $n \geq 1$, *we have a commutative diagram*:

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \arrow[r, "\Gamma_n"] \arrow[dr, "\Gamma_{n-1}"'] & \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \arrow[d, "\mathfrak{Res}_{n-1,n}"] \\ & \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}} \end{tikzcd}$$

That is, *there is a natural isomorphism* $\Gamma_{n-1} \cong \mathfrak{Res}_{n-1,n} \circ \Gamma_n$.

Corollary 3.3.7. *The functors* $\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}$ *are symmetric monoidal functors.*

3.4. ***Restricted inverse limit of categories*** $\mathbf{Rep}(\mathfrak{gl}_n)_{\mathbf{poly}}$**.** This subsection gives a description of the category $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$ as a "restricted" inverse limit of categories $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ (see the Appendix and [Entova Aizenbud 2015b] for definitions and details).

We will use the framework developed in [Entova Aizenbud 2015b] for the inverse limits of categories with $\mathbb{Z}_+$-filtrations on objects, and the restricted inverse limits of finite-length categories (abelian categories in which every object admits a Jordan–Hölder filtration). The necessary definitions (such as $\mathbb{Z}_+$-filtered functors and shortening functors) can be found in the Appendix.

We define a $\mathbb{Z}_+$-filtration on the objects of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ for each $n \in \mathbb{Z}_+$:

Notation 3.4.1. For each $k \in \mathbb{Z}_+$, let $\mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})$ be the full additive subcategory of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ generated by $S^\lambda \mathbb{C}^n$ such that $\ell(\lambda) \leq k$.

Clearly the subcategories $\mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})$ give us a $\mathbb{Z}_+$-filtration on the objects of the category $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$. Furthermore, by Lemma 3.3.5, the functors $\mathfrak{Res}_{n-1,n}$ are $\mathbb{Z}_+$-filtered functors, i.e., they induce functors

$$\mathfrak{Res}^k_{n-1,n} : \mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}) \to \mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}).$$

This allows us to consider the inverse limit

$$\varprojlim_{n\in\mathbb{Z}_+,\mathbb{Z}_+\text{-filtr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \cong \varinjlim_{k\in\mathbb{Z}_+} \varprojlim_{n\in\mathbb{Z}_+} \mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}).$$

This is an abelian category (with a natural $\mathbb{Z}_+$-filtration on objects).

By Lemma 3.3.5, the functors $\mathfrak{Res}_{n-1,n}$ are shortening functors; furthermore, the system $((\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})_{n\in\mathbb{Z}_+}, (\mathfrak{Res}_{n-1,n})_{n\geq 1})$ satisfies the conditions listed in Proposition A.5.1, and therefore the category $\varprojlim_{n\in\mathbb{Z}_+,\mathbb{Z}_+\text{-filtr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ is also equivalent to the restricted inverse limit of this system, $\varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$.

Remark 3.4.2. The functors $\mathfrak{Res}_{n-1,n}$ are symmetric monoidal functors, so the category $\varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ is a symmetric monoidal category.

The following proposition is relatively straightforward. Its detailed proof can be found in [Entova Aizenbud 2015b].

Proposition 3.4.3. *We have an equivalence of symmetric monoidal Karoubian categories*

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$$

induced by the symmetric monoidal functors

$$\Gamma_n = (\,\cdot\,)^{\mathfrak{gl}_n^\perp} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

4. Parabolic category O

In this section, we describe a version of the parabolic category O for $\mathfrak{gl}_N$ which we are going to work with. We give a definition which describes the relevant category for both $\mathfrak{gl}_n$ and $\mathfrak{gl}_\infty$.

4.1. For the benefit of the reader, we will start by giving a definition for $\mathfrak{gl}_N$ when N is a positive integer; this definition is analogous to the usual definition of the category O. The generic definition will then be just a slight modification of the first to accommodate the case $N=\infty$. This version of the parabolic category O is attached to a pair: a vector space V and a fixed nonzero vector $\mathbb{1}$ in it. Such a pair is called a *unital vector space*. In our case, we will just consider $V=\mathbb{C}^N$, with the standard basis $e_1, e_2, \dots,$ and the chosen vector $\mathbb{1} := e_1$. Fix $N\in\mathbb{Z}$ with $N\geq 1$.

Notation 4.1.1. The following notation will be used throughout the paper:

- We denote by $\mathfrak{p}_N \subset \mathfrak{gl}_N$ the parabolic Lie subalgebra which consists of all the endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\phi(\mathbb{1}) \in \mathbb{C}\mathbb{1}$. In terms of matrices this is $\mathrm{span}\{E_{1,1}, E_{i,j} \mid j>1\}$.
- $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{p}_N$ denotes the algebra of endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\mathrm{Im}\,\phi \subset \mathbb{C}\mathbb{1} \subset \mathrm{Ker}\,\phi$. In terms of matrices, $\mathfrak{u}^+_{\mathfrak{p}_N} = \mathrm{span}\{E_{1,j} \mid j>1\}$.

Denote $U_N := \operatorname{span}\{e_2, e_3, \dots, e_N\}$. We have a splitting $\mathfrak{gl}_N \cong \mathfrak{p}_N \oplus \mathfrak{u}^-_{\mathfrak{p}_N}$, where $\mathfrak{u}^-_{\mathfrak{p}_N} \cong U_N = \operatorname{span}\{E_{i,1} \mid i > 1\}$). This gives us an analogue of the triangular decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}\,\mathrm{Id}_{\mathbb{C}^N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N} \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{gl}(U_N).$$

We can now give a precise definition of the parabolic category O we will use:

Definition 4.1.2. We define the category $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ to be the full subcategory of $\mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ whose objects M satisfy the following conditions:

- Viewed as a $\mathcal{U}(\mathfrak{gl}(U_N))$-module, M is a direct sum of polynomial $\mathcal{U}(\mathfrak{gl}(U_N))$-modules (that is, M belongs to $\text{Ind-Rep}(\mathfrak{gl}(U_N))_{\text{poly}}$).
- M is locally finite over $\mathfrak{u}^+_{\mathfrak{p}_N}$.
- M is a finitely generated $\mathcal{U}(\mathfrak{gl}_N)$-module.

Remark 4.1.3. One can replace the requirement that $\mathfrak{u}^+_{\mathfrak{p}_N}$ act locally finitely on M by the requirement that $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ act locally nilpotently on M.

Remark 4.1.4. One can, in fact, give an equivalent definition of the category $O^{\mathfrak{p}}_V$ corresponding to a finite-dimensional unital vector $(V, \mathbb{1})$ without choosing a splitting (cf. [Entova Aizenbud 2015a, Section 5] and the Introduction).

Definition 4.1.5. A module M over the Lie algebra $\mathfrak{gl}_N$ will be said to be of degree $K \in \mathbb{C}$ if $\mathrm{Id}_{\mathbb{C}^N} \in \mathfrak{gl}_N$ acts by $K\,\mathrm{Id}_M$ on M.

We will denote by $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ the full subcategory of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ whose objects are modules of degree ν. To say a module M of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ is of degree ν is the same as to require that $E_{1,1}$ acts on each subspace $S^\lambda U_N$ of M by the scalar $\nu - |\lambda|$.

Definition 4.1.6. Let $\nu \in \mathbb{C}$. Define the functor $\deg_\nu : \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ by letting $\deg_\nu(E)$ be the maximal submodule of E of degree ν (see Definition 4.1.5). For a morphism $f : E \to E'$ of $\mathfrak{gl}_N$-modules, we put $\deg_\nu(f) := f|_{\deg_\nu(E)}$.

Let $E \in \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$. The maximal submodule of E of degree ν is well defined: it is the subspace of E consisting of all vectors on which $\mathrm{Id}_{\mathbb{C}^N}$ acts by the scalar ν, and it is a $\mathfrak{gl}_N$-submodule since $\mathrm{Id}_{\mathbb{C}^N}$ lies in the center of $\mathfrak{gl}_N$.

One can show that the functor $\deg_\nu : \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ is left-exact. Moreover, it is easy to show that the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ is a direct summand of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$, and the functor $\deg_\nu : O^{\mathfrak{p}_N}_{\mathbb{C}^N} \to O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ is exact.

4.2. ***Parabolic category O for $\mathfrak{gl}_N$.*** We now give a definition of the parabolic category O which for $\mathfrak{gl}_N$. Again, we let $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.

Consider a unital vector space $(\mathbb{C}^N, \mathbb{1})$, where $\mathbb{1} := e_1$. Put

$$U_N := \operatorname{span}_{\mathbb{C}}(e_2, e_3, \dots) \subset \mathbb{C}^N,$$

so that we have a splitting $\mathbb{C}^N = \mathbb{C}e_1 \oplus U_N$. We also denote $U_{N,*} := \mathrm{span}(e_2^*, e_3^*, \dots)$ (so $U_{N,*} = U_N^*$ whenever $N \in \mathbb{Z}$). We have a decomposition

$$\mathfrak{gl}_N \cong \mathfrak{gl}(U_N) \oplus \mathfrak{gl}_1 \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N}.$$

Of course, for any N, we have $\mathfrak{u}^-_{\mathfrak{p}_N} \cong U_N$; moreover, $\mathfrak{u}^+_{\mathfrak{p}_N} \cong U_{N,*}$. We will also use the isomorphisms $\mathfrak{gl}(U_N) \cong \mathfrak{gl}_1^\perp \cong \mathfrak{gl}_{N-1}$.

Definition 4.2.1.

- Define the category $\mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ to be the category of $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ lies in Ind-Rep$(\mathfrak{gl}_{U_N})_{\mathrm{poly}}$; that is, $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ is a (perhaps infinite) direct sum of Schur functors applied to U_N. The morphisms would be $\mathfrak{gl}_N$-equivariant maps.
- We say that an object $M \in \mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ is of *degree* ν ($\nu \in \mathbb{C}$) if on every summand $S^\lambda U_N \subset M$, the element $E_{1,1} \in \mathfrak{gl}_N$ acts by $(\nu - |\lambda|)\,\mathrm{Id}_{S^\lambda U_N}$.
- Let $M \in \mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$. We have a commutative algebra $\mathrm{Sym}(U_N) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^-_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a $\mathrm{Sym}(U_N)$-module structure. We say that M is *finitely generated* over $\mathrm{Sym}(U_N)$ if M is a quotient of a "free finitely generated $\mathrm{Sym}(U_N)$-module"; that is, as a $\mathrm{Sym}(U_N)$-module, M is a quotient (in Ind-Rep$(\mathfrak{gl}_N)_{\mathrm{poly}}$) of $\mathrm{Sym}(U_N) \otimes E$ for some $E \in \mathrm{Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}}$.
- Let $M \in \mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$. We have a commutative algebra $\mathrm{Sym}(U_{N,*}) \cong \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a $\mathrm{Sym}(U_{N,*})$-module structure. We say that M is *locally nilpotent* over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ if for any $v \in M$, there exists $m \geq 0$ such that for any $A \in \mathrm{Sym}^m(U_{N,*})$ we have $A.v = 0$.

