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IWAHORI–HECKE ALGEBRAS FOR KAC–MOODY GROUPS OVER LOCAL FIELDS

NICOLE BARDY-PANSE, STÉPHANE GAUSSENT AND GUY ROUSSEAU

We define the Iwahori–Hecke algebra ${}^I\mathcal{H}$ for an almost split Kac–Moody group G over a local nonarchimedean field. We use the hovel $\mathscr{I}$ associated to this situation, which is the analogue of the Bruhat–Tits building for a reductive group. The fixer K_I of some chamber in the standard apartment plays the role of the Iwahori subgroup. We can define ${}^I\mathcal{H}$ as the algebra of some K_I-bi-invariant functions on G with support consisting of a finite union of double classes. As two chambers in the hovel are not always in a same apartment, this support has to be in some large subsemigroup G^+ of G. In the split case, we prove that the structure constants of ${}^I\mathcal{H}$ are polynomials in the cardinality of the residue field, with integer coefficients depending on the geometry of the standard apartment. We give a presentation of this algebra , similar to the Bernstein–Lusztig presentation in the reductive case, and embed it in a greater algebra ${}^{BL}\mathcal{H}$, algebraically defined by the Bernstein–Lusztig presentation. In the affine case, this algebra ${}^{BL}\mathcal{H}$ contains the Cherednik's double affine Hecke algebra. Actually, our results apply to abstract "locally finite" hovels, so that we can define the Iwahori–Hecke algebra with unequal parameters.

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Introduction

A bit of history. Iwahori–Hecke algebras were first introduced in number theory by Erich Hecke [1937]. He defined an algebra, now called the Hecke algebra, generated by some operators on modular forms. Then, based on an idea of André Weil, Goro Shimura [1959] defined an algebra attached to a group containing a subgroup (under some hypotheses) as the algebra spanned by some double cosets and recovered Hecke's algebra. Nagayoshi Iwahori [1964] showed that, in the case of a Chevalley group over a finite field containing a Borel subgroup, Shimura's algebra can be defined in terms of bi-invariant functions on the group. He further gave a presentation by generators and relations of this algebra. Examples of such groups containing a suitable subgroup are given by BN-pairs and the theory of buildings. Nagayoshi Iwahori and Hideya Matsumoto [1965] found a famous instance in a Chevalley group over a p-adic field corresponding to the Bruhat–Tits building associated to the situation. In fact, it is possible to define these algebras only in terms of building theory; see, e.g., [Parkinson 2006] for a contemporary treatment.

In a previous article, Gaussent and Rousseau [2008] introduced the analogue of the Bruhat–Tits building in Kac–Moody theory, and called it, a hovel. Guy Rousseau [2011] developed the notion further and gave an axiomatic definition, applicable in a broader context.

In this paper, we first define, in terms of the hovel, the Iwahori–Hecke algebra associated to a Kac–Moody group over a local field containing the equivalent of the Iwahori subgroup. Then, we study the properties of this algebra, like the structure constants of the product, some presentations by generators and relations, and an interesting example where we recover the double affine Hecke algebras.

For the rest of the introduction, we give a more detailed account of our work.

The case of simple algebraic groups. To begin, we recall the situation in the finite dimensional case. Let $\mathcal{K}$ be a local nonarchimedean field, with residue field $\mathbb{F}_q$. Suppose G is a split, simple and simply connected algebraic group over $\mathcal{K}$ and K an open compact subgroup. The space $\mathcal{H}_K$ of complex functions on G, bi-invariant by K and with compact support, is an algebra for the natural convolution product. Ichiro Satake [1963] studied such algebras to define the spherical functions and proved, in particular, that $\mathcal{H}_K$ is commutative for a good choice K_s of K, maximal compact. The corresponding convolution algebra $\mathcal{H}_{K_s} = {}^s\mathcal{H}(G)$ is now called the spherical Hecke algebra. From [Iwahori and Matsumoto 1965], we know that there exists an interesting open subgroup K_I, so called the Iwahori subgroup, of K_s with a Bruhat decomposition $G = K_I . W . K_I$, where W is an infinite Coxeter group. The corresponding convolution algebra $\mathcal{H}_{K_I} = {}^{\mathrm{I}}\mathcal{H}(G)$, called the Iwahori–Hecke algebra, may be described as the abstract Hecke algebra associated to this Coxeter group and the parameter q. There is another presentation of this Hecke algebra, stated by Joseph Bernstein and proved in the most general case by George Lusztig [1989]. This presentation emphasizes the role of the translations in W and uses new relations, now often called the Bernstein–Lusztig relations. In the building-like definition of these algebras, the group K_s (resp., K_I) is the fixer of a special vertex (resp., a chamber) for the action of G on the Bruhat–Tits building $\mathscr{I}$ [Bruhat and Tits 1972].

The Kac–Moody setting. Kac–Moody groups are interesting generalizations of semisimple groups, hence it is natural to define the Iwahori–Hecke algebras also in the Kac–Moody setting.

So, from now on, let G be a Kac–Moody group over $\mathcal{K}$, assumed minimal or "algebraic", i.e., as studied by Jacques Tits [1987] in the split case and by Bertrand Rémy [2002] in the almost split case. Unfortunately there is, up to now, no good topology on G and no good compact subgroup, so the "convolution product" has to be defined by other means. Alexander Braverman and David Kazhdan [2011] succeeded in defining geometrically such a spherical Hecke algebra, when G is split and untwisted affine; see also the survey [Braverman and Kazhdan 2013]. We were able, in [Gaussent and Rousseau 2014], to generalize their construction to any Kac–Moody group over $\mathcal{K}$. Using results of [Garland 1995; Braverman et al. 2014], Braverman, Kazhdan and Manish Patnaik [Braverman et al. 2016] constructed the spherical Hecke algebra and the Iwahori–Hecke algebra by algebraic computations in the Kac–Moody group, still assumed split and untwisted affine (and even simply laced for some statements). Those algebras are convolution algebras of functions on G bi-invariant under some analogue group K_s or K_I (contained in K_s), but there are two new features: the support has to be in a subsemigroup G^+ of G and the description of the Iwahori–Hecke algebra has to use Bernstein–Lusztig type relations since W is no longer a Coxeter group.

Iwahori–Hecke algebras in the Kac–Moody setting. Similar to [Gaussent and Rousseau 2014], our idea is to define the Iwahori–Hecke algebra using the hovel associated to the almost split Kac–Moody group G that we built in [Gaussent and Rousseau 2008; Rousseau 2011; 2010]. This hovel $\mathscr{I}$ is a set with an action of G and a covering by subsets called apartments. They are in one-to-one correspondence with the maximal split subtori, hence permuted transitively by G. Each apartment A is a finite dimensional real affine space. Its stabilizer N in G acts on A via a generalized affine Weyl group $W = W^v \ltimes Y$, where $Y \subset \vec{A}$ is a discrete subgroup of translations. The group W stabilizes a set $\mathcal{M}$ of affine hyperplanes called walls. So, $\mathscr{I}$ looks much like the Bruhat–Tits building of a reductive group. But as the root system Φ is infinite, the set of walls $\mathcal{M}$ is not locally finite. Further, two points in $\mathscr{I}$ are not always in a same apartment. This is why $\mathscr{I}$ is called a hovel. However, there exists on $\mathscr{I}$ a G-invariant preorder $\leq$ which induces on each apartment A the preorder given by the Tits cone $\mathcal{T} \subset \vec{A}$.

Now, we consider the fixer K_I in G of some (local) chamber C_0^+ in a chosen standard apartment $\mathbb{A}$; it is our Iwahori subgroup. Fix a ring R. The Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ will be defined as the space of some K_I-bi-invariant functions on G with values in R. In other words, it will be the space ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ of some G-invariant functions on $\mathcal{C}_0^+ \times \mathcal{C}_0^+$, where $\mathcal{C}_0^+ = G/K_I$ is the orbit of C_0^+ in the set $\mathcal{C}$ of chambers of $\mathscr{I}$. The convolution product is easy to guess from this point of view:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z \in \mathcal{C}_0^+} \varphi(C_x, C_z)\psi(C_z, C_y)$$

(if this sum means something). As for points, two chambers in $\mathscr{I}$ are not always in a same apartment, i.e., the Bruhat–Iwahori decomposition fails: $G \neq K_I.N.K_I$. So, we have to consider pairs of chambers $(C_x, C_y) \in \mathcal{C}_0^+ \times_{\leq} \mathcal{C}_0^+$, i.e., $C_x \in \mathcal{C}_0^+$ has x for vertex, $C_y \in \mathcal{C}_0^+$ has y for vertex, and $x \leq y$. This implies that C_x, C_y are in a same apartment. For ${}^{\mathrm{I}}\mathcal{H}_R$, this means that the support of $\varphi \in \mathcal{H}_R$ has to be in $K_I \backslash G^+ / K_I$ where $G^+ = \{g \in G \mid 0 \leq g.0\}$ is a semigroup. We suppose moreover this support to be finite. In addition, $K_I \backslash G^+ / K_I$ is in bijective correspondence with the subsemigroup $W^+ = W^v \ltimes Y^+$ of W, where $Y^+ = Y \cap \mathcal{T}$.

With this definition we are able to prove that ${}^{\mathrm{I}}\mathcal{H}_R$ is really an algebra, which generalizes the known Iwahori–Hecke algebras in the semisimple case; see Section 2.

The structure constants. The structure constants of ${}^{\mathrm{I}}\mathcal{H}_R$ are the nonnegative integers $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$, for $\boldsymbol{w}, \boldsymbol{v}, \boldsymbol{u} \in W^+$, such that

$$T_{\boldsymbol{w}} * T_{\boldsymbol{v}} = \sum_{\boldsymbol{u} \in W^+} a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} T_{\boldsymbol{u}},$$

where $T_{\boldsymbol{w}}$ is the characteristic function of $K_I.\boldsymbol{w}.K_I$ and the sum is finite. Each chamber in $\mathscr{I}$ has only a finite number of adjacent chambers along a given panel. These numbers are called the parameters of $\mathscr{I}$ and form a finite set $\mathcal{Q}$. In the split case, there is only one parameter q: the number of elements of the residue field of $\mathcal{K}$. We conjecture that each $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is a polynomial in these parameters with integral coefficients depending only on the geometry of the model apartment $\mathbb{A}$ and on W. We prove this only partially: this is true if G is split or if we replace "polynomial" by "Laurent polynomial" (see Section 6.7); this is also true for $\boldsymbol{w}, \boldsymbol{v}$ "generic" (see Section 3.8). Actually in the generic case, we give, in Section 3, an explicit formula for $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$.

Generators and relations. If the parameters in $\mathcal{Q}$ are invertible in the ring R, we are able, in Section 4, to deduce from the geometry of $\mathscr{I}$ a set of generators and some relations in ${}^{\mathrm{I}}\mathcal{H}_R$. The family $(T_\lambda * T_w)_{\lambda \in Y^+, w \in W^v}$ is an R-basis of ${}^{\mathrm{I}}\mathcal{H}_R$. And the subalgebra $\sum_{w \in W^v} R.T_w$ is the abstract Hecke algebra $\mathcal{H}_R(W^v)$ associated to the Coxeter group W^v, generated by the $T_i = T_{r_i}$, where the r_i are the fundamental reflections in W^v. So, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module. We get also some commuting relations between the T_λ and the T_w, including some relations of Bernstein–Lusztig type (see Theorem 4.8).

From all these relations, we deduce algebraically in Section 5 that there exists a new basis $(X^\lambda * T_w)_{\lambda\in Y^+, w\in W^v}$ of ${}^{\mathrm{I}}\mathcal{H}_R$, associated to some new elements $X^\lambda \in {}^{\mathrm{I}}\mathcal{H}_R$. These elements satisfy $X^\lambda = T_\lambda$ for $\lambda \in Y^{++} = Y \cap \overline{C^v_f}$, where C^v_f is the fundamental Weyl chamber, and $X^\lambda * X^\mu = X^{\lambda+\mu} = X^\mu * X^\lambda$ for $\lambda, \mu \in Y^+$. As, for any $\lambda \in Y^+$, there is $\mu \in Y^{++}$ with $\lambda + \mu \in Y^{++}$; these X^λ are some quotients of some elements T_μ. The Bernstein–Lusztig type relations may be translated to this new basis. When R contains sufficiently high roots of the parameters in $\mathcal{Q}$ (e.g., if $R \supset \mathbb{R}$), we may replace the T_w and X^λ by some $R^\times$-multiples H_w and Z^λ. We get a new basis $(Z^\lambda * H_w)_{\lambda\in Y^+l, w\in W^v}$ of ${}^{\mathrm{I}}\mathcal{H}_R$, satisfying a set of relations very close to the Bernstein–Lusztig presentation in the semisimple case; see Section 5.7.

In Section 6, we define the Bernstein–Lusztig–Hecke algebra ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ algebraically: it is the free module with basis written $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$ over the algebra $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma_i'^{\pm 1})_{i\in I}]$, where σ_i, σ_i' are indeterminates (with some identifications). The product $*$ is given by the same relations as above for the $Z^\lambda * H_w$; one just extends $\lambda \in Y^+$ to $\lambda \in Y$ and replaces $\sqrt{q_i}, \sqrt{q_i'}$ by σ_i, σ_i'. We prove then that, up to a change of scalars, ${}^{\mathrm{I}}\mathcal{H}_R$ may be identified to a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$. This Bernstein–Lusztig algebra may be considered as a ring of quotients of the Iwahori–Hecke algebra.

Ordered affine hovel. Actually, this article is written in a more general framework, explained in Section 1: we work with $\mathcal{I}$ an abstract ordered affine hovel (as defined in [Rousseau 2011]), and we take G to be a strongly transitive group of (positive, "vectorially Weyl") automorphisms. In Section 7, we drop the assumption that G is vectorially Weyl to define extended versions ${}^{\mathrm{I}}\widetilde{\mathcal{H}}$ and ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}$ of ${}^{\mathrm{I}}\mathcal{H}$ and ${}^{\mathrm{BL}}\mathcal{H}$. In the affine case, we prove that they are graded algebras and that the summand of degree 0 of ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}$ is very close to Cherednik's double affine Hecke algebra.

1. General framework

1.1. ***Vectorial data.*** We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where V is a finite dimensional real vector space, W^v a subgroup of $\mathrm{GL}(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a family in V, and $(\alpha_i)_{i\in I}$ a family in the dual V^*. We suppose this family free, i.e., the set $\{\alpha_i \mid i \in I\}$ linearly independent and ask these data to satisfy the conditions of [Rousseau 2011, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V which is an element in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete, we consider the Kac–Moody case of [op. cit., 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac–Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v,\ i \in I\} \subset Q = \bigoplus_{i\in I} \mathbb{Z}.\alpha_i.$$

We set $\Phi^{\pm} = \Phi \cap Q^{\pm}$, where $Q^{\pm} = \pm\big(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i\big)$. Also,

$$Q^{\vee} = \Big(\bigoplus_{i\in I} \mathbb{Z}.\alpha_i^{\vee}\Big), \quad \text{and} \quad Q^{\vee}_{\pm} = \pm\Big(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i^{\vee}\Big).$$

We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$,

$$r_\alpha = w.r_i.w^{-1} \quad \text{and} \quad r_\alpha(v) = v - \alpha(v)\alpha^{\vee},$$

where the coroot $\alpha^{\vee} = w(\alpha_i^{\vee})$ depends only on α.

The set Φ is an (abstract, reduced) real root system in the sense of [Moody and Pianzola 1989; 1995; Bardy 1996]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta^+_{\mathrm{im}} \cup \Delta^-_{\mathrm{im}}$ of all roots (with $-\Delta^-_{\mathrm{im}} = \Delta^+_{\mathrm{im}} \subset Q^+,\ W^v$-stable) defined in [Kac 1990]. It is an (abstract, reduced) root system in the sense of [Bardy 1996].

The *fundamental positive chamber* is $C^v_f = \{v \in V \mid \alpha_i(v) > 0,\ \text{for all } i \in I\}$. Its closure $\overline{C^v_f}$ is the disjoint union of the vectorial faces

$$F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \text{ for all } i \in J,\ \text{and } \alpha_i(v) > 0 \text{ for all } i \in I \setminus J\}$$

for $J \subset I$. We set $V_0 = F^v(I)$. The positive and negative vectorial faces are the sets $w.F^v(J)$ and $-w.F^v(J)$, respectively, for $w \in W^v$ and $J \subset I$. The support of such a face is the vector space it generates. The set J or the face $w.F^v(J)$ or an element of this face is called *spherical* if the group $W^v(J)$ generated by $\{r_i \mid i \in J\}$ is finite. An element of a vectorial chamber $\pm w.C^v_f$ is called *regular*.

The *Tits cone* $\mathcal{T}$ is the (disjoint) union of the positive vectorial faces. Its interior $\mathcal{T}^{\circ}$ consists of those faces that are also spherical. It is a W^v-stable convex cone in V.

We say that $\mathbb{A}^v = (V, W^v)$ is a *vectorial apartment*. A *positive automorphism* of $\mathbb{A}^v$ is a linear bijection $\varphi : \mathbb{A}^v \to \mathbb{A}^v$ stabilizing $\mathcal{T}$ and permuting the roots and corresponding coroots; so it normalizes W^v and permutes the vectorial walls $M^v(\alpha) = \mathrm{Ker}(\alpha)$. As W^v acts simply transitively on the positive (resp., negative) vectorial chambers, any subgroup $\widetilde{W}^v$ of the group $\mathrm{Aut}^+(\mathbb{A}^v)$ (of positive automorphisms of $\mathbb{A}^v$) containing W^v may be written $\widetilde{W}^v = \Omega \ltimes W^v$, where Ω is the stabilizer in $\widetilde{W}^v$ of C^v_f (resp., $-C^v_f$). This group Ω induces a group of permutations of I (by $\omega(\alpha_i) = \alpha_{\omega(i)}$ and $\omega(\alpha_i^{\vee}) = \alpha^{\vee}_{\omega(i)}$); but it may be greater than the whole group of permutations in general, even infinite if $\bigcap \mathrm{Ker}\,\alpha_i \neq \{0\}$.

1.2. ***The model apartment.*** As in [Rousseau 2011, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by $\mathrm{Ker}(\alpha)$ for $\alpha \in \Phi$.

We ask this apartment to be *semidiscrete* and the origin 0 to be *special*. This means that these walls are the hyperplanes defined as

$$M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \quad \text{for } \alpha \in \Phi \text{ and } k \in \Lambda_\alpha,$$

with $\Lambda_\alpha = k_\alpha.\mathbb{Z}$ a nontrivial discrete subgroup of $\mathbb{R}$. Using [Gaussent and Rousseau 2014, Lemma 1.3] (i.e., by replacing Φ by another system Φ_1) we may (and shall) assume that $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$.

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$, and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed points the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; we assume that W^a stabilizes $\mathcal{M}$. We know that $W^a = W^v \ltimes Q^\vee$ and we write $W^a_{\mathbb{R}} = W^v \ltimes V$; here $Q^\vee$ and V have to be understood as groups of translations.

An automorphism of $\mathbb{A}$ is an affine bijection $\varphi : \mathbb{A} \to \mathbb{A}$ stabilizing the set of pairs $(M, \alpha^\vee)$ of a wall M and the coroot associated with $\alpha \in \Phi$ such that $M = M(\alpha, k)$, $k \in \mathbb{Z}$. We write $\overrightarrow{\varphi} : V \to V$ the linear application associated to φ. The group $\mathrm{Aut}(\mathbb{A})$ of these automorphisms contains W^a and normalizes it. We consider also the group $\mathrm{Aut}^W_{\mathbb{R}}(\mathbb{A}) = \{\varphi \in \mathrm{Aut}(\mathbb{A}) \mid \overrightarrow{\varphi} \in W^v\} = \mathrm{Aut}(\mathbb{A}) \cap W^a_{\mathbb{R}}$.

For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$ is an half-space; it is called an *half-apartment* if $k \in \mathbb{Z}$. We write $D(\alpha, \infty) = \mathbb{A}$.

The Tits cone $\mathcal{T}$ and its interior $\mathcal{T}^o$ are convex and W^v-stable cones; therefore, we can define two W^v-invariant preorder relations on $\mathbb{A}$:

$$x \leq y \;\Leftrightarrow\; y - x \in \mathcal{T} \quad \text{and} \quad x \overset{o}{<} y \;\Leftrightarrow\; y - x \in \mathcal{T}^o.$$

If W^v has no fixed point in $V \setminus \{0\}$ and no finite factor, then they are orders; but, in general, they are not.

1.3. ***Faces, sectors, and chimneys.*** The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments. As in [Bruhat and Tits 1972], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$. For the definition of that notion and its properties, see [loc. cit.] or [Gaussent and Rousseau 2008].

If F is a subset of $\mathbb{A}$ containing an element x in its closure, the germ of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of $\mathbb{A}$ which contain intersections of F and neighborhoods of x. In particular, if $x \neq y \in \mathbb{A}$, we denote the germ in x of the segment $[x, y]$ by $[x, y)$ and the germ in x of the segment $]x, y]$ by $]x, y)$.

Given F a filter of subsets of $\mathbb{A}$, its *enclosure* $\mathrm{cl}_{\mathbb{A}}(F)$ is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$, where $k_\alpha \in \mathbb{Z} \cup \{\infty\}$. Its *closure* $\overline{F}$ is the filter made of the subsets of $\mathbb{A}$ containing the closure $\overline{S}$ of some $S \in F$.

A *local face* F in the apartment $\mathbb{A}$ is associated to its vertex, a point $x \in \mathbb{A}$, and its direction, a vectorial face F^v in V. It is defined as $F = \mathrm{germ}_x(x + F^v)$ and we denote it by $F = F^\ell(x, F^v)$. Its closure is $\overline{F^\ell}(x, F^v) = \mathrm{germ}_x(x + \overline{F^v})$

There is an order on the local faces: in fact, the three assertions F is a face of F', F' covers F, and $F \leq F'$ are by definition equivalent to $F \subset \overline{F'}$. The dimension of a local face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the support of F; if $F = F^\ell(x, F^v)$, then $\mathrm{supp}(F) = x + \mathrm{supp}(F^v)$.

A local face $F = F^\ell(x, F^v)$ is spherical if the direction of its support meets the open Tits cone (i.e., when F^v is spherical), then its pointwise stabilizer W_F in W^a is finite. We shall actually speak only of local faces here, and sometimes forget the word local.

Any point $x \in \mathbb{A}$ is contained in a unique face $F(x, V_0) \subset \mathrm{cl}_{\mathbb{A}}(\{x\})$ which is minimal of positive and negative direction (but seldom spherical). For any local face $F^\ell = F^\ell(x, F^v)$, there is a unique face F (as defined in [Rousseau 2011]) containing F^ℓ. Then $\overline{F^\ell} \subset \overline{F} = \mathrm{cl}_{\mathbb{A}}(F^\ell) = \mathrm{cl}_{\mathbb{A}}(F)$ is also the enclosure of any interval-germ $]x, y) = \mathrm{germ}_x(]x, y])$ included in F^ℓ.

A *local chamber* is a maximal local face, i.e., a local face $F^\ell(x, \pm w.C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local chamber* is $C_0^+ = \mathrm{germ}_0(C^v_f)$.

A *(local) panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension $n-1$; its support is a wall.

A *sector* in $\mathbb{A}$ is a V-translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w.C^v_f$, with $w \in W^v$. The point x is its *base point* and C^v its *direction*. Two sectors have the same direction if and only if they are conjugate by V-translation, and if and only if their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a V-translate of $\mathfrak{s}$; it is well determined by the direction C^v. So, the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V, and the set of sector-germs in $\mathbb{A}$ are in canonical bijection. We denote the sector-germ associated to the negative fundamental vectorial chamber $-C^v_f$ by $\mathfrak{S}_{-\infty}$.

A *sector-face* in $\mathbb{A}$ is a V-translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm w.F^v(J)$. The sector-face-germ of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (i.e., $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

A *chimney* in $\mathbb{A}$ is associated to a face $F = F(x, F^v_0)$, called its basis, and to a vectorial face F^v, its direction; it is the filter

$$\mathfrak{r}(F, F^v) = \mathrm{cl}_{\mathbb{A}}(F + F^v).$$

A chimney $\mathfrak{r} = \mathfrak{r}(F, F^v)$ is *splayed* if F^v is spherical; it is *solid* if its support (as a filter, i.e., the smallest affine subspace containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid. The enclosure of a sector-face $\mathfrak{f} = x + F^v$ is a chimney.

A ray δ with origin in x and containing $y \neq x$ (or the interval $]x, y]$, the segment $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $x \overset{o}{<} y$ or $y \overset{o}{<} x$. With these new notions, a chimney can be defined as the enclosure of a preordered ray and a preordered segment-germ sharing the same origin. The chimney is splayed if and only if the ray is generic.

1.4. *The hovel.* In this section, we recall the definition and some properties of an ordered affine hovel given by Rousseau [2011].

1.4.1. An apartment of type $\mathbb{A}$ is a set A endowed with a set $\mathrm{Isom}^W(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that, if $f_0 \in \mathrm{Isom}^W(\mathbb{A}, A)$, then $f \in \mathrm{Isom}^W(\mathbb{A}, A)$ if and only if there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism (resp., a Weyl-isomorphism, a vectorially Weyl isomorphism) between two apartments $\varphi : A \to A'$ is a bijection such that, for any $f \in \mathrm{Isom}^W(\mathbb{A}, A)$, $f' \in \mathrm{Isom}^W(\mathbb{A}, A')$, $f'^{-1} \circ \varphi \circ f$ is contained in $\mathrm{Aut}(\mathbb{A})$ (resp., in W^a, in $\mathrm{Aut}^W_{\mathbb{R}}(\mathbb{A})$); the group of these isomorphisms is written $\mathrm{Isom}(A, A')$ (resp., $\mathrm{Isom}^W(A, A')$, $\mathrm{Isom}^W_{\mathbb{R}}(A, A')$). As the filters in $\mathbb{A}$ defined in Section 1.3 above (e.g., local faces, sectors, walls, etc.) are permuted by $\mathrm{Aut}(\mathbb{A})$, they are well defined in any apartment of type $\mathbb{A}$ and exchanged by any isomorphism.

Definition. An *ordered affine hovel of type* $\mathbb{A}$ (or, for short, a *masure of type* $\mathbb{A}$) is a set $\mathscr{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that:

(MA1) any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$;

(MA2) if F is a point, a germ of a preordered interval, a generic ray, or a solid chimney in an apartment A, and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$;

(MA3) if $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and F;

(MA4) if two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then their intersection contains $\mathrm{cl}_A(\mathfrak{R} \cup F)$ and there exists a Weyl-isomorphism from A onto A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$;

(MA5) if x, y are two points contained in two apartments A and A', and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

We ask here that $\mathscr{I}$ be thick of *finite thickness*: the number of local chambers containing a given (local) panel has to be finite and at least 3. This number is the same for any panel in a given wall M [Rousseau 2011, 2.9]; we denote it by $1 + q_M$.

An automorphism (resp., a Weyl-automorphism, a vectorially Weyl automorphism) of $\mathscr{I}$ is a bijection $\varphi : \mathscr{I} \to \mathscr{I}$ such that $A \in \mathcal{A} \iff \varphi(A) \in \mathcal{A}$ and then $\varphi|_A : A \to \varphi(A)$ is an isomorphism (resp., a Weyl-isomorphism, a vectorially Weyl isomorphism).

1.4.2. For $x \in \mathscr{I}$, the set $\mathcal{T}_x^+\mathscr{I}$ (resp. $\mathcal{T}_x^-\mathscr{I}$) of segment-germs $[x, y)$ for $y > x$ (resp., $y < x$) may be considered as a building, the positive (resp., negative) tangent building. The corresponding faces are the local faces of positive (resp., negative) direction and of vertex x. The associated Weyl group is W^v. If the W-distance (calculated in $\mathcal{T}_x^\pm\mathscr{I}$) of two local chambers is $d^W(C_x, C'_x) = w \in W^v$, to any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$ corresponds a unique minimal gallery from C_x to C'_x of type $(i_1, \dots, i_n)$. We shall say, by abuse of notation, that this gallery is of type w.

The buildings $\mathcal{T}_x^+\mathscr{I}$ and $\mathcal{T}_x^-\mathscr{I}$ are actually twinned. The codistance $d^{*W}(C_x, D_x)$ of two opposite sign chambers C_x and D_x is the W-distance $d^W(C_x, \operatorname{op} D_x)$, where $\operatorname{op} D_x$ denotes the opposite chamber to D_x in an apartment containing C_x and D_x. Similarly two segment-germs $\eta \in \mathcal{T}_x^+\mathscr{I}$ and $\zeta \in \mathcal{T}_x^-\mathscr{I}$ are said opposite if they are in a same apartment A and opposite in this apartment (i.e., in the same line, with opposite directions).

Lemma [Rousseau 2011, 2.9]. *Let D be a half-apartment in $\mathscr{I}$ and $M = \partial D$ its wall (i.e., its boundary). One considers a panel F in M and a local chamber C in $\mathscr{I}$ covering F. Then there is an apartment containing D and C.*

1.4.3. We assume that $\mathscr{I}$ has a strongly transitive group of automorphisms G, i.e., all isomorphisms involved in the above axioms are induced by elements of G; see [Rousseau 2012, 4.10; Ciobotaru and Rousseau 2015]. We choose in $\mathscr{I}$ a fundamental apartment which we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathscr{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ in G induces a group $W = \nu(N) \subset \operatorname{Aut}(\mathbb{A})$ of affine automorphisms of $\mathbb{A}$ which permutes the walls, local faces, sectors, sector-faces, etc., and contains the affine Weyl group $W^a = W^v \ltimes Q^\vee$ [Rousseau 2012, 4.13.1].

We denote the stabilizer of $0 \in \mathbb{A}$ in G by K and the pointwise stabilizer (or fixer) of C_0^+ by K_I; this group K_I is called the *Iwahori subgroup*.

1.4.4. We ask $W = \nu(N)$ to be *positive* and *vectorially Weyl* for its action on the vectorial faces. This means that the associated linear map $\vec{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $W = \nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that: $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$. An element $\boldsymbol{w} \in W$ will often be written $\boldsymbol{w} = \lambda.w$, with $\lambda \in Y$ and $w \in W^v$.

We ask Y to be *discrete* in V. This is clearly satisfied if Φ generates V^*, i.e., $(\alpha_i)_{i \in I}$ is a basis of V^*.

1.4.5. Note that there is only a finite number of constants q_M as in the definition of thickness. Indeed, we must have $q_{wM} = q_M$, $\forall w \in \nu(N)$ and $w.M(\alpha,k) = M(w(\alpha),k), \forall w \in W^v$. So now, fix $i \in I$, as $\alpha_i(\alpha_i^\vee) = 2$ the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i, k)$ (for $k \in \mathbb{Z}$) with two orbits. So, $Q^\vee \subset W^a$ has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$: one containing the $q_i = q_{M(\alpha_i,0)}$ and the other containing the $q'_i = q_{M(\alpha_i,\pm 1)}$. Hence, the number of (possibly) different q_M is at most $2.|I|$. We denote this set of parameters by $\mathcal{Q} = \{q_i, q'_i \mid i \in I\}$.

If $\alpha_i(\alpha_j^\vee)$ is odd for some $j \in I$, the translation by $\alpha_j^\vee$ exchanges the two walls $M(\alpha_i, 0)$ and $M(\alpha_i, -\alpha_i(\alpha_j^\vee))$; so $q_i = q'_i$. More generally, we see that $q_i = q'_i$ when $\alpha_i(Y) = \mathbb{Z}$, i.e., $\alpha_i(Y)$ contains an odd integer. If $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$, one knows that the element $r_i r_j r_i$ of $W^v(\{i, j\})$ exchanges α_i and $-\alpha_j$, so $q_i = q'_i = q_j = q'_j$.

Actually many of the following results (in sections 2, 3) are true without assuming the existence of G: we have only to assume that the parameters q_M satisfy the above conditions.

1.4.6. The main examples of all the above situation are provided by the hovels of almost split Kac–Moody groups over fields complete for a discrete valuation and with a finite residue field, see Section 7.2 below.

1.4.7. *Remarks.* (a) In the following, we sometimes use results of [Gaussent and Rousseau 2008] even though, in this paper we deal with split Kac–Moody groups and residue fields containing $\mathbb{C}$. But the cited results are easily generalizable to our present framework, using the above references.

(b) All isomorphisms in [Rousseau 2011] are Weyl-isomorphisms, and, when G is strongly transitive, all isomorphisms constructed in that reference are induced by an element of G.

1.5. ***Type* 0 *vertices.*** The elements of Y, through the identification $Y = N.0$, are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$ and $Y^{++} = Y \cap \overline{C^v_f}$. The type 0 vertices in $\mathcal{I}$ are the points on the orbit $\mathcal{I}_0$ of 0 by G. This set $\mathcal{I}_0$ is often called the affine Grassmannian as it is equal to G/K, where $K = \mathrm{Stab}_G(\{0\})$. But in general, G is not equal to $KYK = KNK$ [Gaussent and Rousseau 2008, 6.10], i.e., $\mathcal{I}_0 \neq K.Y$.

We know that $\mathcal{I}$ is endowed with a G-invariant preorder $\leq$ which induces the known one on $\mathbb{A}$. Moreover, if $x \leq y$, then x and y are in a same apartment [Rousseau 2011, 5.9]. We set $\mathcal{I}^+ = \{x \in \mathcal{I} \mid 0 \leq x\}$, $\mathcal{I}_0^+ = \mathcal{I}_0 \cap \mathcal{I}^+$, and $G^+ = \{g \in G \mid 0 \leq g.0\}$; so $\mathcal{I}_0^+ = G^+.0 = G^+/K$. As $\leq$ is a G-invariant preorder, G^+ is a semigroup.

If $x \in \mathcal{I}_0^+$ there is an apartment A containing 0 and x (by definition of $\leq$) and all apartments containing 0 are conjugated to $\mathbb{A}$ by K (see (MA2)); so $x \in K.Y^+$ as $\mathcal{I}_0^+ \cap \mathbb{A} = Y^+$. But $\nu(N \cap K) = W^v$ and $Y^+ = W^v.Y^{++}$, with uniqueness of the element in Y^{++}. So $\mathcal{I}_0^+ = K.Y^{++}$, more precisely $\mathcal{I}_0^+ = G^+/K$ is the union of the

KyK/K for $y \in Y^{++}$. This union is disjoint, for the above construction does not depend on the choice of A; see Section 1.9(a).

Hence, we have proved that the map $Y^{++} \to K\backslash G^+/K$ is one-to-one and onto.

1.6. ***Vectorial distance and $Q^\vee$-order.*** For x in the Tits cone $\mathcal{T}$, we denote by x^{++} the unique element in $\overline{C^v_f}$ conjugated by W^v to x.

Let $\mathscr{I} \times_\leq \mathscr{I} = \{(x, y) \in \mathscr{I} \times \mathscr{I} \mid x \leq y\}$ be the set of increasing pairs in $\mathscr{I}$. Such a pair (x, y) is always in a same apartment $g.\mathbb{A}$; so $(g^{-1}).y - (g^{-1}).x \in \mathcal{T}$ and we define the *vectorial distance* $d^v(x, y) \in \overline{C^v_f}$ by $d^v(x, y) = ((g^{-1}).y - (g^{-1}).x)^{++}$. It does not depend on the choices we made (by Section 1.9.a below).

For $(x, y) \in \mathscr{I}_0 \times_\leq \mathscr{I}_0 = \{(x, y) \in \mathscr{I}_0 \times \mathscr{I}_0 \mid x \leq y\}$, the vectorial distance $d^v(x, y)$ takes values in Y^{++}. Actually, as $\mathscr{I}_0 = G.0$, K is the stabilizer of 0 and $\mathscr{I}^+_0 = K.Y^{++}$ (with uniqueness of the element in Y^{++}), the map d^v induces a bijection between the set $\mathscr{I}_0 \times_\leq \mathscr{I}_0/G$ of G-orbits in $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ and Y^{++}.

Further, d^v gives the inverse of the map $Y^{++} \to K\backslash G^+/K$, as any $g \in G^+$ is in $K.d^v(0, g.0).K$.

For $x, y \in \mathbb{A}$, we say that $x \leq_{Q^\vee} y$ when $y - x \in Q^\vee_+$, and $x \leq_{Q^\vee_\mathbb{R}} y$ when

$$y - x \in Q^\vee_{\mathbb{R}+} = \sum_{i \in I} \mathbb{R}_{\geq 0}.\alpha_i^\vee.$$

We get thus a preorder which is an order at least when $(\alpha_i^\vee)_{i\in I}$ is free or $\mathbb{R}_+$-free, i.e., $\sum a_i \alpha_i^\vee = 0$, $a_i \geq 0$ implies $a_i = 0$, for all i.

1.7. ***Paths.*** We consider piecewise linear continuous paths $\pi : [0, 1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v.\lambda$ for some $\lambda \in \overline{C^v_f}$. Such a path is called a λ-*path*; it is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$.

For any $t \neq 0$ (resp., $t \neq 1$), we let $\pi'_-(t)$ (resp., $\pi'_+(t)$) denote the derivative of π at t from the left (resp., from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda$-class such that $\pi'_\pm(t) = w_\pm(t).\lambda$, where $(W^v)_\lambda$ is the stabilizer in W^v of λ.

Hecke paths of shape λ (with respect to the sector germ $\mathfrak{S}_{-\infty} = \mathrm{germ}_\infty(-C^v_f)$) are λ-paths satisfying some further precise conditions, see [Kapovich and Millson 2008, 3.27] or [Gaussent and Rousseau 2014, 1.8]. For us their interest will appear just below in Section 1.8.

But to give a formula for the structure constants of the forthcoming Iwahori–Hecke algebra, we will need slightly different Hecke paths whose definition is detailed in Section 3.3.

1.8. ***Retractions onto Y^+.*** For all $x \in \mathscr{I}^+$ there is an apartment containing x and $C^-_0 = \mathrm{germ}_0(-C^v_f)$ [Rousseau 2011, 5.1] and this apartment is conjugated to $\mathbb{A}$ by an element of K fixing C_0^-; see (MA2). So, by the usual arguments, as well as [op. cit., 5.5] (see below Proposition 1.10(a)), we can define the retraction $\rho_{C_0^-}$ of $\mathscr{I}^+$ into $\mathbb{A}$ with center C_0^-; its image is $\rho_{C_0^-}(\mathscr{I}^+) = \mathcal{T} = \mathscr{I}^+ \cap \mathbb{A}$ and $\rho_{C_0^-}(\mathscr{I}_0^+) = Y^+$.

Using axioms (MA3) and (MA4) [Gaussent and Rousseau 2008, 4.4], we may also define the retraction $\rho_{-\infty}$ of $\mathscr{I}$ onto $\mathbb{A}$ with center the sector-germ $\mathfrak{S}_{-\infty}$.

More generally, we may define the retraction ρ of $\mathscr{I}$ (resp., of the subset $\mathscr{I}_{\geq z} = \{y \in \mathscr{I} \mid y \geq z\}$, for a fixed z) onto an apartment A with center any sector germ (resp., any local chamber of negative direction with vertex z). For any such retraction ρ, the image of any segment $[x, y]$ with $(x, y) \in \mathscr{I} \times_{\leq} \mathscr{I}$ and $d^v(x, y) = \lambda \in \overline{C_f^v}$ (resp., and moreover $x, y \in \mathscr{I}_{\geq z}$) is a λ-path [Gaussent and Rousseau 2008, 4.4]. In particular, $\rho(x) \leq \rho(y)$.

Actually, the image by $\rho_{-\infty}$ of any segment $[x, y]$ with $(x, y) \in \mathscr{I} \times_{\leq} \mathscr{I}$ and $d^v(x, y) = \lambda \in Y^{++}$ is a Hecke path of shape λ with respect to $\mathfrak{S}_{-\infty}$ [Gaussent and Rousseau 2008, th. 6.2], and we have the following lemma.

Lemma. (a) *For $\lambda \in Y^{++}$ and $w \in W^v$, we have $w.\lambda \in \lambda - Q_+^\vee$, i.e., $w.\lambda \leq_{Q^\vee} \lambda$.*

(b) *Let π be a Hecke path of shape $\lambda \in Y^{++}$ with respect to $\mathfrak{S}_{-\infty}$, from $y_0 \in Y$ to $y_1 \in Y$. Then, for $0 \leq t < t' < 1$,*

$$\begin{aligned} &\lambda = \pi'_+(t)^{++} = \pi'_-(t')^{++}; \\ &\pi'_+(t) \leq_{Q^\vee} \pi'_-(t') \leq_{Q^\vee} \pi'_+(t') \leq_{Q^\vee} \pi'_-(1); \\ &\pi'_+(0) \leq_{Q^\vee} \lambda; \\ &\pi'_+(0) \leq_{Q^\vee_{\mathbb{R}}} (y_1 - y_0) \leq_{Q^\vee_{\mathbb{R}}} \pi'_-(1) \leq_{Q^\vee} \lambda; \\ &y_1 - y_0 \leq_{Q^\vee} \lambda. \end{aligned}$$

Moreover $y_1 - y_0$ is in the convex hull $\mathrm{conv}(W^v.\lambda)$ *of all $w.\lambda$ for $w \in W^v$, more precisely in the convex hull* $\mathrm{conv}(W^v.\lambda, \geq \pi'_+(0))$ *of all $w'.\lambda$ for $w' \in W^v$, $w' \leq w$, where w is the element with minimal length such that $\pi'_+(0) = w.\lambda$.*

(c) *If, moreover, $(\alpha_i^\vee)_{i\in I}$ is free, we may replace above $\leq_{Q^\vee_{\mathbb{R}}}$ by $\leq_{Q^\vee}$.*

(d) *If $x \leq z \leq y$ in $\mathscr{I}_0$, then $d^v(x, y) \leq_{Q^\vee} d^v(x, z) + d^v(z, y)$.*

N.B. In the following, we always assume $(\alpha_i^\vee)_{i\in I}$ free.

Proof. Everything is proved in [Gaussent and Rousseau 2014, 2.4], except the second paragraph of (b). Actually we see in [loc. cit.] that $y_1 - y_0$ is the integral of the locally constant vector-valued function $\pi'_+(t) = w_+(t).\lambda$, where $w_+(t)$ is decreasing for the Bruhat order [op. cit., 5.4], hence the result. □

1.9. ***Chambers of type* 0.** Let $\mathscr{C}_0^\pm$ be the set of all local chambers with vertices of type 0 and positive or negative direction. A local chamber of vertex $x \in \mathscr{I}_0$ will often be written C_x and its direction C_x^v. We consider $\mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+ = \{(C_x, C_y) \in \mathscr{C}_0^+ \times \mathscr{C}_0^+ \mid x \leq y\}$. We sometimes write $C_x \leq C_y$ when $x \leq y$.

Proposition [Rousseau 2011, 5.4 and 5.1]. *Let $x, y \in \mathscr{I}$ with $x \le y$. We consider two local faces F_x, F_y with respective vertices x, y.*

(a) *$\{x, y\}$ is included in an apartment and two such apartments A, A' are isomorphic by a Weyl-isomorphism in G, fixing $\mathrm{cl}_A(\{x, y\}) = \mathrm{cl}_{A'}(\{x, y\}) \supset [x, y]$.*

(b) *There is an apartment containing F_x and F_y, unless F_x and F_y are respectively of positive and negative direction. In this case we have to assume moreover $x \overset{o}{<} y$ or $x = y$ to get the same result.*

Consequences. (1) We define $W^+ = W^v \ltimes Y^+$ which is a subsemigroup of W.

If $C_x \in \mathscr{C}_0^+$, we know by (b) above, that there is an apartment A containing C_0^+ and C_x. But all apartments containing C_0^+ are conjugated to $\mathbb{A}$ by K_I (MA2), so there is $k \in K_I$ with $k^{-1}.C_x \subset \mathbb{A}$. Now the vertex $k^{-1}.x$ of $k^{-1}.C_x$ satisfies $k^{-1}.x \ge 0$, so there is $\boldsymbol{w} \in W^+$ such that $k^{-1}.C_x = \boldsymbol{w}.C_0^+$.

When $g \in G^+$, we have $g.C_0^+ \in \mathscr{C}_0^+$ and there are $k \in K_I$, $\boldsymbol{w} \in W^+$ satisfying $g.C_0^+ = k.\boldsymbol{w}.C_0^+$, i.e., $g \in K_I.W^+.K_I$. We have proved the *Bruhat decomposition* $G^+ = K_I.W^+.K_I$.

(2) Let $x \in \mathscr{I}_0$ and $C_y \in \mathscr{C}_0^+$ with $x \le y$, $x \ne y$. We consider an apartment A containing x and C_y (by (b) above) and write $C_y = F(y, C_y^v)$ in A. For $y' \in y + C_y^v$ sufficiently close to y, $\alpha(y' - x) \ne 0$ for any root α. So $]x, y')$ is in a unique local chamber $\mathrm{pr}_x(C_y)$ of vertex x; this chamber satisfies $[x, y) \subset \overline{\mathrm{pr}_x(C_y)} \subset \mathrm{cl}_A(\{x, y'\})$ and does not depend on the choice of y'. Moreover, if A' is another apartment containing x and C_y, we may suppose $y' \in A \cap A'$ and $]x, y')$, $\mathrm{cl}_A(\{x, y'\})$, $\mathrm{pr}_x(C_y)$ are the same in A'. The local chamber $\mathrm{pr}_x(C_y)$ is well determined by x and C_y; it is the projection of C_y in $\mathcal{T}_x^+\mathscr{I}$.

The same things may be done changing accordingly $+$ to $-$ and $\le$ to $\ge$. But, in the above situation, if $C_x \in \mathscr{C}_0^+$, we have to assume $x \overset{o}{<} y$ to define the analogous $\mathrm{pr}_y(C_x) \in \mathscr{C}_0^+$.

Proposition 1.10. *In the setting of Section 1.9,*

(a) *If $x \overset{o}{<} y$ or F_x and F_y are, respectively, of negative and positive direction, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G fixing the convex hull of F_x and F_y (in A or A').*

(b) *If $x = y$ and the directions of F_x and F_y have the same sign, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G, $\varphi : A \to A'$, fixing F_x and F_y. If moreover F_x is a local chamber, any minimal gallery from F_x to F_y is fixed by φ (and in $A \cap A'$).*

(c) *If F_x and F_y are of positive directions and F_y is spherical, any two apartments A and A' containing F_x and F_y are isomorphic by a Weyl-isomorphism in G fixing F_x and F_y.*

Remark. The conclusion in (c) above is less precise than in (a) or in Section 1.9(a). We may actually improve it when the hovel is associated to a very good family of parahorics, as defined in [Rousseau 2012] and already used in [Gaussent and Rousseau 2008]. Then, using the notion of half-good fixers, we may assume that the isomorphism in (c) above fixes some kind of enclosure of F_x and F_y (containing the convex hull). This particular case includes the case of an almost split Kac–Moody group over a local field.

Proof. The assertions (a) and (b) are Propositions 5.5 and 5.2 of [Rousseau 2011], respectively. To prove (c) we improve a little the proof of 5.5 in that reference and use the classical trick that says that it is enough to assume that either F_x or F_y is a local chamber. We assume now that $F_x = C_x$ is a local chamber; the other case is analogous.

We consider an element Ω_x (resp., Ω_y) of the filter C_x (resp., F_y) contained in $A \cap A'$. We have $x \in \overline{\Omega_x}$, $y \in \overline{\Omega_y}$, and one may suppose Ω_x is open and Ω_y is open in the support of F_y. There is an isomorphism $\varphi : A \to A'$ fixing Ω_x. Let $y' \in \Omega_y$; we want to prove that $\varphi(y') = y'$. As F_y is spherical, $x \le y \overset{o}{<} y'$; hence, $x \overset{o}{<} y'$. So $x' \le y'$ for any $x' \in \Omega_x$ (Ω_x sufficiently small). Moreover $[x', y'] \cap \Omega_x$ is an open neighborhood of x' in $[x', y']$. By the following lemma, we get $\varphi(y') = y'$. □

Lemma. *Let us consider two apartments A, A' in $\mathscr{I}$, a subset $\Omega \subset A \cap A'$, a point $z \in A \cap A'$ and an isomorphism $\varphi : A \to A'$ fixing (pointwise) Ω. We assume that there is $z' \in \Omega$ with $z' \le z$ or $z' \ge z$ and $[z', z] \cap \Omega$ open in $[z', z]$. Then $\varphi(z) = z$.*

N.B. This lemma asserts, in particular, that any isomorphism $\varphi : A \to A'$ fixing a local facet $F \subset A \cap A'$ fixes $\overline{F}$.

Proof. Note that $\varphi|_{[z',z]}$ is an affine bijection of $[z', z]$ onto its image in A', which is the identity in a neighborhood of z'. But Section 1.9(a) shows that $[z', z] \subset A \cap A'$ and the identity of $[z', z]$ is an affine bijection (for the affine structures induced by A and A'). Hence $\varphi|_{[z',z]}$ coincides with this affine bijection; in particular $\varphi(z) = z$. □

1.11. ***W-distance.*** Let $(C_x, C_y) \in \mathscr{C}_0^+ \times_\le \mathscr{C}_0^+$; there is an apartment A containing C_x and C_y. We identify $(\mathbb{A}, C_0^+)$ with (A, C_x), i.e., we consider the unique $f \in \mathrm{Isom}^W_{\mathbb{R}}(\mathbb{A}, A)$ such that $f(C_0^+) = C_x$. Then $f^{-1}(y) \ge 0$ and there is $\boldsymbol{w} \in W^+$ such that $f^{-1}(C_y) = \boldsymbol{w} . C_0^+$. By Proposition 1.10(c), $\boldsymbol{w}$ does not depend on the choice of A.

We define the *W-distance* between the two local chambers C_x and C_y to be this unique element: $d^W(C_x, C_y) = \boldsymbol{w} \in W^+ = Y^+ \rtimes W^v$. If $\boldsymbol{w} = \lambda . w$, with $\lambda \in Y^+$ and $w \in W^v$, we write also $d^W(C_x, y) = \lambda$. As $\le$ is G-invariant, the W-distance is also G-invariant. When $x = y$, this definition coincides with the one in Section 1.4.2.

If $C_x, C_y, C_z \in \mathscr{C}_0^+$, with $x \le y \le z$, are in a same apartment, we have the Chasles relation: $d^W(C_x, C_z) = d^W(C_x, C_y) . d^W(C_y, C_z)$.

When $C_x = C_0^+$ and $C_y = g\,.\,C_0^+$ (with $g \in G^+$), $d^W(C_x, C_y)$ is the only $\boldsymbol{w} \in W^+$ such that $g \in K_I\,.\,\boldsymbol{w}\,.\,K_I$. We have thus proved the uniqueness in the Bruhat decomposition: $G^+ = \coprod_{\boldsymbol{w}\in W^+} K_I\,.\,\boldsymbol{w}\,.\,K_I$.

The W-distance classifies the orbits of K_I on $\{C_y \in \mathscr{C}_0^+ \mid y \geq 0\}$, hence also the orbits of G on $\mathscr{C}_0^+ \times_\leq \mathscr{C}_0^+$.

2. Iwahori–Hecke Algebras

Throughout this section, we assume that $(\alpha_i^\vee)_{i\in I}$ is free and we consider any commutative ring with unity R. To each $\boldsymbol{w} \in W^+$, we associate a function $T_{\boldsymbol{w}}$ from $\mathscr{C}_0^+ \times_\leq \mathscr{C}_0^+$ to R defined by

$$T_{\boldsymbol{w}}(C, C') = \begin{cases} 1 & \text{if } d^W(C, C') = \boldsymbol{w}, \\ 0 & \text{otherwise.} \end{cases}$$

Now we consider the following free R-module

$${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}} = \left\{ \varphi = \sum_{\boldsymbol{w}\in W^+} a_{\boldsymbol{w}} T_{\boldsymbol{w}} \;\middle|\; a_{\boldsymbol{w}} \in R,\ a_{\boldsymbol{w}} = 0 \text{ except for a finite number of } \boldsymbol{w} \right\},$$

We endow this R-module with the convolution product:

$$(\varphi * \psi)(C_x, C_y) = \sum_{C_z} \varphi(C_x, C_z)\,\psi(C_z, C_y).$$

where $C_z \in \mathscr{C}_0^+$ is such that $x \leq z \leq y$. It is clear that this product is associative and R-bilinear. We prove below that this product is well defined.

As in [Gaussent and Rousseau 2014, 2.1], we see easily that ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ can be identified with the natural convolution algebra of functions $G^+ \to R$, bi-invariant under K_I and with finite support.

Lemma 2.1. *Let $\mathfrak{S}^- \subset A$ be a sector-germ with negative direction in an apartment A, let $\rho_- : \mathscr{I} \to A$ be the corresponding retraction, and let $\boldsymbol{w} \in W^+$. Then the set*

$$P = \{d^W(\rho_-(C_x), \rho_-(C_y)) \in W^+ \mid \textit{for all } (C_x, C_y) \in \mathscr{C}_0^+ \times_\leq \mathscr{C}_0^+,\ d^W(C_x, C_y) = \boldsymbol{w}\}$$

is finite and included in a finite subset P' of W^+ depending only on $\boldsymbol{w}$ and on the position of C_x with respect to $\mathfrak{S}^-$ (i.e., on the codistance $w_x \in W^v$ from C_x to the local chamber C_x^- in x of direction $\mathfrak{S}^-$).

Let us write $\boldsymbol{w} = \lambda\,.\,w$ for $\lambda \in Y^+$ and $w \in W^v$. If we assume C_x and $\mathfrak{S}^-$ are opposite (i.e., $w_x = 1$), then any $\boldsymbol{v} = \mu\,.\,v \in P'$ satisfies $\lambda \leq_{Q^\vee} \mu \leq_{Q^\vee} \lambda^{++}$ and μ is in $\mathrm{conv}(W^v.\lambda^{++})$. More precisely μ is in the convex hull $\mathrm{conv}(W^v.\lambda^{++}, \geq \lambda)$ of all $w'.\lambda^{++}$ for $w' \in W^v$, $w' \leq w_\lambda$, where w_λ is the element with minimal length such that $\lambda = w_\lambda\,.\,\lambda^{++}$.

If moreover $\lambda \in Y^{++}$*, then* $\mu = \lambda$ *and* $v \le w$*. In particular, for* $\boldsymbol{w} = \lambda \in Y^{++}$*,* $P = \{\boldsymbol{w}\} = \{\lambda\}$.

Proof. We consider an apartment A_1 containing C_x and C_y. We set $C'_y = C_x + (y - x)$ in A_1. By identifying $(\mathbb{A}, C_0^+)$ with (A_1, C_x), we have $y = x + \lambda$, and by identifying $(\mathbb{A}, C_0^+)$ with (A_1, C'_y), we have $C_y = wC'_y$.

We have to prove that the possibilities for $\rho_-(C_y)$ vary in a finite set determined by $\rho_-(C_x)$, $\boldsymbol{w}$, and w_x. We shall prove this by successively showing the same kind of result for $\rho_-([x, y))$, $\rho_-(y)$, and $\rho_-(C'_y)$. Up to isomorphism, one may suppose that $C_x \subset A$.

(a) Fixing a reduced decomposition for w_λ gives a minimal gallery between C_x and $[x, y)$. By retraction, we get a gallery with the same type from $\rho_-(C_x)$ to $\rho_-([x, y))$. The possible foldings of this gallery determine the possibilities for $\rho_-([x, y))$. More precisely, $\rho_-([x, y)) = x + w'(\lambda_A^{++})[0, 1)$ for $w' \le w_\lambda$ and λ_A^{++} the image in A of λ^{++} by the identification of $(\mathbb{A}, C_0^+)$ with (A, C_x).

(b) Now fix $\rho_-([x, y))$. By Section 1.8(b), $\rho_-([x, y])$ is a Hecke path π of shape λ^{++} (with respect to $\mathfrak{S}^-$). Its derivative $\pi'_+(0)$ is well determined by $\rho_-([x, y))$. We identify A with $\mathbb{A}$ in such a way that $\mathfrak{S}^-$ has direction $-C_f^v$. Then $\lambda_A^{++} = w_x(\lambda^{++})$ and $\pi'_+(0) = w'w_x(\lambda^{++})$, with w' as above. By Section 1.8(b),

$$\pi'_+(0) \le_{Q^\vee} \rho_-(y) - \rho_-(x) \le_{Q^\vee} \lambda^{++}.$$

So there are a finite number of possibilities for $\rho_-(y)$.

(c) Now fix $\rho_-([x, y))$ and $\rho_-(y)$, and investigate the possibilities for $\rho_-(C'_y)$. We shall use a segment $[x', y']$ in A_1 parallel to $[x, y]$ and prove successively that there are a finite number of possibilities for $\rho_-(x')$, $\rho_-([x', y'))$, $\rho_-(y')$, and $\rho_-(C'_y)$. So we choose $\xi \in Y^{++}$ and in the interior of the fundamental chamber C_f^v. In the apartment A_1, with (A_1, C_x) identified with $(\mathbb{A}, C_0^+)$, we consider $x' = x + \xi$ and $y' = y + \xi$ (hence, $y' = x' + \lambda$).

As in (a) and (b) above, we get that there are a finite number of possibilities for $\rho_-(x')$. So we fix $\rho_-(x')$.

(c1) On one side, we may also enlarge in A_1 the segment $[x, x']$ by considering the segment $[x', x'']$, where $x'' = x' + \varepsilon\xi = x + (1 + \varepsilon)\xi$, with $\varepsilon > 0$ small.

On the other side, $[x, x']$ can be described as a path $\pi_1 : [0, 1] \to A_1$, defined by $\pi_1(t) = x + t\xi$. The retracted path $\pi = \rho_-(\pi_1)$ satisfies

$$\rho_-(x') - \rho_-(x) \le_{Q^\vee} \pi'_+(1) \le_{Q^\vee} \lambda^{++},$$

again by Section 1.8. So there are a finite number of possibilities for $\pi'_+(1)$, i.e., for $\rho_-([x', x''))$. But there exists (in A_1) a minimal gallery of the type of a reduced decomposition of w_λ from the unique local chamber $(C_x + \xi)$ containing $[x', x'')$

to $[x', y')$. Hence, there exists a gallery of the same type between (a local chamber containing) $\rho_-([x', x''))$ and $\rho_-([x', y'))$. Therefore, there is a finite number of possibilities for $\rho_-([x', y'))$.

As in (b), we deduce that there are a finite number of possibilities for $\rho_-(y')$.

(c2) The path $\rho_-([y, y'])$ is a Hecke path of shape ξ from $\rho_-(y)$ to $\rho_-(y')$. By [Gaussent and Rousseau 2008, Corollary 5.9], there exist a finite number of such paths. In particular, there are a finite number of possibilities for the segment-germ $\rho_-([y, y'))$ and for $\rho_-(C'_y)$.

(d) Next, we fix $\rho_-(C'_y)$. Fixing a reduced decomposition for w gives a minimal gallery between C'_y and C_y, hence a gallery of the same type between $\rho_-(C'_y)$ and $\rho_-(C_y)$. So, the number of possible $\rho_-(C_y)$ is finite and $d^W(\rho_-(C'_y), \rho_-(C_y)) \leq w$.

(e) Finally, let us consider the case $w_x = 1$; hence, $\lambda_A^{++} = \lambda^{++}$. So, in (b), we get $\pi'_+(0) = w'(\lambda^{++})$ with $w' \leq w_\lambda$; hence, $\pi'_+(0) \geq_{Q^\vee} w_\lambda(\lambda^{++}) = \lambda$ and $\lambda \leq_{Q^\vee} \pi'_+(0) \leq_{Q^\vee} \rho_-(y) - \rho_-(x) = \mu \leq_{Q^\vee} \lambda^{++}$. If, moreover, λ is in Y^{++}, then $\lambda = \lambda^{++}$ and $\mu = \lambda$. The Hecke path $\rho_-([x, y])$ is of shape λ and equal to the segment $[\rho_-(x), \rho_-(x)+\lambda]$. Its dual dimension is 0 [op. cit., 5.7]. By [op. cit., 6.3], there is one and only one segment in $\mathscr{I}$ with end y that retracts onto this Hecke path: any apartment containing y and $\mathfrak{S}^-$ contains $[x, y]$. But C_x is in the enclosure of x and $C'_y = C_y$ (computation in A_1). So, any apartment containing $\mathfrak{S}^-$ and C'_y contains C_x. Therefore, we have $\lambda = d^W(C_x, C'_y) = d^W(\rho_-(C_x), \rho_-(C'_y))$.

The end of the proof of the lemma follows then from (d) above. □

Proposition 2.2. *Let $C_x, C_y, C_z \in \mathscr{C}_0^+$ be such that $x \leq z \leq y$ and*

$$d^W(C_x, C_z) = \boldsymbol{w} \in W^+ \quad \text{and} \quad d^W(C_z, C_y) = \boldsymbol{v} \in W^+.$$

Then $d^W(C_x, C_y)$ varies in a finite subset $P_{\boldsymbol{w},\boldsymbol{v}}$ of W^+, depending only on $\boldsymbol{w}$ and $\boldsymbol{v}$.

Let us write $\boldsymbol{w} = \lambda.w$ and $\boldsymbol{v} = \mu.v$ for $\lambda, \mu \in Y^+$ and $w, v \in W^v$. If we assume $\lambda = \lambda^{++}$ and $w = 1$, then any $\boldsymbol{w}' = \nu.u \in P_{\boldsymbol{w},\boldsymbol{v}}$ satisfies $\lambda + \mu \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda + \mu^{++}$ and $\nu - \lambda \in \mathrm{conv}(W^v.\mu^{++}, \geq \mu) \subset \mathrm{conv}(W^v.\mu^{++})$.

If, moreover, $\mu = \mu^{++} \in Y^{++}$, then $\nu = \lambda + \mu$ and $u \leq v$. In particular, for $\boldsymbol{w} = \lambda$ and $\boldsymbol{w}' = \mu$ in Y^{++}, we have $P_{\boldsymbol{w},\boldsymbol{v}} = \{\lambda + \mu\}$.

Proof. Consider any apartment A containing C_x, the sector-germ $\mathfrak{S}^-$ opposite C_x and the retraction ρ_- as in Lemma 2.1. Then $\rho_-(C_x) = C_x$ and $d^W(C_x, \rho_-(C_z))$ varies in a finite subset P_x of W^+ depending on $\boldsymbol{w}$, by Lemma 2.1. If

$$d^W(C_x, \rho_-(C_z)) = \lambda'.w',$$

then the relative position $w_z \in W^v$ of C_z and $\mathfrak{S}^-$ is equal to w'. Applying once more Lemma 2.1 to C_z and C_y, we get that $d^W(\rho_-(C_z), \rho_-(C_y))$ varies in a finite subset $P_{w'}$ of W^+ depending only on $\boldsymbol{v}$ and w'. Finally, $d^W(C_x, \rho_-(C_y))$ varies in the finite subset

$$P_{\boldsymbol{w},\boldsymbol{v}} = \{\boldsymbol{w}'.\boldsymbol{v}' \in W^+ \mid \boldsymbol{w}' = \lambda'.w' \in P_x \text{ and } \boldsymbol{v}' \in P_{w'}\}.$$

Taking now A containing C_x and C_y, we get $d^W(C_x, C_y) = d^W(C_x, \rho_-(C_y)) \in P_{\boldsymbol{w},\boldsymbol{v}}$.

To finish, suppose $\lambda = \lambda^{++}$ and $w = 1$. By Lemma 2.1, $P_1 = \{\lambda\}$; so, $w' = w_z = 1$. By Lemma 2.1 again, every $\boldsymbol{v}' = \mu'.v' \in P_{w'}$ satisfies $\mu \leq_{Q^\vee} \mu' \leq_{Q^\vee} \mu^{++}$. Therefore, any $\boldsymbol{w}'' = \nu.u$ in $P_{\boldsymbol{w},\boldsymbol{v}}$ is equal to $(\lambda + \mu').v'$ for $\mu'.v' \in P_{w'} = P_1$; hence

$$\lambda + \mu \leq_{Q^\vee} \nu = \lambda + \mu' \leq_{Q^\vee} \lambda + \mu^{++}.$$

If moreover $\mu \in Y^{++}$, then $\nu = \lambda + \mu$ and $u \leq v$. The last particular case is now clear. □

Proposition 2.3. *Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \leq y$ and $d^W(C_x, C_y) = \boldsymbol{u} \in W^+$. We consider $\boldsymbol{w}$ and $\boldsymbol{v}$ in W^+. Then the number $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}}$ of $C_z \in C_0^+$ with $x \leq z \leq y$, $d^W(C_x, C_z) = \boldsymbol{w}$ and $d^W(C_z, C_y) = \boldsymbol{v}$ is finite, i.e., in $\mathbb{N}$.*

If we assume $\boldsymbol{w} = \lambda$, $\boldsymbol{v} = \mu$ and $\boldsymbol{u} = \nu$, then $a_{\boldsymbol{w},\boldsymbol{v}}^{\boldsymbol{u}} = a_{\lambda,\mu}^{\nu} \geq 1$ (resp., $a_{\lambda,\mu}^{\nu} = 1$) when $\lambda \in Y^{++}$, $\mu \in Y^+$ (resp., $\lambda, \mu \in Y^{++}$) and $\nu = \lambda + \mu$.

N.B. From the above conditions, we get $d^v(x, z) = \lambda^{++}$ and $d^v(z, y) = \mu^{++}$. By [Gaussent and Rousseau 2014, 2.5], the number of points z satisfying these conditions is finite.

Proof. According to the above note, we may fix z and count now the possible C_z.

Let C_z' be the local chamber in z containing $[z, y)$ and $[z, y')$ for y' in a sufficiently small element of the filter C_y. By convexity, C_z' is well determined by z and C_y. But in an apartment containing C_y and C_z (hence also C_z'), we see that $d^W(C_z', C_z)$ is well determined by $\boldsymbol{v}$. So there is a gallery (of a fixed type) from C_z' to C_z, thus the number of possible C_z is finite.

Assume now that $\boldsymbol{w} = \lambda \in Y^{++}$, $\boldsymbol{v} = \mu \in Y^+$, and $\boldsymbol{u} = \lambda + \mu$. Taking an apartment A_1 containing C_x and C_y, it is clear that the local chamber C_z in A_1 such that $d^W(C_x, C_z) = \lambda$ satisfies also $d^W(C_z, C_y) = \mu$ (as $d^W(C_x, C_y) = \lambda + \mu$). So $a_{\lambda,\mu}^{\lambda+\mu} \geq 1$. We consider now any C_z satisfying the conditions, with moreover $\mu \in Y^{++}$.

As in Proposition 2.2, we choose A containing C_x and $\mathfrak{S}^-$ opposite C_x. We saw in Lemma 2.1(e) that any apartment containing C_z and $\mathfrak{S}^-$ contains C_x and $d^W(C_x, \rho_-(C_z)) = \lambda$. With the same lemma applied to C_z and C_y, we see that any apartment containing C_z and $\mathfrak{S}^-$ contains C_y. In particular, there is an apartment A_1 containing C_x, C_z, C_y; so $d^W(C_x, C_z) = \lambda$, $d^W(C_z, C_y) = \mu$, and $d^W(C_x, C_y) = \lambda + \mu$. But $\lambda, \mu \in Y^{++}$, so C_z is in the enclosure of C_x and C_y. Therefore, C_z is unique: any other apartment A_2 containing C_x and C_y also contains x, y (with $x \leq y$) and $x' = x + \xi$, $y' = y + \xi$ (with $x' \leq y'$), for $\xi \in C_x^v = C_y^v$ small; by Section 1.9(a), A_2 contains $z \in \mathrm{cl}_{A_1}(\{x, y\})$ and $z' = z + \xi \in \mathrm{cl}_{A_1}(\{x', y'\})$, hence also $C_z \subset \mathrm{cl}_{A_1}(]z, z'))$. □

Theorem 2.4. *For any ring R, ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ is an algebra with identity* $\mathrm{Id} = T_1$ *such that*

$$T_{\boldsymbol{w}} * T_{\boldsymbol{v}} = \sum_{\boldsymbol{u} \in P_{\boldsymbol{w},\boldsymbol{v}}} a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} T_{\boldsymbol{u}}$$

and $T_\lambda * T_\mu = T_{\lambda+\mu}$ for $\lambda, \mu \in Y^{++}$.

Proof. It follows from Propositions 2.2 and 2.3, as the map $T_{\boldsymbol{w}} * T_{\boldsymbol{v}} : \mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+ \to R$ is clearly G-invariant. □

Definition 2.5. The algebra ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ is the Iwahori–Hecke algebra associated to $\mathscr{I}$ with coefficients in R.

The structure constants $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ are nonnegative integers. We conjecture that they are polynomials in the parameters q_i, q_i' with coefficients in $\mathbb{Z}$ and that these polynomials depend only on $\mathbb{A}$ and W. We prove this in the following section for $\boldsymbol{w}$, $\boldsymbol{v}$ generic, see the precise hypothesis just below. We get also this conjecture for some $\mathbb{A}$, W when all q_i, q_i' are equal; in the general case we get only that they are Laurent polynomials, see Section 6.7.

Geometrically, it is possible to get more information about $T_\lambda * T_\mu$ when $\lambda \in Y^{++}$ and $\mu \in Y^+$, but we shall obtain them algebraically; see Corollary 5.3.

3. Structure constants

In this section, we compute the structure constants $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ of the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$, assuming that $\boldsymbol{v} = \mu . v$ is regular and $\boldsymbol{w} = \lambda . w$ is spherical, i.e., μ is regular and λ is spherical; see Section 1.1 for the definitions. We will adapt some results obtained in the spherical case in [Gaussent and Rousseau 2014] to our situation.

These structure constants depend on the shape of the standard apartment $\mathbb{A}$ and on the numbers q_M of Section 1.4. Recall that the number of (possibly) different parameters is at most $2|I|$. We denote by $\mathcal{Q} = \{q_1, \dots, q_l, q_1' = q_{l+1}, \dots, q_l' = q_{2l}\}$ this set of parameters.

3.1. ***Centrifugally folded galleries of chambers.*** Let z be a point in the standard apartment $\mathbb{A}$. We have twinned buildings $\mathcal{T}_z^+\mathscr{I}$ (resp., $\mathcal{T}_z^-\mathscr{I}$). We consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp., closed chambers) are the local chambers $C = \mathrm{germ}_z(z + C^v)$ (resp., local closed chambers $\overline{C} = \mathrm{germ}_z(z + \overline{C^v})$), where C^v is a vectorial chamber; see [Gaussent and Rousseau 2008, 4.5] or [Rousseau 2011, §5]. The distances (resp., codistances) between these chambers are written d^W (resp., d^{*W}). To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_z = (\mathbb{A}_z^-, \mathbb{A}_z^+)$.

Choose in $\mathbb{A}_z^-$ a negative (local) chamber C_z^- and denote by C_z^+ its opposite in $\mathbb{A}_z^+$. Consider the system of positive roots Φ^+ associated to C_z^+. Actually, $\Phi^+ = w . \Phi_f^+$ if Φ_f^+ is the system Φ^+ defined in Section 1.1 and $C_z^+ = \mathrm{germ}_z(z + w . C_f^v)$. Denote by $(\alpha_i)_{i\in I}$ the corresponding basis of Φ and by $(r_i)_{i\in I}$ the corresponding generators of W^v. Note that this change of notation is limited to Section 3.

Fix a reduced decomposition of an element $w \in W^v$, $w = r_{i_1} \dots r_{i_r}$, and let $\boldsymbol{i} = (i_1, \dots, i_r)$ be the type of the decomposition. Now consider galleries of (local) chambers $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r)$ in the apartment $\mathbb{A}_z^-$ starting at C_z^- and of type $\boldsymbol{i}$.

The set of all these galleries is in bijection with $\Gamma(\boldsymbol{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$ via the map

$$(c_1, \dots, c_r) \mapsto (C_z^-, c_1 C_z^-, \dots, c_1 \dots c_r C_z^-).$$

Let $\beta_j = -c_1 \cdots c_j(\alpha_{i_j})$; then β_j is the root corresponding to the common limit hyperplane $M_j = M(\beta_j, -\beta_j(z))$ of type i_j of

$$C_{j-1} = c_1 \cdots c_{j-1} C_z^- \quad \text{and} \quad C_j = c_1 \cdots c_j C_z^-$$

and satisfying $\beta_j(C_j) \geq \beta_j(z)$.

Definition. Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_z^+$. A gallery $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r) \in \Gamma(\boldsymbol{i})$ is said to be centrifugally folded with respect to $\mathfrak{Q}$ if $C_j = C_{j-1}$ implies that M_j is a wall and separates $\mathfrak{Q}$ from $C_j = C_{j-1}$. We denote this set of centrifugally folded galleries by $\Gamma_{\mathfrak{Q}}^+(\boldsymbol{i})$.

3.2. ***Liftings of galleries.*** Next, let $\rho_{\mathfrak{Q}} : \mathcal{T}_z\mathcal{I} \to \mathbb{A}_z$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\boldsymbol{c} = (C_z^-, C_1, \dots, C_r)$ in $\Gamma(\boldsymbol{i})$, one can associate the set of all galleries of type $\boldsymbol{i}$ starting at C_z^- in $\mathcal{T}_z^-\mathcal{I}$ that retract onto $\boldsymbol{c}$; we denote this set by $\mathcal{C}_{\mathfrak{Q}}(\boldsymbol{c})$. We denote the set of minimal galleries (i.e., $C_{j-1} \neq C_j$) in $\mathcal{C}_{\mathfrak{Q}}(\boldsymbol{c})$ by $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$. Recall from [Gaussent and Rousseau 2014, Proposition 4.4], that the set $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$ is nonempty if and only if the gallery $\boldsymbol{c}$ is centrifugally folded with respect to $\mathfrak{Q}$. Recall also from [op. cit., Corollary 4.5], that if $\boldsymbol{c} \in \Gamma_{\mathfrak{Q}}^+(\boldsymbol{i})$, then the number of elements in $\mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c})$ is

$$\sharp\, \mathcal{C}_{\mathfrak{Q}}^m(\boldsymbol{c}) = \prod_{j\in J_1} (q_j - 1) \times \prod_{j \in J_2} q_j$$

where $q_j = q_{M_j} \in \mathcal{Q}$,

$$J_1 = \{ j \in \{1, \dots, r\} \mid c_j = 1 \}$$

and

$$J_2 = \{ j \in \{1, \dots, r\} \mid c_j = r_{i_j} \text{ and } M_j \text{ is a wall separating } \mathfrak{Q} \text{ from } C_j \}.$$

3.3. ***Liftings of Hecke paths.*** The Hecke paths we consider here are slight modifications of those used in [Gaussent and Rousseau 2014]. Let us fix a local positive chamber $C_x \in \mathscr{C}_0^+ \cap \mathbb{A}$. Actually, a Hecke path of shape μ^{++} with respect to C_x in $\mathbb{A}$ is a μ^{++}-path in $\mathbb{A}$ that we denote by $\pi = [z' = z_0, z_1, \dots, z_{\ell_\pi}, y]$ and that satisfies the following assumptions.

For all $z = \pi(t)$, $z \neq z_0 = \pi(0)$, we ask that $x \overset{o}{<} z$. Then we choose the local negative chamber C_z^- as $C_z^- = \mathrm{pr}_z(C_x)$. This means that $\overline{C_z^-}$ contains $[z, x)$ and $[z, x')$ for x' in a sufficiently small element of the filter C_x. Then we assume moreover that for all $k \in \{1, \dots, \ell_\pi\}$, there exists a $(W^v_{z_k}, C^-_{z_k})$-chain from $\pi'_-(t_k)$ to $\pi'_+(t_k)$, where $z_k = \pi(t_k)$. More precisely, this means that, for all $k \in \{1, \dots, \ell_\pi\}$, there exist finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s)$ of real roots such that, for all $j = 1, \dots, s$:

(i) $r_{\beta_j}(\xi_{j-1}) = \xi_j$,

(ii) $\beta_j(\xi_{j-1}) < 0$,

(iii) $r_{\beta_j} \in W^v_{\pi(t_k)}$, i.e., $\beta_j(\pi(t_k)) \in \mathbb{Z}$,

(iv) each β_j is positive with respect to C_x, i.e., $\beta_j(z_k - C_x) > 0$.

The centrifugally folded galleries are related to the lifting of Hecke paths by the following lemma that we proved in [Gaussent and Rousseau 2014, Lemma 4.6].

Suppose that $z \in \mathbb{A}$ with $x \overset{o}{<} z$. Let ξ and η be two segment-germs in $\mathbb{A}_z^+$. Let $-\eta$ and $-\xi$ be opposite, respectively, of η and ξ in $\mathbb{A}_z^-$. Let $\boldsymbol{i}$ be the type of a minimal gallery between C_z^- and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $d^W(C_z^-, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}_z^+$ containing η. Suppose that ξ and η are conjugated by W_z^v.

Lemma. *The following conditions are equivalent*:

(i) *There exists an opposite ζ to η in $\mathcal{T}_z^- \mathscr{I}$ such that $\rho_{\mathbb{A}_z, C_z^-}(\zeta) = -\xi$.*

(ii) *There exists a gallery $\boldsymbol{c} \in \Gamma^+_{\mathfrak{Q}}(\boldsymbol{i})$ ending in $-\eta$.*

(iii) *There exists a (W_z^v, C_z^-)-chain from ξ to η.*

Moreover the possible ζ are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{\mathfrak{Q}}(\boldsymbol{c})$ for $\boldsymbol{c}$ in the set $\Gamma^+_{\mathfrak{Q}}(\boldsymbol{i}, -\eta)$ of galleries in $\Gamma^+_{\mathfrak{Q}}(\boldsymbol{i})$ ending in $-\eta$.

For a Hecke path as above and for $k \in \{1, \dots, \ell_\pi\}$, we define the segment-germs $\eta_k = \pi_+(t_k) = \pi(t_k) + \pi'_+(t_k).[0, 1)$ and $-\xi_k = \pi_-(t_k) = \pi(t_k) - \pi'_-(t_k).[0, 1)$. As above, $\boldsymbol{i}_k$ is the type of a minimal gallery between $C^-_{z_k}$ and $C_{-\xi_k}$, where $C_{-\xi_k}$ is the negative (local) chamber such that $-\xi_k \subset \overline{C_{-\xi_k}}$ and $d^W(C^-_{z_k}, C_{-\xi_k})$ is of minimal length. Let $\mathfrak{Q}_k$ be a fixed chamber in $\mathbb{A}^+_{z_k}$ containing η_k and $\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)$ be the set of all the galleries $(C^-_{z_k}, C_1, \dots, C_r)$ of type $\boldsymbol{i}_k$ in $\mathbb{A}^-_{z_k}$, centrifugally folded with respect to $\mathfrak{Q}_k$ and with $-\eta_k \in \overline{C_r}$.

Let us denote the retraction $\rho_{\mathbb{A}, C_x} : \mathscr{I}_{\geq x} \to \mathbb{A}$ simply by ρ and recall that $y = \pi(1)$. Let $S_{C_x}(\pi, y)$ be the set of all segments $[z, y]$ such that $\rho([z, y]) = \pi$, in particular, $\rho(z) = z'$. The following two theorems are proved in the same way as Theorems 4.8 and 4.12 of [Gaussent and Rousseau 2014]; in particular, we lift the path π step by step starting from the end of π.

Theorem 3.4. *The set $S_{C_x}(\pi, y)$ is nonempty if and only if π is a Hecke path with respect to C_x. Then, we have a bijection*

$$S_{C_x}(\pi, y) \simeq \prod_{k=1}^{\ell_\pi} \coprod_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k,-\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}).$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ depending only on $\mathbb{A}$.

Theorem 3.5. *Let $\lambda, \mu, \nu \in Y^{++}$ with λ spherical. Then, the number $m_{\lambda,\mu}(\nu)$ of points z in $\mathscr{I}$ with $d^v(0, z) = \lambda$ and $d^v(z, \nu) = \mu$ is equal to*

$$m_{\lambda,\mu}(\nu) = \sum_{w\in W^v/(W^v)_\lambda} \sum_{\pi} \prod_{k=1}^{\ell_\pi} \sum_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k,-\eta_k)} \sharp\mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}), \tag{1}$$

where π runs over the set of Hecke paths of shape μ with respect to C_x from $w\,.\,\lambda$ to ν and ℓ_π, $\Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)$, and $\mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c})$ are defined as above for each such π.

Remark. In Theorems 3.4 and 3.5 above and in [Gaussent and Rousseau 2014], it is interesting to note that if $t_{\ell_\pi} = 1$, i.e., $z_{\ell_\pi} = y$, then, in the above formulas, $-\eta_{\ell_\pi}$ and $\mathfrak{Q}_{\ell_\pi}$ are not well defined: $\pi_+(1)$ does not exist. We have to understand that

$$\coprod_{\boldsymbol{c}\in\Gamma^+_{\mathfrak{Q}_{\ell_\pi}}(\boldsymbol{i}_{\ell_\pi},-\eta_{\ell_\pi})} \mathcal{C}^m_{\mathfrak{Q}_{\ell_\pi}}(\boldsymbol{c})$$

is the set of all minimal galleries of type $\boldsymbol{i}_{\ell_\pi}$ starting from C_y^-, whose cardinality is $\prod_{j=1}^r q_{i_j}$ if $\boldsymbol{i}_{\ell_\pi} = (i_1, \dots, i_r)$.

3.6. ***The formula.*** Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \le y$ and $d^W(C_x, C_y) = \boldsymbol{u} \in W^+$. We consider $\boldsymbol{w}$ and $\boldsymbol{v}$ in W^+. Then we know that the number $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ of $C_z \in \mathscr{C}_0^+$ with $x \le z \le y$, $d^W(C_x, C_z) = \boldsymbol{w}$, and $d^W(C_z, C_y) = \boldsymbol{v}$ is finite; see Proposition 2.3. In order to obtain a formula for that number, we first use equivalent conditions on the W-distance between the chambers.

Lemma. (1) *Assume λ is spherical. Let $C_z^- = \mathrm{pr}_z(C_x)$ and let w_λ^+ be the longest element such that $w_\lambda^+.\lambda \in \overline{C^v_f}$. Then*

$$d^W(C_x, C_z) = \lambda\,.\,w \iff \begin{cases} d^W(C_x, z) = \lambda, \\ d^{*W}(C_z^-, C_z) = w_\lambda^+ w. \end{cases}$$

(2) *Assume μ is regular. Let $C_z^+ = \mathrm{pr}_z(C_y)$ and let w_μ be the unique element such that $\mu^{++} = w_\mu\,.\,\mu \in \overline{C^v_f}$. Then*

$$d^W(C_z, C_y) = \mu\,.\,v \iff \begin{cases} d^W(C_z, C_z^+) = w_\mu^{-1}, \\ d^W(C_z^+, C_y) = \mu^{++} w_\mu v. \end{cases}$$

As we assume μ regular, $C'_y = \mathrm{pr}_y(C_z)$ is the unique local chamber in y containing $[y, z)$, and $C_z^+ = \mathrm{pr}_z(C_y)$ is the unique local chamber in z containing $[z, y)$. Also,

$$d^W(C_z^+, C_y) = \mu^{++} w_\mu v \iff d^v(z, y) = \mu^{++} \text{ and } d^{*W}(C'_y, C_y) = w_\mu v.$$

Proof. (1) By convexity, C_z^- is in any apartment containing C_x and C_z. Let us fix such an apartment A and identify (A, C_x) with $(\mathbb{A}, \mathrm{germ}_0(C_f^v))$. By definition, we have $d^W(C_x, z) = d^W(C_x, z + C_x)$. Then, of course, $d^W(C_x, z) = \lambda$. Next as λ is supposed spherical, the stabilizer $(W^v)_\lambda$ is finite, so w_λ^+ is well defined and $x \overset{o}{<} z$, so C_z^- is well defined. Moreover, $d^W(\mathrm{op}_A\, C_z^-, z + C_x) = w_\lambda^+$ and $d^W(z + C_x, C_z) = w$. Therefore, by Chasles, we get $d^W(\mathrm{op}_A\, C_z^-, C_z) = w_\lambda^+ w$, but, by definition, $d^{*W}(C_z^-, C_z) = d^W(\mathrm{op}_A\, C_z^-, z + C_z)$.

(2) The first assertion is the Chasles' relation, as C_z, C_y, C_z^+, (and C'_y) are in a same apartment A'. The second comes from the fact that, if μ is regular, then $d^W(C_z^+, C_{zy}^+) = d^v(z, y) \in Y^{++}$, where C_{zy}^+ is opposite C'_y at y in A'. Moreover, $d^{*W}(C'_y, C_y) = d^W(C_{zy}^+, C_y) \in W^v$ by definition, so we conclude by Chasles. $\square$

Theorem 3.7. *Assume μ is regular and λ is spherical. We choose the standard apartment $\mathbb{A}$ containing C_x and C_y. Then*

$$a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} = \sum_{\pi, t_{\ell_\pi}=1} \left(\prod_{k=1}^{\ell_\pi - 1} \sum_{\boldsymbol{c} \in \Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)} \sharp \mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}) \right) \left(\sum_{\boldsymbol{d} \in \Gamma^+_{C_y}(\boldsymbol{i}_\ell, \tilde{C}_y)} \sharp \mathcal{C}^m_{C_y}(\boldsymbol{d}) \right) \left(\sum_{\boldsymbol{e} \in \Gamma^+_{C^-_{z_0}}(\boldsymbol{i}, C'_{z_0})} \sharp \mathcal{C}^m_{C^-_{z_0}}(\boldsymbol{e}) \right)$$
$$+ \sum_{\pi, t_{\ell_\pi} < 1} \left(\prod_{k=1}^{\ell_\pi} \sum_{\boldsymbol{c} \in \Gamma^+_{\mathfrak{Q}_k}(\boldsymbol{i}_k, -\eta_k)} \sharp \mathcal{C}^m_{\mathfrak{Q}_k}(\boldsymbol{c}) \right) \left(\sum_{\boldsymbol{e} \in \Gamma^+_{C^-_{z_0}}(\boldsymbol{i}, C'_{z_0})} \sharp \mathcal{C}^m_{C^-_{z_0}}(\boldsymbol{e}) \right),$$

where the π in the first sum runs over the set of all Hecke paths in $\mathbb{A}$ with respect to C_x of shape μ^{++} from $x + \lambda = z_0$ to $x + \nu = y$ such that $t_{\ell_\pi} = 1$; whereas, in the second sum, the paths have to satisfy $t_{\ell_\pi} < 1$ and $d^{*W}(C_y^-, C_y) = w_\mu v$, where $C_y^- = \mathrm{pr}_y(C_x)$ is the local chamber in y containing $[y, x)$ and $[y, x')$ for x' in a sufficiently small element of the filter C_x.

Moreover, $\boldsymbol{i}$ is a reduced decomposition of w_μ, C'_{z_0} is the local chamber at z_0 in $\mathbb{A}$ defined by $d^{*W}(C^-_{z_0}, C'_{z_0}) = w_\lambda^+ w$, $\boldsymbol{i}_\ell$ is the type of a minimal gallery from C_y^- to the local chamber C_y^* at y in $\mathbb{A}$ containing the segment-germ $\pi_-(y) = y - \pi'_-(1).[0, 1)$, and $\tilde{C}_y$ is the unique local chamber at y in $\mathbb{A}$ such that $d^{*W}(\tilde{C}_y, C_y) = w_\mu v$. The rest of the notation is as defined above.

Proof. Recall that in order to compute the structure constants, we use the retraction $\rho = \rho_{\mathbb{A}, C_x} : \mathscr{I} \to \mathbb{A}$, where C_x and C_y are fixed and in $\mathbb{A}$. We have $y = \rho(y) = x + \nu$, and the condition $d^W(C_x, z) = \lambda$ is equivalent to $\rho(z) = x + \lambda = z_0$. We want to prove a formula of the form

$$a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}} = \sum_{\pi} \big(\text{number of liftings of } \pi\big) \times \big(\text{number of } C_z\big),$$

where π runs over some set of Hecke paths with respect to C_x of shape μ^{++} from $x+\lambda$ to $x+\nu$. It is possible to calculate like that for, in the case of a regular μ^{++}, $\rho(C_z^+)$ is well determined by π. Hence, the number of C_z only depends on π and not on the lifting of π.

The local chambers C_z satisfying $d^{*W}(C_z^-, C_z) = w_\lambda^+ w$ and $d^W(C_z, C_z^+) = w_\mu^{-1}$ are at the end of a minimal gallery starting at C_z^+ of type $\boldsymbol{i}$ and retracting by ρ_{A', C_z^-} onto the local chamber C_z' at z defined by $d^{*W}(C_z^-, C_z') = w_\lambda^+ w$ in a fixed apartment A' containing C_x and C_z^+. So their number is given by the number of minimal galleries starting at C_z^+ of type $\boldsymbol{i}$ and retracting on a centrifugally folded gallery $\boldsymbol{e}$ of type $\boldsymbol{i}$ ending in C_z'. In other words, their number is given by the cardinality of the set $\mathcal{C}^m_{C_z^-}(\boldsymbol{e})$, for each $\boldsymbol{e} \in \Gamma^+_{C_z^-}(\boldsymbol{i}, C_z')$. Using an isomorphism fixing C_x and sending A' to $\mathbb{A}$, we may replace in this formula z, C_z^-, C_z', and C_z^+ by z_0, $C_{z_0}^-$, C_{z_0}', and the unique local chamber $C_{z_0}^+$ in $\mathbb{A}$ containing the segment-germ $\pi_+(0) = z_0 + \pi_+'(0).[0, 1)$. Hence,

$$\text{number of } C_z = \sum_{\boldsymbol{e} \in \Gamma^+_{C_{z_0}^-}(\boldsymbol{i}, C_{z_0}')} \sharp \mathcal{C}^m_{C_{z_0}^-}(\boldsymbol{e}).$$

Now, we compute the number of liftings of a Hecke path π starting from the formula in Theorem 3.5 and according to the two conditions $d^W(C_x, z) = \lambda$ and $d^W(C_z^+, C_y) = \mu^{++} w_\mu v$. The first one fixes one element in the set $W^v/(W^v)_\lambda$, namely the coset of w_λ^+, i.e., $\pi(0) = x + \lambda$. The second one is equivalent to the fact that the segment $[z, y]$ is of type μ^{++} and $d^{*W}(C_y', C_y) = w_\mu v$, as we have seen in the lemma above.

Further, we have that $t_{\ell_\pi} < 1$ if and only if $\pi_-(y) \in C_y^-$. If $\pi_-(y) \in C_y^-$ then $\rho(C_y') = C_y' = C_y^-$, whence, $d^{*W}(C_y^-, C_y) = w_\mu v$. Since we lift the Hecke path into a segment backwards starting with its behavior at $y = \pi(1)$, there is nothing more to count.

If $t_{\ell_\pi} = 1$, then $\pi_-(y) \in C_y^* = \rho(C_y') \neq C_y^-$. We want to lift the path but with the condition that $d^{*W}(C_y', C_y) = w_\mu v$, which may be translated in $\rho'(C_y') = \tilde{C}_y$, for $\rho' = \rho_{\mathbb{A}, C_y}$. Since μ^{++} is regular, to find $[y, z)$ it is enough to find C_y', i.e., to lift $\tilde{C}_y$ with respect to ρ'. The liftings of $\tilde{C}_y$ are then given by the liftings of all the centrifugally folded galleries in $\mathbb{A}$ with respect to C_y of type $\boldsymbol{i}_\ell$ from C_y^- to $\tilde{C}_y$ to minimal galleries. Therefore, their number is given by the cardinality of the set $\mathcal{C}^m_{C_y}(\boldsymbol{d})$, for each $\boldsymbol{d} \in \Gamma^+_{C_y}(\boldsymbol{i}_\ell, \tilde{C}_y)$. The rest of the lifting procedure is the same as in the proof of Theorem 4.12 in [Gaussent and Rousseau 2014]. □

3.8. *Consequence.* The above explicit formula, together with the formula for $\sharp\mathcal{C}^m_{\mathfrak{Q}}(\boldsymbol{c})$ in Section 3.2, tell us that the structure constant $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is a polynomial in the parameters $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ and that this polynomial depends only on $\mathbb{A}$, W, $\boldsymbol{w}$, $\boldsymbol{v}$, and $\boldsymbol{u}$. So we have proved the conjecture following Definition 2.5 in this generic case: when λ is spherical and μ regular.

4. Relations

Here we study the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}^{\mathcal{I}}_R$ as a module over $\mathcal{H}_R(W^v)$ and we prove the first instance of the Bernstein–Lusztig relation. For short, we write ${}^{\mathrm{I}}\mathcal{H}_R = {}^{\mathrm{I}}\mathcal{H}^{\mathcal{I}}_R$ and $T_i = T_{r_i}$ (when $i \in I$).

Proposition 4.1. *Let $\lambda \in Y^+$, $w \in W^v$, and $i \in I$. Then*:

(1) $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$ *if and only if either* $(w(\alpha_i))(\lambda) < 0$ *or* $(w(\alpha_i))(\lambda) = 0$ *and* $\ell(wr_i) > \ell(w)$. *Otherwise* $T_{\lambda.w} * T_i = (q_i - 1)T_{\lambda.w} + q_i T_{\lambda.wr_i}$.

(2) $T_i * T_{\lambda.w} = T_{r_i(\lambda).r_iw}$ *if and only if either* $\alpha_i(\lambda) > 0$ *or* $\alpha_i(\lambda) = 0$ *and* $\ell(r_iw) > \ell(w)$. *Otherwise* $T_i * T_{\lambda.w} = (q_i - 1)T_{\lambda.w} + q_i T_{r_i(\lambda).r_iw}$.

Proof. We consider local chambers C_x, C_z, C_y with $x \le z \le y$ and $d^W(C_x, C_z) = \lambda.w$, $d^W(C_z, C_y) = r_i$. So there is an apartment A containing C_x, C_z (but perhaps not C_y) and, if we identify (A, C_x) to $(\mathbb{A}, C_0^+)$, we have $C_z = (\lambda.w)(C_x) = w(C_x) + \lambda$.

Moreover, $y = z$, $C_z \ne C_y$, and C_z, C_y share a panel F_i of type i. We write D for the half-apartment of A containing C_x and with wall ∂D containing F_i.

Actually the equation of ∂D in A is $(w(\alpha_i))(x') = (w(\alpha_i))(z)$. As $\alpha_i > 0$ on C_x, we have $(w(\alpha_i))(C_z) > (w(\alpha_i))(z)$. And so $(w(\alpha_i))(z) = (w(\alpha_i))(\lambda) < 0 = (w(\alpha_i))(x)$ (resp., $> 0 = (w(\alpha_i))(x)$) if and only if C_z is strictly on the same side (resp., the opposite side) of ∂D as x, hence as C_x; i.e., $C_z \subset D$ (resp., $C_z \not\subset D$). If now $(w(\alpha_i))(\lambda) = 0$, we may argue as if $\lambda = 0$, i.e., $C_z = w(C_x)$, then it is well known that $C_z \subset D$ if and only if $\ell(wr_i) > \ell(w)$. So,

$$C_z \subset D \iff \big((w(\alpha_i))(\lambda) < 0\big) \text{ or } \big((w(\alpha_i))(\lambda) = 0 \text{ and } \ell(wr_i) > \ell(w)\big).$$

Then, by Section 1.4.2, there exists an apartment A' containing C_y and D, hence also C_x, C_z, C_y. So $d^W(C_x, C_y) = \lambda.wr_i$. The panel $F_i = F^\ell(z, F_i^v) \subset A$ is a spherical local face, so, for any $p \in z + F_i^v \subset A$, we have $z \overset{o}{<} p$, hence $x \overset{o}{<} p$. By Proposition 1.10(a), any apartment A'' containing C_x and F_i contains C_z; moreover C_z is well determined by F_i and C_x. The number $a^{\lambda.wr_i}_{\lambda.w,r_i}$ of Proposition 2.3 is equal to 1 and we have proved that $T_{\lambda.w} * T_i = T_{\lambda.wr_i}$.

