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**GRADIENT ESTIMATES  
FOR A NONLINEAR LICHNEROWICZ EQUATION  
UNDER GENERAL GEOMETRIC FLOW  
ON COMPLETE NONCOMPACT MANIFOLDS**

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# GRADIENT ESTIMATES FOR A NONLINEAR LICHNEROWICZ EQUATION UNDER GENERAL GEOMETRIC FLOW ON COMPLETE NONCOMPACT MANIFOLDS

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We study gradient estimates for positive solutions to the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + cu^{-\alpha}$$

under general geometric flow on complete noncompact manifolds, where  $\alpha, c$  are two real constants and  $\alpha > 0$ . As an application, we give the corresponding Harnack inequality.

## 1. Introduction

Recently, there has been active interest in the study of gradient estimates for partial differential equations on noncompact manifolds. Wu [2010] gave a local Li–Yau type gradient estimate for positive solutions to a general nonlinear parabolic equation

$$u_t = \Delta u - \nabla\varphi\nabla u - au \log u - qu$$

in  $M \times [0, \tau]$ , where  $a \in \mathbb{R}$ ,  $\varphi$  is a  $C^2$ -smooth function and  $q = q(x, t)$  is a function, which generalizes many previous well-known gradient estimates. Zhu [2011] investigated the fast diffusion equation

$$(1-1) \quad u_t = \Delta u^\alpha \quad (0 < \alpha < 1).$$

**Theorem 1.1** [Zhu 2011]. *Let  $M$  be a Riemannian manifold of dimension  $n \geq 2$  with  $\text{Ric } M \geq -k$  for some  $k \geq 0$ . Suppose that  $v = -(\alpha/(\alpha - 1))u^{\alpha-1}$  is any positive solution to (1-1) in  $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$ . Suppose also that  $v \leq \tilde{M}$  in  $Q_{R,T}$ . Then there exists a constant  $C = C(\alpha, M)$  such that*

$$\frac{|\nabla v|}{v^{1/2}} \leq C \tilde{M}^{1/2} \left( \frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right)$$

in  $Q_{\frac{R}{2}, \frac{T}{2}}$ .

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Later, Huang and Li [2014] studied the generalized equation

$$u_t = \Delta_f u^\alpha \quad (\alpha > 0)$$

on Riemannian manifolds and got some interesting gradient estimates, where  $f$  is a smooth function and  $\Delta_f$  is defined by

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

For the elliptic case, Zhang and Ma [2011] considered the equation

$$(1-2) \quad \Delta_f u + cu^{-\alpha} = 0 \quad (\alpha > 0)$$

on complete noncompact manifolds when the constant  $N$  is finite and the  $N$ -Bakry–Émery Ricci tensor is bounded from below, obtaining the following gradient estimate.

**Theorem 1.2** [Zhang and Ma 2011]. *Suppose  $(M, g)$  is a complete noncompact  $n$ -dimensional Riemannian manifold with  $N$ -Bakry–Émery Ricci tensor bounded from below by the constant  $-K =: -K(2R)$ , where  $R > 0$  and  $K(2R) > 0$  in the metric ball  $B_{2R}(p)$  around  $p \in M$ . Let  $u$  be a positive solution of (1-2). Then*

(1) *if  $c > 0$ , we have*

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq & \frac{(N+n)(N+n+2)c_1^2}{R^2} + \frac{(N+n)((N+n-1)c_1 + c_2)}{R^2} \\ & + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + 2(N+n)K, \end{aligned}$$

(2) *if  $c < 0$ , we have*

$$\begin{aligned} \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \leq & (A + \sqrt{A})|c| \left( \inf_{B_p(2R)} u \right)^{-\alpha-1} + \frac{(N+n)((N+n-1)c_1 + c_2)}{R^2} \\ & + \frac{(N+n)c_1^2}{R^2} \left( n + N + 2 + \frac{n+N}{2\sqrt{A}} \right) \\ & + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + \left( 2 + \frac{1}{\sqrt{A}} \right) (n+N)K, \end{aligned}$$

where  $A = (N+n)(\alpha+1)(\alpha+2)$  and  $c_1, c_2$  are absolute positive constants.

For interesting gradient estimates on manifolds with fixed metric, see [Chen and Chen 2009; 2010; Li 2005; Ma 2006; 2010; Zhao 2013; 2014].