Recall the natural $\mathbb{Z}_+$-grading on the object of Ind-Rep$(\mathfrak{gl}_N)_{\mathrm{poly}}$. For each $M \in \mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$, the above definition implies that $\mathfrak{gl}(U_N)$ acts by operators of degree zero, and that $U_{N,*}$ acts by operators of degree 1. We now define the parabolic category O for $\mathfrak{gl}_N$ which we will use throughout the paper:

Definition 4.2.2. We define the category $O^{\mathfrak{p}_N}_{\nu, \mathbb{C}^N}$ to be the full subcategory of $\mathrm{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ whose objects M satisfy the following requirements:

- M is of degree ν.
- M is finitely generated over $\mathrm{Sym}(U_N)$.
- M is locally nilpotent over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$.

Of course, for a positive integer N, this is just the category $O^{\mathfrak{p}_N}_{\nu, \mathbb{C}^N}$ we defined in the beginning of this section.

We will also consider the localization of the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by its Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν; such modules exist if and only if $\nu \in \mathbb{Z}_+$. This localization will be denoted by

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$$

and will play an important role when we consider the Schur–Weyl duality in complex rank.

4.3. ***Duality in category O.*** Let $n \in \mathbb{Z}_+$. Recall that in the category O for $\mathfrak{gl}_n$ we have the notion of a duality (cf. [Humphreys 2008, Section 3.2]): namely, given a $\mathfrak{gl}_n$-module M with finite-dimensional weight spaces, we can consider the twisted action of $\mathfrak{gl}_n$ on the dual space M^*, given by $A.f := f \circ A^T$, where A^T means the transpose of $A \in \mathfrak{gl}_n$. This makes M^* a $\mathfrak{gl}_n$-module. We then take $M^\vee$ to be the maximal submodule of M^* lying in category O.

More explicitly, considering M as a direct sum of its finite-dimensional weight spaces

$$M = \bigoplus_{\lambda} M_\lambda$$

we can consider the restricted twisted dual

$$M^\vee := \bigoplus_{\lambda} M_\lambda^*$$

(that is, we take the dual to each weight space separately). The action of $\mathfrak{gl}_n$ is given by $A.f := f \circ A^T$ for any $A \in \mathfrak{gl}_n$. The module $M^\vee$ is called the dual of M, and we get an exact functor $(\,\cdot\,)^\vee : O^{\mathrm{op}} \to O$.

Proposition 4.3.1. *The category $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ is closed under taking duals, and the duality functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_n}_{\mathbb{C}^n})^{\mathrm{op}} \to O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ is an equivalence of categories.*

In fact, a similar construction can be made for $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. All modules M in $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ are weight modules with respect to the subalgebra of diagonal matrices in $\mathfrak{gl}_\infty$, and the weight spaces are finite-dimensional (due to the polynomiality condition in the definition of $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$). This allows one to construct the restricted twisted dual $M^\vee$ in the same way as before, and obtain an exact functor

$$(\,\cdot\,)^\vee : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}.$$

Remark 4.3.2. It is obvious that for $n \in \mathbb{Z}_+$, the functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_n}_{\mathbb{C}^n})^{\mathrm{op}} \to O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ takes finite-dimensional (polynomial) modules to finite-dimensional (polynomial) modules. In fact, one can easily check that the functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ takes polynomial modules to polynomial modules as well.

4.4. *Structure of the category $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.* In this subsection, we present some facts about the category $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ which will be used later on. The material of this section is discussed in more detail in [Entova Aizenbud 2015a, Section 5] and is mostly based on [Humphreys 2008, Chapter 9].

Fix $\nu \in \mathbb{C}$, and fix $n \in \mathbb{Z}_+$. We denote by $e_1, e_2, \dots, e_n$ the standard basis of $\mathbb{C}^n$, and put $\mathbb{1} := e_1$ and $U_n := \operatorname{span}\{e_2, e_3, \dots, e_n\}$. We will consider the category $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ for the unital vector space $(\mathbb{C}^n, \mathbb{1})$ and the splitting $\mathbb{C}^n = \mathbb{C}\mathbb{1} \oplus U_n$.

Proposition 4.4.1. *The categories $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ and* Ind-$O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ *are closed under taking duals, direct sums, submodules, quotients and extensions in $O_{\mathfrak{gl}_n}$, as well as tensoring with finite-dimensional $\mathfrak{gl}_n$-modules.*

The category $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ decomposes into blocks (each of the blocks is an abelian category in its own right). To each $\stackrel{\nu}{\sim}$-class of Young diagrams corresponds a block of $O^{\mathfrak{p}}_{\nu,\mathbb{C}^n}$. If all Young diagrams λ in this $\stackrel{\nu}{\sim}$-class have length at least n, then the corresponding block is zero. To each nonzero block of $O^{\mathfrak{p}}_{\nu,\mathbb{C}^n}$ corresponds a unique $\stackrel{\nu}{\sim}$-class.

Moreover, the blocks corresponding to trivial $\stackrel{\nu}{\sim}$-classes are either semisimple (i.e., equivalent to the category $\mathrm{Vect}_{\mathbb{C}}$) or zero.

We now discuss standard objects in $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$.

Definition 4.4.2. Let λ be a Young diagram. The generalized Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is defined to be the $\mathfrak{gl}_n$-module

$$\mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} S^\lambda U_n,$$

where $\mathfrak{gl}(U_n)$ acts naturally on $S^\lambda U_n$, $\mathrm{Id}_{\mathbb{C}^n} \in \mathfrak{p}_n$ acts on $S^\lambda U_n$ by scalar ν, and $\mathfrak{u}^+_{\mathfrak{p}_n}$ acts on $S^\lambda U_n$ by zero. Thus $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is the parabolic Verma module for $(\mathfrak{gl}_n, \mathfrak{p}_n)$ with highest weight $(\nu - |\lambda|, \lambda)$ if and only if $n - 1 \geq \ell(\lambda)$, and zero otherwise.

Definition 4.4.3. $L(\nu - |\lambda|, \lambda)$ is defined to be zero if $n \geq \ell(\lambda)$, or the simple module for $\mathfrak{gl}_n$ of highest weight $(\nu - |\lambda|, \lambda)$ otherwise.

The following basic lemma will be very helpful.

Lemma 4.4.4. *Let λ be a Young diagram such that $\ell(\lambda) < n$. We then have an isomorphism of $\mathfrak{gl}(U_n)$-modules*:

$$M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda) \cong \mathrm{Sym}(U_n) \otimes S^\lambda U_n.$$

We will also use the following lemma.

Lemma 4.4.5. *Let $\{\lambda^{(i)}\}_i$ be a nontrivial $\stackrel{\nu}{\sim}$-class, and $i \geq 0$ be such that $\ell(\lambda^{(i)}) < n$. Then there is a short exact sequence*

$$0 \to L(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}) \to M_{\mathfrak{p}_n}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to L(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0.$$

Corollary 4.4.6. *The isomorphism classes of the generalized Verma modules and the simple polynomial modules in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ form a basis for the Grothendieck group of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.*

5. Stable inverse limit of parabolic categories O

5.1. *Restriction functors.*

Definition 5.1.1. Let $n \geq 1$. Define the functor

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}^{\perp}_{n-1}}.$$

Again, the subalgebras $\mathfrak{gl}_{n-1}, \mathfrak{gl}^{\perp}_{n-1} \subset \mathfrak{gl}_n$ commute, and therefore the subspace of $\mathfrak{gl}^{\perp}_{n-1}$-invariants of a $\mathfrak{gl}_n$-module automatically carries an action of $\mathfrak{gl}_{n-1}$.

We need to check that this functor is well defined. In order to do so, consider the functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_{n-1})}$. This functor is well defined, and we will show that the objects in the image lie in the full subcategory $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ of $\mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_{n-1})}$.

The functor $\mathfrak{Res}_{n-1,n}$ can alternatively be defined as follows: for a module M in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, we restrict the action of $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$, and then only take the vectors in M attached to specific central characters. More specifically, we have:

Lemma 5.1.2. *The functor $\mathfrak{Res}_{n-1,n}$ is naturally isomorphic to the composition* $\deg_\nu \circ \mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ (the functor $\deg_\nu$ was defined in Definition 4.1.6).

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. For any vector $m \in M$, we know that

$$\mathrm{Id}_{\mathbb{C}^n}.m = (E_{1,1} + E_{2,2} + \cdots + E_{n,n}).m = \nu m.$$

Then the requirement that

$$\mathrm{Id}_{\mathbb{C}^{n-1}}.m = (E_{1,1} + E_{2,2} + \cdots + E_{n-1,n-1}).m = \nu m$$

is equivalent to the requirement that $E_{n,n}.m = 0$, namely that $m \in M^{\mathfrak{gl}^{\perp}_{n-1}}$. □

We will now use this information to prove the following result:

Lemma 5.1.3. *The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is well defined.*

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, and consider the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$. By definition, this is a module of degree ν. We will show that it lies in $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$.

First of all, consider the inclusion $\mathfrak{gl}(U_{n-1})^{\perp} \oplus \mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$. This inclusion gives us the restriction functor (see Definition 3.3.3)

$$\mathfrak{Res}_{U_{n-1},U_n} : \mathrm{Rep}(\mathfrak{gl}(U_n))_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}(U_{n-1}))_{\mathrm{poly}}, \quad \mathfrak{Res}_{U_{n-1},U_n} := (\,\cdot\,)^{\mathfrak{gl}(U_{n-1})^{\perp}}.$$

The latter is an additive functor between semisimple categories, and takes polynomial representations of $\mathfrak{gl}(U_n)$ to polynomial representations of $\mathfrak{gl}(U_{n-1})$.

Now, the restriction to $\mathfrak{gl}(U_{n-1})$ of the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$ is isomorphic to $\mathfrak{Res}_{U_{n-1},U_n}(M|_{\mathfrak{gl}(U_n)})$, and thus is a polynomial representation of $\mathfrak{gl}(U_{n-1})$.

Secondly, $\mathfrak{Res}_{n-1,n}(M)$ is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}})$, since M is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$ and $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}}) \subset \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$.

It remains to check that given $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, the module $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\mathrm{Sym}(U_{n-1})$. Indeed, we know that there exists a polynomial $\mathfrak{gl}(U_n)$-module E and a surjective $\mathfrak{gl}(U_n)$-equivariant morphism of $\mathrm{Sym}(U_n)$-modules $\mathrm{Sym}(U_n) \otimes E \twoheadrightarrow M$. Taking the $\mathfrak{gl}(U_{n-1})^{\perp}$-invariants and using Corollary 3.3.7, we conclude that there is a surjective $\mathfrak{gl}(U_{n-1})$-equivariant morphism of $\mathrm{Sym}(U_{n-1})$-modules

$$\mathrm{Sym}(U_{n-1}) \otimes E^{\mathfrak{gl}(U_{n-1})^{\perp}} \twoheadrightarrow \mathfrak{Res}_{n-1,n}(M).$$

Thus $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\mathrm{Sym}(U_{n-1})$. □

Lemma 5.1.4. *The functor* $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ *is exact.*

Proof. We use Lemma 5.1.2. The functor $\deg_\nu : O^{\mathfrak{p}_{n-1}}_{\mathbb{C}^{n-1}} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is exact, so the functor $\mathrm{Res}_{n-1,n}$ is obviously exact as well. □

Lemma 5.1.5. *The functor* $\mathfrak{Res}_{n-1,n}$ *takes parabolic Verma modules either to parabolic Verma modules or to zero*:

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda).$$

(*Recall that the latter is a parabolic Verma module for* $\mathfrak{gl}_{n-1}$ *if and only if* $\ell(\lambda) \leq n-2$, *and zero otherwise*).

