Pacific Journal of Mathematics

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JAE-CHEON JOO

Volume 285 No. 1 November 2016

PACIFIC JOURNAL OF MATHEMATICS
Vol. 285, No. 1, 2016
dx.doi.org/10.2140/pjm.2016.285.225

A GENERALIZATION OF THE GREENE–KRANTZ THEOREM FOR THE SEMICONTINUITY PROPERTY OF AUTOMORPHISM GROUPS

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We give a CR version of the Greene–Krantz theorem (*Math. Ann.* 261:4 (1982), 425–446) for the semicontinuity of complex automorphism groups. This is not only a generalization but also an intrinsic interpretation of the Greene–Krantz theorem.

1. Introduction

By upper semicontinuity, or simply semicontinuity, in geometry, we mean the property that the set of symmetries of a geometric structure should not decrease at a limit of a sequence of the structures. For instance, a sequence of ellipses in the Euclidean plane can converge to a circle, while a sequence of circles cannot converge to a noncircular ellipse. This property seems as natural as the second law of thermodynamics in physics, but we still need to make it clear in mathematical terminology. A symmetry for a geometric structure is described as a transformation on a space with the geometric structure. The set of transformations becomes a group with respect to the composition operator. Therefore, semicontinuity can be understood as a nondecreasing property of the transformation group at the limit of a sequence of geometric structures. One of the strongest descriptions of semicontinuity was obtained by Ebin for the Riemannian structures on compact manifolds in terms of conjugations by diffeomorphisms.

Theorem 1.1 [Ebin 1970]. *Let M be a C^∞-smooth compact manifold and let $\{g_j : j = 1, 2, \dots\}$ be a sequence of C^∞-smooth Riemannian structures which converges to a Riemannian metric g_0 in the C^∞ sense. Then for each sufficiently large j, there exists a diffeomorphism $\phi_j : M \to M$ such that $\phi_j \circ I_j \circ \phi_j^{-1}$ is a Lie subgroup of I_0, where I_j and I_0 represent the isometry groups for g_j and g_0, respectively.*

The group of holomorphic automorphisms on a complex manifold plays the role of the group of symmetries with respect to the complex structure. By Cartan's

MSC2010: 32M05, 32V05, 53C56.
Keywords: semicontinuity property, CR structures, CR Yamabe equation, automorphism groups.

theorem (cf. [Greene et al. 2011]), the automorphism group of a bounded domain in the complex Euclidean space turns out to be a Lie group with the compact-open topology on the domain. Greene and Krantz proved the following theorem for the semicontinuity property of automorphism groups of bounded strongly pseudoconvex domains.

Theorem 1.2 [Greene and Krantz 1982]. *Let Ω_j $(j = 1, 2, \ldots)$ and Ω_0 be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^∞-smooth boundary. Suppose that Ω_j converges to Ω_0 in the C^∞ sense, that is, there exists a diffeomorphism ψ_j defined on a neighborhood of $\bar{\Omega}_0$ into $\mathbb{C}^n$ such that $\psi_j(\Omega_0) = \Omega_j$ and $\psi_j \to \mathrm{Id}$ in the C^∞ sense on $\bar{\Omega}_0$. Then for every sufficiently large j, there exists a diffeomorphism $\phi_j : \Omega_j \to \Omega_0$ such that $\phi_j \circ \mathrm{Aut}(\Omega_j) \circ \phi_j^{-1}$ is a Lie subgroup of* $\mathrm{Aut}(\Omega_0)$.

Unlike the isometry group of a compact Riemannian manifold, the holomorphic automorphism group on a bounded strongly pseudoconvex domain can be noncompact, so the proof of Theorem 1.2 is divided into two cases: either $\mathrm{Aut}(\Omega_0)$ is compact or it is not. It turns out that the latter case is relatively simple, which is the case of deformations of the unit ball by the Wong–Rosay theorem [Rosay 1979; Wong 1977]. The main part of the proof of Theorem 1.2 is thus devoted to the case when $\mathrm{Aut}(\Omega_0)$ is compact. Greene and Krantz proved this case by constructing a compact Riemannian manifold (M, g_j) which includes Ω_j as a relatively compact subset and whose isometry group contains the automorphism group of Ω_j. Then Ebin's theorem yields the conclusion. The Riemannian manifold (M, g_j) is called a *metric double* of Ω_j.

The idea of this proof is applicable to more general cases. One reasonable generalization is to prove the semicontinuity property for a more general class of domains. In a recent paper [Greene et al. 2013], the authors generalized Theorem 1.2 to finitely differentiable cases. Greene and Kim [2014] proved that a partial generalization is also possible even for some classes of nonstrongly pseudoconvex domains. See also [Krantz 2010] for this line of generalization.

The aim of the present paper is to obtain another generalization of Theorem 1.2. According to Hamilton's theorem [1977; 1979], deformations of a bounded strongly pseudoconvex domain with C^∞-smooth boundary coincide with deformations of a complex structure on a given domain and they give rise to deformations of the CR structure of the boundary. Fefferman's extension theorem [1974] shows that every holomorphic automorphism on a bounded strongly pseudoconvex domain with C^∞-smooth boundary extends to a diffeomorphism up to the boundary and hence gives rise to a CR automorphism on the boundary. Conversely, a CR automorphism on the boundary extends to a holomorphic automorphism on the domain by the Bochner–Hartogs extension theorem. It is also known that the compact-open topology of the automorphism group of the domain coincides with the C^∞-topology of the

CR automorphism group of the boundary (cf. [Bell 1987]) if the holomorphic automorphism group of the domain is compact. In this observation, it is natural to think of the semicontinuity property for abstract strongly pseudoconvex CR manifolds under deformations of CR structures as a generalization of Theorem 1.2. We prove the following theorem for CR automorphism groups when the limit structure has a compact CR automorphism group.

Theorem 1.3. *Let $\{J_k : k = 1, 2, \dots\}$ be a sequence of C^∞-smooth strongly pseudoconvex CR structures on a compact differentiable manifold M of dimension $2n+1$ which converges to a C^∞-smooth strongly pseudoconvex CR structure J_0 on M in the C^∞ sense. Suppose that the CR automorphism group $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact. Then there exists $N > 0$ and a diffeomorphism $\phi_k : M \to M$ for each $k > N$ such that $\phi_k \circ \mathrm{Aut}_{\mathrm{CR}}(M, J_k) \circ \phi_k^{-1}$ is a Lie subgroup of $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$.*

According to Schoen's theorem [1995], $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact if and only if (M, J_0) is not CR equivalent to the sphere S^{2n+1} with the standard CR structure. One should notice that this condition is not necessary if $2n + 1 \geq 5$. Boutet de Monvel [1975] showed that a CR structure on M which is sufficiently close to the standard structure on S^{2n+1} is also embeddable in $\mathbb{C}^{n+1}$ if $2n+1 \geq 5$, in contrast with the 3-dimensional case (see [Burns and Epstein 1990; Lempert 1992; Nirenberg 1974; Rosay 1979]). Therefore, if $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is noncompact and $2n + 1 \geq 5$, then the situation is reduced to the case of deformations of the unit ball and follows immediately from Theorem 1.2.