If C_z is not in D, we denote by C'_z the local chamber in D with panel F_i. By the above argument, C'_z is well determined by F_i and C_x; moreover $d^W(C_x, C'_z) = \lambda.wr_i$. There are two cases: either $C_y = C'_z$ or not. If $C_y = C'_z$, then $d^W(C_x, C_y) = \lambda.wr_i$, and if C_x, C_y are given, there are q_i possibilities for C_z (all local chambers covering F_i

and different from C'_z): $a_{\lambda.w,r_i}^{\lambda.wr_i} = q_i$. If $C_y \neq C'_z$, then $d^W(C_x, C_y) = \lambda.w$ and, if C_x, C_y are given, there are $q_i - 1$ possibilities for C_z (all local chambers covering F_i and different from C'_z, C_y): $a_{\lambda.w,r_i}^{\lambda.w} = q_i - 1$.

We have proved (1) and we leave to the reader the similar proof of (2). □

4.2. ***The subalgebra $\mathcal{H}_R(W^v)$.*** We consider the R-submodule $\mathcal{H}_R(W^v)$ of ${}^{\mathrm{I}}\mathcal{H}_R$ with basis $(T_w)_{w\in W^v}$. As $d^W(C_x, C_y) \in W^v$ if and only if $x = y$, it is clearly a subalgebra of ${}^{\mathrm{I}}\mathcal{H}_R$. Actually $\mathcal{H}_R(W^v)$ is the Iwahori–Hecke algebra of the tangent building $\mathcal{T}_x^+\mathcal{I}$ for any $x \in \mathcal{I}$.

By Proposition 4.1,

$$\begin{cases} T_w * T_i = T_{wr_i} & \text{if } \ell(wr_i) > \ell(w), \\ T_w * T_i = (q_i - 1)T_w + q_i T_{wr_i} & \text{otherwise,} \end{cases}$$

and

$$\begin{cases} T_i * T_w = T_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ T_i * T_w = (q_i - 1)T_w + q_i T_{r_i w} & \text{otherwise.} \end{cases}$$

In particular, $T_i^2 = (q_i - 1)T_i + q_i \operatorname{Id}$, and $T_w = T_{i_1} \cdots T_{i_n}$ for any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$.

Therefore, the algebra $\mathcal{H}_R(W^v)$ is the well known Hecke algebra associated to the Coxeter system $(W^v, \{r_i \mid i \in I\})$ with (in general unequal) parameters $(q_i)_{i\in I}$ and coefficients in the ring R. It is generated, as an R-algebra, by the T_i, for $i \in I$.

Suppose each q_i is invertible in R. Then, as is well known,

$$T_i^{-1} = q_i^{-1}\big(T_i - (q_i - 1)\operatorname{Id}\big) \in \mathcal{H}_R(W^v)$$

is the inverse of T_i. In particular any T_w is invertible: $T_w^{-1} = T_{i_n}^{-1} \cdots T_{i_1}^{-1}$ for any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$.

Remark. Assuming that q_i is invertible, it is easy to see from Proposition 4.1 that either $T_{\lambda.wr_i} = T_{\lambda.w} * T_i$ or $T_{\lambda.wr_i} = T_{\lambda.w} * T_i^{-1}$, and either $T_{r_i(\lambda).r_i w} = T_i * T_{\lambda.w}$ or $T_{r_i(\lambda).r_i w} = T_i^{-1} * T_{\lambda.w}$.

Corollary 4.3. *Suppose each q_i invertible in R and consider $\lambda \in Y^+$. We may write $\lambda = w.\lambda^{++}$, with $w \in W^v$. Then $T_\lambda = T_w * T_{\lambda^{++}} * T_w^{-1}$.*

Proof. Consider a reduced decomposition $w = r_{i_n} \cdots r_{i_1}$ and argue by induction on n. So, for $w' = r_{i_{n-1}} \cdots r_{i_1}$ and $\lambda' = w'.\lambda^{++}$, we have $T_{\lambda'} = T_{w'} * T_{\lambda^{++}} * T_{w'}^{-1}$. Now consider

$$T_w * T_{\lambda^{++}} * T_w^{-1} = T_{i_n} * T_{\lambda'} * T_{i_n}^{-1}.$$

But $\ell(r_{i_n} w') > \ell(w')$ and $\lambda^{++} \in Y^{++} \subset \overline{C_f^v}$, so $\alpha_{i_n}(w'.\lambda^{++}) \geq 0$, i.e., $\alpha_{i_n}(\lambda') \geq 0$. We get $T_{i_n} * T_{\lambda'} = T_{r_{i_n}(\lambda').r_{i_n}}$ by Proposition 4.1(2), and then $T_{i_n} * T_{\lambda'} * T_{i_n}^{-1} = T_{r_{i_n}(\lambda')} = T_\lambda$ by Proposition 4.1(1) (and the above remark). □

Corollary 4.4. *Let $\lambda \in Y^+$ and $w, w' \in W^v$. Then we may write*

$$T_{\lambda.w'} * T_w = \sum_{w'' \leq w} a_{\lambda.w',w}^{\lambda.w'w''} T_{\lambda.w'w''},$$

where each $a_{\lambda.w',w}^{\lambda.w'w''}$ is a polynomial in the q_i with coefficients in $\mathbb{Z}$, and, when $w' = 1$, $a_{\lambda,w}^{\lambda.w} > 0$ is a primitive monomial. This polynomial $a_{\lambda.w',w}^{\lambda.w'w''}$ depends only on $\mathbb{A}$ and on W.

Proof. Write $w = r_{i_1} \cdots r_{i_n}$ and argue by induction on n. The result is then clear from Proposition 4.1(1). Actually, $a_{\lambda,w}^{\lambda.w}$ is the product of certain q_{i_j}, $1 \leq j \leq n$. $\square$

4.5. *The Iwahori–Hecke algebra as a right $\mathcal{H}_R(W^v)$-module.* We assume here that each q_i is invertible in R.

Given $\lambda \in Y^+$, we can conclude from Corollary 4.4 that $\{T_\lambda * T_w \mid w \in W^v\}$ and $\{T_{\lambda.w} \mid w \in W^v\}$ are two bases of the same R-module. The base-change matrix is triangular with respect to the Bruhat order on W^v and the coefficients are Laurent polynomials in the q_i, with coefficients in $\mathbb{Z}$ (primitive Laurent monomials on the diagonal). These polynomials depend only on $\mathbb{A}$ and W.

As $\{T_{\lambda.w} \mid \lambda \in Y^+, w \in W^v\}$ is an R-basis of ${}^{\mathrm{I}}\mathcal{H}_R$ and $\{T_w \mid w \in W^v\}$ is an R-basis of $\mathcal{H}_R(W^v)$, in particular, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module with basis $\{T_\lambda \mid \lambda \in Y^+\}$.

The R-algebra ${}^{\mathrm{I}}\mathcal{H}_R$ is generated by the T_i (for $i \in I$) and the T_λ (for $\lambda \in Y^+$) and even by the T_i (for $i \in I$) and the T_λ (for $\lambda \in Y^{++}$), as we see from Corollary 4.3.

Lemma 4.6. *Let $C_1, C_2 \in \mathcal{C}_0^+$ with vertices x_1, x_2 be such that $d^W(C_1, C_2) = \lambda \in Y^{++}$. We consider $i \in I$, F_1^i (resp., F_2^i) the panel of type i of C_1 (resp., C_2). In an apartment A_1 (resp., A_2) containing C_1 (resp., C_2), we consider the sector panel $\mathfrak{f}_1^-$ (resp., $\mathfrak{f}_2^+$) with base point x_1 (resp., x_2) and direction opposite the direction of F_1^i (resp., equal to the direction of F_2^i).*

Then there is an apartment A containing $\mathfrak{f}_1^-$, $\mathfrak{f}_2^+$, C_1, C_2 and, in this apartment A, the directions of $\mathfrak{f}_1^-$ and $\mathfrak{f}_2^+$, F_2^i and $\mathfrak{f}_1^-$ (resp., F_1^i and $\mathfrak{f}_2^+$) are opposite (resp., equal).

Proof. Choose $\lambda_i \in F^v(\{i\}) \cap Y \subset Y^{++}$, and write $\mathfrak{F}_j^\pm$ for the germ of $\mathfrak{f}_j^\pm$ and $F_j^{\pm v}$ for its direction in A_j. In A_1 (resp., A_2) we consider the splayed chimney $\mathfrak{r}_1^- = \mathfrak{r}(C_1, F_1^{-v})$ (resp., $\mathfrak{r}_2^+ = \mathfrak{r}(C_2, F_2^{+v})$) containing $\mathfrak{f}_1^-$ (resp., $\mathfrak{f}_2^+$) and, for $n \in \mathbb{N}$, the chamber of type 0: $C_1(-n) = C_1 - n\lambda_i \subset \mathfrak{r}_1^-$ (resp., $C_2(+n) = C_2 + n\lambda_i \subset \mathfrak{r}_2^+$); actually we identify $(\mathbb{A}, C_0^+)$ with (A_1, C_1) (resp., (A_2, C_2)) to consider λ_i in $\vec{A}_1$ (resp., $\vec{A}_2$).

Then $d^W(C_1(-n), C_1) = d^W(C_2, C_2(+n)) = n\lambda_i$ and $d^W(C_1, C_2) = \lambda$, both in Y^{++}. By (MA3) there is an apartment A containing the germs $\mathfrak{R}_1^-$ of $\mathfrak{r}_1^-$ and $\mathfrak{R}_2^+$ of $\mathfrak{r}_2^+$; hence, $C_1(-n)$ and $C_2(+n)$ for n great. By Proposition 2.2 and the last paragraph of the proof of Proposition 2.3, $d^W(C_1(-n), C_2(+n)) = \lambda + 2n\lambda_i \in Y^{++}$

and A contains C_1, C_2. By (MA4) A contains also $\mathfrak{f}_1^- \subset \mathfrak{r}_1^- \subset \mathrm{cl}_{A_1}(C_1, \mathfrak{R}_1^-)$ and $\mathfrak{f}_2^+ \subset \mathfrak{r}_2^+ \subset \mathrm{cl}_{A_2}(C_2, \mathfrak{R}_2^+)$. So all assertions of the lemma are satisfied. □

Proposition 4.7. *Let $C_1, C_2, C_3, C_4 \in \mathcal{C}_0^+$ be such that $d^W(C_1, C_2) = \lambda \in Y^{++}$, $d^W(C_2, C_3) = r_i$, and $d^W(C_3, C_4) = \mu \in Y^{++}$. Then there is a direction of wall* (*i.e., a parallel class of walls*) *M_i^∞* (*see* [Rousseau 2011, §4] *or* [Gaussent and Rousseau 2014, 5.5]), *chosen according to C_1, C_2* (*but independently from C_3, C_4*), *such that C_1, C_2, C_3, C_4 are in the extended tree $\mathcal{I}(M_i^\infty)$.*

Proof. We denote by $x_1, x_2 = x_3, x_4$ the three vertices of C_1, C_2, C_3, C_4 and by $F_1^i, F_2^i = F_3^i, F_4^i$ their panels of type i. We choose $\mathfrak{f}_1^-$ associated to C_1 and F_1^i in an apartment A_1 (resp., $\mathfrak{f}_4^+$ associated to C_4 and F_4^i in an apartment A_4), as in Lemma 4.6. By this lemma, using C_1 and C_2, the direction of $\mathfrak{f}_1^-$ opposites that of $F_2^i = F_3^i$ in some apartment A_2 and, using C_3 and C_4, the direction of $\mathfrak{f}_4^+$ is the same as that of $F_2^i = F_3^i$ in some apartment A_3. In A_3 (resp., A_2) we consider the sector face $\mathfrak{f}_3^+$ (resp., $\mathfrak{f}_2^-$) with base point $x_2 = x_3$ and same direction as $\mathfrak{f}_4^+$ or $F_2^i = F_3^i$ (resp., same direction as $\mathfrak{f}_1^-$ and opposite $F_2^i = F_3^i$).

We may use the lemma for $C_1, C_2, \mathfrak{f}_1^-, \mathfrak{f}_3^+$; so the directions of $\mathfrak{f}_1^-$ (or $\mathfrak{f}_2^-$) and $\mathfrak{f}_3^+$ (or $\mathfrak{f}_4^+$) are opposite and C_1, C_2 are in a same apartment A_5 of $\mathcal{I}(M_i^\infty)$, if we consider the direction of wall M_i^∞ associated to the directions of $\mathfrak{f}_1^-$ and $\mathfrak{f}_4^+$. Using now the lemma for $C_3, C_4, \mathfrak{f}_2^-, \mathfrak{f}_4^+$, we see that these filters are in a same apartment A_6 of $\mathcal{I}(M_i^\infty)$. □

Theorem 4.8. *Let $\lambda, \mu \in Y^{++}$ and $i \in I$, write $N = \inf(\alpha_i(\lambda), \alpha_i(\mu)) \in \mathbb{N}$, and, for $n \in \mathbb{N}$, $q_i^{*n} = q_i q_i' q_i q_i' \cdots$, with n terms in this product.*

(a) *If $N = \alpha_i(\mu) \le \alpha_i(\lambda)$, then $T_\lambda * T_i * T_\mu = T_{\lambda+\mu} * T_i$ for $N = 0$ and, for $N > 0$,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_{\lambda+\mu-N\alpha_i^\vee} * T_i + (q_i^{*N} - q_i^{*N-1}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}.$$

(b) *If $N = \alpha_i(\lambda) \le \alpha_i(\mu)$, then $T_\lambda * T_i * T_\mu = T_i * T_{\lambda+\mu}$ for $N = 0$ and, for $N > 0$,*

$$T_\lambda * T_i * T_\mu = q_i^{*N} T_i * T_{\lambda+\mu-N\alpha_i^\vee} + (q_i^{*N} - q_i^{*N-1}) T_{\lambda+\mu-(N-1)\alpha_i^\vee} + \cdots + (q_i^{*2} - q_i) T_{\lambda+\mu-\alpha_i^\vee} + (q_i - 1) T_{\lambda+\mu}.$$

Remarks. (1) The case (b) is less interesting for us, as we try to express any element in the basis of Section 4.5 for ${}^{\mathrm{I}}\mathcal{H}_R$ considered as a right $\mathcal{H}_R(W^v)$-module.

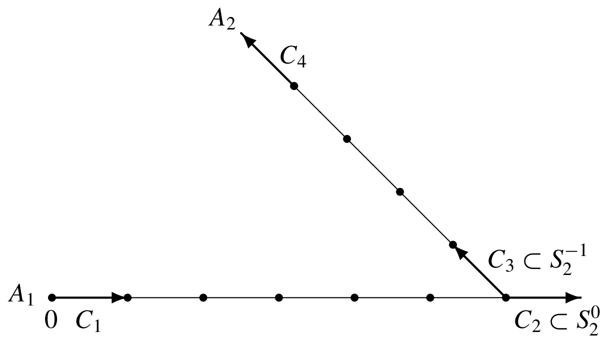
(2) In the case (a) we have $\mu - N\alpha_i^\vee = r_i(\mu)$ and $\lambda + \mu - N\alpha_i^\vee \in Y^{++}$, as $\alpha_i(\lambda + \mu - N\alpha_i^\vee) = \alpha_i(\lambda) - N$ and $\alpha_j(\lambda + \mu - N\alpha_i^\vee) \ge \alpha_j(\lambda) + \alpha_j(\mu)$ for $j \ne i$. So all ν such that T_ν appears on the right of the formula are in the $\alpha_i^\vee$-chain between $\lambda + \mu$ and $\lambda + r_i(\mu)$; in particular they are all in Y^{++}.

(3) We call relation (a) or relation (b) the Bernstein–Lusztig relation for the T_λ, (BLT) for short. We shall use it essentially when $\lambda = \mu$.

(4) When $\alpha_i(\lambda)$ or $\alpha_i(\mu)$ is odd, we know that $q_i' = q_i$; see Section 1.4.5.

Proof. We consider C_1, C_2, C_3, C_4, and M_i^∞ as in Proposition 4.7. When $N = 0$ the results come from Proposition 4.1. We concentrate on the case $0 < N = \alpha_i(\mu) \leq \alpha_i(\lambda)$; the other case is left to the reader. We have to evaluate $d^W(C_1, C_4)$ and, given C_1, C_4 satisfying $d^W(C_1, C_4) = \boldsymbol{u}$, to count the number of possible C_2, C_3. By Proposition 4.7 everything is in the extended tree $\mathscr{I}(M_i^\infty)$, which is semihomogeneous with thicknesses $1 + q_i$, $1 + q_i'$. By Proposition 4.1(2), C_3 is well determined by C_2, C_4 and lies in any apartment containing C_2, C_4; moreover $d^W(C_2, C_4) = r_i(\mu).r_i$.

We consider an apartment A_1 (resp., A_2) of $\mathscr{I}(M_i^\infty)$ containing C_1 and C_2 (resp., C_2 and C_4, hence also C_3), as illustrated in the figure:



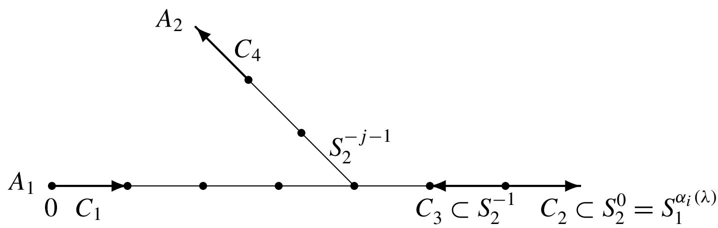
We identify (A_1, C_1) and (A_2, C_2) with $(\mathbb{A}, C_0^+)$; we consider the retraction ρ_1 (resp., ρ_2) of $\mathscr{I}(M_i^\infty)$ onto A_1 (resp., A_2) with center C_1 (resp., C_2). The closed chambers in an apartment of $\mathscr{I}(M_i^\infty)$ are stripes limited by walls of direction M_i^∞. In $A_1 = \mathbb{A}$, these walls are $M(\alpha_i, n)$, $n \in \mathbb{Z}$ and we write S_1^k the stripe $S_1^k = \{x \mid k \leq \alpha_i(x) \leq k+1\}$, in particular $C_1 \subset S_1^0$ and $C_2 \subset S_1^{\alpha_i(\lambda)}$. In $A_2 = \mathbb{A}$, we get also stripes $S_2^k = \{x \mid k \leq \alpha_i(x) \leq k+1\}$ such that $C_2 \subset S_2^0 = S_1^{\alpha_i(\lambda)}$, $C_3 \subset S_2^{-1}$ and $C_4 \subset S_2^{-N-1}$.

We have $C_2 = C_1 + \lambda$ in A_1 and $\rho_2(C_4) = C_3 + r_i(\mu)$ in A_2. To find $d^W(C_1, C_4)$ we have to determine the image of C_4 under ρ_1. It depends actually on the highest number j such that S_2^{-j} (hence also $S_2^0, \ldots, S_2^{-j+1}$) is in A_1. A classical result for affine buildings (clear for extended trees and generalized to hovels in [Rousseau 2011, 2.9.2]) tells, then, that there is an apartment containing the stripes $S_2^{-j-1}, \ldots, S_2^{-N-1}$ and the half-apartment $\bigcup_{k \leq \alpha_i(\lambda)-j-1} S_1^k$.

If $j = 0$, then S_2^{-1} or C_3 is not in A_1, so $\rho_1(C_3) = C_2$ and, more generally, $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda)+k-1}$, for $k \geq 1$. This is the case illustrated in the figure above. We get $\rho_1(C_4) = C_2 + \mu$ and $d^W(C_1, C_4) = \lambda + \mu$. When C_1 and C_4 are fixed with this W-distance, we have to count the number of possible C_2. But $C_3 \subset S_2^{-1}$ is in the enclosure of $C_1 \subset S_1^0$ and $C_4 \subset S_2^{-N-1}$: it is well determined by C_1 and C_4. Now

C_2 has to share its panel of type i with C_3 and to be neither in S_2^{-1} nor in $S_1^{\alpha_i(\lambda)-1}$; so there are $q_i - 1$ possibilities.

If $1 \le j \le N-1$, then A_1 contains $S_2^0 = S_1^{\alpha_i(\lambda)}$, $S_2^{-1} = S_1^{\alpha_i(\lambda)-1}, \dots, S_2^{-j} = S_1^{\alpha_i(\lambda)-j}$, but not $S_2^{-j-1}, \dots, S_2^{-N-1}$; this is the case illustrated below:



So $\rho_1(S_2^{-k}) = S_1^{\alpha_i(\lambda)-2j+k}$, for $k \ge j$. As in the proof of Proposition 4.7, we write $x_1, x_2 = x_3, x_4$ for the vertices of the local chambers C_1, C_2, C_3, C_4. The image of the line segment $[x_2, x_4] = [x_2, x_2 + \mu]$ under ρ_1 is

$$\rho_1([x_2, x_4]) = \left[x_2,\ x_2 + \frac{j}{N} r_i(\mu)\right] \cup \left[x_2 + \frac{j}{N} r_i(\mu),\ x_2 + \frac{j}{N} r_i(\mu) + \frac{N-j}{N}\mu\right].$$

As $N = \alpha_i(\mu)$ and $r_i(\mu) = \mu - N\alpha_i^\vee$, this means that $\rho_1(C_4) = C_2 + \mu - j\alpha_i^\vee$. When C_1 and C_4 are fixed with this W-distance, we have to count the number of possible C_2. As $S_1^0, \dots, S_1^{\alpha_i(\lambda)-j-1}, S_2^{-j-1}, \dots, S_2^{-N-1}$ are well determined by C_1, C_4, we have to count the possibilities for $(S_1^{\alpha_i(\lambda)-j}, \dots, S_1^{\alpha_i(\lambda)})$. As above, there are $q_i - 1$ possibilities for $S_1^{\alpha_i(\lambda)-j}$ (or $q_i' - 1$ if j is odd) and then q_i' (or q_i) possibilities for $S_1^{\alpha_i(\lambda)-j+1}$, etc. Finally the total number of possibilities is $(q_i - 1)q_i' q_i q_i' \cdots$ or $(q_i' - 1)q_i q_i' q_i \cdots$ (according to j being even or odd) with $j+1$ terms in the product. The last factor is necessarily q_i, so this total number is $(q_i^{*j+1} - q_i^{*j})$.

It is convenient to look at the cases $j = N$ or $j = N+1$ simultaneously. This means that $S_2^{-N} = S_1^{\alpha_i(\lambda)-N}$ is in A_1; in particular the panel F_4^i of type i of C_4 is in A_1, in the wall $\{x \mid \alpha_i(x) = \alpha_i(\lambda) - N\}$. More precisely F_4^i is the panel of type i of $C_4' = C_1 + \lambda + r_i(\mu) \subset A_1$. This means that $(T_{\lambda + r_i(\mu)} * T_i)(C_1, C_4) \ge 1$.

Conversely if C_1, C_4 are fixed satisfying this condition, we can find C_2, C_3 with the required W-distances. We have now to count the number of possibilities for C_2, C_3, i.e., for C_2 or for $(S_1^{\alpha_i(\lambda)-N}, \dots, S_1^{\alpha_i(\lambda)})$. The number of possibilities for $S_1^{\alpha_i(\lambda)-N}$ is exactly $(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$. Then the number of possibilities for $S_1^{\alpha_i(\lambda)-N+1}, \dots, S_1^{\alpha_i(\lambda)}$ is alternatively q_i or q_i'. Finally the total number of possibilities for C_2 is $q_i^{*N}(T_{\lambda+r_i(\mu)} * T_i)(C_1, C_4)$ (as, when N is odd, $q_i = q_i'$). □

5. New basis

In this section, we prove that left multiplication by T_μ, for $\mu \in Y^{++}$, is injective. That allows us to introduce a new basis of the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ in terms of $(T_w)_{w \in W^v}$ and $(X^\lambda)_{\lambda \in Y^+}$. From now on the main arguments are algebraic.

We suppose $\mathbb{Z} \subset R$ and each q_i, q_i' in $R^\times$, the set of invertible elements in R. As we saw in Section 4.5, ${}^{\mathrm{I}}\mathcal{H}_R$ is a free right $\mathcal{H}_R(W^v)$-module with basis $\{T_\lambda \mid \lambda \in Y^+\}$.

For $\lambda \in Y^{++}$ and $H \in \mathcal{H}_R(W^v)$, we say that $T_\lambda * H$ is of degree λ.

For $i \in I$ and Ω a subset of the model apartment $\mathbb{A}$, we write $c(i)(\Omega)$ the convex hull of $\Omega \cup r_i(\Omega)$. For $(i_1, i_2, \dots, i_h) \in I^h$ and $(\lambda_0, \lambda_1, \dots, \lambda_h) \in (Y^{++})^{h+1}$, we define: $D(i_h)(\lambda_{h-1}, \lambda_h) = \lambda_{h-1} + c(i_h)(\lambda_h)$ and, by induction for k from $h-1$ to 1, $D(i_k, \dots, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_h) = \lambda_{k-1} + c(i_k)(D(i_{k+1}, \dots, i_h)(\lambda_k, \lambda_{k+1}, \dots, \lambda_h))$, and of course, $c(i_h)(\lambda_h) = c(i_h)(\{\lambda_h\})$.

Lemma 5.1. *With notation as above*:

(a) *If* $\lambda'_{h-1} \in D(i_h)(\lambda_{h-1}, \lambda_h)$, *then*

$$D(i_k, \dots, i_{h-2}, i_{h-1})(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-2}, \lambda'_{h-1}) \subset D(i_k, \dots, i_{h-1}, i_h)(\lambda_{k-1}, \lambda_k, \dots, \lambda_{h-1}, \lambda_h).$$

(b) *If* $r_{i_1} r_{i_2} \cdots r_{i_h}$ *is a reduced word in* W^v *and* $\lambda \in D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h)$, *then* $\lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_h}(\lambda_h) \leq_{Q^\vee_{\mathbb{R}}} \lambda$.

Remark. If the expression $r_{i_1} r_{i_2} \cdots r_{i_h}$ is reduced, we get

$$D(i_1, \dots, i_h)(0, 0, \dots, 0, \lambda_h) = \mathrm{conv}(\{w(\lambda_h) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_h}\})$$

where $\leq_B$ denotes the Bruhat order.

Proof. The proof of (a) is easy.

(b) We have

$$D(i_1, \dots, i_h)(\lambda_0, \lambda_1, \dots, \lambda_h) \subset \lambda_0 + c(i_1)(\lambda_1) + c(i_1, i_2)(\lambda_2) + \cdots + c(i_1, i_2, \dots, i_h)(\lambda_h),$$

with

$$\begin{aligned} c(i_1, i_2, \dots, i_k)(\lambda_k) &= c(i_1)\big(c(i_2)\big(\cdots\big(c(i_k)(\lambda_k)\big)\cdots\big)\big) \\ &= \mathrm{conv}(\{w(\lambda_k) \mid w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}\}), \end{aligned}$$

where $0 \leq k \leq h$ and $\leq_B$ denotes the Bruhat order. For $w \leq_B r_{i_1} r_{i_2} \cdots r_{i_k}$, there is a sequence $w = w_0, w_1, \dots, w_r = r_{i_1} r_{i_2} \cdots r_{i_k}$ such that, for each $1 \leq i < r$, there is a reduced decomposition $w_{i+1} = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p} r_{j_{p+1}} \cdots r_{j_q}$ with $w_i = r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_{p+1}} \cdots r_{j_q}$. Then

$$w_i(\lambda_k) = w_{i+1}(\lambda_k) + \alpha_{j_p}\big(r_{j_{p+1}} \cdots r_{j_q}(\lambda_k)\big) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha^\vee_{j_p})$$

and $Q^\vee_+$ contains the term $\big(r_{j_q} \cdots r_{j_{p+1}}(\alpha_{j_p})\big)(\lambda_k) r_{j_1} r_{j_2} \cdots r_{j_{p-1}}(\alpha^\vee_{j_p})$ by minimality of the expressions $r_{j_1} r_{j_2} \cdots r_{j_{p-1}} r_{j_p}$ and $r_{j_q} \cdots r_{j_{p+1}} r_{j_p}$. So by induction,

$$w(\lambda_k) \geq_{Q^\vee} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k) \quad \text{and} \quad w(\mu) \geq_{Q^\vee_{\mathbb{R}}} r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k)$$

for any $\mu \in c(i_1, \dots, i_k)(\lambda_k)$. The expected result is now clear. □

Proposition 5.2. *For any expression* $H_k = T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ *with* $\lambda_i \in Y^{++}$, $H \in \mathcal{H}_{\mathbb{Z}}(W^v)$, *and any* $\mu \in Y^{++}$ *sufficiently great, the product* $T_\mu * H_k$ *may be written as an* R*-linear combination of elements* $T_\nu * H_\nu$ *with* $\nu \in \mu + D(i_1, \ldots, i_k)(\lambda_0, \lambda_1, \ldots, \lambda_k)$ *and* $H_\nu \in \mathcal{H}_R(W^v)$.

Moreover, if $r_{i_1} r_{i_2} \cdots r_{i_k}$ *is a reduced word and*

$$\nu_0 = \mu + \lambda_0 + r_{i_1}(\lambda_1) + r_{i_1} r_{i_2}(\lambda_2) + \cdots + r_{i_1} r_{i_2} \cdots r_{i_k}(\lambda_k),$$

then $H_{\nu_0} \in R^\times T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H$ *and, more precisely, the constant in* $R^\times$ *is a primitive monomial in the* q_i, q_i'. *Further,* H_{ν_0} *is the only* H_ν *in*

$$(R \setminus \{0\}) \,.\, T_{i_1} * T_{i_2} * \cdots * T_{i_k} * H.$$

N.B. So one may write $T_\mu * H_k = \sum_{\nu,w} a_{\nu,w} T_\nu * T_w$, with $a_{\nu,w} \in R$, ν running in $\mu + D(i_1, \ldots, i_k)(\lambda_0, \lambda_1, \ldots, \lambda_k)$, and w in W^v. Moreover we get from the following proof, that each $a_{\nu,w}$ is a Laurent polynomial in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$; these polynomials depend only on the expression H_k, on $\mathbb{A}$, and on W.

Proof. The proof is easy in the following special case (I).

(I) We say that the expression of H_k is *normalizable of length* k when it satisfies the following properties:

(i) $\lambda_{k-1} - \lambda_k \in Y^{++}$,

(ii) For all h from k to 2, $\lambda_{h-2} - D(i_h, \ldots, i_k)(\lambda_{h-1}, \lambda_h, \ldots, \lambda_k) \subset \overline{C^v_f}$.

For such an expression, we write $D(H_k) = D(i_1, \ldots, i_k)(\lambda_0, \lambda_1, \ldots, \lambda_k)$.

We will then prove that $T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * T_{i_2} * \cdots * T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k} * H$ is a $\mathbb{Z}[q_i, q_i']$-linear combination of normalizable elements H'_{k-1} of length $k-1$ such that $D(H'_{k-1}) \subset D(H_k)$.

Using the fact $\lambda_{k-1} - \lambda_k \in Y^{++}$ and Theorem 4.8, or (BLT), for $T_{\lambda_{k-1}} * T_{i_k} * T_{\lambda_k}$,

$$\text{(E)} \quad H_k = q_{i_k}^{*(\alpha_{i_k}(\lambda_k))} T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{i_{k-1}} * T_{\lambda_{k-1}^{(\alpha_{i_k}(\lambda_k))}} * (T_{i_k} * H) + \sum_{h=0}^{\alpha_{i_k}(\lambda_k)-1} (q_i^{*(h+1)} - q_i^{*(h)}) T_{\lambda_0} * T_{i_1} * T_{\lambda_1} * \cdots * T_{i_{k-1}} * T_{\lambda_{k-1}^{(h)}} * H,$$

with $\lambda_{k-1}^{(h)} = \lambda_{k-1} + \lambda_k - h\alpha_{i_k}^\vee$, and in particular, $\lambda_{k-1}^{(\alpha_{i_k}(\lambda_k))} = \lambda_{k-1} + r_{i_k}(\lambda_k)$. Let us consider $\lambda_i' = \lambda_i$ for $i \leq k-2$ and $\lambda'_{k-1} = \lambda_{k-1}^{(h)}$ for each $0 \leq h \leq \alpha_{i_k}(\lambda_k)$. Then $(\lambda_0', \ldots, \lambda'_{k-1})$ satisfies $\lambda'_{k-2} - \lambda'_{k-1} \in Y^{++}$, by (ii) above for $h = k$ and $\lambda'_{k-1} \in D(i_k)(\lambda_{k-1}, \lambda_k)$, and, for all h from $k-1$ to 2,

$$\lambda'_{h-2} - D(i_h, \ldots, i_{k-1})(\lambda'_{h-1}, \ldots, \lambda'_{k-1}) \subset \overline{C^v_f}.$$

This last result comes from (ii) $\lambda'_{h-2}-D(i_h,\dots,i_k)(\lambda_{h-1},\lambda_h,\dots,\lambda_k)\subset\overline{C^v_f}$ and the inclusion $D(i_h,\dots,i_{k-1})(\lambda'_{h-1},\lambda'_h,\dots,\lambda'_{k-1})\subset D(i_h,\dots,i_k)(\lambda_{h-1},\lambda_h,\dots,\lambda_k)$, coming from Lemma 5.1(a). Since $T_{i_k}*H\in\mathcal{H}_R(W^v)$, every term of the right hand side of (E) is a normalizable element H'_{k-1} of length $k-1$ with $D(H'_{k-1})\subset D(H_k)$.

By induction on each term, after k steps, we obtain H_k as a $\mathbb{Z}[q_i,q'_i]$-linear combination of $T_\nu*H_\nu$, with $\nu\in D(H_k)$ and $H_\nu\in\mathcal{H}_R(W^v)$.

Moreover, if the decomposition $r_{i_1}r_{i_2}\cdots r_{i_k}$ is reduced, we take

$$\nu_0=\lambda_0+r_{i_1}(\lambda_1)+r_{i_1}r_{i_2}(\lambda_2)+\cdots+r_{i_1}r_{i_2}\cdots r_{i_k}(\lambda_k)$$

and look more carefully at the decomposition (E). For $0\le h<\alpha_{i_k}(\lambda_k)$, we have $\nu_0\notin D(T_{\lambda_0}*T_{i_1}*T_{\lambda_1}*\cdots*T_{\lambda^{(h)}_{k-1}}*H)\subset D(H_k)$ by Lemma 5.1(b). Indeed, if

$$\lambda\in D(T_{\lambda_0}*T_{i_1}*T_{\lambda_1}*\cdots*T_{\lambda^{(h)}_{k-1}}*H),$$

then, by minimality of $r_{i_1}r_{i_2}\cdots r_{i_k}$, we have $\nu_0\le_{Q^\vee}\nu_0^{(h)}\le_{Q^\vee}\lambda$ with

$$\nu_0^{(h)}=\lambda_0+r_{i_1}(\lambda_1)+r_{i_1}r_{i_2}(\lambda_2)+\cdots+r_{i_1}r_{i_2}\cdots r_{i_{k-1}}(\lambda^{(h)}_{i_{k-1}})\ne\nu_0.$$

So the unique term of degree ν_0 of the final decomposition comes from the term of first kind (i.e., obtained like the first term of the right hand side of (E)) in every step of the reduction and is also the only term containing all the T_{i_j}. And so, we prove that, in front of the term $T_{\nu_0}*T_{i_1}*T_{i_2}*\cdots*T_{i_k}*H$ obtained for ν_0, the constant is equal to the primitive monomial

$$C=q_{i_k}^{*(\alpha_{i_k}(\lambda_k))}q_{i_{k-1}}^{*(\alpha_{i_{k-1}}(\lambda_{k-1}+r_{i_k}(\lambda_k)))}\cdots q_{i_1}^{*(\alpha_{i_1}(\lambda_1+r_{i_2}(\lambda_2)+\cdots+r_{i_2}\cdots r_{i_k}(\lambda_k)))}.$$

Let us consider now the general case but first prove the following result:

(II) If $H_k=T_{\lambda_0}*T_{i_1}*T_{\lambda_1}*T_{i_2}*\cdots*T_{\lambda_{k-1}}*T_{i_k}*T_{\lambda_k}*H$, with $\lambda_i\in Y^{++}$ and $H\in\mathcal{H}_R(W^v)$, we can choose $\mu_0\in Y^{++}$ such that $T_{\mu_0}*H_k$ can be written as an R-linear combination of normalizable expressions H'_k of length at most k and with $D(H'_k)\subset\mu_0+D(i_1,\dots,i_k)(\lambda_0,\lambda_1,\dots,\lambda_k)$.

We prove this result for $H_{k-h}=T_{\lambda_h}*T_{i_{h+1}}*T_{\lambda_{h+1}}*\cdots*T_{\lambda_{k-1}}*T_{i_k}*T_{\lambda_k}*H$ by decreasing induction on $0\le h\le k-1$. For $h=k-1$, we have $H_1=T_{\lambda_{k-1}}*T_{i_k}*T_{\lambda_k}*H$. Choose $\mu_{k-1}=\lambda_k$; then, $T_{\mu_{k-1}}*H_1$ is normalizable of length 1 and

$$D(T_{\mu_{k-1}}*H_1)\subset\mu_{k-1}+D(i_k)(\lambda_{k-1},\lambda_k).$$

Let $0\le h\le k-2$ and suppose that we can choose $\mu_{h+1}\in Y^{++}$ such that $T_{\mu_{h+1}}*H_{k-(h+1)}=T_{\mu_{h+1}}*T_{\lambda_{h+1}}*T_{i_{h+2}}*\cdots*T_{i_k}*T_{\lambda_k}*H$ can be written as an R-linear combination of normalizable expressions $H'_{k-(h+1)}$ of length at most $k-(h+1)$ and with $D(H'_{k-(h+1)})\subset\mu_{h+1}+D(i_{h+2},\dots,i_k)(\lambda_{h+1},\dots,\lambda_k)$. Write these normalizable expressions $H'_{k-(h+1)}=T_{\lambda'_0}*T_{i'_1}*T_{\lambda'_1}*T_{i'_2}*\cdots*T_{i'_{k'}}*T_{\lambda'_{k'}}*H'$, where $k'\le k-(h+1)$ and $(\lambda'_0,\dots,\lambda'_{k'})$ satisfies (i) and (ii). Consider $\mu_h^{\min}\in Y^{++}$

such that $\mu_h^{\min} - D(i'_1, \dots, i'_{k'})(\lambda'_0, \lambda'_1, \dots, \lambda'_{k'}) \subset \overline{C_f^v}$ for all these expressions. We take $\mu_h = \mu_h^{\min} + 2\mu_{h+1} + r_{i_{h+1}}(\mu_{h+1})$. Then

$$\begin{aligned} T_{\mu_h} * H_{k-h} &= T_{\mu_h} * T_{\lambda_h} * T_{i_{h+1}} * H_{k-(h+1)} \\ &= T_{\mu_h^{\min}+\lambda_h+\mu_{h+1}} * T_{\mu_{h+1}+r_{i_{h+1}}(\mu_{h+1})} * T_{i_{h+1}} * H_{k-(h+1)}. \end{aligned}$$

By (BLT), we have:

$$\begin{aligned} \text{(E}'\text{)} \quad & q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h} \\ & = T_{\mu_h^{\min}+\lambda_h+2\mu_{h+1}} * T_{i_{h+1}} * T_{\mu_{h+1}} * H_{k-(h+1)} \\ & \quad - \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\lambda_h+\mu_h^{\min}+2\mu_{h+1}-j\alpha_{i_{h+1}}^\vee} * T_{\mu_{h+1}} * H_{k-(h+1)}. \end{aligned}$$

The choice of $\mu_h^{\min}$ and the hypothesis on $T_{\mu_{h+1}} * H_{k-(h+1)}$ allow us to say that we have written $T_{\mu_h} * H_{k-h}$ as an R-linear combination of normalizable expressions H'_{k-h} of length at most $k-h$ with

$$D(H'_{k-h}) \subset \mu_h^{\min} + 2\mu_{h+1} + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$$

for the first term and

$$D(H'_{k-h}) \subset \mu_h^{\min} + 2\mu_{h+1} - j\alpha_{i_{h+1}}^\vee + D(i_{h+1}, \dots, i_k)(\lambda_h, \lambda_{h+1} + \mu_{h+1}, \dots, \lambda_k)$$

for the others. We need to be more precise to prove

$$D(H'_{k-h}) \subset \mu_h + D(i_{h+1}, \dots, i_k)(\lambda_h, \dots, \lambda_k).$$

By part (I) of this proof and the hypothesis on $T_{\mu_{h+1}} * H_{k-(h+1)}$, we know that this element can be written $\sum_\Lambda c_\Lambda T_\Lambda * H^\Lambda$ with $\Lambda = \mu_{h+1} + \Lambda'$, where $\Lambda' \in D(i_{h+2}, \dots, i_k)(\lambda_{h+1}, \dots, \lambda_k) c_\Lambda \in R$ and $H^\Lambda \in \mathcal{H}_R(W^v)$. The first term of the right hand side of (E′) becomes

$$T_{\mu_h^{\min}+\lambda_h+2\mu_{h+1}} * T_{i_{h+1}} * \left(\sum_\Lambda c_\Lambda T_\Lambda * H^\Lambda\right) = T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda T_{\mu_h^{\min}} * T_{i_{h+1}} * T_\Lambda * H^\Lambda\right).$$

By the condition on $\mu_h^{\min}$ and (BLT), we write it

$$\begin{aligned} & T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda \left(q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\mu_h^{\min}+r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda\right)\right) \\ & \qquad + T_{\lambda_h+2\mu_{h+1}} * \left(\sum_\Lambda c_\Lambda \left(\sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)}) T_{\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda\right)\right). \end{aligned}$$

The first term of this sum will be

$$\sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda$$

and $\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}}(\Lambda)=\lambda_h+2\mu_{h+1}+\mu_h^{\min}+r_{i_{h+1}}(\mu_{h+1})+r_{i_{h+1}}(\Lambda')=\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')$ is an element of $\lambda_h+\mu_h+r_{i_{h+1}}(D(i_{h+2},\ldots,i_k)(\lambda_{h+1},\ldots,\lambda_k))$ which is included, as expected, in $\mu_h+D(i_{h+1},i_{h+2},\ldots,i_k)(\lambda_h,\lambda_{h+1},\ldots,\lambda_k)$.

The second term is

$$\sum_{\Lambda} c_\Lambda \Bigg(\sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)})\, T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda \Bigg).$$

And we see that in fact (E′) becomes (E″):

$$\begin{aligned}
\text{(E}''\text{)}\quad & q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\mu_{h+1}))} T_{\mu_h} * H_{k-h} \\
&= \sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^\Lambda \\
&\qquad + \sum_{\Lambda} c_\Lambda \sum_{j=0}^{\alpha_{i_{h+1}}(\Lambda)-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)})\, T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda \\
&\qquad - \sum_{\Lambda} c_\Lambda \sum_{j=0}^{\alpha_{i_{h+1}}(\mu_{h+1})-1} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)})\, T_{\lambda_h+\mu_h^{\min}+2\mu_{h+1}-j\alpha_{i_{h+1}}^\vee} * T_\Lambda * H^\Lambda \\
&= \sum_{\Lambda} c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h+\mu_h+r_{i_{h+1}}(\Lambda')} * T_{i_{h+1}} * H^\Lambda \\
&\qquad + \sum_{\Lambda} c_\Lambda \varepsilon_\Lambda \sum_{j} (q_{i_{h+1}}^{*(j+1)} - q_{i_{h+1}}^{*(j)})\, T_{\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee} * H^\Lambda,
\end{aligned}$$

where $\alpha_{i_{h+1}}(\mu_{h+1}) \le j \le \alpha_{i_{h+1}}(\Lambda)-1$ and $\varepsilon_\Lambda=+1$ if $\alpha_{i_{h+1}}(\mu_{h+1}) \le \alpha_{i_{h+1}}(\Lambda)$ (i.e., $\alpha_{i_{h+1}}(\Lambda') \ge 0$), and where $\alpha_{i_{h+1}}(\Lambda) \le j \le \alpha_{i_{h+1}}(\mu_{h+1})-1$ and $\varepsilon_\Lambda=-1$ if $\alpha_{i_{h+1}}(\mu_{h+1}) \ge \alpha_{i_{h+1}}(\Lambda)$ (i.e., $\alpha_{i_{h+1}}(\Lambda') \le 0$). For these values of j, by using $\Lambda - j\alpha_{i_{h+1}}^\vee = r_{i_{h+1}}(\mu_{h+1}) + j'\alpha_{i_{h+1}}^\vee + \Lambda'$ with $j'=\alpha_{i_{h+1}}(\mu_{h+1})-j$, we have

$$\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee = \lambda_h+\mu_h+j'\alpha_{i_{h+1}}^\vee+\Lambda'.$$

If $\alpha_{i_{h+1}}(\mu_{h+1}) \le \alpha_{i_{h+1}}(\Lambda)$, then $\alpha_{i_{h+1}}(\mu_{h+1})-\alpha_{i_{h+1}}(\Lambda)+1 \le j' \le 0$; that is, $-\alpha_{i_{h+1}}(\Lambda')+1 \le j' \le 0$. On the other hand, if $\alpha_{i_{h+1}}(\mu_{h+1}) \ge \alpha_{i_{h+1}}(\Lambda)$, then $\alpha_{i_{h+1}}(\mu_{h+1})-\alpha_{i_{h+1}}(\Lambda) \ge j' \ge 1$; that is $-\alpha_{i_{h+1}}(\Lambda') \ge j' \ge 1$. In all cases, $j'\alpha_{i_{h+1}}^\vee+\Lambda'$ is between Λ' and $r_{i_{h+1}}(\Lambda')$ and so, as expected,

$$\lambda_h+2\mu_{h+1}+\mu_h^{\min}+\Lambda-j\alpha_{i_{h+1}}^\vee \in \mu_h+D(i_{h+1},i_{h+2},\ldots,i_k)(\lambda_h,\lambda_{h+1},\ldots,\lambda_k).$$

So we have proved that $T_{\mu_0} * H_k$ can be written as an R-linear combination of normalizable expressions H'_k of length at most k and with

$$D(H'_k) \subset \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k).$$

By (I) of the proof, we can write it as an R-linear combination of elements $T_\nu * H_\nu$ with $\nu \in \mu_0 + D(i_1, \dots, i_k)(\lambda_0, \lambda_1, \dots, \lambda_k)$ and $H_\nu \in \mathcal{H}_R(W^v)$.

As in (I), if the decomposition $r_{i_1} r_{i_2} \cdots r_{i_k}$, moreover, is reduced, then only the term

$$\sum_\Lambda c_\Lambda q_{i_{h+1}}^{*(\alpha_{i_{h+1}}(\Lambda))} T_{\lambda_h + 2\mu_{h+1} + \mu_h^{\min} + r_{i_{h+1}(\Lambda)}} * T_{i_{h+1}} * H^\Lambda$$

(which contains $T_{i_{h+1}}$) in (E″) can give us a term of lowest degree

$$\mu_h + \lambda_h + r_{i_{h+1}}(\lambda_{h+1}) + \cdots + r_{i_{h+1}} \cdots r_{i_k}(\lambda_k).$$

More precisely, the term of lowest degree comes from the term with

$$\Lambda_0 = \mu_{h+1} + \lambda_{h+1} + r_{i_{h+2}}(\lambda_{h+2}) + \cdots + r_{i_{h+2}} \cdots r_{i_k}(\lambda_k)$$

for which we have $\alpha_{i_{k+1}}(\Lambda_0) \geq \alpha_{i_{k+1}}(\mu_{h+1})$. So, it's easy to see by induction that the coefficient of that term is a primitive monomial in the q_i, q'_i. □

Corollary 5.3. (a) *For $\lambda \in Y^+$ and $\mu \in Y^{++}$ sufficiently great, we have $T_\mu * T_\lambda = \sum_{\lambda \leq_{Q^\vee} \nu \leq_{Q^\vee} \lambda^{++}} T_{\mu+\nu} * H^\nu$ with $H^\nu \in \mathcal{H}_R(W^v)$.*

(b) *More precisely, if $H^\nu \neq 0$ then $\mu + \nu \in Y^{++}$ and ν is in the convex hull* $\mathrm{conv}(W^v.\lambda^{++})$ *of $W^v.\lambda^{++}$ or, better, in the convex hull* $\mathrm{conv}(W^v.\lambda^{++}, \geq \lambda)$ *of all $w'.\lambda^{++}$ for $w' \leq_B w_\lambda$, with w_λ the smallest element of W^v such that $\lambda = w_\lambda.\lambda^{++}$.*

(c) *For $\nu = \lambda$, H^λ is a strictly positive integer a_λ which may be written as a primitive monomial in q_i, q'_i, $i \in I$ (depending only on $\mathbb{A}$).*

(d) *In (a) above, we may write $H^\nu = \sum_{w \in W^v} a_{\mu,\lambda}^{\nu,w} T_w$ and, then each $a_{\mu,\lambda}^{\nu,w}$ is a Laurent polynomial in the parameters q_i, q'_i with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W.*

Proof. Only the result (c) is new (see Propositions 2.2 and 2.3), and we already saw that the constant term in H^λ is in $\mathbb{Z}_{>0}$. We have to prove that $H^\lambda \in \mathcal{H}_R(W^v)$ is actually a constant (for μ sufficiently great). Write $\lambda = w_\lambda(\lambda^{++})$ (with w_λ minimal in W^v for this property), choose a minimal decomposition $w_\lambda = r_{i_1} r_{i_2} \cdots r_{i_k}$, by Corollary 4.3 we have

$$T_\lambda = T_{i_1} * T_{i_2} * \cdots * T_{i_k} * T_{\lambda^{++}} * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}.$$

Then, by Proposition 5.2, for μ great, $T_\mu * T_\lambda$ may be written as an R-linear combination of elements $T_{\mu+\nu} * (H_1^\nu * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1})$ with $\nu \in D(i_1, \dots, i_k)(0, \dots, 0, \lambda^{++})$

and $H_1^\nu \in \mathcal{H}_R(W^v)$ with term of lowest degree $\nu_0 = \lambda$. Moreover,

$$H^\lambda = H_1^\lambda * T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$$

is a primitive monomial in the q_i, q_i'.

To prove (d), notice that $T_{i_k}^{-1} * \cdots * T_{i_1}^{-1}$ may be written $\sum_{w \in W^v} a_w T_w$ with $a_w \in \mathbb{Z}[(q_i^{\pm 1})_{i \in I}]$, and apply Proposition 5.2 with $H = T_w$. □

Corollary 5.4. *In ${}^{\mathrm{I}}\mathcal{H}_R$, for $\mu \in Y^{++}$ the left multiplication by T_μ is injective.*

Proof. As $T_{\mu_1+\mu_2} = T_{\mu_1} * T_{\mu_2}$ for $\mu_1, \mu_2 \in Y^{++}$, we may assume μ sufficiently great. Let $H \in {}^{\mathrm{I}}\mathcal{H}_R \setminus \{0\}$. We may write $H = \sum_{j \in J} T_{\lambda_j} * H^j$ with $\lambda_j \in Y^+$ and $0 \neq H^j \in \mathcal{H}_R(W^v)$. We choose λ_{j_0} minimal among the λ_j for $\leq_{Q^\vee}$. Then

$$T_\mu * H = \sum_{j \in J} \sum_{\mu + \lambda_j \leq_{Q^\vee} \nu_j} T_{\nu_j} * H^{\nu_j, j} * H^j.$$

Hence $\nu_{j_0} = \mu + \lambda_{j_0}$ is minimal for $\leq_{Q^\vee}$ and $H^{\nu_{j_0}, j_0}$ is a monomial in q_i, q_i'; so $H^{\nu_{j_0}, j_0} * H^{j_0} \neq 0$ and $T_\mu * H \neq 0$. □

Theorem 5.5. (1) *For any $\lambda \in Y^+$, there is a unique $X^\lambda \in {}^{\mathrm{I}}\mathcal{H}_R$ such that for all $\mu \in Y^{++}$ with $\lambda + \mu \in Y^{++}$, we have $T_\mu * X^\lambda = T_{\lambda+\mu}$.*

(2) *More precisely,*

$$X^\lambda = b_\lambda T_\lambda + \sum_\nu T_\nu * H'^\nu,$$

where $H'^\nu \in \mathcal{H}_R(W^v)$, $\nu \in \mathrm{conv}(W^v.\lambda^{++}, \geq \lambda) \setminus \{\lambda\}$ and b_λ is a primitive monomial in $q_i^{-1}, q_i'^{-1}$.

(3) *For $\lambda \in Y^{++}$, we have $X^\lambda = T_\lambda$, and for $\lambda, \lambda' \in Y^+$,*

$$X^\lambda * X^{\lambda'} = X^{\lambda+\lambda'} = X^{\lambda'} * X^\lambda.$$

Remarks. (a) We have two bases for the free right $\mathcal{H}_R(W^v)$-module ${}^{\mathrm{I}}\mathcal{H}_R$,

$$\{T_\lambda \mid \lambda \in Y^+\} \quad \text{and} \quad \{X^\lambda \mid \lambda \in Y^+\}.$$

The change of basis matrix is triangular (for the order $\geq_{Q^\vee}$) with diagonal coefficients primitive monomials in $q_i^{-1}, q_i'^{-1}$. From Corollary 5.3(d), we get that all coefficients of this matrix are Laurent polynomials in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and on W.

(b) By (1) above and Corollary 5.4, it is clear that the left multiplication by X^λ is injective, for any $\lambda \in Y^+$.

Proof. By Corollary 5.4, the uniqueness is clear and (3) follows from the relation $T_\lambda * T_\mu = T_{\lambda+\mu}$ of the Theorem 2.4. We have just to prove (1) and (2) for a $\mu \in Y^{++}$ (chosen sufficiently great).

We argue by induction on the height $\mathrm{ht}(\lambda^{++}-\lambda)$ of $\lambda^{++}-\lambda$ with respect to the free family $(\alpha_i^\vee)$ in $Q^\vee$. When the height is 0, $\lambda=\lambda^{++}$ and $X^\lambda=T_\lambda$. By Corollary 5.3, we write

$$T_\mu * T_\lambda = a_\lambda T_{\mu+\lambda} + \sum_{\substack{\lambda\le_{Q^\vee}\nu\le_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} T_{\mu+\nu} * H^\nu$$

with $H^\nu\in\mathcal{H}_R(W^v)$ and $\nu\in\mathrm{conv}(W^v.\lambda^{++})$; hence, $\nu^{++}\in\mathrm{conv}(W^v.\lambda^{++})$ (in particular, $\nu^{++}\le_{Q^\vee}\lambda^{++}$); see Section 1.8(a).

So $\mathrm{ht}(\nu^{++}-\nu)<\mathrm{ht}(\lambda^{++}-\lambda)$. By induction and for μ sufficiently great, we can consider the element X^ν such that $T_{\mu+\nu}=T_\mu * X^\nu$; we can write it

$$X^\nu = \sum_{\nu\le_{Q^\vee}\nu'\le_{Q^\vee}\nu^{++}} T_{\nu'} * H^{\nu',\nu}$$

and we may take

$$\begin{aligned} X^\lambda &= a_\lambda^{-1}T_\lambda - \Bigg(\sum_{\substack{\lambda\le_{Q^\vee}\nu\le_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} X^\nu * H^\nu\Bigg)\\ &= a_\lambda^{-1}T_\lambda - \Bigg(\sum_{\substack{\lambda\le_{Q^\vee}\nu\le_{Q^\vee}\lambda^{++}\\ \lambda\neq\nu}} \Bigg(\sum_{\nu\le_{Q^\vee}\nu'\le_{Q^\vee}\nu^{++}} T_{\nu'} * H^{\nu',\nu}\Bigg) * H^\nu\Bigg). \end{aligned}$$ □

Proposition 5.6. *For $\lambda\in Y^+$ and $i\in I$ we have the following relations*:

(a) *If $\alpha_i(\lambda)\ge 0$, then*

$$T_i * X^\lambda = q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\lambda - h\alpha_i^\vee}.$$

(b) *If $\alpha_i(\lambda)<0$, then*

$$T_i * X^\lambda = \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big(q_i^{*(-\alpha_i(\lambda)+h+1)} - q_i^{*(-\alpha_i(\lambda)+h)}\big) X^{\lambda - h\alpha_i^\vee}.$$

N.B. These are the Bernstein–Lusztig relations for the X^λ, (BLX) for short.

Proof. If $\lambda\in Y^{++}$, by Theorem 4.8(a), we know that $X^\lambda * T_i * X^\lambda = X^{\lambda+\lambda} * T_i$ when $\alpha_i(\lambda)=0$ and, when $\alpha_i(\lambda)>0$,

$$X^\lambda * T_i * X^\lambda = q_i^{*\alpha_i(\lambda)} X^{\lambda + r_i(\lambda)} * T_i + (q_i^{*(\alpha_i(\lambda))} - q_i^{*(\alpha_i(\lambda)-1)}) X^{\lambda+\lambda-(\alpha_i(\lambda)-1)\alpha_i^\vee} + \cdots + (q_i^{*2}-q_i) X^{\lambda+\lambda-\alpha_i^\vee} + (q_i-1)X^{\lambda+\lambda},$$

so we have the result.

In the general case, $\lambda \in Y^+$, we write $\lambda = \mu - \nu$ with μ, ν chosen in Y^{++}. By Theorem 5.5, $X^\nu * X^\lambda = X^\mu$. From (BLX) for X^μ and X^ν,

$$T_i * X^\mu = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i + \sum_{h=0}^{\alpha_i(\lambda+\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}$$

which can also be written

$$\begin{aligned} T_i * X^{\nu+\lambda} &= (T_i * X^\nu) * X^\lambda \\ &= \left(q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i + \sum_{h=0}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu-h\alpha_i^\vee}\right) * X^\lambda \\ &= q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda + \sum_{h=0}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}. \end{aligned}$$

If $\alpha_i(\lambda) \geq 0$, then

$$q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\mu)} * T_i + \sum_{h=\alpha_i(\nu)}^{\alpha_i(\lambda+\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}.$$

We take $h' = h - \alpha_i(\nu)$, then

$$X^{\nu+\lambda-h\alpha_i^\vee} = X^{\nu-\alpha_i(\nu)\alpha_i^\vee+\lambda-h'\alpha_i^\vee} = X^{r_i(\nu)+\lambda-h'\alpha_i^\vee}$$

and $q_i^{*(\alpha_i(\nu)+h')} = q_i^{*\alpha_i(\nu)} q_i^{*h'}$ (by $q_i = q_i'$ if $\alpha_i(\nu)$ is odd, and by an easy calculation if $\alpha_i(\nu)$ is even). So,

$$\begin{aligned} &q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda \\ &\quad = q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * \left(q_i^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h'=0}^{\alpha_i(\lambda)-1} \big(q_i^{*(h'+1)} - q_i^{*(h')}\big) X^{\lambda-h'\alpha_i^\vee}\right). \end{aligned}$$

And we are done, thanks to the injectivity of left multiplication by $X^{r_i(\nu)}$.

If $\alpha_i(\lambda) < 0$, we obtain

$$\begin{aligned} &q_i^{*(\alpha_i(\nu))} X^{r_i(\nu)} * T_i * X^\lambda \\ &\quad = q_i^{*(\alpha_i(\lambda+\nu))} X^{r_i(\lambda+\nu)} * T_i - \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee}. \end{aligned}$$

We have $q_i^{*(\alpha_i(\nu))} = q_i^{*(-\alpha_i(\lambda))} q_i^{*(\alpha_i(\lambda+\nu))}$ by an easy calculation if $\alpha_i(\nu)$ and $\alpha_i(\lambda)$ are even and because $q_i = q_i'$ whenever $\alpha_i(\nu)$ or $\alpha_i(\lambda)$ is odd. So,

$$\begin{aligned} &X^{r_i(\nu)} * T_i * X^\lambda \\ &\quad = \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda+\nu)} * T_i - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\nu+\lambda-h\alpha_i^\vee} \end{aligned}$$

and we have (because of the injectivity of the left multiplication by $X^{r_i(\nu)}$)

$$\begin{aligned} &T_i * X^\lambda \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i \\ &\qquad - \frac{1}{q_i^{*(\alpha_i(\nu))}} \sum_{h=\alpha_i(\lambda+\nu)}^{\alpha_i(\nu)-1} \big(q_i^{*(h+1)} - q_i^{*(h)}\big) X^{\lambda+(\alpha_i(\nu)-h)\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i \\ &\qquad - \frac{1}{q_i^{*(\alpha_i(\nu))} q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big(q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h+1)} - q_i^{*(\alpha_i(\nu)-\alpha_i(\lambda)+h)}\big) X^{\lambda-h\alpha_i^\vee} \\ &= \frac{1}{q_i^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i \\ &\qquad - \frac{1}{q_i^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big(q_i^{*(-\alpha_i(\lambda)+h+1)} - q_i^{*(-\alpha_i(\lambda)+h)}\big) X^{\lambda-h\alpha_i^\vee}. \qquad \square \end{aligned}$$

5.7. *The classical Bernstein–Lusztig relation.* The *module* $\delta : Q^\vee \to R$ is defined by

$$\delta\Big(\sum_{i\in I} a_i \alpha_i^\vee\Big) = \prod_{i\in I} (q_i q_i')^{a_i}$$

[Gaussent and Rousseau 2014, 5.3.2]. After replacing eventually R by a bigger ring R' containing some square roots $\sqrt{q_i}, \sqrt{q_i'}$ of q_i, q_i' (with $\sqrt{q_i} = \sqrt{q_i'}$, if $q_i = q_i'$), we assume moreover that there exists a homomorphism $\delta^{1/2} : Y \to R^\times$, such that $\delta(\lambda) = (\delta^{1/2}(\lambda))^2$ for any $\lambda \in Q^\vee$ and $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i}\cdot\sqrt{q_i'}$. In particular $\sqrt{q_i}^{\pm 1}$ and $\sqrt{q_i'}^{\pm 1}$ are well defined in $R^\times$. In the common example where $R = \mathbb{R}$ or $R = \mathbb{C}$, these expressions are chosen to be the classical ones: $\delta^{1/2}(Y) \subset \mathbb{R}_+^*$.

We define $H_i = \big(\sqrt{q_i}\big)^{-1} T_i$ and $Z^\lambda = \delta^{-1/2}(\lambda) X^\lambda$ for $\lambda \in Y^+$. When $w = r_{i_1} \cdots r_{i_n}$ is a reduced decomposition, we set $H_w = H_{i_1} * \cdots * H_{i_n}$; this does not depend on the chosen decomposition of w.

We may translate the relations (BLX) for these elements.

Proposition. *For $\lambda \in Y^{++}$, we have the relation*

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{k=0}^{\lfloor(\alpha_i(\lambda)-1)/2\rfloor} \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big) Z^{\lambda-(2k)\alpha_i^\vee} + \sum_{k=0}^{\lfloor\alpha_i(\lambda)/2\rfloor-1} \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big) Z^{\lambda-(2k+1)\alpha_i^\vee}.$$

Remarks. (1) This is the Bernstein–Lusztig relation for the Z^λ, (BLZ) for short.

(2) In the following section, we shall consider an algebra containing ${}^{\mathrm{I}}\mathcal{H}_R$ and, for any $i \in I$, an element $Z^{-\alpha_i{}^\vee}$ satisfying $Z^{\lambda-h\alpha_i{}^\vee} = Z^\lambda * (Z^{-\alpha_i{}^\vee})^h$ for $h \in \mathbb{N}$, $\lambda, \lambda - h\alpha_i^\vee \in Y^+$. In such an algebra the relation (BLZ) may be rewritten (using that $\sqrt{q_i} = \sqrt{q_i'}$ if $\alpha_i(\lambda)$ is odd) as the classical Bernstein–Lusztig relation (BL):

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)\frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)\frac{Z^{\lambda-\alpha_i^\vee} - Z^{r_i(\lambda)-\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}},$$

i.e., $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b\big(\sqrt{q_i}, \sqrt{q_i'}; Z^{-\alpha_i{}^\vee}\big)(Z^\lambda - Z^{r_i(\lambda)})$, where

$$b(t, u; z) = \frac{t - t^{-1} + (u - u^{-1})z}{1 - z^2}.$$

This is the same relation as in [Macdonald 2003, 4.2], up to the order; see (3).

(3) Actually this relation (BLZ) is still true when $\lambda \in Y^+$ and $\alpha_i(\lambda) \geq 0$ (same proof as below). If $\alpha_i(\lambda) < 0$, we leave to the reader the proof of the relation

$$T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i - \left(\sum_{\substack{2 \leq h \leq -\alpha_i(\lambda)\\ h \text{ even}}} (q_i - 1) Z^{\lambda+h\alpha_i^\vee} + \sum_{\substack{1 \leq h \leq -\alpha_i(\lambda)\\ h \text{ odd}}} \left(\sqrt{q_i \cdot q_i'} - \frac{\sqrt{q_i \cdot q_i'}}{q_i'}\right) Z^{\lambda+h\alpha_i^\vee}\right).$$

In the situation of (2) above, it may be rewritten

$$\begin{aligned} H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i &= \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)\frac{Z^\lambda - Z^{r_i(\lambda)}}{1 - Z^{-2\alpha_i^\vee}} + \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)\frac{Z^{\lambda-\alpha_i^\vee} - Z^{r_i(\lambda)-\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}} \\ &= b\big(\sqrt{q_i}, \sqrt{q_i'}; Z^{-\alpha_i{}^\vee}\big)(Z^\lambda - Z^{r_i(\lambda)}). \end{aligned}$$

It is the same relation (BLZ) as above. Moreover, it's easy to see in the first equality that $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = Z^\lambda * H_i - H_i * Z^{r_i(\lambda)}$. Actually we shall see in Section 6 that this same relation is true for any $\lambda \in Y$ in a greater algebra containing elements Z^λ for $\lambda \in Y$.

Proof. From $Z^\lambda = \delta^{-1/2}(\lambda)X^\lambda$ and $\delta^{1/2}(\alpha_i^\vee) = \sqrt{q_i \,.\, q_i'}$, we get

$$\begin{aligned} Z^{\lambda-h\alpha_i^\vee} &= \delta^{-1/2}(\lambda - h\alpha_i^\vee)X^{\lambda-h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda)(\delta^{1/2}(\alpha_i^\vee))^h X^{\lambda-h\alpha_i^\vee} \\ &= \delta^{-1/2}(\lambda)\big(\sqrt{q_i \,.\, q_i'}\big)^h X^{\lambda-h\alpha_i^\vee}. \end{aligned}$$

By $\alpha_i(\lambda) \geq 0$ and (BLX),

$$\begin{aligned} &T_i * Z^\lambda \\ &\quad = q_i^{*(\alpha_i(\lambda))}\big(\sqrt{q_i \,.\, q_i'}\big)^{-\alpha_i(\lambda)} Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i^{*(h+1)} - q_i^{*h})\big(\sqrt{q_i \,.\, q_i'}\big)^{(-h)} Z^{\lambda-h\alpha_i^\vee}. \end{aligned}$$

Moreover, $q_i^{*h} = q_i q_i' q_i \cdots$ with h terms in the product, so $q_i^{*h} = \big(\sqrt{q_i \,.\, q_i'}\big)^h$ if h is even and $q_i^{*h} = q_i\big(\sqrt{q_i \,.\, q_i'}\big)^{(h-1)}$ if h is odd. So, if $\alpha_i(\lambda)$ is even, then

$$\begin{aligned} &T_i * Z^\lambda \\ &\quad = Z^{r_i(\lambda)} * T_i + \sum_{k=0}^{(\alpha_i(\lambda)-2)/2} (q_i - 1)Z^{\lambda-(2k)\alpha_i^\vee} + \sum_{k=0}^{(\alpha_i(\lambda)-2)/2} (q_i q_i' - q_i)\big(\sqrt{q_i q_i'}\big)^{-1} Z^{\lambda-(2k+1)\alpha_i^\vee}. \end{aligned}$$

If $\alpha_i(\lambda)$ is odd, then $q_i = q_i'$ and

$$T_i * Z^\lambda = Z^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} (q_i - 1)Z^{\lambda-h\alpha_i^\vee}.$$

In both cases, by $H_i = (\sqrt{q_i})^{-1}T_i$,

$$\begin{aligned} H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i &+ \sum_{k=0}^{\lfloor(\alpha_i(\lambda)-1)/2\rfloor} \big(\sqrt{q_i} - \sqrt{q_i}^{-1}\big)Z^{\lambda-(2k)\alpha_i^\vee} \\ &+ \sum_{k=0}^{\lfloor\alpha_i(\lambda)/2\rfloor-1} \big(\sqrt{q_i'} - \sqrt{q_i'}^{-1}\big)Z^{\lambda-(2k+1)\alpha_i^\vee}. \quad \square \end{aligned}$$

6. Bernstein–Lusztig–Hecke Algebras

The aim of this section is to define, in a formal way, an associative algebra ${}^{\mathrm{BL}}\mathcal{H}_R$, called the Bernstein–Lusztig–Hecke algebra. This construction by generators and relations is motivated by the results obtained in the previous section (in particular Proposition 5.6) and we will be able next to identify ${}^{\mathrm{I}}\mathcal{H}_R$ and a subalgebra of ${}^{\mathrm{BL}}\mathcal{H}_R$ (up to some hypotheses on R).

We use the same notation as before, even if the objects are somewhat different. This choice will be justified by the identification obtained at the end of this section.

We consider $\mathbb{A}$ as in Section 1.2 and $\mathrm{Aut}(\mathbb{A}) \supset W = W^v \ltimes Y \supset W^a$, with Y a discrete group of translations.

6.1. ***The module*** $^{\mathrm{BL}}\mathcal{H}_{R_1}$. We consider now the ring $R_1 = \mathbb{Z}[(\sigma_i^{\pm 1}, \sigma_i'^{\pm 1})_{i\in I}]$ where the indeterminates σ_i, σ_i' satisfy the following relations (as q_i and q_i' in Section 1.4.5 because in the further identification, σ_i, σ_i' will play the role of $\sqrt{q_i}$ and $\sqrt{q_i'}$).

If $\alpha_i(Y) = \mathbb{Z}$, then $\sigma_i = \sigma_i'$.

If r_i and r_j are conjugated (i.e., if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$), then $\sigma_i = \sigma_j = \sigma_i' = \sigma_j'$.

We denote by $^{\mathrm{BL}}\mathcal{H}_{R_1}$ the free R_1-module with basis $(Z^\lambda H_w)_{\lambda\in Y,\, w\in W^v}$. For short, we write $H_i = H_{r_i}$, $H_w = Z^0 H_w$ and $Z^\lambda = Z^\lambda H_e$, where e is the unit element in W^v (and $H_e = Z^0$ will be the multiplicative unit element in $^{\mathrm{BL}}\mathcal{H}_{R_1}$).

Theorem 6.2. *There exists a unique multiplication $*$ on $^{\mathrm{BL}}\mathcal{H}_{R_1}$ which makes it an associative unitary R_1-algebra with unity H_e and satisfies the following conditions*:

(1) $Z^\lambda * H_w = Z^\lambda H_w$ *for all* $\lambda \in Y,\ w \in W^v$,

(2) $H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i - \sigma_i^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) < \ell(w), \end{cases}$ *for all* $i \in I,\ w \in W^v$,

(3) $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$ *for all* $\lambda \in Y,\ \mu \in Y$,

(4) $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)})$ *for all* $\lambda \in Y,\ i \in I$, *where*

$$b(t, u; z) = \frac{(t - t^{-1}) + (u - u^{-1})z}{1 - z^2}.$$

Remarks 6.3. (1) It is already known (see, e.g., [Humphreys 1990, Theorem 7.1] or [Bourbaki 1968, IV §2, exercise 23]) that the free submodule with basis $(H_w)_{w\in W^v}$ can be equipped, in a unique way, with a multiplication $*$ that satisfies (2) and gives it a structure of an associative unitary algebra called the "Hecke algebra of the group W^v over R_1" and denoted by $\mathcal{H}_{R_1}(W^v)$.

(2) The submodule $\mathcal{H}_{R_1}(Y)$ with basis $(Z^\lambda)_{\lambda\in Y}$ will be a commutative subalgebra.

(3) When all σ_i, σ_i' are equal, the existence of this algebra $^{\mathrm{BL}}\mathcal{H}$ is stated in [Garland and Grojnowski 1995] and justified by an action on some Grothendieck group.

(4) This R_1-algebra depends only on $\mathbb{A}$ and Y (i.e., $\mathbb{A}$ and W). We call it the Bernstein–Lusztig–Hecke algebra over R_1 (associated to $\mathbb{A}$ and W).

6.4. ***Proof of Theorem 6.2.***

6.4.1. The uniqueness of the multiplication $*$ is clear: by associativity and distributivity, we have only to identify $H_w * Z^\mu$. If $w = r_{i_1} r_{i_2} \cdots r_{i_n}$ is a reduced decomposition, then, by (2), (4), and Remark 6.3(1),

$$H_w * Z^\mu = H_{i_1} * (H_{i_2} * (\cdots * (H_{i_n} * Z^\mu) \cdots))$$

has to be a well-defined linear combination of terms $Z^\nu H_u$: $H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$ with $a_k \in R_1$, $\nu_k \in Y$, and $u_k \in W^v$.

6.4.2. Construction of $*$. We define $H_w * Z^\mu$ as above and we have to prove that it does not depend on the reduced decomposition $w = r_{i_1} r_{i_2} \cdots r_{i_n}$.

(a) We define $L_i \in \mathrm{End}_{R_1}({}^{\mathrm{BL}}\mathcal{H}_{R_1})$ by

$$L_i(Z^\mu H_w) = H_i * (Z^\mu H_w) = Z^{r_i(\mu)}(H_i * H_w) + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^\mu - Z^{r_i(\mu)}) * H_w,$$

where

$$H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i - \sigma_i{}^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) < \ell(w). \end{cases}$$

By Matsumoto's theorem [Bourbaki 1968, IV §1.5, Proposition 5], the expected independence will be a consequence of the braid relations, i.e.,

$$(*) \qquad L_i(L_j(L_i(\cdots(Z^\lambda H_w)\cdots))) = L_j(L_i(L_j(\cdots(Z^\lambda H_w)\cdots)))$$

(with $m_{i,j}$ factors L on each side), whenever the order $m_{i,j}$ of $r_i r_j$ is finite.

As $\mathcal{H}_{R_1}(W^v)$ is known to be an algebra, it is enough to prove $(*)$ for $w = 1$. We may also suppose $\alpha_j(\alpha_i^\vee) \neq 0$ as otherwise L_i and L_j commute clearly.

We choose $i, j \in I$ with $m_{i,j}$ finite; then $\pm\alpha_i$, $\pm\alpha_j$ generate a finite root system $\Phi_{i,j}$ of rank 2 (or 1 if $i = j$). Moreover, $Y' = \ker(\alpha_i) \cap \ker(\alpha_j) \cap Y$ is cotorsion free in Y. Let Y'' be a supplementary module containing $\alpha_i^\vee$ and $\alpha_j^\vee$; Y'' is a lattice (of rank 2 or 1) between the lattices $Q_{i,j}^\vee$ of coroots and $P_{i,j}^\vee$ of coweights, associated to $\Phi_{i,j}$.

Any $\lambda \in Y$ may be written $\lambda = \lambda' + \lambda''$ with $\lambda' \in Y'$ and $\lambda'' \in Y''$. By (4), $L_i(Z^{\lambda'}) = Z^{\lambda'} H_i$ and $L_j(Z^{\lambda'}) = Z^{\lambda'} H_j$. So we have to prove $(*)$ for $\lambda = \lambda'' \in Y''$. We shall do it by comparing with some Macdonald's results.

(b) Macdonald [2003] builds affine Hecke algebras $\mathcal{H}(W(R, L'))$ over $\mathbb{R}$, associated to any finite irreducible root system R and any lattice L' between the lattices of coroots and coweights; more precisely this algebra is associated to the extended affine Weyl group $W(R, L') = W(R) \ltimes L'$. It is defined by generators and relations, but it is proven that it is endowed with a basis $(Y^\lambda T(w))_{\lambda \in L', w \in W(R)}$ [op. cit., 4.2.7] and satisfies relations analogous to (1)–(4) as above. There are parameters $(\tau_i)_{i \in I}$ and τ_0 which are reals (but may be algebraically independent over $\mathbb{Q}$, so may be considered as indeterminates) and satisfy $\tau_i = \tau_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$. The relation (4) is satisfied with $\sigma_i = \tau_i$ and $\sigma_i' = \tau_i$ when $\alpha_i(L') = \mathbb{Z}$, $\sigma_i' = \tau_0$ when $\alpha_i(L') = 2\mathbb{Z}$.

(c) In the case $R = \Phi_{i,j}$, irreducible, $L' = Y''$, we may choose τ_i, τ_j, and τ_0 such that the relations (4) are the same, for us and Macdonald: either $\alpha_i(\alpha_j^\vee) = -1$ or $\alpha_j(\alpha_i^\vee) = -1$, so $\tau_0 = \sigma_i'$ or $\tau_0 = \sigma_j'$. In particular R_1 may be identified with a subring of $\mathbb{R}$. The operators L_i and L_j of both theories coincide on the elements

$Z^\lambda H_v$ (identified with $Y^\lambda T(v)$ in Macdonald's work) for $\lambda \in L' = Y''$ and $v \in \langle r_i, r_j \rangle$. So $(*)$ is satisfied as $\mathcal{H}(W(R, L'))$ is an associative algebra.

(d) So, if $H_w * Z^\mu = \sum_k a_k Z^{\nu_k} H_{u_k}$, with $a_k \in R_1$, $\nu_k \in Y$, $u_k \in W^v$, we define the product of $Z^\lambda H_w$ and $Z^\mu H_v$ by

$$(Z^\lambda H_w) * (Z^\mu H_v) = \sum_k a_k Z^{\lambda + \nu_k} * (H_{u_k} * H_v).$$

We get a distributive multiplication on ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ with unit H_e.

6.4.3. *Associativity.* (a) Using the associativity in $\mathcal{H}_{R_1}(Y)$ and $\mathcal{H}_{R_1}(W^v)$ and the formula in 6.4.2(d) above, it is clear that, for any $\lambda \in Y$, $w \in W^v$, $E_1, E_2 \in {}^{\mathrm{BL}}\mathcal{H}_{R_1}$,

(R1) $Z^\lambda * (E_1 * E_2) = (Z^\lambda * E_1) * E_2$,

(R2) $E_1 * (E_2 * H_w) = (E_1 * E_2) * H_w$.

We need also to prove (for $\lambda_1, \lambda_2 \in Y$, $w, w_1, w_2 \in W^v$, $E \in {}^{\mathrm{BL}}\mathcal{H}_{R_1}$),

(A) $H_w * (Z^{\lambda_1} * Z^{\lambda_2}) = (H_w * Z^{\lambda_1}) * Z^{\lambda_2}$,

(B) $H_{w_1} * (H_{w_2} * E) = (H_{w_1} * H_{w_2}) * E$.

Then the general associativity will follow: using (R1), (R2), (A), (B), and the formula in 6.4.2(d) for the product, it is not too difficult (and left to the reader) to prove that

$$\begin{aligned}(Z^{\lambda_1} H_{w_1}) * \big((Z^{\lambda_2} H_{w_2}) * (Z^{\lambda_3} H_{w_3})\big) &= Z^{\lambda_1} * (H_{w_1} * \big((Z^{\lambda_2} H_{w_2}) * Z^{\lambda_3})\big) * H_{w_3} \\ &= Z^{\lambda_1} * \big((H_{w_1} * Z^{\lambda_2}) * (H_{w_2} * Z^{\lambda_3})\big) * H_{w_3} \\ &= Z^{\lambda_1} * \big((H_{w_1} * (Z^{\lambda_2} H_{w_2})) * Z^{\lambda_3}\big) * H_{w_3} \\ &= \big((Z^{\lambda_1} H_{w_1}) * (Z^{\lambda_2} H_{w_2})\big) * (Z^{\lambda_3} H_{w_3}).\end{aligned}$$

(b) Proof of (B). This condition is equivalent to the fact that left multiplication by $\mathcal{H}_{R_1}(W^v)$ on ${}^{\mathrm{BL}}\mathcal{H}_{R_1}$ is an action. But the associative algebra $\mathcal{H}_{R_1}(W^v)$ is generated by the H_i with relations consisting of the braid relations and $H_i^2 = (\sigma_i - \sigma_i^{-1}) H_i + H_e$. As L_i is left multiplication by H_i, we have (B) if and only if these L_i satisfy the relation $(*)$ and

$$(**) \qquad L_i(L_i(Z^\lambda H_v)) = (\sigma_i - \sigma_i^{-1}) L_i(Z^\lambda H_v) + Z^\lambda H_v.$$

As in 6.4.2(b), we reduce the verification of $(**)$ to the case $v = 1$ and $\lambda \in Y''$ (associated to $i = j$), i.e., $\lambda \in Y'' = \mathbb{Q}\alpha_i^\vee \cap Y$. Then we look at Macdonald's construction of $\mathcal{H}(W(\{\pm\alpha_i\}, Y''))$ with $\tau_i = \sigma_i$, $\tau_0 = \sigma_i'$. We conclude, as in 6.4.2(c) that $(**)$ is satisfied.

(c) The proof of (A) is by induction on $\ell(w)$. If $w = r_i$,

$$\begin{aligned}
(H_i * Z^{\lambda_1}) * Z^{\lambda_2} &= (Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + \big(b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2} \\
&= Z^{r_i(\lambda_1)} * \big(Z^{r_i(\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_2} - Z^{r_i(\lambda_2)})\big) \\
&\qquad + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1)+\lambda_2}) \\
&= Z^{r_i(\lambda_1+\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{r_i(\lambda_1)+\lambda_2} - Z^{r_i(\lambda_1)+r_i(\lambda_2)}) \\
&\qquad + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1)+\lambda_2}) \\
&= Z^{r_i(\lambda_1+\lambda_2)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1+\lambda_2} - Z^{r_i(\lambda_1+\lambda_2)}) \\
&= H_i * (Z^{\lambda_1} * Z^{\lambda_2})
\end{aligned}$$

If the result is known when $\ell(w) = n$, let us consider $w = w' r_i$ with $\ell(w) = n+1$ and $\ell(w') = n$. Then

$$\begin{aligned}
&H_w * (Z^{\lambda_1} * Z^{\lambda_2}) \\
&\qquad = H_{w'} * (H_i * Z^{\lambda_1+\lambda_2}) \\
&\qquad = H_{w'} * \big((H_i * Z^{\lambda_1}) * Z^{\lambda_2}\big) \\
&\qquad = H_{w'} * \big((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2} + \big(b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2}\big),
\end{aligned}$$

where the first equality is because left multiplication by $\mathcal{H}_{R_1}(W^v)$ is an action, and the second equality is the case $\ell(w) = 1$. On the other hand,

$$\begin{aligned}
&(H_w * Z^{\lambda_1}) * Z^{\lambda_2} \\
&\quad = (H_{w'} * (H_i * Z^{\lambda_1})) * Z^{\lambda_2} \\
&\quad = \big(H_{w'} * (Z^{r_i(\lambda_1)} H_i + b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})\big) * Z^{\lambda_2} \\
&\quad = (H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} + \big(H_{w'} * (b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)}))\big) * Z^{\lambda_2}.
\end{aligned}$$

The second term of the right hand side is an R_1-linear combination of

$$(H_{w'} * Z^{\lambda_1 + k\alpha_i^\vee}) * Z^{\lambda_2}$$

and we see by induction that it is the same as

$$H_{w'} * \big((b(\sigma_i, \sigma_i'; Z^{-\alpha_i^\vee})(Z^{\lambda_1} - Z^{r_i(\lambda_1)})) * Z^{\lambda_2}\big)$$

in $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$.

In the first term, $(H_{w'} * (Z^{r_i(\lambda_1)} H_i)) * Z^{\lambda_2} = ((H_{w'} * Z^{r_i(\lambda_1)}) * H_i)) * Z^{\lambda_2}$, we can write

$$H_{w'} * Z^{r_i(\lambda_1)} = \sum_k c_k Z^{\lambda_k} H_{w_k},$$

and we will use later in the same way

$$H_i * Z^{\lambda_2} = \sum_h a_h Z^{\mu_h} H_{v_h}$$

with $c_k, a_h \in R_1,\ \lambda_k, \mu_h \in Y$, and $w_k, v_h \in W^v$. So, we have

$$\begin{aligned}
&\left(\left(\sum_k c_k Z^{\lambda_k} H_{w_k}\right) * H_i\right) * Z^{\lambda_2} \\
&= \left(\sum_k c_k (Z^{\lambda_k} * (H_{w_k} * H_i))\right) * Z^{\lambda_2} && \text{(by (R2))} \\
&= \sum_k c_k Z^{\lambda_k} * ((H_{w_k} * H_i) * Z^{\lambda_2}) && \text{(by 6.4.2(d))} \\
&= \sum_k c_k Z^{\lambda_k} * (H_{w_k} * (H_i * Z^{\lambda_2})) && \text{(by (B))} \\
&= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * (H_i * Z^{\lambda_2}) && \text{(by (R1))} \\
&= \sum_k c_k (Z^{\lambda_k} * H_{w_k}) * \left(\sum_h a_h Z^{\mu_h} H_{v_h}\right) \\
&= \sum_{k,h} c_k a_h (Z^{\lambda_k} * H_{w_k}) * (Z^{\mu_h} * H_{v_h}) \\
&= \sum_{k,h} c_k a_h (((Z^{\lambda_k} * H_{w_k}) * Z^{\mu_h}) * H_{v_h}) && \text{(by (R2))} \\
&= \sum_h a_h (((H_{w'} * Z^{r_i(\lambda_1)}) * Z^{\mu_h}) * H_{v_h}) \\
&= \sum_h a_h ((H_{w'} * (Z^{r_i(\lambda_1)} * Z^{\mu_h})) * H_{v_h}) && \text{(by induction)} \\
&= \sum_h a_h H_{w'} * ((Z^{r_i(\lambda_1)} * Z^{\mu_h}) * H_{v_h}) && \text{(by (R2))} \\
&= H_{w'} * (Z^{r_i(\lambda_1)} * (H_i * Z^{\lambda_2})). && \text{(by (R1))}
\end{aligned}$$

This corresponds to the term $H_{w'} * ((Z^{r_i(\lambda_1)} H_i) * Z^{\lambda_2})$ in $H_w * (Z^{\lambda_1} * Z^{\lambda_2})$ so we obtain the equality when $\ell(w) = n+1$.

6.5. *Change of scalars.*

6.5.1. Suppose that we are given a morphism φ from R_1 to a ring R. Then we are able to consider, by extension of scalars, ${}^{\mathrm{BL}}\mathcal{H}_R = R \otimes_{R_1} {}^{\mathrm{BL}}\mathcal{H}_{R_1}$ as an R-associative algebra. The family $(Z^\lambda H_w)_{\lambda \in Y,\, w \in W^v}$ is still a basis of the R-module ${}^{\mathrm{BL}}\mathcal{H}_R$.

6.5.2. In order to consider elements similar to the X^λ of Section 4, we are going to define a ring R_3 containing R_1 such that there exists a group homomorphism $\delta^{1/2} : Y \to R_3^\times$ with $\delta(\lambda) = \delta^{1/2}(\lambda)^2$ for any $\lambda \in Q^\vee$ and $\delta^{1/2}(\alpha_i^\vee) = \sigma_i . \sigma_i'$.

Since $Q^\vee$ is a submodule of the free $\mathbb{Z}$-module Y, by the elementary divisor theorem, if we denote by m the biggest elementary divisor, then $m\mu \in Q^\vee$ for any $\mu \in Y \cap (Q^\vee \otimes_{\mathbb{Z}} \mathbb{R})$. Let us consider the ring $R_3 = \mathbb{Z}[(\tau_i^{\pm 1}, \tau_i'^{\pm 1})_{i \in I}]$ (with τ_i, τ_i' satisfying conditions similar to those of Section 6.1) and the identification of R_1 as a subring of R_3 given by $\tau_i^m = \sigma_i$ and $\tau_i'^m = \sigma_i'$. Then, for $\lambda \in Y$ we have $m\lambda = \sum_{i \in I} a_i \alpha_i^\vee + \lambda_0$ with the $a_i \in \mathbb{Z}$ and $\lambda_0 \notin Q^\vee \otimes_{\mathbb{Z}} \mathbb{R}$, and we can define

$$\delta^{1/2}(\lambda) = \prod_{i \in I} (\tau_i \tau_i')^{a_i}$$

and obtain a group homomorphism from Y to R_3, with the wanted properties.

In ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$, let us consider $X^\lambda = \delta^{1/2}(\lambda) Z^\lambda$ for $\lambda \in Y$ and $T_i = \sigma_i H_i = (\tau_i)^m H_i$. It's easy to see that $T_w = T_{i_1} * T_{i_2} * \cdots * T_{i_n}$ is independent of the choice of a reduced decomposition $r_{i_1} r_{i_2} \cdots r_{i_n}$ of w. It is clear that the family $(X^\lambda * T_w)_{\lambda \in Y,\, w \in W^v}$ is a new basis of the R_3-module ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$.

6.5.3. We can give new formulas to define $*$ in terms of these generators. The relation (4) of the definition of ${}^{\mathrm{BL}}\mathcal{H}_{R_3}$ can be written as previously:

If $\alpha_i(\lambda) \geq 0$, then

$$\text{(BLZ+)} \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + \sum_{\substack{0 \leq k \leq \alpha_i(\lambda) - 1 \\ k \text{ even}}} (\sigma_i - \sigma_i^{-1}) Z^{\lambda - k\alpha_i^\vee} + \sum_{\substack{0 \leq k \leq \alpha_i(\lambda) - 1 \\ k \text{ odd}}} (\sigma_i' - \sigma_i'^{-1}) Z^{\lambda - k\alpha_i^\vee}.$$

If $\alpha_i(\lambda) < 0$, then

$$\text{(BLZ}-\text{)} \quad H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i - \sum_{\substack{2 \leq k \leq -\alpha_i(\lambda) \\ k \text{ even}}} (\sigma_i - \sigma_i^{-1}) Z^{\lambda + k\alpha_i^\vee} - \sum_{\substack{1 \leq k \leq -\alpha_i(\lambda) \\ k \text{ odd}}} (\sigma_i' - \sigma_i'^{-1}) Z^{\lambda + k\alpha_i^\vee}.$$

With the same arguments as in Section 5.7, these relations (after changing variables and writing $(\sigma_i^2)^{*n} = \sigma_i^2 \sigma_i'^2 \sigma_i^2 \sigma_i'^2 \cdots$ with n terms in this product) become:

If $\alpha_i(\lambda) \geq 0$, then

$$\text{(BLX+)} \quad T_i * X^\lambda = (\sigma_i^2)^{*(\alpha_i(\lambda))} X^{r_i(\lambda)} * T_i + \sum_{h=0}^{\alpha_i(\lambda)-1} \big((\sigma_i^2)^{*(h+1)} - (\sigma_i^2)^{*(h)} \big) X^{\lambda - h\alpha_i^\vee}.$$

If $\alpha_i(\lambda) < 0$, then

$$(\mathrm{BLX}-)\quad T_i * X^\lambda = \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} X^{r_i(\lambda)} * T_i \\ - \frac{1}{(\sigma_i^2)^{*(-\alpha_i(\lambda))}} \sum_{h=\alpha_i(\lambda)}^{-1} \big((\sigma_i^2)^{*(-\alpha_i(\lambda)+h+1)} - (\sigma_i^2)^{*(-\alpha_i(\lambda)+h)}\big) X^{\lambda - h\alpha_i^\vee}.$$

The other formulas easily give:

$$(2')\quad T_i * T_w = \begin{cases} T_{r_i w} & \text{if } \ell(r_i w) > \ell(w), \\ (\sigma_i^2 - 1) T_w + \sigma_i^2 T_{r_i w} & \text{if } \ell(r_i w) < \ell(w), \end{cases} \quad \text{for all } i \in I,\ w \in W^v,$$

$$(3')\quad X^\lambda * X^\mu = X^{\lambda+\mu} \quad \text{for all } \lambda \in Y,\ \mu \in Y.$$

In all these relations, we can see that the coefficients are in the subring $R_2 = \mathbb{Z}[(\sigma_i^{\pm 2}, \sigma_i'^{\pm 2})_{i \in I}]$ of R_1. So, if we consider ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$ the R_2-submodule with basis $(X^\lambda * T_w)_{\lambda \in Y,\, w \in W^v}$, the multiplication $*$ gives it a structure of associative unitary algebra over R_2.

6.6. ***The positive Bernstein–Lusztig–Hecke algebra.*** If we consider in ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$, the submodule with basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$, it is stable by multiplication $*$ (in (BLX+) and (BLX−) if $\lambda \in Y^+$ all the $\lambda \pm h\alpha_i^\vee$ written are also in Y^+). We denote by ${}^{\mathrm{BL}}\mathcal{H}_{R_2}^+$ this R_2-subalgebra of ${}^{\mathrm{BLX}}\mathcal{H}_{R_2}$. Actually, we can define such positive Hecke subalgebras inside all algebras in Section 6.5.

Like before, if we are given a morphism θ from R_2 to a ring R, we are able to consider, by extension of scalars, ${}^{\mathrm{BL}}\mathcal{H}_R^+ = R \otimes_{R_2} {}^{\mathrm{BL}}\mathcal{H}_{R_2}^+$. Let us consider the ring R of the Section 4 (such that $\mathbb{Z} \subset R$ and all q_i, q_i' are invertible in R); we can construct a morphism θ from R_2 to R by $\theta(\sigma_i^2) = q_i$ and $\theta(\sigma_i'^2) = q_i'$. So, we obtain an algebra ${}^{\mathrm{BL}}\mathcal{H}_R^+$ with basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$ and the same relations as in ${}^{\mathrm{I}}\mathcal{H}_R$. So:

Proposition. *Over R, the Iwahori–Hecke algebra ${}^{\mathrm{I}}\mathcal{H}_R$ and the positive Bernstein–Lusztig–Hecke algebra ${}^{\mathrm{BL}}\mathcal{H}_R^+$ are isomorphic.*

Remark. ${}^{\mathrm{BLX}}\mathcal{H}_R$ is a ring of quotients of ${}^{\mathrm{BL}}\mathcal{H}_R^+ \simeq {}^{\mathrm{I}}\mathcal{H}_R$, as we added in it inverses of the $X^\lambda = T_\lambda$ for $\lambda \in Y^{++}$. Actually, from Proposition 5.2, Corollary 5.4, and similar results, one may prove that $S = \{T_\lambda \mid \lambda \in Y^{++}\}$ satisfies the right and left Ore condition and that the map from ${}^{\mathrm{BL}}\mathcal{H}_R^+$ to the corresponding quotient ring is injective; see, e.g., [McConnell and Robson 2001, 2.1.6 and 2.1.12].

6.7. ***Structure constants.*** Using Section 6.6, the structure constants of the convolution product $*$ of ${}^{\mathrm{I}}\mathcal{H}_R$, in the basis $(X^\lambda * T_w)_{\lambda \in Y^+,\, w \in W^v}$, are Laurent polynomials in the parameters q_i, q_i', with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W. By Theorem 5.5(a), we get the same result for the structure constants in the basis

$(T_\lambda * T_w)_{\lambda\in Y^+,\, w\in W^v}$ and then still the same result for the structure constants $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ in the basis $(T_{\boldsymbol{w}})_{\boldsymbol{w}\in W^+}$ (by Section 4.5).

This last result is not as precise as the one expected in the conjecture of Section 2. But there is at least one case where we can prove it:

Remark. Suppose $\mathscr{I}$ is the hovel associated to a split Kac–Moody group G over a local field $\mathcal{K}$; see [Gaussent and Rousseau 2014, §3]. Then all parameters q_i, q_i' are equal to the cardinality q of the residue field; moreover, we know that each $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is an integer and a Laurent polynomial in q, with coefficients in $\mathbb{Z}$, depending only on $\mathbb{A}$ and W. But, as G is split, the same thing is true (without changing $\mathbb{A}$ and W) for all unramified extensions of the field $\mathcal{K}$, hence for infinitely many q. So the Laurent polynomial $a^{\boldsymbol{u}}_{\boldsymbol{w},\boldsymbol{v}}$ is an integer for infinitely many integral values of the variable q; hence, it has to be a true polynomial. This result was also obtained independently by D. Muthiah [2015], when G is untwisted affine.

7. Extended affine cases and DAHAs

In this section, we define the extended Iwahori–Hecke algebras and explore their relationship with the double affine Hecke algebras introduced by Cherednik.