Proof. Consider the parabolic Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$, where the Young diagram λ has length at most $n-1$. By definition, we have

$$M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda) = \mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} S^\lambda U_n.$$

The branching rule for $\mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$ tells us that

$$(S^\lambda U_n)|_{\mathfrak{gl}(U_{n-1})} \cong \bigoplus_{\lambda'} S^{\lambda'} U_{n-1},$$

where the sum is taken over the set of all Young diagrams obtained from λ by removing several boxes, no two in the same column. So

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong \left(\bigoplus_{\lambda' \subset \lambda} M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda') \right) \otimes \mathcal{U}\big(\mathfrak{u}^-_{\mathfrak{p}_n} / \mathfrak{u}^-_{\mathfrak{p}_{n-1}}\big).$$

Here:

- $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda')$ is either a parabolic Verma module for $\mathfrak{gl}_{n-1}$ of highest weight $(\nu - |\lambda|, \lambda')$ (note that it is of degree $\nu - |\lambda| + |\lambda'|$) or zero.
- $\mathfrak{gl}(U_{n-1})$ acts trivially on the space $\mathcal{U}\big(\mathfrak{u}^-_{\mathfrak{p}_n} / \mathfrak{u}^-_{\mathfrak{p}_{n-1}}\big)$. This space is isomorphic, as a $\mathbb{Z}_+$-graded vector space, to $\mathbb{C}[t]$ (ν standing for $E_{n,1} \in \mathfrak{gl}_n$) and $E_{1,1}$ acts on it by derivations $-t\frac{d}{dt}$.

Thus $\mathrm{Id}_{\mathbb{C}^{n-1}} \in \mathfrak{gl}_n$ acts on the subspace $M_{\mathfrak{p}_{n-1}}(\nu-|\lambda|,\lambda')\otimes t^k \subset M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)$ by the scalar $\nu-|\lambda|+|\lambda'|-k$.

We now apply the functor $\deg_\nu$ to the module $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda))$. To see which subspaces $M_{\mathfrak{p}_{n-1}}(\nu-|\lambda'|,\lambda')\otimes t^k$ of $M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)$ will survive after applying $\deg_\nu$, we require that $|\lambda|-|\lambda'|+k=0$. But we are only considering Young diagrams λ' such that $\lambda'\subset\lambda$, and $k\in\mathbb{Z}_+$, which means that the only relevant case is $\lambda'=\lambda,\ k=0$. We conclude that

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_N}(\nu-|\lambda|,\lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu-|\lambda|,\lambda). \qquad \square$$

Lemma 5.1.6. *Given a simple $\mathfrak{gl}_n$-module $L_n(\nu-|\lambda|,\lambda)$,*

$$\mathfrak{Res}_{n-1,n}(L_n(\nu-|\lambda|,\lambda)) \cong L_{n-1}(\nu-|\lambda|,\lambda).$$

(*Recall that the latter is a simple $\mathfrak{gl}_{n-1}$-module if and only if $\ell(\lambda)\le n-2$, and zero otherwise*).

Proof. The statement follows immediately from Lemma 5.1.5 when λ lies in a trivial $\overset{\nu}{\sim}$-class; for a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$, we have short exact sequences (see Lemma 4.4.5):

$$0 \to L_n(\nu-|\lambda^{(i+1)}|,\lambda^{(i+1)}) \to M_{\mathfrak{p}_n}(\nu-|\lambda^{(i)}|,\lambda^{(i)}) \to L_n(\nu-|\lambda^{(i)}|,\lambda^{(i)}) \to 0.$$

Using the exactness of $\mathfrak{Res}_{n-1,n}$, we can prove the required statement for $L_n(\nu-|\lambda^{(i)}|,\lambda^{(i)})$ by induction on i, provided the statement is true for $i=0$. So it remains to check that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu-|\lambda|,\lambda)) \cong L_{n-1}(\nu-|\lambda|,\lambda)$$

for the minimal Young diagram λ in any nontrivial $\overset{\nu}{\sim}$-class. Recall that in that case, $L_n(\nu-|\lambda|,\lambda)=S^{\tilde{\lambda}(\nu)}\mathbb{C}^n$ is a finite-dimensional simple representation of $\mathfrak{gl}_n$. The branching rule for $\mathfrak{gl}_n$, $\mathfrak{gl}_{n-1}$ implies that

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(S^{\tilde{\lambda}(\nu)}\mathbb{C}^n) \cong \bigoplus_{\mu} S^{\mu}\mathbb{C}^{n-1},$$

where the sum is taken over the set of all Young diagrams obtained from $\tilde{\lambda}(\nu)$ by removing several boxes, no two in the same column. Considering only the summands of degree ν, we see that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu-|\lambda|,\lambda)) \cong S^{\tilde{\lambda}(\nu)}\mathbb{C}^{n-1} \cong L_{n-1}(\nu-|\lambda|,\lambda). \qquad \square$$

The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ clearly takes polynomial modules to polynomial modules; together with Lemma 5.1.4, this means that $\mathfrak{Res}_{n-1,n}$ factors through an exact functor

$$\widehat{\mathfrak{Res}}_{n-1,n} : \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \hat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}},$$

i.e., we have a commutative diagram

$$\begin{tikzcd} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\mathfrak{Res}_{n-1,n}"] \arrow[d, "\hat{\pi}_n"'] & O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \arrow[d, "\hat{\pi}_{n-1}"] \\ \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\widehat{\mathfrak{Res}}_{n-1,n}"] & \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \end{tikzcd}$$

(see Section 4.2 for the definition of the localizations $\hat{\pi}_n$).

5.2. *Specialization functors.*

Definition 5.2.1. Let $n \geq 1$. Define the functor

$$\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \quad \Gamma_n := (\,\cdot\,)^{\mathfrak{gl}_n^\perp}.$$

As before, the subalgebras $\mathfrak{gl}_n, \mathfrak{gl}_n^\perp \subset \mathfrak{gl}_\infty$ commute, and therefore the subspace of $\mathfrak{gl}_n^\perp$-invariants of a $\mathfrak{gl}_\infty$-module automatically carries an action of $\mathfrak{gl}_n$.

Lemma 5.2.2. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is well defined.*

Proof. The proof is essentially the same as that in Lemma 5.1.3. □

Lemma 5.2.3. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is exact.*

Proof. The definition of Γ_n immediately implies that this functor is left-exact. Consider the inclusion $\mathfrak{gl}(U_n) \oplus \mathfrak{gl}(U_n)^\perp \subset \mathfrak{gl}(U_\infty)$. We then have an isomorphism of $\mathfrak{gl}(U_n)$-modules

$$(M|_{\mathfrak{gl}(U_\infty)})^{\mathfrak{gl}(U_n)^\perp} \cong (M^{\mathfrak{gl}_n^\perp})|_{\mathfrak{gl}(U_n)}.$$

The exactness of Γ_n then follows from the additivity of the functor

$$(\,\cdot\,)^{\mathfrak{gl}(U_n)^\perp} : \mathrm{Rep}(\mathfrak{gl}(U_\infty))_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}(U_n))_{\mathrm{poly}},$$

which is an additive functor between semisimple categories. □

The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ clearly takes polynomial $\mathfrak{gl}_\infty$-modules to polynomial $\mathfrak{gl}_n$-modules; together with Lemma 5.2.3, this means that Γ_n factors through an exact functor

$$\widehat{\Gamma}_n : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n},$$

i.e., we have a commutative diagram

$$\begin{tikzcd} O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\Gamma_n"] \arrow[d, "\hat{\pi}_\infty"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[d, "\hat{\pi}_n"] \\ \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\widehat{\Gamma}_n"] & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \end{tikzcd}$$

5.3. ***Stable inverse limit of categories $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ and the category $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$.*** The restriction functors

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}^{\perp}_{n-1}}$$

described in Section 5.1 allow us to consider the inverse limit of the system $((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2})$, and similarly for $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2})$. Let $n \geq 1$.

Notation 5.3.1. For each $k \in \mathbb{Z}_+$, let $\mathrm{Fil}_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ (resp., $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$) be the Serre subcategory of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (resp., $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$) generated by simple modules $L_n(\nu - |\lambda|, \lambda)$ (respectively, $\hat{\pi}_n(L_n(\nu - |\lambda|, \lambda))$), with $\ell(\lambda) \leq k$.

This defines $\mathbb{Z}_+$-filtrations on the objects of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ and $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, i.e.,

$$O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k\in\mathbb{Z}_+} \mathrm{Fil}_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}), \quad \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k\in\mathbb{Z}_+} \mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}).$$

Lemma 5.3.2. *The functors*

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

and

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

are both shortening and $\mathbb{Z}_+$-filtered functors between finite-length categories with $\mathbb{Z}_+$-filtrations on objects (see the Appendix for the relevant definitions). *Moreover, the systems $(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \mathfrak{Res}_{n-1,n})$ and $(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \widehat{\mathfrak{Res}}_{n-1,n})$ satisfy the conditions appearing in Section A.5, and thus for each of these, their restricted inverse limit coincides with their inverse limit as $\mathbb{Z}_+$-graded categories.*

Proof. These statements follow directly from Lemma 5.1.6, which tells us that $\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$, and the fact that $L_n(\nu - |\lambda|, \lambda) = 0$ whenever $\ell(\lambda) > n - 1$. □

We can now consider the inverse limits of the $\mathbb{Z}_+$-filtered systems

$$((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2}), \quad ((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2}).$$

By Proposition A.5.1, these limits are equivalent to the respective restricted inverse limits

$$\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \quad \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

The functors Γ_n described above induce exact functors

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{\Gamma}_{\lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proposition 5.3.3. *The functors Γ_n induce an equivalence*

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. First of all, we need to check that this functor is well defined. Namely, we need to show that for any $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$, the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ stabilizes. In fact, it is enough to show that this sequence is bounded (since it is obviously increasing).

Recall that we have a surjective map of $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$-modules $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_\infty}) \otimes E \twoheadrightarrow M$ for some $E \in \mathrm{Rep}(\mathfrak{gl}(U_\infty))_{\mathrm{poly}}$. Since Γ_{n+1} is exact, it gives us a surjective map $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_{n+1}}) \otimes \Gamma_{n+1}(E) \twoheadrightarrow \Gamma_{n+1}(M)$ for any $n \geq 0$, with $\Gamma_{n+1}(E)$ being a polynomial $\mathfrak{gl}(U_{n+1})$-module.

Now,

$$\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_{n+1}})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E)).$$

The sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E))\}_{n\geq 0}$ is bounded by Proposition 3.4.3, and thus the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ is bounded as well.