The rest of this paper will be devoted to proving Theorem 1.3. Since we are thinking about abstract CR manifolds, we need to develop an intrinsic way of proving this. Therefore, the main interest of Theorem 1.3 is not only in the generalization but also in the intrinsic verification of the Greene–Krantz theorem. The main tool of the proof is the solution for the CR Yamabe problem about the construction of pseudohermitian structures with constant Webster scalar curvature, which is intensively studied in, for instance, [Cheng et al. 2014; Gamara 2001; Gamara and Yacoub 2001; Jerison and Lee 1987; 1989]. The subellipticity of the CR Yamabe equation turned out quite useful in obtaining estimates of derivatives of CR automorphisms in [Schoen 1995]. We make use of various solutions for the CR Yamabe problem — minimal solutions, local scalar flattening solutions and the blowing-up solutions given by the Green functions — developed in [Fischer-Colbrie and Schoen 1980; Jerison and Lee 1987; 1989; Schoen 1995].

2. Strongly pseudoconvex CR manifolds

In this section, we summarize fundamental facts on strongly pseudoconvex CR manifolds and pseudohermitian structures. The summation convention is always assumed.

CR and pseudohermitian structures. Let M be a smooth manifold of dimension $2n+1$ for some positive integer n. A *CR structure* on M is a smooth complex structure J on a subbundle H of the rank $2n$ of the tangent bundle TM which satisfies the integrability condition. More precisely, the restriction of J on a fiber H_p for a point $p \in M$ is an endomorphism $J_p : H_p \to H_p$ which satisfies $J_p \circ J_p = -\mathrm{Id}_{H_p}$, varying smoothly as p varies, and the bundle of i-eigenspace $H^{1,0}$ of J in the complexification $\mathbb{C} \otimes H$ satisfies the Frobenius integrability condition

$$[\Gamma(H^{1,0}), \Gamma(H^{1,0})] \subset \Gamma(H^{1,0}).$$

The subbundle H is called the *CR distribution* of J. A *CR automorphism* on M is a smooth diffeomorphism F from M onto itself such that $F_* H^{1,0} = H^{1,0}$. We denote by $\mathrm{Aut}_{\mathrm{CR}}(M)$ the group of all CR automorphisms on M. A CR structure is said to be *strongly pseudoconvex* if its CR distribution H is a contact distribution and for a contact form θ, the *Levi form* $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta(Z, \overline{W}) := -i\, d\theta(Z, \overline{W})$$

for $Z, W \in H^{1,0}$ is positive definite. It is known that the C^0-topology of $\mathrm{Aut}_{\mathrm{CR}}(M)$ coincides with the C^∞-topology for a compact strongly pseudoconvex CR manifold M if $\mathrm{Aut}_{\mathrm{CR}}(M)$ is compact with respect to the C^0-topology. See [Schoen 1995] for the proof.

We call a fixed contact form for the CR distribution of a strongly pseudoconvex CR structure a *pseudohermitian structure*. Let $\{W_\alpha : \alpha = 1, \dots, n\}$ be a local frame; that is, the W_α are sections of $H^{1,0}$ which form a pointwise basis for $H_{1,0}$. We call a collection of 1-forms $\{\theta^\alpha\}$ the *admissible coframe* of $\{W_\alpha\}$ if they are sections of $(H^{1,0})^*$ and satisfy

$$\theta^\alpha(W_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T) = 0,$$

where T is the vector field uniquely determined by

$$\theta(T) = 1, \quad T \lrcorner\, d\theta = 0,$$

which is called the *characteristic vector field* for θ. Let $g_{\alpha\bar\beta} = \mathcal{L}_\theta(W_\alpha, W_{\bar\beta})$. Then

$$d\theta = 2i g_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta},$$

where $\{\theta^\alpha\}$ is the admissible coframe for $\{W_\alpha\}$.

Theorem 2.1 [Webster 1978]. *There exist a local* 1*-form* $\omega = (\omega_\beta{}^\alpha)$ *and local functions* $A^\alpha{}_\beta$ *uniquely determined by*

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + A^\alpha{}_{\bar\beta}\, \theta \wedge \theta^{\bar\beta},$$
$$dg_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}, \quad A_{\alpha\beta} = A_{\beta\alpha}.$$

Here and in the sequel, we lower or raise an index by $(g_{\alpha\bar\beta})$ and $(g^{\alpha\bar\beta}) = (g_{\alpha\bar\beta})^{-1}$. A connection ∇ defined by

$$\nabla W_\alpha = \omega_\alpha{}^\beta \otimes W_\beta, \quad \nabla T = 0$$

is called the *pseudohermitian connection* or the *Webster connection* for θ. The functions $A^\alpha{}_\beta$ are called the coefficients of the *torsion tensor* $\boldsymbol{T}$. Let

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \equiv R_\alpha{}^\beta{}_{\gamma\bar\sigma}\, \theta^\gamma \wedge \theta^{\bar\sigma} \quad \bmod \theta,\ \theta^\gamma \wedge \theta^\sigma,\ \theta^{\bar\gamma} \wedge \theta^{\bar\sigma}.$$

We call $R_\alpha{}^\beta{}_{\gamma\bar\sigma}$ the coefficients of the *Webster curvature tensor* $\boldsymbol{R}$. Contracting indices, we obtain the coefficients $R_{\alpha\bar\beta}$ of the Webster Ricci curvature Ric and the Webster scalar curvature S:

$$R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}, \quad S = R_{\alpha\bar\beta}\, g^{\alpha\bar\beta}.$$

The norm of the Webster curvature $|\boldsymbol{R}|_\theta$ is defined by

$$|\boldsymbol{R}|_\theta^2 = \sum_{\alpha,\beta,\gamma,\sigma} |R_\alpha{}^\beta{}_{\gamma\bar\sigma}|^2,$$

where the frame is chosen so that $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. We similarly define the norm of the torsion tensor $|\boldsymbol{T}|_\theta$.