However, in the above works, the authors considered gradient estimates for positive solutions to nonlinear equations on complete noncompact manifolds with fixed metric, so it is natural to ask how gradient estimates vary if the metric on a manifold evolves with time. In Perelman’s breakthrough work [2002] on the

Poincaré conjecture, the author showed the gradient estimate for the fundamental solution of the conjugate heat equation

$$\Delta u - Ru + \partial_t u = 0$$

under Ricci flow on a closed Riemannian manifold  $M$ , where  $R$  is the scalar curvature. Since then, a large amount work has been done to study gradient estimates along geometric flow for the solution of the nonlinear equation. Kuang and Zhang [2008] established the corresponding pointwise gradient estimate. For the heat equation under Ricci flow, Liu [2009] got first-order gradient estimates for its positive solutions and derived Harnack inequalities and second-order gradient estimates. Later, Sun [2011] extended it to general geometric flow.

Since gradient estimates often lead to Liouville type theorems and Harnack inequalities, which played an important role in the proof of the Poincaré conjecture, for nonlinear heat equations on manifolds, to get good control of suitable Harnack quantities (depending on nonlinear terms), one may need the key lower bound assumption about Ricci curvature. The results of Theorem 1.2 are about gradient estimates for the elliptic equation (1-2). In this paper, we will extend these results to the parabolic variant of the problem. Thus, we consider the equation

$$(1-3) \quad \frac{\partial u}{\partial t} = \Delta u + cu^{-\alpha}$$

on complete noncompact manifolds  $M$  with evolving metric, where  $\alpha, c$  are two real constants and  $\alpha > 0$ . The motivation for this paper is that (1-3) can be viewed as a simple parabolic Lichnerowicz equation. It is well known that the Lichnerowicz equation arises from the Hamiltonian constraint equation for the Einstein-scalar field. Since (1-3) contains a negative power nonlinearity, it is interesting to discuss gradient estimates for it.

We state our main results about (1-3) as follows.

**Theorem 1.3.** *Let  $(M, g(t))$  be a smooth one-parameter family of complete Riemannian manifolds evolving by*

$$(1-4) \quad \frac{\partial}{\partial t} g = 2h$$

*for  $t$  in some time interval  $[0, T]$ . Let  $M$  be complete under the initial metric  $g(0)$ . Given  $x_0 \in M$  and  $R > 0$ , let  $u$  be a positive solution to the nonlinear equation*

$$\frac{\partial u}{\partial t} = \Delta u + cu^{-\alpha}$$

*in the cube  $Q_{2R, T} := \{(x, t) \mid d(x, x_0, t) \leq 2R, 0 \leq t \leq T\}$ . Suppose that there exist constants  $K_1, K_2, K_3, K_4 \geq 0$  such that*

$$\text{Ric} \geq -K_1 g, \quad -K_2 g \leq h \leq K_3 g, \quad |\nabla h| \leq K_4$$

on  $Q_{2R,T}$ . Then for  $(x, t) \in Q_{R,T}$  and positive constants  $c_1, c_2$ ,

(1) if  $c < 0$  and for a positive constant  $\tilde{M}$ ,  $u^{-(\alpha+1)} \leq \tilde{M}$  for all  $(x, t) \in M \times [0, T]$ , we have

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq H_1 + H_2 + \frac{n}{\beta} \frac{1}{t},$$

where

$$H_1 = \frac{n}{\beta} \left( \frac{(n-1)(1 + \sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2 - c(\alpha+1)\tilde{M} + \frac{nc_1^2}{2\beta(1-\beta)R^2} \right),$$

$$H_2 = \left( \frac{n^2}{4\beta^2(1-\beta)^2} (2\beta K_1 + 2(1-\beta)K_3 - c(\beta+\alpha)(\alpha+1)\tilde{M} + \frac{3}{2}K_4)^2 + \frac{n^2}{\beta} ((K_2 + K_3)^2 + \frac{3}{2}K_4) \right)^{\frac{1}{2}},$$

(2) if  $c > 0$ , we have

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \tilde{H}_1 + \tilde{H}_2 + \frac{n}{\beta} \frac{1}{t},$$

where

$$\tilde{H}_1 = \frac{n}{\beta} \left( \frac{(n-1)(1 + \sqrt{K_1}R)c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2 + \frac{nc^2}{2\beta(1-\beta)R^2} \right),$$

$$\tilde{H}_2 = \left( \frac{n^2}{4\beta^2(1-\beta)^2} (2\beta K_1 + 2(1-\beta)K_3 + \frac{3}{2}K_4)^2 + \frac{n^2}{\beta} ((K_2 + K_3)^2 + \frac{3}{2}K_4) \right)^{\frac{1}{2}}.$$

Here  $0 < \beta < 1$ ,  $c_1, c_2, c_3$  are positive constants.

**Remark.** In fact, our result is the parabolic version of [Theorem 1.2](#) under the evolving metric.

Letting  $R \rightarrow \infty$ , we can get the following global gradient estimate for the nonlinear parabolic equation [\(1-3\)](#).