We now show that $\Gamma_{\lim}$ is an equivalence. A construction similar to the one appearing in [Entova Aizenbud 2015b, Section 7.5] gives a left adjoint to the functor $\Gamma_{\lim}$; that is, we will define a functor

$$\Gamma^*_{\lim} : \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}.$$

Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The isomorphisms $\phi_{n-1,n} : \mathfrak{Res}_{n-1,n}(M_n) \xrightarrow{\sim} M_{n-1}$ define $\mathfrak{gl}_{n-1}$-equivariant inclusions $M_{n-1} \hookrightarrow M_n$. Consider the vector space

$$M := \bigcup_{n\geq 1} M_n,$$

which has a natural action of $\mathfrak{gl}_\infty$ on it. It is easy to see that the obtained $\mathfrak{gl}_\infty$-module M is a direct sum of polynomial $\mathfrak{gl}(U_\infty)$-modules, and is locally nilpotent over the algebra

$$\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_\infty}) \cong \mathrm{Sym}(U_{\infty,*}) \cong \bigcup_{n\geq 1} \mathrm{Sym}(U_n^*).$$

Sublemma 5.3.4. *Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. Then $M := \bigcup_{n\geq 1} M_n$ is a finitely generated module over*

$$\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty}).$$

Proof. In Proposition A.2.2, we show that all the objects in the abelian category $\varprojlim_{n\geq 1,\,\text{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ have finite length, and that the simple objects in this category are exactly those of the form $((L_n(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ for a fixed Young diagram λ. So we only need to check that applying the above construction to these simple objects gives rise to finitely generated modules over $\operatorname{Sym}(U_\infty)\cong\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

Using Corollary 4.4.6 we now reduce the proof of the sublemma to proving the following two statements:

- Let λ be a fixed Young diagram and $((L_n(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be a simple object in $\varprojlim_{n\geq 1,\,\text{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ in which $L_n(\nu-|\lambda|,\lambda)$ is polynomial for every n (i.e., λ is minimal in its nontrivial $\overset{\nu}{\sim}$-class). Then $L:=\bigcup_{n\geq 1} L_n(\nu-|\lambda|,\lambda)$ is a polynomial $\mathfrak{gl}_\infty$-module (in particular, a finitely generated module over $\operatorname{Sym}(U_\infty)\cong\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$).
- Let λ be a fixed Young diagram and let $((M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\text{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (this is a sequence of "compatible" parabolic Verma modules). Then
$$M:=\bigcup_n M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)$$
is a finitely generated module over $\operatorname{Sym}(U_\infty)\cong\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

The first statement follows immediately from Proposition 3.4.3. To prove the second statement, recall from Lemma 4.4.4 that
$$M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)\cong\operatorname{Sym}(U_n)\otimes S^\lambda U_n.$$

So
$$M:=\bigcup_n M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)\cong\bigcup_n \operatorname{Sym}(U_n)\otimes S^\lambda U_n\cong\operatorname{Sym}(U_\infty)\otimes S^\lambda U_\infty,$$
which is clearly a finitely generated module over $\operatorname{Sym}(U_\infty)\cong\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$. □

This allows us to define the functor $\Gamma^*_{\lim}$ by setting
$$\Gamma^*_{\lim}((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2}):=\bigcup_{n\geq 1} M_n$$
and requiring that it act on morphisms accordingly. The definition of $\Gamma^*_{\lim}$ gives us a natural transformation
$$\Gamma^*_{\lim}\circ\Gamma_{\lim}\xrightarrow{\sim}\operatorname{Id}_{\mathcal{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}}.$$
Restricting the action of $\mathfrak{gl}_\infty$ to $\mathfrak{gl}(U_\infty)$ and using Proposition 3.4.3, we conclude that this natural transformation is an isomorphism.

Notice that the definition of $\Gamma^*_{\lim}$ implies that this functor is faithful. Thus we conclude that the functor $\Gamma^*_{\lim}$ is an equivalence of categories, and so is $\Gamma_{\lim}$. □

Proposition 5.3.5. *The functors $\widehat{\Gamma}_n$ induce an equivalence*

$$\widehat{\Gamma}_{\lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. Let $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. First of all, we need to check that the functor $\widehat{\Gamma}_{\lim}$ is well defined; that is, we need to show that the sequence $\{\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M)))\}_{n\geq 1}$ is bounded from above.

Indeed,

$$\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M))) \leq \ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M)).$$

But the sequence $\{\ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M))\}_{n\geq 1}$ is bounded from above by Proposition 5.3.3, so the original sequence is bound from above as well.

Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu} \arrow[r] \arrow[d, "\Gamma_{\lim}"'] & O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\hat{\pi}_\infty"] \arrow[d, "\Gamma_{\lim}"'] & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[d, "\widehat{\Gamma}_{\lim}"'] \\
\varprojlim_{n\geq 1,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},\nu} \arrow[r] & \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\hat{\pi}_{\lim}=\varprojlim_n \hat{\pi}_n"] & \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}
\end{tikzcd}$$

where $\mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly},\nu}$ is the Serre subcategory of $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ consisting of all polynomial modules of degree ν. The rows of this commutative diagram are "exact" (in the sense that $\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ is the Serre quotient of the category $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ by the Serre subcategory $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu}$, and similarly for the bottom row).

The functors

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},\nu}$$

and

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

are equivalences of categories (by Propositions 3.4.3 and 5.3.3), and thus the functor $\widehat{\Gamma}_{\lim}$ is an equivalence as well. □

6. Complex tensor powers of a unital vector space

In this section we describe the construction of a complex tensor power of the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$ (again, $N \in \mathbb{Z}_+ \cup \{\infty\}$). A general construction of the complex tensor power of a unital vector space is given in [Entova Aizenbud 2015a, Section 6].

Again, we denote $U_N := \mathrm{span}\{e_2, e_3, \dots\}$, and $U_{N*} := \mathrm{span}\{e_2^*, e_3^*, \dots\} \subset \mathbb{C}^N_*$. As before, we have a decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}\,\mathrm{Id}_{\mathbb{C}^N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N} \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{gl}(U_N)$$

such that $U_N \cong \mathfrak{u}^-_{\mathfrak{p}_N}$, $U_{N*} \cong \mathfrak{u}^+_{\mathfrak{p}_N}$, and if N is finite, we have $U_N^* \cong U_{N*}$.

Fix $\nu \in \mathbb{C}$. Recall from [Entova Aizenbud 2015a, Section 4] that for any $\nu \in \mathbb{C}$, in the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ we have the objects Δ_k ($k \in \mathbb{Z}_+$). These objects interpolate the representations $\mathbb{C}\,\mathrm{Inj}(\{1, \dots, k\}, \{1, \dots, n\}) \cong \mathrm{Ind}^{S_n \times S_k}_{S_{n-k} \times S_k \times S_k} \mathbb{C}$ of the symmetric groups S_n; in fact, for any $n \in \mathbb{Z}_+$ we have

$$\mathcal{S}_n(\Delta_k) \cong \mathbb{C}\,\mathrm{Inj}(\{1, \dots, k\}, \{1, \dots, n\}),$$

where $\mathcal{S}_n : \underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ is the monoidal functor discussed in Section 2.1.

Definition 6.0.1 (complex tensor power). Define the object $(\mathbb{C}^N)^{\underline{\otimes} \nu}$ of the category $\text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu, \mathbb{C}^N}\big)$ by setting

$$(\mathbb{C}^N)^{\underline{\otimes} \nu} := \bigoplus_{k \geq 0} (U_N^{\otimes k} \otimes \Delta_k)^{S_k}.$$

The action of $\mathfrak{gl}_N$ on $(\mathbb{C}^N)^{\underline{\otimes} \nu}$ is given as follows:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "U_N"] & U_N \otimes \Delta_1 \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & (U_N^{\otimes 2} \otimes \Delta_2)^{S_2} \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & (U_N^{\otimes 3} \otimes \Delta_3)^{S_3} \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & \cdots \arrow[l, bend left, "U_{N*}"]
\end{tikzcd}$$

- $E_{1,1} \in \mathfrak{gl}_N$ acts by scalar $\nu - k$ on each summand $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$.
- $A \in \mathfrak{gl}(U_N) \subset \mathfrak{gl}_N$ acts on $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$ by
$$\sum_{1 \leq i \leq k} A^{(i)}|_{U_N^{\otimes k}} \otimes \mathrm{Id}_{\Delta_k} : (U_N^{\otimes k} \otimes \Delta_k)^{S_k} \to (U_N^{\otimes k} \otimes \Delta_k)^{S_k}.$$
- $u \in U_N \cong \mathfrak{u}^-_{\mathfrak{p}_N}$ acts by morphisms of degree 1, which are given explicitly in [Entova Aizenbud 2015a, Section 6.2].
- $f \in U_{N*} \cong \mathfrak{u}^+_{\mathfrak{p}_N}$ acts by morphisms of degree -1, which are given explicitly in [Entova Aizenbud 2015a, Section 6.2].

Remark 6.0.2. The actions of the elements of $\mathfrak{u}^+_{\mathfrak{p}_N}$ and $\mathfrak{u}^-_{\mathfrak{p}_N}$, though not written here explicitly, are in fact uniquely determined by the actions of $E_{1,1}$ and $\mathfrak{gl}(U_N)$.

To see this, note that the ideal in the Lie algebra $\mathfrak{gl}_N$ generated by the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$ is the entire $\mathfrak{gl}_N$. Given two $\mathfrak{gl}_N$-modules M_1, M_2 and an isomorphism $M_1 \to M_2$ which is equivariant with respect to the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$, the above fact implies that this isomorphism is also $\mathfrak{gl}_N$-equivariant.

In other words, if there exists a way to define an action of $\mathfrak{gl}_N$ whose restriction to the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$ is given by the formulas above, then such an action of $\mathfrak{gl}_N$ is unique.

Remark 6.0.3. The proof that the object $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ lies in Ind-$\big(\underline{\mathrm{Rep}}(S_\nu)\boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\big)$ is an easy check, and can be found in [Entova Aizenbud 2015a]. In particular, it means that the action of the mirabolic subalgebra Lie $\overline{\mathfrak{P}}_{\mathbb{1}}$ on the complex tensor power $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ integrates to an action of the mirabolic subgroup $\overline{\mathfrak{P}}_{\mathbb{1}}$, thus making $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ a Harish-Chandra module in Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ for the pair $(\mathfrak{gl}_N, \overline{\mathfrak{P}}_{\mathbb{1}})$.

The definition of the complex tensor power is compatible with the usual notion of a tensor power of a unital vector space (see [Entova Aizenbud 2015a, Section 6]):

Proposition 6.0.4. *Let $d \in \mathbb{Z}_+$. Consider the functor*

$$\hat{\mathcal{S}}_d : \text{Ind-}\big(\underline{\mathrm{Rep}}(S_{\nu=d})\boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N}\big) \to \text{Ind-}\big(\mathrm{Rep}(S_d)\boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N}\big)$$

induced by the functor

$$\mathcal{S}_d : \underline{\mathrm{Rep}}(S_{\nu=d}) \to \mathrm{Rep}(S_n)$$

described in Section 2.1. Then $\hat{\mathcal{S}}_d((\mathbb{C}^N)^{\underline{\otimes}d}) \cong (\mathbb{C}^N)^{\otimes d}$.