A pseudohermitian structure defines a sub-Riemannian structure. The distance function induced by a sub-Riemannian metric is called the *Carnot–Carathéodory distance* (cf. [Strichartz 1986]). We denote by $B_\theta(x, r)$ the Carnot–Carathéodory ball with respect to the pseudohermitian structure θ of radius $r > 0$ centered at $x \in M$.

The *Heisenberg group* $\mathcal{H}^n$ is a strongly pseudoconvex CR manifold $\mathbb{C}^n \times \mathbb{R}$ with the CR structure whose $H^{1,0}$ bundle is spanned by

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i z^{\bar\alpha} \frac{\partial}{\partial t}, \quad \alpha = 1, \dots, n, \tag{2-1}$$

where $(z, t) = (z^1, \dots, z^n, t)$ is the standard coordinate system of $\mathbb{C}^n \times \mathbb{R}$. It is well known that $\mathcal{H}^n$ is CR equivalent to the sphere in $\mathbb{C}^{n+1}$ minus a single point. If we put

$$\vartheta_0 = dt - i z^{\bar\alpha}\, dz^\alpha + i z^\alpha\, dz^{\bar\alpha}, \tag{2-2}$$

then it turns out the curvature and torsion tensors vanish identically. The converse also follows from the solution of the Cartan equivalence problem.

Proposition 2.2. *If the curvature and the torsion tensors of a pseudohermitian manifold (M, θ) vanish identically, then the pseudohermitian structure of M is locally equivalent to that of $(\mathcal{H}^n, \vartheta_0)$. If we further assume that M is simply connected and complete in the sense that every Carnot–Carathéodory ball is relatively compact in M, then (M, θ) is globally equivalent to $(\mathcal{H}^n, \vartheta_0)$,* For a given pseudohermitian manifold (M, θ), we can extend the CR structure J to a smooth section of endomorphism $\hat{J}$ on TM by putting $\hat{J}(T) = 0$, where T is the characteristic vector field of θ. Let J_k, $k = 1, 2, \dots$, and J_0 be strongly pseudoconvex CR structures on M with CR distributions H_k and H_0, respectively. We say that J_k converges to J_0 in the C^l sense ($l = 0, 1, 2, \dots, \infty$), if there exist pseudohermitian structures θ_k and θ_0 for (M, J_k) and (M, J_0) such that $\theta_k \to \theta_0$ and $\hat{J}_k \to \hat{J}_0$ in the C^l sense as tensors on M.

Pseudoconformal change of structures and the CR Yamabe equation. Let (M, θ) be a $(2n+1)$-dimensional pseudohermitian manifold and let $\tilde{\theta} = e^{2f}\theta$ be a pseudoconformal change, where f is a smooth real-valued function. Let $\{\theta^\alpha\}$ be an admissible coframe for θ satisfying $d\theta = 2i g_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}}$. Then it turns out

$$\tilde{\theta}^\alpha = e^f(\theta^\alpha + i f^\alpha \theta), \quad \alpha = 1, \dots, n,$$

form an admissible coframe for $\tilde{\theta}$ which satisfies

$$d\tilde{\theta} = 2i g_{\alpha\bar{\beta}}\, \tilde{\theta}^\alpha \wedge \tilde{\theta}^{\bar{\beta}}.$$

Let $R_\alpha{}^\beta{}_{\gamma\bar{\sigma}}$ and $\widetilde{R_\alpha{}^\beta}{}_{\gamma\bar{\sigma}}$ be coefficients of the Webster curvatures for θ and $\tilde{\theta}$ evaluated in the coframes $\{\theta^\alpha\}$ and $\{\tilde{\theta}^\alpha\}$, respectively. Then they are related as

$$\begin{aligned}(2\text{-}3)\quad \widetilde{R_\alpha{}^\beta}{}_{\gamma\bar{\sigma}} = e^{-2f}\big\{ & R_\alpha{}^\beta{}_{\gamma\bar{\sigma}} - \delta_\alpha{}^\beta(f_{\gamma\bar{\sigma}} + f_{\bar{\sigma}\gamma}) - 2g_{\alpha\bar{\sigma}} f^\beta{}_\gamma - 2f_{\alpha\bar{\sigma}}\delta^\beta{}_\gamma \\ & -(f^\beta{}_\alpha + f_\alpha{}^\beta)g_{\gamma\bar{\sigma}} - 4(\delta_\alpha{}^\beta g_{\gamma\bar{\sigma}} + g_{\alpha\bar{\sigma}}\,\delta^\beta{}_\gamma)f^\lambda f_\lambda\big\},\end{aligned}$$

where $f_{\alpha\bar{\beta}}$, $f_\alpha{}^\beta$ and $f^\beta{}_\alpha$ are components of the second covariant derivatives of f of the pseudohermitian manifold (M, θ) (cf. Proposition 4.14 in [Joo and Lee 2015] for the more general case). Contracting indices, we obtain the following transformation formula for the Webster scalar curvatures:

$$(2\text{-}4)\qquad \widetilde{S} = e^{-2f}\big\{S + 2(n+1)\Delta_\theta f - 4n(n+1) f^\lambda f_\lambda\big\},$$

where $\Delta_\theta f = -(f_\alpha{}^\alpha + f_{\bar{\alpha}}{}^{\bar{\alpha}})$. The operator Δ_θ is called the *sublaplacian* for θ.

Let u be a positive smooth function on M defined by $u^{p-2} = e^{2f}$, where $p = 2 + 2/n$. Then (2-4) changes into the following nonlinear equation for u:

$$(2\text{-}5)\qquad L_\theta u := (b_n\Delta_\theta + S)u = \widetilde{S} u^{p-1},$$

where $b_n = 2 + 2/n$ (see [Jerison and Lee 1987; 1989; Lee 1986]). Equation (2-5) is called the *CR Yamabe equation* and the subelliptic linear operator L_θ is called the *CR Laplacian* for θ. The *CR Yamabe problem* is to find a positive smooth function u which makes $\widetilde{S}$ constant.

Let $A^\alpha{}_{\bar\beta}$ and $\widetilde{A^\alpha}{}_{\bar\beta}$ be the coefficients of the torsion tensors for θ and $\tilde\theta$ in the coframes $\{\theta^\alpha\}$ and $\{\tilde\theta^\alpha\}$, respectively. Then in turns out that

$$\widetilde{A^\alpha}{}_{\bar\beta} = e^{-2f}\big(A^\alpha{}_{\bar\beta} - i f^\alpha{}_{\bar\beta} + 2i f^\alpha f_{\bar\beta}\big). \tag{2-6}$$

See [Lee 1986] for details.