**Corollary 1.4.** *Let  $(M, g(0))$  be a complete noncompact Riemannian manifold without boundary, and suppose  $g(t)$  evolves by [\(1-4\)](#) for  $t \in [0, T]$  and satisfies*

$$\text{Ric} \geq -K_1g, \quad -K_2g \leq h \leq K_3g, \quad |\nabla h| \leq K_4.$$

If  $u$  is a positive solution to [\(1-3\)](#), then for  $(x, t) \in M \times [0, T]$ ,

(1) if  $c < 0$  and  $u^{-(\alpha+1)} \leq \tilde{M}$  for all  $(x, t) \in M \times [0, T]$ , we have

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \bar{H}_1 + H_2 + \frac{n}{\beta} \frac{1}{t},$$

where

$$\bar{H}_1 = \sqrt{c_3}K_2 - \frac{n}{\beta}c(\alpha+1)\tilde{M},$$

(2) if  $c > 0$ , we have

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \leq \widehat{H}_1 + \widetilde{H}_2 + \frac{n}{\beta} \frac{1}{t},$$

where  $\widehat{H}_1 = \sqrt{c_3}K_2$  and  $H_2, \widetilde{H}_2$  are the same as in [Theorem 1.3](#).

As an application, we get the following Harnack inequality.

**Theorem 1.5.** *Let  $(M, g(0))$  be a complete noncompact Riemannian manifold without boundary, and suppose  $g(t)$  evolves by (1-4) for  $t \in [0, T]$  and satisfies*

$$\text{Ric} \geq -K_1 g, \quad -K_2 g \leq h \leq K_3 g, \quad |\nabla h| \leq K_4.$$

Let  $u$  be a positive solution to (1-3) with  $u^{-(\alpha+1)} \leq \widetilde{M}$  for all  $(x, t) \in M \times (0, T]$ . Then for any points  $(x_1, t_1)$  and  $(x_2, t_2)$  on  $M \times (0, T]$  with  $0 < t_1 < t_2$ , we have the following Harnack inequality:

(1) if  $c < 0$ , we have

$$u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{n/\beta} e^{\varphi(x_1, x_2, t_1, t_2) + (\overline{H}_1 + H_2)(t_2 - t_1)},$$

(2) if  $c > 0$ , we have

$$u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{n/\beta} e^{\varphi(x_1, x_2, t_1, t_2) + (c\widetilde{M} + \widehat{H}_1 + \widetilde{H}_2)(t_2 - t_1)},$$

where  $\varphi(x_1, x_2, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt$  and  $\gamma$  is any spacetime path joining  $(x_1, t_1)$  and  $(x_2, t_2)$ .

## 2. Proof of [Theorem 1.3](#)

Let  $u$  be a positive solution to (1-3). Set  $w = \log u$ ; then  $w$  satisfies

$$(2-1) \quad w_t = \Delta w + |\nabla w|^2 + ce^{-w(\alpha+1)}.$$

**Lemma 2.1** [[Sun 2011](#)]. *Suppose the metric evolves by (1-4). Then, for any smooth function  $w$ , we have*

$$\frac{\partial}{\partial t} |\nabla w|^2 = -2h(\nabla w, \nabla w) + 2\nabla w \nabla w_t$$

and

$$\frac{\partial}{\partial t} \Delta w = \Delta \frac{\partial}{\partial t} w - 2h\nabla^2 w - 2\nabla w (\text{div } h - \frac{1}{2}\nabla(\text{tr}_g h)),$$

where  $\text{div } h$  is the divergence of  $h$ .

**Lemma 2.2.** *Assume  $(M, g(t))$  satisfies the hypotheses of [Theorem 1.3](#). We have*

$$\begin{aligned} \left(\Delta - \frac{\partial}{\partial t}\right)F &\geq -2\nabla w \nabla F + t \left( \frac{\beta}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 \right. \\ &\quad \left. + ((\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)} + 2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4) |\nabla w|^2 \right. \\ &\quad \left. - n \left( \frac{1}{\beta} (K_2 + K_3)^2 + \frac{3}{2}K_4 \right) \right) \\ &\quad + c(\alpha+1)e^{-w(\alpha+1)} F - \frac{F}{t}, \end{aligned}$$

where  $F = t(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t)$  and  $0 < \beta < 1$ .