The construction of the complex tensor power is also compatible with the functors $\mathfrak{Res}_{n,n+1}$ and Γ_n defined in Definitions 5.1.1 and 5.2.1. These properties can be seen as special cases of the following statement (when $N = n+1$ and $N = \infty$, respectively):

Proposition 6.0.5. *Let $n \geq 1$, and let $N \geq n$, $N \in \mathbb{Z}_{\geq 1}\cup\{\infty\}$. Recall that we have an inclusion $\mathfrak{gl}_n \oplus \mathfrak{gl}_n^{\perp} \subset \mathfrak{gl}_N$, and consider the functor*

$$(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : \text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)\boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\big) \to \text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)\boxtimes O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\big)$$

induced by the functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}}$ then takes $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ to $(\mathbb{C}^n)^{\underline{\otimes}\nu}$.

Proof. The functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ induces an endofunctor of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$. We would like to say that we have an isomorphism of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$-objects

$$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \overset{?}{\cong} (\mathbb{C}^n)^{\underline{\otimes}\nu}$$

and that the action of $\mathfrak{gl}_n \subset \mathfrak{gl}_N$ on $((\mathbb{C}^N)^{\underline{\otimes}\nu})$ corresponds to the action of $\mathfrak{gl}_n$ on $(\mathbb{C}^n)^{\underline{\otimes}\nu}$. In order to do this, we first consider $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ as an object in Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ with an action of $\mathfrak{gl}(U_N)$:

$$(\mathbb{C}^N)^{\underline{\otimes}\nu} \cong \bigoplus_{k\geq 0}(\Delta_k \otimes U_N^{\otimes k})^{S_k}.$$

If we consider only the actions of $\mathfrak{gl}(U_N)$, $\mathfrak{gl}(U_n)$, the functor Γ_n is induced by the additive monoidal functor $(\,\cdot\,)^{\mathfrak{gl}(U_n)^{\perp}} : \text{Ind-Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}} \to \text{Ind-Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}}$. This shows that we have an isomorphism of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$-objects

$$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \cong \bigoplus_{k\geq 0}(\Delta_k \otimes U_n^{\otimes k})^{S_k} \cong (\mathbb{C}^n)^{\underline{\otimes}\nu}$$

and the actions of $\mathfrak{gl}(U_n)$ on both sides are compatible. From the definition of the complex tensor power (Definition 6.0.1) one immediately sees that the actions of $E_{1,1}$ on both sides are compatible as well. Remark 6.0.2 now completes the proof. □

7. Schur–Weyl duality in complex rank: the Schur–Weyl functor and the finite-dimensional case

We fix $\nu \in \mathbb{C}$ and $N \in \mathbb{Z}_+ \cup \{\infty\}$. Again, we consider the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$ and the complement $U_N := \mathrm{span}\{e_2, e_3, \dots\}$.

7.1. *Schur–Weyl functor.*

Definition 7.1.1. Define the Schur–Weyl contravariant functor

$$\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$$

by

$$\mathrm{SW}_\nu := \mathrm{Hom}_{\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)}(\,\cdot\,, (\mathbb{C}^N)^{\underline{\otimes}\nu}).$$

Remark 7.1.2. The functor $\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ is a contravariant $\mathbb{C}$-linear additive left-exact functor.

It turns out that the image of the functor $\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ lies in $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ (cf. Remark 6.0.3).

We can now define another Schur–Weyl functor which we will consider: the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$. Recall from Section 4.2 that

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} := O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} / \mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly},\nu}$$

is the Serre quotient of $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by the Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν. We then define

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N} := \hat{\pi}_N \circ \mathrm{SW}_{\nu,\mathbb{C}^N}.$$

7.2. ***The finite-dimensional case.*** Let $n \in \mathbb{Z}_+$. We then have the following theorem, which can be found in [Entova Aizenbud 2015a, Section 7]:

Theorem 7.2.1. *The contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is exact and essentially surjective. Moreover, the induced contravariant functor*

$$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) / \mathrm{Ker}(\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}) \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

is an antiequivalence of abelian categories, thus making $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ a Serre quotient of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$.

We will show that a similar result holds in the infinite-dimensional case, when the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ is in fact an antiequivalence of categories.

In the proof of Theorem 7.2.1 we established the following fact (see [Entova Aizenbud 2015a, Theorem 7.2.3]):

Lemma 7.2.2. *The functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ takes a simple object to either a simple object, or zero. More specifically*:

- *Let λ be a Young diagram lying in a trivial $\overset{\nu}{\sim}$-class. Then*
$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(\boldsymbol{L}(\lambda)) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)).$$
- *Consider a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\geq 0}$. Then*
$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(\boldsymbol{L}(\lambda^{(i)})) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu-|\lambda^{(i+1)}|,\lambda^{(i+1)}))$$
whenever $i\geq 0$.

Remark 7.2.3. Recall that $L_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)$ is zero if $\ell(\lambda)\geq n$.

8. Classical Schur–Weyl duality and the restricted inverse limit

8.1. ***A short overview of the classical Schur–Weyl duality.*** Let V be a vector space over $\mathbb{C}$, and let $d\in\mathbb{Z}_+$. The symmetric group S_d acts on $V^{\otimes d}$ by permuting the factors of the tensor product (the action is semisimple, by Maschke's theorem):
$$\sigma.(v_1\otimes v_2\otimes\cdots\otimes v_d) := v_{\sigma^{-1}(1)}\otimes v_{\sigma^{-1}(2)}\otimes\cdots\otimes v_{\sigma^{-1}(d)}.$$

The actions of $\mathfrak{gl}(V)$ and S_d on $V^{\otimes d}$ commute, which allows us to consider a contravariant functor
$$\mathtt{SW}_{d,V}:\mathrm{Rep}(S_d)\to\mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}},\quad \mathtt{SW}_{d,V}:=\mathrm{Hom}_{S_d}(\,\cdot\,,V^{\otimes d}).$$
The contravariant functor $\mathtt{SW}_{d,V}$ is $\mathbb{C}$-linear and additive, and sends a simple representation λ of S_d to the $\mathfrak{gl}(V)$-module $S^\lambda V$.

Next, consider the contravariant functor
$$\mathtt{SW}_V:\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)\to\mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}},\quad \mathtt{SW}_V:=\bigoplus_d \mathtt{SW}_{d,V}.$$
This functor $\mathtt{SW}_V$ is clearly essentially surjective and full (this is easy to see, since $\mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}}$ is a semisimple category with simple objects $S^\lambda V\cong\mathtt{SW}(\lambda)$). The kernel of the functor $\mathtt{SW}_V$ is the full additive subcategory (direct factor) of $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$ generated by simple objects λ such that $\ell(\lambda)>\dim V$.

8.2. ***Classical Schur–Weyl duality: inverse limit.*** In this subsection, we prove that the classical Schur–Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$ make the category $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$ dual (antiequivalent) to the category
$$\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}\cong\varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}}\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

The contravariant functor $\mathtt{SW}_{\mathbb{C}^N}$ sends the Young diagram λ to the $\mathfrak{gl}_N$-module $S^\lambda \mathbb{C}^N$.

Let $n \in \mathbb{Z}_+$. We start by noticing that the functors $\mathfrak{Res}_{n,n+1}$ and the functors Γ_n (defined in Section 3) are compatible with the classical Schur–Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$:

Lemma 8.2.1. *We have natural isomorphisms*

$$\mathfrak{Res}_{n,n+1} \circ \mathtt{SW}_{\mathbb{C}^{n+1}} \cong \mathtt{SW}_{\mathbb{C}^n}$$

and

$$\Gamma_n \circ \mathtt{SW}_{\mathbb{C}^\infty} \cong \mathtt{SW}_{\mathbb{C}^n}$$

for any $n \geq 0$.

Proof. It is enough to check this on simple objects in $\bigoplus_{d \in \mathbb{Z}_+} \mathrm{Rep}(S_d)$, in which case the statement follows directly from the definitions of $\mathfrak{Res}_{n,n+1}$ and Γ_n together with the fact that $\mathtt{SW}_{\mathbb{C}^N}(\lambda) \cong S^\lambda \mathbb{C}^N$ for any $N \in \mathbb{Z}_+ \cup \{\infty\}$. □

The above lemma implies that we have a commutative diagram

$$\begin{tikzcd}
 & \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} & \\
\bigoplus_{d \in \mathbb{Z}_+} \mathrm{Rep}(S_d)^{\mathrm{op}} \arrow[ur, "\mathtt{SW}_{\mathbb{C}^n}"] \arrow[r, "\mathtt{SW}_{\mathrm{lim}}"'] \arrow[dr, "\mathtt{SW}_{\mathbb{C}^\infty}"'] & \varprojlim_{n \geq 1,\ \mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \arrow[u, "\mathrm{Pr}_n"'] & \\
 & \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \arrow[u, "\Gamma_{\mathrm{lim}}"'] \arrow[uu, bend right=80, "\Gamma_n"'] &
\end{tikzcd}$$

with the functor Γ_{lim} being an equivalence of categories (by Proposition 3.4.3), and Pr_n being the canonical projection functor.

Proposition 8.2.2. *The contravariant functors*

$$\mathtt{SW}_\infty : \bigoplus_{d \in \mathbb{Z}_+} \mathrm{Rep}(S_d) \to \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$$

and

$$\mathtt{SW}_{\mathrm{lim}} : \bigoplus_{d \in \mathbb{Z}_+} \mathrm{Rep}(S_d) \to \varprojlim_{n \in \mathbb{Z}_+,\ \mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$$

are antiequivalences of semisimple categories.

Proof. As was said in Section 8.1, the functor $\mathtt{SW}_N$ is full and essentially surjective for any N. In this case, the functor $\mathtt{SW}_\infty$ is also faithful, since the simple object λ in $\bigoplus_{d \in \mathbb{Z}_+} \mathrm{Rep}(S_d)$ is taken by the functor $\mathtt{SW}_\infty$ to the simple object $S^\lambda \mathbb{C}^\infty \neq 0$. This proves that the contravariant functor $\mathtt{SW}_\infty$ is an antiequivalence of categories. The commutative diagram above then implies that the contravariant functor $\mathtt{SW}_{\mathrm{lim}}$ is an antiequivalence as well. □

9. $\underline{\mathbf{Rep}}^{\mathbf{ab}}(S_\nu)$ and the inverse limit of categories $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$

9.1. In this section we prove that the Schur–Weyl functors defined in Section 7.1 give us an equivalence of categories between $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and the restricted inverse limit $\varprojlim_{N\in\mathbb{Z}_+,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$. We fix $\nu\in\mathbb{C}$.