Folland–Stein spaces and subelliptic estimates. Roughly speaking, a *normal coordinate system* of a pseudohermitian manifold (M,θ) of dimension $2n+1$ is a local approximation by the standard pseudohermitian structure on the Heisenberg group $(\mathcal{H}^n,\theta_0)$. For $p\in M$, let $W_1,\dots,W_n$ be a local frame defined on a neighborhood V of p such that the coefficients of the Levi form for θ are given by $g_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}$. Such a frame is called a *unitary frame*. We denote by T the characteristic vector field for θ. Let (z,t) be the standard coordinates of $\mathcal{H}^n$ and let $|(z,t)|=(|z|^4+t^2)^{1/4}$ be the Heisenberg group norm. We define Z_α and θ_0 on $\mathcal{H}^n$ as (2-1) and (2-2).

Theorem 2.3 [Folland and Stein 1974]. *There is a neighborhood of the diagonal $\Omega\subset V\times V$ and a C^∞-smooth mapping $\Theta:\Omega\to\mathcal{H}^n$ satisfying*:

(a) *We have $\Theta(\xi,\eta)=-\Theta(\eta,\xi)=\Theta(\eta,\xi)^{-1}$. (In particular, $\Theta(\xi,\xi)=0$.)*

(b) *Let $\Theta_\xi(\eta)=\Theta(\xi,\eta)$. Then Θ_ξ is a diffeomorphism of a neighborhood Ω_ξ of ξ onto a neighborhood of the origin in $\mathcal{H}^n$. Denote by $y=(z,t)=\Theta(\xi,\eta)$ the coordinates of $\mathcal{H}^n$. Denote by O^k $(k=1,2,\dots)$ a C^∞ function f of ξ and y such that for each compact set $K\subset V$, there is a constant C_K with $f(\xi,y)\le C_K|y|^k$ (Heisenberg norm) for $\xi\in K$. Then we have the following approximation formula*:

$$(\Theta_\xi^{-1})^*\theta = \theta_0 + O^1dt + \sum_{\alpha=1}^n (O^2dz^\alpha + O^2dz^{\bar\alpha}),$$
$$(\Theta_\xi^{-1})^*(\theta\wedge d\theta^n) = (1+O^1)\theta_0\wedge d\theta_0^n,$$
$$\Theta_{\xi*}W_\alpha = Z_\alpha + O^1\mathcal{E}(\partial_z) + O^2\mathcal{E}(\partial_t),$$
$$\Theta_{\xi*}T = \partial/\partial t + O^1\mathcal{E}(\partial_z,\partial_t),$$
$$\Theta_{\xi*}\Delta_\theta = \Delta_{\theta_0} + \mathcal{E}(\partial_z) + O^1\mathcal{E}(\partial_t,\partial_z^2) + O^2\mathcal{E}(\partial_z\partial_t) + O^3\mathcal{E}(\partial_t^2).$$

Here $O^k\mathcal{E}$ indicates an operator involving linear combinations of the indicated derivatives with smooth coefficients in O^k, and we have used ∂_z to denote any of the derivatives $\partial/\partial z^\alpha$, $\partial/\partial z^{\bar\alpha}$.

The smooth map Θ_ξ is called the *Folland–Stein normal coordinates* centered at ξ with respect to the frame $\{W_\alpha\}$. (This coordinate system depends on the choice of local unitary frame. Another construction of pseudohermitian normal coordinates which does not depend on local frames is given in [Jerison and Lee 1989].) Here and in the sequel, we use the term *frame constants* to mean bounds on finitely many derivatives of the coefficients in the $O^k\mathcal{E}$ terms in Theorem 2.3.

Let V be an open neighborhood of a point $p \in M$ with a fixed local unitary frame $W_1, \dots, W_n$ and let U be a relatively compact open neighborhood of p in V such that Ω_ξ in Theorem 2.3 contains $\bar{U}$ for every $\xi \in \bar{U}$. Let $X_\alpha = \operatorname{Re} W_\alpha$ and $X_{\alpha+n} = \operatorname{Im} W_\alpha$ for $\alpha = 1, \dots, n$. For a multi-index $A = (\alpha_1, \dots, \alpha_k)$, with $1 \le \alpha_j \le 2n$, $j = 1, \dots, k$, we denote k by $\ell(A)$ and write $X^A f = X_{\alpha_1} \cdots X_{\alpha_k} f$ for a smooth function f on U. The $S^p_k(U)$-norm of a smooth function f on U is

$$\|f\|_{S^p_k(U)} = \sup_{\ell(A) \le k} \|X^A f\|_{L^p(U)},$$

where $\|g\|_{L^p(U)} = \left(\int_U |g|^p \,\theta \wedge d\theta^n\right)^{1/p}$ is the L^p-norm of g on U with respect to the volume element induced by θ. The completion of $C^\infty_0(U)$ with respect to $\|\cdot\|_{S^p_k(U)}$ is denoted by $S^p_k(U)$.

Hölder type spaces suited to Δ_θ are defined as follows. For $x, y \in U$, let $\rho(x, y) = |\Theta(x, y)|$ (Heisenberg norm). For a positive real number $0 < s < 1$,

$$\Gamma_s(U) = \{f \in C^0(\bar{U}) : |f(x) - f(y)| \le C\rho(x, y)^s \text{ for some constant } C > 0\}.$$

If s is a positive nonintegral real number such that $k < s < k + 1$ for some integer $k \ge 1$, then

$$\Gamma_s(U) = \{f \in C^0(\bar{U}) : X^A f \in \Gamma_{s-k}(U),\ \ell(A) \le k\}.$$

Then the $\Gamma_s(U)$-norm for $f \in \Gamma_s(U)$ is defined by

$$\|f\|_{\Gamma_s(U)} = \sup_{x\in U} |f(x)| + \sup\left\{\frac{|X^A f(x) - X^A f(y)|}{\rho(x, y)^{s-k}} : x, y \in U,\ x \ne y,\ \ell(A) \le k\right\}.$$

The function spaces $S^p_k(U)$ and $\Gamma_s(U)$ are called the *Folland–Stein spaces* on U. We denote by $\Lambda_s(U)$ the Euclidean Hölder space when we regard U as a subset of $\mathbb{R}^{2n+1}$.