*Proof.* Define  $F = t(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t)$ . It is well known that for the Ricci tensor, we have the Bochner formula:

$$\Delta|\nabla w|^2 \geq 2|\nabla^2 w|^2 + 2\nabla w \nabla(\Delta w) - 2K_1|\nabla w|^2.$$

Noting that

$$\begin{aligned} \Delta w_t &= (\Delta w)_t + 2h\nabla^2 w + 2\nabla w (\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)) \\ &= -(|\nabla w|^2)_t + c(\alpha+1)e^{-w(\alpha+1)}w_t + w_{tt} + 2h\nabla^2 w + 2\nabla w (\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)) \\ &= 2h(\nabla w, \nabla w) - 2\nabla w \nabla w_t + c(\alpha+1)e^{-w(\alpha+1)}w_t \\ &\quad + w_{tt} + 2h\nabla^2 w + 2\nabla w (\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)), \\ \Delta w &= -|\nabla w|^2 - ce^{-w(\alpha+1)} + w_t = \left(1 - \frac{1}{\beta}\right) \left(-ce^{-w(\alpha+1)} + w_t\right) - \frac{F}{\beta t}, \end{aligned}$$

we have

$$\begin{aligned} \Delta F &= t(\beta\Delta|\nabla w|^2 + c\Delta e^{-w(\alpha+1)} - \Delta w_t) \\ &= t(\beta\Delta|\nabla w|^2) + tc((\alpha+1)^2 e^{-w(\alpha+1)}|\nabla w|^2 - (\alpha+1)e^{-w(\alpha+1)}\Delta w) - t\Delta w_t \\ &\geq t \left( 2\beta|\nabla^2 w|^2 + 2\beta\nabla w \nabla(\Delta w) - 2K_1\beta|\nabla w|^2 + c(\alpha+1)^2 e^{-w(\alpha+1)}|\nabla w|^2 \right. \\ &\quad \left. - c(\alpha+1)e^{-w(\alpha+1)} \left( \left(1 - \frac{1}{\beta}\right) \left(-ce^{-w(\alpha+1)} + w_t\right) - \frac{F}{\beta t} \right) + (|\nabla w|^2)_t \right. \\ &\quad \left. - c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt} - 2h\nabla^2 w - 2\nabla w (\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)) \right) \\ &= t \left( 2\beta|\nabla^2 w|^2 - \frac{2}{t}\nabla w \nabla F + 2\beta\nabla w \nabla w_t - 2h(\nabla w, \nabla w) \right. \\ &\quad \left. + ((2\beta + \alpha - 1)c(\alpha+1)e^{-w(\alpha+1)} - 2K_1\beta)|\nabla w|^2 \right. \\ &\quad \left. + c^2(\alpha+1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} + c\left(\frac{1}{\beta} - 2\right)(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt} \right. \\ &\quad \left. - 2h\nabla^2 w - 2\nabla w (\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h)) + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta t} \right), \end{aligned}$$

and by [Lemma 2.1](#), we get

$$\begin{aligned} F_t &= (\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t) + t(\beta(|\nabla w|^2)_t - c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt}) \\ &= \frac{F}{t} + t(2\beta\nabla w\nabla w_t - 2\beta h(\nabla w, \nabla w) - c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt}). \end{aligned}$$

Therefore, it follows that

$$\begin{aligned} &\left(\Delta - \frac{\partial}{\partial t}\right)F \\ &\geq -2\nabla w\nabla F + t\left(2\beta|\nabla^2 w|^2 + ((2\beta + \alpha - 1)c(\alpha + 1)e^{-w(\alpha+1)})|\nabla w|^2\right. \\ &\quad \left.+ c^2(\alpha + 1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} - \frac{\beta-1}{\beta}c(\alpha + 1)e^{-w(\alpha+1)}w_t\right. \\ &\quad \left.- 2h\nabla^2 w + 2(\beta - 1)K_3|\nabla w|^2\right. \\ &\quad \left.- 2K_1\beta|\nabla w|^2 - 2\nabla w(\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h))\right) \\ &\quad + c(\alpha + 1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\ &= -2\nabla w\nabla F + t\left(2\beta|\nabla^2 w|^2\right. \\ &\quad \left.+ ((\beta - 1)c(\alpha + 1)e^{-w(\alpha+1)})\left(|\nabla w|^2 + \frac{ce^{-w(\alpha+1)}}{\beta} - \frac{1}{\beta}w_t\right)\right. \\ &\quad \left.+ (\beta + \alpha)c(\alpha + 1)e^{-w(\alpha+1)}|\nabla w|^2 + 2(\beta - 1)K_3|\nabla w|^2\right. \\ &\quad \left.- 2K_1\beta|\nabla w|^2 - 2h\nabla^2 w - 2\nabla w(\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h))\right) \\ &\quad + c(\alpha + 1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\ &= -2\nabla w\nabla F + t\left(2\beta|\nabla^2 w|^2 + (\beta + \alpha)c(\alpha + 1)e^{-w(\alpha+1)}|\nabla w|^2\right. \\ &\quad \left.+ (\beta - 1)c(\alpha + 1)e^{-w(\alpha+1)}\frac{F}{\beta t} + 2(\beta - 1)K_3|\nabla w|^2\right. \\ &\quad \left.- 2K_1\beta|\nabla w|^2 - 2h\nabla^2 w - 2\nabla w(\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h))\right) \\ &\quad + c(\alpha + 1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\ &= -2\nabla w\nabla F + t\left(2\beta|\nabla^2 w|^2 + (\beta + \alpha)c(\alpha + 1)e^{-w(\alpha+1)}|\nabla w|^2\right. \\ &\quad \left.+ 2(\beta - 1)K_3|\nabla w|^2 - 2K_1\beta|\nabla w|^2\right. \\ &\quad \left.- 2h\nabla^2 w - 2\nabla w(\operatorname{div} h - \frac{1}{2}\nabla(\operatorname{tr}_g h))\right) \\ &\quad + c(\alpha + 1)e^{-w(\alpha+1)}F - \frac{F}{t}. \end{aligned}$$