Proposition 9.1.1. *The functor $\mathfrak{Res}_{n-1,n}$ satisfies $\mathfrak{Res}_{n-1,n}\circ \mathrm{SW}_{\nu,\mathbb{C}^n}\cong \mathrm{SW}_{\nu,\mathbb{C}^{n-1}}$; i.e., there exists a natural isomorphism $\eta_n:\mathfrak{Res}_{n-1,n}\circ \mathrm{SW}_{\nu,\mathbb{C}^n}\to \mathrm{SW}_{\nu,\mathbb{C}^{n-1}}$.*

Proof. This follows directly from Proposition 6.0.5. □

Corollary 9.1.2. $\widehat{\mathfrak{Res}}_{n-1,n}\circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}\cong \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$; *i.e., there exists a natural isomorphism $\hat{\eta}_n:\widehat{\mathfrak{Res}}_{n-1,n}\circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}\to \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$.*

Proof. By the definitions of $\widehat{\mathfrak{Res}}_{n-1,n}$ and $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$, together with Proposition 9.1.1, we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[r, "\mathrm{SW}_{\nu,\mathbb{C}^n}"'] \arrow[rr, bend left, "\mathrm{SW}_{\nu,\mathbb{C}^{n-1}}"] \arrow[dr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\mathfrak{Res}_{n-1,n}"'] \arrow[d, "\hat{\pi}_n"] & O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \arrow[d, "\hat{\pi}_{n-1}"] \\
& \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\widehat{\mathfrak{Res}}_{n-1,n}"'] & \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}
\end{tikzcd}$$

Since $\hat{\pi}_{n-1}\circ \mathrm{SW}_{\nu,\mathbb{C}^{n-1}} =: \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$, we get $\widehat{\mathfrak{Res}}_{n-1,n}\circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}\cong \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$. □

Notation 9.1.3. For each $k\in\mathbb{Z}_+$, $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ is defined to be the Serre subcategory of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ generated by the simple objects $\boldsymbol{L}(\lambda)$ such that the Young diagram λ satisfies either of the following conditions:

- λ belongs to a trivial $\overset{\nu}{\sim}$-class, and $\ell(\lambda)\le k$.
- λ belongs to a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\ge 0}$, $\lambda=\lambda^{(i)}$, and $\ell(\lambda^{(i+1)})\le k$.

This defines a $\mathbb{Z}_+$-filtration on the objects of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$. That is,

$$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)\cong \varinjlim_{k\in\mathbb{Z}_+}\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)).$$

Lemma 9.1.4. *The functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are $\mathbb{Z}_+$-filtered shortening functors* (see the Appendix for the relevant definitions).

Proof. This result follows from the fact that the $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are exact, together with Lemma 7.2.2. □

This lemma, together with Corollary 9.1.2, implies that there is a canonical contravariant ($\mathbb{Z}_+$-filtered shortening) functor

$$\begin{aligned}\widehat{\mathrm{SW}}_{\nu,\lim} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) &\to \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n},\\ X &\mapsto \left(\{\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}, \{\hat{\eta}_n(X)\}_{n\geq 2}\right),\\ (f : X\to Y) &\mapsto \{\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(f) : \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(Y) \to \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}.\end{aligned}$$

This functor is given by the universal property of the restricted inverse limit described in Proposition A.2.7[1] and makes the diagram below commutative:

$$\begin{tikzcd} & \hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[ur, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[r, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] & \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[u, "\mathrm{Pr}_n"'] \end{tikzcd}$$

(here Pr_n is the canonical projection functor).

We show there is an equivalence of categories $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$ and $\varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Theorem 9.1.5. *The Schur–Weyl contravariant functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ induce an anti-equivalence of abelian categories, given by the* (*exact*) *contravariant functor*

$$\widehat{\mathrm{SW}}_{\nu,\lim} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. The functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are exact for each $n \geq 1$, which means that the functor $\widehat{\mathrm{SW}}_{\nu,\lim}$ is exact as well.

To see that it is an antiequivalence, we will use Proposition A.4.2. All we need to check is that the functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ satisfy the stabilization condition (Condition A.4.1): that is, for each $k \in \mathbb{Z}_+$, there exists $n_k \in \mathbb{Z}_+$ such that

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)) \to \mathrm{Fil}_k(\hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an antiequivalence of categories for any $n \geq n_k$.

Indeed, let $k \in \mathbb{Z}_+$, and let $n \geq k+1$. The category $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ decomposes into blocks (corresponding to the blocks of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$), and the category $\mathrm{Fil}_k(\hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ decomposes into blocks corresponding to the blocks of $\hat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

The requirement $n \geq k+1$ together with Lemma 7.2.2 means that for any semisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$, the simple object $\boldsymbol{L}(\lambda)$ corresponding to this block is not sent to zero under $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$. This, in turn, implies that $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$

[1] Alternatively, one can use Proposition A.3.3, since we already stated that in our setting the two notions of inverse limit coincide.

induces an antiequivalence between each semisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ and the corresponding semisimple block of $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$.

Fix a nonsemisimple block $\mathcal{B}_\lambda$ of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$, and denote by $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ the corresponding nonsemisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$. We denote by $\mathfrak{B}_{\lambda,n}$ the corresponding block in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The corresponding block of $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ will then be $\hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$.

We now check that the contravariant functor

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}|_{\mathrm{Fil}_k(\mathcal{B}_\lambda)} : \mathrm{Fil}_k(\mathcal{B}_\lambda) \to \hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$$

is an antiequivalence of categories when $n \geq k+1$.

Since $n \geq k+1$, the Serre subcategories $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ and $\mathrm{Ker}(\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n})$ of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ have trivial intersection (see Lemma 7.2.2), which means that the restriction of $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ to the Serre subcategory $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ is both faithful and full (the latter follows from Theorem 7.2.1).

It remains to establish that the functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}|_{\mathrm{Fil}_k(\mathcal{B}_\lambda)}$ is essentially surjective when $n \geq k+1$. This can be done by checking that this functor induces a bijection between the sets of isomorphism classes of indecomposable projective objects in $\mathrm{Fil}_k(\mathcal{B}_\lambda)$, $\hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$ respectively (see [Entova Aizenbud 2015a, proof of Theorem 7.2.7], where we use a similar technique). The latter fact follows from the proof of [Entova Aizenbud 2015a, Theorem 7.2.7].

Thus $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\mathcal{B}_\lambda) \to \mathrm{Fil}_k(\hat{\pi}(\mathfrak{B}_{\lambda,n}))$ is an antiequivalence of categories for $n \geq k+1$, and

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)) \to \mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an antiequivalence of categories for $n \geq k+1$. □

10. Schur–Weyl duality for $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and $\mathfrak{gl}_\infty$

10.1. Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \ldots$. Fix $\mathbb{1} := e_1$ and $U_\infty := \mathrm{span}_{\mathbb{C}}(e_2, e_3, \ldots)$.

Lemma 10.1.1. *We have a commutative diagram*

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[rr, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] \arrow[drr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & & \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\
& & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_{\lim}"']
\end{tikzcd}$$

Namely, there is a natural isomorphism $\hat{\eta} : \widehat{\Gamma}_{\lim} \circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty} \to \widehat{\mathrm{SW}}_{\nu,\lim}$.

Proof. To prove this statement, we will show that for any $n \geq 1$, the following diagram is commutative:

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{ab}(S_\nu)^{op} \arrow[r, "\mathrm{SW}_{\nu,\mathbb{C}^n}"] \arrow[dr, "\mathrm{SW}_{\nu,\mathbb{C}^\infty}"] \arrow[rr, bend left, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[drr, bend right, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\hat{\pi}_n"'] & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\
& O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\Gamma_n"'] \arrow[r, "\hat{\pi}_\infty"] & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_n"']
\end{tikzcd}$$

The commutativity of this diagram follows from the existence of a natural isomorphism $\Gamma_n \circ \mathrm{SW}_{\nu,\mathbb{C}^\infty} \xrightarrow{\sim} \mathrm{SW}_{\nu,\mathbb{C}^n}$ (due to Proposition 6.0.5) and a natural isomorphism $\widehat{\Gamma}_n \circ \hat{\pi}_\infty \cong \hat{\pi}_n \circ \Gamma_n$ (see proof of Proposition 5.3.5). □

Thus we obtain a commutative diagram

$$\begin{tikzcd}
& \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} & \\
\underline{\mathrm{Rep}}^{ab}(S_\nu)^{op} \arrow[ur, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[r, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] \arrow[dr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & \varprojlim_{n\geq 1,\, \mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[u, "\mathrm{Pr}_n"'] & \\
& \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_{\lim}"'] \arrow[uu, bend right=80, "\widehat{\Gamma}_n"'] &
\end{tikzcd}$$

Theorem 10.1.2. *The contravariant functor* $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty} : \underline{\mathrm{Rep}}^{ab}(S_\nu) \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ *is an antiequivalence of abelian categories.*

Proof. The functor $\widehat{\Gamma}_{\lim}$ is an equivalence of categories (see Proposition 5.3.5), and the functor $\widehat{\mathrm{SW}}_{\nu,\lim}$ is an antiequivalence of categories (see Theorem 9.1.5). The commutative diagram above implies that the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ is an antiequivalence of categories as well. □

11. Schur–Weyl functors and duality structures

11.1. Let $n \in \mathbb{Z}_+$. Recall the contravariant duality functor $(\,\cdot\,)^\vee_n : (O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})^{op} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ discussed in Section 4.3. This functor takes polynomial modules to polynomial modules, and therefore descends to a duality functor $\widehat{(\,\cdot\,)}^\vee_n : (\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})^{op} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Next, the definition of duality functor in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ implies that the duality functors commute with the restriction functors $\mathfrak{Res}_{n-1,n}$, namely, that for any $n \geq 2$,

$$(\,\cdot\,)^\vee_{n-1} \circ \mathfrak{Res}^{op}_{n-1,n} \cong \mathfrak{Res}^{op}_{n-1,n} \circ (\,\cdot\,)^\vee_n.$$

This allows us to define duality functors

$$(\,\cdot\,)^{\vee}_{\lim} : \Big(\varprojlim_{n\geq 1,\,\text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \Big)^{\text{op}} \to \varprojlim_{n\geq 1,\,\text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{(\,\cdot\,)}^{\vee}_{\lim} : \Big(\varprojlim_{n\geq 1,\,\text{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \Big)^{\text{op}} \to \varprojlim_{n\geq 1,\,\text{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Under the equivalence $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\,\text{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ established in Section 5.3, the functor $(\,\cdot\,)^{\vee}_{\lim}$ corresponds to the duality functor $(\,\cdot\,)^{\vee}_{\infty} : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\text{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ discussed in Section 4.3. Again, this functor descends to a contravariant duality functor $\widehat{(\,\cdot\,)}^{\vee}_{\infty} : (\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\text{op}} \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$.

As a corollary of Theorem 7.2.1, a connection was established between the notions of duality in the Deligne category $\underline{\text{Rep}}^{\text{ab}}(S_\nu)$ and duality in the category $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ for $N \in \mathbb{Z}_+$ (see [Entova Aizenbud 2015a, Section 7.3]). The above construction allows us to extend this connection to the case when $N = \infty$. Namely, Theorems 9.1.5 and 10.1.2, together with [Entova Aizenbud 2015a, Section 7.3], imply the next result.