Theorem 2.4 [Folland and Stein 1974]. *For each positive real number s which is not an integer*, *each $1 < r < \infty$ and each integer $k \ge 1$*, *there exists a constant $C > 0$ such that for every $f \in C^\infty_0(U)$*,

(a) $\|f\|_{\Gamma_s(U)} \le C\|f\|_{S^r_k(U)}$, *where* $1/r = (k - s)/(2n + 2)$,

(b) $\|f\|_{\Lambda_{s/2}(U)} \le C\|f\|_{\Gamma_s(U)}$,

(c) $\|f\|_{S^r_2(U)} \le C(\|\Delta_\theta f\|_{L^r(U)} + \|f\|_{L^r(U)})$,

(d) $\|f\|_{\Gamma_{s+2}(U)} \le C(\|\Delta_\theta f\|_{\Gamma_s(U)} + \|f\|_{\Gamma_s(U)})$.

Moreover the constant C depends only on frame constants.

One should notice that the constants C in the theorem above depend on frame constants rather than the pseudohermitian structure itself. Therefore, if $\mathcal{U}$ is a small neighborhood (J_0, θ_0) in the C^∞-topology, then we can choose constants C in Theorem 2.4 which are independent of the choice of $(J, \theta) \in \mathcal{U}$.

If M is compact, we can choose a finite open covering $U_1, \dots, U_m$, each of which is contained in a normal coordinate. Let $\phi_1, \dots, \phi_m$ be a partition of unity subordinate to this covering. Then the spaces of $S_k^p(M)$ and $\Gamma_s(M)$ are defined as spaces of a function u such that $\phi_j u \in S_k^p(U_j)$ or $\phi_j u \in \Gamma_s(U_j)$, respectively, for every $j = 1, \dots, m$.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the following fundamental fact about the semicontinuity property of compact group actions proved by Ebin [1970] for Theorem 1.1. We denote by Diff(M) the group of C^∞-smooth diffeomorphisms. Recall that the C^∞-topology on Diff(M) is metrizable. We denote a metric inducing the C^∞-topology by d.

Theorem 3.1 ([Ebin 1970]; cf. [Greene et al. 2011; 2013; Grove and Karcher 1973; Kim 1987]). *Let M be a compact C^∞-smooth manifold and let G_k $(k = 1, 2, \dots)$ and G_0 be compact subgroups of* Diff(M). *Suppose $G_j \to G_0$ in the C^∞-topology as $j \to \infty$; that is, for every $\epsilon > 0$, there exists an integer N such that $d(f, G_0) := \inf_{g \in G_0} d(f, g) < \epsilon$ for every $f \in G_j$, whenever $j > N$. Then G_j is isomorphic to a subgroup of G_0 for every sufficiently large j. Moreover, the isomorphism can be obtained by the conjugation by a diffeomorphism ϕ_j of M which converges to the identity map in the C^∞ sense.*

Therefore, it suffices to prove the following proposition for the conclusion of Theorem 1.3.

Proposition 3.2. *Let $\{J_k : k = 1, 2, \dots\}$ be a sequence of strongly pseudoconvex CR structures on a compact manifold M which tends to a strongly pseudoconvex CR structure J_0 as in Theorem 1.3. Suppose that* $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ *is compact. Then* $\mathrm{Aut}_{\mathrm{CR}}(M, J_k)$ *is also compact for every sufficiently large k. Furthermore, every sequence* $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k) : k = 1, 2, \dots\}$ *admits a subsequence converging to an element* $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ *in the C^∞ sense.*

We will make use of the solutions of the CR Yamabe problem for the proof of Proposition 3.2. According to the variational approach introduced by Jerison and Lee [1987; 1989], it is very natural to consider the sign of the CR Yamabe invariant defined as follows: Let (M, θ) be a compact pseudohermitian manifold. For a C^∞-smooth real-valued function u, let

$$A(\theta; u) := \int_M u\, L_\theta u\, \theta \wedge d\theta^n = \int_M (b_n |du|_\theta^2 + R u^2)\, \theta \wedge d\theta^n$$

and

$$B(\theta; u) := \int_M |u|^p \,\theta \wedge d\theta^n.$$

Then the *CR Yamabe invariant* $Y(M)$ is defined by

$$Y(M) := \inf\{A(\theta; u) : u \in C^\infty(M),\ B(\theta; u) = 1\}.$$

It is well known that $Y(M)$ does not depend on the choice of contact form θ. Let J_k be a sequence of strongly pseudoconvex CR structures on M tending to a strongly pseudoconvex CR structure J_0 as $k \to \infty$. We denote by Y_k the CR Yamabe invariant of (M, J_k). For the proof, we may assume either that $Y_k \le 0$ for every k or that $Y_k > 0$ for every k.

Case $Y_k \le 0$. In this case, we use the minimal solution of the Yamabe problem.

Theorem 3.3 [Jerison and Lee 1987]. *Let M be a compact strongly pseudoconvex CR manifold of dimension $2n+1$.*

(i) $Y(S^{2n+1}) > 0$, *where $Y(S^{2n+1})$ is the CR Yamabe invariant for the sphere S^{2n+1} with the standard structure.*

(ii) $Y(M) \le Y(S^{2n+1})$.

(iii) *If $Y(M) < Y(S^{2n+1})$, then there exists a positive C^∞-smooth function u which satisfies $B(\theta; u) = 1$ and $A(\theta; u) = Y(M)$ for a given pseudohermitian structure θ. This function u satisfies*

$$L_\theta u = Y(M) u^{p-1}.$$

That is, the pseudohermitian structure $\tilde{\theta} = u^{p-2}\theta$ has a constant Webster scalar curvature $\widetilde{R} = Y(M)$.

It is known from [Jerison and Lee 1989] that $Y(M) < Y(S^{2n+1})$ if M is not locally spherical and $2n+1 \ge 5$. The cases that $2n+1 = 3$ or that M is spherical are dealt with in [Gamara 2001; Gamara and Yacoub 2001].

Proposition 3.4 [Jerison and Lee 1987, Theorem 7.1]. *If $Y(M) \le 0$, then a pseudohermitian structure with constant Webster scalar curvature is unique up to constant multiples. As a consequence, there is a unique pseudohermitian structure with constant Webster scalar curvature under the unit volume condition, if $Y(M) \le 0$.*

Proposition 3.5 [Jerison and Lee 1987, Theorem 5.15]. *Let M be a compact strongly pseudoconvex CR manifold of dimension $2n+1$ and let θ be a pseudohermitian structure. Suppose that $f, g \in C^\infty(M)$, $u \ge 0$, $u \in L^r$ for some $r > p = 2 + 2/n$ and*

$$\Delta_\theta u + gu = f u^{q-1}$$

in the distribution sense for some $2 \le q \le p$. *Then* $u \in C^\infty(M)$, $u > 0$. *Furthermore*, $\|u\|_{C^k}$ *depends only on* $\|u\|_{L^r}$, $\|f\|_{C^k}$, $\|g\|_{C^k}$ *and frame constants*, *but not on* q.