7.1. ***Extended groups of automorphisms.*** We may consider a group $\widetilde{G}$ containing the group G of Section 1.4 and an extension to $\widetilde{G}$ of the action of G on $\mathscr{I}$. We assume that $\widetilde{G}$ permutes the apartments and induces isomorphisms between them, hence $\widetilde{G}$ is equal to $G.\widetilde{N}$, where $\widetilde{N}\supset N$ is the stabilizer of $\mathbb{A}$ in $\widetilde{G}$. This group $\widetilde{N}$ has almost the same properties as the group N described in Section 1.4.4. But we assume now that $\widetilde{W}=\nu(\widetilde{N})\subset \mathrm{Aut}(\mathbb{A})$ is only positive for its action on the vectorial faces; this means that the associated linear map $\vec{w}$ of any $w\in\widetilde{W}$ is in $\mathrm{Aut}^+(\mathbb{A}^v)$. We assume moreover that $\widetilde{W}$ may be written $\widetilde{W}=\widetilde{W}^v\ltimes Y$, where $\widetilde{W}^v$ fixes the origin 0 of $\mathbb{A}$ and Y is the same group of translations as for G; see Section 1.4.4. In particular, $\widetilde{W}^v$ is isomorphic to the group $\{\vec{w}\mid w\in\widetilde{W}\}$ and may be written $\widetilde{W}^v=\Omega\ltimes W^v$ (see Section 1.1); moreover $\widetilde{W}=\Omega\ltimes W$, where Ω is the stabilizer of C^v_f in $\widetilde{W}$. Finally, we assume that G contains the fixer $\mathrm{Ker}\,\nu$ of $\mathbb{A}$ in $\widetilde{G}$ so that $G\lhd\widetilde{G}$ is the subgroup of all vectorially Weyl automorphisms in $\widetilde{G}$ and $\widetilde{G}/G\simeq\Omega$.

As $\widetilde{W}$ is positive, $\widetilde{G}$ preserves the preorder $\le$ on $\mathscr{I}$. So $\widetilde{G}^+=\{g\in\widetilde{G}\mid 0\le g.0\}$ is a semigroup with $\widetilde{G}^+\cap G=G^+$. And $\widetilde{W}^+=\Omega\ltimes W^+=\widetilde{W}^v\ltimes Y^+\subset\widetilde{W}$ is also a semigroup, with $\widetilde{W}^+\cap W=W^+$.

7.2. ***Examples: Kac–Moody and loop groups.***

7.2.1. One considers a field $\mathcal{K}$, complete for a normalized, discrete valuation with a finite residue field (of cardinality q). If $\mathfrak{G}$ is an almost split Kac–Moody group scheme over $\mathcal{K}$, then the Kac–Moody group $G=\mathfrak{G}(\mathcal{K})$ acts on an affine ordered hovel $\mathscr{I}$, with the properties described in Section 1.4. See [Rousseau 2010;

Gaussent and Rousseau 2014, §3] in the split case (where all q_i, q_i' are equal to q) and [Charignon 2009; 2010; Rousseau 2012] in general.

7.2.2. Let $\mathfrak{G}_0$ be a simply connected, almost simple, split, semisimple algebraic group of rank r over $\mathcal{K}$. Its fundamental maximal torus $\mathfrak{T}_0$ is $Q_0^\vee \otimes_{\mathbb{Z}} \mathfrak{Mult}$, where $Q_0^\vee$ and $P_0^\vee$ are the coroot lattice and coweight lattice, respectively, of the root system $\Phi_0 \subset V_0^*$ with Weyl group W_0^v.

Some central extension of (a subgroup of) the loop group $\mathfrak{G}_0(\mathcal{K}[t,t^{-1}]) \rtimes \mathcal{K}^\times$ by $\mathcal{K}^\times$ (where $x \in \mathcal{K}^\times$ acts on $\mathfrak{G}_0(\mathcal{K}[t,t^{-1}])$ via $t \mapsto xt$) is $G = \mathfrak{G}(\mathcal{K})$ for the most popular example $\mathfrak{G}$ of an untwisted, affine, split, Kac–Moody group scheme over $\mathcal{K}$. Its fundamental, maximal torus $\mathfrak{T}$ is $\mathfrak{Mult} \times \mathfrak{T}_0 \times \mathfrak{Mult} = Y \otimes_{\mathbb{Z}} \mathfrak{Mult}$, with cocharacter group $Y = \mathbb{Z}\mathfrak{c} \oplus Q_0^\vee \oplus \mathbb{Z}d$, where $\mathfrak{c}$ is the canonical central element and d is the scaling element.

The set Φ of real roots is $\{\alpha_0 + n\delta \mid \alpha_0 \in \Phi_0,\ n \in \mathbb{Z}\}$ in the dual V^* of

$$V = Y \otimes_{\mathbb{Z}} \mathbb{R} = \mathbb{R}\mathfrak{c} \oplus V_0 \oplus \mathbb{R}d,$$

where $\delta(a\mathfrak{c} + v_0 + bd) = b$ and $\alpha_0(a\mathfrak{c} + v_0 + bd) = \alpha_0(v_0)$. The corresponding Weyl group W^v is actually the affine Weyl group $W_0^a = W_0^v \ltimes Q_0^\vee$ acting linearly on V; its action on the hyperplane $d + V_0$ of $V/\mathbb{R}\mathfrak{c}$ is affine: W_0^v acts linearly on V_0 and $Q_0^\vee$ acts by translations. The group G is generated by $T = \mathfrak{T}(\mathcal{K})$ and root groups $U_\alpha \simeq \mathcal{K} = \mathfrak{Add}(\mathcal{K})$ for $\alpha \in \Phi$; if $\alpha = \alpha_0 + n\delta$, then $U_\alpha = \mathfrak{U}_{\alpha_0}(t^n.\mathcal{K})$.

The fundamental apartment $\mathbb{A}$ of the associated hovel is as described in Section 1.2 with $W = W^v \ltimes Y$ containing the affine Weyl group $W^a = W^v \ltimes Q^\vee$, where $Q^\vee = \mathbb{Z}\mathfrak{c} \oplus Q_0^\vee$.

This is the situation considered in [Braverman et al. 2016]. We saw in [Gaussent and Rousseau 2014, Remark 3.4] that our group K is the same as the K of [Braverman et al. 2016]. It is clear that the Iwahori group I of [op. cit.] is included in our group K_I. But from Section 1.4.2 and [op. cit., 3.1.2], we get two Bruhat decompositions $K = \bigsqcup_{w \in W^v} K_I\,.\,w\,.\,K_I = \bigsqcup_{w \in W^v} I\,.\,w\,.\,I$. So $K_I = I$ and, in this case, our results are the same as those of [op. cit.].

7.2.3. Let us consider a central schematic quotient $\mathfrak{G}_{00}$ of $\mathfrak{G}_0$. It is determined by the cocharacter group Y_{00} of its fundamental torus $\mathfrak{T}_{00}$: $Q_0^\vee \subset Y_{00} \subset P_0^\vee$ and $\mathfrak{T}_{00} = Y_{00} \otimes_{\mathbb{Z}} \mathfrak{Mult}$. The root system $\Phi_0 \subset V_0^*$ and the Weyl group $W_0^v \subset \mathrm{GL}(V_0)$ are the same.

We get a more general untwisted, affine, split Kac–Moody scheme $\mathfrak{G}_1$ by "amalgamating" $\mathfrak{G}$ and the $\mathcal{K}$-split torus $\mathfrak{T}_1 = Y_1 \otimes_{\mathbb{Z}} \mathfrak{Mult}$ (with $Y_1 = \mathbb{Z}\mathfrak{c} \oplus Y_{00} \oplus \mathbb{Z}d$) along $\mathfrak{T}$. A little more precisely, the Kac–Moody group $G_1 = \mathfrak{G}_1(\mathcal{K})$ is a quotient of the free product of G and $T_{00} = \mathfrak{T}_{00}(\mathcal{K}) = Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}^\times$ by some relations; essentially, T_{00} normalizes T and each U_α (hence also G) and one identifies both copies of T_0;

see [Rousseau 2010, 1.8]. The new fundamental torus is $\mathfrak{T}_1$. We keep the same V, Φ, W^v, $\mathbb{A}$, and $\mathscr{I}$, but now $W_1 = W^v \ltimes Y_1 \supset W \supset W^a$.

7.2.4. We may consider a central extension by $\mathcal{K}^\times$ of (a subgroup of) the loop group $\mathfrak{G}_{00}(\mathcal{K}[t,t^{-1}]) \rtimes \mathcal{K}^\times$. We get thus an extended Kac–Moody group $\widetilde{G}_2$ (not among the Kac–Moody groups of [Tits 1987] or [Rousseau 2010]) which may also be described by amalgamation: $\widetilde{G}$ is a quotient of the free product of G and $Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}[t,t^{-1}]^*$ by relations similar to those above; in particular the conjugation by $\lambda \otimes xt^n$ sends $U_{\alpha_0+p\delta}$ to $U_{\alpha_0+(p+n\alpha(\lambda))\delta}$. The group $\widetilde{G}_2$ contains G_1 as a normal subgroup; its fundamental torus is $T_1 = Y_1 \otimes_{\mathbb{Z}} \mathcal{K}^\times$, with normalizer $\widetilde{N}_2 = N_{\widetilde{G}_2}(T_1)$ containing $Y_{00} \otimes_{\mathbb{Z}} \mathcal{K}[t,t^{-1}]^* \supset Y_{00} \otimes_{\mathbb{Z}} t^{\mathbb{Z}} =: t^{Y_{00}}$.

The group $\widetilde{G}_2$ is generated by $t^{Y_{00}}$ and G_1 (which contains $N_1 = N_2 \cap G_1 \supset t^{Q_0^\vee}$); in particular $\widetilde{G}_2/G_1 \simeq Y_{00}/Q_0^\vee$. We keep the same V and Φ, but now the corresponding vectorial Weyl group is $\widetilde{W}_2^v = N_2/T_1 = W_0^v \ltimes Y_{00}$. As in Section 1.1, we may also write $\widetilde{W}_2^v = \Omega_2 \ltimes W^v$, where Ω_2 is the stabilizer in $\widetilde{W}_2^v$ of C_f^v. It is well known that Ω_2 is a finite group isomorphic to $Y_{00}/Q_0^\vee$; it is isomorphic to its image in the permutation group of the affine Dynkin diagram of $\mathfrak{G}_{00}$ or $\mathfrak{G}_0$ (indexed by I) and acts simply transitively on the special vertices of this diagram.

It is not too difficult to extend to $\widetilde{G}_2$ the action of G_1 on the hovel $\mathscr{I}$. The group $\widetilde{N}_2$ is the stabilizer of $\mathbb{A}$; it acts through $\widetilde{W}_2 = \widetilde{W}_2^v \ltimes Y_1 \supset W \supset W^a$. We are exactly in the situation of Section 7.1 with $(\widetilde{G}_2, G_1)$.

7.2.5. We may get new couples $(\widetilde{G}_j, G_j)$ satisfying Section 7.1 for the same hovel $\mathscr{I}$:

We may enlarge $\widetilde{G}_2$ and G_1 by amalgamating them with $T_3 = Y_3 \otimes_{\mathbb{Z}} \mathcal{K}^\times$ along T_1 (or with $T_{000} = Y_{000} \otimes_{\mathbb{Z}} \mathcal{K}^\times$ along T_{00}), where $Y_{00} \subset Y_{000} \subset P_0^\vee$ and

$$Y_3 = \mathbb{Z}.\tfrac{1}{m}.\mathfrak{c} \oplus Y_{000} \oplus \mathbb{Z}d,$$

with $m \in \mathbb{Z}_{>0}$. Then $\widetilde{W}_3^v = \widetilde{W}_2^v$, $\Omega_3 = \Omega_2$, $\widetilde{W}_3 = \widetilde{W}_2^v \ltimes Y_3$, and G_3 is still a Kac–Moody group with maximal torus T_3.

We may keep G_1 (or G_3) and take a semidirect product of $\widetilde{G}_2$ (or $\widetilde{G}_3$) by a group Γ of automorphisms of the Dynkin diagram of $\mathfrak{G}_0$, stabilizing Y_{00} (or Y_{00} and Y_{000}). Then $\widetilde{W}_4^v = \Gamma \ltimes \widetilde{W}_2^v$, $\Omega_4 = \Gamma \ltimes \Omega_2$, and $\widetilde{W}_4 = \widetilde{W}_4^v \ltimes Y_2$ (or $\widetilde{W}_4 = \widetilde{W}_4^v \ltimes Y_3$).

7.2.6. We may also add a split torus as direct factor to any of the preceding groups $\widetilde{G}_i$ or G_i, enlarge $\mathscr{I}$ by a trivial euclidean factor of the same dimension as the torus and add to $\widetilde{W}^v$ and Ω, as a direct factor, any automorphism group (possibly infinite) of this torus.

7.3. ***Marked chambers.*** We come back to the general situation of Section 7.1. We want a set of "geometric objects" in $\mathscr{I}$ on which $\widetilde{G}$ acts with the Iwahori subgroup K_I as one of the isotropy groups.

7.3.1. A *marked chamber* in the hovel $\mathscr{I}$ is the equivalence class of an isomorphism $\varphi : \mathbb{A} \to A \in \mathcal{A}$ sending the fundamental chamber C_0^+ to some local chamber C_x, modulo the equivalence

$$\varphi_1 \simeq \varphi_2 \iff \exists S \in C_0^+ \text{ such that } \varphi_1|_S = \varphi_2|_S.$$

It is simply written $\varphi : C_0^+ \to C_x$; this does not depend on A.

The group $\widetilde{G}$ permutes the marked chambers; for $g \in \widetilde{G}$ and φ as above, $g.\varphi = \varphi$ if and only if g fixes (pointwise) C_x. In particular, the isotropy group in $\widetilde{G}$ of $\widetilde{C}_0^+ = \mathrm{Id} : C_0^+ \to C_0^+ \subset \mathbb{A} \subset \mathscr{I}$ is $K_I \subset G$.

A local chamber of type 0, $C_x \in \mathscr{C}_0^+$ determines a unique marked chamber $\widetilde{C}_x^0 : C_0^+ \to C_x$ (called *normalized*) which is the restriction of some $\varphi \in \mathrm{Isom}_{\mathbb{R}}^W(\mathbb{A}, A)$; see Section 1.11. These normalized marked chambers are permuted transitively by G.

7.3.2. A marked chamber is said *of type* 0 if it is in the orbit under $\widetilde{G}$ of any of those $\widetilde{C}_x^0$. So the set $\widetilde{\mathscr{C}}_0^+$ of marked chambers of type 0 is $\widetilde{G}/K_I$.

By hypothesis $\widetilde{G}$ may be written $G.\widetilde{\Omega}$, where $\widetilde{\Omega} = \nu^{-1}(\Omega) \subset \widetilde{N}$ stabilizes C_0^+ (considered as in $\mathscr{I}$) and induces Ω on it. So $\widetilde{\mathscr{C}}_0^+ = \{\widetilde{C}_x = \widetilde{C}_x^0 \circ \omega^{-1} \mid C_x \in \mathscr{C}_0^+,\ \omega \in \Omega\}$.

7.4. $\widetilde{W}$-*distance.*

7.4.1. Let $\widetilde{C}_x : C_0^+ \to C_x$, $\widetilde{C}_y : C_0^+ \to C_y$ be in $\widetilde{\mathscr{C}}_0^+$ with $x \le y$. There is an apartment A containing C_x and C_y so $\widetilde{C}_x, \widetilde{C}_y$ may be extended to $\varphi, \psi \in \mathrm{Isom}(\mathbb{A}, A)$. We "identify" $(\mathbb{A}, C_0^+)$ with (A, C_x) via φ. Then $\varphi^{-1}(y) \ge 0$ and, as $\widetilde{C}_x, \widetilde{C}_y$ are in a same orbit of $\widetilde{G}$, there is $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$ such that $\psi = \varphi \circ \widetilde{\boldsymbol{w}}$. This $\widetilde{\boldsymbol{w}}$ does not depend on the choice of A by Proposition 1.10(c).

We define the $\widetilde{W}$-*distance* between the marked chambers $\widetilde{C}_x$ and $\widetilde{C}_y$ as this unique element: $d^W(\widetilde{C}_x, \widetilde{C}_y) = \widetilde{\boldsymbol{w}} \in \widetilde{W}^+$. So we get a $\widetilde{G}$-invariant map

$$d^W : \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+ = \{(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times \widetilde{\mathscr{C}}_0^+ \mid x \le y\} \to \widetilde{W}^+.$$

7.4.2. For $(C_x, C_y) \in \mathscr{C}_0^+ \times_{\le} \mathscr{C}_0^+$, we have $d^W(\widetilde{C}_x^0, \widetilde{C}_y^0) = d^W(C_x, C_y)$ and, more generally, for $\omega_x, \omega_y \in \Omega$, we have $(\widetilde{C}_x^0 \circ \omega_x^{-1}, \widetilde{C}_y^0 \circ \omega_y^{-1}) \in \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+$ and

$$d^W(\widetilde{C}_x^0 \circ \omega_x^{-1}, \widetilde{C}_y^0 \circ \omega_y^{-1}) = \omega_x . d^W(C_x, C_y) . \omega_y^{-1} \in \widetilde{W}^+.$$

For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_{\le} \widetilde{\mathscr{C}}_0^+$ and $\omega_x, \omega_y \in \Omega$, we have also

$$d^W(\widetilde{C}_x \circ \omega_x^{-1}, \widetilde{C}_y \circ \omega_y^{-1}) = \omega_x . d^W(\widetilde{C}_x, \widetilde{C}_y) . \omega_y^{-1} \in \widetilde{W}^+.$$

We deduce from this some interesting consequences:

7.4.3. If $\widetilde{C}_x, \widetilde{C}_y, \widetilde{C}_z$, with $x \le y \le z$, are in the same apartment, we have a Chasles relation:

$$d^W(\widetilde{C}_x, \widetilde{C}_z) = d^W(\widetilde{C}_x, \widetilde{C}_y) . d^W(\widetilde{C}_y, \widetilde{C}_z).$$

7.4.4. For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_{\leq} \widetilde{\mathscr{C}}_0^+$, if $\widetilde{C}_x$ is normalized then $d^W(\widetilde{C}_x, \widetilde{C}_y) \in W^+$ if and only if $\widetilde{C}_y$ is normalized. The same is true with the roles of $\widetilde{C}_x$ and $\widetilde{C}_y$ reversed.

7.4.5. For $(\widetilde{C}_x, \widetilde{C}_y) \in \widetilde{\mathscr{C}}_0^+ \times_{\leq} \widetilde{\mathscr{C}}_0^+$,

$$d^W(\widetilde{C}_x, \widetilde{C}_y) = \omega \in \Omega \iff \widetilde{C}_y = \widetilde{C}_x \circ \omega.$$

In particular, $\widetilde{C}_y$ is uniquely determined by $\widetilde{C}_x$ and ω; moreover, $C_y = C_x$.

7.4.6. If $(C_x, C_y) \in \mathscr{C}_0^+ \times_{\leq} \mathscr{C}_0^+$ and $d^W(C_x, C_y) = r_i \in W^v$ (resp., $\lambda \in Y^+$) and $\omega \in \Omega$, then $d^W(\widetilde{C}_x^0 \circ \omega^{-1}, \widetilde{C}_y^0 \circ \omega^{-1}) = \omega . r_i . \omega^{-1} = r_{\omega(i)}$ (resp., $\omega(\lambda) \in Y^+$), where we consider the action of Ω on I (resp., Y).

7.4.7. When $\widetilde{C}_x = \widetilde{C}_0^+$ and $\widetilde{C}_y = g . \widetilde{C}_0^+$ (with $g \in \widetilde{G}^+$), then $d^W(\widetilde{C}_x, \widetilde{C}_y)$ is the only $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$ such that $g \in K_I . \widetilde{\boldsymbol{w}} . K_I$. There is a Bruhat decomposition

$$\widetilde{G}^+ = \bigsqcup_{\widetilde{\boldsymbol{w}} \in \widetilde{W}^+} K_I . \widetilde{\boldsymbol{w}} . K_I .$$

The $\widetilde{W}$-distance classifies the orbits of K_I on $\{\widetilde{C}_y \in \widetilde{\mathscr{C}}_0^+ \mid y \geq 0\}$, hence also the orbits of $\widetilde{G}$ on $\widetilde{\mathscr{C}}_0^+ \times_{\leq} \widetilde{\mathscr{C}}_0^+$.

7.5. *The extended Iwahori–Hecke algebra.*

7.5.1. We define this extended algebra for $\widetilde{G}$ as we did in Section 2 for G:

To each $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$, we associate a function $T_{\widetilde{\boldsymbol{w}}} : \widetilde{\mathscr{C}}_0^+ \times_{\leq} \widetilde{\mathscr{C}}_0^+ \to R$, defined by

$$T_{\widetilde{\boldsymbol{w}}}(\widetilde{C}, \widetilde{C}') = \begin{cases} 1 & \text{if } d^W(\widetilde{C}, \widetilde{C}') = \widetilde{\boldsymbol{w}}, \\ 0 & \text{otherwise.} \end{cases}$$

And we consider the following free R-module of functions $\widetilde{\mathscr{C}}_0^+ \times_{\leq} \widetilde{\mathscr{C}}_0^+ \to R$:

$$^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = \left\{ \varphi = \sum_{\widetilde{\boldsymbol{w}} \in \widetilde{W}^+} a_{\widetilde{\boldsymbol{w}}} T_{\widetilde{\boldsymbol{w}}} \;\middle|\; a_{\widetilde{\boldsymbol{w}}} \in R,\ a_{\widetilde{\boldsymbol{w}}} = 0 \text{ except for a finite number} \right\},$$

We endow this R-module with the convolution product given by

$$(\varphi * \psi)(\widetilde{C}_x, \widetilde{C}_y) = \sum_{\widetilde{C}_z} \varphi(\widetilde{C}_x, \widetilde{C}_z) \psi(\widetilde{C}_z, \widetilde{C}_y).$$

where $\widetilde{C}_z \in \widetilde{\mathscr{C}}_0^+$ is such that $x \leq z \leq y$. This product is associative and R-bilinear. We prove below that it is well defined.

As in Section 2, we see easily that $^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is the natural convolution algebra of the functions $\widetilde{G}^+ \to R$, bi-invariant under K_I and with finite support.

7.5.2. For $\omega \in \Omega$, $\widetilde{\boldsymbol{w}} \in \widetilde{W}^+$, the products $T_\omega * T_{\widetilde{\boldsymbol{w}}}$ and $T_{\widetilde{\boldsymbol{w}}} * T_\omega$ are well defined: actually $T_\omega * T_{\widetilde{\boldsymbol{w}}} = T_{\omega . \widetilde{\boldsymbol{w}}}$ and $T_{\widetilde{\boldsymbol{w}}} * T_\omega = T_{\widetilde{\boldsymbol{w}} . \omega}$; see Sections 7.4.3 and 7.4.5.

7.5.3. As the formula for $\varphi * \psi$ is clearly $\widetilde{G}$-invariant, we may fix $\widetilde{C}_x$ normalized to calculate $\varphi * \psi$. From Section 7.4.4, we deduce that, when $\boldsymbol{w}, \boldsymbol{v} \in W^+$, the product $T_{\boldsymbol{w}} * T_{\boldsymbol{v}}$ may be computed using only normalized marked chambers. So it is well defined and the same as in ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$.

From Section 7.5.2 we deduce now that the convolution product is well defined in ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$:

Proposition. *For any ring R, ${}^{I}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is an algebra; it contains ${}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$ as a subalgebra.*

Definition. The algebra ${}^{I}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ is the *extended Iwahori–Hecke algebra* associated to $\mathscr{I}$ and $\widetilde{G}$ with coefficients in R.

7.6. *Relations.*

7.6.1. From Section 7.5 we see that ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$ contains the algebra

$$R[\Omega] = \bigoplus_{\omega \in \Omega} R\,.\,T_\omega$$

of the group Ω. Moreover, as an R-module, ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = R[\Omega] \otimes_R {}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$: we identify $T_{\omega.\boldsymbol{w}} = T_\omega * T_{\boldsymbol{w}}$ and $T_\omega \otimes T_{\boldsymbol{w}}$ for $\omega \in \Omega$ and $\boldsymbol{w} \in W^+$.

The multiplication in this tensor product is semidirect:

$$\begin{aligned}(T_\omega \otimes T_{\boldsymbol{w}})\,.\,(T_{\omega'} \otimes T_{\boldsymbol{v}}) &= T_\omega * T_{\boldsymbol{w}} * T_{\omega'} * T_{\boldsymbol{v}} = T_{\omega.\boldsymbol{w}.\omega'} * T_{\boldsymbol{v}} \\ &= T_{\omega.\omega'.\boldsymbol{w}'} * T_{\boldsymbol{v}} = T_{\omega.\omega'} * T_{\boldsymbol{w}'} * T_{\boldsymbol{v}} = T_{\omega.\omega'} \otimes (T_{\boldsymbol{w}'} * T_{\boldsymbol{v}}),\end{aligned}$$

where $\boldsymbol{w}' = \omega'^{-1}\,.\,\boldsymbol{w}\,.\,\omega' =: \omega'^{-1}(\boldsymbol{w}) \in W^+$.

In particular, we get the following relations among some elements:

7.6.2. For $\omega \in \Omega$ and $\boldsymbol{w} \in W^+$,

$$T_\omega * T_{\boldsymbol{w}} * T_\omega^{-1} = T_{\omega(\boldsymbol{w})}.$$

If, moreover, $\boldsymbol{w} = r_i \in W^v$, then $\omega(r_i) = r_{\omega(i)}$ and

$$T_\omega * T_i * T_\omega^{-1} = T_{\omega(i)}.$$

If now $\boldsymbol{w} = \lambda \in Y^+$, then

$$T_\omega * T_\lambda * T_\omega^{-1} = T_{\omega(\lambda)},$$

with $\omega(\lambda) \in Y^+$.

7.6.3. From Theorem 5.5(1) and (2) above, it is clear that $T_\omega * X^\lambda * T_\omega^{-1} = X^{\omega(\lambda)}$ if $\omega \in \Omega$ and $\lambda \in Y^+$ (as Ω stabilizes $Y^{++} = Y \cap C_f^v$).

7.6.4. As the action of Ω on $\mathbb{A}$ is induced by automorphisms of $\mathscr{I}$, we have $q_i = q_{\omega(i)}$ and $q_i' = q_{\omega(i)}'$ for $\omega \in \Omega$ and $i \in I$. We may also choose the homomorphism $\delta^{1/2} : Y \to R^*$ of Section 5.7 invariant by Ω (for R great enough). So, for $\omega \in \Omega$, $w, r_i \in W^v$, and $\lambda \in Y$,

$$T_\omega * H_w * T_\omega^{-1} = H_{\omega(w)}, \quad T_\omega * H_i * T_\omega^{-1} = H_{\omega(i)}, \quad T_\omega * Z^\lambda * T_\omega^{-1} = Z^{\omega(\lambda)}.$$

7.7. *The extended Bernstein–Lusztig–Hecke algebra.* Notation from Section 7.1 is still in use. But we no longer assume the existence of a group $\widetilde{G}$ or G. The group $W = W^v \ltimes Y \lhd \widetilde{W}$ satisfies $\widetilde{W} = \Omega \ltimes W$ and the conditions of Section 6.

We consider the ring $\tilde{R} = \mathbb{Z}[(\tilde{\sigma}_i^{\pm 1}, (\tilde{\sigma}_i')^{\pm 1})_{i \in I}]$, where the indeterminates $\tilde{\sigma}_i, \tilde{\sigma}_i'$ satisfy the same relations as σ_i, σ_i' in Section 6.1 and the additional relation (see Section 7.6.4 above)

$$\tilde{\sigma}_i = \tilde{\sigma}_j \text{ and } \tilde{\sigma}_i' = \tilde{\sigma}_j' \quad \text{if } \omega(i) = j \text{ for some } \omega \in \Omega.$$

We denote by ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ the free $\tilde{R}$-module with basis $(T_\omega Z^\lambda H_w)_{\omega \in \Omega,\, \lambda \in Y,\, w \in W^v}$ and write

$$H_w = T_1 Z^0 H_w, \quad H_i = T_1 Z^0 H_i, \quad Z^\lambda = T_1 Z^\lambda H_e, \text{ and } T_\omega = T_\omega Z^0 H_e.$$

Proposition. *There exists a unique multiplication $*$ on ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ which makes it an associative, unitary $\tilde{R}$-algebra with unity $H_e = T_1 = Z^0$ and satisfies the conditions* (1)–(4) *of Theorem 6.2 plus*

(5) *$T_\omega * T_{\omega'} = T_{\omega.\omega'}$, $T_\omega * T_i * T_\omega^{-1} = T_{\omega(i)}$, and $T_\omega * T_\lambda * T_\omega^{-1} = T_{\omega(\lambda)}$ for $\omega, \omega' \in \Omega$, $i \in I$, and $\lambda \in Y$.*

Proof. As $\tilde{R}$-modules, ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}} = \tilde{R}[\Omega] \otimes {}^{\mathrm{BL}}\mathcal{H}_{\tilde{R}}$, where the homomorphism $R_1 \to \tilde{R}$ is given by $\sigma_i \mapsto \tilde{\sigma}_i$, $\sigma_i' \mapsto \tilde{\sigma}_i'$. Now the multiplication is classical on $\tilde{R}[\Omega]$, given by Theorem 6.2 on ${}^{\mathrm{BL}}\mathcal{H}_{\tilde{R}}$, and semidirect for general elements. □

Definition. This $\tilde{R}$-algebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ depends only on $\mathbb{A}$, Y and Ω (i.e., on $\mathbb{A}$ and $\widetilde{W}$). We call it the *extended Bernstein–Lusztig–Hecke algebra* associated to $\mathbb{A}$ and $\widetilde{W}$ with coefficients in $\tilde{R}$.

As in Section 6.6, we may identify, up to an extension of scalars, a subalgebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^+$ of ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$ with the extended Iwahori–Hecke algebra ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_{R}^{\mathscr{I}}$.

7.8. *The affine case.*

7.8.1. We suppose now that $(\mathbb{A}^v, W^v)$ is affine. So there is a smallest positive imaginary root $\delta = \sum a_i \alpha_i \in \Delta_{\mathrm{im}}^+ \subset Q^+$ satisfying $\delta(\alpha_i^\vee) = 0$ for all $i \in I$, and a canonical central element $\mathfrak{c} = \sum a_i^\vee \alpha_i^\vee \in Q_+^\vee$ satisfying $\alpha_i(\mathfrak{c}) = 0$ for all $i \in I$. In particular, δ and $\mathfrak{c}$ are fixed by W^v and $\widetilde{W}^v$.

As $\delta \in Q^+$, it takes integral values on Y. For $n \in \mathbb{Z}$, we define

$$Y^n = \{\lambda \in Y \mid \delta(\lambda) = n\},$$

which is stable under W^v and $\widetilde{W}^v$. We have $Y = \bigsqcup_{n\in\mathbb{Z}} Y^n$ and $Y^+ = \big(\bigsqcup_{n>0} Y^n\big) \sqcup Y_c^0$, with $Y_c^0 = Y^0 \cap Y^+ = Y \cap \mathbb{Q}\mathfrak{c}$. We write $\lambda_c = \frac{1}{m}\mathfrak{c}$ a generator of Y_c^0 (with $m \in \mathbb{Z}_{>0}$). As $\delta(Q^\vee) = 0$, we have $\delta(\lambda) = \delta(\mu)$ whenever $\mu \leq_{Q^\vee} \lambda$ or $\mu \leq_{Q^\vee_{\mathbb{R}}} \lambda$ in Y.

7.8.2. Considering Proposition 2.2 and Theorem 5.5(2), the algebra is graded (for a suitable R) by

$$^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}} = \bigoplus_{n\geq 0} {}^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}n},$$

where $^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}n}$ has for R-basis $\{T_\lambda * T_w \mid \lambda \in Y^n,\ w \in W^v\}$ if $n > 0$ and $\{T_\lambda * T_w \mid \lambda \in Y_c^0,\ w \in W^v\}$ if $n = 0$. For some rings R, we may replace each $T_\lambda * T_w$ by $X^\lambda * T_w$ or by $Z^\lambda * H_w$ to find new bases. Also,

$$^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}} = \bigoplus_{n\geq 0} {}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}n},$$

where $^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}n}$ has for R-basis $\{T_\omega * T_\lambda * T_w \mid \omega \in \Omega,\ \lambda \in Y^n,\ w \in W^v\}$ if $n > 0$ and $\{T_\omega * X^\lambda * T_w \mid \omega \in \Omega,\ \lambda \in Y_c^0,\ w \in W^v\}$ if $n = 0$. For some rings R, we may replace each $T_\omega * T_\lambda * T_w$ by $T_\omega * X^\lambda * T_w$ or by $T_\omega * Z^\lambda * H_w$ to find new bases. Furthermore,

$$^{\mathrm{BL}}\mathcal{H}_{R_1} = \bigoplus_{n\in\mathbb{Z}} {}^{\mathrm{BL}}\mathcal{H}_{R_1}^n,$$

where $^{\mathrm{BL}}\mathcal{H}_{R_1}^n$ has for R_1-basis the $Z^\lambda H_w$ for $\lambda \in Y^n$ and $w \in W^v$, and

$$^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}} = \bigoplus_{n\in\mathbb{Z}} {}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^n,$$

where $^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^n$ has for $\tilde{R}$-basis the $T_\omega Z^\lambda H_w$ for $\omega \in \Omega,\ \lambda \in Y^n$, and $w \in W^v$.

These gradations are compatible with the identifications explained in Section 6.6 or Section 7.7.

7.8.3. For any $\widetilde{C}_x \in \widetilde{\mathscr{C}}_0^+$ and any $\lambda \in Y_c^0 = \mathbb{Z}\lambda_c$, there is a unique $\widetilde{C}_y \in \widetilde{\mathscr{C}}_0^+$ with $d^W(\widetilde{C}_x, \widetilde{C}_y) = \lambda$: the translation by λ in $\mathbb{A}$ stabilizes all enclosed sets and extends to $\mathscr{I}$ as a translation in any apartment. From this we see that

$$\begin{cases} T_\lambda * T_\mu = T_{\lambda+\mu} = T_\mu * T_\lambda & \text{for } \mu \in Y^+, \\ T_\lambda * X^\mu = X^{\lambda+\mu} = X^\mu * T_\lambda & \text{for } \mu \in Y, \\ T_\lambda * T_w = T_{\lambda.w} = T_{w.\lambda} = T_w * T_\lambda & \text{for } w \in W^v. \end{cases}$$

Such a T_λ is central and invertible in $^{\mathrm{I}}\mathcal{H}_R^{\mathscr{I}}$, $^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathscr{I}}$, $^{\mathrm{BL}}\mathcal{H}_{R_1}$, or $^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}$.

Actually ${}^{\mathrm{I}}\mathcal{H}_R^{\mathcal{I}0}$ is the tensor product $R[Y_c^0]\otimes_R\mathcal{H}_R(W^v)$ with a direct multiplication (factor by factor) and ${}^{\mathrm{I}}\widetilde{\mathcal{H}}_R^{\mathcal{I}0} = R[Y_c^0]\otimes_R (R[\Omega]\otimes_R \mathcal{H}_R(W^v))$ with a semidirect multiplication.

7.9. ***The double affine Hecke algebra.*** The subalgebra ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^0$ is well known as Cherednik's double affine Hecke algebra (DAHA). More precisely, Cherednik [1992; 1995] considers an untwisted affine root system, as in [Kac 1990, Chapter 7]; but, as he works with roots instead of coroots, we write $\Phi^\vee$ for this system. He considers the case where $\widetilde{W}^v$ is the full extended Weyl group ($\widetilde{W}^v = W_0^v \ltimes P_0^\vee$ with the notation of Section 7.2), i.e., $\Omega \simeq P_0^\vee/Q_0^\vee$ acts on the extended Dynkin diagram, simply transitively on its "special" vertices. His choice for Y^0 is $Y^0 = \mathbb{Z}\cdot\frac{1}{m}.\mathfrak{c}\oplus P_0^\vee \subset P^\vee$ (and, e.g., $Y = Y^0\oplus\mathbb{Z}d$), where $m\in\mathbb{Z}_{\geq 1}$ is suitably chosen. He then defines the DAHA as an algebra over a field of rational functions $\mathbb{C}(\underline{\delta}, (q_\nu)_{\nu\in\nu_R})$ with generators $(T_i)_{i\in I}$, $(X_\beta)_{\beta\in P_0^\vee}$ and some relations. It is easy to see that this DAHA is, up to scalar changes, a ring of quotients of our ${}^{\mathrm{BL}}\widetilde{\mathcal{H}}_{\tilde{R}}^0$ (for $\mathbb{A}$, $\widetilde{W}$ as described above): actually $\underline{\delta}$ stands for our Z^{λ_c}. Here is a partial dictionary to translate from [Cherednik 1992; Cherednik 1995] to our article: roots $\leftrightarrow$ coroots, $X_\beta\mapsto Z^\beta$, $T_i\mapsto H_i$, $q_i\mapsto\sigma_i$, $\Pi\mapsto\Omega$, $\pi_r\mapsto T_\omega$, $\underline{\delta}\mapsto T_{\lambda_c}$ and $\underline{\Delta}=\underline{\delta}^m\mapsto T_{\mathfrak{c}}$.

In [Cherednik 1992] there is another presentation of the same DAHA using the Bernstein presentation of $\mathcal{H}_R(W^v)$. This is also the point of view of [Macdonald 2003], where the framework is more general.

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A CLASSIFICATION OF SPHERICAL CONJUGACY CLASSES

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Let G be a simple algebraic group over an algebraically closed field k. We complete the classification of the spherical conjugacy classes of G begun by Carnovale (*Pacific J. Math.* 245 (2010), 25–45) and the author (*Trans. Amer. Math. Soc.* 364 (2012), 1997–2019).

1. Introduction

Let G be a simple algebraic group over an algebraically closed field k. In this paper we complete the classification of the spherical conjugacy classes of G (recalling that a conjugacy class $\mathcal{O}$ in G is called *spherical* if a Borel subgroup of G has a dense orbit on $\mathcal{O}$). There has been a lot of work related to this field, beginning with the work of D. Panyushev [1994; 1999], who classified spherical nilpotent orbits in the Lie algebra of G, when the base field is $\mathbb{C}$. R. Fowler and G. Röhrle [2008] classified spherical nilpotent orbits over an algebraically closed field of good characteristic. Then G. Carnovale [2010], exploiting the characterizations of spherical conjugacy classes in terms of the Weyl group given in [Cantarini et al. 2005; Carnovale 2008; 2009], classified the spherical conjugacy classes of G in zero or good, odd characteristic. In [Costantini 2012], we obtained the classification of spherical unipotent conjugacy classes when the characteristic of k is bad, and for characteristic 2 in case A_n. In the present paper we complete the classification, dealing with nonunipotent conjugacy classes when the characteristic of k is bad, and when G is of type A_n and the characteristic is 2.

The second goal of this paper is the characterization of spherical conjugacy classes in terms of the *dimension formula*: we prove in Theorem 4.1 that a conjugacy class $\mathcal{O}$ of G is spherical if and only if $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$, where $w_{\mathcal{O}}$ is a certain element of the Weyl group attached to $\mathcal{O}$, as defined in the next section. This characterization was obtained over $\mathbb{C}$ in [Cantarini et al. 2005] and in good, odd characteristic in [Carnovale 2008]. An elegant proof was obtained in [Lu 2011] in zero characteristic.

We finally deduce further consequences of the classification.

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2. Preliminaries

We denote by $\mathbb{C}$ the complex numbers, by $\mathbb{R}$ the reals, and by $\mathbb{Z}$ the integers.

Let G be a simple algebraic group of rank n over k, where k is an algebraically closed field. We fix a maximal torus T of G, a Borel subgroup B containing T, the unipotent radical U of B and the Borel subgroup B^- opposite to B with unipotent radical U^-. Then Φ is the set of roots relative to T, and B determines the set of positive roots Φ^+ and the simple roots $\Delta = \{\alpha_1, \dots, \alpha_n\}$. We write s_α for the simple reflection associated to $\alpha \in \Phi$. We use the numbering and the description of the simple roots in terms of the canonical basis $(e_1, \dots, e_k)$ of an appropriate $\mathbb{R}^k$ as in [Bourbaki 1981, Planches I–IX]. For the exceptional groups, we write $\beta = (m_1, \dots, m_n)$ for $\beta = m_1\alpha_1 + \dots + m_n\alpha_n$. We identify the Weyl group W with N/T, where N is the normalizer of T. We denote by w_0 the longest element of W. The real space $E = \mathbb{R}\Phi$ is a Euclidean space, endowed with the W-invariant scalar product $(\alpha_i, \alpha_j) = d_i a_{ij}$. Here $\{d_1, \dots, d_n\}$ are relatively prime positive integers such that if D is the diagonal matrix with entries $d_1, \dots, d_n$, then DA is symmetric for $A = (a_{ij})$ the Cartan matrix.

We put $\Pi = \{1, \dots, n\}$, and let ϑ be the symmetry of Π induced by $-w_0$. We denote by ℓ the usual length function on W, and by $\mathrm{rk}(1 - w)$ the rank of $1 - w$ in the geometric representation of W.

We use the notation $x_\alpha(\xi)$ and $h_\alpha(z)$ as in [Steinberg 1968; Carter 1989], for $\alpha \in \Phi$, $\xi \in k$, and $z \in k^*$. For $\alpha \in \Phi$ we put $X_\alpha = \{x_\alpha(\xi) \mid \xi \in k\}$, the root-subgroup corresponding to α, and $H_\alpha = \{h_\alpha(z) \mid z \in k^*\}$. Given an element $w \in W$ we denote a representative of w in N by $\dot{w}$. We choose the x_α so that, for all $\alpha \in \Phi$, $n_\alpha = x_\alpha(1)x_{-\alpha}(-1)x_\alpha(1)$ lies in N and has image the reflection s_α in W. Then

$$\text{(2-1)}\quad \begin{aligned} x_\alpha(\xi)x_{-\alpha}(-\xi^{-1})x_\alpha(\xi) &= h_\alpha(\xi)n_\alpha, & n_\alpha^2 &= h_\alpha(-1) \\ n_\alpha x_\alpha(x) n_\alpha^{-1} &= x_{-\alpha}(-x), & h_\alpha(\xi)x_\beta(x)h_\alpha(\xi)^{-1} &= x_\beta(\xi^{\langle\beta,\alpha\rangle}x) \end{aligned}$$

for every $\xi \in k^*$, $x \in k$ and $\alpha, \beta \in \Phi$, where $\langle\beta, \alpha\rangle = 2(\beta, \alpha)/(\alpha, \alpha)$ [Springer 1998a, Proposition 11.2.1]. The family $(x_\alpha)_{\alpha\in\Phi}$ is called a *realization* of Φ in G.

We set $T^w = \{t \in T \mid wtw^{-1} = t\}$ and $T_2 = \{t \in T \mid t^2 = 1\}$. In particular $T^w = T_2$ if $w = w_0 = -1$. We also put $S^w = \{t \in T \mid wtw^{-1} = t^{-1}\}$.

For algebraic groups we use the notation in [Humphreys 1975; Carter 1985]. In particular, for $J \subseteq \Pi$, we have $\Delta_J = \{\alpha_j \mid j \in J\}$, Φ_J is the corresponding root system, W_J the Weyl group, P_J the standard parabolic subgroup of G, and $L_J = T\langle X_\alpha \mid \alpha \in \Phi_J\rangle$ the standard Levi subgroup of P_J. For $z \in W$ we put $U_z = U \cap z^{-1}U^-z$. Then the unipotent radical R_uP_J of P_J is $U_{w_0w_J}$, where w_J is the longest element of W_J. Moreover $U \cap L_J = U_{w_J}$ is a maximal unipotent subgroup of L_J (of dimension $\ell(w_J)$), and $T_J = T \cap L'_J$ is a maximal torus of L'_J. For unipotent classes in exceptional groups we use the notation in [Carter 1985;

Spaltenstein 1982]. We use the description of centralizers of involutions as in [Iwahori 1970].

If X is a G-variety and $x \in X$, we denote by $G.x$ the G-orbit of x and by G_x the isotropy subgroup of x in G. We say that X is *spherical* if a Borel subgroup of G has a dense orbit on X. It is well known (see [Brion 1986; Vinberg 1986] in characteristic 0, [Grosshans 1992; Knop 1995] in positive characteristic) that X is spherical if and only if the set $\mathcal{V}$ of B-orbits in X is finite. If H is a closed subgroup of G and the homogeneous space G/H is spherical, we say that H is a spherical subgroup of G.

Let g be an element of G with Jordan decomposition $g = su$, for s semisimple and u unipotent. Using a terminology slightly different from the usual, we say that g is *mixed* if $s \notin Z(G)$ and $u \neq 1$. For each conjugacy class $\mathcal{O}$ in G, $w = w_{\mathcal{O}}$ is the unique element of W such that $BwB \cap \mathcal{O}$ is open dense in $\mathcal{O}$.

If x is an element of a group K and $H \leq K$, we denote by $C(x)$ the centralizer of x in K, and by $C_H(x)$ the centralizer of x in H. If $x, y \in K$, then $x \sim y$ means that x and y are conjugate in K.

If H is an algebraic group, we denote by $B(H)$ a Borel subgroup of H. We denote the identity matrix of order r by I_r. Finally, in the remainder of the paper we denote by p the characteristic of k (hence p may be 0).

3. The classification

We recall that the bad primes for the individual types of simple groups are as follows:

- none when G has type A_n;
- $p = 2$ when G has type B_n, C_n or D_n;
- $p = 2$ or 3 when G has type G_2, F_4, E_6 or E_7;
- $p = 2$, 3 or 5 when G has type E_8.

For convenience we assume G simply connected, so that centralizers of semisimple elements are connected [Carter 1985, Theorem 3.5.6]. However the classification of spherical conjugacy classes in G is independent of the isogeny class. More precisely, let $D \leq Z(G)$ and $\overline{G} = G/D$. For the canonical projection $\pi : G \to \overline{G}$ and $g \in G$, put $\bar{g} = \pi(g)$. Then it is clear that the conjugacy class of $\bar{g}$ in $\overline{G}$ is spherical if and only if the conjugacy class of g in G is spherical; see also the discussion at the beginning of [Costantini 2010, §6].

We put $\widetilde{\Pi} = \Pi \cup \{0\}$ and $\widetilde{\Delta} = \Delta \cup \{\alpha_0\}$, where $\alpha_0 = -\beta$ for the highest root β of Φ^+. Thus $\widetilde{\Pi}$ labels the vertices of the extended Dynkin diagram of the root system Φ. For $J \subset \widetilde{\Pi}$, let $\Phi_J = \mathbb{Z}\{\alpha_i \mid i \in J\} \cap \Phi$ and

$$L_J = \langle T, X_\alpha \mid \alpha \in \Phi_J \rangle.$$

This is called a *pseudo-Levi subgroup* of G (in the sense of [Sommers 1998]). Then the following holds:

Proposition 3.1 [McNinch and Sommers 2003, Propositions 30 and 32]. *Let t in G be semisimple. Then $C(t)$ is conjugate to a subgroup L_J for some $J \subset \widetilde{\Pi}$.*

Suppose that the characteristic of k is good for G. Let $J \subset \widetilde{\Pi}$. Then there is $t \in G$ such that $L_J = C(t)$. □

We recall some basic facts which have been proved for zero or good, odd characteristic.

Theorem 3.2. *Let $p \neq 2$, and let $\mathcal{O}$ be a spherical conjugacy class of a connected reductive algebraic group. If $\mathcal{O} \cap BwB$ is nonempty, then $w^2 = 1$.*

Proof. If p is zero or good and odd then this is [Carnovale 2008, Theorem 2.7]. The same proof holds as long as $p \neq 2$; see also [Carnovale and Costantini 2013, Theorem 2.1]. □

Remark 3.3. Let $M(W)$ denote the Richardson–Springer monoid, i.e., the monoid generated by the symbols r_α for $\alpha \in \Delta$, subject to the braid relations and the relation $r_\alpha^2 = r_\alpha$ for $\alpha \in \Delta$. Given a spherical G-variety, there is an $M(W)$-action on the set $\mathcal{V}$ of its B-orbits. Under additional conditions, one can also define an action of W on $\mathcal{V}$. These actions have been introduced in [Richardson and Springer 1990] and [Knop 1995], respectively, and they have been further analyzed and applied in [Brion 2001; Mars and Springer 1998, §4.1; Springer 1998b]. The actions of $M(W)$ and W have been used to prove [Carnovale 2008, Theorem 2.7]. By [Knop 1995, Theorem 4.2(b)], a case in which the action of W is defined is when $p \neq 2$. This allows one to extend the proof of [Carnovale 2008, Theorem 2.7] to the case $p \neq 2$, as done in [Carnovale and Costantini 2013, Theorem 2.1]. We shall come back to this point after the achievement of the classification of spherical conjugacy classes in characteristic 2.

Let $\mathcal{O}$ be a conjugacy class of G and let $\mathcal{V}$ be the set of B-orbits in $\mathcal{O}$. There is a natural map $\phi : \mathcal{V} \to W$ associating to $v \in \mathcal{V}$ the element w in the Weyl group of G for which $v \subseteq BwB$ (equivalently, for which $v \cap BwB \neq \varnothing$).

Theorem 3.4. *Let $p \neq 2$, and let $\mathcal{O}$ be a conjugacy class in a connected reductive algebraic group. If $\mathrm{Im}(\phi)$ contains only involutions in W, then $\mathcal{O}$ is spherical.*

Proof. If p is zero, or good and odd this is [Carnovale 2009, Theorem 5.7]. The same proof holds as long as $p \neq 2$, once it is noticed again that the action of W on $\mathcal{V}$ is defined. □

Theorem 3.5 [Cantarini et al. 2005, Theorem 25; Carnovale 2008, Theorem 4.4]. *A class $\mathcal{O}$ in a connected reductive algebraic group G over an algebraically closed field of zero or good odd characteristic is spherical if and only if there exists v in $\mathcal{V}$*

such that $\ell(\phi(v)) + \mathrm{rk}(1-\phi(v)) = \dim \mathcal{O}$. *If this is the case*, v *is the dense* B*-orbit in* $\mathcal{O}$ *and* $\phi(v) = w_{\mathcal{O}}$ (*and* $v = \mathcal{O} \cap B w_{\mathcal{O}} B$). □

For any conjugacy class $\mathcal{O}$, the element $w_{\mathcal{O}}$ of the Weyl group is an involution, i.e., $w_{\mathcal{O}}^2 = 1$, is the unique maximal element in its conjugacy class and is of the form $w_{\mathcal{O}} = w_0 w_J$, for a certain ϑ-invariant subset J of Π such that $w_0(\alpha) = w_J(\alpha)$ for every $\alpha \in \Delta_J$ [Carnovale 2008, Lemma 3.5; Chan et al. 2010, Corollary 2.11; Perkins and Rowley 2002].

We indicate the strategy we followed to determine the classification. Let $G_{\mathbb{C}}$ be the corresponding group over $\mathbb{C}$. We have shown in [Cantarini et al. 2005] that for every spherical conjugacy class $\mathcal{C}$ of $G_{\mathbb{C}}$ there exists an involution $w = w(\mathcal{C})$ in W such that $\dim \mathcal{C} = \ell(w) + \mathrm{rk}(1-w)$, with $\mathcal{C} \cap BwB \neq \varnothing$ (in fact even $\mathcal{C} \cap BwB \cap B^- \neq \varnothing$). For each group G we introduce a certain set $\mathcal{O}(G)$ of semisimple or mixed conjugacy classes; this set is suggested by the classification in characteristic zero. For each $\mathcal{O} \in \mathcal{O}(G)$ there is a certain spherical conjugacy class $\mathcal{C}$ in $G_{\mathbb{C}}$ such that $\dim \mathcal{O} = \dim \mathcal{C}$. Let $w = w_{\mathcal{C}}$. Our aim is to show that $\mathcal{O} \cap BwB \neq \varnothing$, so that $\mathcal{O}$ is in fact spherical by the following proposition. Finally we show that any conjugacy class not in $\mathcal{O}(G)$ is not spherical.

For convenience of the reader we shall give tables for the nonunipotent spherical conjugacy classes. In the tables we give a representative g of the spherical conjugacy class $\mathcal{O}$, the subset J of Π for which $w_{\mathcal{O}} = w_0 w_J$, the decomposition of $w_{\mathcal{O}}$ into the product of orthogonal reflections, the type of $C(g)$ when g is semisimple and the dimension of $\mathcal{O}$.

We recall the following result, proved in [Cantarini et al. 2005, Theorem 5] over $\mathbb{C}$, but which is valid with the same proof over any algebraically closed field.

Proposition 3.6. *Suppose that* $\mathcal{O}$ *contains an element* $x \in BwB$. *Then*

$$\dim B.x \geq \ell(w) + \mathrm{rk}(1-w).$$

In particular, $\dim \mathcal{O} \geq \ell(w) + \mathrm{rk}(1-w)$. *If in addition* $\dim \mathcal{O} \leq \ell(w) + \mathrm{rk}(1-w)$, *then* $\mathcal{O}$ *is spherical*, $w = w_{\mathcal{O}}$ *and* $B.x$ *is the dense* B*-orbit in* $\mathcal{O}$.

If g is in $Z(G)$, then $g \in T$, $\mathcal{O}_g = \{g\}$ and $w_{\mathcal{O}} = 1$. In the remainder of the paper we consider only noncentral conjugacy classes.

We shall use the following result.

Lemma 3.7. *Assume the positive roots* $\beta_1, \dots, \beta_\ell$ *are such that* $[X_{\pm\beta_i}, X_{\pm\beta_j}] = 1$ *for every* $1 \leq i < j \leq \ell$. *Then, for* $g = n_{\beta_1} \cdots n_{\beta_\ell} x_{\beta_1}(1) \cdots x_{\beta_\ell}(1)$ *and* $h \in T$ *such that* $\beta_i(h) \neq 1$ *for* $i = 1, \dots, \ell$, *we have*

$$ghg^{-1} \in BwB \cap B^-$$

where $w = s_{\beta_1} \cdots s_{\beta_\ell}$.

Proof. By [Carter 1989, p. 106], for every positive root α and every $t \in k^*$ we have $x_{-\alpha}(t) = x_\alpha(t^{-1})n_\alpha x_\alpha(t^{-1})h'$ for a certain $h' \in T$, so that $x_{-\alpha}(t) \in Bs_\alpha B \cap B^-$. Hence, for every $i = 1, \dots, \ell$, by (2-1) we have

$$\begin{aligned} n_{\beta_i}x_{\beta_i}(1)h(n_{\beta_i}x_{\beta_i}(1))^{-1} &= n_{\beta_i}x_{\beta_i}(1)hx_{\beta_i}(-1)h^{-1}hn_{\beta_i}^{-1} \\ &= n_{\beta_i}x_{\beta_i}(1-\beta_i(h))n_{\beta_i}^{-1}n_{\beta_i}hn_{\beta_i}^{-1} \\ &= x_{-\beta_i}(\beta_i(h)-1)h_i \\ &\in Bs_{\beta_i}B \cap B^-, \end{aligned}$$

where $h_i = n_{\beta_i}hn_{\beta_i}^{-1} \in T$. Let $t_1, \dots, t_\ell \in k^*$. Then

$$(x_{\beta_1}(t_1^{-1}) \cdots x_{\beta_\ell}(t_\ell^{-1}))^{-1}x_{-\beta_1}(t_1) \cdots x_{-\beta_\ell}(t_\ell)(x_{\beta_1}(t_1^{-1}) \cdots x_{\beta_\ell}(t_\ell^{-1}))$$

lies in $n_{\beta_1}X_{\beta_1} \cdots n_{\beta_\ell}X_{\beta_\ell}T = n_{\beta_1} \cdots n_{\beta_\ell}X_{\beta_1} \cdots X_{\beta_\ell}T \subseteq wB$. Therefore

$$ghg^{-1} = x_{-\beta_1}(\beta_1(h)-1) \cdots x_{-\beta_\ell}(\beta_\ell(h)-1)h_1 \cdots h_\ell \in BwB \cap B^-. \qquad \square$$

The hypothesis of the lemma is satisfied for instance if $\beta_1, \dots, \beta_\ell$ are pairwise orthogonal and long, as in [Costantini 2010, Lemma 4.1]. In characteristic 2, we have $[X_\gamma, X_\delta] = 1$ for every pair (γ, δ) of orthogonal roots.

Let $\mathcal{O}$ be the conjugacy class of $x \in G$. In general the orbit map $\pi : G/C(x) \to \mathcal{O}$ is a bijective morphism, which may not be separable (i.e., an isomorphism). Nevertheless, we have the following result:

Lemma 3.8 [Fowler and Röhrle 2008, Remark 2.14]. *Let $\mathcal{O}$ be a G-orbit with isotropy subgroup H. Then $\mathcal{O}$ is spherical if and only if G/H is spherical.* $\square$

Proposition 3.9. *Let $g \in G$ with Jordan decomposition $g = su$ for s semisimple and u unipotent. If $\mathcal{O}_g$ is spherical then $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical.*

Proof. By Lemma 3.8, $C(g) = C(s) \cap C(u)$ is a spherical subgroup of G. Hence both $C(s)$ and $C(u)$ are spherical subgroups of G and, by Lemma 3.8, $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical. $\square$

For $J \subseteq \Pi$ we put $T_J = T \cap L_J'$, a maximal torus of the derived subgroup L_J' of the standard Levi subgroup L_J, so that $B_J = T_J U_{w_J}$ is a Borel subgroup of L_J'.

Lemma 3.10. *Let $\mathcal{O}$ be a conjugacy class of G and $\mathcal{F} \subseteq \mathcal{O}$. Assume there exists $J \subseteq \Pi$ such that $\mathcal{F} \subseteq L_J$ and $(B_J.x)_{x\in\mathcal{F}}$ is a family of pairwise distinct B_J-orbits. Then the family $(B.x)_{x\in\mathcal{F}}$ consists of pairwise distinct B-orbits.*

Proof. Let x and y be elements of $\mathcal{F}$, and assume $B.x = B.y$. Then there exists $b \in B$ such that $bxb^{-1} = y$, i.e., $bx = yb$. Since $B = TU_{w_J}U_{w_0w_J}$, where $U_{w_0w_J}$ is the unipotent radical of the standard parabolic subgroup P_J, we can write $b = tu_1u_2$ with $t \in T$, $u_1 \in U_{w_J}$ and $u_2 \in U_{w_0w_J}$, so that $tu_1u_2x = ytu_1u_2$. Since $U_{w_0w_J}$ is normal in P_J, from uniqueness of expression we get $tu_1x = ytu_1$. We may decompose $T = T_J S$ where $S = \left(\bigcap_{i\in J} \ker\alpha_i\right)^\circ$, and $t = t_1t_2$ with $t_1 \in T_J$, $t_2 \in S$. Then $S \le C(L_J)$, so that $t_1u_1x = yt_1u_1$. But t_1u_1 lies in B_J, and we conclude that $B_J.x = B_J.y$. Therefore $x = y$ and we are done. $\square$

Lemma 3.11. *Let x be a semisimple element of G with $C(x) = L_J$, a pseudo-Levi subgroup of G, and assume $\mathcal{O}_x$ is spherical. Let $\tilde{x}$ be a semisimple element in $G_\mathbb{C}$ such that $C(\tilde{x}) = L_J$ (in $G_\mathbb{C}$). Then $\mathcal{O}_{\tilde{x}}$ is spherical.*

Proof. First we note that such an $\tilde{x}$ exists, by Proposition 3.1. By Lemma 3.8 and [Brundan 1998, Theorem 2.2(i)], it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_\mathbb{C}$. $\square$

Type A_n, $n \geq 1$. For every $i = 1, \ldots, \left[\frac{1}{2}(n+1)\right]$, we denote the root $e_i - e_{n+2-i}$ by β_i.

Proposition 3.12. *Let $G = \mathrm{SL}(2)$, any characteristic. Let $\mathcal{O}$ be a conjugacy class of G. Then $\mathcal{O} \cap Bw_\mathcal{O}B \cap B^-$ is nonempty, $\dim\mathcal{O} = \ell(w_\mathcal{O}) + rk(1 - w_\mathcal{O})$ and $\mathcal{O}$ is spherical.*

Proof. We may work (and usually do) up to a central element, hence we may assume $\mathcal{O} = \mathcal{O}_x$, x either unipotent or semisimple. If x is unipotent then the result follows from [Cantarini et al. 2005, Proposition 11], whose proof is characteristic-free. If x is semisimple, then either x is central, or x is regular. In the first case $C(x) = G$, and in the second case we may assume $C(x) = T$. Now

$$x = \begin{pmatrix} f & 0 \\ 0 & 1/f \end{pmatrix}$$

for a certain $f \neq \pm 1$. Let

$$g = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}.$$

Then $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$ by Lemma 3.7, where $w = w_0$, with $\dim\mathcal{O} = 2 = \ell(w_0) + \mathrm{rk}(1 - w_0)$. We conclude by Proposition 3.6. $\square$

Lemma 3.13. *Let H be connected and reductive, any characteristic. Then H has a regular spherical conjugacy class if and only if the semisimple part of H is of type A_1^r. In this case every conjugacy class is spherical.*

Proof. Without loss of generality we may assume $H = Z \times G_1 \times \cdots \times G_r$, where $Z = Z(H)^\circ$ and G_i is simple for each $i = 1, \ldots, r$. Let $n_i = \mathrm{rk}\, G_i$ and N_i the number of positive roots of G_i for $i = 1, \ldots, r$. Let $x = (z, x_1, \ldots, x_r)$ be an element of H and $\mathcal{O} = \mathcal{O}_x$. Then $\mathcal{O}$ is spherical if and only if each $G_i.x_i$ is spherical in G_i, and x is regular if and only if each x_i is regular in G_i. Moreover, a spherical G_i-conjugacy class in G_i has dimension at most $n_i + N_i$, while $G_i.x_i$ is regular in G_i if and only if its dimension is $2N_i$.

If the semisimple part of H is of type A_1^r, then every conjugacy class of H is spherical by Proposition 3.12.

Suppose there exists a regular spherical conjugacy class. Then $2N_i \le n_i + N_i$ for every i, which is possible if and only if $N_i = n_i = 1$ for every i. Hence the semisimple part of H is of type A_1^r. □

Lemma 3.14. *Let $H = GL(3)$, any characteristic, g a regular element of H. Then there exists a subset $\mathcal{F} = \{x_m \mid m \in k^*\}$ of $\mathcal{O}_g$ such that $(B(H).x_m)_{m\in k^*}$ consists of pairwise distinct $B(H)$-orbits.*

Proof. For $m, a, b, c \in k^*$, let

$$x_m = x_m(a,b,c) = \begin{pmatrix} 0 & 0 & \dfrac{abc}{m} \\ 0 & -m & -\dfrac{(a+m)(b+m)(c+m)}{m} \\ 1 & 1 & a+b+c+m \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & \dfrac{abc}{m} \end{pmatrix} \begin{pmatrix} 1 & 1 & a+b+c+m \\ 0 & 1 & \dfrac{(a+m)(b+m)(c+m)}{m^2} \\ 0 & 0 & 1 \end{pmatrix} \in w_0 B.$$

From the uniqueness of Bruhat decomposition, we have $B.x_m \cap w_0 B = T.x_m$; moreover, $C_T(x_m)$ consists of scalar matrices, and

$$S = \left\{ \begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} \middle| \alpha, \beta \in k^* \right\}$$

acts as

$$\begin{pmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 1 \end{pmatrix} .x_m = \begin{pmatrix} 0 & 0 & \alpha\dfrac{abc}{m} \\ 0 & -m & -\beta\dfrac{(a+m)(b+m)(c+m)}{m} \\ \alpha^{-1} & \beta^{-1} & a+b+c+m \end{pmatrix}.$$

Hence

$$T.x_m \cap \mathcal{F} = \{x_m\}.$$

The characteristic polynomial of $x_m(a,b,c)$ is $(X-a)(X-b)(X-c)$. Moreover, $\dim B.x_m(a,b,c) = 5$, so that $\dim \mathcal{O}_{x_m(a,b,c)} = 6$. We have shown that $x_m(a,b,c)$ is regular for every choice of $a, b, c \in k^*$. Now let g be a regular element of GL(3). Since $\mathcal{O}_g$ is determined by the characteristic polynomial of g, there exist $a, b, c \in k^*$ such that $x_m(a,b,c) \in \mathcal{O}_g$ for every $m \in k^*$. We take $x_m = x_m(a,b,c)$ for $m \in k^*$. The set $\mathcal{F} = \{x_m \mid m \in k^*\}$ is the required set. □

Proposition 3.15. *Let s be a semisimple element of* $\mathrm{SL}(n+1)$ *with at most* 2 *eigenvalues, any characteristic, and $\mathcal{O}$ its conjugacy class. Then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty, and $\mathcal{O}$ is spherical.*

Proof. We may assume $s = \mathrm{diag}(aI_k, bI_{n+1-k})$ with $a \neq b$, $1 \leq k \leq \left[\frac{1}{2}(n+1)\right]$. Let $g = n_{\beta_1} \cdots n_{\beta_k} x_{\beta_1}(1) \cdots x_{\beta_k}(1)$. Then, by Lemma 3.7, $gsg^{-1} \in \mathcal{O} \cap BwB \cap B^-$ with $w = w_{\beta_1} \cdots w_{\beta_k}$. As $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$, we conclude by Proposition 3.6. □

Theorem 3.16. *Let g be an element of* $\mathrm{SL}(n+1)$, *any characteristic, $g = su$ its Jordan decomposition and $\mathcal{O}$ its conjugacy class. Then $\mathcal{O}$ is spherical if and only if one of the following holds*:

(a) *$u = 1$ and s has at most* 2 *eigenvalues.*

(b) *$u \neq 1$, $s \in Z(G)$ and u has Jordan blocks of sizes at most* 2.

Proof. Assume that $\mathcal{O}$ is spherical. Suppose that neither (a) nor (b) hold. Since by [Knop 1995, Theorem 2.2] every conjugacy class contained in the closure of $\mathcal{O}$ is spherical, without loss of generality we may assume

$$g = \mathrm{diag}(R, S) \quad \text{for } R \in \mathrm{GL}(3),\ S \in \mathrm{GL}(n-2),\ S \text{ diagonal}$$

with

$$R = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 1 \\ 0 & 0 & a \end{pmatrix} \text{ or } \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & b \end{pmatrix} \text{ or } \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix},$$

a, b and c pairwise distinct. Hence R is regular in $\mathrm{GL}(3)$. Consider the elements

$$g_m = \mathrm{diag}(x_m, S)$$

for $m \in k^*$, where x_m is as defined in Lemma 3.14. We apply Lemma 3.10 with $J = \{1, 2\}$ and $\mathcal{F} = \{g_m \mid m \in k^*\} \subset L_J$. The g_m are all G-conjugate to g, and pairwise not B_J-conjugate. By Lemma 3.10 the family $(B.g_m)_{m \in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence either (a) or (b) holds.

The remaining assertions follow by Proposition 3.15, and from the classification of unipotent classes in zero or odd characteristic ([Carnovale 2010, Theorem 3.2] and in characteristic 2, [Costantini 2012, Table 1]). □

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$\mathrm{diag}(aI_k, bI_{n+1-k})$ $k = 1, \ldots, \left[\frac{1}{2}(n+1)\right]$ $a \neq b$	J_k	$s_{\beta_1} \cdots s_{\beta_k}$	$T_1 A_{k-1} A_{n-k}$	$2k(n+1-k)$

Table 1. Spherical semisimple classes in A_n, where $w_{\mathcal{O}} = w_0 w_J$ and $J_k = \{k+1, \ldots, n-k\}$ for $k = 1, \ldots, \left[\frac{1}{2}(n+1)\right] - 1$, $J_{\left[\frac{1}{2}(n+1)\right]} = \varnothing$.

Type C_n (and B_n), $p = 2$, $n \geq 2$. We put $\beta_i = 2e_i$ for each $i = 1, \dots, n$ and $\gamma_\ell = e_{2\ell-1} + e_{2\ell}$ for $\ell = 1, \dots, \left[\frac{1}{2}n\right]$.

We describe G as the subgroup of $\mathrm{GL}(2n)$ of matrices preserving the bilinear form associated with the matrix $\left(\begin{smallmatrix} 0 & I_n \\ -I_n & 0 \end{smallmatrix}\right)$ with respect to the canonical basis of k^{2n}. We observe that in characteristic 2 the groups of type B_n and C_n are isomorphic as abstract groups, hence we deal only with type C_n.

Proposition 3.17. *Let x be an element of* $\mathrm{Sp}(2n)$, *any characteristic*, $n \geq 2$, *and $\mathcal{O}$ its conjugacy class. If either*

(a) $x = a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ *for $\lambda \neq \pm 1$, or*

(b) $x = c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ *for $\lambda \neq \pm 1$,*

then $\mathcal{O} \cap B w_{\mathcal{O}} B \cap B^-$ is nonempty, $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$ *and $\mathcal{O}$ is spherical.*

Proof. The proof uses the same method as the proof of Proposition 3.15, so we omit it. □

Proposition 3.18. *Let $G = \mathrm{Sp}(2n)$, $p = 2$, $n \geq 2$. The spherical semisimple classes are represented by*

(a) $a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n)$ *for $\lambda \neq 1$,*

(b) $c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ *for $\lambda \neq 1$.*

Proof. Let x be a semisimple element of G, and assume $\mathcal{O} = \mathcal{O}_x$ is spherical. Without loss of generality $C(x) = L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or odd) characteristic [Cantarini et al. 2005, Table 1; Carnovale 2010, Theorem 3.3], it follows that L_J is of type $C_\ell C_{n-\ell}$ for $\ell = 1, \dots, \left[\frac{1}{2}n\right]$, $T_1 C_{n-1}$ or $T_1 \widetilde{A}_{n-1}$. But $Z(C_\ell C_{n-\ell}) = 1$, so that we are left with

$$a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1} I_n) \longleftrightarrow T_1 \widetilde{A}_{n-1},$$
$$c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1}) \longleftrightarrow T_1 C_{n-1},$$

for $\lambda \neq 1$. We conclude by Proposition 3.17. □

We now deal with mixed conjugacy classes.

Lemma 3.19. *Let $H = \mathrm{Sp}(4)$, any characteristic, and*

$$g = \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & \frac{1}{a} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

a mixed regular element of H (so $a \neq \pm 1$). Then there is a subset $\mathcal{F} = \{x_m \mid m \in k^*\}$ of $\mathcal{O}_g$ such that $(B(H).x_m)_{m \in k^*}$ consists of pairwise distinct $B(H)$-orbits.

Proof. For $m \in k^*$, we put

$$x_m = \begin{pmatrix} 0 & 0 & -\dfrac{1}{m} & 0 \\ 0 & 0 & -1 & 1 \\ m & m & \dfrac{a^2+m+1}{a} & \dfrac{m(-2a+m+1)}{a} \\ 0 & -1 & -\dfrac{1}{a} & 2-\dfrac{m}{a} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & \dfrac{1}{m} & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & \dfrac{a^2+m+1}{am} & \dfrac{-2a+m+1}{a} \\ 0 & 1 & \dfrac{1}{a} & \dfrac{m}{a}-2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{pmatrix} \in w_0 B.$$

The characteristic polynomial of x_m is $(X-1)^2(X-a)(X-1/a)$, and the 1-eigenspace has dimension 1. Hence x_m is H-conjugate to g. Suppose $x_m, x_{m'}$ are B-conjugate. Then $x_m, x_{m'}$ are T-conjugate and, from a direct calculation, it follows that $T.x_m \cap \mathcal{F} = \{x_m\}$, hence $m = m'$. □

Proposition 3.20. *Let $\mathcal{O}$ be the conjugacy class of a mixed element g of* $\mathrm{Sp}(2n)$, *$p = 2$. Then $\mathcal{O}$ is not spherical.*

Proof. Let $g = su$, the Jordan decomposition. Assume, for a contradiction, that $\mathcal{O}$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical. By Proposition 3.18, $H = C(s)$ is of type $T_1 C_{n-1}$ or $T_1 \widetilde{A}_{n-1}$. However $\dim T_1 \widetilde{A}_{n-1} = n^2$, and therefore $C_{T_1 \widetilde{A}_{n-1}}(u)$ is not spherical in G. We are left with H of type $T_1 C_{n-1}$, and we may assume $s = c_a = h_{\beta_1}(a)$ for a certain $a \neq 1$.

Since every conjugacy class contained in the closure of $\mathcal{O}$ is spherical, it is enough to deal with the minimal nontrivial spherical unipotent classes in $T_1 C_{n-1}$. From the classification of spherical unipotent classes in characteristic 2 [Costantini 2012, Tables 1 and 2], we may assume

$$g = h_{\beta_1}(a) x_{\alpha_2}(1) \quad \text{if } n = 2,$$
$$g = h_{\beta_{n-1}}(a) x_{\alpha_n}(1) \quad \text{or} \quad g = h_{\beta_1}(a) x_{\alpha_2}(1) \quad \text{if } n \geq 3,$$

since $h_{\beta_{n-1}}(a) = \mathrm{diag}(I_{n-2}, a, 1, I_{n-2}, a^{-1}, 1)$ is conjugate to $h_{\beta_1}(a)$.

Suppose $g = h_{\beta_{n-1}}(a) x_{\alpha_n}(1)$, $n \geq 2$. We apply Lemma 3.10 with $J = \{n-1, n\}$. By considering the corresponding embedding of C_2 into C_n, we may assume that the family $\mathcal{F} = \{x_m \mid m \in k^*\}$, introduced in Lemma 3.19, is a subset of L_J. The

x_m are all G-conjugate to g, and pairwise not B_J-conjugate. By Lemma 3.10, the family $(B.x_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence the class of $g = h_{\beta_{n-1}}(a)x_{\alpha_n}(1)$ is not spherical.

Suppose $g = h_{\beta_1}(a)x_{\alpha_2}(1)$, $n \geq 3$. Then

$$g = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}, \qquad A = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $(x_m)_{m\in k^*}$ be the family introduced in Lemma 3.14, such that x_m is GL(3)-conjugate to A for every $m \in k^*$. We put

$$g_m = \begin{pmatrix} x_m & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & {}^tx_m^{-1} & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}.$$

The g_m are all Sp($2n$)-conjugate to g. By Lemma 3.10 with $J = \{1, 2\}$, the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence the class of $g = h_{\beta_1}(a)x_{\alpha_2}(1)$ is not spherical. □

Theorem 3.21. *Let $G = \mathrm{Sp}(2n)$, $p = 2$, $n \geq 2$. The spherical classes are either semisimple or unipotent. The semisimple classes are represented in Table 2 and the unipotent classes are represented in Table 2 of* [Costantini 2012]. □

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$c_\lambda = \mathrm{diag}(\lambda, I_{n-1}, \lambda^{-1}, I_{n-1})$ $\lambda \neq 1$	J_2	$s_{\beta_1}s_{\beta_2}$	T_1C_{n-1}	$4n-2$
$a_\lambda = \mathrm{diag}(\lambda I_n, \lambda^{-1}I_n)$ $\lambda \neq 1$	$\varnothing$	$w_0 = s_{\beta_1}\cdots s_{\beta_n}$	$T_1\widetilde{A}_{n-1}$	n^2+n

Table 2. Spherical semisimple classes in C_n, $n \geq 2$, $p = 2$. Here $w_{\mathcal{O}} = w_0w_J$, $J_2 = \varnothing$ if $n = 2$ and $J_2 = \{3, \dots, n\}$ if $n \geq 3$.

Type D_n, $p = 2$, $n \geq 4$. Let $r = \left[\frac{1}{2}n\right]$. We put $\beta_\ell = e_{2\ell-1} + e_{2\ell}$ and $\delta_\ell = e_{2\ell-1} - e_{2\ell}$ for $\ell = 1, \dots, r$. Also, we set $J_1 = \{3, \dots, n\}$, $K_r = \{1, 3, \dots, 2r-1\}$ and, if n is even, $K'_r = \{1, 3, \dots, n-3, n\}$.

In this section we deal with groups G of type D_n. We recall that we are assuming G simply connected. Since $p = 2$, the covering map $\pi : G \to \mathrm{SO}(2n)$ is an isomorphism of abstract groups. We describe SO($2n$) as the connected component of the subgroup of $\mathrm{Sp}(2n)$ of matrices preserving the quadratic form associated with $\left(\begin{smallmatrix} 0 & I_n \\ I_n & 0 \end{smallmatrix}\right)$ with respect to the canonical basis of k^{2n} [Carter 1989, §1.6].

Proposition 3.22. *Let x be an element of $G = D_n$, any characteristic, $n \geq 4$, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x = c_\lambda = h_{\beta_1}(\lambda) h_{\delta_1}(\lambda)$ *for $\lambda \neq \pm 1$,*

(b) $x = a_\lambda = h_{\beta_1}(\lambda) \cdots h_{\beta_r}(\lambda)$ *for $\lambda \neq \pm 1$, or*

(c) $x = a'_\lambda = h_{\beta_1}(\lambda) \cdots h_{\beta_{r-1}}(\lambda) h_{\alpha_{n-1}}(\lambda)$ *for $\lambda \neq \pm 1$, n even,*

holds, then $\mathcal{O} \cap B w_{\mathcal{O}} B \cap B^-$ is nonempty, $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$ and $\mathcal{O}$ is spherical.

Proof. Assume $x = c_\lambda$ with $\lambda \neq \pm 1$. Let $g = n_{\beta_1} n_{\delta_1} x_{\beta_1}(1) x_{\delta_1}(1)$. Then we have $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$, with $w = s_{\beta_1} s_{\delta_1}$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - w)$.

Similarly, assume $x = a_\lambda$ with $\lambda \neq \pm 1$. Let $g = n_{\beta_1} \cdots n_{\beta_r} x_{\beta_1}(1) \cdots x_{\beta_r}(1)$. Then $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$, with $w = s_{\beta_1} \cdots s_{\beta_r}$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - w)$.

The case (c) follows from (b) by using the graph automorphism of G exchanging $n - 1$ and n. We conclude by Proposition 3.6. □

Proposition 3.23. *Let $G = D_n$, $p = 2$, $n \geq 4$. The spherical semisimple classes are represented by*

(a) $x = c_\lambda = h_{\beta_1}(\lambda) h_{\delta_1}(\lambda)$ *for $\lambda \neq 1$,*

(b) $x = a_\lambda = h_{\beta_1}(\lambda) \cdots h_{\beta_m}(\lambda)$ *for $\lambda \neq 1$,*

(c) $x = a'_\lambda = h_{\beta_1}(\lambda) \cdots h_{\beta_{m-1}}(\lambda) h_{\alpha_{n-1}}(\lambda)$ *for $\lambda \neq 1$, n even.*

Proof. The proof uses the same method as the proof of Proposition 3.18, so we omit it. □

We now deal with mixed conjugacy classes.

Proposition 3.24. *Let $\mathcal{O}$ be the conjugacy class of a mixed element g in D_n, $p = 2$. Then $\mathcal{O}$ is not spherical.*

Proof. We work with $\mathrm{SO}(2n)$ via π. Let $g = su$, the Jordan decomposition. Assume that $\mathcal{O}$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and we may assume, up to conjugation and graph automorphism, that for a certain $a \neq 1$,

$$s = \mathrm{diag}(aI_{n-1}, a^{-1}, a^{-1}I_{n-1}, a) \quad \text{or} \quad s = \mathrm{diag}(I_{n-3}, a, I_2, I_{n-3}, a^{-1}, I_2).$$

Assume $s = \mathrm{diag}(aI_{n-1}, a^{-1}, a^{-1}I_{n-1}, a)$ for a certain $a \neq 1$. Without loss of generality we may assume $u = x_{\alpha_{n-2}}(a^{-1})$, so that

$$g = \begin{pmatrix} aI_{n-3} & 0 & 0 & 0 \\ 0 & A & 0 & 0 \\ 0 & 0 & a^{-1}I_{n-3} & 0 \\ 0 & 0 & 0 & {}^tA^{-1} \end{pmatrix}, \qquad A = \begin{pmatrix} a & 1 & 0 \\ 0 & a & 0 \\ 0 & 0 & a^{-1} \end{pmatrix}.$$

Let $(x_m)_{m\in k^*}$ be the family introduced in Lemma 3.14, such that x_m is GL(3)-conjugate to A for every $m \in k^*$ and $(B(\mathrm{GL}(3)).x_m)_{m\in k^*}$ consists of pairwise distinct $B(\mathrm{GL}(3))$-orbits.

We put

$$g_m = \begin{pmatrix} aI_{n-3} & 0 & 0 & 0 \\ 0 & x_m & 0 & 0 \\ 0 & 0 & a^{-1}I_{n-3} & 0 \\ 0 & 0 & 0 & {}^t x_m^{-1} \end{pmatrix}.$$

The g_m are all SO$(2n)$-conjugate to g. By Lemma 3.10 with $J = \{n-1, n-2\}$, the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. This settles the cases when n is odd and $C(s)$ is of type T_1A_{n-1}, and when n is even and $C(s)$ is of type $(T_1A_{n-1})'$. Upon application of the graph automorphism exchanging n and $n-1$, this also settles the case when n is even and $C(s)$ is of type T_1A_{n-1}.

Assume $s = \mathrm{diag}(a, I_{n-1}, a^{-1}, I_{n-1})$ for a certain $a \neq 1$. Without loss of generality we may assume $u = x_{\alpha_2}(1)$, so that

$$g = \begin{pmatrix} A & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & {}^tA^{-1} & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}, \qquad A = \begin{pmatrix} a & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $(x_m)_{m\in k^*}$ be the family introduced in Lemma 3.14, such that x_m is GL(3)-conjugate to A for every $m \in k^*$ and $(B(\mathrm{GL}(3)).x_m)_{m\in k^*}$ consists of pairwise distinct $B(\mathrm{GL}(3))$-orbits.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim\mathcal{O}$
$c_\lambda = h_{\beta_1}(\lambda)h_{\delta_1}(\lambda)$ $\lambda \neq 1$ $\mathrm{diag}(\lambda^2, I_{n-1}, \lambda^{-2}, I_{n-1})$	J_1	$s_{\beta_1}s_{\delta_1}$	T_1D_{n-1}	$4(n-1)$
$a_\lambda = h_{\beta_1}(\lambda)\cdots h_{\beta_r}(\lambda)$ $\lambda \neq 1$ $\mathrm{diag}(\lambda I_n, \lambda^{-1}I_n)$	K_r	$s_{\beta_1}\cdots s_{\beta_r}$	T_1A_{n-1}	n^2-n
$a'_\lambda = h_{\beta_1}(\lambda)\cdots h_{\beta_{r-1}}(\lambda)h_{\alpha_{n-1}}(\lambda)$ $\lambda \neq 1$ $\mathrm{diag}(\lambda I_{n-1}, \lambda^{-1}, \lambda^{-1}I_{n-1}, \lambda)$	K'_r	$s_{\beta_1}\cdots s_{\beta_{r-1}}s_{\alpha_{n-1}}$	$(T_1A_{n-1})'$	n^2-n

Table 3. Spherical semisimple classes in D_n, $p=2$, $n\geq 4$, $n=2r$, where $w_{\mathcal{O}} = w_0w_J$.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim\mathcal{O}$
$c_\lambda = h_{\beta_1}(\lambda)h_{\delta_1}(\lambda)$ $\lambda \neq 1$ $\mathrm{diag}(\lambda^2, I_{n-1}, \lambda^{-2}, I_{n-1})$	J_1	$s_{\beta_1}s_{\delta_1}$	T_1D_{n-1}	$4(n-1)$
$a_\lambda = h_{\beta_1}(\lambda)\cdots h_{\beta_r}(\lambda)$ $\lambda \neq 1$ $\mathrm{diag}(\lambda I_n, \lambda^{-1}I_n)$	K_r	$s_{\beta_1}\cdots s_{\beta_r}$	T_1A_{n-1}	n^2-n

Table 4. Spherical semisimple classes in D_n, $p=2$, $n \geq 5$, $n=2r+1$, where $w_{\mathcal{O}} = w_0w_J$.

Set

$$g_m = \begin{pmatrix} x_m & 0 & 0 & 0 \\ 0 & I_{n-3} & 0 & 0 \\ 0 & 0 & {}^tx_m^{-1} & 0 \\ 0 & 0 & 0 & I_{n-3} \end{pmatrix}.$$

The g_m are all $\mathrm{SO}(2n)$-conjugate to g. By Lemma 3.10 with $J = \{1, 2\}$, the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. This settles the case when $C(s)$ is of type T_1D_{n-1}, and we are done. □

Theorem 3.25. *Let $G = D_n$, $p=2$, $n \geq 4$. The spherical classes are either semisimple or unipotent. The semisimple classes are represented in Tables 3 and 4, and the unipotent classes in Tables 3 and 4 of [Costantini 2012].* □

Type E_6. We put

$$\beta_1 = (1, 2, 2, 3, 2, 1), \qquad \beta_2 = (1, 0, 1, 1, 1, 1),$$
$$\beta_3 = (0, 0, 1, 1, 1, 0), \qquad \beta_4 = (0, 0, 0, 1, 0, 0).$$

Proposition 3.26. *Let x be an element of E_6, any characteristic, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x = h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1)$,

(b) $x = h(z) = h_{\alpha_1}(z^4)h_{\alpha_2}(z^3)h_{\alpha_3}(z^5)h_{\alpha_4}(z^6)h_{\alpha_5}(z^4)h_{\alpha_6}(z^2)$ *for* $z^3 \neq 1$,

holds, then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty, $\dim\mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1-w_{\mathcal{O}})$, *and $\mathcal{O}$ is spherical.*

Proof. (a) If $p=2$, then $x=1$, and there is nothing to prove. So assume $p \neq 2$. In G there are two classes of involutions: one has centralizer of type A_1A_5 and dimension 40, the other has centralizer of type D_5T_1 and has dimension 32. Let $y = n_{\beta_1}\cdots n_{\beta_4} \in w_0B$, $w = s_{\beta_1}\cdots s_{\beta_4} = w_0$. Then $y^2 = h_{\beta_1}(-1)\cdots h_{\beta_4}(-1) = 1$, and $\dim\mathcal{O}_y \geq 40$ by Proposition 3.6. Since $C(x)$ is of type A_1A_5, we conclude that $x \sim y$, so that $\mathcal{O} \cap Bw_0B$ is nonempty, $\dim \mathcal{O} = \ell(w_0) + \mathrm{rk}(1 - w_0)$ and $\mathcal{O}$ is spherical. It is a general fact that if t is semisimple and $\mathcal{O}_t \cap BwB \neq \varnothing$, then $\mathcal{O}_t \cap BwB \cap B^- \neq \varnothing$ [Cantarini et al. 2005, Lemma 14].

(b) In this case $C(x)$ is of type D_5T_1 (note that $C(x) = C(h(-1))$ if $p \neq 2$). Let $g = n_{\beta_1}n_{\beta_2}x_{\beta_1}(1)x_{\beta_2}(1)$. Then $gxg^{-1} \in \mathcal{O} \cap Bs_{\beta_1}s_{\beta_2}B \cap B^-$, with $w = s_{\beta_1}s_{\beta_2}$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - w)$. We conclude by Proposition 3.6. □

Proposition 3.27. *Let $G = E_6$. The spherical semisimple classes are represented by*

$$h(z) = h_{\alpha_1}(z^4)h_{\alpha_2}(z^3)h_{\alpha_3}(z^5)h_{\alpha_4}(z^6)h_{\alpha_5}(z^4)h_{\alpha_6}(z^2),\ z^3 \neq 1, \qquad \textit{for } p = 2,$$

$$\left.\begin{array}{l} h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1), \\ h(z) = h_{\alpha_1}(z^4)h_{\alpha_2}(z^3)h_{\alpha_3}(z^5)h_{\alpha_4}(z^6)h_{\alpha_5}(z^4)h_{\alpha_6}(z^2),\ z \neq 1, \end{array}\right\} \quad \textit{for } p = 3.$$

Proof. Let x be a semisimple element of G, and assume $\mathcal{O} = \mathcal{O}_x$ is spherical. Without loss of generality $C(x) = L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or good odd) characteristic [Cantarini et al. 2005, Table 2; Carnovale 2010, Theorem 3.6], it follows that L_J is of type A_1A_5 or D_5T_1.

Let $p = 2$. Then $Z(A_1A_5) = Z(G)$ (of order 3), so that we are left with $h(z)$, for $z^3 \neq 1$.

Let $p = 3$. Then $Z(G) = 1$, and we conclude by Proposition 3.26. □

We have established the information in Tables 5 and 6, where $w_{\mathcal{O}} = w_0w_J$.

Proposition 3.28. *Let $\mathcal{O}$ be the conjugacy class of a mixed element g in E_6, $p = 2$ or 3. Then $\mathcal{O}$ is not spherical.*

Proof. Let $g = su$, the Jordan decomposition. Assume that $\mathcal{O}$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and therefore $C(s)$ is of type A_1A_5 or D_5T_1. A dimensional argument rules out all the possibilities except the case that $C(s)$ is of type A_1A_5 and u is a nonidentical unipotent element in the component A_1 of $C(s)$ (hence $p = 3$). Therefore, without loss of generality we may assume $g = h_{\alpha_1}(-1)x_{\alpha_1}(1)$, which is a regular element of the standard Levi subgroup L_J, for $J = \{1, 2\}$. By Lemma 3.14, there is an infinite family $\mathcal{F} = \{g_m \mid m \in k^*\} \subset L_J$ such that the g_m

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h(z) = h_{\alpha_3}(z^5)h_{\alpha_4}(z^6)h_{\alpha_5}(z^4)h_{\alpha_6}(z^2)$ $z^3 \neq 1$	$\{3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}$	D_5T_1	32

Table 5. Spherical semisimple classes in E_6, $p = 2$.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h(z) = h_{\alpha_1}(z^4)h_{\alpha_2}(z^3)h_{\alpha_3}(z^5) \cdot h_{\alpha_4}(z^6)h_{\alpha_5}(z^4)h_{\alpha_6}(z^2)$ $z \neq 1$	$\{3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}$	D_5T_1	32
$h_{\alpha_1}(-1)h_{\alpha_4}(-1)h_{\alpha_6}(-1) \sim h_{\alpha_1}(-1)$	$\varnothing$	w_0	A_1A_5	40

Table 6. Spherical semisimple classes in E_6, $p = 3$.

are all L_J-conjugate (hence G-conjugate) to g, and pairwise not B_J-conjugate. By Lemma 3.10 the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence $\mathcal{O}$ is not spherical. □

Theorem 3.29. *Let $G = E_6$, $p = 2$ or 3. The spherical classes are either semisimple or unipotent, up to a central element if $p = 2$. The semisimple classes are represented in Tables 5 and 6, and the unipotent classes are represented in Tables 6 and 7 of [Costantini 2012].* □

Type E_7. Here $Z(G) = \langle\tau\rangle$, where $\tau = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1)$. We put

$$\beta_1 = (2, 2, 3, 4, 3, 2, 1), \quad \beta_2 = (0, 1, 1, 2, 2, 2, 1), \quad \beta_3 = (0, 1, 1, 2, 1, 0, 0),$$
$$\beta_4 = \alpha_7, \qquad \beta_5 = \alpha_5, \qquad \beta_6 = \alpha_3, \qquad \beta_7 = \alpha_2.$$

Proposition 3.30. *Let x be an element of E_7, any characteristic, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x = h_{\alpha_2}(-\zeta)h_{\alpha_5}(\zeta)h_{\alpha_6}(-1)h_{\alpha_7}(-\zeta)$ *for* $\zeta^2 = -1$, $p \neq 2$,

(b) $x = h_{\alpha_1}(-1)$ *for* $p \neq 2$,

(c) $x = h(z) = h_{\alpha_1}(z^2)h_{\alpha_2}(z^3)h_{\alpha_3}(z^4)h_{\alpha_4}(z^6)h_{\alpha_5}(z^5)h_{\alpha_6}(z^4)h_{\alpha_7}(z^3)$ *for* $z \neq \pm 1$,

holds, then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty, $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$, *and $\mathcal{O}$ is spherical.*

Proof. (a) Let Y be the set of elements y of order 4 of T such that $y^2 = \tau$. Then Y is the disjoint union of 2 W-classes Y_1 and Y_2: $C(y)$ is of type A_7 if $y \in Y_1$, and of type E_6T_1 if $y \in Y_2$. A representative for Y_1 is $h_{\alpha_2}(-\zeta)h_{\alpha_5}(\zeta)h_{\alpha_6}(-1)h_{\alpha_7}(-\zeta)$ where ζ is a square root of -1.

Let $y = n_{\beta_1} \cdots n_{\beta_7} \in w_0B$, $w = s_{\beta_1} \cdots s_{\beta_7} = w_0$. Then $y^2 = h_{\beta_1}(-1) \cdots h_{\beta_7}(-1) = \tau$, and $\dim \mathcal{O}_y \geq \dim B$ by Proposition 3.6. Since $C(x)$ is of type A_7, we conclude that $x \sim y$, so that $\mathcal{O} \cap Bw_0B$ is nonempty, $\dim \mathcal{O} = \ell(w_0) + \mathrm{rk}(1 - w_0)$ and $\mathcal{O}$ is spherical. As above, $\mathcal{O} \cap BwB \cap B^- \neq \varnothing$.

(b) The group G has 2 classes of noncentral involutions: $\mathcal{O}_{h_{\beta_1}(-1)}$ and $\mathcal{O}_{h_{\beta_1}(-1)\tau}$. In fact there are 127 involutions in T, and τ is central. The remaining 126 fall in 2 classes: $\{h_\alpha(-1) \mid \alpha \in \Phi^+\}$ and $\{h_\alpha(-1)\tau \mid \alpha \in \Phi^+\}$. Let $y = n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_3} \in wB$, where $w = s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\alpha_3}$. Then $y^2 = h_{\beta_1}(-1)h_{\beta_2}(-1)h_{\beta_3}(-1)h_{\alpha_3}(-1) = 1$, so that y is a (noncentral) involution. We conclude that $x \sim y$ or $x \sim y\tau$, so that (in either case) $\mathcal{O} \cap BwB$ is nonempty, $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$ and $\mathcal{O}$ is spherical. As above, $\mathcal{O} \cap BwB \cap B^- \neq \varnothing$. (In fact we have $n_\alpha \sim h_\alpha(\zeta)$ already in $\langle X_\alpha, X_{-\alpha}\rangle$ for every root α, hence $n_{\beta_1}n_{\beta_2}n_{\beta_3}n_{\alpha_3} \sim h_{\beta_1}(\zeta)h_{\beta_2}(\zeta)h_{\beta_3}(\zeta)h_{\alpha_3}(\zeta) = h_\gamma(-1)$, where $\gamma = \beta_1 - \alpha_1$. Therefore $x \sim y$.)

(c) (any characteristic) We have $C(x)$ of type E_6T_1. Let

$$g = n_{\beta_1}n_{\beta_2}n_{\alpha_7}x_{\beta_1}(1)x_{\beta_2}(1)x_{\alpha_7}(1).$$

Then $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$, with $w = s_{\beta_1}s_{\beta_2}s_{\alpha_7}$, and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$. We conclude by Proposition 3.6. □

Proposition 3.31. *Let $G = E_7$. The spherical semisimple classes are represented by*

$$h(z) = h_{\alpha_1}(z^2)h_{\alpha_2}(z^3)h_{\alpha_3}(z^4)h_{\alpha_4}(z^6)h_{\alpha_5}(z^5)h_{\alpha_6}(z^4)h_{\alpha_7}(z^3),\ z \neq 1, \qquad \textit{for } p = 2,$$

$$\left.\begin{array}{l} h_{\alpha_2}(-\zeta)h_{\alpha_5}(\zeta)h_{\alpha_6}(-1)h_{\alpha_7}(-\zeta),\ \zeta^2 = -1, \\ h_{\alpha_1}(-1),\ h_{\alpha_1}(-1)\tau, \\ h(z) = h_{\alpha_1}(z^2)h_{\alpha_2}(z^3)h_{\alpha_3}(z^4)h_{\alpha_4}(z^6)h_{\alpha_5}(z^5)h_{\alpha_6}(z^4)h_{\alpha_7}(z^3),\ z \neq \pm 1, \end{array}\right\} \textit{for } p = 3.$$

Proof. Let x be a semisimple element of G, and assume $\mathcal{O} = \mathcal{O}_x$ is spherical. Without loss of generality $C(x) = L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or good odd) characteristic [Cantarini et al. 2005, Table 2; Carnovale 2010, Theorem 3.7], it follows that L_J is of type E_6T_1, D_6A_1 or A_7.