Proposition 11.1.1. *Let $N \in \mathbb{Z}_+ \cup \{\infty\}$ and $\nu \in \mathbb{C}$. There is an isomorphism of (covariant) functors*

$$\widehat{\text{SW}}_{\nu,\mathbb{C}^N} \circ (\,\cdot\,)^* \to \widehat{(\,\cdot\,)}^{\vee}_{N} \circ \text{SW}_{\nu,\mathbb{C}^N}.$$

Appendix: Restricted inverse limit of categories

We describe the main elements of the framework for the notion of a restricted inverse limit of categories. A detailed description of this framework has been given in the note [Entova Aizenbud 2015b]; this appendix contains the results which are necessary for understanding the Schur–Weyl duality in complex rank. In particular, [Entova Aizenbud 2015b] provides some motivation behind the definitions given below.

Given a system of categories $\mathcal{C}_i$ (with i running through the set $\mathbb{Z}_+$) and functors $\mathcal{F}_{i-1,i} : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for each $i \geq 1$, we define the inverse limit category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ to be the following category:

- The objects are pairs $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ where $C_i \in \mathcal{C}_i$ for each $i \in \mathbb{Z}_+$ and $\phi_{i-1,i} : \mathcal{F}_{i-1,i}(C_i) \xrightarrow{\sim} C_{i-1}$ for any $i \geq 1$.
- A morphism f between objects $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1}), (\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$ is a set of arrows $\{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$ satisfying some obvious compatibility conditions.

This category is an inverse limit of the system $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ in the $(2, 1)$-category of categories with functors and natural isomorphisms. We will denote by Pr_i the projection functors $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i \to \mathcal{C}_i$ (and similarly the projection functors from other inverse limits defined below).

A.1. ***Restricted inverse limit of finite-length categories.*** To define the restricted inverse limit, we work with categories $\mathcal{C}_i$ which are finite-length categories, namely, abelian categories where each object has a (finite) Jordan–Hölder filtration. We require that the functors $\mathcal{F}_{i-1,i}$ be shortening in the following sense:

Definition A.1.1. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two finite-length categories is shortening if it is exact and given an object $C \in \mathcal{C}$, we have

$$\ell_{\mathcal{D}}(\mathcal{F}(C)) \leq \ell_{\mathcal{C}}(C).$$

Since $\mathcal{F}$ is exact, this is equivalent to requiring that for any simple object $L \in \mathcal{A}_1$, the object $\mathcal{F}(L)$ is either simple or zero.

Example A.1.2. The functors

$$\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}} \quad \text{and} \quad \Gamma_n : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$$

(see Section 3.1 for definitions) are examples of shortening functors.

Given a system $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ of finite-length categories and shortening functors, it makes sense to consider the full subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, with $\{\ell_{\mathcal{C}_n}(C_n)\}_{n\geq 0}$ being a bounded sequence (the condition on the functors implies that this sequence is weakly increasing).

This subcategory will be called the *restricted inverse limit* of categories $\mathcal{C}_i$ and will be denoted by $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. It is the inverse limit of the categories $\mathcal{C}_i$ in the (2, 1)-category of finite-length categories and shortening functors.

Example A.1.3. Consider the restricted inverse limit of the system

$$\big((\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})_{n\geq 0}, (\mathfrak{Res}_{n-1,n})_{n\geq 1}\big).$$

We obtain a functor

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \varprojlim_{n\geq 0,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

It is easy to see that $\Gamma_{\lim}$ is an equivalence.

A.2. ***Properties of the restricted inverse limit.*** The category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is an abelian category. In fact, it is a finite-length category, and one can describe its simple objects. We start by introducing some notation.

Notation A.2.1. Denote by $\mathrm{Irr}(\mathcal{C}_i)$ the set of isomorphism classes of irreducible objects in $\mathcal{C}_i$, and define the pointed set

$$\mathrm{Irr}_*(\mathcal{C}_i) := \mathrm{Irr}(\mathcal{C}_i) \sqcup \{0\}.$$

The shortening functors $\mathcal{F}_{i-1,i}$ then define maps of pointed sets

$$f_{i-1,i} : \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_{i-1}).$$

Similarly, we define $\mathrm{Irr}\big(\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i\big)$ to be the set of isomorphism classes of irreducible objects in $\mathcal{C}$, and define the pointed set

$$\mathrm{Irr}_*(\mathcal{C}) := \mathrm{Irr}(\mathcal{C}) \sqcup \{0\}.$$

Denote by $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ the inverse limit of the system $(\{\mathrm{Irr}_*(\mathcal{C}_i\}_{i\geq 0}, \{f_{i-1,i}\}_{i\geq 1})$. We will also denote by $\mathrm{pr}_j : \varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_j)$ the projection maps.

The elements of the set $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ are just sequences $(L_i)_{i\geq 0}$ such that $L_i \in \mathrm{Irr}_*(\mathcal{C}_i)$, and $f_{i-1,i}(L_i) \cong L_{i-1}$.

Proposition A.2.2. *Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a system of finite-length categories and shortening functors. The category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, and its objects have finite length. The set of isomorphism classes of simple objects in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is in bijection with the set $\big(\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)\big) \setminus \{0\}$. That is, we have a natural bijection*

$$\mathrm{Irr}_*(\mathcal{C}) \cong \varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i).$$

Proof. Let

$$C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1}),$$
$$C' := (\{C'_j\}_{j\in\mathbb{Z}_+}, \{\phi'_{j-1,j}\}_{j\geq 1}),$$
$$C'' := (\{C''_j\}_{j\in\mathbb{Z}_+}, \{\phi''_{j-1,j}\}_{j\geq 1})$$

be objects in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, together with morphisms $f : C' \to C$ and $g : C \to C''$, such that the sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

is exact.

If C lies in the subcategory $\mathcal{C}$, then the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is bounded from above, and stabilizes. Denote its maximum by N. For each i, the sequence

$$0 \longrightarrow C'_i \xrightarrow{f_i} C_i \xrightarrow{g} C''_i \longrightarrow 0$$

is exact. Therefore, $\ell_{\mathcal{C}_i}(C'_i), \ell_{\mathcal{C}_i}(C''_i) \leq N$ for each i, and so C', C'' lie in $\mathcal{C}$ as well.

Vice versa, assuming C', C'' lie in $\mathcal{C}$, denote by N', N'' the maximums of the sequences $\{\ell_{\mathcal{C}_i}(C'_i)\}_i, \{\ell_{\mathcal{C}_i}(C''_i)\}_i$, respectively. Then $\ell_{\mathcal{C}_i}(C_i) \leq N' + N''$ for any $i \geq 0$, and so C lies in the subcategory $\mathcal{C}$ as well.

Thus $\mathcal{C}$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$.

Sublemma A.2.3. *Given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have*

$$\ell_{\mathcal{C}}(C) \leq \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}.$$

Proof. Let C lie in $\mathcal{C}$. We would like to say that C has finite length. Denote by N the maximum of the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$. It is easy to see that C has length at most N; indeed, if $\{C', C'', \dots, C^{(n)}\}$ is a subset of $\mathrm{JH}_{\mathcal{C}}(C)$, then for some $i \gg 0$, we have $\mathrm{Pr}_i(C^{(k)}) \neq 0$ for any $k = 1, 2, \dots, n$. The objects $\mathrm{Pr}_i(C^{(k)})$ are distinct Jordan–Hölder components of C_i, so $n \leq \ell_{\mathcal{C}_i}(C_i) \leq N$. In particular, we see that

$$\ell_{\mathcal{C}}(C) \leq N = \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}. \qquad \square$$

Now, let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\mathcal{C}$. We denote by $\mathrm{JH}(C_j)$ the multiset of the Jordan–Hölder components of C_j, and let

$$\mathrm{JH}_*(C_j) := \mathrm{JH}(C_j) \sqcup \{0\}.$$

The corresponding set lies in $\mathrm{Irr}_*(\mathcal{C}_j)$, and we have maps of (pointed) multisets

$$f_{j-1,j} : \mathrm{JH}_*(C_j) \to \mathrm{JH}_*(C_{j-1}).$$

Sublemma A.2.4. *Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be in $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. Then*

$$C \in \mathrm{Irr}_*(\mathcal{C}) \iff \mathrm{Pr}_j(C) = C_j \in \mathrm{Irr}_*(\mathcal{C}_j)\ \forall j.$$

In other words, C is a simple object (that is, C has exactly two distinct subobjects: zero and itself) if and only if $C \neq 0$, and for any $j \geq 0$, the component C_j is either a simple object in $\mathcal{C}_j$, or zero.

Proof. The direction $\Leftarrow$ is obvious, so we will only prove the direction $\Rightarrow$.

Let n_0 be a position in which the maximum of the weakly increasing integer sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is obtained. By definition of n_0, for $j > n_0$, the functors $\mathcal{F}_{j-1,j}$ do not kill any Jordan–Hölder components of C_j.

Now, consider the socles of the objects C_j for $j \geq n_0$. For any $j > 0$, we have

$$\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j)) \xhookrightarrow{\phi_{j-1,j}} \mathrm{socle}(C_{j-1}),$$

and thus for $j > n_0$, we have

$$\ell_{\mathcal{C}_j}(\mathrm{socle}(C_j)) = \ell_{\mathcal{C}_{j-1}}(\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j))) \leq \ell_{\mathcal{C}_{j-1}}(\mathrm{socle}(C_{j-1})).$$

Thus the sequence $\{\ell_{\mathcal{C}_j}(\mathrm{socle}(C_j))\}_{j\geq n_0}$ is a weakly decreasing sequence and stabilizes. Denote its stable value by N. We conclude that there exists $n_1 \geq n_0$ such that

$$\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j)) \xrightarrow{\phi_{j-1,j}} \mathrm{socle}(C_{j-1})$$

is an isomorphism for every $j > n_1$.

Now, set

$$D_j := \begin{cases} \mathcal{F}_{j,n_1}(\mathrm{socle}(C_{n_1})) & \text{if } j < n_1, \\ \mathrm{socle}(C_j) & \text{if } j \geq n_1 \end{cases}$$

(here $\mathcal{F}_{j,n_1} : \mathcal{C}_{n_1} \to \mathcal{C}_j$ are our shortening functors, with n_1 fixed and j varying). We put $D := ((D_j)_{j\geq 0}, (\phi_{j-1,j})_{j\geq 1})$ (this is a subobject of C in the category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$). Of course, $\ell_{\mathcal{C}_j}(D_j) \leq N$ for any j, so D is an object in the full subcategory $\mathcal{C}$ of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. Furthermore, since $C \neq 0$, we have that for $j \gg 0$, $\mathrm{socle}(C_j) \neq 0$, and thus $0 \neq D \subset C$. Thus D is a semisimple object $\mathcal{C}$, with simple summands corresponding to the elements of the inverse limit of the multisets $\varprojlim_{j\in\mathbb{Z}_+} \mathrm{JH}_*(D_j)$.

We conclude that $D = C$, and that $\mathrm{socle}(C_j) = C_j$ has length at most one for any $j \geq 0$. □

Remark A.2.5. The latter multiset is equivalent to the inverse limit of multisets $\mathrm{JH}_*(\mathrm{socle}(C_j))$, so D is, in fact, the socle of C.

This completes the proof of Proposition A.2.2. □

In particular, given an object $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$, we have $\mathrm{JH}_*(C) = \varprojlim_{i\in\mathbb{Z}_+} \mathrm{JH}_*(C_i)$ (an inverse limit of the system of multisets $\mathrm{JH}_*(C_j)$ and maps $f_{j-1,j}$).