By our assumption, we have

$$-(K_2 + K_3)g \leq h \leq (K_2 + K_3)g,$$

which implies that

$$|h|^2 \leq (K_2 + K_3)^2 |g|^2 = n(K_2 + K_3)^2.$$

Applying those bounds and Young's inequality yields

$$|h \nabla^2 w| \leq \frac{\beta}{2} |\nabla^2 w|^2 + \frac{1}{2\beta} |h|^2 \leq \frac{\beta}{2} |\nabla^2 w|^2 + \frac{n}{2\beta} (K_2 + K_3)^2.$$

On the other hand,

$$|\operatorname{div} h - \frac{1}{2} \nabla(\operatorname{tr}_g h)| = |g^{ij} \nabla_i h_{jl} - \frac{1}{2} g^{ij} \nabla_l h_{ij}| \leq \frac{3}{2} |g| |\nabla h| \leq \frac{3}{2} \sqrt{n} K_4.$$

Finally, with the help of the inequality

$$|\nabla^2 w|^2 \geq \frac{1}{n} (\operatorname{tr} \nabla^2 w)^2 = \frac{1}{n} (\Delta w)^2 = \frac{1}{n} (|\nabla w|^2 + c(\alpha + 1)e^{-w(\alpha+1)} - w_t)^2,$$

we get

$$\begin{aligned} & \left( \Delta - \frac{\partial}{\partial t} \right) F \\ & \geq -2\nabla w \nabla F + t \left( \frac{\beta}{n} |\Delta w|^2 + (\beta + \alpha)c(\alpha + 1)e^{-w(\alpha+1)} |\nabla w|^2 + 2(\beta - 1)K_3 |\nabla w|^2 \right. \\ & \quad \left. - 2K_1 \beta |\nabla w|^2 - \frac{n}{\beta} (K_2 + K_3)^2 - 3\sqrt{n} K_4 |\nabla w| \right) \\ & \quad + c(\alpha + 1)e^{-w(\alpha+1)} F - \frac{F}{t}. \end{aligned}$$

Since

$$3\sqrt{n} K_4 |\nabla w| \leq 3K_4 \left( \frac{1}{2}n + \frac{1}{2} |\nabla w|^2 \right),$$

we have

$$\begin{aligned} & \left( \Delta - \frac{\partial}{\partial t} \right) F \\ & \geq -2\nabla w \nabla F + t \left( \frac{\beta}{n} (|\nabla w|^2 + c(\alpha + 1)e^{-w(\alpha+1)} - w_t)^2 \right. \\ & \quad \left. + ((\beta + \alpha)c(\alpha + 1)e^{-w(\alpha+1)} + 2(\beta - 1)K_3 - 2\beta K_1 - \frac{3}{2}K_4) |\nabla w|^2 \right. \\ & \quad \left. - n \left( \frac{1}{\beta} (K_2 + K_3)^2 + \frac{3}{2}K_4 \right) \right) \\ & \quad + c(\alpha + 1)e^{-w(\alpha+1)} F - \frac{F}{t}. \end{aligned}$$

This completes the proof of [Lemma 2.2](#). □

**Remark.** If the general geometric flow is Ricci flow, that is, if  $h = \text{const} \cdot \text{Ric } g$ , the term  $\text{div } h - \frac{1}{2} \nabla(\text{tr}_g h)$  in the above proof will vanish.