Let $p = 2$. Then $Z(G) = Z(D_6A_1) = Z(A_7) = 1$, so that we are left with $h(z)$, for $z \neq 1$.

For $p = 3$, we conclude by Proposition 3.30. □

We have established the information in Tables 7 and 8, where $w_{\mathcal{O}} = w_0w_J$.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h(z) = h_{\alpha_1}(z^2)h_{\alpha_2}(z^3)h_{\alpha_3}(z^4) \cdot h_{\alpha_4}(z^6)h_{\alpha_5}(z^5)h_{\alpha_6}(z^4)h_{\alpha_7}(z^3)$, $z \neq 1$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\alpha_7}$	E_6T_1	54

Table 7. Spherical semisimple classes in E_7, $p = 2$.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h(z) = h_{\alpha_1}(z^2)h_{\alpha_2}(z^3)h_{\alpha_3}(z^4) \cdot h_{\alpha_4}(z^6)h_{\alpha_5}(z^5)h_{\alpha_6}(z^4)h_{\alpha_7}(z^3)$ $z \neq \pm 1$	$\{2, 3, 4, 5\}$	$s_{\beta_1}s_{\beta_2}s_{\alpha_7}$	E_6T_1	54
$h_{\alpha_1}(-1), h_{\alpha_1}(-1)\tau$	$\{2, 5, 7\}$	$s_{\beta_1}s_{\beta_2}s_{\beta_3}s_{\alpha_3}$	D_6A_1	64
$h_{\alpha_2}(-\zeta)h_{\alpha_5}(\zeta)h_{\alpha_6}(-1)h_{\alpha_7}(-\zeta)$ $\zeta^2 = -1$	$\varnothing$	w_0	A_7	70

Table 8. Spherical semisimple classes in E_7, $p = 3$.

Proposition 3.32. *Let $\mathcal{O}$ be the conjugacy class of a mixed element g in E_7, $p = 2$ or 3. Then $\mathcal{O}$ is not spherical.*

Proof. Let $g = su$, the Jordan decomposition. Assume that $\mathcal{O}$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and therefore $C(s)$ is of type E_6T_1, D_6A_1 or A_7. A dimensional argument rules out all the possibilities except the case that $C(s)$ is of type D_6A_1 and u is a nonidentical unipotent element in the component A_1 of $C(s)$ (hence $p = 3$). Therefore, without loss of generality we may assume $g = h_{\alpha_7}(-1)x_{\alpha_7}(1)$, which is a regular element of the standard Levi subgroup L_J, for $J = \{6, 7\}$. By Lemma 3.14, there is an infinite family $\mathcal{F} = \{g_m \mid m \in k^*\} \subset L_J$ such that the g_m are all L_J-conjugate (hence G-conjugate) to g, and pairwise not B_J-conjugate. By Lemma 3.10 the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence $\mathcal{O}$ is not spherical. □

Theorem 3.33. *Let $G = E_7$, $p = 2$ or 3. The spherical classes are either semisimple or unipotent, up to a central element if $p = 3$. The semisimple classes are represented in Tables 7 and 8, and the unipotent classes are represented in Tables 8 and 9 of [Costantini 2012].* □

Type E_8. We put

$$\beta_1 = (2, 3, 4, 6, 5, 4, 3, 2), \qquad \beta_2 = (2, 2, 3, 4, 3, 2, 1, 0),$$
$$\beta_3 = (0, 1, 1, 2, 2, 2, 1, 0), \qquad \beta_4 = (0, 1, 1, 2, 1, 0, 0, 0),$$
$$\beta_5 = \alpha_7, \qquad \beta_6 = \alpha_5, \qquad \beta_7 = \alpha_3, \qquad \beta_8 = \alpha_2.$$

Proposition 3.34. *Let x be an element of E_8, $p \neq 2$, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x = h_{\alpha_2}(-1)h_{\alpha_3}(-1)$,

(b) $x = h_{\alpha_2}(-1)h_{\alpha_5}(-1)h_{\alpha_7}(-1) \sim h_{\alpha_8}(-1)$,

holds, then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty, $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$ *and $\mathcal{O}$ is spherical.*

Proof. The group E_8, for $p \neq 2$, has 2 classes of involutions.

(a) Let $y = n_{\beta_1} \cdots n_{\beta_8} \in w_0 B$, $w = s_{\beta_1} \cdots s_{\beta_8} = w_0$. Then

$$y^2 = h_{\beta_1}(-1) \cdots h_{\beta_8}(-1) = 1,$$

and $\dim \mathcal{O}_y \geq \dim B$ by Proposition 3.6. Since $C(x)$ is of type D_8, we conclude that $x \sim y$, so that $\mathcal{O} \cap B w_0 B$ is nonempty, $\dim \mathcal{O} = \ell(w_0) + \mathrm{rk}(1 - w_0)$ and $\mathcal{O}$ is spherical. As above, $\mathcal{O} \cap BwB \cap B^- \neq \varnothing$.

(b) Let $x = h_{\alpha_8}(-1)$, so that $C(x)$ is of type $E_7 A_1$. Let

$$g = n_{\beta_1} n_{\beta_2} n_{\beta_3} n_{\alpha_7} x_{\beta_1}(1) x_{\beta_2}(1) x_{\beta_3}(1) x_{\alpha_7}(1).$$

Then $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$, with $w = s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_7}$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1-w)$. We conclude by Proposition 3.6. □

Proposition 3.35. *Let $G = E_8$. The spherical (nontrivial) semisimple classes are represented by*

$$\begin{array}{ll} \textit{none}, & \textit{for } p = 2, \\[1ex] \left.\begin{array}{l} h_{\alpha_2}(-1) h_{\alpha_3}(-1), \\ h_{\alpha_8}(-1), \end{array}\right\} & \textit{for } p = 3 \textit{ or } 5. \end{array}$$

Proof. Let x be a semisimple element of G, and assume $\mathcal{O} = \mathcal{O}_x$ is spherical. Without loss of generality $C(x) = L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or good odd) characteristic [Cantarini et al. 2005, Table 2; Carnovale 2010, Theorem 3.8], it follows that L_J is of type $E_7 A_1$ or D_8.

Let $p = 2$. Then $Z(E_7 A_1) = Z(D_8) = 1$, so there are no nontrivial spherical semisimple classes.

For $p = 3$ or 5, we conclude by Proposition 3.34. □

We have established the information in Table 9, where $w_{\mathcal{O}} = w_0 w_J$.

Proposition 3.36. *Let $\mathcal{O}$ be the conjugacy class of a mixed element g in E_8, $p = 2$, 3 or 5. Then $\mathcal{O}$ is not spherical.*

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h_{\alpha_8}(-1)$	$\{2, 3, 4, 5\}$	$s_{\beta_1} s_{\beta_2} s_{\beta_3} s_{\alpha_7}$	$E_7 A_1$	112
$h_{\alpha_2}(-1) h_{\alpha_3}(-1)$	$\varnothing$	w_0	D_8	128

Table 9. Spherical semisimple classes in E_8, $p = 3$ or 5.

Proof. Let $g = su$, the Jordan decomposition. Assume that $\mathcal{O}$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and therefore $C(s)$ is of type E_7A_1 or A_7. A dimensional argument rules out all the possibilities except the case that $C(s)$ is of type E_7A_1 and u is a nonidentical unipotent element in the component A_1 of $C(s)$ (hence $p = 3$ or 5). Therefore, without loss of generality we may assume $g = h_{\alpha_8}(-1)x_{\alpha_8}(1)$, which is a regular element of the standard Levi subgroup L_J, for $J = \{7, 8\}$. By Lemma 3.14, there is an infinite family $\mathcal{F} = \{g_m \mid m \in k^*\} \subset L_J$ such that the g_m are all L_J-conjugate (hence G-conjugate) to g, and pairwise not B_J-conjugate. By Lemma 3.10 the family $(B.g_m)_{m\in k^*}$ is an infinite family of (distinct) B-orbits, a contradiction. Hence $\mathcal{O}$ is not spherical. □

Theorem 3.37. *Let $G = E_8$, $p = 2$, 3 or 5. The spherical classes are either semisimple or unipotent. The semisimple classes are represented in Table 9, and the unipotent classes are represented in [Costantini 2012,* Tables 10 and 11]. □

Type F_4. We put

$$\beta_1 = (2, 3, 4, 2), \quad \beta_2 = (0, 1, 2, 2), \quad \beta_3 = (0, 1, 2, 0), \quad \beta_4 = (0, 1, 0, 0).$$

Also, γ_1 is the highest short root $(1, 2, 3, 2)$.

Proposition 3.38. *Let x be an element of F_4, $p \neq 2$, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x = h_{\alpha_2}(-1)h_{\alpha_4}(-1) \sim h_{\alpha_1}(-1)$,

(b) $x = h_{\alpha_4}(-1)$,

holds, then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty, $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$ *and $\mathcal{O}$ is spherical.*

Proof. The group F_4, for $p \neq 2$, has 2 classes of involutions.

(a) Let $y = n_{\beta_1} \cdots n_{\beta_4} \in w_0 B$, $w = s_{\beta_1} \cdots s_{\beta_4} = w_0$. Then

$$y^2 = h_{\beta_1}(-1) \cdots h_{\beta_4}(-1) = 1,$$

and $\dim \mathcal{O}_y \geq \dim B$ by Proposition 3.6. Since $C(x)$ is of type A_1C_3, we conclude that $x \sim y$, so that $\mathcal{O} \cap Bw_0B$ is nonempty, $\dim \mathcal{O} = \ell(w_0) + \mathrm{rk}(1 - w_0)$ and $\mathcal{O}$ is spherical. As above, $\mathcal{O} \cap BwB \cap B^- \neq \varnothing$.

(b) We have $C(x)$ of type B_4. Let $g = n_{\gamma_1}x_{\gamma_1}(1)$. Then $gxg^{-1} \in \mathcal{O} \cap BwB \cap B^-$, with $w = s_{\gamma_1}$ and $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - w)$. We conclude by Proposition 3.6. □

Proposition 3.39. *Let $G = F_4$. The spherical (nontrivial) semisimple classes are represented by*

$$\begin{array}{ll} \textit{none}, & \textit{for } p = 2, \\[1ex] \left.\begin{array}{l} h_{\alpha_1}(-1), \\ h_{\alpha_4}(-1), \end{array}\right\} & \textit{for } p = 3. \end{array}$$

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim\mathcal{O}$
$h_{\alpha_4}(-1)$	$\{1, 2, 3\}$	s_{γ_1}	B_4	16
$h_{\alpha_1}(-1)$	$\varnothing$	w_0	C_3A_1	28

Table 10. Spherical semisimple classes in F_4, $p = 3$.

Proof. Let x be a semisimple element of G, and assume $\mathcal{O} = \mathcal{O}_x$ is spherical. Without loss of generality $C(x) = L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or good odd) characteristic [Cantarini et al. 2005, Table 2; Carnovale 2010, Theorem 3.9], it follows that L_J is of type C_3A_1 or B_4.

Let $p = 2$. Then $Z(C_3A_1) = Z(B_4) = 1$, so there are no nontrivial spherical semisimple classes.

For $p = 3$, we conclude by Proposition 3.38. □

We have established the information in Table 10, where $w_{\mathcal{O}} = w_0 w_J$.

We finally deal with mixed classes in F_4. We recall that over the complex numbers the principal model orbit is a mixed conjugacy class; see [Luna 2007, 3.3(6) and p. 300], and also [Costantini 2010, Table 24].

Proposition 3.40. *Let $x = h_{\alpha_4}(-1)x_{\alpha_1}(1)$ in F_4, $p \neq 2$, and $\mathcal{O}$ its conjugacy class. Then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$ is nonempty*, $\dim\mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$ *and $\mathcal{O}$ is spherical.*

Proof. This is the mixed class in F_4 which is spherical in zero or good, odd characteristic. We can deal with this class with the same method used in the proof of [Cantarini et al. 2005, Theorem 23], and corrected in the proof of [Carnovale 2010, Theorem 3.9], to show that $Bw_0B \cap \mathcal{O} \neq \varnothing$, so that $w_{\mathcal{O}} = w_0$, $\dim\mathcal{O} = \ell(w_0) + \mathrm{rk}(1 - w_0)$ and $\mathcal{O}$ is spherical. The (correct) argument at the end of the proof of [Cantarini et al. 2005, Theorem 23] shows that $\mathcal{O} \cap Bw_0B \cap B^- \neq \varnothing$. □

Proposition 3.41. *Let $G = F_4$. The spherical mixed classes are represented by*

$$\begin{array}{ll} \textit{none}, & \textit{for } p = 2, \\ h_{\alpha_4}(-1)x_{\alpha_1}(1), & \textit{for } p = 3. \end{array}$$

Proof. Let $g = su$, the Jordan decomposition of a mixed element g. Assume that $\mathcal{O} = \mathcal{O}_g$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and therefore $C(s)$ is of type C_3A_1 or B_4. A dimensional argument rules out all the possibilities except the case that $C(s)$ is of type B_4 and u is in the minimal unipotent class of $C(s)$ (hence $p = 3$). Therefore, without loss of generality we may assume $g = h_{\alpha_4}(-1)x_{\alpha_1}(1)$. We conclude by Proposition 3.40. □

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$\dim \mathcal{O}$
$h_{\alpha_4}(-1)x_{\alpha_1}(1)$	$\varnothing$	w_0	28

Table 11. Spherical mixed classes in F_4, $p=3$.

Theorem 3.42. *Let $G=F_4$, $p=2$ or 3. If $p=2$, the spherical classes are unipotent and are represented in Table 13 of [Costantini 2012]. If $p=3$, the spherical semisimple classes are represented in Table 10, the spherical unipotent classes are represented in Table 12 of [Costantini 2012] and the spherical mixed classes are represented in Table 11.* □

Type G_2. We put $\beta_1=(3,2)$ and $\beta_2=\alpha_1$. Also, γ_1 is the highest short root (2, 1).

Proposition 3.43. *Let x be an element of G_2, any characteristic, and $\mathcal{O}$ its conjugacy class. If one of*

(a) $x=h_{\alpha_1}(-1)$, $p\neq 2$,

(b) $x=h_{\alpha_1}(\zeta)$, ζ *a primitive* 3*rd root of* 1, $p\neq 3$,

holds, then $\mathcal{O}\cap Bw_{\mathcal{O}}B\cap B^-$ is nonempty, $\dim\mathcal{O}=\ell(w_{\mathcal{O}})+\mathrm{rk}(1-w_{\mathcal{O}})$ *and $\mathcal{O}$ is spherical.*

Proof. (a) For $p\neq 2$, the group G_2 has 1 class of involutions. Let

$$y=n_{\beta_1}n_{\beta_2}\in w_0B,\qquad w=s_{\beta_1}s_{\beta_2}=w_0.$$

Then $y^2=h_{\beta_1}(-1)h_{\beta_2}(-1)=1$, and $\dim\mathcal{O}_y\geq\dim B$ by Proposition 3.6. We conclude that $x\sim y$, so that $\mathcal{O}\cap Bw_0B$ is nonempty, $\dim\mathcal{O}=\ell(w_0)+\mathrm{rk}(1-w_0)$ and $\mathcal{O}$ is spherical. As above, $\mathcal{O}\cap BwB\cap B^-\neq\varnothing$.

(b) For $p\neq 3$, let $g=n_{\gamma_1}x_{\gamma_1}(1)$. Then $gxg^{-1}\in\mathcal{O}\cap BwB\cap B^-$, with $w=s_{\gamma_1}$ and $\dim\mathcal{O}=\ell(w)+\mathrm{rk}(1-w)$. We conclude by Proposition 3.6. □

Proposition 3.44. *Let $G=G_2$. The spherical semisimple classes are represented by*

$$h_{\alpha_1}(-1),\qquad \textit{for } p=3,$$

$$h_{\alpha_1}(\zeta),\ \zeta \textit{ a primitive 3rd root of 1},\qquad \textit{for } p=2.$$

Proof. Let x be a semisimple element of G, and assume $\mathcal{O}=\mathcal{O}_x$ is spherical. Without loss of generality $C(x)=L_J$, a pseudo-Levi subgroup of G. There exists a semisimple element $\tilde{x}$ in $G_{\mathbb{C}}$ such that $C(\tilde{x})$ is L_J in $G_{\mathbb{C}}$. By Lemma 3.11, it follows that $\mathcal{O}_{\tilde{x}}$ is a spherical semisimple conjugacy class in $G_{\mathbb{C}}$, and therefore, from the classification of semisimple spherical conjugacy classes in zero (or good odd) characteristic [Cantarini et al. 2005, Table 2; Carnovale 2010, Theorem 3.1], it follows that L_J is of type $A_1\widetilde{A}_1$ or A_2. If $p=2$, then $Z(A_1\widetilde{A}_1)=1$. If $p=3$, then $Z(A_2)=1$ and we conclude by Proposition 3.43. □

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h_{\alpha_1}(\zeta)$ ζ a primitive 3rd root of 1	$\{2\}$	s_{γ_1}	A_2	6

Table 12. Spherical semisimple classes in G_2, $p = 2$.

$\mathcal{O}$	J	$w_{\mathcal{O}}$	$C(g)$	$\dim \mathcal{O}$
$h_{\alpha_1}(-1)$	$\varnothing$	w_0	$A_1\widetilde{A}_1$	8

Table 13. Spherical semisimple classes in G_2, $p = 3$.

Theorem 3.45. *Let $G = G_2$, $p = 2, 3$. The spherical classes are either semisimple or unipotent. The semisimple classes are represented in Tables 12 and 13, and the unipotent classes are represented in Tables 14 and 15 of* [Costantini 2012].

Proof. By the above discussion, we are left to show that no mixed class is spherical. Let $g = su$, the Jordan decomposition. Assume that $\mathcal{O}_g$ is spherical. Then both $\mathcal{O}_s$ and $\mathcal{O}_u$ are spherical, and therefore $C(s)$ is of type $A_1\widetilde{A}_1$ or A_2. A dimensional argument rules out all the possibilities. □

4. Final remarks

Once we have achieved the classification of spherical conjugacy classes and proved that for every spherical conjugacy class $\mathcal{O}$ we have $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$, we can extend to all characteristics the results obtained in [Carnovale 2008; 2009; Cantarini et al. 2005; Lu 2011] for the zero and good odd characteristic cases. In [Cantarini et al. 2005, Theorem 25] we established the characterization of spherical conjugacy classes in terms of the dimension formula: a conjugacy class $\mathcal{O}$ in G is spherical if and only if $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$. This was obtained over the complex numbers, and the same proof works over any algebraically closed field of characteristic zero. Then the same characterization was given in zero, or good odd characteristic in [Carnovale 2008, Theorem 4.4], without the classification of spherical conjugacy classes. Lu gave a very neat proof of the dimension formula (even for twisted conjugacy classes) in [Lu 2011, Theorem 1.1] in characteristic zero. From the results obtained in the previous section, we may state:

Theorem 4.1. *Let $\mathcal{O}$ be a conjugacy class of a simple algebraic group, any characteristic. The following are equivalent*:

(a) *$\mathcal{O}$ is spherical*;

(b) *There exists $w \in W$ such that $\mathcal{O} \cap BwB \neq \varnothing$ and* $\dim \mathcal{O} \leq \ell(w) + \mathrm{rk}(1 - w)$;

(c) $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$. □

Corollary 4.2. *Let $\mathcal{O}$ be a spherical class of G. Then* $\dim \mathcal{O} \le \ell(w_0) + \mathrm{rk}(1 - w_0)$.

Proof. We have $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$, and

$$\ell(w) + \mathrm{rk}(1 - w) \le \ell(w_0) + \mathrm{rk}(1 - w_0)$$

for every $w \in W$ (cf. [Carnovale 2008, Remark 4.14]). □

Proposition 4.3. *Let $\mathcal{O}$ be a spherical conjugacy class and $w = w_{\mathcal{O}} = w_0 w_J$. Then $(T^w)^\circ \le C_T(x) \le T^w$, $C_U(x) = U_{w_J}$ and $(T^w)^\circ U_{w_J} \le C_B(x) \le T^w U_{w_J}$ for every $x \in \mathcal{O} \cap wB$.*

Proof. We choose a representative $\dot{w}$ of w in N such that $x = \dot{w}u$ for $u \in U$. Let $b = tu_1u_2 \in C_B(x)$, where $t \in T$, $u_1 \in U_w$ and $u_2 \in U_{w_J}$. From the Bruhat decomposition, we get $u_1 = 1$ and $t \in T^w$, so that $C_B(x) \le T^w U_{w_J}$. But the dimension formula $\dim \mathcal{O} = \ell(w) + \mathrm{rk}(1 - w)$ implies

$$\dim C_B(x) = n - \mathrm{rk}(1 - w) + \ell(w_J) = \dim T^w U_{w_J}.$$

Hence $(C_B(x))^\circ = (T^w)^\circ U_{w_J}$ and $C_U(x) = U_{w_J}$.

Now assume $b = tu_1 \in C_{TU_w}(x)$, where $t \in T$ and $u_1 \in U_w$. Again from the Bruhat decomposition, we get $u_1 = 1$ and $t \in T^w$, so that $C_{TU_w}(x) = C_T(x) \le T^w$. We have $B.x = TU_wU_{w_J}.x = TU_w.x$, hence $\dim C_{TU_w}(x) = n - \mathrm{rk}(1 - w) = \dim T^w$. It follows that $(T^w)^\circ \le C_T(x) \le T^w$. □

Theorem 4.4. *Let $\mathcal{O}$ be a spherical conjugacy class of a simple algebraic group and $v = \mathcal{O} \cap Bw_{\mathcal{O}}B$ the dense B-orbit. Then $C_U(x)$ is connected and $C_B(x)$ is a split extension of $(C_B(x))^\circ$ by an elementary abelian 2-group for every $x \in v$. If $p = 2$, then $C_U(x)$, $C_T(x)$ and $C_B(x)$ are connected for every $x \in \mathcal{O} \cap Bw_{\mathcal{O}}B$.*

Proof. Let $w = w_{\mathcal{O}}$. We may assume $x \in wB$. From the discussion after [Costantini 2010, Corollary 3.22], we have $T = (T^w)^\circ (S^w)^\circ$, where $S^w = \{t \in T \mid t^w = t^{-1}\}$. Then $T^w = (T^w)^\circ (T^w \cap T_2)$, where $T_2 = \{t \in T \mid t^2 = 1\}$, and $C_T(x) = (T^w)^\circ C_{T_2}(x)$ by Proposition 4.3. There exists a subgroup R of T_2 such that $T^w = (T^w)^\circ \times R$, whence $C_T(x) = (T^w)^\circ \times C_R(x)$. In particular,

$$C_B(x) = ((T^w)^\circ \times C_R(x))U_{w_J} = (C_B(x))^\circ C_R(x).$$

If $p = 2$, then $T_2 = \{1\}$, $T^w = (T^w)^\circ = C_T(x)$ and $C_B(x) = (C_B(x))^\circ$. □

We recall, from Remark 3.3, that there is an action of W on the set $\mathcal{V}$ of B-orbits in $\mathcal{O}$ when $\mathcal{O}$ is a spherical conjugacy class and $p \ne 2$. We are now in the position to prove that this action is also defined for $p = 2$.

Corollary 4.5. *Let $\mathcal{O}$ be a spherical conjugacy class of a simple algebraic group, any characteristic. Then there is an action of the Weyl group W on the set of B-orbits in $\mathcal{O}$* (as defined in [Knop 1995]).

Proof. We have only to deal with $p=2$. By [Knop 1995, Theorem 4.2(c)], the action of W is defined on the set of B-orbits in $\mathcal{O}$ as long as $C_U(x)$ is connected for every $x \in \mathcal{O}$. By Theorem 4.4, $C_U(x)$ is connected for every x in the dense B-orbit; this ensures that $C_U(x)$ is connected for every $x \in \mathcal{O}$ by [Knop 1995, Corollary 3.4]. $\square$

Once the W-action has been defined when $p=2$, we can extend to this case the results obtained by G. Carnovale in zero or good odd characteristic.

Theorem 4.6. *Let $\mathcal{O}$ be a spherical conjugacy class of a simple algebraic group. If $\mathcal{O} \cap BwB$ is nonempty, then $w^2=1$.*

Corollary 4.7. *Let $\mathcal{O}$ be a spherical conjugacy class, and assume $\mathcal{O} \cap BwB \neq \varnothing$ for some $w \in W$. Then $\mathcal{O} \cap BzB \neq \varnothing$ for every conjugate z of w in W.*

Theorem 4.8. *Let $\mathcal{O}$ be a conjugacy class in a simple algebraic group. If*

$$\{w \in W \mid \mathcal{O} \cap BwB \neq \varnothing\} \subseteq \{w \in W \mid w^2 = 1\},$$

then $\mathcal{O}$ is spherical.

Assume $\mathcal{O}$ is a spherical conjugacy class of a simple algebraic group (any characteristic), and v the dense B-orbit in $\mathcal{O}$. Set $P=\{g \in G \mid g.v=v\}$. Then P is a parabolic subgroup of G containing B, and therefore $P=P_K$, the standard parabolic subgroup relative to a certain subset K of Π.

Theorem 4.9. *Let $\mathcal{O}$ be a spherical conjugacy class of a simple algebraic group, any characteristic, $w=w_0w_J$ be the unique element in W such that $\mathcal{O} \cap BwB$ is dense in $\mathcal{O}$, $v=\mathcal{O}\cap BwB$ the dense B-orbit in $\mathcal{O}$ and $P_K=\{g \in G \mid g.v=v\}$. Then $K=J$. If $x \in \mathcal{O} \cap wB$, then L'_J and $(T^w)^\circ$ are contained in $C(x)$ and $C_B(x)^\circ = (T^w)^\circ U_{w_J}$.*

Proof. We have already showed that $C_B(x)^\circ=(T^w)^\circ U_{w_J}$ for every $x \in \mathcal{O} \cap wB$. Let $S=\{i, \vartheta(i)\}$ be a ϑ-orbit in $\Pi \setminus J$ consisting of 2 elements. We define $H_S=\{h_{\alpha_i}(z)h_{\alpha_{\vartheta(i)}}(z^{-1}) \mid z \in k^*\}$. Let $\mathcal{S}_1$ be the set of ϑ-orbits in $\Pi \setminus J$ consisting of 2 elements. Then, by [Costantini 2010, Remark 3.10], $\Delta_J \cup \{\alpha_i - \alpha_{\vartheta(i)}\}_{\mathcal{S}_1}$ is a basis of $\ker(1-w)$ and

$$(T^w)^\circ = \prod_{j \in J} H_{\alpha_j} \times \prod_{S \in \mathcal{S}_1} H_S. \tag{4-1}$$

We put $\Psi_J=\{\beta \in \Phi \mid w(\beta)=-\beta\}$. Then Ψ_J is a root system in $\mathrm{Im}(1-w)$ [Springer 1982, Proposition 2], and $w|_{\mathrm{Im}(1-w)}$ is -1. If $K=C((T^w)^\circ)'$, then K is semisimple with root system Ψ_J and maximal torus $T \cap K=(S^w)^\circ$. Assume $x=\dot{w}u \in v$, with $u \in U$. Then $(T^w)^\circ \leq C(x)$ implies $x \in C((T^w)^\circ)$, and moreover, $\dot{w} \in C(T^w)$, so that $u \in K$. Let $u=\prod_{\alpha \in \Phi^+ \cap \Psi_J} x_\alpha(k_\alpha)$ be the expression of u for any fixed total ordering on Φ^+. If $k_\alpha \neq 0$, then $w(\alpha)=-\alpha$, so that in particular $u \in U_w$. Moreover, if $\beta \in \Phi_J$, then $(\alpha,\beta)=(w\alpha,w\beta)=(-\alpha,\beta)$, so that $\alpha \perp \beta$.

Finally, we have $\vartheta\alpha = -\alpha$, since $w\alpha = -\alpha$ is equivalent to $w_J\alpha = -w_0\alpha$, and $w_J\alpha = \alpha$, since $w_J \in W_J$ and $(\alpha, \alpha_j) = 0$ for every $j \in J$.

From the fact that $U_{w_J} \leq C(x)$, it follows that $U_{w_J} \leq C(\dot{w})$, and therefore $U_{w_J} \leq C(u)$. From the Chevalley commutator formula, we deduce further that $w_J U_{w_J} w_J^{-1} \leq C(u)$, so that $L'_J \leq C(x)$. Then we may argue as in the proof of [Carnovale 2008, Proposition 4.15] to conclude that $K = J$. □

Remark 4.10. Assume G is a connected reductive algebraic group over k. From the classification of spherical conjugacy classes obtained in simple algebraic groups (which is independent of the isogeny class), one gets the classification of spherical conjugacy classes in G. In fact, if $G = ZG_1 \cdots G_r$, where Z is the connected component of the center of G, and $G_1, \ldots, G_r$ are the simple components of G, then the conjugacy class $\mathcal{O}$ in G of $x = zx_1 \cdots x_r$, with $z \in Z$ and $x_i \in G_i$ for $i = 1, \ldots, r$, is spherical if and only if the conjugacy class $\mathcal{O}_i$ of x_i in G_i is spherical for every $i = 1, \ldots, r$.

Remark 4.11. In order to show that a conjugacy class $\mathcal{O}$ is spherical, we showed that $\dim \mathcal{O} = \ell(w_{\mathcal{O}}) + \mathrm{rk}(1 - w_{\mathcal{O}})$. However, in each case we even showed that $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^- \neq \varnothing$. The motivation for this was the proof of the De Concini–Kac–Procesi conjecture for quantum groups at roots of one over spherical conjugacy classes; see [Cantarini et al. 2005]. The fact that $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^- \neq \varnothing$ for every spherical conjugacy class has been proved in characteristic zero in [Cantarini et al. 2005]. It is a general fact that if $\mathcal{O}$ is semisimple, then $\mathcal{O} \cap BwB \neq \varnothing$ implies $\mathcal{O} \cap BwB \cap B^- \neq \varnothing$ for any $w \in W$ [Cantarini et al. 2005, Lemma 14]. For unipotent classes, we showed in [Costantini 2012] for $p = 2$ that $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^- \neq \varnothing$ by exhibiting explicitly an element in $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$. The argument in [Cantarini et al. 2005, Lemma 10] allows one to prove that $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^- \neq \varnothing$ for every spherical unipotent class in good characteristic. However, it is possible to adapt the same proof to the remaining unipotent classes in bad characteristic, due to the fact that we do have the classification, and so we just make a case by case consideration. Assume $\mathcal{O}$ is a spherical mixed class. In all cases, apart from F_4, we have an explicit element in $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^-$. We observed in Proposition 3.40 that the argument used in [Cantarini et al. 2005] holds for every odd characteristic. We conclude that in all characteristics, if $\mathcal{O}$ is a spherical conjugacy class, then $\mathcal{O} \cap Bw_{\mathcal{O}}B \cap B^- \neq \varnothing$.

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AFFINE WEAKLY REGULAR TENSOR TRIANGULATED CATEGORIES

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We prove that the Balmer spectrum of a tensor triangulated category is homeomorphic to the Zariski spectrum of its graded central ring, provided the triangulated category is generated by its tensor unit and the graded central ring is noetherian and regular in a weak sense. There follows a classification of all thick subcategories, and the result extends to the compactly generated setting to yield a classification of all localizing subcategories as well as the analog of the telescope conjecture. This generalizes results of Shamir for commutative ring spectra.

1. Introduction and results

Let $\mathcal{K}$ be an essentially small tensor triangulated category, with symmetric exact tensor product $\otimes$ and tensor unit object $\mathbf{1}$. Balmer [2005] defined a topological space, the *spectrum* $\operatorname{Spc}\mathcal{K}$, that allows for the development of a geometric theory of $\mathcal{K}$, similarly to how the Zariski spectrum captures the intrinsic geometry of commutative rings; see the survey [Balmer 2010b]. Among other uses, Balmer's spectrum encodes the classification of the thick tensor ideals of $\mathcal{K}$ in terms of certain subsets. It is therefore of interest to find an explicit description of the spectrum in the examples, but this is usually a difficult problem requiring some in-depth knowledge of each example at hand.

The goal of this note is to show that in some cases a concrete description of the spectrum can be obtained easily and completely formally. Let us denote by

$$R := \operatorname{End}^*_{\mathcal{K}}(\mathbf{1}) = \bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{K}}(\mathbf{1}, \Sigma^i\mathbf{1})$$

the graded endomorphism ring of the unit, where $\Sigma : \mathcal{K} \to \mathcal{K}$ is the suspension functor. In the terminology of [Balmer 2010a], this is the *graded central ring* of $\mathcal{K}$. It is a graded commutative ring and therefore we can consider its spectrum

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of homogeneous prime ideals, Spec R, equipped with the Zariski topology. As established in that paper, there is always a canonical continuous map

$$\rho : \operatorname{Spc} \mathcal{K} \to \operatorname{Spec} R$$

comparing the two spectra. Under some mild hypotheses, e.g., when R is noetherian, ρ can be shown to be surjective, but it is less frequently injective and, when it is, the proof of injectivity is typically much harder.

Here is our main result:

Theorem 1.1. *Assume that $\mathcal{K}$ satisfies the two following conditions*:

(a) *$\mathcal{K}$ is classically generated by $\mathbf{1}$, i.e., as a thick subcategory*: $\operatorname{Thick}(\mathbf{1}) = \mathcal{K}$.

(b) *R is a (graded) noetherian ring concentrated in even degrees and, for every homogeneous prime ideal $\mathfrak{p}$ of R, the maximal ideal of the local ring $R_{\mathfrak{p}}$ is generated by a (finite) regular sequence of homogeneous non-zero-divisors.*

Then the comparison map $\rho : \operatorname{Spc} \mathcal{K} \xrightarrow{\sim} \operatorname{Spec} R$ is a homeomorphism.

As in the title, we may refer to a tensor triangulated category $\mathcal{K}$ satisfying hypotheses (a) and (b) as being *affine* and *weakly regular*, respectively. Note that R being noetherian implies that $R^0 = \operatorname{End}_{\mathcal{K}}(\mathbf{1})$ is a noetherian ring and that R is a finitely generated R^0-algebra, by [Goto and Yamagishi 1983].

The next result is an easy consequence of the theorem. Here $\operatorname{Supp}_R H^*X$ denotes the (big) Zariski support of the cohomology graded R-module $H^*X := \operatorname{Hom}^*_{\mathcal{K}}(\mathbf{1}, X)$.

Corollary 1.2. *If $\mathcal{K}$ and R are as in the theorem, then there exists a canonical inclusion-preserving bijection*

$$\{\textit{thick subcategories } \mathcal{C} \textit{ of } \mathcal{K}\} \xrightleftharpoons{\sim} \{\textit{specialization closed subsets } V \textit{ of } \operatorname{Spec} R\}$$

mapping a thick subcategory $\mathcal{C}$ to $V = \bigcup_{X \in \mathcal{C}} \operatorname{Supp}_R H^*X$ and a specialization closed subset V to $\mathcal{C} = \{X \in \mathcal{K} \mid \operatorname{Supp}_R H^*X \subseteq V\}$.

In many natural examples, $\mathcal{K}$ occurs as the subcategory $\mathcal{T}^c$ of compact objects in a compactly generated tensor triangulated category $\mathcal{T}$. By the latter we mean a compactly generated triangulated category $\mathcal{T}$ equipped with a symmetric monoidal structure $\otimes$ which preserves coproducts and exact triangles in both variables, and such that the compact objects form a tensor subcategory $\mathcal{T}^c$ (that is, $\mathbf{1}$ is compact and the tensor product of two compact objects is again compact).

In this case, the same hypotheses allow us to classify also the localizing subcategories of $\mathcal{T}$, thanks to the stratification theory of compactly generated categories due to Benson, Iyengar and Krause [Benson et al. 2011]. The *support* $\operatorname{supp}_R X \subseteq \operatorname{Spec} R$ of an object $X \in \mathcal{T}$ is defined in [Benson et al. 2008], and can be described as the set

$$\operatorname{supp}_R X = \{\mathfrak{p} \in \operatorname{Spec} R \mid X \otimes K(\mathfrak{p}) \neq 0\},$$

where the *residue field object* $K(\mathfrak{p})$ of a prime ideal $\mathfrak{p}$ is an object whose cohomology is the graded residue field of R at $\mathfrak{p}$; see Section 3.

Theorem 1.3. *Let $\mathcal{T}$ be a compactly generated tensor triangulated category with compact objects $\mathcal{K} := \mathcal{T}^c$ and graded central ring R satisfying conditions (a) and (b). Then we have the following canonical inclusion-preserving bijection*:

$$\{\textit{localizing subcategories } \mathcal{L} \subseteq \mathcal{T}\} \overset{\sim}{\rightleftarrows} \{\textit{subsets } S \subseteq \operatorname{Spec} R\}.$$

The correspondence sends a localizing subcategory $\mathcal{L}$ to $S = \bigcup_{X \in \mathcal{L}} \operatorname{supp}_R X$, and an arbitrary subset S to $\mathcal{L} = \{X \in \mathcal{T} \mid \operatorname{supp}_R X \subseteq S\}$. Moreover, the bijection restricts to localizing subcategories $\mathcal{L} = \operatorname{Loc}(\mathcal{L} \cap \mathcal{K})$ which are generated by compact objects on the left and to specialization closed subsets $S = \bigcup_{\mathfrak{p} \in S} V(\mathfrak{p})$ on the right.

Note that here the affine condition (a) is equivalent to requiring that $\mathcal{T}$ is generated by $\mathbf{1}$ as a localizing subcategory. As $\operatorname{Supp}_R H^*X = \operatorname{supp}_R X$ for all compact objects $X \in \mathcal{K}$, one sees easily that in the compactly generated case Theorem 1.1 and Corollary 1.2 are also a consequence of Theorem 1.3.

The next corollary is another byproduct of stratification. Recall that a localizing subcategory $\mathcal{L} \subseteq \mathcal{T}$ is *smashing* if the inclusion functor $\mathcal{L} \hookrightarrow \mathcal{T}$ admits a coproduct-preserving right adjoint.

Corollary 1.4 (the telescope conjecture in the affine weakly regular case). *In the situation of Theorem 1.3, every smashing subcategory of $\mathcal{T}$ is generated by a set of compact objects of $\mathcal{T}$.*

A few special cases of our formal results had already been observed, such as when R is even periodic and of global dimension at most one; see [Dell'Ambrogio and Tabuada 2012]. We now consider some more concrete examples.

* * *

Example 1.5. Let A be a commutative dg algebra and $D(A)$ its derived category of dg modules. Then $D(A)$ is an affine compactly generated tensor triangulated category with respect to the standard tensor product $\otimes = \otimes^{\mathrm{L}}_A$, and $R = H^*A$ is the cohomology algebra of A; thus if the latter satisfies (b) all our results apply to $D(A)$. Actually, in this example we can improve our results a little by eliminating the hypothesis that R is even and that the elements of the regular sequences are non-zero-divisors:

Theorem 1.6. *Let A be a commutative dg algebra such that its graded cohomology ring $R = H^*A$ is noetherian and such that every local prime $\mathfrak{p}R_{\mathfrak{p}}$ is generated by a finite regular sequence. Then all the conclusions of Theorems 1.1 and 1.3 and of Corollaries 1.2 and 1.4 hold for $\mathcal{T} = D(A)$ and $\mathcal{K} = D(A)^c$.*

We can apply this, for instance, to a graded polynomial algebra with any choice of grading for the variables, seen as a strictly commutative formal dg algebra.

Example 1.7. Let A be a commutative S-algebra (a.k.a. commutative highly structured ring spectrum), and let $D(A)$ be its derived category. (This covers Example 1.5, as commutative dga's can be seen as commutative S-algebras.) Then $D(A)$ is an affine compactly generated tensor triangulated category, and $R=\pi_*A$ is the stable homotopy algebra of A; thus if the latter satisfies (b) all our results apply to $D(A)$. Shamir [2012] already treated this example under the additional hypothesis that π_*A has finite Krull dimension. Working with ∞-categories and E_∞-rings, Mathew [2015, Theorem 1.4] established the classification of thick subcategories as in Corollary 1.2 for the case when π_*A is even periodic and π_0A regular noetherian. Remarkably, in the special case of S-algebras defined over $\mathbb{Q}$, Mathew was also able to prove the classification of thick subcategories only assuming π_*A noetherian, i.e., without any regularity hypothesis; see [Mathew 2016, Theorem 1.4]. (Note however that, thanks to [Mandell 2012], in order to apply our own results we really only need an E_4-structure on a ring spectrum rather than a fully commutative E_∞-structure.)

The next two well-known examples show that neither hypothesis (a) nor (b) can be weakened with impunity.

Example 1.8. The derived category $\mathcal{T}=D(\mathbb{P}^1_k)$ of the projective line over a field k is an example where $R=\mathrm{End}^*(\mathbf{1})\simeq k$ certainly satisfies (b) but (a) does not hold. Indeed ρ can be identified with the structure map $\mathbb{P}^1_k\to\operatorname{Spec}k$ and is therefore far from injective in this case; see [Balmer 2010a, Remark 8.2].

Example 1.9. If $\mathcal{T}=D(A)$ is the derived category of a commutative (ungraded) ring A, Theorem 1.1 and the classification of thick subcategories always hold by a result of Thomason [1997] (see [Balmer 2010a, Proposition 8.1]); the classification of localizing subcategories and the telescope conjecture hold if A is noetherian by [Neeman 1992a]. On the other hand, Keller [1994] found examples of *nonnoetherian* rings A for which the two latter results fail.

In view of these examples, it would be interesting to know how far our weak regularity hypothesis (b) can be weakened in general. Would noetherian suffice?

2. Preliminaries

Let $\mathcal{K}$ be an essentially small tensor triangulated category.

For any two objects $X,Y\in\mathcal{K}$, consider the $\mathbb{Z}$-graded group $\mathrm{Hom}^*_{\mathcal{K}}(X,Y)=\bigoplus_{i\in\mathbb{Z}}\mathrm{Hom}_{\mathcal{K}}(X,\Sigma^iY)$. Recall that the symmetric tensor product of $\mathcal{K}$ canonically induces on $R:=\mathrm{Hom}^*_{\mathcal{K}}(\mathbf{1},\mathbf{1})$ the structure of a graded commutative[1] ring, and

[1] To be precise, graded commutativity means here that $fg=\epsilon^{|f||g|}gf$ for any two homogeneous elements $f\in\mathrm{Hom}_{\mathcal{K}}(\mathbf{1},\Sigma^{|f|}\mathbf{1})$ and $g\in\mathrm{Hom}_{\mathcal{K}}(\mathbf{1},\Sigma^{|g|}\mathbf{1})$, where $\epsilon\in R^0$ is a constant with $\epsilon^2=1$ induced by the symmetry isomorphism $\Sigma\mathbf{1}\otimes\Sigma\mathbf{1}\xrightarrow{\sim}\Sigma\mathbf{1}\otimes\Sigma\mathbf{1}$. In most cases we have $\epsilon=-1$, e.g., if $\mathcal{K}$ admits a symmetric monoidal model, but usually no extra difficulty arises by allowing the general case. Of course, this is immaterial for R even.

on each $\mathrm{Hom}^*_{\mathcal{K}}(X, Y)$ the structure of a (left and right) graded R-module. The composition of maps in $\mathcal{K}$ and the tensor functor $-\otimes-$ are (graded) bilinear for this action. See [Balmer 2010a, §3] for details.

Since we are using cohomological gradings, we write H^*X for the R-module $\mathrm{Hom}^*_{\mathcal{K}}(\mathbf{1}, X)$ and call it the *cohomology of X*.

Supports for graded modules. We denote by $\mathrm{Spec}\, R$ the Zariski spectrum of all homogeneous prime ideals in R. If M is an R-module (always understood to be graded) and $\mathfrak{p} \in \mathrm{Spec}\, R$, the graded localization of M at $\mathfrak{p}$ is the R-module $M_{\mathfrak{p}}$ obtained by inverting the action of all the *homogeneous* elements in $R \smallsetminus \mathfrak{p}$. The *big support* of M is the following subset of the spectrum:

$$\mathrm{Supp}_R\, M = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid M_{\mathfrak{p}} \neq 0\}.$$

Since our graded ring R is noetherian we also dispose of the *small support*, defined in terms of the indecomposable injective R-modules $E(R/\mathfrak{p})$:

$$\mathrm{supp}_R\, M = \{\mathfrak{p} \mid E(R/\mathfrak{p}) \text{ occurs in the minimal injective resolution of } M\}.$$

We recall from [Benson et al. 2008, §2] some well-known properties of supports. In general we have $\mathrm{supp}_R\, M \subseteq \mathrm{Supp}_R\, M$. If M is finitely generated, these two sets are equal and also coincide with the Zariski closed set $V(\mathrm{Ann}_R\, M)$. For a general M, $\mathrm{Supp}_R\, M$ is always *specialization closed*: if it contains any point $\mathfrak{p}$ then it must contain its closure $V(\mathfrak{p}) = \{\mathfrak{q} \mid \mathfrak{p} \subseteq \mathfrak{q}\}$. In fact $\mathrm{Supp}_R\, M$ is equal to the specialization closure of $\mathrm{supp}_R\, M$: $\mathrm{Supp}_R\, M = \bigcup_{\mathfrak{p} \in \mathrm{supp}_R M} V(\mathfrak{p})$. The small support plays a fundamental role in the Benson–Iyengar–Krause stratification theory, but in this note it will only appear implicitly.

The next lemma follows by a standard induction on the length of the objects.

Lemma 2.1. *If $\mathcal{K} = \mathrm{Thick}(\mathbf{1})$ is affine and R is noetherian, the graded R-module $\mathrm{Hom}^*_{\mathcal{K}}(X, Y)$ is finitely generated for all $X, Y \in \mathcal{K}$.* □

The comparison map of spectra. Recall from [Balmer 2005] that, as a set, the spectrum $\mathrm{Spc}\, \mathcal{K}$ is defined to be the collection of all proper thick subcategories $\mathcal{P} \subsetneq \mathcal{K}$ which are *prime tensor ideals*: $X \otimes Y \in \mathcal{P} \iff X \in \mathcal{P}$ or $Y \in \mathcal{P}$. For every $\mathcal{P} \in \mathrm{Spc}\, \mathcal{K}$, let $\rho_{\mathcal{K}}(\mathcal{P})$ denote the ideal of $\mathrm{Spec}\, R$ generated by the set of homogeneous elements $\{f : \mathbf{1} \to \Sigma^{|f|}\mathbf{1} \mid \mathrm{cone}(f) \notin \mathcal{P}\}$. By [Balmer 2010a, Theorem 5.3], the assignment $\mathcal{P} \mapsto \rho_{\mathcal{K}}(\mathcal{P})$ defines a continuous map $\rho_{\mathcal{K}} : \mathrm{Spc}\, \mathcal{K} \to \mathrm{Spec}\, R$, natural in $\mathcal{K}$. Moreover, the two spaces $\mathrm{Spc}\, \mathcal{K}$ and $\mathrm{Spec}\, R$ are *spectral* in the sense of Hochster [1969], and $\rho_{\mathcal{K}}$ is a *spectral map* in that the preimage of a compact open set is again compact.

Lemma 2.2. *If $\rho_{\mathcal{K}}$ is bijective then it is a homeomorphism.*

Proof. This is an immediate consequence of [Hochster 1969, Proposition 15], which says that for a spectral map of spectral topological spaces to be a homeomorphism it suffices that it is an order isomorphism for the specialization order of the two spaces. Recall that the specialization order is defined for the points of any topological space by $x \geq y \iff x \in \overline{\{y\}}$. Indeed $\rho := \rho_{\mathcal{K}}$ is inclusion reversing, $\mathcal{Q} \subseteq \mathcal{P} \iff \rho(\mathcal{Q}) \supseteq \rho(\mathcal{P})$, hence it maps the closure $\overline{\{\mathcal{P}\}} = \{\mathcal{Q} \mid \mathcal{Q} \subseteq \mathcal{P}\}$ in $\operatorname{Spc} \mathcal{K}$ of any point $\mathcal{P}$ to the Zariski closure $V(\rho(\mathcal{P})) = \{\mathfrak{q} \mid \mathfrak{q} \supseteq \rho(\mathcal{P})\}$ in $\operatorname{Spec} R$ of the corresponding point. $\square$

Central localization. For every prime ideal $\mathfrak{p}$ of the graded central ring R of $\mathcal{K}$, there exists by [Balmer 2010a, Theorem 3.6] a tensor triangulated category $\mathcal{K}_{\mathfrak{p}}$ having the same objects as $\mathcal{K}$ and such that its graded Hom modules are the localizations

$$\operatorname{Hom}^*_{\mathcal{K}_{\mathfrak{p}}}(X, Y) = \operatorname{Hom}^*_{\mathcal{K}}(X, Y)_{\mathfrak{p}}.$$

In particular the graded central ring of $\mathcal{K}_{\mathfrak{p}}$ is the local ring $R_{\mathfrak{p}}$. There is a canonical exact functor $q_{\mathfrak{p}} : \mathcal{K} \to \mathcal{K}_{\mathfrak{p}}$, which is in fact the Verdier quotient by the thick tensor ideal generated by $\{\operatorname{cone}(f) \in \mathcal{K} \mid f \in R \smallsetminus \mathfrak{p} \text{ homogeneous}\}$. For emphasis, we will sometimes write $X_{\mathfrak{p}}$ for $X = q_{\mathfrak{p}} X$ when considered as an object of $\mathcal{K}_{\mathfrak{p}}$.

Clearly if $\mathcal{K}$ is generated by $\mathbf{1}$ then $\mathcal{K}_{\mathfrak{p}}$ is generated by $\mathbf{1}_{\mathfrak{p}}$. Later we will use the fact that if a tensor triangulated category is generated by its unit then every thick subcategory is automatically a tensor ideal.

Let $\ell_{\mathfrak{p}} : R \to R_{\mathfrak{p}}$ denote the localization map between the graded central rings of the two categories. By [Balmer 2010a, Theorem 5.4], we have a pullback square of spaces

$$\begin{tikzcd} \operatorname{Spc}(\mathcal{K}_{\mathfrak{p}}) \arrow[r, hook, "\operatorname{Spc}(q_{\mathfrak{p}})"] \arrow[d, "\rho_{\mathcal{K}_{\mathfrak{p}}}"'] & \operatorname{Spc}(\mathcal{K}) \arrow[d, "\rho_{\mathcal{K}}"] \\ \operatorname{Spec}(R_{\mathfrak{p}}) \arrow[r, hook, "\operatorname{Spec}(\ell_{\mathfrak{p}})"] & \operatorname{Spec}(R) \end{tikzcd} \tag{2.3}$$

where the horizontal maps are injective.

Koszul objects. We adapt some convenient notation from [Benson et al. 2008]. For any object $X \in \mathcal{K}$ and homogeneous element $f \in R$, let $X/\!\!/f := \operatorname{cone}(f \cdot X)$ be any choice of mapping cone for the map $f \cdot X : \Sigma^{-|f|} X \to X$ given by the R-action. If $f_1, \ldots, f_n$ is a finite sequence of homogeneous elements, define recursively $X_0 := X$ and $X_i := X/\!\!/(f_1, \ldots, f_i) := (X/\!\!/(f_1, \ldots, f_{i-1}))/\!\!/f_i$ for $i \in \{1, \ldots, n\}$. Thus by construction we have exact triangles

$$\Sigma^{-|f_i|} X_{i-1} \xrightarrow{f_i \cdot X_{i-1}} X_{i-1} \longrightarrow X_i \longrightarrow \Sigma^{-|f_i|+1} X_{i-1}, \tag{2.4}$$

and moreover, since the tensor product is exact, we have isomorphisms

$$X/\!\!/(f_1,\ldots,f_i)\simeq X\otimes \mathbf{1}/\!\!/f_1\otimes\cdots\otimes\mathbf{1}/\!\!/f_i$$

for all $i\in\{1,\ldots,n\}$. In the following, we will perform this construction *inside the* $\mathfrak{p}$*-local category* $\mathcal{K}_\mathfrak{p}$.

We need the following triangular version of the Nakayama lemma, for $\mathcal{K}$ affine.

Lemma 2.5. *If $X\in\mathcal{K}_\mathfrak{p}$ is any object and $f_1,\ldots,f_n$ is a set of homogeneous generators for $\mathfrak{p}R_\mathfrak{p}$, then in $\mathcal{K}_\mathfrak{p}$ we have $X=0$ if and only if $X/\!\!/(f_1,\ldots,f_n)=0$.*

Proof. Since $\mathcal{K}$ and thus $\mathcal{K}_\mathfrak{p}$ are generated by their tensor unit, it suffices to show that $H^*X_\mathfrak{p}=0$ if and only if $H^*(X/\!\!/(f_1,\ldots,f_n))_\mathfrak{p}=0$, and the latter can be proved as in [Benson et al. 2008, Lemma 5.11 (3)]. We give the easy argument for completeness.

With the above notation, by taking cohomology $H^*=\mathrm{Hom}^*_{\mathcal{K}_\mathfrak{p}}(\mathbf{1}_\mathfrak{p},-)$ of the triangle (2.4) of $\mathcal{K}_\mathfrak{p}$ we obtain the long exact sequence of $R_\mathfrak{p}$-modules

$$\cdots\longrightarrow H^{*-|f_i|}X_{i-1}\xrightarrow{f_i}H^*X_{i-1}\longrightarrow H^*X_i\longrightarrow H^{*-|f_i|+1}X_{i-1}\longrightarrow\cdots,$$

where each module is finitely generated by Lemma 2.1. Since $f_i\in\mathfrak{p}$, if $H^*X_{i-1}\neq 0$ the first map in the sequence is not invertible by the Nakayama lemma, hence $H^*X_i\neq 0$. The evident recursion shows that $H^*X\neq 0$ implies $H^*X_n\neq 0$. The very same exact sequences also show that if $H^*X=0$ then $H^*X_n=0$. □

3. Thick subcategories

Assume from now on that $\mathcal{K}$ satisfies conditions (a) and (b) of Theorem 1.1.

Residue field objects. By hypothesis, for every prime ideal $\mathfrak{p}\in\operatorname{Spec}R$ there exists a regular sequence $f_1,\ldots,f_n$ of homogeneous non-zero-divisors of $R_\mathfrak{p}$ which generate the ideal $\mathfrak{p}R_\mathfrak{p}$. Choose one such sequence once and for all, and construct the associated Koszul object

$$K(\mathfrak{p}):=\mathbf{1}_\mathfrak{p}/\!\!/(f_1,\ldots,f_n)\simeq\mathbf{1}_\mathfrak{p}/\!\!/f_1\otimes\cdots\otimes\mathbf{1}_\mathfrak{p}/\!\!/f_n$$

in the $\mathfrak{p}$-local tensor triangulated category $\mathcal{K}_\mathfrak{p}$.

Lemma 3.1. *For every object $X\in\mathcal{K}_\mathfrak{p}$ and every $i\in\{1,\ldots,n\}$, each element f of the ideal $(f_1,\ldots,f_i)\subset R_\mathfrak{p}$ acts as zero on $X/\!\!/(f_1,\ldots,f_i)$, i.e., $f\cdot X/\!\!/(f_1,\ldots,f_i)=0$.*

Proof. Recall that, as an immediate consequence of the $R_\mathfrak{p}$-bilinearity of the composition in $\mathcal{K}_\mathfrak{p}$, the elements of $R_\mathfrak{p}$ acting as zero on an object Y form an ideal (coinciding with the annihilator of the $R_\mathfrak{p}$-module $\mathrm{Hom}^*_{\mathcal{K}_\mathfrak{p}}(Y,Y)$). Thanks to the isomorphism $X/\!\!/(f_1,\ldots,f_i)\simeq X\otimes\mathbf{1}_\mathfrak{p}/\!\!/f_1\otimes\cdots\otimes\mathbf{1}_\mathfrak{p}/\!\!/f_i$ and the $R_\mathfrak{p}$-linearity of the tensor product, it will therefore suffice to prove that f_i acts as zero on $\mathbf{1}_\mathfrak{p}/\!\!/f_i$. Consider the commutative diagram

$$\begin{tikzcd}
\Sigma^{-|f_i|}\mathbf{1}_\mathfrak{p} \arrow[r, "f_i"] & \mathbf{1}_\mathfrak{p} \arrow[r, "g"] \arrow[d, "f_i"'] \arrow[dr, "0"'] & \mathbf{1}_\mathfrak{p}/\!\!/f_i \arrow[r] \arrow[d, "f_i"] & \Sigma^{-|f_i|+1}\mathbf{1}_\mathfrak{p} \arrow[dl, dotted, "h"] \\
& \Sigma^{|f_i|}\mathbf{1}_\mathfrak{p} \arrow[r, "g"'] & \Sigma^{|f_i|}\mathbf{1}_\mathfrak{p}/\!\!/f_i &
\end{tikzcd}$$

where the top row is the exact triangle defining $\mathbf{1}_\mathfrak{p}/\!\!/f_i$. Being the composite of two consecutive maps in a triangle, gf_i is zero. Up to a suspension, this is also the diagonal map in the square. Hence $f_i \cdot \mathbf{1}_\mathfrak{p}/\!\!/f_i$ factors through a map h as pictured. Since R is even by hypothesis, we have that $R_\mathfrak{p}$ is even, and we claim that also

$$H^n(\mathbf{1}_\mathfrak{p}/\!\!/f_i) = 0 \quad \text{for all odd } n. \tag{3.2}$$

This implies $h = 0$ and therefore $f_i \cdot \mathbf{1}_\mathfrak{p}/\!\!/f_i = 0$, as required. To prove the claim, note that the defining triangle of $\mathbf{1}_\mathfrak{p}/\!\!/f_i$ induces the exact sequence

$$R_\mathfrak{p}^{n-|f_i|} \xrightarrow{f_i} R_\mathfrak{p}^n \longrightarrow H^n(\mathbf{1}_\mathfrak{p}/\!\!/f_i) \longrightarrow R_\mathfrak{p}^{n-|f_i|+1} \xrightarrow{f_i} R_\mathfrak{p}^{n+1},$$

where the first and last maps are injective by the hypothesis that f_i is a non-zero-divisor in $R_\mathfrak{p}$. Thus (3.2), and even $H^*(\mathbf{1}_\mathfrak{p}/\!\!/f_i) \simeq R_\mathfrak{p}/(f_i)$, follows immediately. □

Corollary 3.3. *$H^*(X \otimes K(\mathfrak{p}))$ is a graded $k(\mathfrak{p})$-vector space for every $X \in \mathcal{K}_\mathfrak{p}$.*

Proof. By Lemma 3.1 together with the R-linearity of the tensor product, each $f \in \mathfrak{p}R_\mathfrak{p}$ acts as zero on $X \otimes K(\mathfrak{p}) \simeq X \otimes \mathbf{1}_\mathfrak{p}/\!\!/(f_1, \ldots, f_n)$. Therefore all such f also act as zero on $H^*(X \otimes K(\mathfrak{p}))$ by the R-linearity of composition. □

Lemma 3.4. *There is an isomorphism $H^*(\mathbf{1}_\mathfrak{p}/\!\!/(f_1, \ldots, f_i)) \simeq R_\mathfrak{p}/(f_1, \ldots, f_i)$ of R-modules for all $i \in \{1, \ldots, n\}$. In particular $H^*K(\mathfrak{p})$ is isomorphic to the residue field $k(\mathfrak{p}) := R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.*

Proof. Write $C_0 = \mathbf{1}_\mathfrak{p}$ and $C_i := \mathbf{1}_\mathfrak{p}/\!\!/(f_1, \ldots, f_i)$ for short. Then $K(\mathfrak{p}) = C_n$, and for all $i \in \{1, \ldots, n\}$ we have exact triangles

$$\Sigma^{-|f_i|}C_{i-1} \xrightarrow{f_i \cdot C_{i-1}} C_{i-1} \longrightarrow C_i \longrightarrow \Sigma^{-|f_i|+1}C_{i-1}.$$

The claim follows by recursion on i. Indeed $H^*C_0 = R_\mathfrak{p}$, and assume that $H^*C_{i-1} \simeq R_\mathfrak{p}/(f_1, \ldots, f_{i-1})$. Then the above triangle induces an exact sequence

$$H^{*-|f_i|}C_{i-1} \xrightarrow{f_i} H^*C_{i-1} \longrightarrow H^*C_i \longrightarrow H^{*-|f_i|+1}C_{i-1} \xrightarrow{f_i} H^{*+1}C_{i-1},$$

where the first and last maps are injective because by hypothesis f_i is a non-zero-divisor in the ring $R_\mathfrak{p}/(f_1, \ldots, f_{i-1})$. We thus obtain a short exact sequence $0 \to f_i R_\mathfrak{p}/(f_1, \ldots, f_{i-1}) \to R_\mathfrak{p}/(f_1, \ldots, f_{i-1}) \to H^*C_i \to 0$, proving the claim for i. □

Remark 3.5. Of the weak regularity hypothesis (b), the proof of Lemma 3.4 only uses that $f_1, \ldots, f_n$ is a regular sequence, while the proof of Lemma 3.1 only uses that the f_i are non-zero-divisors in $R_{\mathfrak{p}}$ and that the ring R is even. These are the only places where we make use of these assumptions (the noetherian hypothesis, on the other hand, will be needed on several occasions). Note that, although we already know by Corollary 3.3 that $H^*K(\mathfrak{p})$ is a $k(\mathfrak{p})$-vector space, for the next proposition we also need it to be one-dimensional as per Lemma 3.4.

Proposition 3.6. *For all $\mathfrak{p} \in \operatorname{Spec} R$ and $X \in \mathcal{K}_{\mathfrak{p}}$, the tensor product $X \otimes K(\mathfrak{p})$ decomposes into a coproduct of shifted copies of the residue field object*:

$$\coprod_{\alpha} \Sigma^{n_\alpha} K(\mathfrak{p}) \xrightarrow{\sim} X \otimes K(\mathfrak{p}).$$

Proof. By Corollary 3.3 we know that $H^*(X \otimes K(\mathfrak{p}))$ is a graded $k(\mathfrak{p})$-vector space. Choose a graded basis $\{x_\alpha\}_\alpha$, corresponding to a morphism $\coprod_\alpha \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} \to X \otimes K(\mathfrak{p})$. We will show that this map extends nontrivially to the Koszul object

$$\Big(\coprod_{\alpha} \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}}\Big) /\!\!/ (f_1, \ldots, f_n) = \coprod_{\alpha} \big(\Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} /\!\!/ (f_1, \ldots, f_n)\big).$$

For this, it will suffice to extend each individual map $x_\alpha : \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} \to X \otimes K(\mathfrak{p})$. As before, we proceed recursively along the regular sequence $f_1, \ldots, f_n$. Consider the commutative diagram

$$\begin{tikzcd}
\Sigma^{n_\alpha - |f_1|} \mathbf{1}_{\mathfrak{p}} \arrow[r, "f_1"] \arrow[d, "\Sigma^{-|f_1|} x_\alpha"'] \arrow[dr, "0"] & \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} \arrow[r] \arrow[d, "x_\alpha"] & \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} /\!\!/ f_1 \arrow[r] \arrow[dl, dotted, "x_\alpha^1"] & {} \\
\Sigma^{-|f_1|} X \otimes K(\mathfrak{p}) \arrow[r, "f_1 = 0"'] & X \otimes K(\mathfrak{p}) & &
\end{tikzcd}$$

where the top row is a rotation of the defining triangle for $\mathbf{1}_{\mathfrak{p}} /\!\!/ f_1$. The left-bottom composite vanishes because f_1 acts trivially on $X \otimes K(\mathfrak{p})$ by Lemma 3.1. Hence we obtain the map x_α^1 on the right. Note that $x_\alpha^1 \neq 0$ because $x_\alpha \neq 0$. Now we repeat the procedure for $i = 2, \ldots, n$, using the triangle

$$\Sigma^{-|f_i|} \mathbf{1}_{\mathfrak{p}} /\!\!/ (f_1, \ldots, f_{i-1}) \xrightarrow{f_i} \mathbf{1}_{\mathfrak{p}} /\!\!/ (f_1, \ldots, f_{i-1}) \longrightarrow \mathbf{1}_{\mathfrak{p}} /\!\!/ (f_1, \ldots, f_i) \longrightarrow$$

to extend x_α^{i-1} to a nonzero map $x_\alpha^i : \Sigma^{n_\alpha} \mathbf{1}_{\mathfrak{p}} /\!\!/ (f_1, \ldots, f_i) \to X \otimes K(\mathfrak{p})$ hitting the same element in cohomology. In particular we obtain the announced extension $x_\alpha^n : \Sigma^{n_\alpha} K(\mathfrak{p}) \to X \otimes K(\mathfrak{p})$. As a nonzero map on a one-dimensional $k(\mathfrak{p})$-vector space (Lemma 3.4), the induced map $H^*(x_\alpha^n)$ must be injective. Hence, collectively, the maps $\{x_\alpha^n\}_\alpha$ yield an isomorphism as required. $\square$

Proposition 3.7. *For every $\mathfrak{p}$, the thick subcategory* Thick$(K(\mathfrak{p}))$ *of $\mathcal{K}_{\mathfrak{p}}$ is* minimal, *meaning that it contains no proper nonzero thick subcategories.*

Proof. Note that for every nonzero object X of $\mathcal{K}_\mathfrak{p}$ we have $X \otimes K(\mathfrak{p}) \neq 0$. Indeed if $X \otimes K(\mathfrak{p}) = X/\!\!/(f_1, \dots, f_n) = 0$ then $X_\mathfrak{p} = 0$ by Lemma 2.5.

Let $\mathcal{C}$ be a thick subcategory of $\mathrm{Thick}(K(\mathfrak{p}))$. Because $\mathcal{C}$ is a tensor ideal, if it contains a nonzero object X then it also contains $X \otimes K(\mathfrak{p})$, which is again nonzero by the above observation. Therefore $\mathcal{C}$ must contain a shifted copy of $K(\mathfrak{p})$ by Proposition 3.6, hence $\mathcal{C} = \mathrm{Thick}(K(\mathfrak{p}))$. □

Proof of Theorem 1.1. Now we show how to deduce our main result from the minimality of the thick subcategories $\mathrm{Thick}(K(\mathfrak{p}))$. By Lemma 2.2 it will suffice to show that the map $\rho_\mathcal{K} : \mathrm{Spc}\,\mathcal{K} \to \mathrm{Spec}\,R$ is bijective. Since R is graded noetherian, $\rho_\mathcal{K}$ is surjective by [Balmer 2010a, Theorem 7.3]. It remains to prove it is injective.

Let $\mathfrak{p} \in \mathrm{Spec}\,R$ be any homogeneous prime. We must show that the fiber of the comparison map $\rho_\mathcal{K} : \mathrm{Spc}\,\mathcal{K} \to \mathrm{Spec}\,R$ over $\mathfrak{p}$ consists of a single prime tensor ideal. By the pullback square (2.3), every point of $\mathrm{Spc}\,\mathcal{K}$ lying over $\mathfrak{p}$ must belong to $\mathrm{Spc}\,\mathcal{K}_\mathfrak{p}$. Hence it will suffice to show that the fiber of $\rho := \rho_{\mathcal{K}_\mathfrak{p}}$ over the maximal ideal $\mathfrak{m} := \mathfrak{p}R_\mathfrak{p}$ of $R_\mathfrak{p}$ consists of a single point. In fact a stronger statement is true: if $\mathcal{P} \in \mathrm{Spc}\,\mathcal{K}_\mathfrak{p}$ is such that $\rho(\mathcal{P}) = \mathfrak{m}$, then $\mathcal{P} = \{0\}$. Let us prove this.

By definition of the comparison map we have

$$\rho(\mathcal{P}) = \langle \{ f \in R_\mathfrak{p} \mid f \text{ is homogeneous and } \mathbf{1}_\mathfrak{p}/\!\!/f \notin \mathcal{P} \} \rangle,$$

and as $\rho(\mathcal{P}) \subseteq \mathfrak{m}$ always holds by the maximality of $\mathfrak{m}$, the hypothesis $\rho(\mathcal{P}) = \mathfrak{m}$ precisely means that $\mathbf{1}_\mathfrak{p}/\!\!/f \notin \mathcal{P}$ for all homogeneous elements $f \in \mathfrak{m}$. In particular $\mathbf{1}_\mathfrak{p}/\!\!/f_i \notin \mathcal{P}$ for the elements f_i in the chosen regular sequence for $\mathfrak{m}$. As $\mathcal{P}$ is a tensor prime, we deduce further that

$$K(\mathfrak{p}) \simeq \mathbf{1}_\mathfrak{p}/\!\!/f_1 \otimes \cdots \otimes \mathbf{1}_\mathfrak{p}/\!\!/f_n \notin \mathcal{P}. \tag{3.8}$$

Now let $X \in \mathcal{P}$ and assume that $X \neq 0$. Then $X \otimes K(\mathfrak{p}) \neq 0$ by Lemma 2.5, hence

$$\mathrm{Thick}(X \otimes K(\mathfrak{p})) = \mathrm{Thick}(K(\mathfrak{p})) \tag{3.9}$$

by the minimality of $\mathrm{Thick}(K(\mathfrak{p}))$, Proposition 3.7. As $\mathcal{P}$ is a thick tensor ideal we also have $X \otimes K(\mathfrak{p}) \in \mathcal{P}$ and therefore $K(\mathfrak{p}) \in \mathcal{P}$ by (3.9), but this contradicts (3.8). Therefore $X = 0$ and we conclude that $\mathcal{P} = \{0\}$, proving the claim. This concludes the proof of Theorem 1.1.

Proof of Corollary 1.2. To deduce Corollary 1.2 from the theorem, we must verify that the homeomorphism $\rho_\mathcal{K}$ identifies $\mathrm{Supp}_R\, H^*X \subseteq \mathrm{Spec}\,R$, the ring-theoretic support of an object $X \in \mathcal{K}$, with $\mathrm{supp}\,X := \{\mathcal{P} \in \mathrm{Spc}\,\mathcal{K} \mid X \notin \mathcal{P}\}$, the universal support datum of X:

Lemma 3.10. *We have* $\mathrm{Supp}_R\, H^*X = \rho_\mathcal{K}(\mathrm{supp}\,X)$ *for all* $X \in \mathcal{K}$.

Proof. Let $\mathfrak{p} = \rho_{\mathcal{K}}(\mathcal{P})$. It follows from (2.3) that $X \in \mathcal{P}$ if and only if $X_{\mathfrak{p}} \in \mathcal{P}_{\mathfrak{p}}$, where $\mathcal{P}_{\mathfrak{p}}$ denotes $\mathcal{P}$ seen as an element of $\operatorname{Spc}\mathcal{K}_{\mathfrak{p}}$. We have just proved that $\rho_{\mathcal{K}_{\mathfrak{p}}} : \operatorname{Spc}\mathcal{K}_{\mathfrak{p}} \xrightarrow{\sim} R_{\mathfrak{p}}$ is a bijection sending $\{0\}$ to $\mathfrak{p}R_{\mathfrak{p}}$, so we must have $\mathcal{P}_{\mathfrak{p}} = \{0\}$. Therefore

$$\mathfrak{p} \in \operatorname{Supp}_R H^*X \iff H^*X_{\mathfrak{p}} \neq 0 \iff X_{\mathfrak{p}} \neq 0 \iff X_{\mathfrak{p}} \notin \mathcal{P}_{\mathfrak{p}} \iff \mathcal{P} \in \operatorname{supp} X. \quad \square$$

Now it suffices to appeal to the abstract classification theorem [Balmer 2005, Theorem 4.10]. Indeed, since R is noetherian, the space $\operatorname{Spec} R$ is noetherian and therefore its specialization closed subsets and its Thomason subsets coincide (cf. [Balmer 2005, Remark 4.11]). Moreover, since $\mathcal{K}$ is generated by its tensor unit, all its objects are dualizable (because dualizable objects form a thick subcategory and $\mathbf{1}$ is dualizable) and therefore all its thick tensor ideals are radical (see [Balmer 2007, Proposition 2.4]). Hence by Theorem 1.1 and Lemma 3.10 the classification of [Balmer 2005, Theorem 4.10] immediately translates into the classification described in Corollary 1.2, as wished.

4. Localizing subcategories

Assume from now on that $\mathcal{T}$ is a compactly generated tensor triangulated category such that its subcategory $\mathcal{K} := \mathcal{T}^c$ of compact objects satisfies hypotheses (a) and (b) of Theorem 1.1. Thus in particular $\mathcal{T}$ is generated as a localizing subcategory by the tensor unit: $\operatorname{Loc}(\mathbf{1}) = \mathcal{T}$. It follows that every localizing subcategory of $\mathcal{T}$ is automatically a tensor ideal.

Since $\mathcal{T}$ is compactly generated, the (Verdier) $\mathfrak{p}$-localization functor $q_{\mathfrak{p}} : \mathcal{K} \to \mathcal{K}_{\mathfrak{p}}$ we used so far can be extended to a finite (Bousfield) localization functor

$$(-)_{\mathfrak{p}} : \mathcal{T} \to \mathcal{T}.$$

We briefly recall its properties, referring for all proofs to [Benson et al. 2011, §2] or [Dell'Ambrogio 2010, §2]. Let

$$\mathcal{L} = \operatorname{Loc}(\{\operatorname{cone}(f) \mid f \in R \smallsetminus \mathfrak{p} \text{ homogeneous}\}).$$

Then the Verdier quotient $Q : \mathcal{T} \to \mathcal{T}/\mathcal{L} =: \mathcal{T}_{\mathfrak{p}}$ has a fully faithful right adjoint, $I : \mathcal{T}_{\mathfrak{p}} \hookrightarrow \mathcal{T}$, and the functor $(-)_{\mathfrak{p}}$ can be defined to be the composite $(-)_{\mathfrak{p}} := I \circ Q$. As $\mathcal{L}$ is generated by a tensor ideal of dualizable objects, we have $X_{\mathfrak{p}} \cong X \otimes \mathbf{1}_{\mathfrak{p}}$ for all $X \in \mathcal{T}$. Moreover, the unit $X \to X_{\mathfrak{p}}$ of the (Q, I)-adjunction induces a natural map $\operatorname{Hom}^*_{\mathcal{T}}(Y, X)_{\mathfrak{p}} \to \operatorname{Hom}^*_{\mathcal{T}}(Y, X_{\mathfrak{p}})$ which is an isomorphism whenever $Y \in \mathcal{K}$ (see [Benson et al. 2011, Proposition 2.3] or [Dell'Ambrogio 2010, Theorem 2.33 (h)]). In particular we have the identification

$$(H^*X)_{\mathfrak{p}} \xrightarrow{\sim} H^*(X_{\mathfrak{p}})$$

for all $X \in \mathcal{T}$. It follows also that the restriction of Q to compact objects $X, Y \in \mathcal{K}$ agrees with $q_\mathfrak{p}$, so that we may identify $\mathcal{K}_\mathfrak{p}$ with the full subcategory $I(\mathcal{K}_\mathfrak{p})$ of $\mathcal{T}$ (and thereby eliminate the slight ambiguity of the notation "$X_\mathfrak{p}$").

Recall the residue field objects $K(\mathfrak{p})$ defined in Section 3:

$$K(\mathfrak{p}) := \mathbf{1}_\mathfrak{p}/\!\!/(f_1, \dots, f_n) \simeq \mathbf{1}_\mathfrak{p}/\!\!/f_1 \otimes \cdots \otimes \mathbf{1}_\mathfrak{p}/\!\!/f_n \in \mathcal{T}$$

(as before, $f_1, \dots, f_n$ denotes the chosen regular sequence of non-zero-divisors generating the prime $\mathfrak{p}$).

The main point of this section is that the crucial minimality result of Proposition 3.7 can be extended to localizing subcategories of $\mathcal{T}$, as we verify next.

Lemma 4.1. *For every object $X \in \mathcal{T}$ and every $i \in \{1, \dots, n\}$, each element f of $(f_1, \dots, f_i) \subset R$ acts as zero on $X_\mathfrak{p}/\!\!/(f_1, \dots, f_i)$, i.e., $f \cdot X_\mathfrak{p}/\!\!/(f_1, \dots, f_i) = 0$. In particular, the R-module $H^*(X \otimes K(\mathfrak{p}))$ is a graded $k(\mathfrak{p})$-vector space.*

Proof. Exactly the same as for Lemma 3.1 and Corollary 3.3. (Use that $X \otimes K(\mathfrak{p}) = X_\mathfrak{p} \otimes K(\mathfrak{p})$ to work inside the big $\mathfrak{p}$-local category $\mathcal{T}_\mathfrak{p}$.) □

Proposition 4.2. *For all $\mathfrak{p} \in \operatorname{Spec} R$ and $X \in \mathcal{T}$, the tensor product $X \otimes K(\mathfrak{p})$ decomposes into a coproduct of shifted copies of the residue field object:*

$$\coprod_\alpha \Sigma^{n_\alpha} K(\mathfrak{p}) \xrightarrow{\sim} X \otimes K(\mathfrak{p}).$$

Proof. Exactly the same as for Proposition 3.6, using Lemma 4.1. □

Proposition 4.3. *For every $\mathfrak{p}$, the localizing subcategory $\operatorname{Loc}(K(\mathfrak{p}))$ of $\mathcal{T}$ is minimal, meaning that it contains no proper nonzero localizing subcategories.*

Proof. This follows from Proposition 4.2 precisely as in the proof of Proposition 3.7, except that we cannot use Lemma 2.5 to show that $X \otimes K(\mathfrak{p}) \neq 0$ for every nonzero object $X \in \operatorname{Loc}(K(\mathfrak{p}))$. Instead, we may use the following argument.

First note that $X \otimes K(\mathfrak{q}) = 0$ for all $\mathfrak{q} \in \operatorname{Spec} R \smallsetminus \{\mathfrak{p}\}$. Indeed, this property holds for $X = K(\mathfrak{p})$ by Lemma 4.1 (because if $\mathfrak{p} \neq \mathfrak{q}$ then some homogeneous element of R must act on $K(\mathfrak{p}) \otimes K(\mathfrak{q})$ both as zero and invertibly) and is stable under taking coproducts and mapping cones (as the latter are preserved by $- \otimes K(\mathfrak{p})$); hence it must hold for all objects of $\operatorname{Loc}(K(\mathfrak{p}))$, as wished. Now combine this with Proposition 4.5 below. □

Lemma 4.4. *Let M be any nonzero module, possibly infinitely generated, over a noetherian $\mathbb{Z}$-graded commutative ring S. Then there exists a minimal prime in $\operatorname{Supp}_S M := \{\mathfrak{p} \in \operatorname{Spec} S \mid M_\mathfrak{p} \neq 0\}$, the big Zariski support of M.*

Proof. If $M \neq 0$ then $M_\mathfrak{p} \neq 0$ for some prime $\mathfrak{p}$, so the support is not empty. Moreover, it suffices to prove the claim for the nonzero module $M_\mathfrak{p}$ over $S_\mathfrak{p}$, because a minimal prime of $\operatorname{Supp}_{S_\mathfrak{p}} M_\mathfrak{p}$ yields a minimal prime in $\operatorname{Supp}_S M$; hence we may assume that S is local. By Zorn's lemma it suffices to show that in $\operatorname{Supp}_S M$ every chain of primes admits a minimum. Indeed, each such chain must stabilize, because a local commutative noetherian ring has finite Krull dimension. In the ungraded case, the latter is a well-known corollary of Krull's principal ideal theorem. A proof of the analogous result for graded rings can be found in [Bruns and Herzog 1993, Theorem 1.5.8] or [Park and Park 2011, Theorem 3.5]. □

Proposition 4.5. *If an object $X \in \mathcal{T}$ is such that $X \otimes K(\mathfrak{p}) = 0$ for all $\mathfrak{p} \in \operatorname{Spec} R$ then $X = 0$.*

Proof. We prove the contrapositive. Assume that $X \neq 0$. Then $H^*X \neq 0$, hence for some $\mathfrak{p} \in \operatorname{Spec} R$ we must have $H^*(X_\mathfrak{p}) = (H^*X)_\mathfrak{p} \neq 0$ and therefore $X_\mathfrak{p} \neq 0$. By Lemma 4.4, we may choose a prime $\mathfrak{p}$ which is minimal among the primes with this property. Thus the big support of the R-module $H^*X_\mathfrak{p}$ consists precisely of the prime $\mathfrak{p}$. We are going to recursively show that $X_i := X_\mathfrak{p}/\!\!/(f_1, \ldots, f_i)$ satisfies $\operatorname{Supp}_R H^*X_i = \{\mathfrak{p}\}$ for all $i \in \{1, \ldots, n\}$. Thus in particular $X \otimes K(\mathfrak{p}) = X_n \neq 0$, which proves the proposition. We already know $\operatorname{Supp}_R H^*X_0 = \{\mathfrak{p}\}$ for $X_0 := X_\mathfrak{p}$, and suppose we have shown that $\operatorname{Supp}_R H^*X_{i-1} = \{\mathfrak{p}\}$. The exact triangle

$$\Sigma^{-|f_i|}X_{i-1} \xrightarrow{f_i} X_{i-1} \longrightarrow X_i \longrightarrow \Sigma^{-|f_i|+1}X_{i-1}$$

implies that $\operatorname{Supp}_R H^*X_i \subseteq \{\mathfrak{p}\}$. Hence $X_i \neq 0$ is equivalent to $\operatorname{Supp}_R H^*X_i = \{\mathfrak{p}\}$. By the triangle again, if $X_i = 0$ were the case f_i would act invertibly on X_{i-1} and thus on H^*X_{i-1}. This implies $H^*X_{i-1} = (H^*X_{i-1})[f_i^{-1}]$, and since $f_i \in \mathfrak{p}$ we would conclude that $\mathfrak{p} \notin \operatorname{Supp}_R H^*X_{i-1}$, in contradiction with the induction hypothesis. Therefore $X_i \neq 0$, as claimed. □

Proof of Theorem 1.3. The result now follows easily from the machinery developed by Benson, Iyengar and Krause [Benson et al. 2008; 2011]. Indeed, by [Benson et al. 2011, Theorem 4.2], to obtain the claimed classification of localizing subcategories it suffices to verify that *the action of R stratifies $\mathcal{T}$*. By definition, this means that the following two axioms are satisfied:

- *The local-global principle:* For every object $X \in \mathcal{T}$ we have the equality

$$\operatorname{Loc}(X) = \operatorname{Loc}(\{\Gamma_\mathfrak{p} X \mid \mathfrak{p} \in \operatorname{Spec} R\})$$

  of localizing subcategories of $\mathcal{T}$.

- *Minimality:* For every $\mathfrak{p} \in \operatorname{Spec} R$ the localizing subcategory $\Gamma_\mathfrak{p}\mathcal{T}$ of $\mathcal{T}$ is minimal or zero.

The functors $\Gamma_\mathfrak{p} : \mathcal{T} \to \mathcal{T}$ are introduced in [Benson et al. 2008], but we don't need to know how they are defined. In our context, i.e., where $\mathcal{T}$ is a tensor category and the action of R is the canonical one of the central ring, the local-global principle always holds by [Benson et al. 2011, Theorem 7.2] (see also [Stevenson 2013, Theorem 6.8]). Moreover $\Gamma_\mathfrak{p} X = X \otimes \Gamma_\mathfrak{p}\mathbf{1}$ for all $X \in \mathcal{T}$, which implies that $\Gamma_\mathfrak{p}\mathcal{T} = \mathrm{Loc}(\Gamma_\mathfrak{p}\mathbf{1})$ since $\mathcal{T}$ is generated by $\mathbf{1}$. Therefore the remaining minimality condition follows from Proposition 4.3, because $\mathrm{Loc}(K(\mathfrak{p})) = \mathrm{Loc}(\Gamma_\mathfrak{p}\mathbf{1})$ by [Benson et al. 2011, Lemma 3.8 (2)] (indeed, by construction $K(\mathfrak{p})$ is a particular instance of the objects collectively denoted by $\mathbf{1}(\mathfrak{p})$ in [loc. cit.]). This establishes the first bijection in Theorem 1.3.

The claimed identification of the Benson–Iyengar–Krause support, $\mathrm{supp}_R X = \{\mathfrak{p} \in \mathrm{Spec}\, R \mid X \otimes \Gamma_\mathfrak{p}\mathbf{1} \neq 0\}$, with the set $\{\mathfrak{p} \in \mathrm{Spec}\, R \mid X \otimes K(\mathfrak{p}) \neq 0\}$ is an easy consequence of the equality $\mathrm{Loc}(K(\mathfrak{p})) = \mathrm{Loc}(\Gamma_\mathfrak{p}\mathbf{1})$ mentioned above.

It remains to verify the moreover part of Theorem 1.3. Let us begin by noting that, if $X \in \mathcal{K}$ is a compact object, we have

$$\mathrm{supp}_R X = \mathrm{supp}_R H^*X = \mathrm{Supp}_R H^*X \tag{4.6}$$

by [Benson et al. 2008, Theorem 5.5 (1)] and Lemma 2.1. Now let $\mathcal{L} \subseteq \mathcal{T}$ be such that $\mathcal{L} = \mathrm{Loc}(\mathcal{L} \cap \mathcal{K})$. Then

$$\bigcup_{X \in \mathcal{L}} \mathrm{supp}_R X = \bigcup_{X \in \mathcal{L}\cap\mathcal{K}} \mathrm{supp}_R X = \bigcup_{X \in \mathcal{L}\cap\mathcal{K}} \mathrm{Supp}_R H^*X$$

by (4.6), and the latter is a specialization closed subset of the spectrum. Conversely, if $S \subseteq \mathrm{Spec}\, R$ is specialization closed the corresponding localizing subcategory $\{X \in \mathcal{T} \mid \mathrm{supp}_R X \subseteq S\}$ is generated by compact objects by [Benson et al. 2008, Theorem 6.4], hence $\mathcal{L} = \mathrm{Loc}(\mathcal{L} \cap \mathcal{K})$. This concludes the proof of the theorem.

It is well known that the assignments $\mathcal{C} \mapsto \mathrm{Loc}(\mathcal{C})$ and $\mathcal{L} \mapsto \mathcal{L} \cap \mathcal{K}$ are mutually inverse bijections between thick subcategories $\mathcal{C} \subseteq \mathcal{K}$ and localizing subcategories $\mathcal{L} \subseteq \mathcal{T}$ which are generated by compact objects of $\mathcal{T}$ (see [Neeman 1992b]). Together with (4.6), this shows how to deduce the classification of thick subcategories of Corollary 1.2 from Theorem 1.3.

Finally, there are several ways to derive the telescope conjecture of Corollary 1.4 from the previous results. For instance, we may proceed as in [Benson et al. 2011, §6.2].

Remark 4.7. Using the theory of coherent functors, Benson, Iyengar and Krause have recently developed in [Benson et al. 2015] an analogue of their stratification theory of compactly generated categories that can be applied to general essentially small triangulated categories. Their theory, and more specifically [Benson et al. 2015, Theorem 7.4], provides an alternative way to derive Theorem 1.1 from Proposition 3.7.

The case of commutative dg algebras. We still owe readers a proof of Theorem 1.6. Let A be a commutative dg algebra and let $D(A)$ be the derived category of (left, say) dg-A-modules. The following elementary fact was pointed out to us by the referee.

Lemma 4.8. *Every $f \in H^*A$ acts as zero on its own mapping cone $C(f)$.*

Proof. A (homogeneous) element $f \in H^*A$ of degree $|f| = -n$ is (represented by) a morphism $f : \Sigma^n A \to A$ of left dg-A-modules. Let us write $\mathrm{s}a$ ($a \in A$) for a generic element of degree $|a| - 1$ in the suspension $\mathrm{s}A := \Sigma A$; here we use the Koszul sign convention and treat s as a symbol of degree -1. The cone $C(f)$ has elements $(a, \mathrm{s}^{n+1}b)$ (for $a, b \in A$). Then f acts on $C(f)$ by a morphism $\mathrm{s}^n C(f) \to C(f)$ which, under the isomorphism $\mathrm{s}^n C(f) \cong C(\mathrm{s}^n f)$, is written as follows:

$$g : C(\mathrm{s}^n f) \to C(f), \quad g(\mathrm{s}^n a, \mathrm{s}^{2n+1} b) = (f(\mathrm{s}^n a), \mathrm{s}^{n+1} f(\mathrm{s}^n b))$$

(recall that the suspension $\mathrm{s}h : \mathrm{s}B \to \mathrm{s}C$ of a morphism $h : B \to C$ is given by $(\mathrm{s}h)(\mathrm{s}b) = \mathrm{s}(h(b))$). With these notations, the map $H : C(\mathrm{s}^n f) \to C(f)$ defined by $H(\mathrm{s}^n a, \mathrm{s}^{2n+1} b) := (0, \mathrm{s}^{n+1} a)$ is easily seen to satisfy $H(tx) = (-1)^{|t|} t H(x)$ (for $t \in A$, $x \in C(\mathrm{s}^n f)$) and $dH + Hd = -g$; in other words, H is a homotopy $g \sim 0$ defined over A. □

As noted in Remark 3.5, Lemma 3.1 was the only place in all of our arguments where we made use of the hypothesis that R is concentrated in even degrees and that in the regular sequences we may choose the elements to be non-zero-divisors. But if we consider the example $\mathcal{K} := D(A)^c$, $\mathcal{T} := D(A)$ and $R := H^*A$, we see immediately that the conclusion of the lemma also follows from the above result (Lemma 4.8). Hence in this case we can get rid of the extra hypotheses, while the rest of our arguments go through unchanged. This proves Theorem 1.6.

Indeed, in general in condition (b) of Theorem 1.1 we could similarly renounce the evenness of R if we substitute the requirement that all elements f_i of the regular sequences be non-zero-divisors with the requirement that $f_i \cdot \mathbf{1}/\!\!/f_i = 0$.

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INVOLUTIVE AUTOMORPHISMS OF $N_\circ^\circ$-GROUPS OF FINITE MORLEY RANK

ADRIEN DELORO AND ÉRIC JALIGOT

Ma Virgilio n'avea lasciati scemi
di sé, Virgilio dolcissimo patre,
Virgilio a cui per mia salute die' mi.

We classify a large class of small groups of finite Morley rank: $N_\circ^\circ$-groups which are the infinite analogues of Thompson's N-groups. More precisely, we constrain the 2-structure of groups of finite Morley rank containing a definable, normal, nonsoluble, $N_\circ^\circ$-subgroup.

1. Introduction

This is the final item in the series [*DJ 2012; 2010; 2013], a collaboration interrupted by the demise of Jaligot. The present article has a sad story but at least it has the merit to exist: it was started in 2007 with hope and then never completed, started again in 2013 as a brave last sally and then lost, and then started over again by the first endorser alone.

So for the last time let us deal with $N_\circ^\circ$-groups of finite Morley rank. And although we have just used some phrases that our prospective reader may not know we hope our work to be of interest to the experts in finite group theory as many ideas and methods will seem familiar to them. Efforts were made in that direction and that of self-containedness.

1.1. *The context.*

Groups of finite Morley rank. Let us first say a few words about groups of finite Morley rank. We shall remain deliberately vague as we only hope to catch the

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reader's attention (possibly through provocation). Should we succeed we can suggest three books. The first monograph dealing with groups of finite Morley rank, among other groups, was [Poi 1987]. An excellent and thorough reference textbook is [BN 1994b] which has no pictures but many exercises instead. The more recent [ABC 2008] quickly focuses on the specific topic of the classification of the infinite simple groups of finite Morley rank of so-called even or mixed type, a technical assumption. For the moment let us be quite unspecific.

Morley rank is a notion invented by model theorists for the purposes of pure mathematical logic, and turned out to be an abstract form of the Zariski dimension in algebraic geometry. It was then natural to investigate the relations between groups of finite Morley rank and algebraic groups.

More precisely (we shall keep this facultative paragraph short and direct the brave to [Poi 1987]), the rank introduced by Morley for his categoricity theorem was quickly understood to be a central notion in mathematical logic, enabling a more algebraic treatment of model-theoretic phenomena, and hopefully allowing closer interactions with classical mathematics. This was confirmed when Zilber's "ladder" analysis of uncountably categorical theories revealed towers of atomic pieces bound to each other by some definable groups, similar to differential Galois groups in (Kolchin–)Picard–Vessiot theory, and therefore of utmost relevance even as abstract groups. It is expected that understanding the structure of such groups would shed further light on the nature of uncountably categorical theories, which would please model theorists, and other mathematicians as well.

But because of their very nature, groups of finite Morley rank cannot be studied with the techniques of algebraic geometry, and only elementary (in both the naive and model-theoretic senses of the term) methods apply, which results in massive technological smuggling from finite group theory to model theory.

To make a long story short: some abstract groups arose in one part of mathematics; it would be good to classify the simple ones; logicians need finite group theorists.

Groups with a dimension. And now for the sake of the introduction we shall suggest a completely different, anachronistic, and self-contained motivation.

The classification of the simple Lie groups, the classification of the simple algebraic groups, and the classification of the finite simple groups are facets of a single truth: in certain categories, simple groups are matrix groups in the classical sense. The case of the finite simple groups reminds us that we are at the level of an erroneous truth, but still there must be something common to Lie groups, algebraic groups, and finite groups beyond the mere group structure that forces them to fall into the same class.

In a sense, groups of finite Morley rank describe this phenomenon; Morley rank is a form of common structural layer, or methodological least common denominator to the Lie-theoretic, algebraic geometric, and finite group-theoretic worlds. Our groups are equipped with a dimension function on subsets enabling the most basic computations; the expert in finite group theory will be delighted to read that matching involutions against cosets, for instance, is possible. On the other hand, no analysis, no geometry, and no number theory are available. But the existence of a rudimentary dimension function is a common though thin structural layer extending the pure group structure.

It remains to say which sets are subject to having a dimension. These sets are called the definable sets; the definable class is the model-theoretic analogue of the constructible class in algebraic geometry. In a group G with no extra structure, one would consider the collection of subsets of the various G^n obtained by allowing group equations, (finite) boolean combinations, projections, and then by also allowing quotients by equivalence relations of the same form. This setting is a little too tight in general and model theorists enlarge the basic case of group equations by allowing other primary relations, that is, by working in an abstract structure extending the group structure. In particular, the natural structure on an affine algebraic group is richer than its pure group structure.

A group of finite Morley rank is such an extended group structure with an integer-valued dimension function on its definable sets. As for the properties of the dimension function itself, they are so natural they do not need to be described.

Although we have given no definition we hope to have motivated the Cherlin–Zilber conjecture, which surmises that infinite simple groups of finite Morley rank are groups of points of algebraic groups. The conjecture goes back to the seventies.

Relations with finite group theory. A consequence of the classification of the simple, periodic, linear groups [Thomas 1983] (also [Bender 1984; Hartley and Shute 1984]) is the locally finite version of the Cherlin–Zilber conjecture: infinite simple *locally finite* groups of finite Morley rank are algebraic. The fact that [Thomas 1983] heavily relies on the classification of the finite simple groups means that conventional group theory can help elucidate problems in model theory.

A proof of the classification of the simple, periodic, linear groups in odd characteristic without using the classification of the finite simple groups but some of its methods, such as component analysis and signaliser functors, is in [Borovik 1984]. Similar techniques carried to the model-theoretic context provide the *locally finite* version of the Cherlin–Zilber conjecture under an assumption standing for characteristic oddness [Bor 1995], still without using the classification of the finite simple groups. Let us now forget about local finiteness. All this suggests to ask whether conversely to the above, model theory may shed light on conventional group theory, and whether finite group theorists can learn something from logicians.