It is now obvious that the projection functors $\mathrm{Pr}_i : \mathcal{C} \to \mathcal{C}_i$ are shortening as well, and induce the maps $\mathrm{pr}_i : \mathrm{Irr}_*(\mathcal{C}) \to \mathrm{Irr}_*(\mathcal{C}_i)$.

Corollary A.2.6. *Given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have*

$$\ell_{\mathcal{C}}(C) = \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}.$$

It is now easy to see that the restricted inverse limit has the following universal property:

Proposition A.2.7. *Let $\mathcal{A}$ be a finite-length category, together with a set of shortening functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ with the property that for any $i \geq 1$, there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}.$$

Then $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is universal among such categories; that is, we have a shortening functor

$$\mathcal{G} : \mathcal{A} \to \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i,$$

$$A \mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1}),$$

$$f : A_1 \to A_2 \mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+}$$

and $\mathcal{G}_i \cong \mathrm{Pr}_i \circ \mathcal{G}$ for every $i \in \mathbb{Z}_+$.

Proof. Consider the functor $\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ induced by the functors $\mathcal{G}_i$. We would like to say that for any $A \in \mathcal{A}$, the object $\mathcal{G}(A)$ lies in the subcategory $\varprojlim_{i \in \mathbb{Z}_+, \text{restr}} \mathcal{C}_i$, i.e., that the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above.

Indeed, since $\mathcal{G}_i$ are shortening functors, we have $\ell_{\mathcal{C}_i}(\mathcal{G}_i(A)) \leq \ell_{\mathcal{A}}(A)$. Thus the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above by $\ell_{\mathcal{A}}(A)$.

Now, using Corollary A.2.6, we obtain

$$\ell_{\mathcal{C}}(\mathcal{G}(A)) = \max_{i \geq 0}\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\} \leq \ell_{\mathcal{A}}(A)$$

and we conclude that $\mathcal{G}$ is a shortening functor. □

A.3. ***Inverse limit of categories with filtration.*** We now define the inverse limit of categories in a different setting, a priori not related to the restricted inverse limit defined above. The new inverse limit is defined in the setting of categories with filtrations, and is sometimes more convenient to use. We will later give a sufficient condition for the two notions of inverse limit to coincide.

Fix a directed partially ordered set $(K, \leq)$, where "directed" means that for any $k_1, k_2 \in K$, there exists $k \in K$ such that $k_1, k_2 \leq k$.

Definition A.3.1 (categories with K-filtrations). We say that a category $\mathcal{A}$ has a K-filtration if for each $k \in K$ we have a full subcategory $\mathcal{A}^k$ of $\mathcal{A}$, and these subcategories satisfy the following conditions:

(1) $\mathcal{A}^k \subset \mathcal{A}^l$ whenever $k \leq l$.

(2) $\mathcal{A}$ is the union of $\mathcal{A}^k$, $k \in K$: that is, for any $A \in \mathcal{A}$, there exists $k \in K$ such that $A \in \mathcal{A}^k$.

A functor $\mathcal{F} : \mathcal{A}_1 \to \mathcal{A}_2$ between categories with K-filtrations $\mathcal{A}_1, \mathcal{A}_2$ is called a *K-filtered functor* if for any $k \in K$, $\mathcal{F}(\mathcal{A}_1^k)$ is a subcategory of $\mathcal{A}_2^k$.

Note that if we consider abelian categories and exact functors, we should require that the subcategories be Serre subcategories in order for the constructions to work nicely.

Consider a system $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ of categories with K-filtrations and K-filtered functors between them. We can define a full subcategory $\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ of $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$ such that there exists $k \in K$ for which $C_i \in \text{Fil}_k(\mathcal{C}_i)$ for any $i \geq 0$. The category $\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ is automatically a category with a K-filtration on objects. It is the inverse limit of the categories $\mathcal{C}_i$ in the (2, 1)-category of categories with K-filtrations on objects, and functors respecting these filtrations:

Example A.3.2. Consider the $\mathbb{Z}_+$-filtration on the objects of $\text{Rep}(\mathfrak{gl}_N)_{\text{poly}}$ where $S^\lambda \mathbb{C}^N$ lies in the component $|\lambda|$ of the filtration. The functors $\mathfrak{Res}_{n-1,n}$ respect this filtration, and we obtain a functor

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \varprojlim_{n\geq 0, \mathbb{Z}_+\text{-filtr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

One can show that this is an equivalence.

We have the following universal property, whose proof is straightforward:

Proposition A.3.3. *Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a system with a K-filtration as above, and let $\mathcal{A}$ be a category with a K-filtration, together with a set of K-filtered functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ such that for any $i \geq 1$ there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}.$$

Then $\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ is universal among such categories; that is, we have a functor

$$\begin{aligned} \mathcal{G} : \mathcal{A} &\to \varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i, \\ A &\mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1}), \\ f : A_1 \to A_2 &\mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+} \end{aligned}$$

which is obviously K-filtered and satisfies $\mathcal{G}_i \cong \mathrm{Pr}_i \circ \mathcal{G}$ for every $i \in \mathbb{Z}_+$.

A.4. ***Stabilizing inverse limit.*** Working in the setting of categories with K-filtrations and K-filtered functors, we consider the case when $\mathcal{A}$, $\{\mathcal{G}_i\}_{i\in\mathbb{Z}_+}$ satisfy the following stabilization condition (this is the case in Theorem 9.1.5):

Condition A.4.1. For every $k \in K$, there exists $i_k \in \mathbb{Z}_+$ such that $\mathcal{G}_j : \mathcal{A}^k \to \mathcal{C}_j^k$ is an equivalence of categories for any $j \geq i_k$.

In this setting, the following proposition holds:

Proposition A.4.2. *The functor $\mathcal{G} : \mathcal{A} \to \varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ is an equivalence of categories with K-filtrations.*

Proof. To prove that $\mathcal{G}$ is an equivalence of categories with K-filtrations, we need to show that

$$\mathcal{G} : \mathcal{A}^k \to \mathrm{Fil}_k\big(\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big)$$

is an equivalence of categories for any $k \in K$. Recall that

$$\mathrm{Fil}_k\big(\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big) \cong \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i^k.$$

By Condition A.4.1, for any $i > i_k$ we have a commutative diagram where all arrows are equivalences:

$$\begin{tikzcd} \mathcal{A}^k \arrow[r, "\mathcal{G}_i"] \arrow[d, "\mathcal{G}_{i_k}"'] & \mathcal{C}_i^k \arrow[dl, "\mathcal{F}_{i-1,i}"] \\ \mathcal{C}_{i-1}^k & \end{tikzcd}$$

Since for any fixed k, $\mathcal{F}_{i-1,i} : \mathcal{C}_i^k \to \mathcal{C}_{i-1}^k$ is an equivalence for $i > i_k$, it is obvious that $\mathrm{Pr}_i : \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i^k \to \mathcal{C}_i^k$ is an equivalence of categories for any $i > i_k$. Thus $\mathcal{G} : \mathcal{A}^k \to \mathrm{Fil}_k\big(\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big)$ is an equivalence of categories. $\square$

A.5. ***Equivalence of inverse limits.*** Finally, we provide a sufficient condition for the two notions of "special" inverse limit to coincide. This is the case in the setting of Theorem 9.1.5.

Let $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ be a system of finite-length categories with K-filtrations and shortening K-filtered functors, whose filtration components are Serre subcategories. We would like to give a sufficient condition on the K-filtration for the inverse limit of a system of categories with K-filtrations to coincide with the restricted inverse limit of these categories.

Recall that since the functors $\mathcal{F}_{i-1,i}$ are shortening, we have maps

$$f_{i-1,i} : \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_{i-1})$$

and we can consider the inverse limit $\varprojlim_{i \in \mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ of the sequence of sets $\mathrm{Irr}_*(\mathcal{C}_i)$ and maps $f_{i-1,i}$; we will denote by $\mathrm{pr}_j : \varprojlim_{i \in \mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_j)$ the projection maps.

Notice that the sets $\mathrm{Irr}_*(\mathcal{C}_i)$ have natural K-filtrations, and the maps $f_{i-1,i}$ respect these filtrations.

Proposition A.5.1. *Assume the following conditions hold*:

(1) *There exists a K-filtration on the set $\varprojlim_{i \in \mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$. That is, we require that for each L in $\varprojlim_{i \in \mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$, there exists $k \in K$ so that $\mathrm{pr}_i(L) \in \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i))$ for any $i \geq 0$. We would then say that such an object L belongs in the k-th filtration component of $\varprojlim_{i \in \mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$.*

(2) *Stabilization condition*: *For any $k \in K$, there exists $N_k \geq 0$ such that the map $f_{i-1,i} : \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i)) \to \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_{i-1}))$ is an injection for any $i \geq N_k$. That is, for any $k \in K$ there exists $N_k \in \mathbb{Z}_+$ such that the* (*exact*) *functor $\mathcal{F}_{i-1,i}$ is faithful for any $i \geq N_k$.*

Then the two full subcategories $\varprojlim_{i \in \mathbb{Z}_+, \mathrm{restr}} \mathcal{C}_i$ and $\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ of $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ coincide.

Proof. Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. As before, we denote by $\mathrm{JH}(C_j)$ the multiset of Jordan–Hölder components of C_j, and let $\mathrm{JH}_*(C_j) := \mathrm{JH}(C_j) \sqcup \{0\}$.

The first condition is natural: giving a K-filtration on the objects of $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is equivalent to giving a K-filtration on the simple objects of $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$, i.e., on the set $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$.

Assume $C \in \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. Let $n_0 \geq 0$ be such that $\ell_{\mathcal{C}_j}(C_j)$ is constant for $j \geq n_0$. Recall that we have

$$\mathrm{JH}_*(C) = \varprojlim_{i\in\mathbb{Z}_+} \mathrm{JH}_*(C_j).$$

Choose k such that all the elements of $\mathrm{JH}_*(C)$ lie in the k-th filtration component of $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$. This is possible due to the first condition.

Then for any $L_j \in \mathrm{JH}(C_j)$, we have that $L_j = \mathrm{pr}_j(L)$ for some $L \in \mathrm{JH}_*(C)$, and thus $L_j \in \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_j))$. We conclude that $C \in \mathrm{Fil}_k(\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i)$.

Thus the first condition of the theorem holds if and only if $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is a full subcategory of $\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i$.

Now, let $C \in \varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i$, and let $k \in K$ be such that $C \in \mathrm{Fil}_k(\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i)$. We would like to show that $\ell_{\mathcal{C}_i}(C_i)$ is constant starting from some i. Indeed, the second condition of the theorem tells us that there exists $N_k \geq 0$ such that the map

$$f_{i-1,i} : \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i)) \to \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_{i-1}))$$

is an injection for any $i \geq N_k$. We claim that for $i \geq N_k$, $\ell_{\mathcal{C}_i}(C_i)$ is constant. Indeed, if it weren't, then there would be some $i \geq N_k + 1$ and some $L_i \in \mathrm{JH}(C_i)$ such that $f_{i-1,i}(L_i) = 0$. But this is impossible, due to the requirement above. □

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