We take a  $C^2$  cutoff function  $\tilde{\varphi}$  defined on  $[0, \infty)$  such that  $\tilde{\varphi}(r) = 1$  for  $r \in [0, 1]$ ,  $\tilde{\varphi}(r) = 0$  for  $r \in [2, \infty)$ , and  $0 \leq \tilde{\varphi}(r) \leq 1$ . Furthermore  $\tilde{\varphi}$  satisfies

$$-\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{1/2}(r)} \leq c_1$$

and

$$\tilde{\varphi}''(r) \geq -c_2$$

for two constants  $c_1, c_2 > 0$ . Set

$$\varphi(x, t) = \tilde{\varphi}\left(\frac{r(x, t)}{R}\right),$$

where  $r(x, t) = d(x, x_0, t)$ . Using an argument of Calabi [1958], we can assume  $\varphi(x, t) \in C^2(M)$  with support in  $Q_{2R, T}$ . Direct calculation shows that on  $Q_{2R, T}$

$$(2-2) \quad \frac{|\nabla\varphi|^2}{\varphi} \leq \frac{c_1^2}{R^2}.$$

By the Laplacian comparison theorem in [Aubin 1982],

$$(2-3) \quad \Delta\varphi \geq -\frac{(n-1)(1 + \sqrt{K_1}R)c_1^2 + c_2}{R^2}.$$

For any  $0 < T_1 < T$ , let  $(x_0, t_0)$  be a point in the cube  $Q_{2R, T_1}$  at which  $\varphi F$  attains its maximum value. We can assume that this value is positive (otherwise the proof is trivial). At the point  $(x_0, t_0)$ , we have

$$\nabla(\varphi F) = 0, \quad \Delta(\varphi F) \leq 0, \quad (\varphi F)_t \geq 0.$$

It follows that

$$0 \geq \left(\Delta - \frac{\partial}{\partial t}\right)(\varphi F) = (\Delta\varphi)F - \varphi_t F + \varphi\left(\Delta - \frac{\partial}{\partial t}\right)F + 2\nabla\varphi\nabla F.$$

By [Sun 2011, p. 494], we know there exists a positive constant  $c_3$  such that

$$-\varphi_t F \geq -\sqrt{c_3}K_2 F.$$

So we obtain

$$\varphi\left(\Delta - \frac{\partial}{\partial t}\right)F + F\Delta\varphi - \varphi_t F - 2F\varphi^{-1}|\nabla\varphi|^2 \leq 0.$$

This inequality, together with the inequalities (2-2) and (2-3), yields

$$\varphi\left(\Delta - \frac{\partial}{\partial t}\right)F \leq AF,$$

where

$$A = \frac{((n-1)(1 + \sqrt{K_1 R}))c_1^2 + c_2 + 2c_1^2}{R^2} + \sqrt{c_3}K_2.$$

At  $(x_0, t_0)$ , by Lemma 2.2, we have

$$\begin{aligned} 0 &\geq \varphi \left( \Delta - \frac{\partial}{\partial t} \right) F - AF \\ &\geq -AF + \varphi \left( -\frac{F}{t_0} + c(\alpha+1)e^{-w(\alpha+1)}F + \frac{\beta t_0}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 \right. \\ &\quad - 2\nabla w \nabla F \\ &\quad \left. + ((\beta + \alpha)c(\alpha+1)e^{-w(\alpha+1)} + 2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4)|\nabla w|^2 t_0 \right. \\ &\quad \left. - n \left( \frac{1}{\beta}(K_2 + K_3)^2 + \frac{3}{2}K_4 \right) t_0 \right). \end{aligned}$$

(1) If  $c < 0$  and  $u^{-(\alpha+1)} \leq \tilde{M}$  for all  $(x, t) \in M \times [0, T]$ , we have

$$\begin{aligned} 0 &\geq -AF - \varphi \frac{F}{t_0} + \varphi \frac{\beta t_0}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 - 2\varphi \nabla w \nabla F \\ &\quad + c\tilde{M}(\alpha+1)\varphi F + ((\beta + \alpha)c(\alpha+1)\tilde{M} + 2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4)|\nabla w|^2 \varphi t_0 \\ &\quad - \left( \frac{1}{\beta}(K_2 + K_3)^2 + \frac{3}{2}K_4 \right) n\varphi t_0. \end{aligned}$$