Altinel, Borovik, and Cherlin [ABC 2008] give a positive answer by proving the Cherlin–Zilber conjecture in even or mixed type, viz., when there is an infinite subgroup of exponent 2, thus obtaining an ideal sketch of a decent fragment of the classification of the finite simple groups. Apart from this case one should not expect the conjecture to be proved in full generality. There is no evidence for a model-theoretic analogue of the Feit–Thompson “odd order” theorem. Simple groups of finite Morley rank with no involutions cause major technical difficulties since most methods in the area heavily rely on 2-local analysis. Actually the experts do not regard the existence of the most dramatic (potential) counterexamples to the conjecture called bad groups as entirely unlikely. But after all, not all finite simple groups are groups of Lie type, so refuting the Cherlin–Zilber conjecture would certainly not show that it is not interesting.

The present work deals with a certain class of *small* groups of finite Morley rank: $N_\circ^\circ$-groups, defined in Section 2 by a condition borrowed from the classification of the finite simple groups. The former were called $*$-locally$_\circ^\circ$ soluble groups in [*DJ 2012; 2010; 2013]; we now change terminology to conform more closely to the standards of finite group theory.

Two notions of smallness. So let us push the analogy with finite group theory further. The classical N- property was introduced in [Thompson 1968] where the full classification of the finite, nonsoluble N-groups was given, and then proved in a series of subsequent papers: an N-group is a group G all of whose so-called local subgroups are soluble, which in the finite case amounts to requiring that $N_G(A)$ be soluble for every abelian subgroup $1 \neq A \leq G$. The decorations in $N_\circ^\circ$ indicate that we shall focus on connected components, making our condition less restrictive than proper N-ness. According to the Cherlin–Zilber conjecture, every connected, nonsoluble $N_\circ^\circ$-group should be isomorphic to $\mathrm{PSL}_2(\mathbb{K})$ or $\mathrm{SL}_2(\mathbb{K})$ with $\mathbb{K}$ an algebraically closed field. We cannot prove this, and our results will look partial when compared to [Thompson 1968].

Another, more restrictive notion of smallness in [Thompson 1968] was minimal simplicity: a minimal simple group is a simple group all of whose proper subgroups are soluble. The full classification of the finite, minimal simple groups is given in [Thompson 1968] as a corollary to that of the finite N-groups. The finite Morley rank analogue is named minimal *connected* simplicity and defined naturally in Section 2. According again to the Cherlin–Zilber conjecture, every minimal connected simple group should be isomorphic to $\mathrm{PSL}_2(\mathbb{K})$ with $\mathbb{K}$ an algebraically closed field; even under the assumption that the group contains involutions, this is an open question.

Minimal connected simple groups of finite Morley rank have already been studied at length as recalled in Sections 1.2 and 1.3. These groups obviously are $N_\circ^\circ$-groups but it is not clear whether one should hope for a converse statement. So transferring the partial, current knowledge from the minimal connected simple to the $N_\circ^\circ$ setting was a nontrivial task, undertaken in [*DJ 2012; 2010; 2013].

This extension will hopefully fit into a revised strategy for the classification of simple groups of finite Morley rank with involutions. The last written account of a master plan was in Burdges' thesis [Bur 2004b, Appendix A] and would need to be updated because of major advances in the general structural theory of groups of finite Morley rank, notably through results on torsion briefly touched upon in Section 2.2. But interestingly enough the theory of $N_\circ^\circ$-groups has already been used and will be used again in another topic: permutation groups of finite Morley rank [Del 2009a; BD 2015].

The present work completes the transition from the minimal connected simple to the $N_\circ^\circ$ setting, and does more. We cannot provide a full classification of $N_\circ^\circ$-groups, but we delineate major cases and give strong restrictions on their groups of automorphisms.

1.2. ***The result and its proof.*** The ideal goal would have been to show that the only nonsoluble $N_\circ^\circ$-groups of finite Morley rank are $\mathrm{PSL}_2(\mathbb{K})$ and $\mathrm{SL}_2(\mathbb{K})$. Under the assumption that there is an infinite elementary abelian 2-subgroup, this is a straightforward corollary or subcase of [ABC 2008]; see [*DJ 2010, Theorem 4]. In general the question is delicate and one should only hope to identify $\mathrm{PSL}_2(\mathbb{K})$ and $\mathrm{SL}_2(\mathbb{K})$ among such groups. This we do, and more, by giving restrictive information on the structure of potential counterexamples. In particular we show that such counterexamples would admit no infinite dihedral groups of automorphisms, which is likely to be of use in a prospective inductive setting.

As a matter of fact, the focus on outer involutive automorphisms, as opposed to inner involutions, became so prominent over the years (see Section 1.3) that we could take involutions out of the configurations, viz., our extra assumptions are not on the structure of the "inner" Sylow 2-subgroup of the $N_\circ^\circ$-group under consideration but on the structure of that of an acting group; incidentally, the inner 2-structure is fairly well understood. Taking involutions out is a pleasant advance, but makes results slightly more complex to state.

Our theorem below thus reads as follows: if a connected, nonsoluble $N_\circ^\circ$-group G is a definable subgroup of some larger group of finite Morley rank (possibly equal to G) with a few assumptions on the action of outer involutions on G, then G is either algebraic or one of four mutually exclusive configurations with common features; in any case the structure of the outer Sylow 2-subgroup is well understood too. The existence of the four said configurations is a presumably difficult open question. But we do not need involutions inside G to run the argument, and we are confident this will allow some form of induction.

The notation used below is all explained in Section 2. The reader will find some informal remarks on methods at the end of the current subsection, and a discussion of the general structure of the proof at the beginning of Section 4.

Theorem. *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Then the Sylow 2-subgroup of G*

has one of the following structures: *isomorphic to that of* $\mathrm{PSL}_2(\mathbb{C})$, *isomorphic to that of* $\mathrm{SL}_2(\mathbb{C})$, *or a* 2*-torus of Prüfer* 2*-rank at most* 2*.*

Suppose in addition that for all involutions $\iota \in I(\hat{G})$, *the group* $C_G^\circ(\iota)$ *is soluble. Then* $m_2(\hat{G}) \leq 2$, *one of* G *or* $\hat{G}/G$ *is* $2^\perp$, *and involutions are conjugate in* $\hat{G}$. *Moreover*, *one of the following cases occurs*:

- $\mathbf{PSL_2}$**:** $G \simeq \mathrm{PSL}_2(\mathbb{K})$ *in characteristic not* 2; $\hat{G}/G$ *is* $2^\perp$.
- $\mathbf{CiBo_\varnothing}$**:** G *is* $2^\perp$; $m_2(\hat{G}) \leq 1$; *for* $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ *is a self-normalising Borel subgroup of* G.
- $\mathbf{CiBo_1}$**:** $m_2(G) = m_2(\hat{G}) = 1$; $\hat{G}/G$ *is* $2^\perp$; *for* $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ *is a self-normalising Borel subgroup of* G.
- $\mathbf{CiBo_2}$**:** $\mathrm{Pr}_2(G) = 1$ *and* $m_2(G) = m_2(\hat{G}) = 2$; $\hat{G}/G$ *is* $2^\perp$; *for* $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ *is an abelian Borel subgroup of* G *inverted by any involution in* $C_G(i) \setminus \{i\}$ *and satisfies* $\mathrm{rk}\, G = 3\, \mathrm{rk}\, C_G^\circ(i)$.
- $\mathbf{CiBo_3}$**:** $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; $\hat{G}/G$ *is* $2^\perp$; *for* $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ *is a self-normalising Borel subgroup of* G; *if* $i \neq j$ *are two involutions of* G *then* $C_G(i) \neq C_G(j)$.

There is at present no hope to kill any of the nonalgebraic configurations of type **CiBo** ("Centralisers of involutions are Borel subgroups"; unlike the cardinal of the same name, these configurations are far from innocent). Three of these configurations were first and very precisely described in [*CJ 2004] under much stronger assumptions of both a group-theoretic and a model-theoretic nature, and the goal of [*BCJ 2007; *Del 2007a; 2007b; 2008] merely was to carry the same analysis with no model-theoretic restrictions. Despite progress in technology, nothing new could be added on the **CiBo** configurations since their appearance in [*CJ 2004]. So it is likely these potential monstrosities will linger for a while; in any case their consistency is not desirable.

Beyond porting the description of nonalgebraic configurations from the minimal connected simple setting [*Del 2007a] to the broader $N_\circ^\circ$ context, our theorem gives strong limitations on how these potential counterexamples would embed into bigger groups. This line of thought goes back to Delahan and Nesin's proof that so-called simple bad groups have no involutive automorphisms [DN 1993; BN 1994b, Proposition 13.4]. The question of involutive automorphisms of minimal connected simple groups has already been addressed in [*BCD 2009; Fré 2010]; we insist that a significant part of our results was not previously known in the minimal connected setting. This is the reason why we believe that our theorem, however partial and technical it may look, will prove relevant to the classification project.

The present result therefore replaces a number of earlier (pre)publications: [*BCJ 2007; *Del 2007a; 2007b; 2008; *DJ 2008; *BCD 2009], the contents of which are described in Section 1.3. (We cannot dismiss Frécon's analysis [Fré 2010, Theorem 3.1] as it heavily uses the solubility of centralisers of p-elements, a property which might fail in the $N_\circ^\circ$ case.)

And now we wish to say a few words about the proof. One cannot adapt [Thompson 1968] and subsequent papers. The expert in finite group theory will appreciate here how little structure there is on a group of finite Morley rank. A finite analogue of $\mathbf{CiBo}_1$, for instance, has a cyclic Sylow 2-subgroup; for a variety of classical reasons it has a normal 2-complement; if an N-group, it is soluble. We would be delighted to see quick arguments removing finite analogues of $\mathbf{CiBo}_2$ and $\mathbf{CiBo}_3$. In any case, however elementary they may seem, such methods are not available in our context. Character theory, remarkably absent from [Thompson 1968], cannot be used either. Even Sylow theory (see Sections 2.1 and 2.2) is rudimentary. From finite group theory there remains of course 2-local analysis, but we are dealing with small cases where one cannot apply the standard machinery, otherwise well acclimatised to the finite Morley rank setting.

The main group-theoretic method is then matching involutions against cosets, in the spirit of Bender as quoted in the beginning of Section 4.2. At times our arguments in this line are rather classical and Proposition 3, for instance, may have a known counterpart in finite group theory, while at other times they are unorthodoxly convoluted as in Proposition 6. But this is our main method mostly because we lack a better option. We also use a variant of local analysis [Bender 1970] developed by Burdges for groups of finite Morley rank (Sections 2.3 and 2.4). This will not surprise the expert.

As for model-theoretic methods, we see two main lines. First, we tend to focus on generic elements of groups, with the effect of smoothing phenomena. The general theory of genericity in model-theoretic contexts owes much to Cherlin and Poizat so one could refer the reader to [Poi 1987], but thanks to the rank function it is a rather obvious notion here. In the same vein we often resort to connectedness arguments which from the point of view of algebraic group theory will always be straightforward. Typical of connectedness methods is Zilber's indecomposability theorem [BN 1994b, Theorem 5.26]. The use of fields is the second essential feature; although Zilber's field theorem [BN 1994b, Theorem 9.1] nominally appears only in the proofs of Propositions 3 and 5, it underlies our knowledge of soluble groups, in particular the unipotence theory of Section 2.3 which is fundamental for the whole analysis.

The *structure* of the proof itself is described in Section 4.

1.3. ***Version history.*** The current subsection will be of little interest to a reader not familiar with the community of groups of finite Morley rank; we include it mostly because the present article marks the voyage's end.

The project of classifying $N_\circ^\circ$-groups with involutions started as early as 2007 under the suggestion of Borovik and yet is only the last chapter of an older story: the identification of $\mathrm{PSL}_2(\mathbb{K})$ among small groups of odd type.

• We could go back to Cherlin's seminal article on groups of small Morley rank [Che 1979] which identified $\mathrm{PSL}_2(\mathbb{K})$, considered bad groups, and formulated the algebraicity conjecture. Other important results on $\mathrm{PSL}_2(\mathbb{K})$ in the finite Morley rank context were found by Hrushovski [Hru 1989] and by Nesin et Ali(i) [Nes 1990a; BDN 1994; DN 1995]. But we shall not go this far.

• Jaligot was the first to do something specifically in so-called odd type [*Jal 2000], adapting computations from [BDN 1994] (we say a bit more in Sections 4.2 and 4.3).

• Another preprint by Jaligot [*Jal 2002], then at Rutgers University, deals with *tame* minimal connected simple groups of Prüfer rank 1. (Tameness is a model-theoretic assumption on fields arising in a group, already used for instance in [DN 1995].) In this context, either the group is isomorphic to $\mathrm{PSL}_2(\mathbb{K})$, or centralisers$^\circ$ of involutions are Borel subgroups.

Quite interestingly the tameness assumption, viz., "no bad fields", appears there in small capitals and bold font each time it is used; it seems clear that Jaligot already thought about removing it.

• Jaligot's time at Rutgers resulted in a monumental article with Cherlin [*CJ 2004] where tame minimal connected simple groups were thoroughly studied and potential nonalgebraic configurations carefully described. The very structure of our theorem reflects the result of [*CJ 2004].

• A collaboration between Burdges, Cherlin, and Jaligot [*BCJ 2007] was significant progress towards removing tameness: minimal connected simple groups have Prüfer rank at most 2.

• Using major advances by Burdges (described in Sections 2.3 and 2.4), the author was able to entirely remove the tameness assumption from [*CJ 2004] and reach essentially the same conclusions. This was the subject of his dissertation [*Del 2007a] under the supervision of Jaligot, published as [*Del 2007b; 2008].

• A few months before the completion of the author's Ph.D., the present project of classifying $N_\circ^\circ$-groups of finite Morley rank was suggested by Borovik, a task the author and Jaligot undertook with great enthusiasm and which over the years resulted in the series [*DJ 2012; 2010; 2013].

A 2008 preprint [*DJ 2008] was close to fully porting [*Del 2007a] to the $N_\circ^\circ$ context. Involutions remained confined inside the group. (This amounts to supposing $\hat{G} = G$ in the theorem.)

• While a postdoc at Rutgers University, the author, in an unpublished joint work with Burdges and Cherlin [*BCD 2009], went back to the minimal connected simple case but with outer involutory automorphisms. (This amounts to supposing G minimal connected simple and $2^\perp$ in the theorem.)

• Delays and shifts in interests postponed both [*DJ 2008] and [*BCD 2009]. In the spring of 2013 the author tried to convince Jaligot that time had come to redo [*DJ 2008] in full generality, that is, with outer involutions. The present theorem was an ideal statement we vaguely dreamt of but we never discussed nor even mentioned to each other anything beyond as it looked distant enough. In March and April of that year we were trying to fix earlier proofs with all possible repair patches, and mixed success.

The author recalls how Jaligot would transcribe those meetings in a small red "Rutgers" notebook when visiting Paris. He did not recover these notes after Jaligot's untimely death.

And this is how a project started with great enthusiasm was completed in grief and sorrow, yet completed. The author feels he is now repaying his debt for the care he received as a student, for an auspicious dissertation topic, and for all the friendly confidence his adviser trusted him with.

In short I hope that the present work is the kind of monument Éric's shadow begs for. I dare print that the article is much better than last envisioned in the spring of 2013. Offended reader, understand that *there* precisely lies my tribute to him.

Such a reconstruction would never have been even imaginable without the hospitality of the Mathematics Institute of NYU Shanghai during the fall of 2013. The good climate and supportive staff made it happen. At various later stages the comments of Gregory Cherlin proved invaluable, as always. Last but not least, and despite the author's lack of taste for mixing genres, Lola's immense patience is most thankfully acknowledged.

2. Prerequisites and facts

We have tried to make the article as self-contained as possible, an uneasy task since the theory of groups of finite Morley rank combines a variety of methods. Reading the prior articles in the series [*DJ 2012; 2010; 2013] is not necessary to understand this one. In the introduction we already mentioned three general references [Poi 1987; BN 1994b; ABC 2008]. Yet we highly recommend the preliminaries of a recent research article, [ABF 2013, §2]; the reader may wish to first look there before picking a book from the shelves.

We denote by $d(X)$ the definable hull of X, i.e., the smallest definable group containing X. If H is a definable group, we denote by H° its connected component.

If H fails to be definable we then set $H^\circ = H \cap d^\circ(H)$. These constructions behave as expected.

One more word on general terminology: the author supports linguistic minorities.

Definition [*DJ 2012, Definition 3.1(4)]. A group G of finite Morley rank is an $N_\circ^\circ$-group if $N_G^\circ(A)$ is soluble for every nontrivial, definable, abelian, connected subgroup $A \leq G$.

Remarks. • The property was named $*$-local$_\circ^\circ$ solubility in [*DJ 2012; 2010; 2013]; the $*$- prefix was a mere warning to the eye in order to distinguish from local conditions in the usual sense, the lower $_\circ$ was supposed to stand for the connectedness assumption on A, and the upper $^\circ$ symbolised the conclusion only being on the connected component $N_G^\circ(A)$.

We preferred to adapt Thompson's N- terminology from [Thompson 1968] by simply adding connectedness symbols.

• We do require full $N_\circ^\circ$-ness in our proofs and apparently cannot restrict to a certain class of local subgroups. For instance, Thompson's classification of the nonsoluble, finite N-groups was extended by [Gorenstein and Lyons 1976] to the nonsoluble, finite groups where only 2-local subgroups are supposed to be soluble (i.e., when A above must in addition be a 2-group).

Such a generalisation looks impossible in our setting as will become obvious during the proof, simply because we must take too many normalisers.

• Many results in the present work will be stated for $N_\circ^\circ$-groups of finite Morley rank. With our definition this is redundant but as other contexts, model-theoretic in particular, give rise to a notion of a connected component, this also is safer.

Remark (and Definition). An extreme case of an $N_\circ^\circ$-group G is that in which all definable, connected, proper subgroups of G are soluble; G is then said to be *minimal connected simple*. As opposed to past work (see Sections 1.2 and 1.3) the present article does not rely on minimal connected simplicity.

As we said in the introduction, there is no hope to prove that $N_\circ^\circ$-groups are close to being minimal connected simple. One could expect many more configurations [*DJ 2012, §3.3].

As one imagines, involutions will play a major role. We denote by $I(G)$ the set of involutions in G; i, j, k, ℓ will stand for some of them. We also use ι, κ, λ for involutions of the bigger, ambient group $\hat{G}$. When a group has no involutions, we call it a $2^\perp$ group. We shall refer to the following as "commutation principles".

Fact 1. *Suppose that there exists some involutive automorphism ι of a semidirect product $H \rtimes K$, where K is 2-divisible, and that ι centralises or inverts H, and inverts K. Then $[H, K] = 1$.*

2.1. ***Semisimplicity.*** In what follows, p stands for a prime number.

Fact 2 (torsion lifting [BN 1994b, Exercise 11 p. 98]). *Let G be a group of finite Morley rank, $H \trianglelefteq G$ be a normal, definable subgroup and $x \in G$ be such that xH is a p-element in G/H. Then $d(x) \cap xH$ contains a p-element of G.*

Apart from the above principle, most of our knowledge of torsion relies either on the assumption that $p = 2$, on some solubility assumption, or on a $U_p^\perp$ assumption explained below.

• To emphasise the case where $p = 2$, recall that in groups of finite Morley rank the maximal 2-subgroups, also known as Sylow 2-subgroups, are conjugate ([BN 1994b, Theorem 10.11], originating in [Bor 1984]). As a matter of fact, their structure is known [BN 1994b, Corollary 6.22]. If S is a Sylow 2-subgroup then $S^\circ = T * U_2$, where T is a 2-torus and U_2 a 2-unipotent group. Let us explain the terminology:

- T is a sum of finitely many copies of the Prüfer 2-group, $T \simeq \mathbb{Z}_{2^\infty}^d$, and d is called the Prüfer 2-rank of T, which we denote by $\mathrm{Pr}_2(T) = d$. By conjugacy, $\mathrm{Pr}_2(G) = \mathrm{Pr}_2(T)$ is well-defined. Interestingly enough, $N_G^\circ(T) = C_G^\circ(T)$ [BN 1994b, Theorem 6.16, "rigidity of tori"]; the latter actually holds for any prime.
- U_2 in turn has bounded exponent. We shall mostly deal with groups having no infinite such subgroups, and we call them $U_2^\perp$ *groups*.

The 2-rank $m_2(G)$ is the maximal rank (in the finite group-theoretic sense) of an elementary abelian 2-subgroup of G; again this is well-defined by conjugacy. A $U_2^\perp$ assumption implies finiteness of $m_2(G)$; one always has $\mathrm{Pr}_2(G) \leq m_2(G)$; see [Del 2012] for a reverse inequality.

• Actually the same holds for any prime p provided that the ambient group of finite Morley rank is soluble ([BN 1994b, Theorem 6.19 and Corollary 6.20], originating in [BP 1990]). In case the ambient group is also connected, then the Sylow p-subgroups are connected [BN 1994b, Theorem 9.29]. We call this fact the structure of torsion in definable, connected, soluble groups.

• Consistently generalising the case $p = 2$, a group of finite Morley rank is said to be $U_p^\perp$ (also, of $p^\perp$-type) if it contains no infinite, elementary abelian p-group. A word on Sylow p-subgroups of $U_p^\perp$ groups is said in Section 2.2.

We often rely either on some specific assumption on involutions, or on solubility, as in the following.

Fact 3 (bigeneration, [BC 2008, special case of Theorem 2.1]). *Let $\hat{G}$ be a $U_p^\perp$ group of finite Morley rank. Suppose that $\hat{G}$ contains a nontrivial, definable, connected, normal subgroup $G \trianglelefteq \hat{G}$ and an elementary abelian p-group $\hat{V} \leq \hat{G}$ of p-rank 2. If G is soluble, or if $p = 2$ and G has no involutions, then $G = \langle C_G^\circ(v) : v \in \hat{V} \setminus \{1\}\rangle$.*

We finish with a property of repeated use.

Fact 4 (Steinberg's torsion theorem, [Del 2009b]). *Let G be a connected, $U_p^\perp$ group of finite Morley rank, and $\zeta \in G$ be a p-element such that $\zeta^{p^n} \in Z(G)$. Then $C_G(\zeta)/C_G^\circ(\zeta)$ has exponent dividing p^n.*

As the argument essentially relies on the connectedness of centralisers of *inner* tori obtained by Altınel and Burdges [AB 2008, Theorem 1], one should not expect anything similar for outer automorphisms of order p, not even for outer toral automorphisms.

2.2. ***Sylow theory.*** By definition, a *Sylow p-subgroup* of a group of finite Morley rank is a maximal, *soluble* p-subgroup. It turns out that for a p-subgroup of a group of finite Morley rank, solubility is equivalent to local solubility (in the usual sense of finitely generated subgroups being soluble) [BN 1994b, Theorem 6.19], so every soluble p-subgroup is contained in some Sylow p-subgroup. But the solubility requirement is not for free: even if a group of finite Morley rank G is assumed to be $U_p^\perp$, it is not known whether every p-subgroup of G is soluble; actually it is still not known whether bad groups of exponent p exist or not. In short, a Sylow p-subgroup is not necessarily a maximal p-subgroup, even in the $U_p^\perp$ case. We now focus on Sylow p-subgroups.

As suggested above, Sylow p-subgroups of a $U_p^\perp$ group of finite Morley rank are toral-by-finite [BN 1994b, Corollary 6.20]. There is more.

Fact 5 [BC 2009, Theorem 4]. *Let G be a $U_p^\perp$ group of finite Morley rank. Then Sylow p-subgroups of G are conjugate.*

Remarks. Let $\hat{G}$ be a $U_p^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, normal subgroup.

• The Sylow p-subgroups of G are exactly the traces of the Sylow p-subgroups of $\hat{G}$. A Sylow p-subgroup of G is obviously the trace of some Sylow p-subgroup of $\hat{G}$. The converse is immediate by conjugacy of the Sylow p-subgroups in the $U_p^\perp$ group $\hat{G}$.

• The Sylow p-subgroups of $\hat{G}/G$ are exactly the images of the Sylow p-subgroups of $\hat{G}$. The following argument was suggested by Gregory Cherlin.

Let φ be the projection modulo G. Suppose that $\hat{S}$ is a Sylow p-subgroup of $\hat{G}$ but $\varphi(\hat{S})$ is not a Sylow p-subgroup of $\hat{G}/G$. Then by the normaliser condition [BN 1994b, Corollary 6.20] there is a p-element $\alpha \in N_{\hat{G}/G}(\varphi(\hat{S})) \setminus \varphi(\hat{S})$, which we lift to a p-element $a \in \hat{G}$. Note $\alpha \in N_{\hat{G}/G}(\varphi(\hat{S}^\circ))$, so $\varphi([a, \hat{S}^\circ G]) = [\alpha, \varphi(\hat{S}^\circ)] \leq \varphi(\hat{S}^\circ G)$ and $a \in N_{\hat{G}}(\hat{S}^\circ G)$.

Now $N = N_{\hat{G}}(\hat{S}^\circ G)$ is definable since it is the inverse image of $N_{\hat{G}/G}(\varphi(\hat{S}^\circ))$, which is definable as the normaliser of a p-torus by the rigidity of tori. In particular, N conjugates its Sylow p-subgroups, and a Frattini argument yields $N \leq \hat{S}^\circ G \cdot N_{\hat{G}}(\hat{S}) \leq G N_{\hat{G}}(\hat{S})$. Write $a = gn$ with $g \in G$ and $n \in N_{\hat{G}}(\hat{S})$; n is a p-element modulo G, so lifting torsion there is a p-element $m \in d(n) \cap nG$. Then $m \in N_{\hat{G}}(\hat{S})$ and therefore $m \in \hat{S}$. Hence $a = gn \in nG = mG \subseteq \hat{S}G$ and $\alpha = \varphi(a) \in \varphi(\hat{S})$, a contradiction.

As a consequence the image of any Sylow p-subgroup of $\hat{G}$ is a Sylow p-subgroup of $\hat{G}/G$. The converse is now immediate, conjugating in $\hat{G}/G$.

• Without the $U_p^\perp$ assumption this remains quite obscure. The reader will find in [PW 1993; 2000] a model-theoretic discussion.

We shall refer to the many consequences of the following fact as "torality principles".

Fact 6 [BC 2009, Corollary 3.1]. *Let $\underline{p}$ be a set of primes. Let G be a connected group of finite Morley rank with a $\underline{p}$-element x such that $C(x)$ is $U_{\underline{p}}^\perp$. Then x belongs to any maximal $\underline{p}$-torus of $C(x)$.*

And now for some unrelated remarks involving notions from [Che 2005]. A decent torus is a definable, divisible, abelian subgroup which equals the definable hull of its torsion subgroup. Goodness is the hereditary version of decency: a good torus is a definable, connected subgroup all definable, connected subgroups of which are decent tori.

Remarks. • Let $\hat{G}$ be a connected, $U_p^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected subgroup. If $\hat{T} \leq \hat{G}$ is a maximal p-torus of $\hat{G}$, then $T = \hat{T} \cap G$ is a maximal p-torus of G.

Let $\hat{S} \geq \hat{T}$ be a Sylow p-subgroup of $\hat{G}$. Then $S = \hat{S} \cap G$ is a Sylow p-subgroup of G. So $T = G \cap \hat{T} \leq G \cap \hat{S}^\circ \leq C_S(S^\circ) = S^\circ$ by torality principles. Hence $T \leq S^\circ \leq \hat{S}^\circ \cap G = \hat{T} \cap G = T$.

• This is not true for an arbitrary p-torus $\hat{\tau} \leq \hat{G}$: take two copies T_1, T_2 of $\mathbb{Z}_{2^\infty}$ with respective involutions i and j; now let $\hat{G} = (T_1 \times T_2)/\langle ij \rangle$ and G be the image of T_1. Then the intersection of (the image of) T_2 with G is $\langle \bar{i} \rangle$.

• This is not true if $\hat{G}$ is not $U_p^\perp$. Take for instance two Prüfer p-groups $T \simeq T' \simeq \mathbb{Z}_{p^\infty}$, an infinite elementary abelian p-group A, and a central product $K = T' * A$ with $T' \cap A = \langle a \rangle \neq \{1\}$. Set $G = T \times A$ and $\hat{G} = T \times K$. One will find $\hat{T} = T \times T'$, but $\hat{T} \cap G = T \times \langle a \rangle$ is not connected.

• Similarly, if $\hat{\Theta}$ is a good torus of $\hat{G}$ then $(\hat{\Theta} \cap G)^\circ$ is one of G, but connectedness of $\Theta = \hat{\Theta} \cap G$ is not granted even when $\hat{\Theta}$ is maximal; of course connectedness holds if G is $U_p^\perp$ for every prime number p.

• As for maximal decent tori, their connected intersections with subgroups need not be decent tori; in the language of the next subsection, (0, 0)-groups need not be homogeneous.

All this begs for a notion of reductivity, which however is not our present goal.

2.3. ***Unipotence.*** Developing a suitable theory of unipotence in the context of abstract groups of finite Morley rank took some time. One needs to describe a geometric phenomenon in group-theoretic terms. The positive characteristic notion may look straightforward to the hasty reader: when p is a prime number, a *p-unipotent subgroup* is a definable, connected, nilpotent p-group of bounded exponent. Yet the definition is naive only in appearance. First, nilpotence is perhaps not for free, as indicated in Section 2.2. Second, Baudisch has constructed a nonabelian p-unipotent group not interpreting a field [Bau 1996]: as a consequence, the Baudisch group does not belong to algebraic geometry (for more on field interpretation, see [GH 1993]). Despite these technical complications, the notion of unipotence in positive characteristic remains rather intuitive.

Matters are considerably worse in characteristic zero as there is no intrinsic way to distinguish, say, some torsion-free subgroup of $\mathbb{C}^\times$ from the additive group of some other field. Unpublished work by Altseimer and Berkman dated 1998 on so-called “pseudounipotent” and “quasiunipotent” subgroups, two notions which we shall not define, therefore required tameness assumptions on fields arising in the structure (see Section 1.3).

Burdges found a satisfactory unipotence theory; the point (and also the difficulty) is that one has a multiplicity of notions in characteristic zero. We do not wish to describe his construction. For a complete exposition of Burdges’ unipotence theory, see Burdges’ dissertation [Bur 2004b, Chapter 2], its first formally published expositions [Bur 2004a; 2006], or the first article in the present series [*DJ 2012].

A *unipotence parameter* is a pair of the form (p, ∞) where p is a prime, or $(0, d)$ where d is a nonnegative integer. The case $(0, 0)$ describes decent tori. We shall denote unipotence parameters by ρ, σ, τ. For every parameter ρ, there is a notion of a ρ-group, and of the ρ-generated subgroup $U_\rho(G)$ of a group G. Bear in mind that by definition, a ρ-group is always definable, connected, and nilpotent; the latter need not hold of the ρ-generated subgroup even if the ambient group is soluble.

Notation. We order unipotence parameters as follows:

$$(2, \infty) \succ (3, \infty) \succ \cdots \succ (p, \infty) \succ \cdots \succ (0, \mathrm{rk}(G)) \succ \cdots \succ (0, 0)$$

Notation. • For any group of finite Morley rank H, ρ_H will denote the greatest unipotence parameter it admits, i.e., with $U_{\rho_H}(H) \neq 1$; we simply call it *the* parameter of H. (Any infinite group of finite Morley rank admits a parameter, possibly $(0, 0)$; see [Bur 2004b, Theorem 2.19], [Bur 2004a, Theorem 2.15], or [*DJ 2012, Lemma 2.6].)

Be careful to note that the parameter of a group equal to its ρ-generated subgroup can be greater than ρ: take a decent torus which is not good and $\rho = (0, 0)$. (More generally, a definable, connected, soluble group H has parameter (0, 0) if and only if it is a good torus, but $H = U_{(0,0)}(H)$ if and only if H is generated by its decent tori.)

- For ι a definable involutive automorphism of some group of finite Morley rank, let $\rho_\iota = \rho_{C^\circ(\iota)}$.

With this notation at hand let us review a few classical properties. The reader should be familiar with the following before venturing further.

Fact 7. (i) *If N is a connected, nilpotent group of finite Morley rank, then $N = *_\rho U_\rho(N)$ (central product) where ρ ranges over all unipotent parameters* (Burdges' decomposition of nilpotent groups [Bur 2004b, Theorem 2.31; 2006, Corollary 3.6; *DJ 2012, Fact 2.3]).

(ii) *If H is a connected, soluble group of finite Morley rank, one has $U_{\rho_H}(H) \leq F^\circ(H)$* ([Bur 2004b, Theorem 2.21; 2004a, Theorem 2.16; *DJ 2012, Fact 2.8]; incidentally, the connected component of the Fitting subgroup $F^\circ(H)$ is defined and studied in [BN 1994b, §7.2]; one has $H' \leq F^\circ(H)$ [BN 1994b, Corollary 9.9]).

(iii) *If H is as above then $U_{\rho_H}(Z(F^\circ(H))) \neq 1$* [Bur 2004b, Lemma 2.26; 2006, Lemma 2.3].

(iv) *A σ-group V_σ normalises a ρ-group V_ρ with $\rho \preccurlyeq \sigma$ then $V_\rho V_\sigma$ is nilpotent* [Bur 2004b, Lemma 4.10; 2006, Proposition 4.1; *DJ 2012, Fact 2.7].

(v) *The image and preimage of a ρ-group under a definable homomorphism are ρ-groups* (push-forward and pull-back: [Bur 2004b, Lemma 2.12; 2004a, Lemma 2.11]).

(vi) *If G is a soluble group of finite Morley rank, $S \subseteq G$ is any subset, and $H \trianglelefteq G$ is a ρ-subgroup, then $[H, S]$ is a ρ-group* [Bur 2004b, Lemma 2.32; 2006, Corollary 3.7].

(vii) *Generalising the latter, Frécon obtained a remarkable homogeneity result we shall* not *use*:

> *if G is a connected group of finite Morley rank acting definably on a ρ-group, then $[G, H]$ is a homogeneous ρ-group, i.e., all its definable, connected subgroups are ρ-groups* [Fré 2006, Theorem 4.11; *DJ 2012, Fact 2.1].

(The last phenomenon was deemed essential in all earlier versions of the present work, but to our great surprise one actually does not need it. Frécon has developed in [Fré 2006] even subtler notions of unipotence with respect to isomorphism types instead of unipotence parameters.)

By definition, a Sylow ρ-subgroup is a maximal ρ-subgroup. Recall from Burdges' decidedly inspiring thesis ([Bur 2004b, §4.3], oddly published only in

[FJ 2008, §3.2]) that if π denotes a *set* of unipotence parameters, then a Carter π-subgroup of some ambient group G is a definable, connected, nilpotent subgroup L which is U_π-self-normalising, i.e., with $U_\pi(N_G^\circ(L)) = L$ (the π-generated subgroup is defined naturally and always definable and connected). Carter subgroups, i.e., definable, connected, nilpotent, almost-self-normalising subgroups are examples of the latter where π is the set of all unipotence parameters. All this is very well-understood in a soluble context [Wag 1994; Fré 2000a].

2.4. *Borel subgroups and intersections.*

Definition. A Borel subgroup of a group of finite Morley rank is a definable, connected, soluble subgroup which is maximal as such.

We shall refer to the following as "uniqueness principles".

Fact 8 [*DJ 2012, from Corollary 4.3]. *Let G be an $N_\circ^\circ$-group of finite Morley rank and B be a Borel subgroup of G. Let $U \leq B$ be a ρ_B-subgroup of B with $\rho_{C_G^\circ(U)} \preccurlyeq \rho_B$. Then $U_{\rho_B}(B)$ is the only Sylow ρ_B-subgroup of G containing U. Furthermore, B is the only Borel subgroup of G with parameter ρ_B containing U.*

Remarks. • Because of our ordering on unipotence parameters and our definition of ρ_B, the result does hold when $\rho_B = (0, 0)$, i.e., for B a good torus (cf. [*DJ 2012, Remark (3) after Theorem 4.1]). It would actually suffice to preorder parameters by $(0, k+1) \succ (0, k)$, and $(p, \infty) \succ (0, 0)$ for any prime number p.

• In particular, if $G \trianglelefteq \hat{G}$ where $\hat{G}$ is another (not necessarily $N_\circ^\circ$-) group of finite Morley rank, then $N_{\hat{G}}(U) \leq N_{\hat{G}}(B)$.

• If $1 < U \trianglelefteq B$ is a nontrivial, normal ρ_B-subgroup of B the result applies. We shall often use this with $U = U_{\rho_B}(Z(F^\circ(B)))$; see Fact 7(iii).

For reference we list below the facts from Burdges' monumental rewriting [Bur 2004b, §9; 2007] of Bender's method [Bender 1970] that we shall use. The method was devised to study intersections of Borel subgroups; it is quite technical. It will play an important role throughout the proof of our main maximality proposition (Proposition 6). As a matter of fact it does not appear elsewhere in the present article apart from Step 1 of Proposition 3.

It must be noted that the Bender method does *not* finish any job; it merely helps treat nonabelian cases on the same footing as the abelian case. This will be clear during Step 4 of Proposition 6. So the reader who feels lost here must keep in mind the following:

• nonabelian intersections of Borel subgroups complicate the details but do not alter in the least the skeleton of the proof of Proposition 6;

• the utter technicality is, in Burdges' own words [Bur 2004b], "motivated by desperation";

- such nonabelian intersections are not supposed to exist in the first place.

Since Burdges' original work was in the context of minimal connected simple groups we need to quote [*DJ 2012], which merely reproduced Burdges' work in the $N_\circ^\circ$ case.

Fact 9 [*DJ 2012, 4.46(2)]. *Let G be an $N_\circ^\circ$-group of finite Morley rank. Then any nilpotent, definable, connected subgroup of G contained in two distinct Borel subgroups is abelian.*

Yet past the nilpotent case it is not always possible to prove abelianity of intersections of Borel subgroups. The purpose of the Bender method is then to extract as much information as possible from nonabelian intersections. Unfortunately "as much as possible" means much more than reasonable. This is the analysis of so-called *maximal pairs* [*DJ 2012, Definition 4.12], a terminology we shall avoid.

Fact 10 (from [*DJ 2012, 4.50]). *Let G be an $N_\circ^\circ$-group of finite Morley rank. Let $B \neq C$ be two distinct Borel subgroups of G. Suppose that $H = (B \cap C)^\circ$ is nonabelian. Then the following are equivalent*:

[*DJ 2012, 4.50(1)]: *B and C are the only Borel subgroups of G containing H.*

[*DJ 2012, 4.50(2)]: *H is maximal among connected components of intersections of distinct Borel subgroups.*

[*DJ 2012, 4.50(3)]: *H is maximal among intersections of the form $(B \cap D)^\circ$, where $D \neq B$ is another Borel subgroup.*

[*DJ 2012, 4.50(6)]: $\rho_B \neq \rho_C$.

In the following, subscripts ℓ and h stand for *light* and *heavy*, respectively.

Fact 11 (from [*DJ 2012, 4.52]). *Let G, B_ℓ, B_h, H be as in the assumptions* and conclusions *of Fact 10. For brevity let $\rho' = \rho_{H'}$, $\rho_\ell = \rho_{B_\ell}$, $\rho_h = \rho_{B_h}$; suppose $\rho_\ell \prec \rho_h$. Then the following hold*:

[*DJ 2012, 4.52(2)]: *Any Carter subgroup of H is a Carter subgroup of B_h.*

[*DJ 2012, 4.38, 4.51(3) and 4.52(3)]: *$U_{\rho'}(F(B_h)) = (F(B_h) \cap F(B_\ell))^\circ$ is ρ'-homogeneous; ρ' is the least unipotence parameter in $F(B_h)$.*

[*DJ 2012, 4.52(6)]: *$U_{\rho'}(H) \leq F^\circ(B_\ell)$ and $N_G^\circ(U_{\rho'}(H)) \leq B_\ell$.*

[*DJ 2012, 4.52(7)]: *$U_\sigma(F(B_\ell)) \leq Z(H)$ for $\sigma \neq \rho'$.*

[*DJ 2012, 4.52(8)]: *Any Sylow ρ'-subgroup of G containing $U_{\rho'}(H)$ is contained in B_ℓ.*

We finish with an addendum.

Lemma A. *Let $\hat{G}$ be a connected group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble, $N_\circ^\circ$-subgroup. Let $B_1 \neq B_2$ be two distinct Borel subgroups of G such that $H = (B_1 \cap B_2)^\circ$ is maximal among connected components of intersections of distinct Borel subgroups* and nonabelian. *Let $Q \leq H$ be a Carter subgroup of H. Then*:

- $N_{\hat{G}}(H) = N_{\hat{G}}(B_1) \cap N_{\hat{G}}(B_2)$;
- $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_1) \cup N_{\hat{G}}(B_2)$.

Proof. By [*DJ 2012, 4.50(1), (2) and (6)], B_1 and B_2 are the only Borel subgroups of G containing H, and they have distinct unipotence parameters. This proves the first item. Let ρ' be the parameter of H' and $Q_{\rho'} = U_{\rho'}(Q)$. Then $N_{\hat{G}}(Q) \leq N_{\hat{G}}(Q_{\rho'}) \leq N_{\hat{G}}(N_G^\circ(Q_{\rho'}))$ and three cases can occur, following [*DJ 2012, 4.51]:

- In case (4a), $N_{\hat{G}}(Q) \leq N_{\hat{G}}(H) = N_{\hat{G}}(B_1) \cap N_{\hat{G}}(B_2)$; we are done.
- In case (4b), B_1 is the only Borel subgroup of G containing $N_G^\circ(Q_{\rho'})$, so that $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_1)$.
- Case (4c) is similar to case (4b) and yields $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B_2)$. □

3. Requisites (general lemmas)

Our theorem requires extending some well-known facts, so let us revisit a few classics. All lemmas below go beyond the $N_\circ^\circ$ setting.

3.1. ***Normalisation principles.*** The results in the present subsection are folklore; it turns out that none was formally published. They originate either in [*Del 2007a, Chapitre 2] or in [Bur 2009]. We shall use them with no reference, merely invoking "normalisation principles".

Lemma B (cf. [*Del 2007a, Lemmes 2.1.1 and 2.1.2; 2007b, §3.4]). *Let $\hat{G}$ be a group of finite Morley rank, $G \leq \hat{G}$ be a definable subgroup, $P \leq G$ be a Sylow p-subgroup of G, and $\hat{S} \leq N_{\hat{G}}(G)$ be a soluble p-subgroup normalising G. If $p \neq 2$, suppose that $\hat{G}$ is $U_p^\perp$. Then some G-conjugate of $\hat{S}$ normalises P.*

Proof. Since G is definable, $d(\hat{S}) \leq N_{\hat{G}}(G)$, so we may assume $\hat{G} = G \cdot d(\hat{S})$ and $G \trianglelefteq \hat{G}$. We may assume that $\hat{S}$ is a Sylow p-subgroup of $\hat{G}$. Recall that $S = \hat{S} \cap G$ is then a Sylow p-subgroup of G (see for instance Section 2.2). Since G is definable and $U_p^\perp$ if $p \neq 2$, it conjugates its Sylow p-subgroups; there is $g \in G$ with $P = S^g$. Hence $\hat{S}^g$ normalises $\hat{S}^g \cap G = S^g = P$. □

Remarks. The argument is slightly subtler than it looks.

- The original version [*Del 2007a, Lemmes 2.1.1 and 2.1.2] made the unnecessary assumption that $\hat{S}$, there denoted K, be definable. Its proof used only conjugacy in $\hat{G}$; but when $K^{\hat{g}} \leq N_{\hat{G}}(P)$ for some $\hat{g} \in \hat{G}$, why should $K^{\hat{g}}$ be a G-conjugate of K? Then [*Del 2007a] used definability of K to continue: we may assume $\hat{G} = G \cdot K \leq G \cdot N_{\hat{G}}(K)$, so $K^{\hat{g}}$ is actually a G-conjugate of K. Alas it is false in general that $d(\hat{S}) \leq N_{\hat{G}}(\hat{S})$ (consider the Sylow 2-subgroup of $\mathrm{PSL}_2(\mathbb{C})$). So without definability of $\hat{S}$ one is forced to use conjugacy *inside* G as we do here.

- In particular, if G is not supposed to be definable (and one then needs to assume $G \trianglelefteq \hat{G}$ to save the beginning of the proof), the statement is not clear at all since an arbitrary subgroup of a $U_p^\perp$ group of finite Morley rank need not conjugate its Sylow p-subgroups; take $\mathrm{PSL}_2(\mathbb{Z}[\sqrt{3}]) \leq \mathrm{PSL}_2(\mathbb{C})$ for instance. But for a *normal* subgroup, we do not know. This could even depend on the Cherlin–Zilber conjecture.

Recall in the following that if π consists of a single parameter ρ, then a Carter π-subgroup is exactly a Sylow ρ-subgroup.

Lemma C [*Del 2007a, Corollaires 2.1.5 and 2.1.6]. *Let $\hat{G}$ be a group of finite Morley rank, $H \leq \hat{G}$ be a* soluble, *definable subgroup, π be a set of unipotence parameters, $L \leq H$ be a Carter π-subgroup of H, and $\hat{S} \leq N_{\hat{G}}(H)$ be a soluble p-subgroup normalising H. Suppose that H is $U_p^\perp$. Then some H-conjugate of $\hat{S}$ normalises L.*

Proof. We first deal with the case where $L = Q$ is a Carter subgroup of H; the last paragraph will handle the general case. We may suppose that H is connected; we may suppose that $\hat{G} = H \cdot d(\hat{S})$ is soluble and that $H \trianglelefteq \hat{G}$; we may suppose that $\hat{S}$ is a Sylow p-subgroup of $\hat{G}$. Since H is soluble it conjugates its Carter subgroups, so $\hat{G} = H \cdot N_{\hat{G}}(Q)$.

First assume that H is $p^\perp$. Let $\hat{R} \leq N_{\hat{G}}(Q)$ be a Sylow p-subgroup of $N_{\hat{G}}(Q)$ and $\hat{R}_2 \leq \hat{G}$ a Sylow p-subgroup of $\hat{G}$ containing $\hat{R}$. Now $\hat{R}H/H$ and $\hat{R}_2H/H$ are both Sylow p-subgroups of $N_{\hat{G}}(Q)H/H = \hat{G}/H$; therefore $\hat{R}H = \hat{R}_2H$. But H is $p^\perp$, hence $\hat{R} = \hat{R}_2$ is a Sylow p-subgroup of $\hat{G}$, and it normalises Q.

If we no longer assume that H is $p^\perp$, then since H is $U_p^\perp$ the structure of torsion in definable, connected, soluble groups implies that Sylow p-subgroups of H are tori. By Lemma B, $\hat{S}$ normalises a Sylow p-subgroup P of H, so it normalises $d(P)$ as well. Up to conjugacy in H, Q contains P and therefore centralises P and $d(P)$ as well. So we may work in $N_{\hat{G}}(d(P))$ and factor out $d(P)$, which reduces to the first case. Then $\hat{S}$ normalises some Carter subgroup $\overline{C}$ of $H/d(P)$, and normalises its preimage $\varphi^{-1}(\overline{C}) \leq H$ which is of the form $\overline{C} = Cd(P)/d(P)$ for some Carter subgroup C of H [Fré 2000a, Corollaire 5.20]. Hence $\hat{S}$ normalises C modulo $d(P) \leq C$, that is, $\hat{S}$ normalises C.

The reader has observed that for the moment, $\hat{S}$ normalises some Carter subgroup of H. But by conjugacy of such groups in H, there is an H-conjugate of $\hat{S}$ normalising Q.

We now go back to the general case of a Carter π-subgroup L of H (see Section 2.3 for the definition). By [FJ 2008, Corollary 5.9] there is a Carter subgroup Q of H with $U_\pi(Q) \le L \le U_\pi(Q) \cdot U_\pi(H')$; by what we just proved and up to conjugating over H we may suppose that Q is $\hat{S}$-invariant. So we consider the generalised centraliser $E = E_H(U_\pi(Q))$ [Fré 2000a, Définition 5.15], a definable, connected, and $\hat{S}$-invariant subgroup of H satisfying $U_\pi(Q) \le F^\circ(E)$ [Fré 2000a, Corollaire 5.17]; by construction of E and nilpotence, $\langle L, Q\rangle \le E$. If $E < H$ then noting that L is a Carter π-subgroup of E we apply induction. So we may suppose $E = H$. But in this case $U_\pi(Q) \le F^\circ(H)$, so actually $L \le U_\pi(F^\circ(H))$ and equality holds as the former is a Carter π-subgroup of H. It is therefore $\hat{S}$-invariant. □

The following lemma is entirely due to Burdges who cleverly adapted the Frécon–Jaligot construction of Carter subgroups [FJ 2005]. We reproduce it here with Burdges' kind permission. The lemma is not used anywhere in the present article but included for possible future reference.

Lemma D [Bur 2009]. *Let $\hat{G}$ be a $U_2^\perp$ group of finite Morley rank, $G \le \hat{G}$ be a definable subgroup, and $\hat{S} \le N_{\hat{G}}(G)$ be a 2-subgroup. Then G has an $\hat{S}$-invariant Carter subgroup.*

Proof. We may assume that every definable, $\hat{S}$-invariant subquotient of G of smaller rank has an $\hat{S}$-invariant Carter subgroup; we may assume that $C_{\hat{S}}(G) = 1$; we may assume that G is connected.

We first find an infinite, definable, abelian, $\hat{S}$-invariant subgroup. Let $\iota \in Z(\hat{S})$ be a central involution; then $C_G^\circ(\iota) < G$. If $C_G^\circ(\iota) = 1$ then G is abelian and there is nothing to prove. So we may suppose that $C_G^\circ(\iota)$ is infinite and find some $\hat{S}$-invariant Carter subgroup of $C_G^\circ(\iota)$ by induction; it contains an infinite, definable, abelian, $\hat{S}$-invariant subgroup.

Let ρ be the minimal unipotence parameter such that there exists a nontrivial $\hat{S}$-invariant ρ-subgroup of G (possibly $\rho = (0, 0)$); this makes sense since there exists an infinite, definable, abelian, $\hat{S}$-invariant subgroup. Let $P \le G$ be a maximal $\hat{S}$-invariant ρ-subgroup; hence $P \ne 1$. Let $N = N_G^\circ(P)$.

If $N < G$ then induction applies: N has an $\hat{S}$-invariant Carter subgroup Q. So far PQ is soluble; moreover, for any parameter σ, $U_\sigma(Q)$ is $\hat{S}$-invariant as well. So by definition of ρ and [*DJ 2012, Fact 2.7], PQ is actually nilpotent; hence $PQ = Q$, $P \le Q$ and $P \le U_\rho(Q)$. By maximality of P, $P = U_\rho(Q)$ is characteristic in Q so $N_G^\circ(Q) \le N_N^\circ(Q) = Q$ and Q is a Carter subgroup of G.

Now suppose that $N = G$, that is, P is normal in G. By induction, $\overline{G} = G/P$ has an $\hat{S}$-invariant Carter subgroup $\overline{C}$. Let H be the preimage of $\overline{C}$ in G; note H is soluble. By Lemma C, H has an $\hat{S}$-invariant Carter subgroup Q. Here again PQ is soluble and even nilpotent, so $P \le Q$. Since H is soluble, $Q/P = PQ/P$ is a Carter subgroup of $H/P = \overline{C}$ [Fré 2000a, Corollaire 5.20], so $Q/P = \overline{C}$ and

$Q = H$. Finally $N_G^\circ(Q)/P \leq N_{\overline{G}}^\circ(\overline{C}) = \overline{C} = Q/P$, so $N_G^\circ(Q) = Q$ and Q is a Carter subgroup of G. □

Remarks. • Burdges left the highly necessary assumption that $\hat{G}$ be $U_2^\perp$ implicit from the title of his prepublication, and the original statement must therefore be taken with care: the Sylow 2-subgroup of $(\overline{\mathbb{F}}_2)_+ \rtimes (\overline{\mathbb{F}}_2)^\times$ certainly does not normalise any Carter subgroup.

• The assumption that $p = 2$ is used only to find an infinite, definable, abelian $\hat{S}$-invariant subgroup. It is not known whether all connected groups of finite Morley rank having a definable automorphism of order $p \neq 2$ with finitely may fixed points are soluble, although this is a classical property of algebraic groups.

3.2. ***Involutive automorphisms.*** The need for the present subsection is the following. In order to provide a decomposition for a connected, soluble group of odd type under an *inner* involutive automorphism, [*DJ 2010, Section 5] collected various well-known facts. But in the present article we shall consider the case of *outer* automorphisms, more precisely the action of abstract 2-tori on our groups. So the basic discussion of [*DJ 2010] must take place in a broader setting; this is what we do here.

Notation. If α is an involutive automorphism of some group G, we let

$$G^+ = C_G(\alpha) = \{g \in G : g^\alpha = g\},$$
$$G^- = \{g \in G : g^\alpha = g^{-1}\}.$$

We also let $\{G, \alpha\} = \{[g, \alpha] : g \in G\}$ (in context there is no risk of confusion with the usual notation for unordered pairs).

If G and α are definable, so are G^+, G^-, and $\{G, \alpha\}$; in general only the first need be a group. However, $\{G, \alpha\}$ is stable under inversion, since $[g^\alpha, \alpha] = [g, \alpha]^{-1}$. Observe that $\{G, \alpha\} \subseteq G^-$ but equality may fail to hold: for instance if α centralises G and G contains an involution i, then $i \in G^+ \cap G^-$ but $i \notin \{G, \alpha\} = \{1\}$. Notice further that $G = G^+ \cdot G^-$ if and only if $\{G, \alpha\} \subseteq (G^-)^{\wedge 2}$, and $G = G^+ \cdot \{G, \alpha\}$ if and only if $\{G, \alpha\} \subseteq \{G, \alpha\}^{\wedge 2}$, where $X^{\wedge 2}$ denotes the set of squares of X. Finally, we remark that $\deg\{G, \alpha\} = \deg \alpha^G \alpha = \deg \alpha^G \leq \deg G$.

Lemma E (cf. [*DJ 2010, Theorem 19]). *Let G be a group of finite Morley rank with Sylow 2-subgroup a (possibly trivial)* central *2-torus S, and α be a definable involutive automorphism of G. Then $G = G^+ \cdot \{G, \alpha\}$ where the fibres of the associated product map are in bijection with $I(\{G, \alpha\}) \cup \{1\} = \Omega_2([S, \alpha])$. Furthermore one has $G = (G^+)^\circ \cdot \{G, \alpha\}$ whenever G is connected.*

Proof. The proof follows that of [*DJ 2010, Theorem 19] closely and for some parts a minor adjustment would suffice, but we prefer to give a complete proof instead. Bear in mind that if $a^b = a^{-1}$ for two elements of our present group G, then a has order at most 2 (this is [*DJ 2010, Lemma 20], an easy consequence of torsion lifting). Also remember from [*DJ 2010, Lemma 18] that G is 2-divisible, essentially because 2-torsion is divisible and central.

Step 1. $S \cap \{G, \alpha\} = [S, \alpha]$.

Proof of Step 1. This is the argument from [*DJ 2010, Theorem 19, Step 1] with one more remark. One inclusion is trivial. Now let $\zeta \in S \cap \{G, \alpha\}$, and write $\zeta = [g, \alpha]$. Since G is 2-divisible we let $h \in H$ satisfy $h^2 = g$. Let $n = 2^k$ be the order of ζ. Then $[h^2, \alpha] = [h, \alpha]^h [h, \alpha] = \zeta \in Z(G)$ so $[h, \alpha]$ and $[h, \alpha]^h$ commute. Hence $1 = \zeta^n = [h, \alpha]^n [h, \alpha]^{nh}$. It follows that h inverts $[h, \alpha]^n$ which must have order at most 2; so $\xi = [h, \alpha]^{-1}$ is a 2-element inverted by α, and since it is central it commutes with h. Finally $[\xi, \alpha] = \xi^{-2} = [h, \alpha]^2 = [h^2, \alpha] = \zeta$. □

It follows that $I(\{G, \alpha\}) \cup \{1\} = \Omega_2([S, \alpha])$, the group generated by involutions of $[S, \alpha]$.

Step 2. *$\{G, \alpha\}$ is 2-divisible and $G = G^+ \cdot \{G, \alpha\}$.*

Proof of Step 2. Here again this is the argument from [*DJ 2010, Theorem 19, Step 2]; 2-divisibility of $\{G, \alpha\}$ was announced but not explicitly proved.

Let $x = [g, \alpha] \in \{G, \alpha\}$. Like in [*DJ 2010, Theorem 19, Step 2], write the definable hull of x as $d(x) = \delta \oplus \langle \gamma \rangle$, where δ is connected and γ has finite order; rewrite $\gamma = \varepsilon\zeta$, where ε has odd order and ζ is a 2-element; let $\Delta = \delta \oplus \langle \varepsilon \rangle$, so that $d(x) = \Delta \oplus \langle \zeta \rangle$, where Δ is 2-divisible and inverted by α. Now let $y \in \Delta$ satisfy $y^4 = x\zeta^{-1}$. Then $[gy^2, \alpha] = [g, \alpha]^{y^2} [y^2, \alpha] = xy^{-4} = \zeta \in S \cap \{G, \alpha\} = [S, \alpha]$ by Step 1, so there is $\xi \in S$ with $[\xi^2, \alpha] = \zeta$. Now $[y^{-1}\xi, \alpha] = [y^{-1}, \alpha]^\xi [\xi, \alpha] = y^2[\xi, \alpha]$ squares to $y^4[\xi, \alpha]^2 = x\zeta^{-1}[\xi^2, \alpha] = x$. The set $\{G, \alpha\}$ is therefore 2-divisible; as observed this implies $G = G^+ \cdot \{G, \alpha\}$. □

Step 3. *Fibres in Step 2 are in bijection with $\Omega_2([S, \alpha])$.*

Proof of Step 3. Let $k = [s, \alpha]$ have order at most 2, where $s \in S$. Fix any decomposition $\gamma = a \cdot [g, \alpha]$ with $a \in G^+$ and $g \in G$. Since α inverts (hence centralises) k, one has $ka \in G^+$. Moreover,

$$[sg, \alpha] = [s, \alpha]^g [g, \alpha] = k^g [g, \alpha] = k[g, \alpha] \in \{G, \alpha\}.$$

So $a[g, \alpha] = (ka) \cdot (k[g, \alpha])$ is yet another decomposition for γ.

Conversely, work as in [*DJ 2010, Theorem 19, Step 3]: suppose that $ax = by$ are two decompositions, with $a, b \in G^+$ and $x = [g, \alpha]$, $y = [h, \alpha] \in \{G, \alpha\}$. Then $(a^{-1}b)^y = (xy^{-1})^y = y^{-1}x = (yx^{-1})^\alpha = (b^{-1}a)^\alpha = b^{-1}a = (a^{-1}b)^{-1}$, so $a^{-1}b$ has order at most 2, say $k = a^{-1}b$. More precisely, $k = xy^{-1} = [g, \alpha][h, \alpha]^{-1} = [g, \alpha]h^{-\alpha}h$ is central, so $k = h[g, \alpha]h^{-\alpha} = [gh^{-1}, \alpha] \in \{G, \alpha\}$; it follows from Step 1 that $k \in \Omega_2([S, \alpha])$. □

Step 4. *Left G^+-translates of the set $(G^+)^\circ \cdot \{G, \alpha\}$ are disjoint or equal.*

Proof of Step 4. As in [*DJ 2010, Theorem 19, Step 4], suppose that for $a, b \in G^+$, the sets $a(G^+)^\circ \cdot \{G, \alpha\}$ and $b(G^+)^\circ \cdot \{G, \alpha\}$ meet, in say $ag_+[g, \alpha] = bh_+[h, \alpha]$ with natural notations. By the proof of Step 3, $k = (ag_+)^{-1}(bh_+)$ is in $\Omega_2([S, \alpha])$, therefore central in G and inverted (hence centralised) by α. So $k = (bh_+)(ag_+)^{-1} = (ag_+)(bh_+)^{-1}$. Hence for any $b\gamma_+[\gamma, \alpha] \in b(G^+)^\circ \cdot \{G, \alpha\}$, one finds

$$b\gamma_+[\gamma, \alpha] = k^2 b\gamma_+[\gamma, \alpha] = a(g_+h_+^{-1}\gamma_+)([\gamma, \alpha]k).$$

Since $k \in \Omega_2([S, \alpha])$, there is $s \in S$ with $k = [s, \alpha]$. So $[\gamma s, \alpha] = [\gamma, \alpha]^s[s, \alpha] = [\gamma, \alpha]k \in \{G, \alpha\}$; hence $b\gamma_+[\gamma, \alpha] \in a(G^+)^\circ \cdot \{G, \alpha\}$. This shows $b(G^+)^\circ\{G, \alpha\} \subseteq a(G^+)^\circ\{G, \alpha\}$ and the converse inclusion holds too. □

Step 5. *At most* $\deg G$ *left G^+-translates of $(G^+)^\circ \cdot \{G, \alpha\}$ cover G. In particular, if G is connected, then $G = (G^+)^\circ \cdot \{G, \alpha\}$.*

Proof of Step 5. Consider such left translates. They all have rank $\operatorname{rk} G$ by Step 3. As they are disjoint or equal by Step 4, at most $\deg G$ of them suffice to cover G. □

This completes the proof of Lemma E. □

Remarks. • Notice the flaw in [*DJ 2010, Theorem 19, Step 5], where "at most" is erroneously replaced by "exactly". The reason is that the degree of α^G need not be 1 in general; all one knows is $\deg \alpha^G \leq \deg G$. For instance, let α invert $\mathbb{Z}/3\mathbb{Z}$. Then $\deg G = 3$ but $(G^+)^\circ \cdot G^- = G$.

• If G is a connected group of finite Morley rank of odd type whose Sylow 2-subgroup S is central, then S is a 2-torus as $S = C_S(S^\circ) = S^\circ$ by torality principles.

• The lemma fails if S is not 2-divisible, even at the abelian level: let α invert $\mathbb{Z}/4\mathbb{Z}$.

As a consequence we deduce another useful decomposition which will be used repeatedly.

Lemma F (cf. [*DJ 2010, Lemma 24]). *Let H be a $U_2^\perp$, connected, soluble group of finite Morley rank, and α be a definable involutive automorphism of H. Suppose that $\{H, \alpha\} \subseteq F^\circ(H)$. Then $H = (H^+)^\circ \cdot \{H, \alpha\}$ with finite fibres.*

Proof. By normalisation principles, H admits an α-invariant Carter subgroup Q; by the theory of Carter subgroups of soluble groups, $H = Q \cdot F^\circ(H)$ [Fré 2000a, Corollaire 5.20]. Now both Q and $F^\circ(H)$ are definable, connected, nilpotent, and $U_2^\perp$, so Lemma E applies to them. Hence $Q = (Q^+)^\circ \cdot \{Q, \alpha\} \subseteq (H^+)^\circ \cdot F^\circ(H)$, and

$$\begin{aligned} H = Q \cdot F^\circ(H) &\subseteq (H^+)^\circ \cdot F^\circ(H) \\ &\subseteq (H^+)^\circ \cdot (F^\circ(H)^+)^\circ \cdot \{F^\circ(H), \alpha\} \subseteq (H^+)^\circ \cdot \{H, \alpha\}. \end{aligned}$$

The fibres are finite: this works as in [*DJ 2010, Lemma 24] since if $c_1b_1 = c_2b_2$ with $c_i \in H^+$, $b_i \in \{H, \alpha\}$, then $c_2^{-1}c_1 = b_2b_1^{-1} \in H^+$ so $b_2b_1^{-1} = b_2^{-1}b_1$ and $b_1^2 = b_2^2$, but by assumption $b_i \in \{H, \alpha\} \subseteq F^\circ(H)$ so b_1 and b_2 differ by an element of $\Omega_2(F^\circ(H))$ (in case of hyperbolic doubt read the next remark). Unlike in Lemma E we cannot be too precise about the cardinality of the fibre. □

Remarks. • We can show $\{H, \alpha\} \subseteq \Omega_2(F^\circ(H)) \cdot \{F^\circ(H), \alpha\}$. Indeed, letting $h \in H$, we then have $[h, \alpha] \in \{H, \alpha\} \subseteq F^\circ(H)$. Applying Lemma E in $F^\circ(H)$, we write $[h, \alpha] = f_+[f, \alpha]$ with $f_+ \in F^\circ(H)^+$ and $f \in F^\circ(H)$. Taking the commutator with α we find $[h, \alpha]^2 = [f, \alpha]^2$. But in $F^\circ(H)$, the equation $x^2 = y^2$ results in

$$x^{-1} \cdot x^{-1}y \cdot x = y^{-1}x = (x^{-1}y)^{-1}$$

and by the first observation in the proof of Lemma E, $x^{-1}y$ has order at most 2. Hence, $[h, \alpha] = k[f, \alpha]$ for some $k \in \Omega_2(F^\circ(H))$.

• Without the crucial assumption that $\{H, \alpha\} \subseteq F^\circ(H)$, one still has

$$H = \{H, \alpha\} \cdot (H^+)^\circ \cdot \{H, \alpha\},$$

and therefore $H = H^- \cdot H^+ \cdot H^-$, but one can hardly say more.

Consider two copies $A_1 = \{a_1 : a \in \mathbb{C}\}$, $A_2 = \{a_2 : a \in \mathbb{C}\}$ of $\mathbb{C}_+$ and let $Q = \{t : t \in \mathbb{C}^\times\} \simeq \mathbb{C}^\times$ act on A_1 by $a_1^t = (t^2a)_1$ and on A_2 by $a_2^t = (t^{-2}a)_2$. Form the group $H = (A_1 \oplus A_2) \rtimes Q$. Let α be the definable, involutive automorphism of H given by

$$(a_1b_2t)^\alpha = b_1a_2t^{-1},$$

that is, "α swaps the ± 2 weight spaces while inverting the torus". The reader may check that α is an automorphism of H, and perform the following computations:

- $[a_1b_2t, \alpha] = (t^2b - t^2a)_1(t^{-2}a - t^{-2}b)_2t^{-2}$ (so $\{H, \alpha\} \not\subseteq F^\circ(H)$);
- $H^+ = \{a_1a_2 \cdot \pm 1 : a \in \mathbb{C}_+\}$ (incidentally $(H^+)^\circ \leq F^\circ(H)$);
- $H^- = \{a_1(-t^2a)_2t : a \in \mathbb{C}_+, t \in \mathbb{C}^\times\}$ (incidentally $H^- = \{H, \alpha\}$);
- $H^+ \cdot H^- = \{(a + b)_1(a - t^2b)_2 \cdot \pm t : a, b \in \mathbb{C}_+, t \in \mathbb{C}^\times\}$ does *not* contain $0_1a_2 \cdot i$ for $a \neq 0$ (here i is a complex root of -1).

• Rewriting [*DJ 2010, Theorem 19] is necessary for the argument; one cannot simply use the idea of Lemma F together with the original decomposition.

Let $Q = \mathbb{C}^\times$ act on $A = \mathbb{C}_+$ by $a^t = (t^2 \cdot a)$ and form $H = A \rtimes Q$. Consider α the involutive automorphism doing $(at)^\alpha = (-a)t$ (α inverts the Fitting subgroup while centralising the Carter subgroup). The reader will check that $H^+ = Q$, $H^- = A \cdot \pm 1$, $\{H, \alpha\} = A$, and of course $H = H^+ \cdot H^-$.

Running the argument in Lemma F using the (naive) $G = G^+ \cdot G^-$ decomposition of [*DJ 2010, Theorem 19], one finds $Q = (Q^+)^\circ \cdot Q^-$, but $Q^- \simeq \mathbb{Z}/2\mathbb{Z}$ is not in

$F^\circ(H)$. One could then wish to apply the decomposition to $F(H)$ instead, but the Sylow 2-subgroup of the latter is not a 2-torus.

Extending [*DJ 2010, Theorem 19] into Lemma E was therefore needed for Lemma F.

3.3. $U_{\underline{p}}^\perp$ *actions and centralisers.* The need for the present subsection is Lemma J below but we shall digress a bit for completeness and future reference. Let $\underline{p}$ denote a set of prime numbers. The class of $U_{\underline{p}}^\perp$ groups is defined naturally. We slightly refine the analysis of [ABC 2008, §I.9.5], which deals with two dual settings:

- soluble, $\underline{p}^\perp$ groups acting on definable, connected, soluble, $U_{\underline{p}}$ groups;
- $\underline{p}$-groups acting on definable, connected, soluble, $\underline{p}^\perp$ groups.

Notation. If A and B are two subgroups of some ambient abelian group, we write $A \oplus B$ to denote the quasidirect sum, i.e., in order to mean that $A \cap B$ is finite.

Lemma G. *In a universe of finite Morley rank, let A be a definable, abelian group and R be a group acting on A by definable automorphisms. Let $A_0 \leq A$ be a definable, R-invariant subgroup. Suppose one of the following*:

(i) *A is a $\underline{p}^\perp$ group and R is a finite, soluble $\underline{p}$-group*;

(ii) *A is a connected, $\underline{p}^\perp$ group, A_0 is connected, and R is a soluble $\underline{p}$-group*;

(iii) *A is a connected, $U_{\underline{p}}^\perp$ group, A_0 is connected, and R is a soluble $\underline{p}$-group*;

(iv) *A is a $U_{\underline{p}}$-group and R is a definable, soluble, $\underline{p}^\perp$ group*;

(v) *A is a connected $U_{\underline{p}}$-group and $R \leq S$ where S is a definable, soluble, $\underline{p}^\perp$ group acting on A.*

Then $C_R(A) = C_R(A_0, A/A_0)$. In cases (i), (ii), (iv), and (v): $A = [A, R] \oplus C_A(R)$, $[A, R] \cap A_0 = [A_0, R]$, and $C_A(R)$ covers $C_{A/A_0}(R)$. In case (iii), the properties hold provided that connected components are added (where not redundant), and $\oplus$ is replaced by $\oplus$. In case (ii), $C_A(R)$ and $C_{A/A_0}(R)$ are connected.

Proof. (i) This is an extension of [ABC 2008, Corollary I.9.14] taking A_0 into account.

We prove that $A = [A, R] + C_A(R)$ by induction on the order of R. By solubility, there exist a proper subgroup $S \lhd R$ and an element $r \in R$ with $R = \langle S, r\rangle$. By induction, $A = [A, S] + C_A(S)$. But r normalises $C = C_A(S)$ which is a definable $\underline{p}^\perp$-group. Consider the definable homomorphisms $\mathrm{ad}_r : C \to C$ and $\mathrm{Tr}_r : C \to C$, respectively given by

$$\mathrm{ad}_r(a) = [a, r] \quad \text{and} \quad \mathrm{Tr}_r(a) = \sum_{r^i \in \langle r\rangle} a^{r^i}.$$

Since $\mathrm{ad}_r \circ \mathrm{Tr}_r = \mathrm{Tr}_r \circ \mathrm{ad}_r = 0$, one has $\operatorname{im} \mathrm{ad}_r \leq \ker \mathrm{Tr}_r$ and $\operatorname{im} \mathrm{Tr}_r \leq \ker \mathrm{ad}_r$. But since $\ker \mathrm{Tr}_r \cap \ker \mathrm{ad}_r$ consists of elements of order dividing $|r|$, it is trivial by assumption. In particular, $\operatorname{im} \mathrm{ad}_r \cap \ker \mathrm{ad}_r = 0$ so

$$C = \operatorname{im} \mathrm{ad}_r + \ker \mathrm{ad}_r = [C, r] + C_C(r) \leq [A, R] + C_A(R).$$

Let us show that $[A, R] \cap C_A(R)$ is trivial. Consider the definable homomorphism $\mathrm{Tr}_R : A \to A$ given by

$$\mathrm{Tr}_R(a) = \sum_{r \in R} a^r.$$

Since Tr_R vanishes on any subgroup of the form $[A, r]$, it vanishes on $[A, R]$; notice that it coincides with multiplication by $|R|$ on $C_A(R)$. It follows that $[A, R] \cap C_A(R)$ consists of elements of order dividing $|R|$, so by assumption it is trivial.

We can say a bit more: $\ker \mathrm{Tr}_R = [A, R]$ and $\operatorname{im} \mathrm{Tr}_R = C_A(R)$. Indeed, $A = [A, R] + C_A(R)$ and $[A, R] \leq \ker \mathrm{Tr}_R$, so $\ker \mathrm{Tr}_R \leq [A, R] + C_{\ker \mathrm{Tr}_R}(R)$. But $C_{\ker \mathrm{Tr}_R}(R)$ consists of elements of order dividing $|R|$, therefore it is trivial. It follows that $\ker \mathrm{Tr}_R = [A, R]$. Again $\operatorname{im} \mathrm{Tr}_R \cap \ker \mathrm{Tr}_R \leq C_{\ker \mathrm{Tr}_R}(R) = 0$, so as above $A = \operatorname{im} \mathrm{Tr}_R + \ker \mathrm{Tr}_R$, proving $C_A(R) \leq \operatorname{im} \mathrm{Tr}_R + C_{\ker \mathrm{Tr}_R}(R) = \operatorname{im} \mathrm{Tr}_R$.

We turn our attention to the definable, R-invariant subgroup $A_0 \leq A$. One sees that

$$[A, R] \cap A_0 = \ker \mathrm{Tr}_R \cap A_0 = \ker(\mathrm{Tr}_R)_{|A_0} = [A_0, R]$$

and, letting φ stand for projection modulo A_0,

$$\varphi(C_A(R)) = \varphi \circ \mathrm{Tr}_R(A) = \mathrm{Tr}_R \circ \varphi(A) = \mathrm{Tr}_R(A/A_0) = C_{A/A_0}(R).$$

Finally, let $S = C_R(A_0, A/A_0)$. We apply our results to the action of S on A and find $A \leq [A, S] + C_A(S) \leq C_A(S)$, so $S = C_R(A)$.

(ii) We reduce to case (i) with the following claim.

In a universe of finite Morley rank, if G is a definable, connected group and R is a locally finite group acting on G, then there is a finite subgroup $R_0 \leq R$ with $C_G(R_0) = C_G(R)$ and $[G, R_0] = [G, R]$.

The first equality is by the descending chain condition on centralisers: there is a finite subset $X \subseteq R$ with $C_G(X) = C_G(R)$. Now by connectedness of G and Zilber's indecomposability theorem, $[G, r]$ is definable and connected for any $r \in R$. By the ascending chain condition on definable, connected subgroups, there is a finite subset $Y \subseteq R$ such that $[G, Y] = [G, R]$. Take $R_0 = \langle X \cup Y \rangle$, a finite subgroup of R.

So taking both actions on A and on A_0 into account we may suppose R to be finite; apply case (i) and see that $A = [A, R] \oplus C_A(R)$ implies connectedness of the latter.

(iii) Here again we may suppose R to be finite. Now read the proof of case (i) again, replacing "trivial" by "finite" and adding connected components where necessary.

(iv) This is essentially [CD 2012, Facts 1.15 and 1.16]; also see [ABC 2008, Corollary I.9.11].

Let $H = A \rtimes R$, a definable, soluble group with $A \leq F(H)$. Then for $q \notin \underline{p}$, $U_q(R) \leq F(H) \leq C_H(A)$, and likewise, $U_{(0,k)}(R) \leq C_H(A)$ for $k > 0$. So R° acts as a good torus which we may replace with a finite, normal subgroup of R; then we may suppose that R itself is finite.

Considering the complement of $\underline{p}$ in the set of primes, we may apply case (i).

(v) We reduce to case (iv) with the following claim.

In a universe of finite Morley rank, if G is a definable, connected group and S is a definable group acting on G, then any subgroup $R \leq S$ satisfies $C_G(R) = C_G(d(R))$ and $[G, R] = [G, d(R)]$.

The first equality is by definability of centralisers. The second is as in [CD 2012, Lemma 1.14]: let $X = \{s \in d(R) : [G, s] \leq [G, R]\}$. Since $[G, R]$ is definable by connectedness of G and Zilber's indecomposability theorem, so is its normaliser in $d(R)$. Hence $d(R)$ normalises $[G, R]$; the definable set X is actually a subgroup of $d(R)$. So $d(R) \leq d(X)$ and $[G, d(R)] = [G, R]$. □

Remark. The lemma fails for $U_{\underline{p}}^\perp$, nonconnected A since it fails at the finite level: let $R = \mathbb{Z}/2\mathbb{Z}$ act by inversion on $A = \mathbb{Z}/4\mathbb{Z}$; one has $C_A(R) = 2A = [A, R]$.

After obtaining the following lemma the author realised it was already proved by Burdges and Cherlin using a different argument.

Lemma H (cf. [ABC 2008, Proposition I.9.12]; also [BC 2008, Lemma 2.5]). *In a universe of finite Morley rank, let G be a definable group, R be a soluble $\underline{p}$-group acting on G by definable automorphisms, and $H \trianglelefteq G$ be a definable, connected, soluble, $U_{\underline{p}}^\perp$, R-invariant subgroup. Then $C^\circ_{G/H}(R) = C^\circ_G(R)H/H$.*

Proof. As in Lemma G, using chain conditions and local finiteness, we may assume that R is finite. Let $L = \varphi^{-1}(C^\circ_{G/H}(R))$, where φ denotes projection modulo H. Since φ is surjective, $\varphi(L) = C^\circ_{G/H}(R)$, which is connected and a finite extension of $\varphi(L^\circ)$; so $\varphi(L) = \varphi(L^\circ)$ and $L = L^\circ H = L^\circ$ by connectedness of H. Hence L itself is connected. We now proceed by induction on the solubility class of H.

First suppose that H is abelian; we proceed by induction on the solubility class of R.

- First suppose that $R = \langle r \rangle$. Be careful to note that the definable map $\mathrm{Tr}_r : G \to G$ given by
$$\mathrm{Tr}_r(g) = \prod_{i=0}^{|r|-1} g^{r^i}$$
is *not* a group homomorphism, but $(\mathrm{Tr}_r)_{|H}$ is one.

Since $[L, r] \leq H \cap \mathrm{Tr}_r^{-1}(0) = \ker(\mathrm{Tr}_r)_{|H}$, one has by connectedness and Zilber's indecomposability theorem $[L, r] \leq \ker^\circ(\mathrm{Tr}_r)_{|H} = [H, r]$ by the proof of Lemma G. Bear in mind that H is abelian; it follows that $L \leq HC_G(r)$, so by connectedness $L \leq HC_G^\circ(r)$, as desired.

• Now suppose $R = \langle S, r\rangle$ with $S \lhd R$. By induction, $L \leq HC_G^\circ(S)$ and since $H \leq L$, one has $L \leq HC_L^\circ(S)$. Let $G_S = C_G^\circ(S)$ and $H_S = C_H^\circ(S)$; also let φ_S be the projection $G_S \to G_S/H_S$, and $L_S = \varphi_S^{-1}(C^\circ_{G_S/H_S}(r))$.

By the cyclic case, $L_S \leq H_S C^\circ_{G_S}(r) \leq HC_G^\circ(R)$. But $[C_L^\circ(S), r] \leq H \cap C_G^\circ(S)$ so by connectedness $[C_L^\circ(S), r] \leq C_H^\circ(S) = H_S$. It follows that $C_L^\circ(S) \leq L_S \leq HC_G^\circ(R)$ and $L \leq HC_L^\circ(S) \leq HC_G^\circ(R)$.

We now let $K = H'$, which is a definable, connected, R-invariant subgroup normal in G. Let $\varphi_K : G \to G/K$ and $\psi : G/K \to G/H$ be the standard projections, so that $\varphi = \psi\varphi_K$. By induction, $\varphi_K(C_G^\circ(R)) = C^\circ_{\varphi_K(G)}(R)$. But $\varphi_K(H) \trianglelefteq \varphi_K(G)$ and $\varphi_K(H)$ is abelian, so by the abelian case we just covered, $\psi(C^\circ_{\varphi_K(G)}(R)) = C^\circ_{\psi\varphi_K(G)}(R)$. Therefore,

$$\varphi(C_G^\circ(R)) = \psi(\varphi_K(C_G^\circ(R))) = \psi(C^\circ_{\varphi_K(G)}(R)) = C^\circ_{\psi\varphi_K(G)}(R) = C^\circ_{\varphi(G)}(R). \quad \square$$

The following inductive consequence will not be used in the present work.

Lemma I (cf. [ABC 2008, Proposition I.9.13]). *In a universe of finite Morley rank, let H be a definable, connected, soluble, $U_{\underline{p}}^\perp$ group and R be a soluble $\underline{p}$-group acting on H by definable automorphisms. Then $H = [H, R]C_H^\circ(R)$.*

Now let ρ denote a unipotence parameter. We wish to generalise [Bur 2004a, Lemma 3.6], relaxing the $\underline{p}^\perp$ assumption to $U_{\underline{p}}^\perp$. This will considerably simplify some arguments; in particular we shall no longer care whether Burdges' unipotent radicals of Borel subgroups contain involutions or not when taking centralisers. This will spare us the contortions of [*Del 2007a, Lemmes 5.2.33, 5.2.39, 5.3.20, 5.3.23].

Lemma J (cf. [Bur 2004a, Lemma 3.6]). *In a universe of finite Morley rank, let U be a definable, $U_{\underline{p}}^\perp$, ρ-group and R be a soluble $\underline{p}$-group acting on U by definable automorphisms. Then $C_U^\circ(R)$ is a ρ-group.*

Proof. The proof is by induction on the nilpotence class of U. First suppose that U is abelian. Then by Lemma G one has $U = [U, R] \oplus C_U^\circ(R)$. Let K stand for the finite intersection. Then $C_U^\circ(R)/K \simeq U/[U, R]$, which by push-forward [Bur 2004a, Lemma 2.11] is a ρ-group. It follows that $C_U^\circ(R)$ itself is a ρ-group. (Since we could not locate a proof of this trivial fact in the literature, here it goes: Let $V = C_U^\circ(R)$ and $\varphi : V \to V/K$ be the standard projection. By pull-back [Bur 2004a, Lemma 2.11], $\varphi(U_\rho(V)) = V/K = \varphi(V)$, and since $\ker\varphi$ is finite, $\mathrm{rk}\, U_\rho(V) = \mathrm{rk}\, V$. By connectedness, $V = U_\rho(V)$.)

Now let $1 < A \lhd U$ be an abelian, definable, connected, characteristic subgroup. By induction, $C_A^\circ(R)$ and $C_{U/A}^\circ(R)$ are ρ-groups. Now by Lemma H,

$$\begin{aligned}
C_{U/A}^\circ(R) &\simeq C_U^\circ(R)A/A \\
&\simeq C_U^\circ(R)/(A \cap C_U^\circ(R)) \\
&\simeq \big(C_U^\circ(R)/C_A^\circ(R)\big)/\big((A \cap C_U^\circ(R))/C_A^\circ(R)\big) \\
&= \big(C_U^\circ(R)/C_A^\circ(R)\big)/L,
\end{aligned}$$

where $L = (A \cap C_U^\circ(R))/C_A^\circ(R)$ is finite. Since $C_{U/A}^\circ(R)$ is a ρ-group, so is $C_U^\circ(R)/C_A^\circ(R)$. But $C_A^\circ(R)$ is a ρ-group, so by pull-back, so is $C_U^\circ(R)$. □

One could of course do the same with a set of unipotence parameters instead of a single parameter ρ.

Remark. As opposed to the usual setting of $\underline{p}^\perp$ groups [Bur 2004a, Lemma 3.6], connectedness of $C_U(R)$ is not granted in the $U_{\underline{p}}^\perp$ case: think of an involutive automorphism inverting a ρ-group which contains a nontrivial 2-torus.

As a consequence, if inside a group of odd type some involution i acts on a σ-group H with $\rho_{C(i)} \prec \sigma$, then i inverts H. We shall use this fact with no reference.

3.4. *Carter π-subgroups.* Section 2.3 recalled the maybe not-so-familiar notion of a Carter π-subgroup. Bear in mind that by definition, π-groups are nilpotent.

Lemma K. *Let H be a connected, soluble group of finite Morley rank, π be a set of parameters such that $U_\pi(H') = 1$, and $L \leq H$ be a maximal π-subgroup. Then there is a Carter subgroup $Q \leq H$ of H with $L = U_\pi(Q)$.*

Proof. It suffices to show that for any π-subgroup $L \leq H$ there is a Carter subgroup Q of H with $L \leq Q$.

If $|\pi| = 1$ then we are actually dealing with a single unipotence parameter ρ, and the result follows from the theory of Sylow ρ-subgroups [Bur 2004b, Lemma 4.19; 2006, Theorem 5.7]. If $|\pi| > 1$, write Burdges' decomposition of $L = L_\rho * M$, where ρ is any unipotence parameter occurring in L, $L_\rho = U_\rho(L)$, and M is a $(\pi \setminus \{\rho\})$-group. By induction there is a Carter subgroup Q of H with $L_\rho \leq Q$.

Now consider the generalised centraliser (a tool we already used in the proof of Lemma C) $E = E_H(L_\rho) \geq \langle Q, M\rangle$. If $E < H$, then by induction on the Morley rank, L is contained in some Carter subgroup of E. Since $Q \leq E$, the former also is a Carter subgroup of H.

So we may assume $E = H$, and therefore $L_\rho \leq F^\circ(H)$ [Fré 2000a, Corollaire 5.17]. Actually we may assume this for any parameter ρ, meaning $L \leq F^\circ(H)$. Now Q acts on $U_\pi(F^\circ(H))$ so

$$[Q, U_\pi(F^\circ(H))] \leq U_\pi(H') = 1 \quad \text{and} \quad L \leq U_\pi(F^\circ(H)) \leq N_H(Q) = Q. \quad \square$$

3.5. $W_p^\perp$ *groups.* Weyl groups of minimal connected simple groups have been abundantly discussed [AB 2008; BC 2009; BD 2010; Fré 2010]. We do not feel utterly interested now; as a consequence we shall not even define Weyl groups. Instead we shall develop a more limited view which will suffice for our purposes. This line is very much in the spirit of [BP 1990], the influence of which on later work should not be concealed.

Notation. Let G be a $U_p^\perp$ group of finite Morley rank. Let $W_p(G) = S/S^\circ$ for any Sylow p-subgroup S of G (these are conjugate by [BC 2009, Theorem 4], our Fact 5, so this is well-defined).

Lemma L. *Let G be a $U_p^\perp$ group of finite Morley rank.*

(i) *If $H \leq G$ is a definable, connected subgroup, then $W_p(H) \hookrightarrow W_p(G)$.*

(ii) *If $H \trianglelefteq G$ is a definable, normal subgroup, then $W_p(G) \twoheadrightarrow W_p(G/H)$.*

(iii) *If $H \trianglelefteq G$ is a definable, connected, normal subgroup, then*

$$W_p(G/H) \simeq W_p(G)/W_p(H).$$

(iv) *If G is connected and $H \leq Z(G)$ is a central subgroup, then*

$$W_p(G/H) \simeq W_p(G).$$

Proof. (i) Let S_H be a Sylow p-subgroup of H and extend it to a Sylow p-subgroup S_G of G. To $w \in W_p(H)$ associate $hS_G^\circ \in W_p(G)$, where $h \in S_H$ is such that $hS_H^\circ = w$. This is a well-defined group homomorphism as $S_H^\circ \leq S_G^\circ$. It is injective since if $h \in S_H \cap S_G^\circ$, then $h \in C_{S_H}(S_H^\circ) = S_H^\circ$ by torality principles and connectedness of H.

(ii) Let $S_H \leq S_G$ be as above and denote projection modulo H by $\overline{}$; we know that $\Sigma = \overline{S_G} \simeq S_G/S_H$ is a Sylow p-subgroup of G/H. To $w \in W_p(G)$ associate $\bar{g}\Sigma^\circ \in W_p(G/H)$, where $g \in S_G$ is such that $gS_G^\circ = w$. This is a well-defined group homomorphism as $\overline{S_G^\circ} = \Sigma^\circ$. It is clearly surjective.

(iii) Suppose in addition that H is connected. With notation as in the argument for claim (ii), if w is in the kernel then $g \in S_G^\circ H$, and we may suppose $g \in H$ (the converse is obvious). Hence the kernel coincides with the image of $W_p(H)$ in $W_p(G)$ given by claim (i).

(iv) By claim (ii) the map $W_p(G) \to W_p(G/H)$ is a surjective group homomorphism; now if $gS_G^\circ \in W_p(G)$ lies in the kernel, since H is central in G one finds $g \in S_G \cap (HS_G^\circ) \leq C_{S_G}(S_G^\circ) = S_G^\circ$ by torality principles and connectedness of G. So the map is injective and $W_p(G) \simeq W_p(G/H)$. □

Remarks. • In claims (i) and (iii), connectedness of H is necessary: consider $\mathbb{Z}/2\mathbb{Z}$ inside $\mathbb{Z}_{2^\infty}$, then inside $\mathrm{SL}_2(\mathbb{C})$.

• As a consequence, if G is connected and $H \trianglelefteq G$ is a definable, normal subgroup, then $W_p(G/H) \simeq W_p((G/H^\circ)/(H/H^\circ)) \simeq W_p(G/H^\circ) \simeq W_p(G)/W_p(H^\circ)$.

• Lemma L could be used as a qualifying test for tentative notions of the Weyl group.

We wish to suggest a bit of terminology.

Definition. A $U_p^\perp$ group of finite Morley rank is $W_p^\perp$ if its Sylow p-subgroups are connected.

As a consequence of Lemma L, when $H \trianglelefteq G$, where both are definable and connected, if H and G/H are $W_p^\perp$ then so is G. We aim at saying a bit more about extending tori. The following result is not used anywhere in the present article.

Lemma M. *Let $\hat{G}$ be a connected, $U_p^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected subgroup. Suppose that $\hat{G}/G$ is $W_p^\perp$. Let $\hat{S} \leq \hat{G}$ be a Sylow p-subgroup and $S = \hat{S} \cap G$. Then there exist*

- *a p-torus $\hat{T} \leq \hat{G}$ with $\hat{S} = S \rtimes \hat{T}$ (semidirect product);*
- *a p-torus $\hat{\Theta} \leq \hat{G}$ with $\hat{S} = S \circledast \hat{\Theta}$ (central product over a finite intersection).*

Proof. We know that S is a Sylow p-subgroup of G and that $\hat{S}/S \simeq \hat{S}G/G$ is a Sylow p-subgroup of $\hat{G}/G$; as the latter is $W_p^\perp$ it is a p-torus. In particular, $\hat{S} = \hat{S}^\circ S$. Note that $S \cap \hat{S}^\circ \leq C_S(S^\circ) = S^\circ$ by torality principles and connectedness of G.

Bear in mind that p-tori are injective as $\mathbb{Z}$-modules. Inside $\hat{S}^\circ$, take a direct complement $\hat{T}$ of S°, so that $\hat{S}^\circ = S^\circ \oplus \hat{T}$. Then $\hat{S} = S\hat{S}^\circ = S\hat{T}$, but $S \cap \hat{T} \leq S \cap \hat{S}^\circ \cap \hat{T} \leq S^\circ \cap \hat{T} = 1$. Hence $\hat{S} = S \rtimes \hat{T}$.

We now consider the action of $\hat{S}$ on $\hat{S}^\circ$; observe that $\hat{S}$ as a pure group has finite Morley rank, so Lemma G applies and yields $\hat{S}^\circ = [\hat{S}^\circ, \hat{S}] \oplus C^\circ_{\hat{S}^\circ}(\hat{S})$. Since $\hat{S}/S$ is a p-torus, it is abelian, so $[\hat{S}^\circ, \hat{S}] \leq \hat{S}' \leq S$, and by Zilber's indecomposability theorem $[\hat{S}^\circ, \hat{S}] \leq S^\circ$. Inside $C^\circ_{\hat{S}^\circ}(\hat{S})$ take a direct complement $\hat{\Theta}$ of $C^\circ_{S^\circ}(\hat{S})$, so that $C^\circ_{\hat{S}^\circ}(\hat{S}) = C^\circ_{S^\circ}(\hat{S}) \oplus \hat{\Theta}$. Then $\hat{S} = S\hat{S}^\circ = SC^\circ_{\hat{S}^\circ}(\hat{S}) = S\hat{\Theta}$, and $\hat{\Theta} \leq C^\circ_{\hat{S}^\circ}(\hat{S})$ commutes with S. Moreover $(S \cap \hat{\Theta})^\circ \leq (C_S(\hat{S}) \cap \hat{\Theta})^\circ \leq C^\circ_{S^\circ}(\hat{S}) \cap \hat{\Theta} = 1$ by construction, so $\hat{S} = S \circledast \hat{\Theta}$. □

Remark. One may not demand that $\hat{S} = S \times \hat{T}$ (direct product). Indeed, consider the two groups $\mathrm{SL}_2(\mathbb{C})$ with involution i and $\mathbb{C}^\times$ with involution j. Let $\hat{G} = (\mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times)/\langle ij \rangle$ and let $\varphi : \mathrm{SL}_2(\mathbb{C}) \times \mathbb{C}^\times \to \hat{G}$ be the standard projection. Let $G = \varphi(\mathrm{SL}_2(\mathbb{C})) \simeq \mathrm{SL}_2(\mathbb{C})$ and $\hat{\Theta} = \varphi(\mathbb{C}^\times) \simeq \mathbb{C}^\times$. Fix any Sylow 2-subgroup $\hat{S}$ of $\hat{G}$. Then with $S = \hat{S} \cap G$ one has $S\hat{\Theta} = S \circledast \hat{\Theta} = \hat{S}$, and $S \cap \hat{\Theta} = \langle \varphi(i) \rangle$.

If one asks for a semidirect complement $\hat{T}$, the latter must contain its own involution, which will be $\varphi(ab)$ (or possibly $\varphi(iab)$, a similar case), where $a \in \varphi^{-1}(S) \leq \mathrm{SL}_2(\mathbb{C})$ satisfies $a^2 = i$ and $b^2 = j$ in $\mathbb{C}^\times$. Remember that inside a fixed Sylow

2-subgroup of $\mathrm{SL}_2(\mathbb{C})$, every element of order four (be it toral inside the fixed Sylow 2-subgroup or not) is inverted by another element of order four. So let $\zeta \in \varphi^{-1}(S)$ invert a. Then

$$\varphi(\zeta^{ab}) = \varphi(\zeta^a) = \varphi(i\zeta) \neq \varphi(\zeta),$$

so the action of $\hat{T}$ on S is always nontrivial.

One may not demand $\hat{S} = S \times \hat{T}$, and in any case nothing can apparently prevent $d(\hat{T})$ from intersecting G nontrivially, so the question is rather pointless.

3.6. *A counting lemma.* The following quite elementary lemma was devised in Cappadocia in 2007 as an explanation of [*Del 2007a, Corollaire 5.1.7] (or [*Del 2008, Corollaire 4.7]). It will be used only once.

Lemma N (Göreme). *Let G be a connected, $U_2^\perp$, $W_2^\perp$ group of finite Morley rank. Then the number of conjugacy classes of involutions is odd* (*or zero*).

Proof. By torality principles, every class is represented in a fixed Sylow 2-subgroup $S = S^\circ$. We group involutions of S° by classes γ_k, and assume we find an even number of these: $I(S^\circ) = \bigsqcup_{k=1}^{2m} \gamma_k$. Since the number of involutions in S° is however odd, some class, say γ, has an even number of involutions. Now $N = N_G(S)$ acts on γ; by definition of a conjugacy class and by a classical fusion control argument [BN 1994b, Lemma 10.22], N acts transitively on γ. Hence $[N : C_N(\gamma)] = |\gamma|$ is even. Lifting torsion, there is a nontrivial 2-element ζ in $N \setminus C_N(\gamma)$. Since $S \trianglelefteq N$, one has $\zeta \in S = S^\circ \leq C_N(\gamma)$, a contradiction. □

The author hoped to be able to use this lemma without any form of bound on the Prüfer 2-rank. He failed as one shall see in Step 6 of the theorem. The general statement remains as a relic of happier times past.