Indeed, a local version of the above lemma is stated in [Jerison and Lee 1987]. But it is obvious it holds globally by taking a partition of unity subordinate to a chart of normal coordinates.

Proposition 3.6 [Jerison and Lee 1987, Proposition 5.5, case $k = 1$, $r = 2$ and $s = p$]. *For a compact pseudohermitian manifold* (M, θ) *of dimension* $2n+1$, *there exists a constant* $C > 0$ *such that*

$$\int_M |v|^p \,\theta \wedge d\theta^n \le C \int_M (|dv|_\theta^2 + |v|^2)\,\theta \wedge d\theta^n$$

for every C^∞*-smooth function* v *on* M.

Since we are considering CR structures converging to the target structure J_0, we can choose also a sequence $\{\theta_k\}$ of contact forms which tends to a target pseudohermitian structure θ_0 in the C^∞ sense. Without loss of generality, we always assume that $\int_M \theta_k \wedge d\theta_k^n = 1$ for every k.

Lemma 3.7. *Suppose that* $Y_k \le 0$ *for every* k. *Let* $u_k > 0$ *be the* (*unique*) *solution as in Theorem 3.3*(*iii*) *with respect to* (J_k, θ_k). *Then for each nonnegative integer* l, *there exists a constant* C *such that* $\|u_k\|_{C^l} \le C$ *for every* k.

Proof. Since u_k satisfies

$$b_n \Delta_{\theta_k} u_k + R_k u_k = Y_k u_k^{p-1}, \tag{3-1}$$

where R_k is the Webster scalar curvature for θ_k, we have

$$\int_M \tfrac{1}{2}(p-1)\, b_n u_k^{p-2} |du_k|_{\theta_k} \,\theta_k \wedge d\theta_k^n \le \int_M |R_k u_k^p|\,\theta_k \wedge d\theta_k^n$$

by integrating after multiplying by u_k^{p-1} on both sides of (3-1), since $Y_k \le 0$. Therefore, the function $w_k := u_k^{p/2}$ satisfies

$$\int_M |dw_k|_{\theta_k}^2 \,\theta_k \wedge d\theta_k^n \le C \int_M w_k^2 \,\theta_k \wedge d\theta_k^n = C \int_M u_k^p \,\theta_k \wedge d\theta_k^n = C,$$

since R_k is bounded uniformly for k. Moreover since $(J_k, \theta_k) \to (J_0, \theta_0)$ in the C^∞ sense, Proposition 3.6 implies that there exists a constant $C > 0$ independent of k such that

$$\int_M w_k^p \,\theta_k \wedge d\theta_k^n \le C \int_M (|dw_k|_{\theta_k}^2 + w_k^2)\,\theta_k \wedge d\theta_k^n,$$

which is uniformly bounded for every k. This implies that $\|u_k\|_{L^r}$ is uniformly bounded as $(J_k, \theta_k) \to (J_0, \theta_0)$, where $r = \frac{1}{2}p^2 > p$. Then the conclusion follows from Proposition 3.5, since frame constants for (J_k, θ_k) are also uniformly bounded as $(J_k, \theta_k) \to (J_0, \theta_0)$ in the C^∞ sense. □

If $Y_k \le 0$ for every $k \ge 1$, then by taking a subsequence, we may assume the sequence $\{u_k\}$ of solutions of the Yamabe problem with respect to (J_k, θ_k) converges to u_0, the solution of the Yamabe problem with respect to (J_0, θ_0) in the C^∞ sense by Lemma 3.7. Replacing θ_k by $u_k^{p-2}\theta_k$, then we may assume the Webster scalar curvature of θ_k is a nonpositive constant for every k. In this case, it is known that the CR automorphism group of (M, J_k) coincides with the pseudohermitian automorphism group for (M, J_k, θ_k). Let g_k be the Riemannian metric on M defined by

$$g_k = \theta_k \otimes \theta_k + d\theta_k(\,\cdot\,, J_k\,\cdot\,)$$

for each k. Then we see that $g_k \to g_0 = \theta_0 \otimes \theta_0 + d\theta_0(\,\cdot\,, J_0\,\cdot\,)$ in the C^∞ sense, and the CR automorphism groups $\mathrm{Aut}_{\mathrm{CR}}(M, J_k)$ and $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ are subgroups of the isometry groups of g_k and g_0, respectively. Then the conclusion follows from the proof of Theorem 1.1.

Case $Y_k > 0$. We will show that if a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$ is divergent, then it generates a single "bubble" which is CR equivalent to (M, J_0). This case should be excluded by proving the CR structure of the bubble is the same as that of the standard sphere, which contradicts the hypothesis that $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact. An essential ingredient for analyzing the bubbling phenomenon is the reparametrization of the pseudohermitian structure by the Green function of the CR Laplacian. The existence of the Green function is guaranteed by the hypothesis $Y_k > 0$ (see, for instance, [Cheng et al. 2014; Gamara 2001]). We discuss the bubbling after the following fundamental lemma on the convergence of CR automorphisms.

Lemma 3.8. *Suppose for a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$, $F_k \to F$ and $F_k^{-1} \to G$ in the C^0 sense for some continuous mappings F and G. Then $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$, $G = F^{-1}$ and $F_k \to F$ in the C^∞ sense.*

Proof. This lemma is a sequential version of Proposition 1.1′ in [Schoen 1995]. Let θ_k and θ_0 be pseudohermitian structures for J_k and J_0, respectively, and suppose $\theta_k \to \theta_0$ in the C^∞ sense. For a given point $p \in M$, let $q_k = F_k(p)$ and $q = F(p)$. Let $q \in \widetilde{U} \Subset \widetilde{V} \Subset \widetilde{W}$ be relatively compact neighborhoods of q. Since $q_k \to q$, we can assume that $q_k \in \widetilde{U}$ for every k. The fact that $Y_k > 0$ implies that the principal eigenvalue of L_{θ_k} on M, and hence the Dirichlet principal eigenvalue of L_{θ_k} on $\widetilde{W}$, is also positive for every k. Then by the local scalar flattening argument of Fischer-Colbrie and Schoen [1980; 1995], we have a positive C^∞-smooth function u_k on $\widetilde{W}$ such that $L_{\theta_k} u_k = 0$ on $\widetilde{W}$ for every k. Multiplying by a positive constant, we may assume that $u_k(q) = 1$ for every k. Then the subelliptic theory in Theorem 2.3 for the sublaplacian and the Harnack principle (cf. Proposition 5.12 in [Jerison and Lee 1987]) imply that $\{u_k\}$ has a convergent subsequence which tends to a positive function u_0 on the closure of $\widetilde{V}$ in the C^∞ sense. We denote the convergent subsequence by $\{u_k\}$ again. Then $\tilde{\theta}_k = u_k^{p-2}\theta_k$ and $\tilde{\theta}_0 = u_0^{p-2}\theta_0$ have the trivial