Set

$$\tilde{C}_1 = -(\beta + \alpha)c(\alpha+1)\tilde{M} + 2(1 - \beta)K_3 + 2\beta K_1 + \frac{3}{2}K_4$$

and

$$\tilde{C}_2 = \frac{1}{\beta}(K_2 + K_3)^2 + \frac{3}{2}K_4.$$

Multiplying by  $\varphi t_0$  on both sides of the above inequality, we get

$$\begin{aligned} 0 &\geq -A\varphi t_0 F - \varphi F + 2t_0 F \varphi \nabla w \nabla \varphi + c\tilde{M}(\alpha+1)\varphi F t_0 \\ &\quad - \tilde{C}_1 |\nabla w|^2 \varphi^2 t_0^2 - \tilde{C}_2 n \varphi^2 t_0^2 + \varphi^2 t_0^2 \frac{\beta t_0}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 \\ &\geq \varphi F (-At_0 - 1 + c\tilde{M}(\alpha+1)t_0) - \frac{2c_1}{R} t_0 F \varphi^{3/2} |\nabla w| \\ &\quad + \frac{\beta t_0^2}{n} \left( \varphi^2 (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 - \frac{n}{\beta} \tilde{C}_1 \varphi^2 |\nabla w|^2 \right) - \tilde{C}_2 n t_0^2, \end{aligned}$$

where the last inequality used

$$-2\varphi \nabla w \nabla F = 2F \nabla w \nabla \varphi \geq -2F |\nabla w| |\nabla \varphi| \geq -\frac{2c_1}{R} \varphi^{1/2} F |\nabla w|.$$

Assume that  $y = \varphi|\nabla w|^2$  and  $z = \varphi(-ce^{-w(\alpha+1)} + w_t)$ . We have

$$0 \geq \varphi F(-At_0 - 1 + c\tilde{M}(\alpha+1)t_0) + \frac{\beta t_0^2}{n} \left( (y-z)^2 - \frac{n}{\beta} \tilde{C}_1 y - 2 \frac{nc_1}{R} y^{1/2} \left( y - \frac{z}{\beta} \right) \right) - \tilde{C}_2 n t_0^2.$$

Using the inequality  $ax^2 - bx \geq -b^2/(4a)$ , valid for  $a, b > 0$ , one obtains

$$\begin{aligned} & \frac{\beta t_0^2}{n} \left( (y-z)^2 - \frac{n}{\beta} \tilde{C}_1 y - 2 \frac{nc_1}{R} y^{1/2} \left( y - \frac{z}{\beta} \right) \right) \\ &= \frac{\beta t_0^2}{n} \left( \beta^2 \left( y - \frac{z}{\beta} \right)^2 + (1-\beta)^2 y^2 - \frac{n}{\beta} \tilde{C}_1 y + \left( 2(\beta-\beta^2)y - 2 \frac{nc_1}{R} y^{1/2} \right) \left( y - \frac{z}{\beta} \right) \right) \\ &\geq \frac{\beta t_0^2}{n} \left( \beta^2 \left( y - \frac{z}{\beta} \right)^2 - \frac{n^2 \tilde{C}_1^2}{4\beta^2(1-\beta)^2} - \frac{n^2 c_1^2}{2R^2(\beta-\beta^2)} \left( y - \frac{z}{\beta} \right) \right) \\ &= \frac{\beta}{n} (\varphi F)^2 - \frac{n \tilde{C}_1^2 t_0^2}{4\beta(1-\beta)^2} - \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)} (\varphi F). \end{aligned}$$

Hence,

$$\begin{aligned} & \frac{\beta}{n} (\varphi F)^2 + \left( -At_0 - 1 + c\tilde{M}(\alpha+1)t_0 - \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)} \right) (\varphi F) \\ & \quad - \frac{n \tilde{C}_1^2 t_0^2}{4\beta(1-\beta)^2} - \tilde{C}_2 n t_0^2 \leq 0. \end{aligned}$$

From the inequality  $Ax^2 - 2Bx \leq C$ , we have  $x \leq 2B/A + \sqrt{C/A}$ . We can get

$$\varphi F \leq \frac{n}{\beta} \left( At_0 + 1 - c\tilde{M}(\alpha+1)t_0 + \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)} \right) + \left( \frac{n}{\beta} \left( \frac{n \tilde{C}_1^2}{4\beta(1-\beta)^2} + \tilde{C}_2 n \right) \right)^{\frac{1}{2}} t_0.$$

If  $d(x, x_0, T_1) < R$ , we have  $\varphi(x, T_1) = 1$ . Then

$$\begin{aligned} & F(x, T_1) \\ &= T_1 (\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t) \\ &\leq \varphi F(x_0, t_0) \\ &\leq \frac{n}{\beta} \left( At_0 + 1 - c\tilde{M}(\alpha+1)t_0 + \frac{nc_1^2 t_0}{2R^2(\beta-\beta^2)} \right) + \left( \frac{n}{\beta} \left( \frac{n \tilde{C}_1^2}{4\beta(1-\beta)^2} + \tilde{C}_2 n \right) \right)^{\frac{1}{2}} t_0. \end{aligned}$$

As  $T_1$  is arbitrary, we can get case (1) of [Theorem 1.3](#).