4. The proof — before the maximality proposition

Theorem. *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Then the Sylow 2-subgroup of G has one of the following structures: isomorphic to that of* $\mathrm{PSL}_2(\mathbb{C})$, *isomorphic to that of* $\mathrm{SL}_2(\mathbb{C})$, *or a 2-torus of Prüfer 2-rank at most 2.*

Suppose in addition that for all involutions $\iota \in I(\hat{G})$, the group $C_G^\circ(\iota)$ is soluble. Then $m_2(\hat{G}) \leq 2$, one of G or $\hat{G}/G$ is $2^\perp$, and involutions are conjugate in $\hat{G}$. Moreover, one of the following cases occurs:

- **PSL_2:** $G \simeq \mathrm{PSL}_2(\mathbb{K})$ *in characteristic not 2; $\hat{G}/G$ is $2^\perp$.*
- **$\mathrm{CiBo}_\varnothing$:** *$G$ is $2^\perp$; $m_2(\hat{G}) \leq 1$; for $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ is a self-normalising Borel subgroup of G.*
- **CiBo_1:** *$m_2(G) = m_2(\hat{G}) = 1$; $\hat{G}/G$ is $2^\perp$; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G.*

- **CiBo$_2$:** $\mathrm{Pr}_2(G) = 1$ *and* $m_2(G) = m_2(\hat{G}) = 2$; $\hat{G}/G$ *is* $2^\perp$; *for* $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ *is an abelian Borel subgroup of* G *inverted by any involution in* $C_G(i) \setminus \{i\}$ *and satisfies* $\operatorname{rk} G = 3 \operatorname{rk} C_G^\circ(i)$.
- **CiBo$_3$:** $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; $\hat{G}/G$ *is* $2^\perp$; *for* $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ *is a self-normalising Borel subgroup of* G; *if* $i \neq j$ *are two involutions of* G *then* $C_G(i) \neq C_G(j)$.

The proof requires eight propositions all strongly relying on the $N_\circ^\circ$ assumption, the deepest of which will be Proposition 6. Let us briefly describe the global outline. More detailed information will be found before each proposition.

In Proposition 1 (Section 4.1) we determine the 2-structure of $N_\circ^\circ$-groups by elementary methods. Proposition 2 (Section 4.2) is a classical rank computation required both by Proposition 3 (Section 4.3) which identifies $\mathrm{PSL}_2(\mathbb{K})$ through reconstruction of its BN-pair, and by Proposition 6 which shows that in nonalgebraic configurations centralisers° of involutions are Borel subgroups. The proof may be of interest to experts in finite group theory; perhaps they will find something unexpected there. Proposition 6 will take all of Section 5 but actually requires two more technical preliminaries: Propositions 4 (Section 4.4) and 5 (Section 4.5), which deal with actions of involutions and torsion, respectively. After Proposition 6 things go faster. We study the action of an infinite dihedral group in Proposition 7 (Section 6.1) and a strong embedding configuration in Proposition 8 (Section 6.2). Both are rather classical, methodologically speaking; Proposition 7 is more involved than Proposition 8; they can be read in any order but both rely on maximality. The final assembling takes place in Section 6.3 where all preliminary Propositions 1, 2, 4 and 5 reappear as independent themes.

The resulting architecture surprised the author. In the original minimal connected simple setting one proceeded by first bounding the Prüfer 2-rank [*BCJ 2007] and then studying the remaining cases [*Del 2007b; 2008]. There maximality propositions had to be proved three times in order to complete the analysis. The reason for such a clumsy treatment, with one part of the proof being repeated over and over again, was that torsion arguments were systematically based on some control on involutions.

Here we do the opposite. By providing careful torsion control in Proposition 5 and relaxing our expectations on conjugacy classes of involutions we shall be able to run maximality without prior knowledge of the Prüfer 2-rank. This seems to be the right level both of elegance and generality. Bounding the Prüfer 2-rank then follows by adapting a small part of [*BCJ 2007].

Before the curtain opens one should note that bounding the Prüfer 2-rank of $\hat{G}$ a priori is possible if one assumes G to be $2^\perp$, as Burdges noted for [*BCD 2009]. We do not follow this line.

4.1. ***The 2-structure proposition.*** Proposition 1 hereafter comes directly from [*Del 2007a, Chapitre 4 and Addendum], published as [*Del 2008, §2]. It is the most elementary of our propositions, and, together with Proposition 8, one of the two not requiring almost-solubility of centralisers of involutions.

Proposition 1 (2-Structure). *Let G be a connected, $U_2^\perp$, $N_\circ^\circ$-group of finite Morley rank. Then the Sylow 2-subgroup of G has the following form*:

- *connected, i.e., a possibly trivial 2-torus*;
- *isomorphic to that of* $\mathrm{PSL}_2(\mathbb{C})$;
- *isomorphic to that of* $\mathrm{SL}_2(\mathbb{C})$, *in which case $C_G^\circ(i)$ is nonsoluble for any involution i of G.*

Proof. If the Prüfer rank is 0 this is a consequence of the analysis of degenerate type groups [BBC 2007]. If it is 1, this is well-known; see for reference [*DJ 2010, Proposition 27]. Notice that if the Sylow 2-subgroup is as in $\mathrm{SL}_2(\mathbb{C})$ and i is any involution, then by torality principles (our Fact 6) all Sylow 2-subgroups of $C_G(i)$ are in $C_G^\circ(i)$, but none is connected: this, and the structure of torsion in connected, soluble groups of finite Morley rank, prevents $C_G^\circ(i)$ from being soluble.

So we suppose that the Prüfer 2-rank is at least 2 and show that a Sylow 2-subgroup S of G is connected. Let G be a minimal counterexample to this statement. Then G is nonsoluble. Since G is an $N_\circ^\circ$-group, $Z(G)$ is finite, but we actually may suppose that G is centreless. For if the result holds of $G/Z(G)$, then $SZ(G)/Z(G)$ is a Sylow 2-subgroup of $G/Z(G)$, and therefore connected, so that $S \leq S^\circ Z(G) \cap S \leq C_S(S^\circ) = S^\circ$ by torality principles. Since $G/Z(G)$ is centreless we may therefore assume $Z(G) = 1$.

Still assuming that the Prüfer 2-rank is at least 2 we let $\zeta \in S \setminus S^\circ$ have minimal order, so that $\zeta^2 \in S^\circ$. Let $\Theta_1 = C_{S^\circ}^\circ(\zeta)$. If $\Theta_1 \neq 1$ then $\langle S^\circ, \zeta\rangle \leq C_G(\Theta_1)$, which is connected by [AB 2008, Theorem 1] and soluble since G is an $N_\circ^\circ$-group. The structure of torsion in such groups yields $\zeta \in S^\circ$, a contradiction. So $\Theta_1 = C_{S^\circ}^\circ(\zeta) = 1$ and ζ therefore inverts S°. In particular ζ centralises the group $\Omega = \Omega_2(S^\circ)$ generated by involutions of S°, and Ω normalises $C_G^\circ(\zeta)$. By normalisation principles Ω normalises a maximal 2-torus T of $C_G^\circ(\zeta)$; by torality principles, $\zeta \in T$ and hence T has the same Prüfer 2-rank as S. Now $|\Omega| \geq 4$ so there is $i \in \Omega$ such that $\Theta_2 = C_T^\circ(i)$ is nontrivial. Then $\langle T, i\rangle \leq C_G(\Theta_2)$, which is soluble and connected as above, implying $i \in T$. This is not a contradiction yet, but now $\zeta \in T \leq C_G^\circ(i)$ and of course $S^\circ \leq C_G^\circ(i)$. Hence $C_G^\circ(i) < G$ is a smaller counterexample, a contradiction. Connectedness is proved. □

Remark. One can show that if $\alpha \in G$ is a 2-element with $\alpha^2 \neq 1$, then $C_G(\alpha)$ is connected.

Let $\alpha \in G$ have order 2^k with $k > 1$. By Steinberg's torsion theorem (our Fact 4), $C_G(\alpha)/C_G^\circ(\alpha)$ has exponent dividing 2^k. Using torality principles, fix a maximal 2-torus T of G containing α. If the Sylow 2-subgroup of G is connected, then T is a Sylow 2-subgroup of G included in $C_G^\circ(\alpha)$; hence $C_G(\alpha) = C_G^\circ(\alpha)$. If the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$ or to that of $\mathrm{SL}_2(\mathbb{C})$, then any 2-element $\zeta \in C_G(\alpha)$ normalising T centralises α of order at least 4, so it also centralises T. It follows from torality principles that $\zeta \in T \le C_G^\circ(\alpha)$, and $C_G(\alpha)$ is connected again.

We shall not use this remark.

4.2. *The genericity proposition.*

> *Considerations concerning the distribution of involutions in the cosets of a given subgroup are often useful in the study of groups of even order.*

So wrote Bender in the beginning of [Bender 1974a]. The first instance of this method in the finite Morley rank context seems to be [BDN 1994, after Lemma 7] which with [BN 1994a] aimed at identifying $\mathrm{SL}_2(\mathbb{K})$ in characteristic 2. Jaligot brought it to the odd type setting [*Jal 2000]. The present subsection is the cornerstone of Propositions 3 and 6 and is used again when conjugating involutions in Step 5 of the final argument. We introduce subsets of a group H describing the distribution of involutions in the translates of H.

Notation. For κ an involutive automorphism and H a subgroup of some ambient group, we let $T_H(\kappa) = \{h \in H : h^\kappa = h^{-1}\}$. (This set is definable as soon as κ and H are.)

The following is completely classical; the proof will not surprise the experts and is included for the sake of self-containedness. It will be applied only when H is a Borel subgroup of G.

Proposition 2 (genericity). *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Suppose that $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus $\hat{S}^\circ$ of $\hat{G}$. Let $\iota \in I(\hat{G})$ and $H \le G$ be a definable, infinite, soluble subgroup of G. Then*

$$K_H = \left\{\kappa \in \iota^{\hat{G}} \setminus N_{\hat{G}}(H) : \operatorname{rk} T_H(\kappa) \ge \operatorname{rk} H - \operatorname{rk} C_G(\iota)\right\}$$

is generic in $\iota^{\hat{G}}$.

Proof. This is vacuous for central ι. Now the statement is invariant under conjugating $\hat{S}^\circ$ so by torality principles we may assume $\iota \in \hat{S}^\circ$; in particular $\iota^{\hat{G}} = \iota^G$. We shall first show that $\iota^{\hat{G}} \setminus N_{\hat{G}}(H)$ is generic in $\iota^{\hat{G}}$. Lemmas 2.16 and 3.33 of [*DJ 2012] were supposed to do this, but they only apply when $\iota \in G$. Minor work must be added.

Suppose that $\iota^{\hat{G}} \setminus N_{\hat{G}}(H)$ is not generic in $\iota^{\hat{G}}$. Then by a degree argument, $\iota^{\hat{G}} \cap N_{\hat{G}}(H)$ is generic in $\iota^{\hat{G}}$. Inside $\hat{G}$ apply [*DJ 2012, Lemma 2.16] with $X = \iota^{\hat{G}}$ and $M = N_{\hat{G}}(H)$: $X \cap M$ contains a definable, $\hat{G}$-invariant subset X_1 which is generic in X. Note that X is infinite as otherwise ι inverts $\hat{G}$, so X_1 is infinite as well; since X has degree 1 by connectedness of $\hat{G}$, so does X_1. We cannot directly apply [*DJ 2012, Lemma 3.33] as $\hat{G}$ itself need not be $N_\circ^\circ$. So let $X_2 = \{\kappa\lambda : \kappa, \lambda \in X_1\}$, which is an infinite, $\hat{G}$-invariant subset of $N_{\hat{G}}(H)$. Since

$$X_1 \subseteq \iota^{\hat{G}} = \iota^G \subseteq \iota G = G\iota,$$

X_2 is actually a subset of G. Hence $X_2 \subseteq N_G(H)$. The latter need not be soluble but is a finite extension of $N_G^\circ(H)$, which is. Since X_2 is infinite and has degree 1 like X_1, there is a generic subset X_3 of X_2 which is contained in some translate $nN_G^\circ(H)$ of $N_G^\circ(H)$, where $n \in N_G(H)$. Then $X_3 \subseteq N_G^\circ(H) \cdot \langle n \rangle$ which is a definable, soluble group we denote by M_2; X_3 itself may fail to be G-invariant. But X_2 is a G-invariant subset such that $X_3 \subseteq X_2 \cap M_2$ is generic in X_2. By [*DJ 2012, Lemma 3.33] applied in $G = G^\circ$ to X_2 and M_2, G is soluble: a contradiction.

The end of the proof is rather worn-out. We consider the definable function $\varphi : \iota^{\hat{G}} \setminus N_{\hat{G}}(H) \to G \cdot \langle \iota \rangle / H$ which maps κ to κH. The domain has rank $\operatorname{rk} \iota^{\hat{G}} = \operatorname{rk} \iota^G = \operatorname{rk} G - \operatorname{rk} C_G(\iota)$. The image set has rank at most $\operatorname{rk} G - \operatorname{rk} H$. So the generic fibre has rank at least $\operatorname{rk} H - \operatorname{rk} C_G(\iota)$. But if κ, λ lie in the same fibre, then $\kappa H = \lambda H$ and $\kappa\lambda \in T_H(\kappa)$. Hence, for generic κ,

$$\operatorname{rk} T_H(\kappa) \geq \operatorname{rk} \varphi^{-1}(\varphi(\kappa)) \geq \operatorname{rk} H - \operatorname{rk} C_G(\iota). \qquad \square$$

As it turns out, the algebraic properties of $T_H(\kappa)$ are not always as good as one may wish, and one then focuses on the following sets instead.

Notation. For κ an involutive automorphism and H a subgroup of some ambient group, we let $\mathbb{T}_H(\kappa) = \{h^2 \in H : h^\kappa = h^{-1}\} \subseteq T_H(\kappa)$. (This set is definable as soon as κ and H are.)

There is no a priori estimate on $\operatorname{rk} \mathbb{T}_H(\kappa)$, and Proposition 5 will remedy this. The $\mathbb{T}$ sets were denoted τ in [*Del 2007a]; interestingly enough, they were already used in [*BCJ 2007, Notation 7.4].

4.3. ***The algebraicity proposition.*** We now return to the historical core of the subject.

Identifying $\mathrm{SL}_2(\mathbb{K})$ is a classical topic in finite group theory. Proposition 3 may be seen as a very weak form of the Brauer–Suzuki–Wall theorem [Brauer et al. 1958] in odd characteristic. However [Brauer et al. 1958] heavily relied on character theory, a tool not available in and perhaps not compatible in spirit with the context of groups of finite Morley rank. (One may even interpret the expected failure of the Feit–Thompson theorem in our context as evidence for this thesis.) A character-free proof of outstanding elegance was found by Goldschmidt. Yet his article [Goldschmidt 1974] dealt only with the characteristic 2 case, and ended on the conclusive remark:

> *Finally, some analogues of Theorem 2* [Goldschmidt's version of BSW] *may hold for odd primes but* [. . .] *this problem seems to be very difficult.*

Bender's investigations in odd characteristic [Bender 1974b; 1981] both require some character theory. We do not know of a general yet elementary identification theorem for $\mathrm{PSL}(2, q)$ with odd q, and hope that the present paper will help ask the question.

In the finite Morley rank context various results identifying $\mathrm{PSL}_2(\mathbb{K})$ exist, starting with Cherlin's very first article in the field [Che 1979] and Hrushovski's generalisation [Hru 1989]. For groups of even type, [BDN 1994; BN 1994a] provide identification using heavy rank computations. In a different spirit, the reworking of Zassenhaus' classic [Zassenhaus 1935] by Nesin [Nes 1990a] and its extension [DN 1995] identify $\mathrm{PSL}_2(\mathbb{K})$ among 3-transitive groups; the latter gives a very handy statement.

Most of the ideas in the proof below are in [*Del 2007b] and in many other articles before. Only two points need be commented on.

- First, we shift from the tradition as in [*CJ 2004; *Del 2007b] of invoking the results on permutation groups Nesin had ported to the finite Morley rank context ([DN 1995], see above).

We decided to use final identification arguments based on the theory of Moufang sets instead. At that point of the analysis the difference may seem essentially cosmetic but the Moufang setting is in our opinion more appropriate as it focuses on the BN-pair. We now rely on recent work by Wiscons [Wis 2011].

(Incidentally, Nesin had started thinking about BN-pairs in prison [Nes 1990b] but was released before reaching an identification theorem for $\mathrm{PSL}_2(\mathbb{K})$ in this context; not returning to gaol he apparently never returned to the topic.)

- Second, we refrained from using Frécon homogeneity. This makes the proof only marginally longer in Step 3. The reasons for doing so were consistency with not using it in Proposition 6, and the mere challenge as it was thought a few years ago to be unavoidable.

Proposition 3 (algebraicity). *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Suppose that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$. Suppose that there exists $\iota \in I(\hat{G})$ such that $C_G^\circ(\iota)$ is contained in two distinct Borel subgroups. Then G has the same Sylow 2-subgroup as* $\mathrm{PSL}_2(\mathbb{K})$. *If in addition $\iota \in G$, then $G \simeq$* $\mathrm{PSL}_2(\mathbb{K})$, *where $\mathbb{K}$ is an algebraically closed field of characteristic not* 2.

Proof. Since $\hat{G}$ is connected, every involution ι is toral: say $\iota \in \hat{S}^\circ$ a 2-torus. We may therefore assume that $\hat{G} = G \cdot d(\hat{S}^\circ)$, so that the standard rank computations of Proposition 2 apply. Moreover, $\hat{G}/G$ is connected and abelian, hence $W_2^\perp$.

Notation. • Let $B \geq C_G^\circ(\iota)$ be a Borel subgroup of G maximising ρ_B; let $\rho = \rho_B$.

• Let $K_B = \{\kappa \in \iota^{\hat{G}} \setminus N_{\hat{G}}(B) : \operatorname{rk} T_B(\kappa) \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota)\}$; by Proposition 2, K_B is generic in $\iota^{\hat{G}}$.

• Let $\kappa \in K_B$.

Note that it is not clear at this point whether ι normalises B.

Step 1. $U_\rho(C_G^\circ(\iota)) = 1$. *If $U \leq B$ is a nontrivial ρ-group, $H \leq G$ is a definable, connected subgroup of G containing U, and $\lambda \in \iota^{\hat{G}}$ normalises H, then λ normalises B.*

Proof of Step 1. Throughout this proof, letting $Y_B = U_\rho(Z(F^\circ(B)))$ will spare a few parentheses; by Fact 7(iii), $Y_B \neq 1$.

Suppose $U_\rho(C_G^\circ(\iota)) \neq 1$. Let $D \neq B$ be a Borel subgroup of G containing $C_G^\circ(\iota)$ and maximising $H = (B \cap D)^\circ$; such a Borel subgroup exists by assumption on $C_G^\circ(\iota)$. By construction $\rho_D \succcurlyeq \rho_\iota = \rho_B \succcurlyeq \rho_D$, so all are equal. If H is not abelian then by [*DJ 2012, 4.50(3) and (6) (our Fact 10)] $\rho_B \neq \rho_D$, a contradiction. Hence H is abelian, and in particular $C_G^\circ(\iota) \leq H \leq C_G^\circ(U_\rho(H))$ which is a soluble group; by definition of B, the parameter of $C_G^\circ(U_\rho(H))$ is ρ. It follows from uniqueness principles (Fact 8) that $U_\rho(H)$ is contained in a unique Sylow ρ-subgroup of G. This must be $U_\rho(B) = U_\rho(D)$, so $B = D$: a contradiction.

We just proved $\rho_\iota \prec \rho$. It follows from Lemma J that for any $\sigma \succcurlyeq \rho$, any ι-invariant σ-group is inverted by ι. Now let U, H, and λ be as in the statement. There is a Sylow ρ-subgroup V of H containing U. By normalisation principles λ has an H-conjugate μ normalising V, so μ inverts $V \geq U$.

Let $C = C_G^\circ(U)$, a definable, connected, soluble group. Since $U \leq U_\rho(B)$, one has $Y_B \leq C$. So there is a Sylow ρ-subgroup W of C containing Y_B. As μ inverts U it normalises C; by normalisation principles μ has a C-conjugate ν normalising W, so ν inverts $W \geq Y_B$. Now ν also inverts $U_{\rho_C}(C)$, and commutation principles (our Fact 1) yield $[U_{\rho_C}(C), Y_B] = 1$, whence $U_{\rho_C}(C) \leq C_G^\circ(Y_B) \leq B$. At this point it is clear that $\rho_C = \rho$ and $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing U by uniqueness principles.

On the other hand μ inverts $U_{\rho_H}(H)$ and U, so by commutation principles $[U_{\rho_H}(H), U] = 1$ and $U_{\rho_H}(H) \leq C$, meaning that $\rho_H = \rho$ as well. Hence λ inverts $U_{\rho_H}(H) = U_\rho(H) \geq U$. Since $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing U, it follows that λ normalises B. $\square$

Notation. Let $L_\kappa = B \cap B^\kappa$ and $\Theta_\kappa = \{\ell \in L_\kappa : \ell\ell^\kappa \in L_\kappa'\}$.

Step 2. *L_κ and Θ_κ are infinite, definable, κ-invariant, abelian-by-finite groups. Moreover, $\Theta_\kappa^\circ \subseteq T_B(\kappa) \subseteq \Theta_\kappa$.*

Proof of Step 2. L'_κ is finite since we otherwise let $H = C_G^\circ(L'_\kappa) \geq U_\rho(Z(F^\circ(B)))$, which is definable, connected, and soluble since G is an $N_\circ^\circ$-group; Step 1 shows that κ normalises B, contradicting its choice in the notation preceding Step 1. It follows that L_κ° is abelian and L_κ is abelian-by-finite. Θ_κ is clearly a definable, κ-invariant subgroup of L_κ, so it is abelian-by-finite as well. By construction $T_B(\kappa) \subseteq \Theta_\kappa$, and Θ_κ is therefore infinite.

We now consider the action of κ on Θ_κ° and find according to Lemma G a decomposition $\Theta_\kappa^\circ = C^\circ_{\Theta_\kappa^\circ}(\kappa) \oplus [\Theta_\kappa^\circ, \kappa]$. Now the definable function $\varphi : C^\circ_{\Theta_\kappa^\circ}(\kappa) \to L'_\kappa$ which maps t to $tt^\kappa = t^2$ is a group homomorphism, so by connectedness and since L'_κ is finite, $C^\circ_{\Theta_\kappa^\circ}(\kappa)$ has exponent 2: it is trivial. So κ inverts Θ_κ°, meaning $\Theta_\kappa^\circ \subseteq T_B(\kappa)$. □

Notation. Let $U \leq [U_\rho(Z(F^\circ(B))), \Theta_\kappa^\circ]$ be a nontrivial, Θ_κ°-invariant ρ-subgroup minimal with these properties.

Step 3. *U exists and $C_U^\circ(\iota) = 1$; $C_{\Theta_\kappa^\circ}(U)$ is finite and there exists an algebraically closed field structure $\mathbb{K}$ with $U \simeq \mathbb{K}_+$ and $\Theta_\kappa^\circ / C_{\Theta_\kappa^\circ}(U) \simeq \mathbb{K}^\times$. Moreover, G has the same Sylow 2-subgroup as* $\mathrm{PSL}_2(\mathbb{K})$.

Proof of Step 3. Here again we let $Y_B = U_\rho(Z(F^\circ(B))) \neq 1$.

If Θ_κ° centralises Y_B then the κ-invariant, definable, connected, soluble group $C_G^\circ(\Theta_\kappa^\circ)$ contains Y_B and Step 1 forces κ to normalise B, against its choice in the notation preceding Step 1. Hence $[Y_B, \Theta_\kappa^\circ] \neq 1$; it is a ρ-group (Fact 7(vi); no need for Frécon homogeneity here).

We show that $C_U^\circ(\iota) = 1$; be careful to note that ι need not normalise U nor even B. Yet if $C_U^\circ(\iota)$ is infinite then Step 1 applied to $C_G^\circ(C_U^\circ(\iota)) \geq Y_B$ forces ι to normalise B, and then ι inverts $U_\rho(B) \geq U \geq C_U^\circ(\iota)$: a contradiction.

Suppose that $C_{\Theta_\kappa^\circ}(U)$ is infinite; Step 1 applied to $C_G^\circ(C_{\Theta_\kappa^\circ}(U)) \geq U$ forces κ to normalise B: a contradiction. We now wish to apply Zilber's field theorem. It may look like we fall short of Θ_κ°-minimality but fear not. Follow for instance the proof in [BN 1994b, Theorem 9.1]. It suffices to check that any nonzero r in the subring of $\mathrm{End}(U)$ generated by Θ_κ° is actually an automorphism. But by push-forward [Bur 2004a, Lemma 2.11], $\operatorname{im} r \simeq U/\ker r$ is a nontrivial, Θ_κ°-invariant ρ-subgroup. By minimality of U as such, r is surjective. In particular $\ker r$ is finite. Suppose it is nontrivial and form, like in [BN 1994b, Theorem 9.1], the chain $(\ker r^n)$. Each term is Θ_κ°-central by connectedness, so $C_U^\circ(\Theta_\kappa^\circ)$ contains an infinite torsion subgroup A. If there is some torsion unipotence then $A = U$ by minimality as a ρ-group, and Θ_κ° centralises U: a contradiction. So A contains a nontrivial q-torus for some prime number q. This means that there is a q-torus in $[Y_B, \Theta_\kappa^\circ] \leq B'$ which contradicts, for instance, [Fré 2000b, Proposition 3.26]. Hence every $r \in \langle \Theta_\kappa^\circ \rangle_{\mathrm{End}(U)}$ is actually an automorphism of U; field interpretation applies (it also follows, a posteriori, that U is Θ_κ°-minimal all right).

A priori $\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U)$ simply embeds into $\mathbb{K}^\times$. But one has, by Step 2 and the definition of κ,

$$\begin{aligned}\operatorname{rk}\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U) = \operatorname{rk}\Theta_\kappa^\circ = \operatorname{rk} T_B(\kappa) \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota) = \operatorname{rk} B - \operatorname{rk} C_B^\circ(\iota) = \operatorname{rk}\iota^B \\ \geq \operatorname{rk}\iota^U = \operatorname{rk} U - \operatorname{rk} C_U(\iota) = \operatorname{rk} U = \operatorname{rk}\mathbb{K}_+.\end{aligned}$$

It follows that $\Theta_\kappa^\circ/C_{\Theta_\kappa^\circ}(U) \simeq \mathbb{K}^\times$. At this point Θ_κ° contains a nontrivial 2-torus. By the 2-structure Proposition 1 and in view of the assumption on centralisers of involutions, the Sylow 2-subgroup of G is either connected or isomorphic to that of $\mathrm{PSL}_2(\mathbb{K})$. Suppose it is connected. Then G is $W_2^\perp$; since $\hat{G}/G$ is as well, so is $\hat{G}$ by Lemma L. This contradicts the fact that κ inverts the 2-torus of Θ_κ°. □

For the rest of the proof we now suppose that ι lies in G. So we may assume $\hat{G} = G$. Bear in mind that since the Prüfer 2-rank is 1 by Step 3, all involutions are conjugate.

Notation. • For consistency of notation, let $i = \iota \in G$ and $k = \kappa \in G$. (By torality principles, $i \in C_G^\circ(i) \leq B$.)

• Let j_k be the involution in Θ_k°.

Since i, j_k are in B they are B-conjugate. In particular $C_G^\circ(j_k) \leq B$.

Step 4. $\Theta_k^\circ = C_G^\circ(j_k)$. *Moreover*, $\operatorname{rk} U = \operatorname{rk} C_G^\circ(i) = \operatorname{rk}\Theta_k$, $\operatorname{rk} B \leq 2\operatorname{rk} U$, *and* $\operatorname{rk} G \leq \operatorname{rk} B + \operatorname{rk} U$.

Proof of Step 4. One inclusion is clear by abelianity of Θ_k° obtained in Step 2. Now let $N = N_G^\circ(C_G^\circ(k, j_k))$. Since L_k° is abelian by Step 2, so are $C_G^\circ(j_k) \leq L_k^\circ$ and its conjugate $C_G^\circ(k)$. Hence $\Theta_k^\circ \leq C_G^\circ(j_k) \leq N$ and by torality $k \in C_G^\circ(k) \leq N$. So N contains a nontrivial 2-torus and an involution inverting it; by the structure of torsion in definable, connected, soluble groups, N is not soluble. Since G is an $N_\circ^\circ$-group, one has $C_G^\circ(k, j_k) = 1$, so k inverts $C_G^\circ(j_k)$. Hence $C_G^\circ(j_k) \leq \Theta_k^\circ$.

We now compute ranks. By Steps 3 and 4, $\operatorname{rk} C_G^\circ(i) = \operatorname{rk}\Theta_k^\circ = \operatorname{rk}\mathbb{K}^\times = \operatorname{rk}\mathbb{K}_+ = \operatorname{rk} U$. By definition of $k \in K_B$ and Step 2, $\operatorname{rk}\Theta_k^\circ = \operatorname{rk} T_B(k) \geq \operatorname{rk} B - \operatorname{rk} C_B(i)$, so $\operatorname{rk} B \leq 2\operatorname{rk} U$.

Now remember that k varies in a set K_B generic in i^G. Let $f : K_B \to i^B$ be the definable function mapping k to j_k. If $j_k = j_\ell$ then $\ell \in C_G(j_k)$, and the latter has the same rank as $C_G(i)$ so we control fibres. Hence,

$$\operatorname{rk} G - \operatorname{rk} C_G(i) = \operatorname{rk} i^G = \operatorname{rk} K_B \leq \operatorname{rk} i^B + \operatorname{rk} C_G(i) = \operatorname{rk} i^B + \operatorname{rk} C_B(i) = \operatorname{rk} B,$$

that is, $\operatorname{rk} G \leq \operatorname{rk} B + \operatorname{rk} C_G(i)$. □

For the end of the proof k will stay fixed; conjugating again in B we may therefore suppose that $j_k = i$.

Notation. Let $N = C_G(i)$ and $H = B \cap N$.

Step 5. (B, N, U) *forms a split BN-pair of rank* 1 (see [Wis 2011] if necessary).

Proof of Step 5. We must check the following:

- $G = \langle B, N\rangle$;
- $[N : H] = 2$;
- for any $\omega \in N \setminus H$, one has $H = B \cap B^\omega$, $G = B \sqcup B\omega B$, and $B^\omega \neq B$;
- $B = U \rtimes H$.

First, $H = B\cap N = C_B(i) = C_B^\circ(i)$ by Steinberg's torsion theorem and the structure of torsion in B. By the structure of the Sylow 2-subgroup obtained in Step 3, $H < N$, so using Steinberg's torsion theorem again $[N : H] = 2$. Hence for any $\omega \in N \setminus H = Hk$ one has $B^\omega = B^k \geq H^k = H$ and $H \leq B \cap B^k$. Now by the structure of torsion in B, the intersection $B \cap B^k$ centralises the 2-torus in the abelian group $(B \cap B^k)^\circ = L_k^\circ$ so $B \cap B^k \leq C_B(i) = H$.

Recall that the action of $H = C_G^\circ(i) = \Theta_k^\circ$ on U induces a field structure; in particular $H \cap U \leq C_U(\Theta_k^\circ) = 1$. So $U \cdot H = U \rtimes H$ has rank $2\operatorname{rk} U \geq \operatorname{rk} B$ by Step 4, and therefore $B = U \rtimes H$.

It remains to obtain the Bruhat decomposition. But first note that if $C_{N_G(B)}(i) > C_B(i)$ then $C_{N_G(B)}(i) = N$ contains k, which contradicts $k \notin N_G(B)$ from the notation preceding Step 1. So $C_{N_G(B)}(i) = C_B(i)$ and since B conjugates its involutions, a Frattini argument yields $N_G(B) \subseteq B \cdot C_{N_G(B)}(i) = B$.

Finally let $g \in G \setminus B$; g does not normalise B. Let $X = (U \cap B^g)^\circ$ and suppose $X \neq 1$. In characteristic p this contradicts uniqueness principles. In characteristic 0, $U \simeq \mathbb{K}_+$ is minimal [Poi 1987, Corollaire 3.3], so $X = U$; at this point $U = U_\rho(B^g) = U^g$, a contradiction again. In any case $X = 1$. In particular UgB has rank $\operatorname{rk} U + \operatorname{rk} B = \operatorname{rk} G$ by Step 4 and UgB is generic in G. This also holds of UkB so $g \in BkB$ and $G = B \sqcup BkB = B \sqcup B\omega B$ for any $\omega \in N \setminus H$. This certainly implies $G = \langle B, N\rangle$. □

We finish the proof with [Wis 2011, Theorem 1.2] or [DMT 2008, Theorem 2.1], depending on the characteristic. If U has exponent p, then $U_p(H) = 1$ as $H \simeq \mathbb{K}^\times$, so [Wis 2011, Theorem 1.2] applies. If not, then U is torsion-free; we use [DMT 2008, Theorem 2.1] instead. In any case, $G/\bigcap_{g\in G} B^g \simeq \mathrm{PSL}_2(\mathbb{K})$ for some field structure $\mathbb{K}$ which a priori need not be the same as in Step 3, but could easily be proved to be. Since $\bigcap_{g\in G} B^g$ is a normal, soluble subgroup, it is finite as G is an $N_\circ^\circ$-group, and therefore central by connectedness. But central extensions of finite Morley rank of quasisimple algebraic groups are known [AC 1999, Corollary 1], so $G \simeq \mathrm{SL}_2(\mathbb{K})$ or $\mathrm{PSL}_2(\mathbb{K})$, and the first is impossible by assumption on the centralisers of involutions. □

Remark. In order to prove nonconnectedness of the Sylow 2-subgroup of G, one only needs solubility of $C^\circ_G(\iota)$ regardless of how centralisers of involutions in other classes may behave. But in order to continue one needs much more.

- One cannot work with j_κ as all our rank computations rely on the equality $\operatorname{rk} C_G(j_\kappa) = \operatorname{rk} C_G(\iota)$, for which there is no better reason than conjugacy with ι. This certainly implies $\iota \in G$ to start with.

- One cannot entirely drop ι and focus on j_κ, since there is no reason why $C^\circ_G(j_\kappa)$ should be soluble.

4.4. ***The Devil's Ladder.*** Proposition 4 comes from [*Del 2007a, Proposition 5.4.9] and was realised (somewhere in Turkey, in 2007) to be more general; the name was given after a Ligeti study. The first lucid uses were in [*DJ 2008; *BCD 2009]. Both the statement and the proof have undergone considerable change since: in 2013 the argument still took three pages.

We shall climb the Ladder three times: in order to control torsion, which is the very purpose of Proposition 5; at a rather convoluted moment in Step 5 of Proposition 6; and in order to conjugate involutions in the very end of the proof of our theorem, Step 5. It may be viewed as an extreme form of Proposition 3, Step 1; the effective contents of the argument are not perfectly intuitive but for a contradiction proof it suffices to stand firm longer than the group.

Proposition 4 (The Devil's Ladder). *Let $\hat{G}$ be a connected, $U_2^\perp$, $W_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N^\circ_\circ$-subgroup. Suppose $C^\circ_G(\iota)$ is soluble for all $\iota \in I(\hat{G})$.*

Let $\kappa, \lambda \in I(\hat{G})$ be two involutions. Suppose that $C^\circ_G(\mu)$ is a Borel subgroup of G for all $\mu \in I(\hat{G})$ such that $\rho_\mu \succ \rho_\kappa$.

Let $B \geq C^\circ_G(\kappa)$ be a Borel subgroup of G and $1 \neq X \leq F^\circ(B)$ be a definable, connected subgroup which is centralised by κ and inverted by λ.

Then $C^\circ_G(X) \leq B$ and B is the only Borel subgroup of G of parameter ρ_B containing $C^\circ_G(X)$; in particular κ and λ normalise B.

Proof. First observe that $\kappa \in C_{\hat{G}}(X)$ which is λ-invariant, so by normalisation principles λ has a $C_{\hat{G}}(X)$-conjugate λ' which normalises some Sylow 2-subgroup of $C_{\hat{G}}(X)$ containing κ. By the $W_2^\perp$ assumption the Sylow 2-subgroup of $\hat{G}$ is abelian, so $[\kappa, \lambda'] = 1$; also observe that λ' inverts X. Let $C = C^\circ_G(X)$, a definable, connected, and soluble group since G is an $N^\circ_\circ$-group.

First suppose $\rho_C \succ \rho_\kappa$. Then κ inverts $U_{\rho_C(C)}$, which is therefore abelian. Since the four-group $\langle \kappa, \lambda' \rangle$ normalises $U_{\rho_C}(C)$, one of the two involutions λ' or $\kappa\lambda'$, call it μ, satisfies $Y = C^\circ_{U_{\rho_C}(C)}(\mu) \neq 1$. Note that Y is a ρ_C-group by Lemma J. Let $D = C^\circ_G(Y) \geq U_{\rho_C}(C)$; it is a definable, connected, soluble, κ-invariant subgroup. Since $\rho_D \succcurlyeq \rho_C \succ \rho_\kappa$, it follows that κ inverts $U_{\rho_D}(D)$. On the other hand, $Y \leq C^\circ_G(\mu)$

so $\rho_\mu \succ \rho_\kappa$ and by assumption, $C_G^\circ(\mu)$ is a Borel subgroup of G, say B_μ. Since κ and μ commute, κ normalises B_μ and since $\rho_\mu \succ \rho_\kappa$, κ inverts $U_{\rho_\mu}(B_\mu) \trianglelefteq B_\mu$. It also inverts $Y \leq B_\mu$, so by commutation principles, $[U_{\rho_\mu}(B_\mu), Y] = 1$ and $U_{\rho_\mu}(B_\mu) \leq C_G^\circ(Y) = D$.

We are still assuming $\rho_C \succ \rho_\kappa$. The involution κ inverts both $U_{\rho_D}(D) \trianglelefteq D$ and $U_{\rho_\mu}(B_\mu) \leq D$; so by commutation principles, $[U_{\rho_\mu}(B_\mu), U_{\rho_D}(D)] = 1$ and $U_{\rho_D}(D) \leq N_G^\circ(U_{\rho_\mu}(B_\mu)) = B_\mu$. At this stage it is clear that $\rho_D = \rho_\mu$ and $U_{\rho_\mu}(B_\mu) = U_{\rho_D}(D)$. In particular $D \leq N_G^\circ(U_{\rho_\mu}(B_\mu)) = B_\mu$. As a conclusion,

$$X \leq C_G^\circ(U_{\rho_C}(C)) \leq C_G^\circ(Y) = D \leq B_\mu = C_G^\circ(\mu),$$

against the fact that μ inverts X.

This contradiction shows that $\rho_C \preccurlyeq \rho_\kappa$. Now $X \leq F^\circ(B)$, so $U_{\rho_B}(Z(F^\circ(B))) \leq C_G^\circ(X) = C$; hence $\rho_B \preccurlyeq \rho_C \preccurlyeq \rho_\kappa \preccurlyeq \rho_B$ and equality holds. Since by uniqueness principles $U_{\rho_B}(B)$ is the only Sylow ρ_B-subgroup of G containing $U_{\rho_B}(Z(F^\circ(B)))$, it also is unique as such containing $U_{\rho_C}(C)$. Hence $N_{\hat{G}}(C) \leq N_{\hat{G}}(U_{\rho_C}(C)) \leq N_{\hat{G}}(B)$. Therefore κ and λ normalise B. □

4.5. *Inductive torsion control.* It will be necessary to control torsion in the $T_B(\kappa)$-sets. In [*Del 2007a] this was redone for each conjugacy class of involutions by ad hoc arguments which could, in high Prüfer rank, get involved (the "birthday lemmas" [*Del 2007a, Lemmes 5.3.9 and 5.3.10] published as [*Del 2008, Lemmes 6.9 and 6.10]). We proceed more uniformly, although some juggling is required. Like in [*Del 2008] the argument will be applied twice: to start the proof of Proposition 6, and later to conjugate involutions in Step 5 of the final argument. This accounts for the disjunction in the statement.

There was nothing equally technical in [*BCD 2009], as controlling involutions there was trivial. An inner version of the argument was found in Yanartaş in the spring of 2007 and added to [*DJ 2008]. Externalising involutions is no major issue.

Proposition 5 (inductive torsion control). *Let $\hat{G}$ be a connected, $U_2^\perp$, $W_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Suppose that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$.*

Let $\iota \in I(\hat{G})$ and $B \geq C_G^\circ(\iota)$ be a Borel subgroup. Suppose that $C_G^\circ(\mu)$ is a Borel subgroup of G for all $\mu \in I(\hat{G})$ such that $\rho_\mu \succ \rho_\iota$. Let $\kappa \in I(\hat{G}) \setminus N_{\hat{G}}(B)$ be such that $T_B(\kappa)$ is infinite.

Suppose either that $B = C_G^\circ(\iota)$ or that ι and κ are $\hat{G}$-conjugate. Then $\mathbb{T}_B(\kappa)$ has the same rank as $T_B(\kappa)$, and contains no torsion elements.

Proof. First remember that since $\hat{G}$ is $W_2^\perp$, if some involution $\omega \in I(\hat{G})$ inverts a toral element $t \in \hat{G}$, then $t^2 = 1$. One may indeed take a maximal decent torus $\hat{T}$ of $\hat{G}$ containing t; then ω normalises $C_{\hat{G}}^\circ(t)$ which contains $\hat{T}$ and its 2-torus $\hat{T}_2$, so by normalisation principles ω has a $C^\circ_{\hat{G}}(t)$-conjugate ω' normalising $\hat{T}_2$. By the $W_2^\perp$ assumption, the latter already is a Sylow 2-subgroup of $\hat{G}$, whence $\omega' \in \hat{T}_2 \leq C^\circ_{\hat{G}}(t)$. It follows that ω centralises t; it also inverts it by assumption, so $t^2 = 1$.

The proof starts here.

We first show that B has no torsion unipotence. The argument is a refinement of Step 2 of Proposition 3. Suppose that there is a prime number p with $U_p(B) \neq 1$. Let $L_\kappa = B \cap B^\kappa$ (be careful to note that we do *not* consider the connected component). Since $C^\circ_G(L'_\kappa)$ contains both $U_p(Z(F^\circ(B)))$ and $U_p(Z(F^\circ(B^\kappa)))$, uniqueness principles imply that L'_κ is finite. Unfortunately L_κ need not be abelian, so let us introduce

$$\Theta_\kappa = \{\ell \in L_\kappa : \ell\ell^\kappa \in L'_\kappa\},$$

which is a definable, κ-invariant subgroup of B containing $T_B(\kappa)$; in particular it is infinite. Also note that Θ°_κ is abelian. Now let $A \leq U_p(B)$ be a Θ°_κ-minimal subgroup. Θ°_κ cannot centralise A since otherwise $C^\circ_G(\Theta^\circ_\kappa) \geq \langle A, A^\kappa\rangle$, against uniqueness principles. So by Zilber's field theorem the action induces an algebraically closed field of characteristic p structure. By Wagner's theorem on fields [Wag 2001, consequence of Corollary 9], Θ°_κ contains a q-torus T_q for some $q \neq p$. Up to taking the maximal q-torus of Θ°_κ we may assume that κ normalises T_q. Write if necessary T_q as the sum of a κ-centralised and a κ-inverted subgroup; by the first paragraph of the proof, κ centralises T_q. So for any $t \in T_q$ one has $tt^\kappa = t^2 \in L'_\kappa$; therefore $T_q \leq L'_\kappa$ against finiteness of the latter.

We have disposed of torsion unipotence inside B, and every element of prime order in B is toral by the structure of torsion in definable, connected, soluble groups. By the first paragraph of the proof, no element of finite order $\neq 2$ of B is inverted by *any* involution (this will be used in the next paragraph with an involution distinct from κ). In particular $d(t^2)$ is torsion-free for any $t \in T_B(\kappa)$; hence the definable hull of any element of $\mathbb{T}_B(\kappa)$ is torsion-free.

We now show that $T_B(\kappa)$ can contain but finitely many involutions (possibly none). Suppose that it contains infinitely many. Since B has only finitely many conjugacy classes of involutions, there are $i, j \in T_B(\kappa)$ which are B-conjugate. Now $i \in B$ so $\{B, i\} \subseteq F^\circ(B)$; by Lemma F (although [*DJ 2010, Lemma 24] would do here) $B = B^{+i} \cdot \{B, i\}$, so there is $x \in \{B, i\} \subseteq (F^\circ(B))^{-i}$ with $j = i^x$. Since i inverts x, $d(x^2)$ is torsion-free. Also, $1 \neq ij = ii^x = x^2 \in F^\circ(B)$. Let $X = d(x^2)$, which is an abelian, definable, connected, infinite subgroup; like ij it is centralised by κ and inverted by i. There are two cases.

- If $B = C^\circ_G(\iota)$ then ι centralises X whereas κi inverts it (yes, κ and i do commute). Since $X \leq F^\circ(B)$ with $C^\circ_G(\iota) \leq B$, The Devil's Ladder, Proposition 4, applied to the pair $(\iota, \kappa i)$, leads to $\kappa i \in N_{\hat{G}}(B)$ and $\kappa \in N_{\hat{G}}(B)$: a contradiction.

• If κ is $\hat{G}$-conjugate to ι, say $\kappa = \iota^\gamma$ for some $\gamma \in \hat{G}$, we work in $B^\gamma \geq C_G^\circ(\kappa)$. Since κ centralises X, we have $X \leq B^\gamma$. Since $i \in C_{\hat{G}}(\kappa) \cap B \leq C_G(\kappa)$, and by connectedness of the Sylow 2-subgroup of $\hat{G}$, one has $i \in C_G^\circ(\kappa) \leq B^\gamma$. Since i inverts X, we have $X \leq F^\circ(B^\gamma)$. Finally, by conjugacy $\rho_\kappa = \rho_\iota$, so climbing The Devil's Ladder for the pair (κ, i) we find $C_G^\circ(X) \leq B^\gamma = B^{\gamma\kappa}$. Since $X \leq F^\circ(B)$ this implies $U_{\rho_B}(Z(F^\circ(B))) \leq C_G^\circ(X) \leq B^\gamma$. Uniqueness principles now yield $B = B^\gamma$. Hence $\kappa \in N_{\hat{G}}(B)$: a contradiction.

We conclude to rank equality. Let $i_1, \ldots, i_n$ be the finitely many involutions in $T_B(\kappa)$ (possibly $n = 0$) and set $i_0 = 1$. If $t \in T_B(\kappa)$ then the torsion subgroup of $d(t)$ is some $\langle i_m \rangle$, so $d(i_m t)$ is 2-divisible, and $i_m t \in \mathbb{T}_B(\kappa)$. Hence $T_B(\kappa) \subseteq \bigcup i_m \mathbb{T}_B(\kappa)$, which proves $\operatorname{rk} \mathbb{T}_B(\kappa) = \operatorname{rk} T_B(\kappa)$. $\square$

Remarks. • One needs $T_B(\kappa)$ to be infinite only to show $U_p(B) = 1$; if one were to assume the latter, the rest of the argument would still work with finite $T_B(\kappa)$, and yield $\mathbb{T}_B(\kappa) = \{1\}$.

• The fact that $U_p(B) = 1$ is a strong indication of the moral inconsistency of the configuration.

5. The proof — the maximality proposition

Proposition 6 is the technical core of the present article; we would be delighted to learn of a finite group-theoretic analogue. It was first devised in the context of minimal connected simple groups of odd type [*Del 2007a], then ported to $N_\circ^\circ$-groups of odd type [*DJ 2008], and to actions on minimal connected simple groups of degenerate type [*BCD 2009]. The main idea and the final contradiction have not changed but every generalisation has required new technical arguments. So neither of the above mentioned adaptations was routine; nor was combining them. We can finally state a general form.

Proposition 6 (maximality). *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, $W_2^\perp$, nonsoluble $N_\circ^\circ$-subgroup. Suppose $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$. Then $C_G^\circ(\iota)$ is a Borel subgroup of G for all $\iota \in I(\hat{G})$.*

Proof. The proof is longer and more demanding than others in the article, but one should be careful to distinguish two levels.

• At a superficial level, all arguments resorting to local analysis in G and to the Bender method (Steps 3 and 4) would be much shorter and more intuitive if one knew that Borel subgroups of G have abelian intersections. There is no hope to prove such a thing but it may be a good idea to have a quick look at the structure of the proof in this ideally behaved case.

• At a deeper level, assuming abelianity of intersections does not make the statement of the proposition obvious and the reader is invited to think about it. Even with abelian intersections of Borel subgroups there would still be something to prove; this certainly uses the $T_B(\kappa)$ sets and rank computations of Section 4.2 as nothing else is available. As a matter of fact, even under abelian assumptions, we cannot think of a better strategy than the following.

The long-run goal (Step 6) is to collapse the configuration by showing that G-conjugates of some subgroup of G generically lie inside B. This form of contradiction was suggested by Jaligot to the author, then his Ph.D. student, for [*Del 2007b]. It is typical of Jaligot's early work in odd type [*Jal 2000, Lemme 2.13]. (The author's original argument based on the distribution of involutions was both doubtful and less elegant; even recently he could feel the collapse in terms of involutions, but failed to write it down properly.)

Controlling generic G-conjugates of an arbitrary subgroup is not an easy task. The surprise (Step 5) is that the $T_B(\kappa)$ sets, or more precisely the $\mathbb{T}_B(\kappa)$ sets, form the desired family. Seeing this requires a thorough analysis of $\mathbb{T}_B(\kappa)$, and embedding it into some abelian subgroup of B with pathological rigidity properties (Step 4). The crux of the argument involves some intersection of Borel subgroups. Interestingly enough, abelian intersections could be removed from [*Del 2007a; 2007b; 2008; *DJ 2008] by a somehow artificial observation on torsion; abelian intersections started playing a nontrivial role in [*BCD 2009] but as a result the global proof then divided into two parallel lines. We could find a more uniform treatment, although the proof of Step 4 still divides into two along the line of abelianity.

The beginning of the argument (Steps 3, 2, 1) simply prepares for the analysis, showing that $\mathbb{T}_B(\kappa)$ behaves like a semisimple group. Of course controlling torsion with Proposition 5 is essential in the first place; studying torsion separately, thus allowing inductive treatment, was the main success of [*DJ 2008]. The proof starts here.

5.1. ***The reactor.*** Since $\hat{G}$ is connected, by torality principles every involution has a conjugate in some fixed 2-torus $\hat{S}^\circ$. We may therefore assume that $\hat{G} = G \cdot d(\hat{S}^\circ)$, so that the standard rank computations of Proposition 2 apply. Moreover, $\hat{G}/G$ is connected and abelian, hence $W_2^\perp$. Since G is $W_2^\perp$ as well, so is $\hat{G}$ by Lemma L.

We then proceed by descending induction on ρ_ι and fix some involution $\iota_0 \in I(\hat{G})$ such that for any $\mu \in I(\hat{G})$ with $\rho_\mu \succ \rho_{\iota_0}$, $C_G^\circ(\mu)$ is a Borel subgroup. Notice that induction will not be used as such in the current proof but merely in order to apply Propositions 4 and 5.

Be warned that there will be some running ambiguity on ι_0 starting from Section 5.2 onwards, the resolution being in the proof of Step 5.

Notation. • Let $B \geq C_G^\circ(\iota_0)$ be a Borel subgroup of G and suppose $B > C_G^\circ(\iota_0)$; let $\rho = \rho_B$.

• Let $K_B = \{\kappa \in \iota_0^{\hat{G}} \setminus N_{\hat{G}}(B) : \operatorname{rk} T_B(\kappa) \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota_0)\}$; by Proposition 2, K_B is generic in $\iota_0^{\hat{G}}$.

• Let $\kappa \in K_B$.

• For the moment we simply write $\mathbb{T} = \mathbb{T}_B(\kappa)$.

By inductive torsion control (Proposition 5), one has $\operatorname{rk} \mathbb{T} \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota_0)$, and $\mathbb{T}$ contains no torsion elements.

Step 1 (uniqueness). (i) *B is the only Borel subgroup of G containing $C_G^\circ(\iota_0)$.*

(ii) *$N_{\hat{G}}(B)$ contains a Sylow 2-subgroup $\hat{S}_0$ of $\hat{G}$.*

(iii) *If $\lambda \in I(\hat{G}) \cap N_{\hat{G}}(B)$, then $[B, \lambda] \leq F^\circ(B)$ and $B = B^{+_\lambda} \cdot (F^\circ(B))^{-_\lambda}$ with finite fibres.*

(iv) $(N_G(B))^{-_\lambda} \subseteq B$.

Proof of Step 1. Since G is $W_2^\perp$, by Proposition 3 there is a unique Borel subgroup of G containing $C_G^\circ(\iota_0)$; in particular $C_{\hat{G}}(\iota_0)$ normalises B. By torality principles, $N_{\hat{G}}(B)$ contains a full Sylow 2-subgroup $\hat{S}_0$ of $\hat{G}$, which is a 2-torus as $\hat{G}$ is $W_2^\perp$. Now let $\lambda \in I(\hat{G}) \cap N_{\hat{G}}(B)$. Conjugating in $N_{\hat{G}}(B)$ we may suppose $\lambda \in \hat{S}_0$. Then $\hat{B} = B \cdot d(\hat{S}_0)$ is a definable, connected, soluble group, so $\hat{B}' \leq F^\circ(\hat{B})$. Using Zilber's indecomposability theorem, $[B, \lambda] \leq [B, \hat{S}_0] \leq (B \cap F^\circ(\hat{B}))^\circ \leq F^\circ(B)$. So Lemma F yields $B = (B^+)^\circ \cdot \{B, \lambda\}$. Of course $\{B, \lambda\} \subseteq (F^\circ(B))^{-_\lambda}$.

It remains to prove (iv). The 2-torus $\hat{S}_0$ also acts on $N_G(B)$, so it centralises the finite set $N_G(B)/B$. It follows that if $n \in (N_G(B))^{-_\lambda}$, then $nB = n^\lambda B = n^{-1}B$, that is, $n^2 \in B$. If G has no involutions then neither does $N_G(B)/B$ by torsion lifting. But if G does have involutions, then by torality principles $B \geq C_G^\circ(\iota_0)$ already contains a maximal 2-torus of G, which is a Sylow 2-subgroup of G; hence in that case again, $N_G(B)/B$ has no involutions. In any case $n \in B$, which proves $(N_G(B))^{-_\lambda} \subseteq B$. □

The most important claims for the moment are (i) and (iii). Claim (iv) will play its role in the sole final step but was more conveniently proved here.

5.2. *The fuel.* Controlling $\iota_0^G \cap N_{\hat{G}}(B)$ was claimed to be essential in [*Del 2008, after Corollaire 5.37]. We can actually do without but this will result in some counterpoint of involutions with a final chord at the very end of the proof of Step 5.

Notation. Let $I_B = \{\iota \in \iota_0^{\hat{G}} : C_G^\circ(\iota) \leq B\}$.

Remarks. $I_B = \iota_0^{N_G(B)}$ and any maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B)$ intersects I_B, two facts we shall use with no reference. A proof and an observation follow.

• If $\iota \in I_B$ then there is $x \in \hat{G} = G \cdot d(\hat{S}_0)$ with $\iota = \iota_0^x$, where $\hat{S}_0$ is a 2-torus containing ι_0; one may clearly assume $x \in G$. Now by uniqueness (Step 1(i)) and definition of I_B, B^x is the only Borel subgroup of G containing $C_G^\circ(\iota) \leq B$, whence $x \in N_G(B)$ and $I_B \subseteq \iota_0^{N_G(B)}$. The converse inclusion is obvious.

By Step 1(ii), $N_{\hat{G}}(B)$ contains a Sylow 2-subgroup of $\hat{G}$, so any maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B)$ is in fact a Sylow 2-subgroup of $N_{\hat{G}}(B)$, and contains an $N_{\hat{G}}(B)$-conjugate ι of ι_0; then $\iota \in \hat{S} \cap I_B$.

• On the other hand it is not clear at all whether equality holds in $I_B \subseteq \iota_0^G \cap N_{\hat{G}}(B)$. As a matter of fact we cannot show that B is self-normalising in G; this is easy when G is $2^\perp$ but not in general. At this point, using $C_G^\circ(\iota) < B$, there is a lovely little argument showing that $C_G(\iota)$ is connected (which is not obvious if $G < \hat{G}$ as Steinberg's torsion theorem no longer applies), but one cannot go further. Moreover, self-normalisation techniques à la [ABF 2013] do not work in the $N_\circ^\circ$ context.

The first claim below will remedy this.

Step 2 (action). (i) *If $\lambda \in \iota_0^G \cap N_{\hat{G}}(B)$ but $\lambda \notin I_B$, then λ inverts $U_\rho(Z(F^\circ(B)))$.*
(ii) $[U_\rho(Z(F^\circ(B))), \mathbb{T}] \neq 1$.

Proof of Step 2. Throughout this proof, letting $Y_B = U_\rho(Z(F^\circ(B)))$ will spare a few parentheses.

Let λ be as in the statement and suppose that $X = C_{Y_B}^\circ(\lambda)$ is nontrivial. Then X is a ρ-group by Lemma J. By Step 1(iii), $B = B^{+\lambda} \cdot (F^\circ(B))^{-\lambda}$; obviously both terms normalise X so $X \trianglelefteq B$. It follows from uniqueness principles that $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing X. Since $X \leq C_G^\circ(\lambda)$ is contained in some conjugate B^x of B, we have $U_\rho(B^x) = U_\rho(B)$ so $C_G^\circ(\lambda) \leq B$ and $\lambda \in I_B$: a contradiction.

We move to the second claim. Suppose that $\mathbb{T}$ centralises Y_B. Let $C = C_G^\circ(\mathbb{T})$, a definable, connected, soluble, κ-invariant subgroup; let U be a Sylow ρ-subgroup of C containing Y_B. By normalisation principles κ has a C-conjugate λ normalising U and inverting $\mathbb{T}$. Since $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing Y_B, λ normalises B. Hence $\lambda \in \iota_0^{\hat{G}} \cap N_{\hat{G}}(B) = \iota_0^G \cap N_{\hat{G}}(B)$. We see two cases.

First suppose $\lambda \notin I_B$. Then by claim (i), λ inverts Y_B. If $\rho_C = \rho$, then apply uniqueness principles: $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing Y_B, so it also is the only Sylow ρ-subgroup of G containing $U_\rho(C)$. As the latter is κ-invariant, so is B: a contradiction. Therefore $\rho_C \succ \rho$. It follows that λ inverts $U_{\rho_C}(C)$, whence $[U_{\rho_C}(C), Y_B] = 1$ by commutation principles. This forces $U_{\rho_C}(C) \leq C_G^\circ(Y_B) \leq B$, against $\rho_C \succ \rho$.

So $\lambda \in I_B$, i.e., $C_G^\circ(\lambda) \leq B$. But by Step 1(iii), $\mathbb{T} \subseteq (F^\circ(B))^{-\lambda}$, and therefore $\mathbb{T} \subseteq F^\circ(B) \cap F^\circ(B)^\kappa$. Since all elements in $\mathbb{T}$ are torsion-free by Proposition 5, one even has $\mathbb{T} \subseteq (F^\circ(B) \cap F^\circ(B)^\kappa)^\circ$. The latter is abelian by [*DJ 2012, 4.46(2) (our Fact 10)], and $\mathbb{T}$ is therefore a definable, connected, abelian subgroup. Now always by the torsion control and genericity propositions (Propositions 5 and 2), and by the decomposition of B obtained in Step 1(iii), one has

$$\begin{aligned}\operatorname{rk} \mathbb{T} = \operatorname{rk} T_B(\kappa) \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota_0) &= \operatorname{rk} B - \operatorname{rk} C_G^\circ(\lambda)\\ &= \operatorname{rk} B - \operatorname{rk} C_B^\circ(\lambda) = \operatorname{rk}(F^\circ(B))^{-\lambda}.\end{aligned}$$

A definable set contains at most one definable, connected, generic subgroup, so $\mathbb{T}$ is the only definable, connected, generic group included in $(F^\circ(B))^{-\lambda}$; hence $N_{\hat{G}}((F^\circ(B))^{-\lambda}) \leq N_{\hat{G}}(\mathbb{T})$ and $B^{+\lambda}$ normalises $\mathbb{T}$. Moreover $\mathbb{T} \cap B^{+\lambda} = 1$ since λ inverts $\mathbb{T}$ and $\mathbb{T}$ contains no torsion elements. So $\mathbb{T} \cdot B^{+\lambda} = \mathbb{T} \rtimes B^{+\lambda}$ is a definable subgroup of rank $\geq \operatorname{rk}(F^\circ(B_\lambda))^{-\lambda} + \operatorname{rk} B^{+\lambda} = \operatorname{rk} B$ by Step 1(iii). Hence $B = \mathbb{T} \rtimes B^{+\lambda}$ normalises $\mathbb{T}$, and $B = N_G^\circ(\mathbb{T})$ since G is an $N_\circ^\circ$-group. In particular κ normalises B: a contradiction. □

Claim (i) will be used only once more, in the next step.

5.3. *The fuel, refined.*

Step 3 (abelianity). (i) *If $\iota \in I_B$ then $\mathbb{T} \cap C_G(\iota) = 1$.*

(ii) *There is no definable, connected, soluble, κ-invariant group containing both $U_\rho(Z(F^\circ(B)))$ and $\mathbb{T}$.*

(iii) *$\mathbb{T}$ is a definable, abelian, torsion-free group.*

Proof of Step 3. The first claim is easy. Let $\iota \in I_B$ and $t \in \mathbb{T} \setminus \{1\}$ be such that $t^\iota = t$. Then $\iota \in C_{\hat{G}}(t)$, which is κ-invariant; by normalisation principles and abelianity of the Sylow 2-subgroup, κ has a $C_{\hat{G}}(t)$-conjugate λ commuting with ι. By Step 1(i) (uniqueness), B is the only Borel subgroup of G containing $C_G^\circ(\iota)$, so λ normalises B. Recall from inductive torsion control (Proposition 5) that $d(t)$ is torsion-free. By Step 1(iii), $t^\lambda = t^\kappa = t^{-1}$ forces $t^2 = [t^{-1}, \lambda] \in F^\circ(B)$ and $t \in F^\circ(B)$. We then apply The Devil's Ladder (Proposition 4) to the action of $\langle \iota, \kappa \rangle$ on $d(t)$ and find that κ normalises B: a contradiction.

As the proof of the second claim is a little involved let us first see how it entails the third one. Suppose that $X = (F^\circ(B) \cap F^\circ(B)^\kappa)^\circ$ is nontrivial and let $H = N_G^\circ(X)$; then G being an $N_\circ^\circ$-group and the second claim yield a contradiction. Hence $X = 1$ which proves abelianity of $(B \cap B^\kappa)^\circ$. Then, since elements of $\mathbb{T} \subseteq B \cap B^\kappa$ contain no torsion in their definable hulls by Proposition 5, one has $\mathbb{T} \subseteq (B \cap B^\kappa)^\circ$ and $\mathbb{T}$ is therefore an abelian group, obviously definable and torsion-free. So we now proceed to proving the second claim. Here again we let $Y_B = U_\rho(Z(F^\circ(B)))$.

Let L be a definable, connected, soluble, κ-invariant group containing Y_B and $\mathbb{T}$. We shall show that Y_B and $\mathbb{T}$ commute, which will contradict Step 2(ii). To do this we proceed piecewise in the following sense. Bear in mind that for $t \in \mathbb{T}$, $d(t)$ is torsion-free by Proposition 5, so one may take Burdges' decomposition of the definable, connected, abelian group $d(t)$. As a result, the set $\mathbb{T}$ is a union of products of various abelian τ-groups for various parameters τ. We shall show that each of them centralises Y_B, which will be the contradiction.

So we let τ be a parameter and Θ be an abelian τ-group included in the set $\mathbb{T}$. If $\tau = \rho$ then we are done as $\Theta \leq U_\rho(B)$. So suppose $\tau \prec \rho$ and prepare to use the Bender method (Section 2.4). Since $L \geq \langle Y_B, \mathbb{T}\rangle$, L is not abelian by Step 2(ii).

Let $C \leq G$ be a Borel subgroup of G containing $N_G^\circ(L') \geq L$ and maximising ρ_C. Notice that

$$U_{\rho_C}(Z(F^\circ(C))) \leq C_G^\circ(F^\circ(C)) \leq C_G^\circ(C') \leq C_G^\circ(L') \leq N_G^\circ(L'),$$

so by uniqueness principles and definition of C, we find that C is actually the only Borel subgroup of G containing $N_G^\circ(L')$. As the latter is κ-invariant, so is C; in particular $C \neq B$. Moreover, $Y_B \leq C$, so uniqueness principles force $\rho_C \succ \rho$, and $H = (B \cap C)^\circ \geq \langle Y_B, \mathbb{T}\rangle$ is nonabelian. So we are under the assumptions of Fact 11 with $B_\ell = B$ and $B_h = C$.

We determine the linking parameter ρ', i.e., the only parameter of the homogeneous group H' [*DJ 2012, 4.51(3) (our Fact 11)]. By Fact 7(vi) (no need for Frécon homogeneity here), the commutator subgroup $[Y_B, \mathbb{T}]$ is a ρ-subgroup of H'. But by Step 2(ii), it is nontrivial. Hence $\rho' = \rho$.

We now construct a most remarkable involution. Let $V_\rho \leq C$ be a Sylow ρ-subgroup of C containing Y_B. Since κ normalises C, it has by normalisation principles a C-conjugate λ normalising V_ρ. By uniqueness principles, $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing Y_B, so λ normalises B. If $\lambda \notin I_B$ then by Step 2(i), λ inverts Y_B; since $\rho_C \succ \rho$ it certainly inverts $U_{\rho_C}(C)$ as well, whence by commutation principles $[Y_B, U_{\rho_C}(C)] = 1$ and $U_{\rho_C}(C) \leq C_G^\circ(Y_B) \leq B$, contradicting $\rho_C \succ \rho$. Hence $\lambda \in I_B$; it normalises B and C (hence H).

We return to our abelian τ-group Θ included in the set $\mathbb{T}$, with $\tau \prec \rho$. Let $V_\tau \leq H$ be a Sylow τ-subgroup of H containing Θ. By normalisation principles λ has an H-conjugate μ normalising V_τ. We shall prove that μ actually centralises V_τ; little work will remain after that. Observe that V_τ is a definable, connected, nilpotent group contained in two different Borel subgroups of G, so by [*DJ 2012, 4.46(2) (our Fact 9)] it is abelian. By the commutator argument of Fact 7(vi) or the simpler push-forward argument of Fact 7(v) (no need for Frécon homogeneity here), $[V_\tau, \mu]$ is a τ-group inverted by μ.

Now note that μ, like λ, is in I_B, and normalises B and C. Moreover, by Step 1(iii), $[V_\tau, \mu] \leq F^\circ(B)$. We shall prove that $[V_\tau, \mu] \leq F^\circ(C)$ as well by making it commute with all of $F^\circ(C)$, checking it on each term of Burdges' decomposition of $F^\circ(C)$. Keep Fact 11 in mind.

First, by [*DJ 2012, 4.38], $\rho' = \rho$ is the least parameter in $F^\circ(C)$; we handle it as follows. Recall that $[V_\tau, \mu] \leq F^\circ(B)$ is a τ-group, so $[V_\tau, \mu] \leq U_\tau(F^\circ(B)))$. By [*DJ 2012, 4.52(7)] and since $\rho' = \rho \neq \tau$, the latter is in $Z(H)$. But by [*DJ

2012, 4.52(3)], $U_\rho(F^\circ(C)) = U_{\rho'}(F^\circ(C)) = (F^\circ(B) \cap F^\circ(C))^\circ \leq H$, so $[V_\tau, \mu]$ does commute with $U_\rho(F^\circ(C))$. Now let $\sigma \succ \rho$ be another parameter. Remember that μ normalises C; since $\mu \in \iota_0^{\hat{G}}$, $\sigma \succ \rho_\mu$ and μ inverts $U_\sigma(F^\circ(C))$. It inverts $[V_\tau, \mu]$ as well so commutation principles force $[V_\tau, \mu]$ to centralise $U_\sigma(F^\circ(C))$.

As a consequence $[V_\tau, \mu] \leq C$ centralises $F^\circ(C)$. Unfortunately this is not quite enough to apply the Fitting subgroup theorem as literally stated in [BN 1994b, Proposition 7.4] due to connectedness issues. The first option is to note that with exactly the same proof as in [BN 1994b, Proposition 7.4]: in any connected, soluble group K of finite Morley rank one has $C_K^\circ(F^\circ(K)) \leq F^\circ(K)$. Another option is to observe that by [*DJ 2012, 4.52(1)], $F^\circ(C)$ has no torsion unipotence: in particular, the torsion in $F(C)$ is central in C [*DJ 2012, 2.14]. Altogether $[V_\tau, \mu]$ commutes with $F(C)$ and we then use the Fitting subgroup theorem stated in [BN 1994b, Proposition 7.4] to conclude $[V_\tau, \mu] \leq F(C)$. Either way we find $[V_\tau, \mu] \leq F^\circ(C)$, and we already knew $[V_\tau, \mu] \leq F^\circ(B)$. By connectedness $[V_\tau, \mu] \leq (F^\circ(B) \cap F^\circ(C))^\circ$. But the latter as we know [*DJ 2012, 4.51(3)] is $\rho' = \rho$-homogeneous; since $\rho \succ \tau$, this shows $[V_\tau, \mu] = 1$.

In particular $\mu \in I_B$ centralises $\Theta \leq V_\tau$. By claim (i), $\Theta = 1$ which certainly commutes with Y_B. This contradiction finishes the proof of claim (ii). □

Remark. It is possible to avoid using the devil's ladder in the proof of claim (i). Postpone and finish the proof of claim (ii) as follows:

> In particular $\mu \in I_B$ centralises Θ, so $\mu \in C_{\hat{G}}(\Theta)$ which is κ-invariant. By normalisation principles and abelianity of the Sylow 2-subgroup, κ has a $C_{\hat{G}}(\Theta)$-conjugate ν commuting with μ. Since $\mu \in I_B$, by uniqueness (Step 1(i)), ν normalises B. By Step 1(iii), $\Theta = [\Theta, \nu] \leq F^\circ(B)$ commutes with Y_B. Hence all of $\mathbb{T}$ commutes with Y_B, against Step 2(ii).

Then prove claim (i):

> Now let $t \in \mathbb{T} \setminus \{1\}$ be centralised by $\iota \in I_B$. Like in the previous paragraph, $t \in F^\circ(B)$; t has infinite order and is inverted by κ. But we proved in the third claim that $(F^\circ(B) \cap F^\circ(B)^\kappa)^\circ = 1$, a contradiction.

Both claims (i) and (iii) are crucial. Claim (ii) is a gadget used in the proof of claim (iii) and in the next step.

5.4. *The core.*

Notation. • Let π be the set of parameters occurring in $\mathbb{T}$.

• Let $J_\kappa = U_\pi(C_B^\circ(\mathbb{T}))$ (one has $\mathbb{T} \leq J_\kappa$ by Step 3(iii)).

We feel extremely uncomfortable with the next step. The question of why to maximise over $C_B^\circ(\mathbb{T})$ is a mystery and always was. Nine years before writing these lines, the author, then a Ph.D. student, produced an incorrect study of some similar maximal intersection configuration, and after noticing a well-hidden flaw had to reassemble his proof by trying all possible maximisations. Exactly the same happened to him again. We feel like one piece of the puzzle is still missing, or more confusingly that we are playing with incomplete sets of pieces from distinct puzzles. There are many ways to get it wrong and the step works by miracle.

Step 4 (rigidity). *J_κ is an abelian Carter π-subgroup of B. There is a maximal 2-torus $\hat{S}$ of $\hat{G}$ contained in $N_{\hat{G}}(B) \cap N_{\hat{G}}(J_\kappa)$, and for any $\iota \in I_B \cap \hat{S}$, one has $C^\circ_{U_\pi(N^\circ_G(J_\kappa))}(\iota) \leq C^\circ_G(\mathbb{T})$.*

Proof of Step 4. First of all, observe that by torality principles there is a maximal 2-torus $\hat{S}_0$ of $\hat{G}$ containing ι_0; by uniqueness (Step 1(i)) $\hat{S}_0$ normalises B. Bear in mind that any maximal 2-torus in $N_{\hat{G}}(B)$ contains an involution in I_B.

We need more structure now, so let $C \neq B$ be a Borel subgroup of G containing $C^\circ_B(\mathbb{T})$ and maximising $H = (B \cap C)^\circ$. There is such a Borel subgroup indeed since $C^\circ_G(\mathbb{T})$ is κ-invariant whereas B is not. As one expects there are two cases and we first deal with the abelian one. The other will be more involved technically, but there will be no more complications of this kind when we are done.

Suppose that H is abelian. Since $H \geq C^\circ_B(\mathbb{T}) \geq \mathbb{T}$ by abelianity of the latter, Step 3(iii), and since H is supposed to be abelian as well, $H = C^\circ_B(\mathbb{T}) \leq N^\circ_G(J_\kappa)$. We now consider $N^\circ_G(J_\kappa)$. It is not clear at all whether B contains $N^\circ_G(J_\kappa)$ but one may ask.

If (H is abelian and) B happens to be the only Borel subgroup of G containing $N^\circ_G(J_\kappa)$, then

$$U_\pi\left(N^\circ_{C^\circ_G(\mathbb{T})}(J_\kappa)\right) \leq U_\pi\left(N^\circ_{C^\circ_B(\mathbb{T})}(J_\kappa)\right) = U_\pi(C^\circ_B(\mathbb{T})) = J_\kappa$$

and $J_\kappa \leq C^\circ_G(\mathbb{T})$ is a Carter π-subgroup of $C^\circ_G(\mathbb{T})$. As the latter is κ-invariant, by normalisation principles κ has a $C^\circ_G(\mathbb{T})$-conjugate λ normalising J_κ. But our current assumption that B is the only Borel subgroup of G containing $N^\circ_G(J_\kappa)$ forces λ to normalise B as well. By Step 1(iii) and since λ, like κ, inverts the 2-divisible group $\mathbb{T}$, we find $\mathbb{T} = [\mathbb{T}, \lambda] \leq F^\circ(B)$, which contradicts Step 2(ii).

So (provided H is abelian) B is not the only Borel subgroup of G containing $N^\circ_G(J_\kappa)$: let $D \neq B$ be one such. Then $C^\circ_B(\mathbb{T}) = H \leq N^\circ_B(J_\kappa) \leq (B \cap D)^\circ$ so by maximality of H, $H = (B \cap D)^\circ = N^\circ_B(J_\kappa)$ and $J_\kappa = U_\pi(C^\circ_B(\mathbb{T})) = U_\pi(H)$ is a Carter π-subgroup of B. By normalisation principles there is a B-conjugate $\hat{S}$ of $\hat{S}_0$ normalising J_κ. For $\iota \in \hat{S} \cap I_B$ one has $C^\circ_G(\iota) \leq B$ and

$$C^\circ_{U_\pi(N^\circ_G(J_\kappa))}(\iota) \leq N^\circ_B(J_\kappa) = H \leq C^\circ_G(\mathbb{T}).$$

It is not easy to say more as $N^\circ_G(J_\kappa)$ need not be nilpotent, but we are done with the proof in the abelian case.

We now suppose that H is not abelian. However $H \geq C_B^\circ(\mathbb{T})$ so if $D \neq B$ is a Borel subgroup of G containing H, one has by definition of the latter $H = (B \cap D)^\circ$. By [*DJ 2012, 4.50(3) and (6) (our Fact 10)], we are under the assumptions of Fact 11. Keep it at hand. Let $Q \leq H$ be a Carter subgroup of H. Let ρ' denote the parameter of the homogeneous group H'. Studying J_κ certainly means asking about ρ' and π.

Here is a useful principle: if σ is a *set* of parameters not containing ρ', $V_\sigma \leq H$ is a σ-subgroup of H, and $\hat{S} \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(V_\sigma) \cap N_{\hat{G}}(C)$ is a 2-torus, then $\hat{S}$ centralises V_σ. It is easily proved: First, V_σ being nilpotent by definition of a σ-group and contained in two distinct Borel subgroups, is abelian by Fact 9. Now let $\hat{B} = B \cdot d(\hat{S})$, a definable, connected, soluble subgroup of $\hat{G}$. Then by Zilber's indecomposability theorem, $[B, \hat{S}] \leq (F^\circ(\hat{B}) \cap B)^\circ \leq F^\circ(B)$ and likewise in C. Hence $[V_\sigma, \hat{S}] \leq (F^\circ(B) \cap F^\circ(C))^\circ$, which is ρ'-homogeneous [*DJ 2012, 4.52(3)]. As $\rho' \notin \sigma$, we have $[V_\sigma, \hat{S}] = 1$ by (Fact 7(v) or (vi)), and $\hat{S}$ centralises V_σ.

The argument really starts here. First, $\rho' \in \pi$. Otherwise, by Lemma K, $\mathbb{T}$ is included in a Carter subgroup of H; we may assume $\mathbb{T} \leq Q$, and in particular, by abelianity of Q (Fact 9), $Q \leq C_G^\circ(\mathbb{T})$. By Lemma A, $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B) \cup N_{\hat{G}}(C)$. So there are two cases (yes, this does work for groups).

- First suppose that ($\rho' \notin \pi$ and) $N_{\hat{G}}(Q) \leq N_{\hat{G}}(C)$. In particular $N_B^\circ(Q) \leq N_H^\circ(Q) = Q$ and Q is a Carter subgroup of B. By normalisation principles, $\hat{S}_0$ has a B-conjugate $\hat{S}$ in $N_{\hat{G}}(B) \cap N_{\hat{G}}(Q) \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(U_\pi(Q)) \cap N_{\hat{G}}(C)$. As we noted $\hat{S}$ must centralise $U_\pi(Q) \geq \mathbb{T}$. But there is an involution $\iota \in \hat{S} \cap I_B$, and this contradicts Step 3(i).

- Hence (still assuming $\rho' \notin \pi$) one has $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B)$. Then

$$N^\circ_{C_G^\circ(\mathbb{T})}(Q) \leq N^\circ_{C_B^\circ(\mathbb{T})}(Q) \leq N_H^\circ(Q) = Q,$$

and $Q \leq C_G^\circ(\mathbb{T})$ is a Carter subgroup of $C_G^\circ(\mathbb{T})$. As the latter is κ-invariant, by normalisation principles κ has a $C_G^\circ(\mathbb{T})$-conjugate λ normalising Q. Now since $N_{\hat{G}}(Q) \leq N_{\hat{G}}(B)$, λ normalises B. Then $\mathbb{T}$ is inverted by λ and 2-divisible, whence $\mathbb{T} = [\mathbb{T}, \lambda] \leq [B, \lambda] \leq F^\circ(B)$ by Step 1(iii), contradicting Step 2(ii).

So we have proved $\rho' \in \pi$. On the other hand $\rho_B = \rho \notin \pi$, as otherwise $C_G^\circ(U_\rho(\mathbb{T}))$ would contradict Step 3(ii). Suppose for a second that $\rho_C \succ \rho_B$; then since $\rho \neq \rho'$, one has $U_\rho(Z(F^\circ(B))) \leq Z(H) \leq C_G^\circ(\mathbb{T})$ [*DJ 2012, 4.52(7)], against Step 2(ii). Since parameters differ [*DJ 2012, 4.50(6)] one has $\rho_B \succ \rho_C$. In particular [*DJ 2012, 4.52(2)], Q is a Carter subgroup of B.

We now show that $\mathbb{T}$ is ρ'-homogeneous, i.e., $\pi = \{\rho'\}$. Let $\sigma = \pi \setminus \{\rho'\}$. Since H' is ρ'-homogeneous, by Lemma K we may assume that $U_\sigma(\mathbb{T}) \leq Q$. Now $U_{\rho'}(H) = U_{\rho'}(F^\circ(H))$ is a Sylow ρ'-subgroup of B [*DJ 2012, implicit but clear in 4.52(6)]. By normalisation principles $\hat{S}_0$ has a B-conjugate $\hat{S}$ in $N_{\hat{G}}(B) \cap N_{\hat{G}}(U_{\rho'}(H)) \leq$

$N_{\hat{G}}(B) \cap N_{\hat{G}}(C)$ [*DJ 2012, 4.52(6)]. Hence $\hat{S}$ normalises H. But Q is a Carter subgroup of H so by normalisation principles over H, $\hat{S}$ has an H-conjugate $\hat{S}_1$ in $N_{\hat{G}}(B) \cap N_{\hat{G}}(C) \cap N_{\hat{G}}(Q)$. By our initial principle, $\hat{S}_1$ centralises $U_\sigma(Q) \geq U_\sigma(\mathbb{T})$. Since $\hat{S}_1$ contains an involution in I_B, we find $U_\sigma(\mathbb{T}) = 1$ by Step 3(i), as desired. Hence $\mathbb{T}$ is ρ'-homogeneous.

As a conclusion $\pi = \{\rho'\}$ and $J_\kappa = U_{\rho'}(C_B^\circ(\mathbb{T})) \leq U_{\rho'}(H)$. The latter is an abelian Sylow ρ'-subgroup of B [*DJ 2012, implicit but clear in 4.52(6) and noted above]. Also, $\mathbb{T} \leq U_{\rho'}(H) \leq C_B^\circ(\mathbb{T})$ and $J_\kappa = U_{\rho'}(H)$. We constructed a maximal 2-torus $\hat{S} \leq N_{\hat{G}}(B) \cap N_{\hat{G}}(J_\kappa)$ a minute ago.

Finally fix $\iota \in \hat{S} \cap I_B$. We aim to show that $C^\circ_{U_{\rho'}(N_G^\circ(J_\kappa))}(\iota) \leq C_G^\circ(\mathbb{T})$. Recall that $\hat{S}$ normalises C. By normalisation principles $\hat{S}$ normalises some Sylow ρ'-subgroup $V_{\rho'}$ of C. Then with Lemma E under the action of ι, $V_{\rho'} = (V_{\rho'}^+)^\circ \cdot \{V_{\rho'}, \iota\}$. Now $(V_{\rho'}^+)^\circ$ is a ρ'-subgroup of $(B \cap C)^\circ = H$, so $(V_{\rho'}^+)^\circ \leq J_\kappa \leq F^\circ(C)$ [*DJ 2012, 4.52(6)]. Letting $\hat{C} = C \cdot d(\hat{S})$ one easily sees as we often did that $\{V_{\rho'}, \iota\} \subseteq F^\circ(C)$. So $V_{\rho'} \leq F^\circ(C)$ and $V_{\rho'} \leq U_{\rho'}(F^\circ(C))$. Conjugating Sylow ρ'-subgroups in C, this means that $U_{\rho'}(F^\circ(C))$ is actually the only Sylow ρ'-subgroup of C. But by [*DJ 2012, 4.52(8)] any Sylow ρ'-subgroup of G containing $U_{\rho'}(H)$ is contained in C. This means that $U_{\rho'}(F^\circ(C))$ is the only Sylow ρ'-subgroup of G containing $U_{\rho'}(H) = J_\kappa$.

As a conclusion, any Sylow ρ'-subgroup of $N_G^\circ(J_\kappa)$ lies in $U_{\rho'}(F^\circ(C))$. Hence, paying attention to the fact that ι normalises the nilpotent ρ'-group $U_{\rho'}(F^\circ(C))$,

$$C^\circ_{U_\pi(N_G^\circ(J_\kappa))}(\iota) \leq C^\circ_{U_{\rho'}(F^\circ(C))}(\iota) \leq U_{\rho'}(H) = J_\kappa \leq C_G^\circ(\mathbb{T}). \qquad \square$$

We shall use the Bender method no more.

5.5. *The reaction.*

Notation. • We now write $\mathbb{T}_\kappa$ for $\mathbb{T}_B(\kappa)$, as the involution κ will vary in K_B.

• Let $Y = \{B, \iota_0\}$.

Step 5 (conjugacy). (i) *Y is a normal subgroup of B.*

(ii) $\operatorname{rk} B = \operatorname{rk} C_G(\iota_0) + \operatorname{rk} Y$.

(iii) *Any element of $Y \setminus \{1\}$ lies in finitely many G-conjugates of Y.*

(iv) *$\mathbb{T}_\kappa$ and Y are G-conjugate.*

Proof of Step 5. As a matter of fact, we let $Y_\iota = \{B, \iota\}$ for any $\iota \in I_B$. Since $I_B = \iota_0^{N_G(B)}$, such sets are $N_G(B)$-conjugate to $Y_{\iota_0} = Y$.

Let $\iota \in \hat{S} \cap I_B$; do not forget that there is such an involution. Under the action of ι we may write $J_\kappa = J_\kappa^+ \oplus [J_\kappa, \iota]$. By Step 3(i), $\mathbb{T}_\kappa \cap J_\kappa^+ = 1$. So using the very definition of $\kappa \in K_B$ this yields the rank estimate

$$\begin{aligned}\operatorname{rk}[J_\kappa, \iota] = \operatorname{rk} J_\kappa - \operatorname{rk} J_\kappa^+ \geq \operatorname{rk} \mathbb{T}_\kappa \geq \operatorname{rk} B - \operatorname{rk} C_G^\circ(\iota_0) \\ = \operatorname{rk} B - \operatorname{rk} C_B^\circ(\iota) = \operatorname{rk} \iota^B \geq \operatorname{rk} \iota^{J_\kappa} = \operatorname{rk}[J_\kappa, \iota].\end{aligned}$$

Equality follows. In particular, $[J_\kappa, \iota] \subseteq Y_\iota$ is generic in Y_ι. Since a definable set of degree 1 contains at most one definable, generic subgroup, one has $C_B(\iota) \leq N_B(Y_\iota) \leq N_B([J_\kappa, \iota])$. On the other hand, since $\hat{G}$ is $W_2^\perp$, $[J_\kappa, \iota]$ has no involutions; it is disjoint from $C_B(\iota)$. Hence $[J_\kappa, \iota] \cdot C_B(\iota) = [J_\kappa, \iota] \rtimes C_B(\iota)$ is a generic subgroup of B. It follows $B = [J_\kappa, \iota] \rtimes C_B(\iota)$. At this stage it is clear that $Y_\iota = [J_\kappa, \iota]$ is a normal subgroup of B contained in $F^\circ(B)$, and the same holds of Y by $N_G(B)$-conjugacy. Moreover $\operatorname{rk} Y_\iota = \operatorname{rk} \mathbb{T}_\kappa$; we are not done.

Consider the definable function $f : \mathbb{T}_\kappa \to Y_\iota$ which maps t to $[t, \iota]$; as J_κ is abelian, it is a group homomorphism. Bearing in mind that $\mathbb{T}_\kappa \cap C_{J_\kappa}(\iota) = 1$ by Step 3(i) and in view of the equality of ranks, f is actually a group isomorphism; we are not done.

Let us show that any nontrivial element of $Y = Y_{\iota_0}$ lies in finitely many G-conjugates. Indeed, if $a \in Y \setminus \{1\}$ then by the isomorphism $\mathbb{T}_\kappa \simeq Y$ and inductive torsion control (Proposition 5), a has infinite order; $C = C_G^\circ(a) \geq \langle U_\rho(Z(F^\circ(B))), Y\rangle$ is therefore soluble and ι_0-invariant. If $\rho_C \succ \rho_B$ then ι_0 inverts both $U_{\rho_C}(C)$ and Y, and commutation principles yield $[U_{\rho_C}(C), Y] = 1$, whence $U_{\rho_C}(C) \leq N_G^\circ(Y) = B$, a contradiction. Hence $\rho_C \preccurlyeq \rho_B$ and equality follows. Now uniqueness principles show that $U_\rho(B)$ is the only Sylow ρ-subgroup of G containing $U_\rho(C)$. If $a \in Y^g$ with $g \in G$ then $U_\rho(B^g)$ is the only Sylow ρ-subgroup of G containing $U_\rho(C)$ likewise, so $g \in N_G(B)$. Since $B \leq N_G(Y) \leq N_G(B)$, this can happen only for a finite number of conjugates of Y; we are not done.

It remains to conjugate $\mathbb{T}_\kappa$ to Y. We claim that $J_\kappa \leq C_G^\circ(\mathbb{T}_\kappa)$ is a Carter π-subgroup of $C_G^\circ(\mathbb{T}_\kappa)$, where π is as in the notation of Section 5.4. Indeed, let $N = U_\pi(N_G^\circ(J_\kappa))$ and $N_1 = U_\pi(N \cap C_G^\circ(\mathbb{T}_\kappa))$. We wish to decompose, under the action of ι, the involution we fixed at the beginning of the proof. Be very careful to note, however, that ι need not normalise N_1. But since $\hat{S}$ normalises J_κ it also normalises N. Then $\hat{N} = N \cdot d(\hat{S})$ is yet another definable, connected, soluble group, so $\{N, \iota\} \subseteq (\hat{N}' \cap N)^\circ \leq F^\circ(N)$, and Lemma F applies to N. Now take $n_1 \in N_1$ and write its decomposition $n_1 = pn$ *inside* N, with $p \in (N^+)^\circ$ and $n \in \{N, \iota\}$. Then $p \in C^\circ_{U_\pi(N_G^\circ(J_\kappa))}(\iota) \leq C_G^\circ(\mathbb{T}_\kappa)$ by Step 4. So $n \in C_G^\circ(\mathbb{T}_\kappa)$. On the other hand, for any $t \in \mathbb{T}_\kappa$ one has, using a famous identity,

$$\begin{aligned}1 &= [[\iota, n^{-1}], t]^n \cdot [[n, t^{-1}], \iota]^t \cdot [[t, \iota], n]^\iota \\ &= [n^{-2}, t]^n \cdot [[t, \iota], n]^\iota \\ &= [[t, \iota], n]^\iota.\end{aligned}$$

Hence n commutes with $[\mathbb{T}_\kappa, \iota] = Y_\iota$ and $n \in N_G(N_G^\circ(Y_\iota)) = N_G(B)$. Because $p \in C_G^\circ(\iota) \leq B$, one has $n_1 = pn \in N_G(B)$, meaning $N_1 \leq N_G^\circ(B) = B$. Now $N_1 \leq U_\pi(N_B^\circ(J_\kappa))$ and since J_κ is a Carter π-subgroup of B, $N_1 \leq J_\kappa$. Therefore J_κ is a Carter π-subgroup of $C_G^\circ(\mathbb{T}_\kappa)$.

Stretto. This extra rigidity has devastating consequences. By normalisation principles, κ has a $C_G^\circ(\mathbb{T}_\kappa)$-conjugate λ normalising J_κ. If λ normalises B then $\mathbb{T}_\kappa \leq [J_\kappa, \lambda] \leq F^\circ(B)$ by Step 1(iii), which contradicts $[U_\rho(Z(F^\circ(B))), \mathbb{T}_\kappa] \neq 1$ from Step 2(ii). So λ does not normalise B. On the other hand $T_\lambda(B)$ contains $\mathbb{T}_\kappa$ so $\lambda \in K_B$. In particular, everything we said so far of κ holds of λ; by rank equality, $\mathbb{T}_\lambda = \mathbb{T}_\kappa$.

By conjugacy of Sylow 2-subgroups, λ has an $N_{\hat{G}}(J_\kappa)$-conjugate μ in $\hat{S}$. Remember that we took $\hat{G} = G \cdot d(\hat{S}^\circ)$, so $N_{\hat{G}}(J_\kappa) = N_G(J_\kappa) \cdot d(\hat{S})$ and $\mu = \lambda^n$ for some $n \in N_G(J_\kappa)$. Moreover $\mu \in \hat{S}$ commutes with the involution ι we fixed earlier in the proof. Let $X = C_{Y_\iota}^\circ(\mu) \leq F^\circ(B)$.

• Suppose $X = 1$. Then μ inverts Y_ι, so

$$Y_\iota \leq [J_\kappa, \mu] = [J_\kappa, \lambda^n] = [J_\kappa, \lambda]^n \leq \mathbb{T}_\lambda^n = \mathbb{T}_\kappa^n$$

and equality follows from the equality of ranks.

• Suppose $X \neq 1$. We apply The Devil's Ladder (Proposition 4) to the action of $\langle \mu, \iota \rangle$ on X inside B_μ, the only Borel subgroup of G containing $C_G^\circ(\mu)$ by uniqueness (Step 1(i)). We find $B_\mu \geq C_G^\circ(X) \geq U_\rho(Z(F^\circ(B)))$. Uniqueness principles force $U_\rho(B_\mu) = U_\rho(B)$, which means $\mu \in I_B \cap \hat{S}$. In particular, everything we said in this proof of ι holds of μ, so

$$Y_\mu = [J_\kappa, \mu] = [J_\kappa, \lambda^n] = [J_\kappa, \lambda]^n \leq \mathbb{T}_\lambda^n = \mathbb{T}_\kappa^n$$

and equality follows from the equality of ranks.

In any case, $\mathbb{T}_\kappa$ is G-conjugate to Y; we are done. □

Notations and steps from Sections 5.2 to 5.4 may be forgotten.

5.6. *Critical mass.*

Step 6 (the collapse).

We first estimate $\operatorname{rk}\{\mathbb{T}_\kappa : \kappa \in K_B\}$. The set under consideration is definable as a subset of $\{Y^g : g \in G\} = G/N_G(Y)$ by Step 5(iv). If $\mathbb{T}_\kappa = \mathbb{T}_\lambda$ then there is $g \in G$ with $\mathbb{T}_\kappa = Y^g$. In particular, κ and λ lie in $N_{\hat{G}}(N_G^\circ(Y^g)) = N_{\hat{G}}(B^g)$ by Step 5(i). Since κ and λ are G-conjugate, $\kappa\lambda \in N_G(B^g)$. Now κ inverts $\kappa\lambda$ so by Step 1(iv), $\kappa\lambda \in B^g$, and $\lambda \in \kappa T_{B^g}(\kappa)$. The latter has the same rank as Y by Proposition 5 and Step 5(iv). It follows that $\operatorname{rk}\{\mathbb{T}_\kappa : \kappa \in K_B\} \geq \operatorname{rk} K_B - \operatorname{rk} Y = \operatorname{rk} G - \operatorname{rk} C_G(\iota) - \operatorname{rk} Y$.

We move to something else. Let $\mathcal{F}$ be a definable family of conjugates of Y. Since an element in Y lies in only finitely many conjugates by Step 5(iii), $\operatorname{rk}\bigcup\mathcal{F} = \operatorname{rk}\mathcal{F} + \operatorname{rk} Y$. We first apply this to $\mathcal{F}_1 = \{\mathbb{T}_\kappa : \kappa \in K_B\}$, finding

$$\operatorname{rk}\bigcup\mathcal{F}_1 = \operatorname{rk}\bigcup_{\kappa\in K_B}\mathbb{T}_\kappa \geq \operatorname{rk} G - \operatorname{rk} C_G(\iota_0) - \operatorname{rk} Y + \operatorname{rk} Y = \operatorname{rk} G - \operatorname{rk} C_G(\iota_0).$$

We now apply it to $\mathcal{F}_2 = \{Y^g : g \in G/N_G(Y)\}$, finding

$$\operatorname{rk}\bigcup\mathcal{F}_2 = \operatorname{rk} Y^G = \operatorname{rk} G - \operatorname{rk} N_G(B) + \operatorname{rk} Y = \operatorname{rk} G - \operatorname{rk} B + \operatorname{rk} Y.$$

Both agree by Step 5(ii), so $\bigcup\mathcal{F}_1$ is generic in $\bigcup\mathcal{F}_2$. However, $\bigcup\mathcal{F}_1 \subseteq \left(\bigcup\mathcal{F}_2 \cap B\right)$, which contradicts [*DJ 2012, Lemma 3.33].

This concludes the proof of Proposition 6. □

6. The proof — after the maximality proposition

6.1. *The dihedral case.* The following is a combination of two different lines of thought: the study of a pathological "$W = 2$" configuration in [*Del 2007a, Chapitre 4] (published as [*Del 2008, §3]) and the final argument in [*BCD 2009]. Since we can quickly focus on the $2^\perp$ case only a few details need be adapted in order to move from minimal connected simple groups to $N_\circ^\circ$-groups, so we feel that the resulting proposition owes much to Burdges and Cherlin. The final contradiction is by constructing two disjoint generic subsets of some definable subset of G.

Proposition 7 (dihedral case). *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Suppose that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$.*

Suppose that the Sylow 2-subgroup of $\hat{G}$ is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$. *Suppose in addition that for $\iota \in I(\hat{G})$, the group $C_G^\circ(\iota)$ is contained in a unique Borel subgroup of G.*

Then $\hat{G}/G$ is $2^\perp$ and $B_\iota = C_G^\circ(\iota)$ is a Borel subgroup of G inverted by any involution $\omega \in C_G(\iota) \setminus \{\iota\}$.