Webster scalar curvatures on $\widetilde{V}$. From the equicontinuity of the sequence $\{F_k\}$, we can choose a neighborhood W of p such that $F_k(W) \Subset \widetilde{U}$ for every k. Let v_k be a positive smooth function on V defined by $F_k^*\tilde{\theta}_k = v_k^{p-2}\theta_k$. Then for every k, we have

$$L_{\theta_k} v_k = 0 \quad \text{on } W. \tag{3-2}$$

We denote by $\mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U})$ the volume of $\widetilde{U}$ with respect to the volume form $\tilde{\theta}_k \wedge d\tilde{\theta}_k^n$. Since $\tilde{\theta}_k$ converges to $\tilde{\theta}_0$ in the C^∞ sense in $\widetilde{V}$, there exists a uniform bound C of $\mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U})$. Therefore, it turns out that

$$\int_W v_k^p\, \theta_k \wedge d\theta_k^n = \int_W F_k^*(\tilde{\theta}_k \wedge d\tilde{\theta}_k) = \mathrm{Vol}_{\tilde{\theta}_k}(F_k(W)) \le \mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U}) \le C$$

for every k. Fix a neighborhood $V \Subset W$ of p. Then the subelliptic mean-value inequality for (3-2) implies that there exists a constant C such that $v_k(x) \le C$ for every $x \in V$. We can also choose this C independently on k by the convergence of structures. Then for a given neighborhood $U \Subset V$ of p and for each positive integer l, there exists a constant C_l which is independent of k such that

$$\|v_k\|_{C^l(U)} \le C_l$$

for every k, by Theorem 2.3. Since each F_k is pseudoconformal, the C^l-norm of F_k on U is completely determined by that of v_k and is uniformly bounded on U. This yields that every subsequence of $\{F_k\}$ contains a subsequence converging in the C^l sense, for every positive integer l. Since F_k converges to F in the C^0 sense on M and since M is compact, we conclude that F_k converges to F in the C^∞ sense. By the same reasoning, $F_k^{-1} \to G$ in the C^∞ sense. It follows immediately that $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ and $G = F^{-1}$. □

For a CR diffeomorphism $F : (M, \theta) \to (\widetilde{M}, \tilde{\theta})$ between two pseudohermitian manifolds, we denote by $|F'|_{\theta,\tilde{\theta}}$ the pseudoconformal factor of F, that is, $F^*\tilde{\theta} = |F'|_{\theta,\tilde{\theta}}\,\theta$. We abbreviate it to $|F'|_\theta$ in case $(M, \theta) = (\widetilde{M}, \tilde{\theta})$.

Lemma 3.9. *Let (M, θ) and $(\widetilde{M}, \tilde{\theta})$ be pseudohermitian manifolds of the same dimension. Let K be a relatively compact subset of M and suppose that the Webster scalar curvature for $\tilde{\theta}$ vanishes on $\widetilde{M}$. Then there exist constants $r_0 > 0$ and $C > 0$ such that for every CR diffeomorphism F on a Carnot–Carathéodory ball $B_\theta(x, r)$ into M,*

$$B_{\tilde{\theta}}(F(x), C^{-1}\lambda r) \subset F(B_\theta(x, r)) \subset B_{\tilde{\theta}}(F(x), C\lambda r)$$

whenever $x \in K$ and $r \le \frac{1}{2} r_0$, where $\lambda = |F'|_{\theta,\tilde{\theta}}(x)$. The constant C depends only on r_0, K and uniform bounds of finite-order derivatives of the CR and pseudohermitian structures of (M, θ).

This lemma is a restatement of Proposition 2.1′(i) in [Schoen 1995], which is a consequence of the subelliptic Harnack principle.

To prove Proposition 3.2, assume the contrary. Then there exists a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$ such that $\sup_{x\in M} |F_k'|_{\theta_k}(x) \to \infty$ as $k \to \infty$, thanks to Lemma 3.8. Let $x_k \in M$ be a point of M with $|F_k'|_{\theta_k}(x_k) = \sup_{x\in M} |F_k'|_{\theta_k}(x)$. Extracting a subsequence, we assume that $x_k \to x_0 \in M$ and $F_k(x_k) \to z_0$ as $k \to \infty$. Choose $r > 0$ small enough that the Carnot–Carathéodory balls satisfy $B_{\theta_k}(x_k, r) \Subset B_{\theta_k}(x_k, 2r) \Subset U$ for each k, where U is a relatively compact neighborhood of x_0 in M, and $2r < r_0$ for r_0 given in Lemma 3.9.

Lemma 3.10. *There exists a subsequence $\{F_{k_j} : j = 1, 2, \dots\}$ of $\{F_k : k = 1, 2, \dots\}$ which admits a point $y_0 \in M$ such that for every compact subset K in $M \setminus \{y_0\}$, there exists $N > 0$ such that $K \subset F_{k_j}(B_{\theta_{k_j}}(x_{k_j}, 2r))$ if $k_j > N$. Moreover, for the subsequence, one can choose the point y_0 independently of $r > 0$ as $r \to 0$.*

Proof. Suppose for every $r > 0$, there exists no sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ such that $d(y_k, F_k(x_k)) > \epsilon$ for any given $\epsilon > 0$, where d is the sub-Riemannian distance induced from θ_0. Then it turns out every sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ converges to z_0. In this case, we just need to put $y_0 = z_0$.