(2) If  $c > 0$ , we have

$$0 \geq -AF + \varphi \left( -\frac{F}{t_0} + \frac{\beta t_0}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 - 2\nabla w \nabla F \right. \\ \left. + (2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4) |\nabla w|^2 t_0 - n \left( \frac{1}{\beta} (K_2 + K_3)^2 + \frac{3}{2}K_4 \right) t_0 \right).$$

Therefore,

$$0 \geq -AF - \varphi \frac{F}{t_0} + \varphi \frac{\beta t_0}{n} (|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t)^2 - 2\varphi \nabla w \nabla F \\ + (2(\beta-1)K_3 - 2\beta K_1 - \frac{3}{2}K_4) |\nabla w|^2 \varphi t_0 - \left( \frac{1}{\beta} (K_2 + K_3)^2 + \frac{3}{2}K_4 \right) n \varphi t_0.$$

Similarly, we can get case (2) of [Theorem 1.3](#). This completes the proof of [Theorem 1.3](#).  $\square$

*Proof of Theorem 1.5.* For any points  $(x_1, t_1)$  and  $(x_2, t_2)$  on  $M \times (0, T]$  with  $0 < t_1 < t_2$ , we take a curve  $\gamma(t)$  parametrized with  $\gamma(t_1) = x_1$  and  $\gamma(t_2) = x_2$ . One gets from [Corollary 1.4](#):

(1) If  $c < 0$ , we have

$$\log u(x_2, t_2) - \log u(x_1, t_1) \\ = \int_{t_1}^{t_2} ((\log u)_t + \langle \nabla \log u, \dot{\gamma} \rangle) dt \\ \geq \int_{t_1}^{t_2} \left( \beta |\nabla \log u|^2 - \frac{n}{\beta t} - cu^{-(\alpha+1)} - \bar{H}_1 - H_2 - |\nabla \log u| |\dot{\gamma}| \right) dt \\ \geq - \int_{t_1}^{t_2} \left( \frac{1}{4\beta} |\dot{\gamma}|^2 + \frac{n}{\beta t} + \bar{H}_1 + H_2 \right) dt \\ = - \left( \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt + \log \left( \frac{t_2}{t_1} \right)^{n/\beta} + (\bar{H}_1 + H_2)(t_2 - t_1) \right),$$

which means that

$$\log \frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt + \log \left( \frac{t_2}{t_1} \right)^{n/\beta} + (\bar{H}_1 + H_2)(t_2 - t_1).$$

Therefore,

$$u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{n/\beta} e^{\varphi(x_1, x_2, t_1, t_2) + (\bar{H}_1 + H_2)(t_2 - t_1)},$$

where  $\varphi(x_1, x_2, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt$ .

(2) If  $c > 0$ , we have

$$\begin{aligned} & \log u(x_2, t_2) - \log u(x_1, t_1) \\ &= \int_{t_1}^{t_2} ((\log u)_t + \langle \nabla \log u, \dot{\gamma} \rangle) dt \\ &\geq \int_{t_1}^{t_2} \left( \beta |\nabla \log u|^2 - \frac{n}{\beta t} - cu^{-(\alpha+1)} - \hat{H}_1 - \tilde{H}_2 - |\nabla \log u| |\dot{\gamma}| \right) dt \\ &\geq - \int_{t_1}^{t_2} \left( \frac{1}{4\beta} |\dot{\gamma}|^2 + \frac{n}{\beta t} + c\tilde{M} + \hat{H}_1 + \tilde{H}_2 \right) dt \\ &= - \left( \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt + \log \left( \frac{t_2}{t_1} \right)^{n/\beta} + (c\tilde{M} + \hat{H}_1 + \tilde{H}_2)(t_2 - t_1) \right), \end{aligned}$$

which means that

$$\log \frac{u(x_1, t_1)}{u(x_2, t_2)} \leq \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt + \log \left( \frac{t_2}{t_1} \right)^{n/\beta} + (c\tilde{M} + \hat{H}_1 + \tilde{H}_2)(t_2 - t_1).$$

Therefore,

$$u(x_1, t_1) \leq u(x_2, t_2) \left( \frac{t_2}{t_1} \right)^{n/\beta} e^{\varphi(x_1, x_2, t_1, t_2) + (c\tilde{M} + \hat{H}_1 + \tilde{H}_2)(t_2 - t_1)},$$

where  $\varphi(x_1, x_2, t_1, t_2) = \inf_{\gamma} \int_{t_1}^{t_2} \frac{1}{4\beta} |\dot{\gamma}|^2 dt$ . □

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