Proof. First observe that by torality principles, all involutions in $\hat{G}$ are conjugate. If one is in $\hat{G} \setminus G$ then all are, and G is $2^\perp$. If one is in G then $\Pr_2(G) = 1$ and $\Pr_2(\hat{G}/G) = 0$; $\hat{G}/G$ is $2^\perp$ by the degenerate type analysis [BBC 2007] and connectedness.

Notation. • Let $V = \{1, \iota, \omega, \iota\omega\} \leq \hat{G}$ be a four-group.

• Let $\hat{T}_\iota$ be a 2-torus containing ι and inverted by ω, and $\hat{T}_\omega$ likewise.

• Let B_ι be the only Borel subgroup of G containing $C_G^\circ(\iota)$, and B_ω likewise (observe that by uniqueness of B_ι over $C_G^\circ(\iota)$, V normalises B_ι and B_ω).

• Let $\rho = \rho_{B_\iota}$.

Here is a small unipotence principle we shall use with no reference: if $L \leq G$ is a definable, connected, soluble, V-invariant subgroup, then $\rho_L \preccurlyeq \rho$. This is obvious as otherwise all involutions in V invert $U_{\rho_L}(L)$. Bigeneration (Fact 3) will also play a growing role in the subsequent pages.

Step 1. $B_\iota \neq B_\omega$.

Proof of Step 1. Suppose not. If G is $2^\perp$, then it is $W_2^\perp$: by Proposition 6, B_ι is a Borel subgroup of G. Hence $C_G^\circ(\iota) = B_\iota = B_\omega = C_G^\circ(\omega)$, and therefore $B_\iota = C_G^\circ(\iota\omega)$ as well. Yet bigeneration, Fact 3, applies to the action of V on the $2^\perp$ group G: a contradiction.

If G is not $2^\perp$ then bigeneration might fail. But now all involutions are in G; by torality principles $\iota \in C_G^\circ(\iota) \leq B_\iota = B_\omega$ so B_ω contains $\hat{T}_\omega \rtimes \langle\iota\rangle$, which contradicts the structure of torsion in connected, soluble groups. □

Notation. Let $H = (B_\iota \cap B_\omega)^\circ$.

Since ω normalises B_ι and vice-versa, H is V-invariant.

Step 2. *H is abelian and $2^\perp$. Moreover, ι centralises $U_\rho(B_\iota)$ and ω inverts it; V centralises H and $N_G^\circ(H) = C_G^\circ(H)$.*

Proof of Step 2. If $H = 1$ then $C_{B_\iota}^\circ(\omega) = 1$ and ω inverts B_ι; since ω inverts $\hat{T}_\iota$ which normalises B_ι, commutation principles yield $[\hat{T}_\iota, B_\iota] = 1$ and $B_\iota \leq C_G^\circ(\iota)$. So $B_\iota = C_G^\circ(\iota)$ is an abelian Borel subgroup inverted by ω and by $\iota\omega$. Hence all our claims hold if $H = 1$. We now suppose $H \neq 1$.

Suppose that H is not abelian and let $L = N_G^\circ(H')$, a definable, connected, soluble, V-invariant group. Then $\rho_L \preccurlyeq \rho$ but since L contains $U_\rho(Z(F^\circ(B_\iota)))$ and $U_\rho(Z(F^\circ(B_\omega)))$, equality holds. Hence $U_\rho(Z(F^\circ(B_\iota))) \leq U_\rho(L)$; by uniqueness principles $U_\rho(B_\iota)$ is the only Sylow ρ-subgroup of G containing $U_\rho(L)$. The same holds of $U_\rho(B_\omega)$, proving equality and $B_\iota = B_\omega$, against Step 1. So H is abelian.

Now suppose that $U_\rho(H) \neq 1$ and let $L = N_G^\circ(U_\rho(H))$. The same causes having the same effects, we reach a contradiction again. Hence $U_\rho(H) = 1$, and it follows that ω inverts $U_\rho(B_\iota)$. The same argument works for $\iota\omega$, so ι centralises $U_\rho(B_\iota)$.

We now claim that V centralises H. Let $K = [H, \iota]$; since H is abelian, using Zilber's indecomposability theorem we see that K is a definable, connected, abelian group inverted by ι; in particular it is 2-divisible. Since ι centralises $U_\rho(B_\iota)$ and inverts $U_\rho(B_\omega)$, commutation principles yield $\langle U_\rho(B_\iota), U_\rho(B_\omega)\rangle \leq C_G^\circ(K)$ and the latter is V-invariant. Uniqueness principles and Step 1 forbid solubility of $C_G^\circ(K)$; this means $K = 1$, and ι centralises H. The same holds of ω.

Suppose that H has involutions. Since it is V-invariant, so is its Sylow 2-subgroup T (no need for normalisation principles here). If $\iota \in T$, then $\iota \in H \leq B_\iota$ and $\omega \in B_\omega$ by conjugacy; hence B_ω contains $\hat{T}_\omega \rtimes \langle\iota\rangle$, against the structure of torsion in connected, soluble groups. So $\iota \notin T$, and by assumption on the structure of the Sylow 2-subgroup of $\hat{G}$, ι inverts T; the same holds of ω and $\iota\omega$, a contradiction.

It remains to show that $N_G^\circ(H) = C_G^\circ(H)$. Let $N = N_G^\circ(H)$. First assume that G is $2^\perp$. Then using Lemma E under the action of ι we write $N = (N^{+_\iota})^\circ \cdot \{N, \iota\}$, where $\{N, \iota\}$ is 2-divisible. Since ι centralises H, commutation principles applied pointwise force $\{N, \iota\} \subseteq C_G(H)$. We turn to the action of ω on $N_1 = (N^{+_\iota})^\circ$; with Lemma E again $N_1 = (N_1^{+_\omega})^\circ \cdot \{N_1, \omega\}$, and here again $\{N_1, \omega\} \subseteq C_G(H)$. Finally, $(N_1^{+_\omega})^\circ \leq C_G^\circ(\iota, \omega) \leq H \leq C_G(H)$ by abelianity, so $N \leq C_G(H)$ and we conclude by connectedness of N.

Now suppose that $\hat{G}/G$ is $2^\perp$; as a consequence, $V \leq G$. It is not quite clear whether N has involutions and whether $\{N, \iota\}$ is 2-divisible, so we argue as follows. By normalisation principles, there is a V-invariant Carter subgroup Q of N. The previous argument applies to Q, so $Q \leq C_G^\circ(H)$; it also applies to $F^\circ(N)$, so $F^\circ(N) \leq C_G^\circ(H)$, and $N = F^\circ(N) \cdot Q \leq C_G^\circ(N)$. □

Step 3. *We may suppose that G is $2^\perp$.*

Proof of Step 3. Suppose that G contains involutions, i.e., $V \leq G$. We shall prove that $H = 1$. So suppose in addition that $H \neq 1$. For the consistency of notations, let $i = \iota \in G$, $w = \omega \in G$, and $T_i = \hat{T}_i$, $T_w = \hat{T}_w$.

We claim that w does *not* invert $F^\circ(B_i)$. If it does, then w inverts $T_i \leq B_i$ and $F^\circ(B_i)$, so by commutation principles $[T_i, F^\circ(B_i)] = 1$. Let $Q \leq B_i$ be a Carter subgroup of B_i containing T_i; then $B_i = F^\circ(B_i) \cdot Q$ centralises T_i, and $T_w \leq Z(B_w)$ by conjugacy. Hence,

$$T_i \rtimes \langle w \rangle \leq \langle T_i, T_w \rangle \leq C_G^\circ(H),$$

against the structure of torsion in connected, soluble groups and G being $N_\circ^\circ$.

Hence $Y_i = C^\circ_{F^\circ(B_i)}(w) \neq 1$. Since $U_\rho(B_i)$ is abelian by Step 2, $U_\rho(B_i) \leq C_G^\circ(Y_i)$; since Y_i is V-invariant, our small unipotence principle and general uniqueness principles force $C_G^\circ(Y_i) \leq B_i$. Hence, by Step 2,

$$N^\circ_{B_w}(H) = C^\circ_{B_w}(H) \leq C^\circ_{B_w}(Y_i) \leq H,$$

which proves that H is a Carter subgroup of B_w. It therefore contains involutions, against Step 2.

This contradiction shows that if G has involutions then $H = 1$. Hence, as in the beginning of Step 2, w inverts $B_i = C_G^\circ(i)$ and so does any other involution in $C_G(i) \setminus \{i\}$; if G has involutions, Proposition 7 is proved. □

From now on, we suppose that G is $2^\perp$; we are after a contradiction. Since G is $W_2^\perp$, Proposition 6 applies and $C_G^\circ(\iota) = B_\iota$ is a Borel subgroup of G. Moreover, since G is $2^\perp$, it admits a decomposition $G = G^{+_\iota} \cdot G^{-_\iota}$ by Lemma E, and the fibres are trivial. From the connectedness of G we deduce that $C_G(\iota) = G^+$ is connected. Finally, since the 2-torus $\hat{T}_\iota$ normalises B_ι, it centralises the finite quotient $N_G(B_\iota)/B_\iota$, and so does ι. Now $N = N_G(B_\iota)$ admits a decomposition $N = N^+ \cdot \{N, \iota\}$ as well; we just proved $N^+ \leq B$ and $\{N, \iota\} \subseteq B$. Hence $B_\iota = C_G(\iota)$ is a self-normalising Borel subgroup of G, which will be used with no reference.

Step 4. *For any involution $\lambda \in C_{\hat{G}}(\iota) \setminus \{\iota\}$, $B_\iota^{-\lambda} = F^\circ(B_\iota)$.*

Proof of Step 4. The claim is actually obvious if $H = 1$, an extreme case in which the below argument remains however valid. Let $X_\iota = C^\circ_{F^\circ(B_\iota)}(\omega)$ and $X_\omega = C^\circ_{F^\circ(B_\omega)}(\iota)$.

Suppose that $X_\iota \neq 1$ *and* $X_\omega \neq 1$. By abelianity of $U_\rho(B_\iota)$ from Step 2, $U_\rho(B_\iota) \leq C^\circ_G(X_\iota)$. As the latter is V-invariant, it has parameter exactly ρ, so $C^\circ_G(X_\iota) \leq N^\circ_G(U_\rho(B_\iota)) = B_\iota$; by uniqueness principles, B_ι is the only Borel subgroup of G with parameter ρ containing $C^\circ_G(X_\iota)$, and likewise for B_ω over $C^\circ_G(X_\omega)$. It follows that $C^\circ_{B_\omega}(H) \leq (B_\iota \cap B_\omega)^\circ = H$ and H is a Carter subgroup of B_ω. The latter is $\hat{T}_\omega \rtimes \langle\iota\rangle$-invariant, so by normalisation principles $N_{\hat{G}}(H)$ contains a Sylow 2-subgroup $\hat{S}$ of $\hat{G}$. Since $V \leq C_{\hat{G}}(H)$ by Step 2, we may assume $V \leq \hat{S}$.

Still assuming that $X_\iota \neq 1$ and $X_\omega \neq 1$, we denote by μ the involution of V which lies in $\hat{S}^\circ = \hat{T}_\mu$ and fix $\nu \in V \setminus \langle\mu\rangle$. Then by assumption on the structure of the Sylow 2-subgroup of $\hat{G}$, ν inverts $\hat{T}_\mu$; it also centralises H, so by commutation principles $\hat{T}_\mu \rtimes \langle\nu\rangle = \hat{S}$ centralises $H \geq \langle X_\iota, X_\omega\rangle$. Since B_ι is the only Borel subgroup of G with parameter ρ containing $C^\circ_G(X_\iota)$ (and likewise for ω), $\hat{S}$ normalises both B_ι and B_ω. Remember that $V = \langle\iota, \omega\rangle = \langle\mu, \nu\rangle$; so up to taking $\nu\mu$ instead of ν, we may suppose that $\hat{S}$ normalises B_ν. Now ν inverts $\hat{T}_\mu$ and centralises B_ν, so by commutation principles $[\hat{T}_\mu, B_\nu] = 1$ and $B_\nu \leq C^\circ_G(\mu) = B_\mu$: a contradiction to Step 1.

All this shows that $X_\iota = 1$ *or* $X_\omega = 1$; we suppose the first. Then ω inverts $F^\circ(B_\iota)$. Using Lemma E we write $B_\iota = B_\iota^{+\omega} \cdot \{B_\iota, \omega\}$. Notice that since B_ι is $2^\perp$, $B_\iota^- = \{B_\iota, \omega\}$ (the sign $-$ refers to the action of ω throughout the present paragraph). Since ω inverts the 2-divisible subgroup $F^\circ(B_\iota)$, one has $F^\circ(B_\iota) \subseteq B_\iota^-$. Since the set B_ι^- is 2-divisible, commutation principles applied pointwise show $F^\circ(B_\iota) \subseteq B_\iota^- \subseteq C_{B_\iota}(F^\circ(B_\iota))$. Hence B_ι^- turns out to be a union of translates of $F^\circ(B_\iota)$. Now $C_{B_\iota}(F^\circ(B_\iota))$ is normal in B_ι and nilpotent, so by definition of the Fitting subgroup, $C_{B_\iota}(F^\circ(B_\iota)) \leq F(B_\iota)$. As a consequence $B_\iota^- \subseteq F(B_\iota)$ is a union of *finitely many* translates of $F^\circ(B_\iota)$. But $\deg B_\iota^- = \deg\{B_\iota, \omega\} = \deg \omega^{B_\iota} = 1$, so $F^\circ(B_\iota) = B_\iota^-$.

The previous paragraph shows that if $X_\iota = 1$, then our desired conclusion holds of $\lambda = \omega$; it then also holds of $\lambda = \iota\omega$. Now any involution $\lambda \in C_{\hat{G}}(\iota) \setminus \{\iota\}$ is a $C_{\hat{G}}(\iota)$-conjugate of ω or $\iota\omega$, say $\lambda = \omega^n$ with $n \in C_{\hat{G}}(\iota) \leq N_{\hat{G}}(B_\iota) \leq N_{\hat{G}}(F^\circ(B_\iota))$, so

$$B_\iota^{-\lambda} = B_\iota^{-\omega^n} = (B_\iota^{-\omega})^n = (F^\circ(B_\iota))^n = F^\circ(B_\iota).$$

Similarly, if $X_\omega = 1$, then $B_\omega^{-\lambda} = F^\circ(B_\omega)$ for any $\lambda \in C_{\hat{G}}(\omega) \setminus \{\omega\}$. We conjugate ω to ι and see that in this case we are done as well. $\square$

Step 5. $\operatorname{rk} G^{-\iota} \leq 2 \operatorname{rk} F^\circ(B_\iota)$.

Proof of Step 5. Let $\kappa = \iota\omega$ and $\check{G} = G \rtimes V$. Observe that in $\check{G}$ the involutions ι, ω, κ are *not* conjugate; one has exactly three conjugacy classes, which also are G-classes. So for $(\omega_1, \kappa_1) \in \omega^G \times \kappa^G$, the definable closure $d(\omega_1\kappa_1)$ contains a unique involution which must be a conjugate ι_1 of ι.

Now consider the definable function from $\omega^G \times \kappa^G$ to ι^G which maps (ω_1, κ_1) to ι_1; we shall compute its fibres. If (ω_2, κ_2) also maps to ι_1 then $\omega_1\omega_2 \in C_G(\iota_1) = B_{\iota_1}$. Hence $\omega_1\omega_2 \in B_{\iota_1}^{-\omega_1} = F^\circ(B_{\iota_1})$ by Step 4, and fibres have rank at most $2\,\mathrm{rk}\, F^\circ(B_\iota)$. As the map is obviously onto, one has $2\,\mathrm{rk}\, F^\circ(B_\iota) \geq \mathrm{rk}\, \check{G} - \mathrm{rk}\, B = \mathrm{rk}\, G^{-\iota}$. □

Step 6. *$(F^\circ(B_\omega))^{F^\circ(B_\iota)}$ and $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$ are generic subsets of $G^{-\iota}$.*

Proof of Step 6. Recall from Step 4 that ι inverts $F^\circ(B_\omega)$ and centralises B_ι. In particular since G is $2^\perp$, one has $F^\circ(B_\omega) \cap B_\iota = 1$; moreover $(F^\circ(B_\omega))^{F^\circ(B_\iota)} \subseteq G^{-\iota}$. We now compute the rank. Consider the definable function from $F^\circ(B_\iota) \times F^\circ(B_\omega)$ to G which maps (a, x) to x^a. Let us prove that it has finite fibres.

Suppose $x^a = y^b$ with $b \in F^\circ(B_\iota)$ and $y \in F^\circ(B_\omega)$; then $x^{ab^{-1}} = y$, and applying ω one finds

$$y = y^\omega = x^{ab^{-1}\omega} = x^{\omega a^{-1} b} = x^{a^{-1}b} = y^{ba^{-2}b}.$$

Since $F^\circ(B_\iota)$ is abelian and G is $2^\perp$, this results in $a^{-1}b \in C_G(y)$ and $x = y$. We now estimate the size of $C_{F^\circ(B_\iota)}(x)$. Suppose $Y = C^\circ_{F^\circ(B_\iota)}(x)$ is infinite. Since Y is V-invariant, so is $C^\circ_G(Y)$, a definable, connected, soluble group containing $F^\circ(B_\iota)$. As we know, $C^\circ_G(Y)$ has unipotence parameter at most ρ, so $C^\circ_G(Y)$ normalises $U_\rho(B_\iota)$ and $C^\circ_G(Y) \leq B_\iota$; as a matter of fact, by uniqueness principles B_ι is the only Borel subgroup of G with parameter ρ containing $C^\circ_G(Y)$. It follows that $x \in N_G(B_\iota)$. Hence $x \in N_G(B_\iota) \cap F^\circ(B_\omega) = C_G(\iota) \cap F^\circ(B_\omega) = 1$.

Thus, fibres are finite; it follows that $\mathrm{rk}(F^\circ(B_\omega))^{F^\circ(B_\iota)} = 2\,\mathrm{rk}\, F^\circ(B_\iota) \geq \mathrm{rk}\, G^{-\iota}$ by Step 5; inclusion forces equality. The same holds of $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$. □

We now finish the proof of Proposition 7. By Step 6, both the sets $(F^\circ(B_\omega))^{F^\circ(B_\iota)}$ and $(F^\circ(B_{\iota\omega}))^{F^\circ(B_\iota)}$ are generic in $G^{-\iota}$. So there is $t \in F^\circ(B_\omega) \cap F^\circ(B_{\iota\omega})^f \setminus \{1\}$ for some $f \in F^\circ(B_\iota)$. Then the involution $(\iota\omega)^f = f^{-1}\iota\omega f = f^\omega \iota\omega f = \iota\omega f^2$ centralises t, whereas $\iota\omega$ inverts it. So $f^2 \in G$ inverts t. This creates an involution in G, against Step 3. □

6.2. ***Strong embedding.*** Strong embedding is a classical topic in finite group theory [Bender 1971]. Recall that a proper subgroup M of a group G is said to be strongly embedded if M contains an involution but $M \cap M^g$ does not for any $g \notin M$. The reader should also keep in mind a few basic facts about strongly embedded configurations [BN 1994b, Theorem 10.19] (checking the apparently missing assumptions would be almost immediate here):

- involutions in M are M-conjugate;

- a Sylow 2-subgroup of M is a Sylow 2-subgroup of G;
- M contains the centraliser of any of its involutions.

We need no more. The study of a minimal connected simple group with a strongly embedded subgroup was carried out in [*BCJ 2007, Theorem 1].

Proposition 8 (strong embedding). *Let G be a connected, $U_2^\perp$, nonsoluble $N_\circ^\circ$-group of finite Morley rank. If G has a definable, soluble, strongly embedded subgroup, then* $\mathrm{Pr}_2(G) \leq 1$.

Our proof will be considerably shorter than [*BCJ 2007]; thanks to Proposition 6 we need only handle the case of central involutions [*BCJ 2007, §4]. Apart from this, our argument is a subset of the one in [*BCJ 2007, §4]: we construct two disjoint generic sets. We only hope to have helped clarify matters in Step 4 below.

(Incidentally, an alternate proof of the noncentral case of [*BCJ 2007, Theorem 1] was suggested using state-of-the-art genericity arguments in minimal connected simple groups [ABF 2013, Theorem 6.1]. Yet this new proof reproduces the central case [*BCJ 2007, §4] and affects only the configuration we need not consider by maximality.)

Proof. We let G be a minimal counterexample, i.e., G satisfies the assumptions but $\mathrm{Pr}_2(G) \geq 2$. By Proposition 1, the Sylow 2-subgroups of G are connected.

Notation. Let $M < G$ be a definable, soluble, strongly embedded subgroup. Let $S \leq M$ be a Sylow 2-subgroup of G and $A = \Omega_2(S^\circ)$ be the group generated by the involutions of S°.

Step 1. *$C_G^\circ(i)$ is soluble for all $i \in I(G)$.*

Proof of Step 1. First observe that $Z(G)$ has no involutions by strong embedding, as they would lie in $S \leq M$ and in any conjugate.

Suppose that there is $i \in A \setminus \{1\}$ with nonsoluble $C_G^\circ(i)$. Fix some 2-torus $\tau_i \leq S$ of Prüfer rank 1 containing i; since $C_G^\circ(\tau_i)$ is soluble because G is an $N_\circ^\circ$-group, there exists by the descending chain condition some $\alpha \in \tau_i$ with $C_G^\circ(\alpha)$ soluble. We take α with minimal order; then $C_G^\circ(\alpha^2)$ is not soluble, and $\alpha^2 \neq 1$ since $\alpha \neq i$.

Let $H = C_G^\circ(\alpha^2)$ and $N = M \cap H$. Since $\alpha^2 \neq 1$ and $Z(G)$ has no 2-elements, $H < G$. Observe how $\alpha \in \tau_i \leq S \leq N$. Let $\overline{H} = H/\langle\alpha^2\rangle$ and $\overline{N} = N/\langle\alpha^2\rangle$. Then $\overline{N}$ is definable, soluble, and strongly embedded in $\overline{H}$, which still has Prüfer rank ≥ 2, against minimality of G as a counterexample. □

Notation. Let $B = M^\circ$.

Step 2. *B is a Borel subgroup of G and $A \leq Z(B)$; the group M/B is nontrivial and has odd order. Moreover, the following hold.*

(i) *Strongly real elements of G which lie in B actually lie in A.*

(ii) *If $i \in I(B)$ inverts $n \in N_G(B)$ then $n \in B$.*

(iii) *$BgI(G)$ is generic in G for any $g \in G$.*

(iv) *$(B \cap B^g)^\circ = 1$ for $g \notin N_G(B)$.*

Proof of Step 2. By Step 1, connectedness of the Sylow 2-subgroup, and the maximality proposition (Proposition 6), $C_G^\circ(i)$ is a Borel subgroup of G for any $i \in I(G)$. But for $i \in A \setminus \{1\}$, $C_G(i) \leq M$ by strong embedding of the latter, so $C_G^\circ(i) \leq B$ and equality follows. In particular, $A \leq Z(B)$.

By structure of the Sylow 2-subgroup, $N_G(B)/B$ has odd order, and so has its subgroup M/B. But M, being strongly embedded, conjugates its (more than one) involutions, which are central in B. This shows $B < M$.

If $b \in B$ is inverted by some $k \in I(G)$ then k normalises $C_G(b) \geq A$; by normalisation principles and structure of the Sylow 2-subgroup, one has $k \in C_G(b)$, so b has order at most 2; this is claim (i). If $i \in I(B)$ inverts $n \in N_G(B)$ then computing modulo B, we get $n^{-1}B = n^iB = nB$ and $n^2 \in B$. Since $N_G(B)/B$ has odd order, $n \in B$, proving (ii).

We move to (iii). Consider the definable function $B \times I(G)$ which maps (b, k) to bk. If $b_1k_1 = b_2k_2$ with the obvious notation, then $b_2^{-1}b_1$ is a strongly real element of G lying in B, and hence has order at most 2 by claim (i). This happens only finitely many times, so fibres are finite and $\operatorname{rk}(B \cdot I(G)) = \operatorname{rk} B + \operatorname{rk} I(G) = \operatorname{rk} B + \operatorname{rk} G - \operatorname{rk} B = \operatorname{rk} G$. Then for any $g \in G$,

$$\operatorname{rk}(BgI(G)) = \operatorname{rk}(gB^gI(G)^g) = \operatorname{rk}(g(BI(G))^g) = \operatorname{rk}(BI(G)) = \operatorname{rk} G.$$

It remains to control intersections of conjugates of B, claim (iv). Suppose that $H = (B \cap B^g)^\circ$ is infinite. Let $Q \leq H$ be a Carter subgroup of H; since A^g centralises $B^g \geq H \geq Q$, it normalises the definable, connected, soluble group $N_G^\circ(Q)$. By bigeneration (Fact 3), $N_G^\circ(Q) \leq \langle C_G^\circ(a^g) : a \in A \setminus \{1\}\rangle = B^g$, so $N_B^\circ(Q) \leq N_H^\circ(Q) = Q$ and Q is actually a Carter subgroup of B. Hence, Q contains a Sylow 2-subgroup of B. Thus $A \leq Q \leq B^g$, and strong embedding guarantees $g \in N_G(B)$. □

Notation. Let $w \in M \setminus B$ (denoted σ in [*BCJ 2007, Notation 4.1(2)]).

Step 3. *We may assume that w is strongly real, in which case the following hold.*

(i) *$C_G(w)$ has no involutions.*

(ii) *If some involution $k \in I(G)$ inverts w, then k inverts $C_G^\circ(w)$.*

(iii) $C_B^\circ(w) = 1$.

Proof of Step 3. By Step 2(iii), both $BI(G)$ and $BwI(G)$ are generic in G, so they intersect. Hence up to translating by an element of B, we may suppose that w is a strongly real element.

Suppose that there is an involution $\ell \in C_G(w)$. Then $w \in C_G(\ell) = C_G^\circ(\ell)$ by Steinberg's torsion theorem and connectedness of the Sylow 2-subgroup; $C_G^\circ(\ell)$ is a conjugate of B (Sylow theory suffices here; no need to invoke strong embedding). But w is strongly real, so by Step 2(i) it is an involution, against the fact that M/B has odd order.

We prove (ii): Let k be an involution inverting w. Then $C_G^\circ(k)$ is a conjugate B_k of B, and $k \in B_k$. Observe how $w \notin N_G(B_k)$ by Step 2(ii). So $C_G^\circ(k, w) \leq (B_k \cap B_k^w)^\circ$ is trivial by Step 2(iv), and k inverts $C_G^\circ(w)$.

Finally, let $H = C_B^\circ(w)$ and suppose $H \neq 1$. Bear in mind that A centralises H, so it normalises the definable, soluble group $N_G^\circ(H)$. By bigeneration (Fact 3), $N_G^\circ(H) \leq B$. But k inverts H, so it normalises $N_G^\circ(H)$ as well. Hence $N_G^\circ(H) \leq B \cap B^k$, and Step 2(iv) forces $k \in N_G(B)$. Now $k \in B$ inverts $w \in N_G(B) \setminus B$, a contradiction to Step 2(ii). This shows that $C_B^\circ(w) = 1$. □

Notation. Let $\check{C} = C_G^\circ(w) \setminus N_G(B)$.

$\check{C}$ is obviously generic in $C_G^\circ(w)$, as $C_{N_G(B)}^\circ(w) \leq C_B^\circ(w) = 1$ by Step 3(iii).

Step 4. *$B\check{C}B$ is generic in G.*

Proof of Step 4. This is the only part where we slightly rewrite the argument given in [*BCJ 2007]. Let $\mathcal{F} = \{(m, \ell) \in Bw \times I(G) : m^\ell = m^{-1}\}$.

Let $m \in Bw$. If m is inverted by some involution in G, then by Step 3(iii), $C_B^\circ(m) = 1$ and $m^B \subseteq Bm$ is generic in Bm. So is w^B, and m is therefore B-conjugate with w. So let us count involutions inverting w. First, there is such an involution k by Step 3. If ℓ is another such, then $k\ell \in C_G(w)$ and $\ell \in kC_G(w)$. Conversely, since k inverts $C_G^\circ(w)$ by Step 3(ii), any element in $kC_G^\circ(w)$ is an involution inverting w. This together shows

$$\operatorname{rk} \mathcal{F} = \operatorname{rk} w^B + \operatorname{rk} C_G^\circ(w) = \operatorname{rk} B + \operatorname{rk} C_G^\circ(w).$$

On the other hand, since $BwI(G)$ and $BI(G)$ are generic in G by Step 2(iii), a generic $\ell \in I(G)$ inverts some element in Bw. Hence $\operatorname{rk} \mathcal{F} \geq \operatorname{rk} I(G) = \operatorname{rk} G - \operatorname{rk} B$.

Finally consider the definable function which maps $(b_1, c, b_2) \in B \times \check{C} \times B$ to $b_1 c b_2$. We claim that all fibres are finite. Since the fibre over $b_1 c_0 b_2$ has the same rank as the fibre over c_0, we compute the latter. Suppose $b_1 c b_2 = c_0$ with the obvious notation. Then, applying w,

$$c_0 = c_0^w = b_1^w c b_2^w = [w, b_1^{-1}] b_1 c b_2 [b_2, w] = [w, b_1^{-1}] c_0 [b_2, w].$$

In particular, $[w, b_1^{-1}]^{c_0} = [b_2, w]^{-1} \in B \cap B^{c_0}$, which is finite by Step 2(iv). Since $C_B^\circ(w) = 1$ by Step 3(iii), there are finitely many possibilities for b_1 and b_2, and c is then determined. So the function has finite fibres, and therefore,

$$\operatorname{rk}(B\check{C}B) = 2 \operatorname{rk} B + \operatorname{rk} C_G^\circ(w) = \operatorname{rk} \mathcal{F} + \operatorname{rk} B \geq \operatorname{rk} G. \qquad \square$$

We now finish the proof of Proposition 8. By Steps 2(iii) and 4, both $BI(G)$ and $B\check{C}B$ are generic in G. So they intersect; there is an involution $k = b_1 c b_2 \in B\check{C}B$. Conjugating by b_1, there is an involution $\ell = cb \in \check{C}B$. Now, applying w, one finds

$$\ell^w = cb^w = cb[b, w] = \ell[b, w],$$

which means that $[b, w] \in B$ is a strongly real element. There are two possibilities. If $[b, w] \neq 1$ then by Step 2(i), $[b, w] \in A \setminus \{1\}$ and $\ell \in C_G([b, w])$, so ℓ and c lie in B: a contradiction. If $[b, w] = 1$ then w centralises b and $cb = \ell$, against Step 3(i). □

6.3. *November.*

Theorem. *Let $\hat{G}$ be a connected, $U_2^\perp$ group of finite Morley rank and $G \trianglelefteq \hat{G}$ be a definable, connected, nonsoluble $N_\circ^\circ$-subgroup. Then the Sylow 2-subgroup of G has one of the following structures: isomorphic to that of* $\mathrm{PSL}_2(\mathbb{C})$, *isomorphic to that of* $\mathrm{SL}_2(\mathbb{C})$, *or a 2-torus of Prüfer 2-rank at most 2.*

Suppose in addition that for all involutions $\iota \in I(\hat{G})$, the group $C_G^\circ(\iota)$ is soluble. Then $m_2(\hat{G}) \leq 2$, one of G or $\hat{G}/G$ is $2^\perp$, and involutions are conjugate in $\hat{G}$. Moreover, one of the following cases occurs:

- **PSL_2:** *$G \simeq \mathrm{PSL}_2(\mathbb{K})$ in characteristic not 2; $\hat{G}/G$ is $2^\perp$.*
- **$\mathrm{CiBo}_\emptyset$:** *G is $2^\perp$; $m_2(\hat{G}) \leq 1$; for $\iota \in I(\hat{G})$, $C_G(\iota) = C_G^\circ(\iota)$ is a self-normalising Borel subgroup of G.*
- **CiBo_1:** *$m_2(G) = m_2(\hat{G}) = 1$; $\hat{G}/G$ is $2^\perp$; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G.*
- **CiBo_2:** $\mathrm{Pr}_2(G) = 1$ *and $m_2(G) = m_2(\hat{G}) = 2$; $\hat{G}/G$ is $2^\perp$; for $i \in I(\hat{G}) = I(G)$, $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$ and satisfies* $\mathrm{rk}\, G = 3\, \mathrm{rk}\, C_G^\circ(i)$.
- **CiBo_3:** $\mathrm{Pr}_2(G) = m_2(G) = m_2(\hat{G}) = 2$; *$\hat{G}/G$ is $2^\perp$; for $i \in I(\hat{G}) = I(G)$, $C_G(i) = C_G^\circ(i)$ is a self-normalising Borel subgroup of G; if $i \neq j$ are two involutions of G then $C_G(i) \neq C_G(j)$.*

Proof.

Step 1. *We may suppose that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$.*

Proof of Step 1. By Proposition 1, the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$ or to that of $\mathrm{SL}_2(\mathbb{C})$, or is connected. Our dividing line is based on the Prüfer 2-rank.

If $\mathrm{Pr}_2(G) \leq 2$ then we are done with the first part of the theorem; since the second and longer part is precisely under the assumption that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$, we may proceed if $\mathrm{Pr}_2(G) \leq 2$.

So, suppose not; we shall prove a contradiction in Step 3 below. We may assume that G is minimal with $\mathrm{Pr}_2(G) \geq 3$, and that $\hat{G} = G$. First note that $G/Z(G)$ has Prüfer rank at least 3 but is centreless. So we may suppose $Z(G) = 1$. In this setting we actually *prove* that $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G})$.

Suppose that there is some involution $i \in G = \hat{G}$ with $C_G^\circ(i)$ nonsoluble. Then as in Step 1 of Proposition 8, we take a 2-torus of rank 1, τ_i containing i, and $\alpha \in \tau_i$ of minimal order with $C_G^\circ(\alpha)$ soluble; note $\alpha^2 \neq 1$. Let $H = C_G^\circ(\alpha^2)$. By torality principles, it has the same Prüfer 2-rank as G; hence, by minimality of G as a counterexample, $H = G$ and $\alpha^2 \in Z(G)$: a contradiction.

So, if G is minimal with $\mathrm{Pr}_2(G) \geq 3$, then $C_G^\circ(\iota)$ is soluble for all $\iota \in I(\hat{G}) = I(G)$. We proceed under the assumption. □

Step 2. *We may suppose that G is $W_2^\perp$.*

Proof of Step 2. Suppose G is not. By Proposition 1 and since centralisers° in G of involutions are soluble, the Sylow 2-subgroup of G is isomorphic to that of $\mathrm{PSL}_2(\mathbb{C})$, that is, $\mathrm{Pr}_2(G) = 1$ and $m_2(G) = 2$. Fix $i \in I(G)$.

If $C_G^\circ(i)$ is contained in at least two Borel subgroups of G, then by Proposition 3, $G \simeq \mathrm{PSL}_2(\mathbb{K})$ for some algebraically closed field of characteristic not 2. The latter has no outer automorphisms [BN 1994b, Theorem 8.4]; by the assumption on centralisers of involutions, $\hat{G}/G$ is $2^\perp$ and we are in case $\mathbf{PSL}_2$.

So we may assume that $C_G^\circ(i)$ is contained in a unique Borel subgroup of G. We then apply Proposition 7 inside $\check{G} = G$ to find that $C_G^\circ(i)$ is an abelian Borel subgroup of G inverted by any involution in $C_G(i) \setminus \{i\}$. By torality principles in G there exist Sylow 2-subgroups of G, say $S_i = S_i^\circ \rtimes \langle w \rangle$ with $i \in S_i^\circ$, and $S_w = S_w^\circ \rtimes \langle i \rangle$ likewise. In order to reach case $\mathbf{CiBo}_2$, one first shows that $\hat{G}/G$ is $2^\perp$; only the rank estimate will remain to prove.

If $\hat{G}/G$ is not $2^\perp$ then S_i is no Sylow 2-subgroup of $\hat{G}$. Let $\hat{S} \leq \hat{G}$ be a Sylow 2-subgroup containing S_i properly; it is folklore that $\mathrm{Pr}_2(\hat{S}) \geq 2$. Since $\hat{S}^\circ$ is 2-divisible and invariant under $\omega \in \hat{S}$, we may apply Maschke's theorem (see for instance [Del 2012, Fact 2]) to find a quasicomplement, i.e., a w-invariant 2-torus $\hat{T}$ with $\hat{S}^\circ = S_i^\circ \mathbin{(+)} \hat{T}$. Then using Zilber's indecomposability theorem, $[\hat{T}, w] \leq (\hat{T} \cap G)^\circ = 1$, that is, w centralises $\hat{T}$. It follows that $\hat{T}$ normalises both $C_G^\circ(i)$ and $C_G^\circ(w)$; by the rigidity of tori, it therefore centralises both S_i° and S_w°. Hence, $S_i^\circ \rtimes \langle w \rangle \leq \langle S_i^\circ, S_w^\circ \rangle \leq C_G^\circ(\hat{T})$, so by the structure of torsion in connected, soluble groups, $C_G^\circ(\hat{T})$ may not be soluble. As $\hat{T} \not\leq G$ this does not contradict G being $N_\circ^\circ$, but this is against the fact that $\hat{T} \neq 1$ contains an involution of $\hat{G}$, which has soluble centraliser° by assumption.

Hence $\hat{G}/G$ is $2^\perp$; we finally show $\operatorname{rk} G = 3 \operatorname{rk} C_G^\circ(i)$. This exactly follows [*Del 2007a, Proposition 4.1.30 and Corollaire 4.1.31] or [*Del 2008, Proposition 3.26 and Corollaire 3.27]: since $C_G(i)$ is not connected for $i \in I(G)$, using the map from [BBC 2007, §5] (some day we shall return to this) one sees that generic, independent $j, k \in I(G)$ are such that $d(jk)$ is not 2-divisible, and we let ℓ be the only involution in $d(jk)$. Then $(j, k) \mapsto \ell$ is a (generically) well-defined, definable function; obvious rank computations yield $\operatorname{rk} G = 3 \operatorname{rk} C_G^\circ(i)$. □

Notation. For $\iota \in I(\hat{G})$, let $B_\iota = C_G^\circ(\iota)$.

By Steps 1 and 2 and Proposition 6, B_ι is a Borel subgroup of G.

Step 3. $\operatorname{Pr}_2(\hat{G}) \leq 2$.

Proof of Step 3. Suppose $\operatorname{Pr}_2(\hat{G}) \geq 3$. We may assume that $\hat{G} = G \cdot d(\hat{S})$ for some maximal 2-torus $\hat{S}$ of $\hat{G}$. In particular $\hat{G}/G$ is $W_2^\perp$. But so is G by Step 2; by Lemma L, so is $\hat{G}$, i.e., $\hat{S}$ is actually a Sylow 2-subgroup of $\hat{G}$. Let $A = \Omega_2(\hat{S})$ be the group generated by the involutions of $\hat{S}$; then $A \leq \hat{G}$ is an elementary abelian 2-group with 2-rank $\operatorname{Pr}_2(\hat{G}) \geq 3$. Let $\rho = \max\{\rho_{B_\iota} : \iota \in A \setminus \{1\}\}$ and $\iota \in A \setminus \{1\}$ be such that $\rho_{B_\iota} = \rho$.

We show that $B_\lambda = B_\iota$ for any involution $\lambda \in A \setminus \{1\}$. Let $\kappa \in A \setminus \langle\iota\rangle$ be such that $X = C^\circ_{U_\rho(Z(F^\circ(B_\iota)))}(\kappa) \neq 1$; this exists as A has rank at least 3. Then $X \leq C_G^\circ(\kappa) = B_\kappa$, so $\rho_\kappa = \rho$ and $X \leq U_\rho(B_\kappa)$. Let as always $\hat{B}_\iota = B_\iota \cdot d(\hat{S})$; one has $\{B_\iota, \kappa\} \subseteq (\hat{B}_\iota' \cap B)^\circ \leq F^\circ(B_\iota)$ so we may apply Lemma F and write $B_\iota = B_\iota^{+_\kappa} \cdot \{B_\iota, \kappa\} \subseteq B_\iota^+ \cdot F^\circ(B_\iota)$. Now both B_ι^+ and $F^\circ(B_\iota)$ normalise X, hence X is normal in B_ι. Uniqueness principles imply that $U_\rho(B_\iota)$ is the only Sylow ρ-subgroup of G containing X. In particular $U_\rho(B_\iota) = U_\rho(B_\kappa)$. Hence $C_G^\circ(\iota) = B_\iota = B_\kappa = C_G^\circ(\kappa) = C_G^\circ(\iota\kappa)$. Turning to an arbitrary $\lambda \in A \setminus \{1\}$, we apply bigeneration (Fact 3) to the action of $V = \langle\iota, \kappa\rangle$ on the soluble group B_λ, and find $B_\lambda \leq \langle C^\circ_{B_\lambda}(\mu) : \mu \in V \setminus \{1\}\rangle \leq B_\iota$. So $B_\lambda = B_\iota$ for any $\lambda \in A \setminus \{1\}$.

We claim that $\operatorname{Pr}_2(G) = 1$. First, if G is $2^\perp$ then bigeneration applies and we find $G = \langle C_G^\circ(\mu) : \mu \in V \setminus \{1\}\rangle = B_\iota$, a contradiction. Therefore G has involutions. In order to bound its Prüfer 2-rank we use Proposition 8. We argue that $M = N_G(B_\iota)$ is strongly embedded in G. Let j be an involution in $S = \hat{S} \cap G$, which is a Sylow 2-subgroup of G; then $j \in N_G(B_\iota)$. But G is $W_2^\perp$ and B_ι contains a maximal 2-torus of G, so $j \in B_\iota$. Let $V = \langle\iota, \kappa\rangle$; recall that V centralises B_ι. In particular V centralises j, and normalises B_j. As the latter is soluble we apply bigeneration (Fact 3) and find $B_j = \langle C^\circ_{B_j}(\lambda) : \lambda \in V \setminus \{1\}\rangle \leq B_\iota$. Now if $j \in M^x$ with $x \in G$, then we argue likewise: $j \in B_\iota^x$, so V^x centralises j, hence V^x normalises B_j, and $B_j = B_\iota^x$. Therefore $x \in N_G(B_\iota)$, and $M = N_G(B_\iota)$ is strongly embedded in G. By Proposition 8, $\operatorname{Pr}_2(G) = 1$, as desired.

Observe that any two commuting involutions of $\hat{G}$ centralise the same Borel subgroup of G: if $\langle\mu, \nu\rangle$ is a four-subgroup of $\hat{G}$ then up to conjugacy, $\langle\mu, \nu\rangle \leq A$, so $B_\mu = B_\nu$. Now any two nonconjugate involutions of $\hat{G}$ commute to a third involution, so they centralise the same Borel subgroup of G. But there are at least two conjugacy classes of involutions in $\hat{G}$, since $\operatorname{Pr}_2(G) = 1$ and $\operatorname{Pr}_2(\hat{G}) \geq 3$. So actually any two involutions of $\hat{G}$ centralise the same Borel subgroup of G. This means that $B_\iota^g = B_\iota$ for any $g \in G$; B_ι is normal in G, which contradicts G being $N_\circ^\circ$. □

Step 4. *Let $\iota \in I(\hat{G})$. If $\iota \in G$ or G is $2^\perp$, then B_ι is self-normalising in G.*

Proof of Step 4. First suppose $i = \iota \in I(G)$. We claim that i is the only involution in $Z(B_i)$. If $\mathrm{Pr}_2(G) = 1$ this is clear by the structure of torsion in connected, soluble groups. If $\mathrm{Pr}_2(G) \geq 2$ (and one has equality by Step 3), then let $k \in I(B_i) \setminus \{i\}$; if $k \in Z(B_i)$ then $B_k = B_i = B_{ik}$ is clearly strongly embedded, against Proposition 8. In particular, $N_G(B_i) \leq B_i \cdot C_G(i) \leq C_G(i) = C_G^\circ(i) = B_i$ by Steinberg's torsion theorem and connectedness of the Sylow 2-subgroup of G (Step 2).

Now suppose that G is $2^\perp$ (this case was already covered in Proposition 7, between Steps 3 and 4). Since $N = N_G(B_\iota) \leq G$ is $2^\perp$, it admits a decomposition $N = N^{+_\iota} \cdot N^{-_\iota}$ under the action of ι. But on the one hand so does G; hence $G = C_G(\iota) \cdot G^{-_\iota}$ with trivial fibres, and by a degree argument $C_G(\iota)$ is connected, so $N^+ \leq B_\iota$. And on the other hand, by torality principles there exists a 2-torus $\hat{S}^\circ$ of $\hat{G}$ containing ι; $\hat{S}^\circ$ normalises B_ι and $N_G(B_\iota)$. By connectedness, $\hat{S}^\circ$ centralises the finite group $N_G(B_\iota)/B_\iota$, and so does ι. So $N^- \subseteq B_\iota$ and therefore $N = B_\iota$. □

Notation. For $\kappa, \lambda \in I(\hat{G})$, let $T_\kappa(\lambda) = T_{B_\kappa}(\lambda)$.

Before reading the following be very careful to note that Proposition 5 requires $\hat{G}$ to be $W_2^\perp$; for the moment only G need be by Step 2.

Step 5 (Antalya). *If $\hat{G}$ is $W_2^\perp$ and $\lambda \notin N_{\hat{G}}(B_\kappa)$ then $T_\kappa(\lambda)$ is finite. If in addition $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus $\hat{S}^\circ \leq \hat{G}$, then $\mathrm{rk}\, C_{\hat{G}}^\circ(\kappa) = \mathrm{rk}\, C_{\hat{G}}^\circ(\lambda)$ and the generic left translate $\hat{g}C_{\hat{G}}^\circ(\lambda)$ contains a conjugate of κ.*

Proof of Step 5. Suppose that $\hat{G}$ is $W_2^\perp$ and $T_\kappa(\lambda)$ is infinite. Then by inductive torsion control (Proposition 5), $\mathbb{T}_\kappa(\lambda)$ is infinite and contains no torsion elements. Then λ inverts $\mathbb{T}_\kappa(\lambda)$ pointwise, and normalises $C_{\hat{G}}(\mathbb{T}_\kappa(\lambda))$; the latter contains κ. By the structure of the Sylow 2-subgroup of $\hat{G}$ and normalisation principles, λ has a $C_{\hat{G}}(\mathbb{T}_\kappa(\lambda))$-conjugate μ commuting with κ. Now μ inverts $\mathbb{T}_\kappa(\lambda)$ and normalises B_κ. Since $N_{\hat{G}}(B_\kappa)$ already contains a Sylow 2-subgroup of $\hat{G}$ which is a 2-torus, μ is toral in $N_{\hat{G}}(B_\kappa)$ by torality principles. Hence $\mathbb{T}_\kappa(\lambda) \subseteq \{B, \mu\} \subseteq F^\circ(B)$. We now take any $t \in \mathbb{T}_\kappa(\lambda) \setminus \{1\}$ and $X = d(t)$, and we climb The Devil's Ladder (Proposition 4): B_κ is the only Borel subgroup of G containing $C_G^\circ(X)$. In particular, λ normalises B_κ, a contradiction.

For the rest of the argument we assume in addition that $\hat{G} = G \cdot d(\hat{S}^\circ)$ for some maximal 2-torus $\hat{S}^\circ \leq \hat{G}$; in particular $\hat{G}$ is $W_2^\perp$ by Step 2 and Lemma L, but also $\hat{G}/G$ is abelian.

Let us introduce the definable maps

$$\begin{array}{rrcl} \pi_{\kappa,\lambda}: & \kappa^{\hat G} \setminus N_{\hat G}(B_\lambda) & \to & \hat G / C^\circ_{\hat G}(\lambda), \\ & \kappa_1 & \mapsto & \kappa_1 C^\circ_{\hat G}(\lambda). \end{array}$$

We shall compute fibres.

Suppose that $\pi_{\kappa,\lambda}(\kappa_1) = \pi_{\kappa,\lambda}(\kappa_2)$. Then by the assumption that $\hat G = G \cdot d(\hat S^\circ)$, G controls $\hat G$-conjugacy of involutions. Hence $\kappa_1\kappa_2 \in C^\circ_{\hat G}(\lambda) \cap G \le C_G(\lambda)$. Be very careful to note that we do not a priori have connectedness of the latter, insofar as there is no outer version of Steinberg's torsion theorem; as a matter of fact, connectedness is immediate only when G is $2^\perp$ or $\lambda \in G$, not in general.

But if $c \in C_G(\lambda)$ is inverted by κ, then κ normalises $C_{\hat G}(c)$, which contains λ; since $\hat G$ is $W_2^\perp$ and by normalisation principles, κ has a $C_{\hat G}(c)$-conjugate μ commuting with λ. Now $\mu \in N_{\hat G}(C_G(\lambda))$, which contains a maximal 2-torus by torality principles; torality principles again provide some maximal 2-torus $T_\mu \le N_{\hat G}(C_G(\lambda))$ containing μ. Then by Zilber's indecomposability theorem, $[c, \mu] \in [c, T_\mu] \le C^\circ_G(\lambda)$, that is, $c^2 \in C^\circ_G(\lambda)$. If G is $2^\perp$ the conclusion comes easily; if G contains involutions, then by torality principles $C^\circ_G(\lambda)$ contains a maximal 2-torus of G which is a Sylow 2-subgroup of G by Step 2, so $c \in C^\circ_G(\lambda)$.

Turning back to our fibre computation, we have $\kappa_1\kappa_2 \in C^\circ_G(\lambda)$, and $\kappa_1\kappa_2 \in T_\lambda(\kappa)$. The latter is finite as first proved. Hence $\pi_{\kappa,\lambda}$ has finite fibres; it follows, keeping Proposition 2 in mind, that

$$\operatorname{rk} \kappa^{\hat G} \le \operatorname{rk} \hat G - \operatorname{rk} C^\circ_{\hat G}(\lambda);$$

that is, $\operatorname{rk} C^\circ_{\hat G}(\lambda) \le \operatorname{rk} C^\circ_{\hat G}(\kappa)$, and vice-versa. So equality holds. By a degree argument, $\pi_{\kappa,\lambda}$ is now generically onto. □

Step 6. *We may suppose that* $\operatorname{Pr}_2(\hat G) = 1$.

Proof of Step 6. Suppose that $\operatorname{Pr}_2(\hat G) \ge 2$; equality follows from Step 3 and we aim at finding case $\mathbf{CiBo}_3$. There seem to be three cases depending on the values of $\operatorname{Pr}_2(G)$ and $\operatorname{Pr}_2(\hat G/G) = 2 - \operatorname{Pr}_2(G)$. We give a common argument. Notice, however, that we rely on Step 3, to the author's great aesthetic discontentment.

Let $\hat S^\circ \le \hat G$ be a maximal 2-torus of $\hat G$ and $\check G = G \cdot d(\hat S^\circ)$. Bear in mind that $\check G$ is $W_2^\perp$ by Step 2 and Lemma L. In particular, $\hat S^\circ$ is a Sylow 2-subgroup of $\hat G$. Let κ, λ, μ be the three involutions in $\hat S^\circ$.

If κ, λ, and μ are not pairwise G-conjugate, then they are not $\check G$-conjugate either. So $\check G$ has at least (hence exactly) three conjugacy classes of involutions by Lemma N: κ, λ, and μ are pairwise not G-conjugate. We apply Step 5 in $\check G$. The generic left-translate $\check g C^\circ_{\check G}(\lambda)$ contains both a conjugate κ_1 of κ and a conjugate μ_1 of μ. Now κ_1 and μ_1 are not $\check G$-conjugate so $d(\kappa_1\mu_1)$ contains an involution ν. By the structure of the Sylow 2-subgroup of $\check G$, ν must be a conjugate λ_1 of λ.

Of course $\lambda_1 \in d(\kappa_1\mu_1) \le C^\circ_{\check{G}}(\lambda)$. By the structure of the Sylow 2-subgroup of $\check{G}$ again, λ is the only conjugate of λ in its centraliser. Hence $\lambda_1 = \lambda$. It follows that $\kappa_1, \mu_1 \in C_{\check{G}}(\lambda)$, and $\check{g} \in C_{\check{G}}(\lambda)$: a contradiction to genericity of $\check{g}C^\circ_{\check{G}}(\lambda)$ in $\check{G}/C^\circ_{\check{G}}(\lambda)$.

So involutions in $\check{G}$ are G-conjugate. This certainly rules out the case where $\mathrm{Pr}_2(G) = 1 = \mathrm{Pr}_2(\check{G}/G)$. Actually this also eliminates the case where $\mathrm{Pr}_2(G) = 0$ and $\mathrm{Pr}_2(\check{G}/G) = 2$. Indeed, in that case κ, λ, μ remain distinct in the quotient $\check{G}/G$, so G cannot conjugate them in $\check{G}$.

Hence $\mathrm{Pr}_2(G) = 2$ and by Step 3, $\hat{G}/G$ is $2^\perp$. We have proved that G conjugates its involutions; by Step 4 their centralisers° in G are self-normalising Borel subgroups. Notice that if $i \neq j$ are two involutions of G with $B_i = B_j$ then $i \in C^\circ_G(j)$, so i and j commute; now $B_i = B_j = B_{ij}$ is strongly embedded in G, against Proposition 8. We recognise case $\mathbf{CiBo}_3$. □

This is the end. If G has involutions then by Steps 2 and 6,

$$m_2(G) = \mathrm{Pr}_2(G) = 1,$$
$$m_2(\hat{G}/G) = \mathrm{Pr}_2(\hat{G}/G) = 0;$$

with a look at Step 4 this is case $\mathbf{CiBo}_1$. So we may suppose that G is $2^\perp$. Since $\mathrm{Pr}_2(\hat{G}) = 1$, Proposition 7 yields $m_2(\hat{G}) = 1$. With a look at Step 4 this is case $\mathbf{CiBo}_\varnothing$. IN MEMORIAM

References

References are divided into three categories:

- Finite groups, with keys of the form [Bender 1984] (author(s), year).
- Groups of finite Morley rank, with keys of the form [AB 2008] (author abbreviation, year).
- Stages of development (chronological order), with keys of the form [*Jal 2000] (*author abbreviation, year).

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SCHUR–WEYL DUALITY FOR DELIGNE CATEGORIES II: THE LIMIT CASE

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This paper is a continuation of a previous paper by the author (*Int. Math. Res. Not.* 2015:18 (2015), 8959–9060), which gave an analogue to the classical Schur–Weyl duality in the setting of Deligne categories.

Given a finite-dimensional unital vector space V (a vector space V with a chosen nonzero vector $\mathbb{1}$), we constructed in that paper a complex tensor power of V: an Ind-object of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ which is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$, where $\overline{\mathfrak{P}}_{\mathbb{1}} \subset \mathrm{GL}(V)$ is the mirabolic subgroup preserving the vector $\mathbb{1}$.

This construction allowed us to obtain an exact contravariant functor $\widehat{\mathrm{SW}}_{\nu,V}$ from the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (the abelian envelope of the category $\underline{\mathrm{Rep}}(S_\nu)$) to a certain localization of the parabolic category O associated with the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$.

In this paper, we consider the case when $V = \mathbb{C}^\infty$. We define the appropriate version of the parabolic category O and its localization, and show that the latter is equivalent to a "restricted" inverse limit of categories $\hat{O}^{\mathfrak{p}}_{\nu,\mathbb{C}^N}$ with N tending to infinity. The Schur–Weyl functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N}$ then give an antiequivalence between this category and the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$.

This duality provides an unexpected tensor structure on the category $\hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$.

1. Introduction

1.1. The Karoubian rigid symmetric monoidal categories $\underline{\mathrm{Rep}}(S_\nu)$, $\nu \in \mathbb{C}$, were defined by P. Deligne [2007] as a polynomial family of categories interpolating the categories of finite-dimensional representations of the symmetric groups; namely, at points $n = \nu \in \mathbb{Z}_+$ the category $\underline{\mathrm{Rep}}(S_{\nu=n})$ allows an essentially surjective additive symmetric monoidal functor onto the standard category $\mathrm{Rep}(S_n)$. The categories $\underline{\mathrm{Rep}}(S_\nu)$ were subsequently studied by Deligne and others (e.g., J. Comes and V. Ostrik [2011; 2014]).

In [Entova Aizenbud 2015a], we gave an analogue to the classical Schur–Weyl duality in the setting of Deligne categories. To do that, we defined the "complex

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tensor power" of a finite-dimensional unital complex vector space (i.e., a vector space V with a distinguished nonzero vector $\mathbb{1}$). This complex tensor power of V, denoted by $V^{\underline{\otimes}\nu}$, is an Ind-object in the category $\underline{\mathrm{Rep}}(S_\nu)$, and comes with an action of $\mathfrak{gl}(V)$ on it; moreover, this Ind-object is a Harish-Chandra module for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$, where $\overline{\mathfrak{P}}_{\mathbb{1}} \subset \mathrm{GL}(V)$ is the mirabolic subgroup preserving the vector $\mathbb{1}$.

The "ν-th tensor power" of V is defined for any $\nu \in \mathbb{C}$; for $n = \nu \in \mathbb{Z}_+$, the functor $\underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ takes this Ind-object of $\underline{\mathrm{Rep}}(S_{\nu=n})$ to the usual tensor power $V^{\otimes n}$ in $\mathrm{Rep}(S_n)$. Moreover, the action of $\mathfrak{gl}(V)$ on the former object corresponds to the action of $\mathfrak{gl}(V)$ on $V^{\otimes n}$.

This let us define an additive contravariant functor, called the Schur–Weyl functor:

$$\mathrm{SW}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to O_V^{\mathfrak{p}}, \quad \mathrm{SW}_{\nu,V} := \mathrm{Hom}_{\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)}(\,\cdot\,, V^{\underline{\otimes}\nu}).$$

Here $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ is the abelian envelope of the category $\underline{\mathrm{Rep}}(S_\nu)$ (this envelope was described in [Comes and Ostrik 2014; Deligne 2007, Chapter 8]). The category $O_V^{\mathfrak{p}}$ is a version of the parabolic category O for $\mathfrak{gl}(V)$ associated with the pair $(V, \mathbb{1})$, which is defined as follows.

We define $O_V^{\mathfrak{p}}$ to be the category of Harish-Chandra modules for the pair $(\mathfrak{gl}(V), \overline{\mathfrak{P}}_{\mathbb{1}})$ on which the group $\mathrm{GL}(V/\mathbb{C}\mathbb{1})$ acts by polynomial maps, and which satisfy some additional finiteness conditions (similar to the ones in the definition of the usual BGG category O).

We now consider the localization of $O_V^{\mathfrak{p}}$ obtained by taking the full subcategory of $O_V^{\mathfrak{p}}$ consisting of modules of degree ν (i.e., modules on which $\mathrm{Id}_V \in \mathrm{End}(V)$ acts by the scalar ν), and localizing by the Serre subcategory of $\mathfrak{gl}(V)$-polynomial modules. This quotient is denoted by $\widehat{O}^{\mathfrak{p}}_{\nu,V}$. It turns out that for any unital finite-dimensional space $(V, \mathbb{1})$ and any $\nu \in \mathbb{C}$, the contravariant functor $\widehat{\mathrm{SW}}_{\nu,V}$ makes $\widehat{O}^{\mathfrak{p}}_{\nu,V}$ a Serre quotient of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$.

In this paper, we will consider the categories $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ for $N \in \mathbb{Z}_+$ and for $N = \infty$. Defining appropriate restriction functors

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

allows us to consider the inverse limit of the system $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 0}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 1})$. Inside this inverse limit we consider a full subcategory which is equivalent to $\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$; this subcategory is the "restricted inverse limit" of $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 0}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 1})$ and will be denoted by $\varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. This category has an intrinsic description, which we give in this paper (intuitively, this is the inverse limit among finite-length categories).

Similarly to [Entova Aizenbud 2015a], we define the complex tensor power of the unital vector space $(\mathbb{C}^\infty, \mathbb{1} := e_1)$, and the corresponding Schur–Weyl contravariant functor $\mathrm{SW}_{\nu,\mathbb{C}^\infty}$. As in the finite-dimensional case, this functor induces an exact contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$, and we have the following commutative diagram:

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[rr, "\widehat{\mathrm{SW}}_{\nu,\mathrm{lim}}"'] \arrow[drr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & & \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\
& & \hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u]
\end{tikzcd}$$

The contravariant functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ and $\widehat{\mathrm{SW}}_{\nu,\mathrm{lim}}$ turn out to be antiequivalences induced by the Schur–Weyl functors $\mathrm{SW}_{\nu,\mathbb{C}^n}$. The antiequivalences $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ and $\widehat{\mathrm{SW}}_{\nu,\mathrm{lim}}$ induce an unexpected rigid symmetric monoidal category structure on

$$\hat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\,\mathrm{restr}} \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

We obtain an interesting corollary: the duality in this category given by the tensor structure will coincide with the one arising from the usual notion of duality in BGG category O.

1.2. ***Notation.*** The base field throughout the paper will be $\mathbb{C}$. The notation and definitions used in this paper can be found in [Entova Aizenbud 2015a, Section 2]. In particular, lowercase Greek letters will denote Young diagrams, and $\ell(\lambda)$ will denote the number of rows in λ, while $|\lambda|$ will denote the number of boxes in λ.

We will use the definition of a finite-length abelian category given below.

Definition 1.2.1. Let $\mathcal{C}$ be an abelian category, and C be an object of $\mathcal{C}$. A *Jordan–Hölder filtration* for C is a finite sequence of subobjects of C

$$0 = C_0 \subset C_1 \subset \cdots \subset C_n = C$$

such that each subquotient C_{i+1}/C_i is simple.

The Jordan–Hölder filtration might not be unique, but the simple factors C_{i+1}/C_i are unique (up to reordering and isomorphisms). Consider the multiset of the simple factors: each simple factor is considered as an isomorphism class of simple objects, and its multiplicity is the multiplicity of its isomorphism class in the Jordan–Hölder filtration of C. This multiset is denoted by $\mathrm{JH}(C)$, and its elements are called the *Jordan–Hölder components* of C. The *length* of the object C, denoted by $\ell_{\mathcal{C}}(C)$, is defined to be the size of the finite multiset $\mathrm{JH}(C)$.

Definition 1.2.2. An abelian category $\mathcal{C}$ is called a *finite-length category* if every object admits a Jordan–Hölder filtration.

1.3. ***Structure of the paper.*** Sections 2 and 3 contain preliminaries on the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$, the categories of polynomial representations of $\mathfrak{gl}_N$ (where $N \in \mathbb{Z}_+ \cup \{\infty\}$) and the parabolic category O for $\mathfrak{gl}_N$. These sections are based on [Entova Aizenbud 2015a; 2015b; Sam and Snowden 2015].

In Section 4, we define the version of the parabolic category O for $\mathfrak{gl}_N$ which we will consider (including the case when $N = \infty$; see Section 4.2), and recall the necessary information about this category.

In Section 5, we give a description of the parabolic category O for $\mathfrak{gl}_\infty$ as a restricted inverse limit of the parabolic categories O for $\mathfrak{gl}_n$ as n tends to infinity.

In Sections 6 and 7, we recall the definition of the complex tensor power $(\mathbb{C}^N)^{\underline{\otimes}\nu}$, and define the functors $\mathrm{SW}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \to O^{\mathfrak{p}}_{\nu,V}$, $\widehat{\mathrm{SW}}_{\nu,V} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \to \widehat{O}^{\mathfrak{p}}_{\nu,V}$ for a unital vector space $(V, \mathbb{1})$ (finite- or infinite-dimensional). In Section 7.2, we recall the finite-dimensional case (studied in [Entova Aizenbud 2015a]).

Section 8 discusses the restricted inverse limit construction in the case of the classical Schur–Weyl duality, which motivates our construction for the Deligne categories. Sections 9 and 10 prove the main results of the paper. Section 11 discusses the relation between the rigidity (duality) in $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and the duality in the parabolic category O for $\mathfrak{gl}_\infty$.

2. Deligne category $\underline{\mathrm{Rep}}(S_\nu)$

A detailed description of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ and its abelian envelope can be found in [Comes and Ostrik 2011; 2014; Deligne 2007; Etingof 2014; Entova Aizenbud 2015a].

2.1. ***General description.*** For any $\nu \in \mathbb{C}$, the category $\underline{\mathrm{Rep}}(S_\nu)$ is generated, as a $\mathbb{C}$-linear Karoubian tensor category, by one object, denoted $\mathfrak{h}$. This object is the analogue of the permutation representation of S_n, and any object in $\underline{\mathrm{Rep}}(S_\nu)$ is a direct summand in a direct sum of tensor powers of $\mathfrak{h}$.

For $\nu \notin \mathbb{Z}_+$, $\underline{\mathrm{Rep}}(S_\nu)$ is a semisimple abelian category.

For $\nu \in \mathbb{Z}_+$, the category $\underline{\mathrm{Rep}}(S_\nu)$ has a tensor ideal $\mathfrak{I}_\nu$, called the ideal of negligible morphisms (this is the ideal of morphisms $f : X \to Y$ such that $\mathrm{tr}(fu) = 0$ for any morphism $u : Y \to X$). In that case, the classical category $\mathrm{Rep}(S_n)$ of finite-dimensional representations of the symmetric group for $n := \nu$ is equivalent to $\underline{\mathrm{Rep}}(S_{\nu=n})/\mathfrak{I}_\nu$ (equivalent as Karoubian rigid symmetric monoidal categories). The full, essentially surjective functor $\underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ defining this equivalence will be denoted by $\mathcal{S}_n$. Note that $\mathcal{S}_n$ sends $\mathfrak{h}$ to the permutation representation of S_n.

The indecomposable objects of $\underline{\mathrm{Rep}}(S_\nu)$, regardless of the value of ν, are parametrized (up to isomorphism) by all Young diagrams (of arbitrary size). We will denote the indecomposable object in $\underline{\mathrm{Rep}}(S_\nu)$ corresponding to the Young diagram τ by X_τ.

For $\nu =: n \in \mathbb{Z}_+$, the partitions λ for which X_λ has a nonzero image in the quotient $\underline{\mathrm{Rep}}(S_{\nu=n})/\mathfrak{I}_{\nu=n} \cong \mathrm{Rep}(S_n)$ are exactly the λ for which $\lambda_1 + |\lambda| \leq n$. If $\lambda_1 + |\lambda| \leq n$, then the image of λ in $\mathrm{Rep}(S_n)$ is the irreducible representation of S_n

corresponding to the Young diagram $\tilde{\lambda}(n)$: the Young diagram obtained by adding a row of length $n-|\lambda|$ on top of λ.

For each ν, we define an equivalence relation $\overset{\nu}{\sim}$ on the set of all Young diagrams: we say that $\lambda \overset{\nu}{\sim} \lambda'$ if the sequence $(\nu-|\lambda|, \lambda_1-1, \lambda_2-2, \dots)$ can be obtained from the sequence $(\nu-|\lambda'|, \lambda'_1-1, \lambda'_2-2, \dots)$ by permuting a finite number of entries. The equivalence classes thus obtained are in one-to-one correspondence with the blocks of the category $\underline{\mathrm{Rep}}(S_\nu)$ (see [Comes and Ostrik 2011]).

We say that a block is *trivial* if the corresponding equivalence class is trivial, i.e., has only one element (in that case, the block is a semisimple category).

The nontrivial equivalence classes (respectively, blocks) are parametrized by all Young diagrams of size ν; in particular, this happens only if $\nu \in \mathbb{Z}_+$. These classes are always of the form $\{\lambda^{(i)}\}_i$, with

$$\lambda^{(0)} \subset \lambda^{(1)} \subset \lambda^{(2)} \subset \cdots$$

(each $\lambda^{(i)}$ can be explicitly described based on the Young diagram of size ν corresponding to this class).

2.2. ***Abelian envelope.*** As was mentioned before, the category $\underline{\mathrm{Rep}}(S_\nu)$ is defined as a Karoubian category. For $\nu \notin \mathbb{Z}_+$, it is semisimple and thus abelian, but for $\nu \in \mathbb{Z}_+$, it is not abelian. Fortunately, it has been shown that $\underline{\mathrm{Rep}}(S_\nu)$ possesses an "abelian envelope", that is, it can be embedded (as a full monoidal subcategory) into an abelian rigid symmetric monoidal category, and this abelian envelope has a universal mapping property (see [Comes and Ostrik 2014, Theorem 1.2; Deligne 2007, Conjecture 8.21.2]). We will denote the abelian envelope of the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ by $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (with $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) := \underline{\mathrm{Rep}}(S_\nu)$ for $\nu \notin \mathbb{Z}_+$).

An explicit construction of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_{\nu=n})$ is given in [Comes and Ostrik 2014], and a detailed description of its structure can be found in [Entova Aizenbud 2015a]. It turns out that the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ is a highest weight category (with infinitely many weights) corresponding to the partially ordered set $(\{\text{Young diagrams}\}, \geq)$, where

$$\lambda \geq \mu \iff \lambda \overset{\nu}{\sim} \mu, \quad \lambda \subset \mu$$

(namely, in a nontrivial $\overset{\nu}{\sim}$-class, $\lambda^{(i)} \geq \lambda^{(j)}$ if $i \leq j$).

Thus the isomorphism classes of simple objects in $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ are parametrized by the set of Young diagrams of arbitrary sizes. We will denote the simple object corresponding to λ by $\boldsymbol{L}(\lambda)$.

We will also use the fact that blocks of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$, just like the blocks of $\underline{\mathrm{Rep}}(S_\nu)$, are parametrized by $\overset{\nu}{\sim}$-equivalence classes. For each $\overset{\nu}{\sim}$-equivalence class, the corresponding block of $\underline{\mathrm{Rep}}(S_\nu)$ is the full subcategory of tilting objects in the corresponding block of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ (see [Comes and Ostrik 2014, Proposition 2.9, Section 4]).

3. $\mathfrak{gl}_\infty$ and the restricted inverse limit of representations of $\mathfrak{gl}_n$

In this section, we discuss the category of polynomial representations of the Lie algebra $\mathfrak{gl}_\infty$ and its relation to the categories of polynomial representations of $\mathfrak{gl}_n$ for $n \geq 0$. The representations of the Lie algebra $\mathfrak{gl}_\infty$ are discussed in detail in [Penkov and Styrkas 2011; Dan-Cohen et al. 2016; Sam and Snowden 2015, Section 3]. Most of the constructions and the proofs of the statements appearing in this section can be found in [Entova Aizenbud 2015b, Section 7].

3.1. ***The Lie algebra*** $\mathfrak{gl}_\infty$**.** Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \dots$. Consider the Lie algebra $\mathfrak{gl}_\infty$ of infinite matrices $A = (a_{ij})_{i,j\geq 1}$ with finitely many nonzero entries. We have a natural action of $\mathfrak{gl}_\infty$ on $\mathbb{C}^\infty$ and on the restricted dual $\mathbb{C}^\infty_* = \mathrm{span}_{\mathbb{C}}(e_1^*, e_2^*, e_3^*, \dots)$ (here e_i^* is the linear functional dual to e_i: $e_i^*(e_j) = \delta_{ij}$).

Let $N \in \mathbb{Z}_+ \cup \{\infty\}$, and let $m \geq 1$. We will consider the Lie subalgebra $\mathfrak{gl}_m \subset \mathfrak{gl}_N$ which consists of matrices $A = (a_{ij})_{1\leq i,j\leq N}$ for which $a_{ij} = 0$ whenever $i > m$ or $j > m$. We will also denote by $\mathfrak{gl}_m^\perp$ the Lie subalgebra of $\mathfrak{gl}_N$ consisting of matrices $A = (a_{ij})_{1\leq i,j\leq N}$ for which $a_{ij} = 0$ whenever $i \leq m$ or $j \leq m$.

Remark 3.1.1. Note that $\mathfrak{gl}_m^\perp \cong \mathfrak{gl}_{N-m}$ for any $m \leq N$.

3.2. ***Categories of polynomial representations of*** $\mathfrak{gl}_N$**.** In this subsection, we take $N \in \mathbb{Z}_+ \cup \{\infty\}$. The notation $\mathbb{C}^N_*$ will stand for $(\mathbb{C}^N)^*$ whenever $N \in \mathbb{Z}_+$, and for $\mathbb{C}^\infty_*$ when $N = \infty$.

Consider the category $\mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ of polynomial representations of $\mathfrak{gl}_N$: this is the category of the representations of $\mathfrak{gl}_N$ which can be obtained as summands of a direct sum of tensor powers of the tautological representation $\mathbb{C}^N$ of $\mathfrak{gl}_N$.

It is easy to see that this is a semisimple abelian category, whose simple objects are parametrized (up to isomorphism) by all Young diagrams of arbitrary sizes whose length does not exceed N: the simple object corresponding to λ is $S^\lambda \mathbb{C}^N$.

Remark 3.2.1. Note that $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$ is the free abelian symmetric monoidal category generated by one object (see [Sam and Snowden 2015, Section 2.2.11]). It has an equivalent definition as the category of polynomial functors of bounded degree, which can be found in [Hong and Yacobi 2013; Sam and Snowden 2015].

Next, we define a natural $\mathbb{Z}_+$-grading on objects in $\text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ (cf. [Sam and Snowden 2015, Section 2.2.2]):

Definition 3.2.2. The objects in $\text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$ have a natural $\mathbb{Z}_+$-grading. Given $M \in \text{Ind-Rep}(\mathfrak{gl}_N)_{\mathrm{poly}}$, we consider the decomposition $M = \bigoplus_\lambda S^\lambda \mathbb{C}^N \otimes \mathrm{mult}_\lambda$ (here

mult_λ is the multiplicity space of $S^\lambda\mathbb{C}^N$ in M), and we define

$$\mathrm{gr}_k(M) := \bigoplus_{\lambda:|\lambda|=k} S^\lambda\mathbb{C}^N \otimes \mathrm{mult}_\lambda.$$

Of course, the morphisms in Ind-Rep$(\mathfrak{gl}_N)_{\mathrm{poly}}$ respect this grading.

3.3. ***Specialization and restriction functors.*** We now define specialization functors from the category of representations of $\mathfrak{gl}_\infty$ to the categories of representations of $\mathfrak{gl}_n$ (cf. [Sam and Snowden 2015, Section 3]):

Definition 3.3.1. We have

$$\Gamma_n : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}, \quad \Gamma_n := (\,\cdot\,)^{\mathfrak{gl}_n^\perp}.$$

One can easily check (cf. [Entova Aizenbud 2015b, Section 7]) that the functor Γ_n is well defined.

Lemma 3.3.2 [Penkov and Styrkas 2011; Sam and Snowden 2015, Section 3]. *The functors Γ_n are additive symmetric monoidal functors between semisimple symmetric monoidal categories. Their effect on the simple objects is described as follows: for any Young diagram λ, we have $\Gamma_n(S^\lambda\mathbb{C}^\infty) \cong S^\lambda\mathbb{C}^n$.*

Definition 3.3.3. Let $n \geq 1$. We define the functors

$$\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}_{n-1}^\perp}.$$

Again, one can easily show that these functors are well defined.

Remark 3.3.4. There is an alternative definition of the functors $\mathfrak{Res}_{n-1,n}$. One can think of the functor $\mathfrak{Res}_{n-1,n}$ acting on a $\mathfrak{gl}_n$-module M as taking the restriction of M to $\mathfrak{gl}_{n-1}$ and then considering only the vectors corresponding to "appropriate" central characters.

More specifically, we say that a $\mathfrak{gl}_n$-module M is of *degree d* if $\mathrm{Id}_{\mathbb{C}^n} \in \mathfrak{gl}_n$ acts by $d\,\mathrm{Id}_M$ on M. Also, given any $\mathfrak{gl}_n$-module M, we may consider the maximal submodule of M of degree d, and denote it by $\deg_d(M)$. This defines an endofunctor $\deg_d$ of Rep$(\mathfrak{gl}_n)_{\mathrm{poly}}$. Note that a simple module $S^\lambda\mathbb{C}^n$ is of degree $|\lambda|$.

The notion of degree gives a decomposition

$$\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \cong \bigoplus_{d\in\mathbb{Z}_+} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},d},$$

where Rep$(\mathfrak{gl}_n)_{\mathrm{poly},d}$ is the full subcategory of Rep$(\mathfrak{gl}_n)_{\mathrm{poly}}$ consisting of all polynomial $\mathfrak{gl}_n$-modules of degree d. Then

$$\mathfrak{Res}_{n-1,n} = \bigoplus_{d\in\mathbb{Z}_+} \mathfrak{Res}_{d,n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}},$$

with

$$\mathfrak{Res}_{d,n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},d} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly},d}, \quad \mathfrak{Res}_{d,n-1,n} := \deg_d \circ \mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}},$$

where $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ is the usual restriction functor for the pair $\mathfrak{gl}_{n-1} \subset \mathfrak{gl}_n$.

Once again, the functors $\mathfrak{Res}_{n-1,n}$ are additive functors between semisimple categories.

Lemma 3.3.5. $\mathfrak{Res}_{n-1,n}(S^\lambda\mathbb{C}^n) \cong S^\lambda\mathbb{C}^{n-1}$ *for any Young diagram* λ (*recall that* $S^\lambda\mathbb{C}^{n-1} = 0$ *if* $\ell(\lambda) > n-1$).

Moreover, these functors are compatible with the functors Γ_n defined before.

Lemma 3.3.6. *For any* $n \geq 1$, *we have a commutative diagram*:

$$\begin{tikzcd} \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \arrow[r, "\Gamma_n"] \arrow[dr, "\Gamma_{n-1}"'] & \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \arrow[d, "\mathfrak{Res}_{n-1,n}"] \\ & \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}} \end{tikzcd}$$

That is, *there is a natural isomorphism* $\Gamma_{n-1} \cong \mathfrak{Res}_{n-1,n} \circ \Gamma_n$.

Corollary 3.3.7. *The functors* $\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}$ *are symmetric monoidal functors.*

3.4. ***Restricted inverse limit of categories* $\mathbf{Rep}(\mathfrak{gl}_n)_{\mathbf{poly}}$.** This subsection gives a description of the category $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$ as a "restricted" inverse limit of categories $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ (see the Appendix and [Entova Aizenbud 2015b] for definitions and details).

We will use the framework developed in [Entova Aizenbud 2015b] for the inverse limits of categories with $\mathbb{Z}_+$-filtrations on objects, and the restricted inverse limits of finite-length categories (abelian categories in which every object admits a Jordan–Hölder filtration). The necessary definitions (such as $\mathbb{Z}_+$-filtered functors and shortening functors) can be found in the Appendix.

We define a $\mathbb{Z}_+$-filtration on the objects of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ for each $n \in \mathbb{Z}_+$:

Notation 3.4.1. For each $k \in \mathbb{Z}_+$, let $\mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})$ be the full additive subcategory of $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$ generated by $S^\lambda\mathbb{C}^n$ such that $\ell(\lambda) \leq k$.

Clearly the subcategories $\mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})$ give us a $\mathbb{Z}_+$-filtration on the objects of the category $\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$. Furthermore, by Lemma 3.3.5, the functors $\mathfrak{Res}_{n-1,n}$ are $\mathbb{Z}_+$-filtered functors, i.e., they induce functors

$$\mathfrak{Res}^k_{n-1,n} : \mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}) \to \mathrm{Fil}_k(\mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}}).$$

This allows us to consider the inverse limit

$$\varprojlim_{n\in\mathbb{Z}_+,\mathbb{Z}_+\text{-filtr}} \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}} \cong \varinjlim_{k\in\mathbb{Z}_+} \varprojlim_{n\in\mathbb{Z}_+} \operatorname{Fil}_k(\operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}).$$

This is an abelian category (with a natural $\mathbb{Z}_+$-filtration on objects).

By Lemma 3.3.5, the functors $\mathfrak{Res}_{n-1,n}$ are shortening functors; furthermore, the system $((\operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}})_{n\in\mathbb{Z}_+}, (\mathfrak{Res}_{n-1,n})_{n\geq 1})$ satisfies the conditions listed in Proposition A.5.1, and therefore the category $\varprojlim_{n\in\mathbb{Z}_+,\mathbb{Z}_+\text{-filtr}} \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}$ is also equivalent to the restricted inverse limit of this system, $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}$.

Remark 3.4.2. The functors $\mathfrak{Res}_{n-1,n}$ are symmetric monoidal functors, so the category $\varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}$ is a symmetric monoidal category.

The following proposition is relatively straightforward. Its detailed proof can be found in [Entova Aizenbud 2015b].

Proposition 3.4.3. *We have an equivalence of symmetric monoidal Karoubian categories*

$$\Gamma_{\lim} : \operatorname{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \to \varprojlim_{n\in\mathbb{Z}_+,\text{ restr}} \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}$$

induced by the symmetric monoidal functors

$$\Gamma_n = (\,\cdot\,)^{\mathfrak{gl}_n^\perp} : \operatorname{Rep}(\mathfrak{gl}_\infty)_{\text{poly}} \to \operatorname{Rep}(\mathfrak{gl}_n)_{\text{poly}}.$$

4. Parabolic category *O*

In this section, we describe a version of the parabolic category O for $\mathfrak{gl}_N$ which we are going to work with. We give a definition which describes the relevant category for both $\mathfrak{gl}_n$ and $\mathfrak{gl}_\infty$.

4.1. For the benefit of the reader, we will start by giving a definition for $\mathfrak{gl}_N$ when N is a positive integer; this definition is analogous to the usual definition of the category O. The generic definition will then be just a slight modification of the first to accommodate the case $N = \infty$. This version of the parabolic category O is attached to a pair: a vector space V and a fixed nonzero vector $\mathbb{1}$ in it. Such a pair is called a *unital vector space*. In our case, we will just consider $V = \mathbb{C}^N$, with the standard basis $e_1, e_2, \ldots$, and the chosen vector $\mathbb{1} := e_1$. Fix $N \in \mathbb{Z}$ with $N \geq 1$.

Notation 4.1.1. The following notation will be used throughout the paper:

- We denote by $\mathfrak{p}_N \subset \mathfrak{gl}_N$ the parabolic Lie subalgebra which consists of all the endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\phi(\mathbb{1}) \in \mathbb{C}\mathbb{1}$. In terms of matrices this is $\operatorname{span}\{E_{1,1}, E_{i,j} \mid j > 1\}$.
- $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{p}_N$ denotes the algebra of endomorphisms $\phi : \mathbb{C}^N \to \mathbb{C}^N$ for which $\operatorname{Im}\phi \subset \mathbb{C}\mathbb{1} \subset \operatorname{Ker}\phi$. In terms of matrices, $\mathfrak{u}^+_{\mathfrak{p}_N} = \operatorname{span}\{E_{1,j} \mid j > 1\}$.

Denote $U_N := \mathrm{span}\{e_2, e_3, \dots, e_N\}$. We have a splitting $\mathfrak{gl}_N \cong \mathfrak{p}_N \oplus \mathfrak{u}^-_{\mathfrak{p}_N}$, where $\mathfrak{u}^-_{\mathfrak{p}_N} \cong U_N = \mathrm{span}\{E_{i,1} \mid i > 1\}$). This gives us an analogue of the triangular decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}\,\mathrm{Id}_{\mathbb{C}^N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N} \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{gl}(U_N).$$

We can now give a precise definition of the parabolic category O we will use:

Definition 4.1.2. We define the category $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ to be the full subcategory of $\mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ whose objects M satisfy the following conditions:

- Viewed as a $\mathcal{U}(\mathfrak{gl}(U_N))$-module, M is a direct sum of polynomial $\mathcal{U}(\mathfrak{gl}(U_N))$-modules (that is, M belongs to $\text{Ind-Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}}$).
- M is locally finite over $\mathfrak{u}^+_{\mathfrak{p}_N}$.
- M is a finitely generated $\mathcal{U}(\mathfrak{gl}_N)$-module.

Remark 4.1.3. One can replace the requirement that $\mathfrak{u}^+_{\mathfrak{p}_N}$ act locally finitely on M by the requirement that $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ act locally nilpotently on M.

Remark 4.1.4. One can, in fact, give an equivalent definition of the category $O^{\mathfrak{p}}_V$ corresponding to a finite-dimensional unital vector $(V, \mathbb{1})$ without choosing a splitting (cf. [Entova Aizenbud 2015a, Section 5] and the Introduction).

Definition 4.1.5. A module M over the Lie algebra $\mathfrak{gl}_N$ will be said to be of degree $K \in \mathbb{C}$ if $\mathrm{Id}_{\mathbb{C}^N} \in \mathfrak{gl}_N$ acts by $K\,\mathrm{Id}_M$ on M.

We will denote by $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ the full subcategory of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ whose objects are modules of degree ν. To say a module M of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$ is of degree ν is the same as to require that $E_{1,1}$ acts on each subspace $S^\lambda U_N$ of M by the scalar $\nu - |\lambda|$.

Definition 4.1.6. Let $\nu \in \mathbb{C}$. Define the functor $\deg_\nu : \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ by letting $\deg_\nu(E)$ be the maximal submodule of E of degree ν (see Definition 4.1.5). For a morphism $f : E \to E'$ of $\mathfrak{gl}_N$-modules, we put $\deg_\nu(f) := f|_{\deg_\nu(E)}$.

Let $E \in \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$. The maximal submodule of E of degree ν is well defined: it is the subspace of E consisting of all vectors on which $\mathrm{Id}_{\mathbb{C}^N}$ acts by the scalar ν, and it is a $\mathfrak{gl}_N$-submodule since $\mathrm{Id}_{\mathbb{C}^N}$ lies in the center of $\mathfrak{gl}_N$.

One can show that the functor $\deg_\nu : \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ is left-exact. Moreover, it is easy to show that the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ is a direct summand of $O^{\mathfrak{p}_N}_{\mathbb{C}^N}$, and the functor $\deg_\nu : O^{\mathfrak{p}_N}_{\mathbb{C}^N} \to O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ is exact.

4.2. *Parabolic category O for $\mathfrak{gl}_N$.* We now give a definition of the parabolic category O which for $\mathfrak{gl}_N$. Again, we let $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$.

Consider a unital vector space $(\mathbb{C}^N, \mathbb{1})$, where $\mathbb{1} := e_1$. Put

$$U_N := \mathrm{span}_{\mathbb{C}}(e_2, e_3, \dots) \subset \mathbb{C}^N,$$

so that we have a splitting $\mathbb{C}^N = \mathbb{C}e_1 \oplus U_N$. We also denote $U_{N,*} := \operatorname{span}(e_2^*, e_3^*, \dots)$ (so $U_{N,*} = U_N^*$ whenever $N \in \mathbb{Z}$). We have a decomposition

$$\mathfrak{gl}_N \cong \mathfrak{gl}(U_N) \oplus \mathfrak{gl}_1 \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N}.$$

Of course, for any N, we have $\mathfrak{u}^-_{\mathfrak{p}_N} \cong U_N$; moreover, $\mathfrak{u}^+_{\mathfrak{p}_N} \cong U_{N,*}$. We will also use the isomorphisms $\mathfrak{gl}(U_N) \cong \mathfrak{gl}_1^\perp \cong \mathfrak{gl}_{N-1}$.

Definition 4.2.1.

- Define the category $\operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ to be the category of $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ lies in $\text{Ind-Rep}(\mathfrak{gl}_{U_N})_{\text{poly}}$; that is, $\mathfrak{gl}_N$-modules whose restriction to $\mathfrak{gl}(U_N)$ is a (perhaps infinite) direct sum of Schur functors applied to U_N. The morphisms would be $\mathfrak{gl}_N$-equivariant maps.
- We say that an object $M \in \operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ is of *degree* ν ($\nu \in \mathbb{C}$) if on every summand $S^\lambda U_N \subset M$, the element $E_{1,1} \in \mathfrak{gl}_N$ acts by $(\nu - |\lambda|)\operatorname{Id}_{S^\lambda U_N}$.
- Let $M \in \operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$. We have a commutative algebra $\operatorname{Sym}(U_N) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^-_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a $\operatorname{Sym}(U_N)$-module structure. We say that M is *finitely generated* over $\operatorname{Sym}(U_N)$ if M is a quotient of a "free finitely generated $\operatorname{Sym}(U_N)$-module"; that is, as a $\operatorname{Sym}(U_N)$-module, M is a quotient (in $\text{Ind-Rep}(\mathfrak{gl}_N)_{\text{poly}}$) of $\operatorname{Sym}(U_N) \otimes E$ for some $E \in \operatorname{Rep}(\mathfrak{gl}(U_N))_{\text{poly}}$.
- Let $M \in \operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$. We have a commutative algebra $\operatorname{Sym}(U_{N,*}) \cong \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ (the enveloping algebra of $\mathfrak{u}^+_{\mathfrak{p}_N} \subset \mathfrak{gl}_N$). The action of $\mathfrak{gl}_N$ on M gives M a $\operatorname{Sym}(U_{N,*})$-module structure. We say that M is *locally nilpotent* over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$ if for any $v \in M$, there exists $m \geq 0$ such that for any $A \in \operatorname{Sym}^m(U_{N,*})$ we have $A.v = 0$.

Recall the natural $\mathbb{Z}_+$-grading on the object of $\text{Ind-Rep}(\mathfrak{gl}_N)_{\text{poly}}$. For each $M \in \operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$, the above definition implies that $\mathfrak{gl}(U_N)$ acts by operators of degree zero, and that $U_{N,*}$ acts by operators of degree 1. We now define the parabolic category O for $\mathfrak{gl}_N$ which we will use throughout the paper:

Definition 4.2.2. We define the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ to be the full subcategory of $\operatorname{Mod}_{\mathfrak{gl}_N, \mathfrak{gl}(U_N)\text{-poly}}$ whose objects M satisfy the following requirements:

- M is of degree ν.
- M is finitely generated over $\operatorname{Sym}(U_N)$.
- M is locally nilpotent over the algebra $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_N})$.

Of course, for a positive integer N, this is just the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ we defined in the beginning of this section.

We will also consider the localization of the category $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by its Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν; such modules exist if and only if $\nu \in \mathbb{Z}_+$. This localization will be denoted by

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$$

and will play an important role when we consider the Schur–Weyl duality in complex rank.

4.3. ***Duality in category O.*** Let $n \in \mathbb{Z}_+$. Recall that in the category O for $\mathfrak{gl}_n$ we have the notion of a duality (cf. [Humphreys 2008, Section 3.2]): namely, given a $\mathfrak{gl}_n$-module M with finite-dimensional weight spaces, we can consider the twisted action of $\mathfrak{gl}_n$ on the dual space M^*, given by $A.f := f \circ A^T$, where A^T means the transpose of $A \in \mathfrak{gl}_n$. This makes M^* a $\mathfrak{gl}_n$-module. We then take $M^\vee$ to be the maximal submodule of M^* lying in category O.

More explicitly, considering M as a direct sum of its finite-dimensional weight spaces

$$M = \bigoplus_{\lambda} M_\lambda$$

we can consider the restricted twisted dual

$$M^\vee := \bigoplus_{\lambda} M^*_\lambda$$

(that is, we take the dual to each weight space separately). The action of $\mathfrak{gl}_n$ is given by $A.f := f \circ A^T$ for any $A \in \mathfrak{gl}_n$. The module $M^\vee$ is called the dual of M, and we get an exact functor $(\,\cdot\,)^\vee : O^{\mathrm{op}} \to O$.

Proposition 4.3.1. *The category $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ is closed under taking duals, and the duality functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_n}_{\mathbb{C}^n})^{\mathrm{op}} \to O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ is an equivalence of categories.*

In fact, a similar construction can be made for $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. All modules M in $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ are weight modules with respect to the subalgebra of diagonal matrices in $\mathfrak{gl}_\infty$, and the weight spaces are finite-dimensional (due to the polynomiality condition in the definition of $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$). This allows one to construct the restricted twisted dual $M^\vee$ in the same way as before, and obtain an exact functor

$$(\,\cdot\,)^\vee : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}.$$

Remark 4.3.2. It is obvious that for $n \in \mathbb{Z}_+$, the functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_n}_{\mathbb{C}^n})^{\mathrm{op}} \to O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ takes finite-dimensional (polynomial) modules to finite-dimensional (polynomial) modules. In fact, one can easily check that the functor $(\,\cdot\,)^\vee : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ takes polynomial modules to polynomial modules as well.

4.4. ***Structure of the category*** $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. In this subsection, we present some facts about the category $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ which will be used later on. The material of this section is discussed in more detail in [Entova Aizenbud 2015a, Section 5] and is mostly based on [Humphreys 2008, Chapter 9].

Fix $\nu \in \mathbb{C}$, and fix $n \in \mathbb{Z}_+$. We denote by $e_1, e_2, \ldots, e_n$ the standard basis of $\mathbb{C}^n$, and put $\mathbb{1} := e_1$ and $U_n := \mathrm{span}\{e_2, e_3, \ldots, e_n\}$. We will consider the category $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ for the unital vector space $(\mathbb{C}^n, \mathbb{1})$ and the splitting $\mathbb{C}^n = \mathbb{C}\mathbb{1} \oplus U_n$.

Proposition 4.4.1. *The categories* $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ *and* Ind-$O^{\mathfrak{p}_n}_{\mathbb{C}^n}$ *are closed under taking duals, direct sums, submodules, quotients and extensions in* $O_{\mathfrak{gl}_n}$, *as well as tensoring with finite-dimensional* $\mathfrak{gl}_n$*-modules.*

The category $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ decomposes into blocks (each of the blocks is an abelian category in its own right). To each $\overset{\nu}{\sim}$-class of Young diagrams corresponds a block of $O^{\mathfrak{p}}_{\nu,\mathbb{C}^n}$. If all Young diagrams λ in this $\overset{\nu}{\sim}$-class have length at least n, then the corresponding block is zero. To each nonzero block of $O^{\mathfrak{p}}_{\nu,\mathbb{C}^n}$ corresponds a unique $\overset{\nu}{\sim}$-class.

Moreover, the blocks corresponding to trivial $\overset{\nu}{\sim}$-classes are either semisimple (i.e., equivalent to the category $\mathrm{Vect}_{\mathbb{C}}$) or zero.

We now discuss standard objects in $O^{\mathfrak{p}_n}_{\mathbb{C}^n}$.

Definition 4.4.2. Let λ be a Young diagram. The generalized Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is defined to be the $\mathfrak{gl}_n$-module

$$\mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} S^\lambda U_n,$$

where $\mathfrak{gl}(U_n)$ acts naturally on $S^\lambda U_n$, $\mathrm{Id}_{\mathbb{C}^n} \in \mathfrak{p}_n$ acts on $S^\lambda U_n$ by scalar ν, and $\mathfrak{u}^+_{\mathfrak{p}_n}$ acts on $S^\lambda U_n$ by zero. Thus $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is the parabolic Verma module for $(\mathfrak{gl}_n, \mathfrak{p}_n)$ with highest weight $(\nu - |\lambda|, \lambda)$ if and only if $n - 1 \geq \ell(\lambda)$, and zero otherwise.

Definition 4.4.3. $L(\nu - |\lambda|, \lambda)$ is defined to be zero if $n \geq \ell(\lambda)$, or the simple module for $\mathfrak{gl}_n$ of highest weight $(\nu - |\lambda|, \lambda)$ otherwise.

The following basic lemma will be very helpful.

Lemma 4.4.4. *Let* λ *be a Young diagram such that* $\ell(\lambda) < n$. *We then have an isomorphism of* $\mathfrak{gl}(U_n)$*-modules*:

$$M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda) \cong \mathrm{Sym}(U_n) \otimes S^\lambda U_n.$$

We will also use the following lemma.

Lemma 4.4.5. *Let* $\{\lambda^{(i)}\}_i$ *be a nontrivial* $\overset{\nu}{\sim}$*-class, and* $i \geq 0$ *be such that* $\ell(\lambda^{(i)}) < n$. *Then there is a short exact sequence*

$$0 \to L(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}) \to M_{\mathfrak{p}_n}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to L(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0.$$

Corollary 4.4.6. *The isomorphism classes of the generalized Verma modules and the simple polynomial modules in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ form a basis for the Grothendieck group of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.*

5. Stable inverse limit of parabolic categories O

5.1. *Restriction functors.*

Definition 5.1.1. Let $n \geq 1$. Define the functor

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}^{\perp}_{n-1}}.$$

Again, the subalgebras $\mathfrak{gl}_{n-1}, \mathfrak{gl}^{\perp}_{n-1} \subset \mathfrak{gl}_n$ commute, and therefore the subspace of $\mathfrak{gl}^{\perp}_{n-1}$-invariants of a $\mathfrak{gl}_n$-module automatically carries an action of $\mathfrak{gl}_{n-1}$.