Now suppose for some $r > 0$, there exists a sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ such that $d(y_k, F_k(x_k)) > \epsilon$ for infinitely many k for some $\epsilon > 0$. Extracting a subsequence, we may assume that $y_k \to y_0 \in M$ and $d(y_k, F_k(x_k)) > \epsilon$ for every k so that the sequence $\{F_k(x_k)\}$ is relatively compact in $M \setminus \{y_0\}$. Let G_k be the Green function for L_{θ_k} with pole at y_k. We normalize G_k by the condition $\min_{M\setminus\{y_k\}} G_k = 1$. Since each $G_k > 0$ and $L_{\theta_k} G_k = 0$ on $M \setminus \{y_k\}$, we may assume $\{G_k : k = 1, 2, \dots\}$ converges to a positive function G_0 on $M \setminus \{y_0\}$ in the local C^∞ sense, by extracting a subsequence if necessary. Let $\tilde\theta_k = G_k^{p-2}\theta_k$. Then $\tilde\theta_k$ is a pseudohermitian structure on $M \setminus \{y_k\}$ which is Webster scalar flat. Therefore, if we denote $\lambda_k = |F_k'|_{\theta_k,\tilde\theta_k}(x_k)$, then Lemma 3.9 implies that there exists a constant C independent of k such that

$$B_{\tilde\theta_k}(F_k(x_k), C^{-1}\lambda_k r) \subset F_k(B_{\theta_k}(x_k, r)) \subset B_{\tilde\theta_k}(F_k(x_k), C\lambda_k r).$$

Since $G_k \geq 1$ and $|F_k'|_{\theta_k}(x_k) \to \infty$, λ_k also tends to infinity as $k \to \infty$. Therefore, a relatively compact subset K in $M \setminus \{y_0\}$ should be included in $F_k(B_{\theta_k}(x_k, r))$ for every sufficiently large k, since $F_k(x_k)$ lies on a fixed relatively compact subset of $M \setminus \{y_0\}$ and $\tilde\theta_k \to \tilde\theta_0 = G_0^{p-2}\theta_0$ in the local C^∞-smooth sense on $M \setminus \{y_0\}$. Note that the choice of the sequence $\{y_k\}$ and y_0 still works for every $r' \leq r$. This yields the independence of y_0 on r as $r \to 0$. □

As a consequence of Lemma 3.10, it turns out that $M \setminus \{y_0\}$ is simply connected and complete with respect to the sub-Riemannian distance induced by $\tilde\theta_0$. In fact, any loop in $M\setminus\{y_0\}$ is contained in $F_k(B_{\theta_k}(x_k, 2r))$ for some large k by Lemma 3.10. Since $B_{\theta_k}(x_k, 2r)$ is simply connected if $r > 0$ is small enough and since F_k is a diffeomorphism, $F_k(B_{\theta_k}(x_k, 2r))$ is simply connected as well. Therefore, the given loop should be contractible. This shows that $M \setminus \{y_0\}$ is simply connected.

Extracting a subsequence, we assume that Lemma 3.10 holds for the entire sequence $\{F_k\}$. Choose $y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))$ which tends to y_0. Let v_k and f_k be real-valued functions on $B_{\theta_k}(x_k, 2r)$ defined by

$$v_k^{p-2} = |F_k'|_{\theta_k, \tilde{\theta}_k} = e^{2f_k},$$

where G_k is the normalized Green function for L_{θ_k} with pole at y_k which converges to a positive function G_0 in the local C^∞-smooth sense on $M \setminus \{y_0\}$ as $k \to \infty$, and $\tilde{\theta}_k = G_k^{p-2}\theta_k$. Since $L_{\theta_k} v_k = 0$, we see that there exists a constant C independent of k such that $|F_k'|_{\theta_k, \tilde{\theta}_k} \geq C\lambda_k$ on $B_{\theta_k}(x_k, r)$ by the Harnack principle, where $\lambda_k = |F_k'|_{\theta_k, \tilde{\theta}_k}(x_k)$. Let $\{Z_k \in \Gamma(H_k^{1,0})\}$ be a sequence of vector fields on U which tends to $Z_0 \in \Gamma(H_0^{1,0})$ as $k \to \infty$, where $H_k^{1,0}$ represents the $(1, 0)$-bundle with respect to J_k. Since $f_k = (1/n) \log v_k$, we have

$$Z_k f_k = \frac{Z_k v_k}{n v_k}$$

for every k. Since $L_{\theta_k} v_k = 0$ on $B_{\theta_k}(x_k, 2r)$, the subelliptic estimates in Theorem 2.4 imply that $Z_k f_k$ is uniformly bounded on $B_{\theta_k}(x_k, r)$ for every k. So is $\bar{Z}_k f_k$, and if W_k is another sequence of vector fields, then $Z_k W_k f_k$ and $Z_k \overline{W}_k f_k$ are all uniformly bounded on $B_{\theta_k}(x_k, r)$ as $k \to \infty$. Therefore, if we denote by $\boldsymbol{R}_k$ and $\widetilde{\boldsymbol{R}}_k$ the Webster curvature tensors for θ_k and $\tilde{\theta}_k$, respectively, then (2-3) implies that

$$|\widetilde{\boldsymbol{R}}_k|^2_{\tilde{\theta}_k}(F_k(x)) \leq C\lambda_k^{-2}\big\{|\boldsymbol{R}_k|^2_{\theta_k}(x) + A_k|\boldsymbol{R}_k|_{\theta_k}(x) + B_k\big\}$$

for every $x \in B_{\theta_k}(x_k, r)$, where A_k and B_k are some functions of the first and second covariant derivatives of f_k with respect to the pseudohermitian structure θ_k which are uniformly bounded on $B_{\theta_k}(x_k, r)$ as $k \to \infty$. Since $\lambda_k \to \infty$ and $|\boldsymbol{R}_k|_{\theta_k}$ is uniformly bounded on $B_{\theta_k}(x_k, r)$ for every k, it turns out that $|\widetilde{\boldsymbol{R}}_k|_{\tilde{\theta}_k} \to 0$ uniformly on every compact subset of $M \setminus \{y_0\}$ by Lemma 3.10. Therefore, we see that the pseudohermitian manifold $(M \setminus \{y_0\}, \tilde{\theta}_0)$ has trivial Webster curvature. A similar argument using (2-6) implies that the torsion tensor of $\tilde{\theta}_0$ is also trivial. Therefore, we can conclude that $(M \setminus \{y_0\}, \tilde{\theta}_0)$ is equivalent to the standard pseudohermitian structure of the Heisenberg group and therefore, (M, J_0) is CR equivalent to the sphere by the removable singularity theorem. This contradicts the hypothesis that $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact and hence yields the conclusion of Proposition 3.2.

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Received February 25, 2015. Revised January 15, 2016.

JAE-CHEON JOO
DEPARTMENT OF MATHEMATICS EDUCATION
INHA UNIVERSITY
100 INHARO
NAM-GU
INCHEON 22212
SOUTH KOREA
jcjoo91@inha.ac.kr

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Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

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PACIFIC JOURNAL OF MATHEMATICS

Volume 285 No. 1 November 2016

0030-8730(201611)285:1;1-Y