We need to check that this functor is well defined. In order to do so, consider the functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_{n-1})}$. This functor is well defined, and we will show that the objects in the image lie in the full subcategory $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ of $\mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_{n-1})}$.

The functor $\mathfrak{Res}_{n-1,n}$ can alternatively be defined as follows: for a module M in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, we restrict the action of $\mathfrak{gl}_n$ to $\mathfrak{gl}_{n-1}$, and then only take the vectors in M attached to specific central characters. More specifically, we have:

Lemma 5.1.2. *The functor $\mathfrak{Res}_{n-1,n}$ is naturally isomorphic to the composition* $\deg_\nu \circ \mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}$ (the functor $\deg_\nu$ was defined in Definition 4.1.6).

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. For any vector $m \in M$, we know that

$$\mathrm{Id}_{\mathbb{C}^n}.m = (E_{1,1} + E_{2,2} + \cdots + E_{n,n}).m = \nu m.$$

Then the requirement that

$$\mathrm{Id}_{\mathbb{C}^{n-1}}.m = (E_{1,1} + E_{2,2} + \cdots + E_{n-1,n-1}).m = \nu m$$

is equivalent to the requirement that $E_{n,n}.m = 0$, namely that $m \in M^{\mathfrak{gl}^{\perp}_{n-1}}$. □

We will now use this information to prove the following result:

Lemma 5.1.3. *The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is well defined.*

Proof. Let $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, and consider the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$. By definition, this is a module of degree ν. We will show that it lies in $O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$.

First of all, consider the inclusion $\mathfrak{gl}(U_{n-1})^{\perp} \oplus \mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$. This inclusion gives us the restriction functor (see Definition 3.3.3)

$$\mathfrak{Res}_{U_{n-1},U_n} : \mathrm{Rep}(\mathfrak{gl}(U_n))_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}(U_{n-1}))_{\mathrm{poly}}, \quad \mathfrak{Res}_{U_{n-1},U_n} := (\,\cdot\,)^{\mathfrak{gl}(U_{n-1})^{\perp}}.$$

The latter is an additive functor between semisimple categories, and takes polynomial representations of $\mathfrak{gl}(U_n)$ to polynomial representations of $\mathfrak{gl}(U_{n-1})$.

Now, the restriction to $\mathfrak{gl}(U_{n-1})$ of the $\mathfrak{gl}_{n-1}$-module $\mathfrak{Res}_{n-1,n}(M)$ is isomorphic to $\mathfrak{Res}_{U_{n-1},U_n}(M|_{\mathfrak{gl}(U_n)})$, and thus is a polynomial representation of $\mathfrak{gl}(U_{n-1})$.

Secondly, $\mathfrak{Res}_{n-1,n}(M)$ is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}})$, since M is locally nilpotent over $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$ and $\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_{n-1}}) \subset \mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_n})$.

It remains to check that given $M \in O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, the module $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\mathrm{Sym}(U_{n-1})$. Indeed, we know that there exists a polynomial $\mathfrak{gl}(U_n)$-module E and a surjective $\mathfrak{gl}(U_n)$-equivariant morphism of $\mathrm{Sym}(U_n)$-modules $\mathrm{Sym}(U_n) \otimes E \twoheadrightarrow M$. Taking the $\mathfrak{gl}(U_{n-1})^{\perp}$-invariants and using Corollary 3.3.7, we conclude that there is a surjective $\mathfrak{gl}(U_{n-1})$-equivariant morphism of $\mathrm{Sym}(U_{n-1})$-modules

$$\mathrm{Sym}(U_{n-1}) \otimes E^{\mathfrak{gl}(U_{n-1})^{\perp}} \twoheadrightarrow \mathfrak{Res}_{n-1,n}(M).$$

Thus $\mathfrak{Res}_{n-1,n}(M)$ is finitely generated over $\mathrm{Sym}(U_{n-1})$. □

Lemma 5.1.4. *The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is exact.*

Proof. We use Lemma 5.1.2. The functor $\deg_\nu : O^{\mathfrak{p}_{n-1}}_{\mathbb{C}^{n-1}} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ is exact, so the functor $\mathrm{Res}_{n-1,n}$ is obviously exact as well. □

Lemma 5.1.5. *The functor $\mathfrak{Res}_{n-1,n}$ takes parabolic Verma modules either to parabolic Verma modules or to zero*:

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda).$$

(*Recall that the latter is a parabolic Verma module for $\mathfrak{gl}_{n-1}$ if and only if $\ell(\lambda) \le n-2$, and zero otherwise*).

Proof. Consider the parabolic Verma module $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$, where the Young diagram λ has length at most $n-1$. By definition, we have

$$M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda) = \mathcal{U}(\mathfrak{gl}_n) \otimes_{\mathcal{U}(\mathfrak{p}_n)} S^\lambda U_n.$$

The branching rule for $\mathfrak{gl}(U_{n-1}) \subset \mathfrak{gl}(U_n)$ tells us that

$$(S^\lambda U_n)|_{\mathfrak{gl}(U_{n-1})} \cong \bigoplus_{\lambda'} S^{\lambda'} U_{n-1},$$

where the sum is taken over the set of all Young diagrams obtained from λ by removing several boxes, no two in the same column. So

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)) \cong \left(\bigoplus_{\lambda' \subset \lambda} M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda')\right) \otimes \mathcal{U}\left(\mathfrak{u}^-_{\mathfrak{p}_n}/\mathfrak{u}^-_{\mathfrak{p}_{n-1}}\right).$$

Here:

- $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda')$ is either a parabolic Verma module for $\mathfrak{gl}_{n-1}$ of highest weight $(\nu - |\lambda|, \lambda')$ (note that it is of degree $\nu - |\lambda| + |\lambda'|$) or zero.
- $\mathfrak{gl}(U_{n-1})$ acts trivially on the space $\mathcal{U}\left(\mathfrak{u}^-_{\mathfrak{p}_n}/\mathfrak{u}^-_{\mathfrak{p}_{n-1}}\right)$. This space is isomorphic, as a $\mathbb{Z}_+$-graded vector space, to $\mathbb{C}[t]$ (ν standing for $E_{n,1} \in \mathfrak{gl}_n$) and $E_{1,1}$ acts on it by derivations $-t\frac{d}{dt}$.

Thus $\mathrm{Id}_{\mathbb{C}^{n-1}} \in \mathfrak{gl}_n$ acts on the subspace $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda') \otimes t^k \subset M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ by the scalar $\nu - |\lambda| + |\lambda'| - k$.

We now apply the functor $\deg_\nu$ to the module $\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda))$. To see which subspaces $M_{\mathfrak{p}_{n-1}}(\nu - |\lambda'|, \lambda') \otimes t^k$ of $M_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ will survive after applying $\deg_\nu$, we require that $|\lambda| - |\lambda'| + k = 0$. But we are only considering Young diagrams λ' such that $\lambda' \subset \lambda$, and $k \in \mathbb{Z}_+$, which means that the only relevant case is $\lambda' = \lambda,\ k = 0$. We conclude that

$$\mathfrak{Res}_{n-1,n}(M_{\mathfrak{p}_N}(\nu - |\lambda|, \lambda)) \cong M_{\mathfrak{p}_{n-1}}(\nu - |\lambda|, \lambda). \qquad \square$$

Lemma 5.1.6. *Given a simple $\mathfrak{gl}_n$-module $L_n(\nu - |\lambda|, \lambda)$,*

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda).$$

(*Recall that the latter is a simple $\mathfrak{gl}_{n-1}$-module if and only if $\ell(\lambda) \le n-2$, and zero otherwise*).

Proof. The statement follows immediately from Lemma 5.1.5 when λ lies in a trivial $\overset{\nu}{\sim}$-class; for a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_i$, we have short exact sequences (see Lemma 4.4.5):

$$0 \to L_n(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}) \to M_{\mathfrak{p}_n}(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to L_n(\nu - |\lambda^{(i)}|, \lambda^{(i)}) \to 0.$$

Using the exactness of $\mathfrak{Res}_{n-1,n}$, we can prove the required statement for $L_n(\nu - |\lambda^{(i)}|, \lambda^{(i)})$ by induction on i, provided the statement is true for $i = 0$. So it remains to check that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$$

for the minimal Young diagram λ in any nontrivial $\overset{\nu}{\sim}$-class. Recall that in that case, $L_n(\nu - |\lambda|, \lambda) = S^{\tilde{\lambda}(\nu)}\mathbb{C}^n$ is a finite-dimensional simple representation of $\mathfrak{gl}_n$. The branching rule for $\mathfrak{gl}_n, \mathfrak{gl}_{n-1}$ implies that

$$\mathrm{Res}^{\mathfrak{gl}_n}_{\mathfrak{gl}_{n-1}}(S^{\tilde{\lambda}(\nu)}\mathbb{C}^n) \cong \bigoplus_\mu S^\mu \mathbb{C}^{n-1},$$

where the sum is taken over the set of all Young diagrams obtained from $\tilde{\lambda}(\nu)$ by removing several boxes, no two in the same column. Considering only the summands of degree ν, we see that

$$\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong S^{\tilde{\lambda}(\nu)}\mathbb{C}^{n-1} \cong L_{n-1}(\nu - |\lambda|, \lambda). \qquad \square$$

The functor $\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$ clearly takes polynomial modules to polynomial modules; together with Lemma 5.1.4, this means that $\mathfrak{Res}_{n-1,n}$ factors through an exact functor

$$\widehat{\mathfrak{Res}}_{n-1,n} : \hat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \hat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}},$$

i.e., we have a commutative diagram

$$\begin{tikzcd} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\mathfrak{Res}_{n-1,n}"] \arrow[d, "\hat{\pi}_n"'] & O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \arrow[d, "\hat{\pi}_{n-1}"] \\ \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\widehat{\mathfrak{Res}}_{n-1,n}"] & \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \end{tikzcd}$$

(see Section 4.2 for the definition of the localizations $\hat{\pi}_n$).

5.2. *Specialization functors.*

Definition 5.2.1. Let $n \geq 1$. Define the functor

$$\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \quad \Gamma_n := (\,\cdot\,)^{\mathfrak{gl}_n^\perp}.$$

As before, the subalgebras $\mathfrak{gl}_n, \mathfrak{gl}_n^\perp \subset \mathfrak{gl}_\infty$ commute, and therefore the subspace of $\mathfrak{gl}_n^\perp$-invariants of a $\mathfrak{gl}_\infty$-module automatically carries an action of $\mathfrak{gl}_n$.

Lemma 5.2.2. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is well defined.*

Proof. The proof is essentially the same as that in Lemma 5.1.3. □

Lemma 5.2.3. *The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is exact.*

Proof. The definition of Γ_n immediately implies that this functor is left-exact. Consider the inclusion $\mathfrak{gl}(U_n) \oplus \mathfrak{gl}(U_n)^\perp \subset \mathfrak{gl}(U_\infty)$. We then have an isomorphism of $\mathfrak{gl}(U_n)$-modules

$$(M|_{\mathfrak{gl}(U_\infty)})^{\mathfrak{gl}(U_n)^\perp} \cong (M^{\mathfrak{gl}_n^\perp})|_{\mathfrak{gl}(U_n)}.$$

The exactness of Γ_n then follows from the additivity of the functor

$$(\,\cdot\,)^{\mathfrak{gl}(U_n)^\perp} : \mathrm{Rep}(\mathfrak{gl}(U_\infty))_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}(U_n))_{\mathrm{poly}},$$

which is an additive functor between semisimple categories. □

The functor $\Gamma_n : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ clearly takes polynomial $\mathfrak{gl}_\infty$-modules to polynomial $\mathfrak{gl}_n$-modules; together with Lemma 5.2.3, this means that Γ_n factors through an exact functor

$$\widehat{\Gamma}_n : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n},$$

i.e., we have a commutative diagram

$$\begin{tikzcd} O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\Gamma_n"] \arrow[d, "\hat{\pi}_\infty"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[d, "\hat{\pi}_n"] \\ \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\widehat{\Gamma}_n"] & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \end{tikzcd}$$

5.3. ***Stable inverse limit of categories*** $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ ***and the category*** $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$**.** The restriction functors

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}, \quad \mathfrak{Res}_{n-1,n} := (\,\cdot\,)^{\mathfrak{gl}^{\perp}_{n-1}}$$

described in Section 5.1 allow us to consider the inverse limit of the system $((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2})$, and similarly for $((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2})$. Let $n \geq 1$.

Notation 5.3.1. For each $k \in \mathbb{Z}_+$, let $\mathrm{Fil}_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ (resp., $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$) be the Serre subcategory of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (resp., $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$) generated by simple modules $L_n(\nu - |\lambda|, \lambda)$ (respectively, $\hat{\pi}_n(L_n(\nu - |\lambda|, \lambda))$), with $\ell(\lambda) \leq k$.

This defines $\mathbb{Z}_+$-filtrations on the objects of $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ and $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$, i.e.,

$$O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k\in\mathbb{Z}_+} \mathrm{Fil}_k(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}), \quad \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \cong \varinjlim_{k\in\mathbb{Z}_+} \mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}).$$

Lemma 5.3.2. *The functors*

$$\mathfrak{Res}_{n-1,n} : O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

and

$$\widehat{\mathfrak{Res}}_{n-1,n} : \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}$$

are both shortening and $\mathbb{Z}_+$-filtered functors between finite-length categories with $\mathbb{Z}_+$-filtrations on objects (see the Appendix for the relevant definitions). *Moreover, the systems $(O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \mathfrak{Res}_{n-1,n})$ and $(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \widehat{\mathfrak{Res}}_{n-1,n})$ satisfy the conditions appearing in Section A.5, and thus for each of these, their restricted inverse limit coincides with their inverse limit as $\mathbb{Z}_+$-graded categories.*

Proof. These statements follow directly from Lemma 5.1.6, which tells us that $\mathfrak{Res}_{n-1,n}(L_n(\nu - |\lambda|, \lambda)) \cong L_{n-1}(\nu - |\lambda|, \lambda)$, and the fact that $L_n(\nu - |\lambda|, \lambda) = 0$ whenever $\ell(\lambda) > n - 1$. □

We can now consider the inverse limits of the $\mathbb{Z}_+$-filtered systems

$$((O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\mathfrak{Res}_{n-1,n})_{n\geq 2}), \quad ((\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})_{n\geq 1}, (\widehat{\mathfrak{Res}}_{n-1,n})_{n\geq 2}).$$

By Proposition A.5.1, these limits are equivalent to the respective restricted inverse limits

$$\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}, \quad \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

The functors Γ_n described above induce exact functors

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{\Gamma}_{\lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proposition 5.3.3. *The functors Γ_n induce an equivalence*

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. First of all, we need to check that this functor is well defined. Namely, we need to show that for any $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$, the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ stabilizes. In fact, it is enough to show that this sequence is bounded (since it is obviously increasing).

Recall that we have a surjective map of $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$-modules $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_\infty}) \otimes E \twoheadrightarrow M$ for some $E \in \mathrm{Rep}(\mathfrak{gl}(U_\infty))_{\mathrm{poly}}$. Since Γ_{n+1} is exact, it gives us a surjective map $\mathrm{Sym}(\mathfrak{u}^-_{\mathfrak{p}_{n+1}}) \otimes \Gamma_{n+1}(E) \twoheadrightarrow \Gamma_{n+1}(M)$ for any $n \geq 0$, with $\Gamma_{n+1}(E)$ being a polynomial $\mathfrak{gl}(U_{n+1})$-module.

Now,

$$\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_{n+1}})}(\Gamma_{n+1}(M)) \leq \ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E)).$$

The sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}(U_{n+1}))}(\Gamma_{n+1}(E))\}_{n\geq 0}$ is bounded by Proposition 3.4.3, and thus the sequence $\{\ell_{\mathcal{U}(\mathfrak{gl}_{n+1})}(\Gamma_{n+1}(M))\}_n$ is bounded as well.

We now show that $\Gamma_{\lim}$ is an equivalence. A construction similar to the one appearing in [Entova Aizenbud 2015b, Section 7.5] gives a left adjoint to the functor $\Gamma_{\lim}$; that is, we will define a functor

$$\Gamma^*_{\lim} : \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}.$$

Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The isomorphisms $\phi_{n-1,n} : \mathfrak{Res}_{n-1,n}(M_n) \xrightarrow{\sim} M_{n-1}$ define $\mathfrak{gl}_{n-1}$-equivariant inclusions $M_{n-1} \hookrightarrow M_n$. Consider the vector space

$$M := \bigcup_{n\geq 1} M_n,$$

which has a natural action of $\mathfrak{gl}_\infty$ on it. It is easy to see that the obtained $\mathfrak{gl}_\infty$-module M is a direct sum of polynomial $\mathfrak{gl}(U_\infty)$-modules, and is locally nilpotent over the algebra

$$\mathcal{U}(\mathfrak{u}^+_{\mathfrak{p}_\infty}) \cong \mathrm{Sym}(U_{\infty,*}) \cong \bigcup_{n\geq 1} \mathrm{Sym}(U^*_n).$$

Sublemma 5.3.4. *Let $((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. Then $M := \bigcup_{n\geq 1} M_n$ is a finitely generated module over*

$$\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty}).$$

Proof. In Proposition A.2.2, we show that all the objects in the abelian category $\varprojlim_{n\geq 1,\,\mathrm{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ have finite length, and that the simple objects in this category are exactly those of the form $((L_n(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ for a fixed Young diagram λ. So we only need to check that applying the above construction to these simple objects gives rise to finitely generated modules over $\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

Using Corollary 4.4.6 we now reduce the proof of the sublemma to proving the following two statements:

- Let λ be a fixed Young diagram and $((L_n(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be a simple object in $\varprojlim_{n\geq 1,\,\mathrm{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ in which $L_n(\nu-|\lambda|,\lambda)$ is polynomial for every n (i.e., λ is minimal in its nontrivial $\overset{\nu}{\sim}$-class). Then $L := \bigcup_{n\geq 1} L_n(\nu-|\lambda|,\lambda)$ is a polynomial $\mathfrak{gl}_\infty$-module (in particular, a finitely generated module over $\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$).
- Let λ be a fixed Young diagram and let $((M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda))_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2})$ be an object of $\varprojlim_{n\geq 1,\,\mathrm{restr}} \mathcal{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ (this is a sequence of "compatible" parabolic Verma modules). Then
$$M := \bigcup_n M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda)$$
is a finitely generated module over $\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$.

The first statement follows immediately from Proposition 3.4.3. To prove the second statement, recall from Lemma 4.4.4 that
$$M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda) \cong \mathrm{Sym}(U_n) \otimes S^\lambda U_n.$$
So
$$M := \bigcup_n M_{\mathfrak{p}_n}(\nu-|\lambda|,\lambda) \cong \bigcup_n \mathrm{Sym}(U_n) \otimes S^\lambda U_n \cong \mathrm{Sym}(U_\infty) \otimes S^\lambda U_\infty,$$
which is clearly a finitely generated module over $\mathrm{Sym}(U_\infty) \cong \mathcal{U}(\mathfrak{u}^-_{\mathfrak{p}_\infty})$. □

This allows us to define the functor $\Gamma^*_{\lim}$ by setting
$$\Gamma^*_{\lim}((M_n)_{n\geq 1}, (\phi_{n-1,n})_{n\geq 2}) := \bigcup_{n\geq 1} M_n$$
and requiring that it act on morphisms accordingly. The definition of $\Gamma^*_{\lim}$ gives us a natural transformation
$$\Gamma^*_{\lim} \circ \Gamma_{\lim} \xrightarrow{\sim} \mathrm{Id}_{\mathcal{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}}.$$
Restricting the action of $\mathfrak{gl}_\infty$ to $\mathfrak{gl}(U_\infty)$ and using Proposition 3.4.3, we conclude that this natural transformation is an isomorphism.

Notice that the definition of $\Gamma^*_{\lim}$ implies that this functor is faithful. Thus we conclude that the functor $\Gamma^*_{\lim}$ is an equivalence of categories, and so is $\Gamma_{\lim}$. □

Proposition 5.3.5. *The functors $\widehat{\Gamma}_n$ induce an equivalence*

$$\widehat{\Gamma}_{\lim} : \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. Let $M \in O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$. First of all, we need to check that the functor $\widehat{\Gamma}_{\lim}$ is well defined; that is, we need to show that the sequence $\{\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M)))\}_{n\geq 1}$ is bounded from above.

Indeed,

$$\ell_{\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\hat{\pi}_n(\Gamma_n(M))) \leq \ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M)).$$

But the sequence $\{\ell_{O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}}(\Gamma_n(M))\}_{n\geq 1}$ is bounded from above by Proposition 5.3.3, so the original sequence is bound from above as well.

Thus we obtain a commutative diagram

$$\begin{tikzcd}
\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu} \arrow[r] \arrow[d, "\Gamma_{\lim}"'] & O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[r, "\hat{\pi}_\infty"] \arrow[d, "\Gamma_{\lim}"'] & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[d, "\widehat{\Gamma}_{\lim}"'] \\
\varprojlim_{n\geq 1,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},\nu} \arrow[r] & \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\hat{\pi}_{\lim}=\varprojlim_n \hat{\pi}_n"] & \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}
\end{tikzcd}$$

where $\mathrm{Rep}(\mathfrak{gl}_N)_{\mathrm{poly},\nu}$ is the Serre subcategory of $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ consisting of all polynomial modules of degree ν. The rows of this commutative diagram are “exact” (in the sense that $\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ is the Serre quotient of the category $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ by the Serre subcategory $\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu}$, and similarly for the bottom row).

The functors

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly},\nu} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly},\nu}$$

and

$$\Gamma_{\lim} : O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

are equivalences of categories (by Propositions 3.4.3 and 5.3.3), and thus the functor $\widehat{\Gamma}_{\lim}$ is an equivalence as well. □

6. Complex tensor powers of a unital vector space

In this section we describe the construction of a complex tensor power of the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$ (again, $N \in \mathbb{Z}_+ \cup \{\infty\}$). A general construction of the complex tensor power of a unital vector space is given in [Entova Aizenbud 2015a, Section 6].

Again, we denote $U_N := \mathrm{span}\{e_2, e_3, \dots\}$, and $U_{N*} := \mathrm{span}\{e_2^*, e_3^*, \dots\} \subset \mathbb{C}^N_*$. As before, we have a decomposition:

$$\mathfrak{gl}_N \cong \mathbb{C}\,\mathrm{Id}_{\mathbb{C}^N} \oplus \mathfrak{u}^-_{\mathfrak{p}_N} \oplus \mathfrak{u}^+_{\mathfrak{p}_N} \oplus \mathfrak{gl}(U_N)$$

such that $U_N \cong \mathfrak{u}^-_{\mathfrak{p}_N}$, $U_{N*} \cong \mathfrak{u}^+_{\mathfrak{p}_N}$, and if N is finite, we have $U_N^* \cong U_{N*}$.

Fix $\nu \in \mathbb{C}$. Recall from [Entova Aizenbud 2015a, Section 4] that for any $\nu \in \mathbb{C}$, in the Deligne category $\underline{\mathrm{Rep}}(S_\nu)$ we have the objects Δ_k ($k \in \mathbb{Z}_+$). These objects interpolate the representations $\mathbb{C}\,\mathrm{Inj}(\{1, \dots, k\}, \{1, \dots, n\}) \cong \mathrm{Ind}^{S_n \times S_k}_{S_{n-k} \times S_k \times S_k} \mathbb{C}$ of the symmetric groups S_n; in fact, for any $n \in \mathbb{Z}_+$ we have

$$\mathcal{S}_n(\Delta_k) \cong \mathbb{C}\,\mathrm{Inj}(\{1, \dots, k\}, \{1, \dots, n\}),$$

where $\mathcal{S}_n : \underline{\mathrm{Rep}}(S_{\nu=n}) \to \mathrm{Rep}(S_n)$ is the monoidal functor discussed in Section 2.1.

Definition 6.0.1 (complex tensor power). Define the object $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ of the category $\text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\big)$ by setting

$$(\mathbb{C}^N)^{\underline{\otimes}\nu} := \bigoplus_{k \geq 0} (U_N^{\otimes k} \otimes \Delta_k)^{S_k}.$$

The action of $\mathfrak{gl}_N$ on $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ is given as follows:

$$\begin{tikzcd}
\mathbb{1} \arrow[r, bend left, "U_N"] & U_N \otimes \Delta_1 \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & (U_N^{\otimes 2} \otimes \Delta_2)^{S_2} \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & (U_N^{\otimes 3} \otimes \Delta_3)^{S_3} \arrow[l, bend left, "U_{N*}"] \arrow[r, bend left, "U_N"] \arrow[loop below, "\mathfrak{gl}(U_N)"] & \cdots \arrow[l, bend left, "U_{N*}"]
\end{tikzcd}$$

- $E_{1,1} \in \mathfrak{gl}_N$ acts by scalar $\nu - k$ on each summand $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$.
- $A \in \mathfrak{gl}(U_N) \subset \mathfrak{gl}_N$ acts on $(U_N^{\otimes k} \otimes \Delta_k)^{S_k}$ by
$$\sum_{1 \leq i \leq k} A^{(i)}|_{U_N^{\otimes k}} \otimes \mathrm{Id}_{\Delta_k} : (U_N^{\otimes k} \otimes \Delta_k)^{S_k} \to (U_N^{\otimes k} \otimes \Delta_k)^{S_k}.$$
- $u \in U_N \cong \mathfrak{u}^-_{\mathfrak{p}_N}$ acts by morphisms of degree 1, which are given explicitly in [Entova Aizenbud 2015a, Section 6.2].
- $f \in U_{N*} \cong \mathfrak{u}^+_{\mathfrak{p}_N}$ acts by morphisms of degree -1, which are given explicitly in [Entova Aizenbud 2015a, Section 6.2].

Remark 6.0.2. The actions of the elements of $\mathfrak{u}^+_{\mathfrak{p}_N}$ and $\mathfrak{u}^-_{\mathfrak{p}_N}$, though not written here explicitly, are in fact uniquely determined by the actions of $E_{1,1}$ and $\mathfrak{gl}(U_N)$.

To see this, note that the ideal in the Lie algebra $\mathfrak{gl}_N$ generated by the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$ is the entire $\mathfrak{gl}_N$. Given two $\mathfrak{gl}_N$-modules M_1, M_2 and an isomorphism $M_1 \to M_2$ which is equivariant with respect to the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$, the above fact implies that this isomorphism is also $\mathfrak{gl}_N$-equivariant.

In other words, if there exists a way to define an action of $\mathfrak{gl}_N$ whose restriction to the Lie subalgebra $\mathbb{C}E_{1,1} \oplus \mathfrak{gl}(U_N)$ is given by the formulas above, then such an action of $\mathfrak{gl}_N$ is unique.

Remark 6.0.3. The proof that the object $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ lies in Ind-$\big(\underline{\mathrm{Rep}}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\big)$ is an easy check, and can be found in [Entova Aizenbud 2015a]. In particular, it means that the action of the mirabolic subalgebra Lie $\overline{\mathfrak{P}}_{\mathbb{1}}$ on the complex tensor power $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ integrates to an action of the mirabolic subgroup $\overline{\mathfrak{P}}_{\mathbb{1}}$, thus making $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ a Harish-Chandra module in Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ for the pair $(\mathfrak{gl}_N, \overline{\mathfrak{P}}_{\mathbb{1}})$.

The definition of the complex tensor power is compatible with the usual notion of a tensor power of a unital vector space (see [Entova Aizenbud 2015a, Section 6]):

Proposition 6.0.4. *Let $d \in \mathbb{Z}_+$. Consider the functor*

$$\hat{\mathcal{S}}_d : \text{Ind-}\big(\underline{\mathrm{Rep}}(S_{\nu=d}) \boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N}\big) \to \text{Ind-}\big(\mathrm{Rep}(S_d) \boxtimes O^{\mathfrak{p}_N}_{d,\mathbb{C}^N}\big)$$

induced by the functor

$$\mathcal{S}_d : \underline{\mathrm{Rep}}(S_{\nu=d}) \to \mathrm{Rep}(S_n)$$

described in Section 2.1. Then $\hat{\mathcal{S}}_d((\mathbb{C}^N)^{\underline{\otimes}d}) \cong (\mathbb{C}^N)^{\otimes d}$.

The construction of the complex tensor power is also compatible with the functors $\mathfrak{Res}_{n,n+1}$ and Γ_n defined in Definitions 5.1.1 and 5.2.1. These properties can be seen as special cases of the following statement (when $N = n + 1$ and $N = \infty$, respectively):

Proposition 6.0.5. *Let $n \geq 1$, and let $N \geq n$, $N \in \mathbb{Z}_{\geq 1} \cup \{\infty\}$. Recall that we have an inclusion $\mathfrak{gl}_n \oplus \mathfrak{gl}_n^{\perp} \subset \mathfrak{gl}_N$, and consider the functor*

$$(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : \text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \boxtimes O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}\big) \to \text{Ind-}\big(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \boxtimes O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\big)$$

induced by the functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}}$ then takes $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ to $(\mathbb{C}^n)^{\underline{\otimes}\nu}$.

Proof. The functor $(\,\cdot\,)^{\mathfrak{gl}_n^{\perp}} : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ induces an endofunctor of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$. We would like to say that we have an isomorphism of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$-objects

$$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \overset{?}{\cong} (\mathbb{C}^n)^{\underline{\otimes}\nu}$$

and that the action of $\mathfrak{gl}_n \subset \mathfrak{gl}_N$ on $((\mathbb{C}^N)^{\underline{\otimes}\nu})$ corresponds to the action of $\mathfrak{gl}_n$ on $(\mathbb{C}^n)^{\underline{\otimes}\nu}$. In order to do this, we first consider $(\mathbb{C}^N)^{\underline{\otimes}\nu}$ as an object in Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ with an action of $\mathfrak{gl}(U_N)$:

$$(\mathbb{C}^N)^{\underline{\otimes}\nu} \cong \bigoplus_{k\geq 0} (\Delta_k \otimes U_N^{\otimes k})^{S_k}.$$

If we consider only the actions of $\mathfrak{gl}(U_N)$, $\mathfrak{gl}(U_n)$, the functor Γ_n is induced by the additive monoidal functor $(\,\cdot\,)^{\mathfrak{gl}(U_n)^{\perp}} : \text{Ind-Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}} \to \text{Ind-Rep}(\mathfrak{gl}(U_N))_{\mathrm{poly}}$. This shows that we have an isomorphism of Ind-$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$-objects

$$((\mathbb{C}^N)^{\underline{\otimes}\nu})^{\mathfrak{gl}_n^{\perp}} \cong \bigoplus_{k\geq 0} (\Delta_k \otimes U_n^{\otimes k})^{S_k} \cong (\mathbb{C}^n)^{\underline{\otimes}\nu}$$

and the actions of $\mathfrak{gl}(U_n)$ on both sides are compatible. From the definition of the complex tensor power (Definition 6.0.1) one immediately sees that the actions of $E_{1,1}$ on both sides are compatible as well. Remark 6.0.2 now completes the proof. $\square$

7. Schur–Weyl duality in complex rank: the Schur–Weyl functor and the finite-dimensional case

We fix $\nu \in \mathbb{C}$ and $N \in \mathbb{Z}_+ \cup \{\infty\}$. Again, we consider the unital vector space $\mathbb{C}^N$ with the chosen vector $\mathbb{1} := e_1$ and the complement $U_N := \operatorname{span}\{e_2, e_3, \dots\}$.

7.1. *Schur–Weyl functor.*

Definition 7.1.1. Define the Schur–Weyl contravariant functor

$$\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$$

by

$$\mathrm{SW}_\nu := \mathrm{Hom}_{\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)}(\,\cdot\,, (\mathbb{C}^N)^{\underline{\otimes}\nu}).$$

Remark 7.1.2. The functor $\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ is a contravariant $\mathbb{C}$-linear additive left-exact functor.

It turns out that the image of the functor $\mathrm{SW}_\nu : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \mathrm{Mod}_{\mathcal{U}(\mathfrak{gl}_N)}$ lies in $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ (cf. Remark 6.0.3).

We can now define another Schur–Weyl functor which we will consider: the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$. Recall from Section 4.2 that

$$\hat{\pi}_N : O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} \to \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} := O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N} / \operatorname{Rep}(\mathfrak{gl}_N)_{\mathrm{poly},\nu}$$

is the Serre quotient of $O^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ by the Serre subcategory of polynomial $\mathfrak{gl}_N$-modules of degree ν. We then define

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N} := \hat{\pi}_N \circ \mathrm{SW}_{\nu,\mathbb{C}^N}.$$

7.2. *The finite-dimensional case.*

Let $n \in \mathbb{Z}_+$. We then have the following theorem, which can be found in [Entova Aizenbud 2015a, Section 7]:

Theorem 7.2.1. *The contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ is exact and essentially surjective. Moreover, the induced contravariant functor*

$$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) / \operatorname{Ker}(\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}) \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

is an antiequivalence of abelian categories, thus making $\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ a Serre quotient of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$.

We will show that a similar result holds in the infinite-dimensional case, when the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ is in fact an antiequivalence of categories.

In the proof of Theorem 7.2.1 we established the following fact (see [Entova Aizenbud 2015a, Theorem 7.2.3]):

Lemma 7.2.2. *The functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ takes a simple object to either a simple object, or zero. More specifically*:

- *Let λ be a Young diagram lying in a trivial $\overset{\nu}{\sim}$-class. Then*
$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(\boldsymbol{L}(\lambda)) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)).$$
- *Consider a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\geq 0}$. Then*
$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(\boldsymbol{L}(\lambda^{(i)})) \cong \hat{\pi}(L_{\mathfrak{p}_n}(\nu - |\lambda^{(i+1)}|, \lambda^{(i+1)}))$$
whenever $i \geq 0$.

Remark 7.2.3. Recall that $L_{\mathfrak{p}_n}(\nu - |\lambda|, \lambda)$ is zero if $\ell(\lambda) \geq n$.

8. Classical Schur–Weyl duality and the restricted inverse limit

8.1. ***A short overview of the classical Schur–Weyl duality.*** Let V be a vector space over $\mathbb{C}$, and let $d \in \mathbb{Z}_+$. The symmetric group S_d acts on $V^{\otimes d}$ by permuting the factors of the tensor product (the action is semisimple, by Maschke's theorem):
$$\sigma.(v_1 \otimes v_2 \otimes \cdots \otimes v_d) := v_{\sigma^{-1}(1)} \otimes v_{\sigma^{-1}(2)} \otimes \cdots \otimes v_{\sigma^{-1}(d)}.$$

The actions of $\mathfrak{gl}(V)$ and S_d on $V^{\otimes d}$ commute, which allows us to consider a contravariant functor
$$\mathtt{SW}_{d,V} : \mathrm{Rep}(S_d) \to \mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}}, \quad \mathtt{SW}_{d,V} := \mathrm{Hom}_{S_d}(\,\cdot\,, V^{\otimes d}).$$
The contravariant functor $\mathtt{SW}_{d,V}$ is $\mathbb{C}$-linear and additive, and sends a simple representation λ of S_d to the $\mathfrak{gl}(V)$-module $S^\lambda V$.

Next, consider the contravariant functor
$$\mathtt{SW}_V : \bigoplus_{d\in\mathbb{Z}_+} \mathrm{Rep}(S_d) \to \mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}}, \quad \mathtt{SW}_V := \bigoplus_d \mathtt{SW}_{d,V}.$$
This functor $\mathtt{SW}_V$ is clearly essentially surjective and full (this is easy to see, since $\mathrm{Rep}(\mathfrak{gl}(V))_{\mathrm{poly}}$ is a semisimple category with simple objects $S^\lambda V \cong \mathtt{SW}(\lambda)$). The kernel of the functor $\mathtt{SW}_V$ is the full additive subcategory (direct factor) of $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$ generated by simple objects λ such that $\ell(\lambda) > \dim V$.

8.2. ***Classical Schur–Weyl duality: inverse limit.*** In this subsection, we prove that the classical Schur–Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$ make the category $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$ dual (antiequivalent) to the category
$$\mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \cong \varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

The contravariant functor $\mathtt{SW}_{\mathbb{C}^N}$ sends the Young diagram λ to the $\mathfrak{gl}_N$-module $S^\lambda\mathbb{C}^N$. Let $n \in \mathbb{Z}_+$. We start by noticing that the functors $\mathfrak{Res}_{n,n+1}$ and the functors Γ_n (defined in Section 3) are compatible with the classical Schur–Weyl functors $\mathtt{SW}_{\mathbb{C}^n}$:

Lemma 8.2.1. *We have natural isomorphisms*

$$\mathfrak{Res}_{n,n+1} \circ \mathtt{SW}_{\mathbb{C}^{n+1}} \cong \mathtt{SW}_{\mathbb{C}^n}$$

and

$$\Gamma_n \circ \mathtt{SW}_{\mathbb{C}^\infty} \cong \mathtt{SW}_{\mathbb{C}^n}$$

for any $n \geq 0$.

Proof. It is enough to check this on simple objects in $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$, in which case the statement follows directly from the definitions of $\mathfrak{Res}_{n,n+1}$ and Γ_n together with the fact that $\mathtt{SW}_{\mathbb{C}^N}(\lambda) \cong S^\lambda\mathbb{C}^N$ for any $N \in \mathbb{Z}_+ \cup \{\infty\}$. □

The above lemma implies that we have a commutative diagram

$$\begin{tikzcd}
 & & \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \\
\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)^{\mathrm{op}} \arrow[urr, "\mathtt{SW}_{\mathbb{C}^n}"] \arrow[rr, "\mathtt{SW}_{\lim}"'] \arrow[drr, "\mathtt{SW}_{\mathbb{C}^\infty}"'] & & \varprojlim_{n\geq 1,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \arrow[u, "\mathrm{Pr}_n"'] \\
 & & \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \arrow[u, "\Gamma_{\lim}"'] \arrow[uu, bend right=60, "\Gamma_n"']
\end{tikzcd}$$

with the functor $\Gamma_{\lim}$ being an equivalence of categories (by Proposition 3.4.3), and Pr_n being the canonical projection functor.

Proposition 8.2.2. *The contravariant functors*

$$\mathtt{SW}_\infty : \bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d) \to \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}}$$

and

$$\mathtt{SW}_{\lim} : \bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d) \to \varprojlim_{n\in\mathbb{Z}_+,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$$

are antiequivalences of semisimple categories.

Proof. As was said in Section 8.1, the functor $\mathtt{SW}_N$ is full and essentially surjective for any N. In this case, the functor $\mathtt{SW}_\infty$ is also faithful, since the simple object λ in $\bigoplus_{d\in\mathbb{Z}_+}\mathrm{Rep}(S_d)$ is taken by the functor $\mathtt{SW}_\infty$ to the simple object $S^\lambda\mathbb{C}^\infty \neq 0$. This proves that the contravariant functor $\mathtt{SW}_\infty$ is an antiequivalence of categories. The commutative diagram above then implies that the contravariant functor $\mathtt{SW}_{\lim}$ is an antiequivalence as well. □

9. $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and the inverse limit of categories $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$

9.1. In this section we prove that the Schur–Weyl functors defined in Section 7.1 give us an equivalence of categories between $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and the restricted inverse limit $\varprojlim_{N\in\mathbb{Z}_+,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$. We fix $\nu \in \mathbb{C}$.

Proposition 9.1.1. *The functor $\mathfrak{Res}_{n-1,n}$ satisfies $\mathfrak{Res}_{n-1,n} \circ \mathrm{SW}_{\nu,\mathbb{C}^n} \cong \mathrm{SW}_{\nu,\mathbb{C}^{n-1}}$; i.e., there exists a natural isomorphism $\eta_n : \mathfrak{Res}_{n-1,n} \circ \mathrm{SW}_{\nu,\mathbb{C}^n} \to \mathrm{SW}_{\nu,\mathbb{C}^{n-1}}$.*

Proof. This follows directly from Proposition 6.0.5. □

Corollary 9.1.2. $\widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} \cong \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$; *i.e., there exists a natural isomorphism $\hat{\eta}_n : \widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} \to \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$.*

Proof. By the definitions of $\widehat{\mathfrak{Res}}_{n-1,n}$ and $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$, together with Proposition 9.1.1, we have a commutative diagram

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[r, "\mathrm{SW}_{\nu,\mathbb{C}^n}"'] \arrow[rr, bend left, "\mathrm{SW}_{\nu,\mathbb{C}^{n-1}}"] \arrow[dr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\mathfrak{Res}_{n-1,n}"'] \arrow[d, "\hat{\pi}_n"] & O^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}} \arrow[d, "\hat{\pi}_{n-1}"] \\
 & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\widehat{\mathfrak{Res}}_{n-1,n}"'] & \widehat{O}^{\mathfrak{p}_{n-1}}_{\nu,\mathbb{C}^{n-1}}
\end{tikzcd}$$

Since $\hat{\pi}_{n-1} \circ \mathrm{SW}_{\nu,\mathbb{C}^{n-1}} =: \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$, we get $\widehat{\mathfrak{Res}}_{n-1,n} \circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} \cong \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^{n-1}}$. □

Notation 9.1.3. For each $k \in \mathbb{Z}_+$, $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ is defined to be the Serre subcategory of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ generated by the simple objects $\boldsymbol{L}(\lambda)$ such that the Young diagram λ satisfies either of the following conditions:

- λ belongs to a trivial $\overset{\nu}{\sim}$-class, and $\ell(\lambda) \le k$.
- λ belongs to a nontrivial $\overset{\nu}{\sim}$-class $\{\lambda^{(i)}\}_{i\ge 0}$, $\lambda = \lambda^{(i)}$, and $\ell(\lambda^{(i+1)}) \le k$.

This defines a $\mathbb{Z}_+$-filtration on the objects of the category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$. That is,

$$\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \cong \varinjlim_{k\in\mathbb{Z}_+} \mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)).$$

Lemma 9.1.4. *The functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are $\mathbb{Z}_+$-filtered shortening functors* (see the Appendix for the relevant definitions).

Proof. This result follows from the fact that the $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are exact, together with Lemma 7.2.2. □

This lemma, together with Corollary 9.1.2, implies that there is a canonical contravariant ($\mathbb{Z}_+$-filtered shortening) functor

$$\begin{aligned}\widehat{\mathrm{SW}}_{\nu,\lim} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n},\\ X \mapsto \big(\{\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}, \{\hat{\eta}_n(X)\}_{n\geq 2}\big),\\ (f : X \to Y) \mapsto \{\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(f) : \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(Y) \to \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}(X)\}_{n\geq 1}.\end{aligned}$$

This functor is given by the universal property of the restricted inverse limit described in Proposition A.2.7[1] and makes the diagram below commutative:

$$\begin{tikzcd} & \widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[ur, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[r, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] & \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[u, "\mathrm{Pr}_n"'] \end{tikzcd}$$

(here Pr_n is the canonical projection functor).

We show there is an equivalence of categories $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}}$ and $\varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Theorem 9.1.5. *The Schur–Weyl contravariant functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ induce an antiequivalence of abelian categories, given by the (exact) contravariant functor*

$$\widehat{\mathrm{SW}}_{\nu,\lim} : \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu) \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Proof. The functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ are exact for each $n \geq 1$, which means that the functor $\widehat{\mathrm{SW}}_{\nu,\lim}$ is exact as well.

To see that it is an antiequivalence, we will use Proposition A.4.2. All we need to check is that the functors $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ satisfy the stabilization condition (Condition A.4.1): that is, for each $k \in \mathbb{Z}_+$, there exists $n_k \in \mathbb{Z}_+$ such that

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)) \to \mathrm{Fil}_k(\widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an antiequivalence of categories for any $n \geq n_k$.

Indeed, let $k \in \mathbb{Z}_+$, and let $n \geq k+1$. The category $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ decomposes into blocks (corresponding to the blocks of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$), and the category $\mathrm{Fil}_k(\widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ decomposes into blocks corresponding to the blocks of $\widehat{\mathcal{O}}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

The requirement $n \geq k+1$ together with Lemma 7.2.2 means that for any semisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$, the simple object $\boldsymbol{L}(\lambda)$ corresponding to this block is not sent to zero under $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$. This, in turn, implies that $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$

[1] Alternatively, one can use Proposition A.3.3, since we already stated that in our setting the two notions of inverse limit coincide.

induces an antiequivalence between each semisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$ and the corresponding semisimple block of $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$.

Fix a nonsemisimple block $\mathcal{B}_\lambda$ of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$, and denote by $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ the corresponding nonsemisimple block of $\mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu))$. We denote by $\mathfrak{B}_{\lambda,n}$ the corresponding block in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$. The corresponding block of $\mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$ will then be $\hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$.

We now check that the contravariant functor

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}|_{\mathrm{Fil}_k(\mathcal{B}_\lambda)} : \mathrm{Fil}_k(\mathcal{B}_\lambda) \to \hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$$

is an antiequivalence of categories when $n \geq k+1$.

Since $n \geq k+1$, the Serre subcategories $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ and $\mathrm{Ker}(\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n})$ of $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ have trivial intersection (see Lemma 7.2.2), which means that the restriction of $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}$ to the Serre subcategory $\mathrm{Fil}_k(\mathcal{B}_\lambda)$ is both faithful and full (the latter follows from Theorem 7.2.1).

It remains to establish that the functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}|_{\mathrm{Fil}_k(\mathcal{B}_\lambda)}$ is essentially surjective when $n \geq k+1$. This can be done by checking that this functor induces a bijection between the sets of isomorphism classes of indecomposable projective objects in $\mathrm{Fil}_k(\mathcal{B}_\lambda)$, $\hat{\pi}(\mathrm{Fil}_k(\mathfrak{B}_{\lambda,n}))$ respectively (see [Entova Aizenbud 2015a, proof of Theorem 7.2.7], where we use a similar technique). The latter fact follows from the proof of [Entova Aizenbud 2015a, Theorem 7.2.7].

Thus $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\mathcal{B}_\lambda) \to \mathrm{Fil}_k(\hat{\pi}(\mathfrak{B}_{\lambda,n}))$ is an antiequivalence of categories for $n \geq k+1$, and

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n} : \mathrm{Fil}_k(\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)) \to \mathrm{Fil}_k(\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})$$

is an antiequivalence of categories for $n \geq k+1$. ∎

10. Schur–Weyl duality for $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and $\mathfrak{gl}_\infty$

10.1. Let $\mathbb{C}^\infty$ be a complex vector space with a countable basis $e_1, e_2, e_3, \ldots$. Fix $\mathbb{1} := e_1$ and $U_\infty := \mathrm{span}_{\mathbb{C}}(e_2, e_3, \ldots)$.

Lemma 10.1.1. *We have a commutative diagram*

$$\begin{tikzcd} \underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)^{\mathrm{op}} \arrow[rr, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] \arrow[drr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & & \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\ & & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_{\lim}"'] \end{tikzcd}$$

Namely, there is a natural isomorphism $\hat{\eta} : \widehat{\Gamma}_{\lim} \circ \widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty} \to \widehat{\mathrm{SW}}_{\nu,\lim}$.

Proof. To prove this statement, we will show that for any $n \geq 1$, the following diagram is commutative:

$$\begin{tikzcd}
\underline{\mathrm{Rep}}^{ab}(S_\nu)^{op} \arrow[r, "\mathrm{SW}_{\nu,\mathbb{C}^n}"] \arrow[rr, bend left, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[dr, "\mathrm{SW}_{\nu,\mathbb{C}^\infty}"] \arrow[drr, bend right, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[r, "\hat{\pi}_n"'] & \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\
& O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\Gamma_n"] \arrow[r, "\hat{\pi}_\infty"] & \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_n"]
\end{tikzcd}$$

The commutativity of this diagram follows from the existence of a natural isomorphism $\Gamma_n \circ \mathrm{SW}_{\nu,\mathbb{C}^\infty} \xrightarrow{\sim} \mathrm{SW}_{\nu,\mathbb{C}^n}$ (due to Proposition 6.0.5) and a natural isomorphism $\widehat{\Gamma}_n \circ \hat{\pi}_\infty \cong \hat{\pi}_n \circ \Gamma_n$ (see proof of Proposition 5.3.5). $\square$

Thus we obtain a commutative diagram

$$\begin{tikzcd}
& \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \\
\underline{\mathrm{Rep}}^{ab}(S_\nu)^{op} \arrow[ur, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^n}"] \arrow[r, "\widehat{\mathrm{SW}}_{\nu,\lim}"'] \arrow[dr, "\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}"'] & \varprojlim_{n\geq 1,\, \mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n} \arrow[u, "\mathrm{Pr}_n"] \\
& \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \arrow[u, "\widehat{\Gamma}_{\lim}"] \arrow[uu, bend right=60, "\widehat{\Gamma}_n"']
\end{tikzcd}$$

Theorem 10.1.2. *The contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty} : \underline{\mathrm{Rep}}^{ab}(S_\nu) \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ is an antiequivalence of abelian categories.*

Proof. The functor $\widehat{\Gamma}_{\lim}$ is an equivalence of categories (see Proposition 5.3.5), and the functor $\widehat{\mathrm{SW}}_{\nu,\lim}$ is an antiequivalence of categories (see Theorem 9.1.5). The commutative diagram above implies that the contravariant functor $\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^\infty}$ is an antiequivalence of categories as well. $\square$

11. Schur–Weyl functors and duality structures

11.1. Let $n \in \mathbb{Z}_+$. Recall the contravariant duality functor $(\,\cdot\,)^\vee_n : (O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})^{op} \to O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ discussed in Section 4.3. This functor takes polynomial modules to polynomial modules, and therefore descends to a duality functor $\widehat{(\,\cdot\,)}^\vee_n : (\widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n})^{op} \to \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$.

Next, the definition of duality functor in $O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ implies that the duality functors commute with the restriction functors $\mathfrak{Res}_{n-1,n}$, namely, that for any $n \geq 2$,

$$(\,\cdot\,)^\vee_{n-1} \circ \mathfrak{Res}^{op}_{n-1,n} \cong \mathfrak{Res}^{op}_{n-1,n} \circ (\,\cdot\,)^\vee_n.$$

This allows us to define duality functors

$$(\,\cdot\,)^{\vee}_{\lim} : \Big(\varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\Big)^{\mathrm{op}} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$$

and

$$\widehat{(\,\cdot\,)}^{\vee}_{\lim} : \Big(\varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}\Big)^{\mathrm{op}} \to \varprojlim_{n\geq 1,\,\mathrm{restr}} \widehat{O}^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}.$$

Under the equivalence $O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty} \cong \varprojlim_{n\geq 1,\,\mathrm{restr}} O^{\mathfrak{p}_n}_{\nu,\mathbb{C}^n}$ established in Section 5.3, the functor $(\,\cdot\,)^{\vee}_{\lim}$ corresponds to the duality functor $(\,\cdot\,)^{\vee}_{\infty} : (O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to O^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$ discussed in Section 4.3. Again, this functor descends to a contravariant duality functor $\widehat{(\,\cdot\,)}^{\vee}_{\infty} : (\widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty})^{\mathrm{op}} \to \widehat{O}^{\mathfrak{p}_\infty}_{\nu,\mathbb{C}^\infty}$.

As a corollary of Theorem 7.2.1, a connection was established between the notions of duality in the Deligne category $\underline{\mathrm{Rep}}^{\mathrm{ab}}(S_\nu)$ and duality in the category $\widehat{O}^{\mathfrak{p}_N}_{\nu,\mathbb{C}^N}$ for $N \in \mathbb{Z}_+$ (see [Entova Aizenbud 2015a, Section 7.3]). The above construction allows us to extend this connection to the case when $N=\infty$. Namely, Theorems 9.1.5 and 10.1.2, together with [Entova Aizenbud 2015a, Section 7.3], imply the next result.

Proposition 11.1.1. *Let $N \in \mathbb{Z}_+ \cup \{\infty\}$ and $\nu \in \mathbb{C}$. There is an isomorphism of (covariant) functors*

$$\widehat{\mathrm{SW}}_{\nu,\mathbb{C}^N} \circ (\,\cdot\,)^{*} \to \widehat{(\,\cdot\,)}^{\vee}_{N} \circ \mathrm{SW}_{\nu,\mathbb{C}^N}.$$

Appendix: Restricted inverse limit of categories

We describe the main elements of the framework for the notion of a restricted inverse limit of categories. A detailed description of this framework has been given in the note [Entova Aizenbud 2015b]; this appendix contains the results which are necessary for understanding the Schur–Weyl duality in complex rank. In particular, [Entova Aizenbud 2015b] provides some motivation behind the definitions given below.

Given a system of categories $\mathcal{C}_i$ (with i running through the set $\mathbb{Z}_+$) and functors $\mathcal{F}_{i-1,i} : \mathcal{C}_i \to \mathcal{C}_{i-1}$ for each $i \geq 1$, we define the inverse limit category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ to be the following category:

- The objects are pairs $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ where $C_i \in \mathcal{C}_i$ for each $i \in \mathbb{Z}_+$ and $\phi_{i-1,i} : \mathcal{F}_{i-1,i}(C_i) \xrightarrow{\sim} C_{i-1}$ for any $i \geq 1$.
- A morphism f between objects $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, $(\{D_i\}_{i\in\mathbb{Z}_+}, \{\psi_{i-1,i}\}_{i\geq 1})$ is a set of arrows $\{f_i : C_i \to D_i\}_{i\in\mathbb{Z}_+}$ satisfying some obvious compatibility conditions.

This category is an inverse limit of the system $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ in the (2, 1)-category of categories with functors and natural isomorphisms. We will denote by Pr_i the projection functors $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i \to \mathcal{C}_i$ (and similarly the projection functors from other inverse limits defined below).

A.1. ***Restricted inverse limit of finite-length categories.*** To define the restricted inverse limit, we work with categories $\mathcal{C}_i$ which are finite-length categories, namely, abelian categories where each object has a (finite) Jordan–Hölder filtration. We require that the functors $\mathcal{F}_{i-1,i}$ be shortening in the following sense:

Definition A.1.1. A functor $\mathcal{F} : \mathcal{C} \to \mathcal{D}$ between two finite-length categories is shortening if it is exact and given an object $C \in \mathcal{C}$, we have

$$\ell_{\mathcal{D}}(\mathcal{F}(C)) \leq \ell_{\mathcal{C}}(C).$$

Since $\mathcal{F}$ is exact, this is equivalent to requiring that for any simple object $L \in \mathcal{A}_1$, the object $\mathcal{F}(L)$ is either simple or zero.

Example A.1.2. The functors

$$\mathfrak{Res}_{n-1,n} : \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_{n-1})_{\mathrm{poly}} \quad \text{and} \quad \Gamma_n : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}$$

(see Section 3.1 for definitions) are examples of shortening functors.

Given a system $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ of finite-length categories and shortening functors, it makes sense to consider the full subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$, with $\{\ell_{\mathcal{C}_n}(C_n)\}_{n\geq 0}$ being a bounded sequence (the condition on the functors implies that this sequence is weakly increasing).

This subcategory will be called the *restricted inverse limit* of categories $\mathcal{C}_i$ and will be denoted by $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. It is the inverse limit of the categories $\mathcal{C}_i$ in the (2, 1)-category of finite-length categories and shortening functors.

Example A.1.3. Consider the restricted inverse limit of the system

$$\big((\mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}})_{n\geq 0}, (\mathfrak{Res}_{n-1,n})_{n\geq 1}\big).$$

We obtain a functor

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \varprojlim_{n\geq 0,\,\mathrm{restr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

It is easy to see that $\Gamma_{\lim}$ is an equivalence.

A.2. ***Properties of the restricted inverse limit.*** The category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is an abelian category. In fact, it is a finite-length category, and one can describe its simple objects. We start by introducing some notation.

Notation A.2.1. Denote by $\mathrm{Irr}(\mathcal{C}_i)$ the set of isomorphism classes of irreducible objects in $\mathcal{C}_i$, and define the pointed set

$$\mathrm{Irr}_*(\mathcal{C}_i) := \mathrm{Irr}(\mathcal{C}_i) \sqcup \{0\}.$$

The shortening functors $\mathcal{F}_{i-1,i}$ then define maps of pointed sets

$$f_{i-1,i} : \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_{i-1}).$$

Similarly, we define $\mathrm{Irr}\big(\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i\big)$ to be the set of isomorphism classes of irreducible objects in $\mathcal{C}$, and define the pointed set

$$\mathrm{Irr}_*(\mathcal{C}) := \mathrm{Irr}(\mathcal{C}) \sqcup \{0\}.$$

Denote by $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ the inverse limit of the system $(\{\mathrm{Irr}_*(\mathcal{C}_i\}_{i\geq 0}, \{f_{i-1,i}\}_{i\geq 1})$. We will also denote by $\mathrm{pr}_j : \varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_j)$ the projection maps.

The elements of the set $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ are just sequences $(L_i)_{i\geq 0}$ such that $L_i \in \mathrm{Irr}_*(\mathcal{C}_i)$, and $f_{i-1,i}(L_i) \cong L_{i-1}$.

Proposition A.2.2. *Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a system of finite-length categories and shortening functors. The category $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, and its objects have finite length. The set of isomorphism classes of simple objects in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is in bijection with the set $\big(\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)\big) \setminus \{0\}$. That is, we have a natural bijection*

$$\mathrm{Irr}_*(\mathcal{C}) \cong \varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i).$$

Proof. Let

$$C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1}),$$
$$C' := (\{C'_j\}_{j\in\mathbb{Z}_+}, \{\phi'_{j-1,j}\}_{j\geq 1}),$$
$$C'' := (\{C''_j\}_{j\in\mathbb{Z}_+}, \{\phi''_{j-1,j}\}_{j\geq 1})$$

be objects in $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$, together with morphisms $f : C' \to C$ and $g : C \to C''$, such that the sequence

$$0 \longrightarrow C' \xrightarrow{f} C \xrightarrow{g} C'' \longrightarrow 0$$

is exact.

If C lies in the subcategory $\mathcal{C}$, then the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is bounded from above, and stabilizes. Denote its maximum by N. For each i, the sequence

$$0 \longrightarrow C'_i \xrightarrow{f_i} C_i \xrightarrow{g} C''_i \longrightarrow 0$$

is exact. Therefore, $\ell_{\mathcal{C}_i}(C'_i), \ell_{\mathcal{C}_i}(C''_i) \leq N$ for each i, and so C', C'' lie in $\mathcal{C}$ as well.

Vice versa, assuming C', C'' lie in $\mathcal{C}$, denote by N', N'' the maximums of the sequences $\{\ell_{\mathcal{C}_i}(C'_i)\}_i, \{\ell_{\mathcal{C}_i}(C''_i)\}_i$, respectively. Then $\ell_{\mathcal{C}_i}(C_i) \leq N' + N''$ for any $i \geq 0$, and so C lies in the subcategory $\mathcal{C}$ as well.

Thus $\mathcal{C}$ is a Serre subcategory of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$.

Sublemma A.2.3. *Given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have*

$$\ell_{\mathcal{C}}(C) \leq \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}.$$

Proof. Let C lie in $\mathcal{C}$. We would like to say that C has finite length. Denote by N the maximum of the sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$. It is easy to see that C has length at most N; indeed, if $\{C', C'', \dots, C^{(n)}\}$ is a subset of $\mathrm{JH}_{\mathcal{C}}(C)$, then for some $i \gg 0$, we have $\mathrm{Pr}_i(C^{(k)}) \neq 0$ for any $k = 1, 2, \dots, n$. The objects $\mathrm{Pr}_i(C^{(k)})$ are distinct Jordan–Hölder components of C_i, so $n \leq \ell_{\mathcal{C}_i}(C_i) \leq N$. In particular, we see that

$$\ell_{\mathcal{C}}(C) \leq N = \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}. \qquad \square$$

Now, let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\mathcal{C}$. We denote by $\mathrm{JH}(C_j)$ the multiset of the Jordan–Hölder components of C_j, and let

$$\mathrm{JH}_*(C_j) := \mathrm{JH}(C_j) \sqcup \{0\}.$$

The corresponding set lies in $\mathrm{Irr}_*(\mathcal{C}_j)$, and we have maps of (pointed) multisets

$$f_{j-1,j} : \mathrm{JH}_*(C_j) \to \mathrm{JH}_*(C_{j-1}).$$

Sublemma A.2.4. *Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be in $\mathcal{C} := \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. Then*

$$C \in \mathrm{Irr}_*(\mathcal{C}) \iff \mathrm{Pr}_j(C) = C_j \in \mathrm{Irr}_*(\mathcal{C}_j)\ \forall j.$$

In other words, C is a simple object (that is, C has exactly two distinct subobjects: zero and itself) if and only if $C \neq 0$, and for any $j \geq 0$, the component C_j is either a simple object in $\mathcal{C}_j$, or zero.

Proof. The direction $\Leftarrow$ is obvious, so we will only prove the direction $\Rightarrow$.

Let n_0 be a position in which the maximum of the weakly increasing integer sequence $\{\ell_{\mathcal{C}_i}(C_i)\}_{i\geq 0}$ is obtained. By definition of n_0, for $j > n_0$, the functors $\mathcal{F}_{j-1,j}$ do not kill any Jordan–Hölder components of C_j.

Now, consider the socles of the objects C_j for $j \geq n_0$. For any $j > 0$, we have

$$\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j)) \xhookrightarrow{\phi_{j-1,j}} \mathrm{socle}(C_{j-1}),$$

and thus for $j > n_0$, we have

$$\ell_{\mathcal{C}_j}(\mathrm{socle}(C_j)) = \ell_{\mathcal{C}_{j-1}}(\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j))) \leq \ell_{\mathcal{C}_{j-1}}(\mathrm{socle}(C_{j-1})).$$

Thus the sequence $\{\ell_{\mathcal{C}_j}(\mathrm{socle}(C_j))\}_{j\geq n_0}$ is a weakly decreasing sequence and stabilizes. Denote its stable value by N. We conclude that there exists $n_1 \geq n_0$ such that

$$\mathcal{F}_{j-1,j}(\mathrm{socle}(C_j)) \xrightarrow{\phi_{j-1,j}} \mathrm{socle}(C_{j-1})$$

is an isomorphism for every $j > n_1$.

Now, set

$$D_j := \begin{cases} \mathcal{F}_{j,n_1}(\mathrm{socle}(C_{n_1})) & \text{if } j < n_1, \\ \mathrm{socle}(C_j) & \text{if } j \geq n_1 \end{cases}$$

(here $\mathcal{F}_{j,n_1} : \mathcal{C}_{n_1} \to \mathcal{C}_j$ are our shortening functors, with n_1 fixed and j varying). We put $D := ((D_j)_{j\geq 0}, (\phi_{j-1,j})_{j\geq 1})$ (this is a subobject of C in the category $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$). Of course, $\ell_{\mathcal{C}_j}(D_j) \leq N$ for any j, so D is an object in the full subcategory $\mathcal{C}$ of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$. Furthermore, since $C \neq 0$, we have that for $j \gg 0$, $\mathrm{socle}(C_j) \neq 0$, and thus $0 \neq D \subset C$. Thus D is a semisimple object $\mathcal{C}$, with simple summands corresponding to the elements of the inverse limit of the multisets $\varprojlim_{j\in\mathbb{Z}_+} \mathrm{JH}_*(D_j)$.

We conclude that $D = C$, and that $\mathrm{socle}(C_j) = C_j$ has length at most one for any $j \geq 0$. □

Remark A.2.5. The latter multiset is equivalent to the inverse limit of multisets $\mathrm{JH}_*(\mathrm{socle}(C_j))$, so D is, in fact, the socle of C.

This completes the proof of Proposition A.2.2. □

In particular, given an object $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$, we have $\mathrm{JH}_*(C) = \varprojlim_{i\in\mathbb{Z}_+} \mathrm{JH}_*(C_i)$ (an inverse limit of the system of multisets $\mathrm{JH}_*(C_j)$ and maps $f_{j-1,j}$).

It is now obvious that the projection functors $\mathrm{Pr}_i : \mathcal{C} \to \mathcal{C}_i$ are shortening as well, and induce the maps $\mathrm{pr}_i : \mathrm{Irr}_*(\mathcal{C}) \to \mathrm{Irr}_*(\mathcal{C}_i)$.

Corollary A.2.6. *Given an object $C := (\{C_i\}_{i\in\mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i\geq 1})$ in $\mathcal{C}$, we have*

$$\ell_{\mathcal{C}}(C) = \max\{\ell_{\mathcal{C}_i}(C_i) \mid i \geq 0\}.$$

It is now easy to see that the restricted inverse limit has the following universal property:

Proposition A.2.7. *Let $\mathcal{A}$ be a finite-length category, together with a set of shortening functors $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ with the property that for any $i \geq 1$, there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}.$$

Then $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is universal among such categories; that is, we have a shortening functor

$$\mathcal{G} : \mathcal{A} \to \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i,$$

$$A \mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1}),$$

$$f : A_1 \to A_2 \mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+}$$

and $\mathcal{G}_i \cong \mathrm{Pr}_i \circ \mathcal{G}$ for every $i \in \mathbb{Z}_+$.

Proof. Consider the functor $\mathcal{G} : \mathcal{A} \to \varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ induced by the functors $\mathcal{G}_i$. We would like to say that for any $A \in \mathcal{A}$, the object $\mathcal{G}(A)$ lies in the subcategory $\varprojlim_{i \in \mathbb{Z}_+, \text{restr}} \mathcal{C}_i$, i.e., that the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above.

Indeed, since $\mathcal{G}_i$ are shortening functors, we have $\ell_{\mathcal{C}_i}(\mathcal{G}_i(A)) \leq \ell_{\mathcal{A}}(A)$. Thus the sequence $\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\}_i$ is bounded from above by $\ell_{\mathcal{A}}(A)$.

Now, using Corollary A.2.6, we obtain

$$\ell_{\mathcal{C}}(\mathcal{G}(A)) = \max_{i \geq 0}\{\ell_{\mathcal{C}_i}(\mathcal{G}_i(A))\} \leq \ell_{\mathcal{A}}(A)$$

and we conclude that $\mathcal{G}$ is a shortening functor. □

A.3. ***Inverse limit of categories with filtration.*** We now define the inverse limit of categories in a different setting, a priori not related to the restricted inverse limit defined above. The new inverse limit is defined in the setting of categories with filtrations, and is sometimes more convenient to use. We will later give a sufficient condition for the two notions of inverse limit to coincide.

Fix a directed partially ordered set $(K, \leq)$, where "directed" means that for any $k_1, k_2 \in K$, there exists $k \in K$ such that $k_1, k_2 \leq k$.

Definition A.3.1 (categories with K-filtrations). We say that a category $\mathcal{A}$ has a K-filtration if for each $k \in K$ we have a full subcategory $\mathcal{A}^k$ of $\mathcal{A}$, and these subcategories satisfy the following conditions:

(1) $\mathcal{A}^k \subset \mathcal{A}^l$ whenever $k \leq l$.

(2) $\mathcal{A}$ is the union of $\mathcal{A}^k$, $k \in K$: that is, for any $A \in \mathcal{A}$, there exists $k \in K$ such that $A \in \mathcal{A}^k$.

A functor $\mathcal{F} : \mathcal{A}_1 \to \mathcal{A}_2$ between categories with K-filtrations $\mathcal{A}_1, \mathcal{A}_2$ is called a *K-filtered functor* if for any $k \in K$, $\mathcal{F}(\mathcal{A}_1^k)$ is a subcategory of $\mathcal{A}_2^k$.

Note that if we consider abelian categories and exact functors, we should require that the subcategories be Serre subcategories in order for the constructions to work nicely.

Consider a system $((\mathcal{C}_i)_{i \in \mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i \geq 1})$ of categories with K-filtrations and K-filtered functors between them. We can define a full subcategory $\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ of $\varprojlim_{i \in \mathbb{Z}_+} \mathcal{C}_i$ whose objects are of the form $(\{C_i\}_{i \in \mathbb{Z}_+}, \{\phi_{i-1,i}\}_{i \geq 1})$ such that there exists $k \in K$ for which $C_i \in \mathrm{Fil}_k(\mathcal{C}_i)$ for any $i \geq 0$. The category $\varprojlim_{i \in \mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ is automatically a category with a K-filtration on objects. It is the inverse limit of the categories $\mathcal{C}_i$ in the (2, 1)-category of categories with K-filtrations on objects, and functors respecting these filtrations:

Example A.3.2. Consider the $\mathbb{Z}_+$-filtration on the objects of $\mathrm{Rep}(\mathfrak{gl}_N)_{\text{poly}}$ where $S^\lambda \mathbb{C}^N$ lies in the component $|\lambda|$ of the filtration. The functors $\mathfrak{Res}_{n-1,n}$ respect this filtration, and we obtain a functor

$$\Gamma_{\lim} : \mathrm{Rep}(\mathfrak{gl}_\infty)_{\mathrm{poly}} \to \varprojlim_{n\geq 0, \mathbb{Z}_+\text{-filtr}} \mathrm{Rep}(\mathfrak{gl}_n)_{\mathrm{poly}}.$$

One can show that this is an equivalence.

We have the following universal property, whose proof is straightforward:

Proposition A.3.3. *Let* $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ *be a system with a* K*-filtration as above, and let* $\mathcal{A}$ *be a category with a* K*-filtration, together with a set of* K*-filtered functors* $\mathcal{G}_i : \mathcal{A} \to \mathcal{C}_i$ *such that for any* $i \geq 1$ *there exists a natural isomorphism*

$$\eta_{i-1,i} : \mathcal{F}_{i-1,i} \circ \mathcal{G}_i \to \mathcal{G}_{i-1}.$$

Then $\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ *is universal among such categories; that is, we have a functor*

$$\begin{aligned} \mathcal{G} : \mathcal{A} &\to \varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i, \\ A &\mapsto (\{\mathcal{G}_i(A)\}_{i\in\mathbb{Z}_+}, \{\eta_{i-1,i}\}_{i\geq 1}), \\ f : A_1 \to A_2 &\mapsto \{f_i := \mathcal{G}_i(f)\}_{i\in\mathbb{Z}_+} \end{aligned}$$

which is obviously K*-filtered and satisfies* $\mathcal{G}_i \cong \mathrm{Pr}_i \circ \mathcal{G}$ *for every* $i \in \mathbb{Z}_+$.

A.4. *Stabilizing inverse limit.* Working in the setting of categories with K-filtrations and K-filtered functors, we consider the case when $\mathcal{A}$, $\{\mathcal{G}_i\}_{i\in\mathbb{Z}_+}$ satisfy the following stabilization condition (this is the case in Theorem 9.1.5):

Condition A.4.1. For every $k \in K$, there exists $i_k \in \mathbb{Z}_+$ such that $\mathcal{G}_j : \mathcal{A}^k \to \mathcal{C}_j^k$ is an equivalence of categories for any $j \geq i_k$.

In this setting, the following proposition holds:

Proposition A.4.2. *The functor* $\mathcal{G} : \mathcal{A} \to \varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ *is an equivalence of categories with* K*-filtrations.*

Proof. To prove that $\mathcal{G}$ is an equivalence of categories with K-filtrations, we need to show that

$$\mathcal{G} : \mathcal{A}^k \to \mathrm{Fil}_k\big(\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big)$$

is an equivalence of categories for any $k \in K$. Recall that

$$\mathrm{Fil}_k\big(\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big) \cong \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i^k.$$

By Condition A.4.1, for any $i > i_k$ we have a commutative diagram where all arrows are equivalences:

$$\begin{tikzcd} \mathcal{A}^k \arrow[r, "\mathcal{G}_i"] \arrow[d, "\mathcal{G}_{i_k}"'] & \mathcal{C}_i^k \arrow[dl, "\mathcal{F}_{i-1,i}"] \\ \mathcal{C}_{i-1}^k & \end{tikzcd}$$

Since for any fixed k, $\mathcal{F}_{i-1,i} : \mathcal{C}_i^k \to \mathcal{C}_{i-1}^k$ is an equivalence for $i > i_k$, it is obvious that $\mathrm{Pr}_i : \varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i^k \to \mathcal{C}_i^k$ is an equivalence of categories for any $i > i_k$. Thus $\mathcal{G} : \mathcal{A}^k \to \mathrm{Fil}_k\big(\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i\big)$ is an equivalence of categories. $\square$

A.5. ***Equivalence of inverse limits.*** Finally, we provide a sufficient condition for the two notions of "special" inverse limit to coincide. This is the case in the setting of Theorem 9.1.5.

Let $((\mathcal{C}_i)_{i\in\mathbb{Z}_+}, (\mathcal{F}_{i-1,i})_{i\geq 1})$ be a system of finite-length categories with K-filtrations and shortening K-filtered functors, whose filtration components are Serre subcategories. We would like to give a sufficient condition on the K-filtration for the inverse limit of a system of categories with K-filtrations to coincide with the restricted inverse limit of these categories.

Recall that since the functors $\mathcal{F}_{i-1,i}$ are shortening, we have maps

$$f_{i-1,i} : \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_{i-1})$$

and we can consider the inverse limit $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$ of the sequence of sets $\mathrm{Irr}_*(\mathcal{C}_i)$ and maps $f_{i-1,i}$; we will denote by $\mathrm{pr}_j : \varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i) \to \mathrm{Irr}_*(\mathcal{C}_j)$ the projection maps.

Notice that the sets $\mathrm{Irr}_*(\mathcal{C}_i)$ have natural K-filtrations, and the maps $f_{i-1,i}$ respect these filtrations.

Proposition A.5.1. *Assume the following conditions hold*:

(1) *There exists a K-filtration on the set $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$. That is, we require that for each L in $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$, there exists $k \in K$ so that $\mathrm{pr}_i(L) \in \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i))$ for any $i \geq 0$. We would then say that such an object L belongs in the k-th filtration component of $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$.*

(2) *Stabilization condition*: *For any $k \in K$, there exists $N_k \geq 0$ such that the map $f_{i-1,i} : \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i)) \to \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_{i-1}))$ is an injection for any $i \geq N_k$. That is, for any $k \in K$ there exists $N_k \in \mathbb{Z}_+$ such that the (exact) functor $\mathcal{F}_{i-1,i}$ is faithful for any $i \geq N_k$.*

Then the two full subcategories $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ and $\varprojlim_{i\in\mathbb{Z}_+, K\text{-filtr}} \mathcal{C}_i$ of $\varprojlim_{i\in\mathbb{Z}_+} \mathcal{C}_i$ coincide.

Proof. Let $C := (\{C_j\}_{j\in\mathbb{Z}_+}, \{\phi_{j-1,j}\}_{j\geq 1})$ be an object in $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. As before, we denote by $\mathrm{JH}(C_j)$ the multiset of Jordan–Hölder components of C_j, and let $\mathrm{JH}_*(C_j) := \mathrm{JH}(C_j) \sqcup \{0\}$.

The first condition is natural: giving a K-filtration on the objects of $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is equivalent to giving a K-filtration on the simple objects of $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$, i.e., on the set $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$.

Assume $C \in \varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$. Let $n_0 \geq 0$ be such that $\ell_{\mathcal{C}_j}(C_j)$ is constant for $j \geq n_0$. Recall that we have

$$\mathrm{JH}_*(C) = \varprojlim_{i\in\mathbb{Z}_+} \mathrm{JH}_*(C_j).$$

Choose k such that all the elements of $\mathrm{JH}_*(C)$ lie in the k-th filtration component of $\varprojlim_{i\in\mathbb{Z}_+} \mathrm{Irr}_*(\mathcal{C}_i)$. This is possible due to the first condition.

Then for any $L_j \in \mathrm{JH}(C_j)$, we have that $L_j = \mathrm{pr}_j(L)$ for some $L \in \mathrm{JH}_*(C)$, and thus $L_j \in \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_j))$. We conclude that $C \in \mathrm{Fil}_k(\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i)$.

Thus the first condition of the theorem holds if and only if $\varprojlim_{i\in\mathbb{Z}_+,\,\mathrm{restr}} \mathcal{C}_i$ is a full subcategory of $\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i$.

Now, let $C \in \varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i$, and let $k \in K$ be such that $C \in \mathrm{Fil}_k(\varprojlim_{i\in\mathbb{Z}_+,K\text{-filtr}} \mathcal{C}_i)$. We would like to show that $\ell_{\mathcal{C}_i}(C_i)$ is constant starting from some i. Indeed, the second condition of the theorem tells us that there exists $N_k \geq 0$ such that the map

$$f_{i-1,i} : \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_i)) \to \mathrm{Fil}_k(\mathrm{Irr}_*(\mathcal{C}_{i-1}))$$

is an injection for any $i \geq N_k$. We claim that for $i \geq N_k$, $\ell_{\mathcal{C}_i}(C_i)$ is constant. Indeed, if it weren't, then there would be some $i \geq N_k + 1$ and some $L_i \in \mathrm{JH}(C_i)$ such that $f_{i-1,i}(L_i) = 0$. But this is impossible, due to the requirement above. $\square$

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A GENERALIZATION OF THE GREENE–KRANTZ THEOREM FOR THE SEMICONTINUITY PROPERTY OF AUTOMORPHISM GROUPS

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We give a CR version of the Greene–Krantz theorem (*Math. Ann.* 261:4 (1982), 425–446) for the semicontinuity of complex automorphism groups. This is not only a generalization but also an intrinsic interpretation of the Greene–Krantz theorem.

1. Introduction

By upper semicontinuity, or simply semicontinuity, in geometry, we mean the property that the set of symmetries of a geometric structure should not decrease at a limit of a sequence of the structures. For instance, a sequence of ellipses in the Euclidean plane can converge to a circle, while a sequence of circles cannot converge to a noncircular ellipse. This property seems as natural as the second law of thermodynamics in physics, but we still need to make it clear in mathematical terminology. A symmetry for a geometric structure is described as a transformation on a space with the geometric structure. The set of transformations becomes a group with respect to the composition operator. Therefore, semicontinuity can be understood as a nondecreasing property of the transformation group at the limit of a sequence of geometric structures. One of the strongest descriptions of semicontinuity was obtained by Ebin for the Riemannian structures on compact manifolds in terms of conjugations by diffeomorphisms.

Theorem 1.1 [Ebin 1970]. *Let M be a C^∞-smooth compact manifold and let $\{g_j : j = 1, 2, \dots\}$ be a sequence of C^∞-smooth Riemannian structures which converges to a Riemannian metric g_0 in the C^∞ sense. Then for each sufficiently large j, there exists a diffeomorphism $\phi_j : M \to M$ such that $\phi_j \circ I_j \circ \phi_j^{-1}$ is a Lie subgroup of I_0, where I_j and I_0 represent the isometry groups for g_j and g_0, respectively.*

The group of holomorphic automorphisms on a complex manifold plays the role of the group of symmetries with respect to the complex structure. By Cartan's

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theorem (cf. [Greene et al. 2011]), the automorphism group of a bounded domain in the complex Euclidean space turns out to be a Lie group with the compact-open topology on the domain. Greene and Krantz proved the following theorem for the semicontinuity property of automorphism groups of bounded strongly pseudoconvex domains.

Theorem 1.2 [Greene and Krantz 1982]. *Let Ω_j $(j=1,2,\dots)$ and Ω_0 be bounded strongly pseudoconvex domains in $\mathbb{C}^n$ with C^∞-smooth boundary. Suppose that Ω_j converges to Ω_0 in the C^∞ sense, that is, there exists a diffeomorphism ψ_j defined on a neighborhood of $\overline{\Omega}_0$ into $\mathbb{C}^n$ such that $\psi_j(\Omega_0)=\Omega_j$ and $\psi_j\to\mathrm{Id}$ in the C^∞ sense on $\overline{\Omega}_0$. Then for every sufficiently large j, there exists a diffeomorphism $\phi_j:\Omega_j\to\Omega_0$ such that $\phi_j\circ\mathrm{Aut}(\Omega_j)\circ\phi_j^{-1}$ is a Lie subgroup of* $\mathrm{Aut}(\Omega_0)$.

Unlike the isometry group of a compact Riemannian manifold, the holomorphic automorphism group on a bounded strongly pseudoconvex domain can be noncompact, so the proof of Theorem 1.2 is divided into two cases: either $\mathrm{Aut}(\Omega_0)$ is compact or it is not. It turns out that the latter case is relatively simple, which is the case of deformations of the unit ball by the Wong–Rosay theorem [Rosay 1979; Wong 1977]. The main part of the proof of Theorem 1.2 is thus devoted to the case when $\mathrm{Aut}(\Omega_0)$ is compact. Greene and Krantz proved this case by constructing a compact Riemannian manifold (M, g_j) which includes Ω_j as a relatively compact subset and whose isometry group contains the automorphism group of Ω_j. Then Ebin's theorem yields the conclusion. The Riemannian manifold (M, g_j) is called a *metric double* of Ω_j.

The idea of this proof is applicable to more general cases. One reasonable generalization is to prove the semicontinuity property for a more general class of domains. In a recent paper [Greene et al. 2013], the authors generalized Theorem 1.2 to finitely differentiable cases. Greene and Kim [2014] proved that a partial generalization is also possible even for some classes of nonstrongly pseudoconvex domains. See also [Krantz 2010] for this line of generalization.

The aim of the present paper is to obtain another generalization of Theorem 1.2. According to Hamilton's theorem [1977; 1979], deformations of a bounded strongly pseudoconvex domain with C^∞-smooth boundary coincide with deformations of a complex structure on a given domain and they give rise to deformations of the CR structure of the boundary. Fefferman's extension theorem [1974] shows that every holomorphic automorphism on a bounded strongly pseudoconvex domain with C^∞-smooth boundary extends to a diffeomorphism up to the boundary and hence gives rise to a CR automorphism on the boundary. Conversely, a CR automorphism on the boundary extends to a holomorphic automorphism on the domain by the Bochner–Hartogs extension theorem. It is also known that the compact-open topology of the automorphism group of the domain coincides with the C^∞-topology of the

CR automorphism group of the boundary (cf. [Bell 1987]) if the holomorphic automorphism group of the domain is compact. In this observation, it is natural to think of the semicontinuity property for abstract strongly pseudoconvex CR manifolds under deformations of CR structures as a generalization of Theorem 1.2. We prove the following theorem for CR automorphism groups when the limit structure has a compact CR automorphism group.

Theorem 1.3. *Let $\{J_k : k = 1, 2, \dots\}$ be a sequence of C^∞-smooth strongly pseudoconvex CR structures on a compact differentiable manifold M of dimension $2n+1$ which converges to a C^∞-smooth strongly pseudoconvex CR structure J_0 on M in the C^∞ sense. Suppose that the CR automorphism group $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact. Then there exists $N > 0$ and a diffeomorphism $\phi_k : M \to M$ for each $k > N$ such that $\phi_k \circ \mathrm{Aut}_{\mathrm{CR}}(M, J_k) \circ \phi_k^{-1}$ is a Lie subgroup of $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$.*

According to Schoen's theorem [1995], $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact if and only if (M, J_0) is not CR equivalent to the sphere S^{2n+1} with the standard CR structure. One should notice that this condition is not necessary if $2n + 1 \geq 5$. Boutet de Monvel [1975] showed that a CR structure on M which is sufficiently close to the standard structure on S^{2n+1} is also embeddable in $\mathbb{C}^{n+1}$ if $2n+1 \geq 5$, in contrast with the 3-dimensional case (see [Burns and Epstein 1990; Lempert 1992; Nirenberg 1974; Rosay 1979]). Therefore, if $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is noncompact and $2n + 1 \geq 5$, then the situation is reduced to the case of deformations of the unit ball and follows immediately from Theorem 1.2.

The rest of this paper will be devoted to proving Theorem 1.3. Since we are thinking about abstract CR manifolds, we need to develop an intrinsic way of proving this. Therefore, the main interest of Theorem 1.3 is not only in the generalization but also in the intrinsic verification of the Greene–Krantz theorem. The main tool of the proof is the solution for the CR Yamabe problem about the construction of pseudohermitian structures with constant Webster scalar curvature, which is intensively studied in, for instance, [Cheng et al. 2014; Gamara 2001; Gamara and Yacoub 2001; Jerison and Lee 1987; 1989]. The subellipticity of the CR Yamabe equation turned out quite useful in obtaining estimates of derivatives of CR automorphisms in [Schoen 1995]. We make use of various solutions for the CR Yamabe problem — minimal solutions, local scalar flattening solutions and the blowing-up solutions given by the Green functions — developed in [Fischer-Colbrie and Schoen 1980; Jerison and Lee 1987; 1989; Schoen 1995].

2. Strongly pseudoconvex CR manifolds

In this section, we summarize fundamental facts on strongly pseudoconvex CR manifolds and pseudohermitian structures. The summation convention is always assumed.

CR and pseudohermitian structures. Let M be a smooth manifold of dimension $2n+1$ for some positive integer n. A *CR structure* on M is a smooth complex structure J on a subbundle H of the rank $2n$ of the tangent bundle TM which satisfies the integrability condition. More precisely, the restriction of J on a fiber H_p for a point $p \in M$ is an endomorphism $J_p : H_p \to H_p$ which satisfies $J_p \circ J_p = -\mathrm{Id}_{H_p}$, varying smoothly as p varies, and the bundle of i-eigenspace $H^{1,0}$ of J in the complexification $\mathbb{C} \otimes H$ satisfies the Frobenius integrability condition

$$[\Gamma(H^{1,0}), \Gamma(H^{1,0})] \subset \Gamma(H^{1,0}).$$

The subbundle H is called the *CR distribution* of J. A *CR automorphism* on M is a smooth diffeomorphism F from M onto itself such that $F_* H^{1,0} = H^{1,0}$. We denote by $\mathrm{Aut}_{\mathrm{CR}}(M)$ the group of all CR automorphisms on M. A CR structure is said to be *strongly pseudoconvex* if its CR distribution H is a contact distribution and for a contact form θ, the *Levi form* $\mathcal{L}_\theta$ defined by

$$\mathcal{L}_\theta(Z, \overline{W}) := -i\, d\theta(Z, \overline{W})$$

for $Z, W \in H^{1,0}$ is positive definite. It is known that the C^0-topology of $\mathrm{Aut}_{\mathrm{CR}}(M)$ coincides with the C^∞-topology for a compact strongly pseudoconvex CR manifold M if $\mathrm{Aut}_{\mathrm{CR}}(M)$ is compact with respect to the C^0-topology. See [Schoen 1995] for the proof.

We call a fixed contact form for the CR distribution of a strongly pseudoconvex CR structure a *pseudohermitian structure*. Let $\{W_\alpha : \alpha = 1, \dots, n\}$ be a local frame; that is, the W_α are sections of $H^{1,0}$ which form a pointwise basis for $H_{1,0}$. We call a collection of 1-forms $\{\theta^\alpha\}$ the *admissible coframe* of $\{W_\alpha\}$ if they are sections of $(H^{1,0})^*$ and satisfy

$$\theta^\alpha(W_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T) = 0,$$

where T is the vector field uniquely determined by

$$\theta(T) = 1, \quad T \lrcorner\, d\theta = 0,$$

which is called the *characteristic vector field* for θ. Let $g_{\alpha\bar\beta} = \mathcal{L}_\theta(W_\alpha, W_{\bar\beta})$. Then

$$d\theta = 2i g_{\alpha\bar\beta}\, \theta^\alpha \wedge \theta^{\bar\beta},$$

where $\{\theta^\alpha\}$ is the admissible coframe for $\{W_\alpha\}$.

Theorem 2.1 [Webster 1978]. *There exist a local 1-form $\omega = (\omega_\beta{}^\alpha)$ and local functions $A^\alpha{}_\beta$ uniquely determined by*

$$d\theta^\alpha = \theta^\beta \wedge \omega_\beta{}^\alpha + A^\alpha{}_{\bar\beta}\, \theta \wedge \theta^{\bar\beta},$$
$$dg_{\alpha\bar\beta} = \omega_{\alpha\bar\beta} + \omega_{\bar\beta\alpha}, \quad A_{\alpha\beta} = A_{\beta\alpha}.$$

Here and in the sequel, we lower or raise an index by $(g_{\alpha\bar\beta})$ and $(g^{\alpha\bar\beta}) = (g_{\alpha\bar\beta})^{-1}$. A connection ∇ defined by

$$\nabla W_\alpha = \omega_\alpha{}^\beta \otimes W_\beta, \quad \nabla T = 0$$

is called the *pseudohermitian connection* or the *Webster connection* for θ. The functions $A^\alpha{}_\beta$ are called the coefficients of the *torsion tensor* $\boldsymbol{T}$. Let

$$d\omega_\alpha{}^\beta - \omega_\alpha{}^\gamma \wedge \omega_\gamma{}^\beta \equiv R_\alpha{}^\beta{}_{\gamma\bar\sigma}\, \theta^\gamma \wedge \theta^{\bar\sigma} \quad \operatorname{mod} \theta,\ \theta^\gamma \wedge \theta^\sigma,\ \theta^{\bar\gamma} \wedge \theta^{\bar\sigma}.$$

We call $R_\alpha{}^\beta{}_{\gamma\bar\sigma}$ the coefficients of the *Webster curvature tensor* $\boldsymbol{R}$. Contracting indices, we obtain the coefficients $R_{\alpha\bar\beta}$ of the Webster Ricci curvature Ric and the Webster scalar curvature S:

$$R_{\alpha\bar\beta} = R_\gamma{}^\gamma{}_{\alpha\bar\beta}, \quad S = R_{\alpha\bar\beta}\, g^{\alpha\bar\beta}.$$

The norm of the Webster curvature $|\boldsymbol{R}|_\theta$ is defined by

$$|\boldsymbol{R}|_\theta^2 = \sum_{\alpha,\beta,\gamma,\sigma} |R_\alpha{}^\beta{}_{\gamma\bar\sigma}|^2,$$

where the frame is chosen so that $g_{\alpha\bar\beta} = \delta_{\alpha\bar\beta}$. We similarly define the norm of the torsion tensor $|\boldsymbol{T}|_\theta$.

A pseudohermitian structure defines a sub-Riemannian structure. The distance function induced by a sub-Riemannian metric is called the *Carnot–Carathéodory distance* (cf. [Strichartz 1986]). We denote by $B_\theta(x, r)$ the Carnot–Carathéodory ball with respect to the pseudohermitian structure θ of radius $r > 0$ centered at $x \in M$.

The *Heisenberg group* $\mathcal{H}^n$ is a strongly pseudoconvex CR manifold $\mathbb{C}^n \times \mathbb{R}$ with the CR structure whose $H^{1,0}$ bundle is spanned by

$$Z_\alpha = \frac{\partial}{\partial z^\alpha} + i z^{\bar\alpha} \frac{\partial}{\partial t}, \quad \alpha = 1, \ldots, n, \tag{2-1}$$

where $(z, t) = (z^1, \ldots, z^n, t)$ is the standard coordinate system of $\mathbb{C}^n \times \mathbb{R}$. It is well known that $\mathcal{H}^n$ is CR equivalent to the sphere in $\mathbb{C}^{n+1}$ minus a single point. If we put

$$\vartheta_0 = dt - i z^{\bar\alpha}\, dz^\alpha + i z^\alpha\, dz^{\bar\alpha}, \tag{2-2}$$

then it turns out the curvature and torsion tensors vanish identically. The converse also follows from the solution of the Cartan equivalence problem.

Proposition 2.2. *If the curvature and the torsion tensors of a pseudohermitian manifold* (M, θ) *vanish identically, then the pseudohermitian structure of* M *is locally equivalent to that of* $(\mathcal{H}^n, \vartheta_0)$. *If we further assume that* M *is simply connected and complete in the sense that every Carnot–Carathéodory ball is relatively compact in* M, *then* (M, θ) *is globally equivalent to* $(\mathcal{H}^n, \vartheta_0)$, For a given pseudohermitian manifold (M, θ), we can extend the CR structure J to a smooth section of endomorphism $\hat{J}$ on TM by putting $\hat{J}(T) = 0$, where T is the characteristic vector field of θ. Let J_k, $k = 1, 2, \ldots$, and J_0 be strongly pseudoconvex CR structures on M with CR distributions H_k and H_0, respectively. We say that J_k converges to J_0 in the C^l sense ($l = 0, 1, 2, \ldots, \infty$), if there exist pseudohermitian structures θ_k and θ_0 for (M, J_k) and (M, J_0) such that $\theta_k \to \theta_0$ and $\hat{J}_k \to \hat{J}_0$ in the C^l sense as tensors on M.

Pseudoconformal change of structures and the CR Yamabe equation. Let (M, θ) be a $(2n+1)$-dimensional pseudohermitian manifold and let $\tilde{\theta} = e^{2f}\theta$ be a pseudoconformal change, where f is a smooth real-valued function. Let $\{\theta^\alpha\}$ be an admissible coframe for θ satisfying $d\theta = 2i g_{\alpha\bar{\beta}}\, \theta^\alpha \wedge \theta^{\bar{\beta}}$. Then it turns out

$$\tilde{\theta}^\alpha = e^f(\theta^\alpha + i f^\alpha \theta), \quad \alpha = 1, \ldots, n,$$

form an admissible coframe for $\tilde{\theta}$ which satisfies

$$d\tilde{\theta} = 2i g_{\alpha\bar{\beta}}\, \tilde{\theta}^\alpha \wedge \tilde{\theta}^{\bar{\beta}}.$$

Let $R_\alpha{}^\beta{}_{\gamma\bar{\sigma}}$ and $\widetilde{R_\alpha{}^\beta}{}_{\gamma\bar{\sigma}}$ be coefficients of the Webster curvatures for θ and $\tilde{\theta}$ evaluated in the coframes $\{\theta^\alpha\}$ and $\{\tilde{\theta}^\alpha\}$, respectively. Then they are related as

$$\begin{aligned}(2\text{-}3)\quad \widetilde{R_\alpha{}^\beta}{}_{\gamma\bar{\sigma}} = e^{-2f}\big\{R_\alpha{}^\beta{}_{\gamma\bar{\sigma}} - \delta_\alpha{}^\beta(f_{\gamma\bar{\sigma}} + f_{\bar{\sigma}\gamma}) - 2g_{\alpha\bar{\sigma}} f^\beta{}_\gamma - 2f_{\alpha\bar{\sigma}}\delta^\beta{}_\gamma \\ -(f^\beta{}_\alpha + f_\alpha{}^\beta)g_{\gamma\bar{\sigma}} - 4(\delta_\alpha{}^\beta g_{\gamma\bar{\sigma}} + g_{\alpha\bar{\sigma}}\,\delta^\beta{}_\gamma) f^\lambda f_\lambda\big\},\end{aligned}$$

where $f_{\alpha\bar{\beta}}$, $f_\alpha{}^\beta$ and $f^\beta{}_\alpha$ are components of the second covariant derivatives of f of the pseudohermitian manifold (M, θ) (cf. Proposition 4.14 in [Joo and Lee 2015] for the more general case). Contracting indices, we obtain the following transformation formula for the Webster scalar curvatures:

$$(2\text{-}4)\qquad \widetilde{S} = e^{-2f}\big\{S + 2(n+1)\Delta_\theta f - 4n(n+1) f^\lambda f_\lambda\big\},$$

where $\Delta_\theta f = -(f_\alpha{}^\alpha + f_{\bar{\alpha}}{}^{\bar{\alpha}})$. The operator Δ_θ is called the *sublaplacian* for θ.

Let u be a positive smooth function on M defined by $u^{p-2} = e^{2f}$, where $p = 2 + 2/n$. Then (2-4) changes into the following nonlinear equation for u:

$$(2\text{-}5)\qquad L_\theta u := (b_n \Delta_\theta + S)u = \widetilde{S} u^{p-1},$$

where $b_n = 2 + 2/n$ (see [Jerison and Lee 1987; 1989; Lee 1986]). Equation (2-5) is called the *CR Yamabe equation* and the subelliptic linear operator L_θ is called the *CR Laplacian* for θ. The *CR Yamabe problem* is to find a positive smooth function u which makes $\widetilde{S}$ constant.

Let $A^{\alpha}{}_{\bar\beta}$ and $\widetilde{A^{\alpha}}_{\bar\beta}$ be the coefficients of the torsion tensors for θ and $\tilde\theta$ in the coframes $\{\theta^{\alpha}\}$ and $\{\tilde\theta^{\alpha}\}$, respectively. Then in turns out that

$$\widetilde{A^{\alpha}}_{\bar\beta} = e^{-2f}\big(A^{\alpha}{}_{\bar\beta} - i f^{\alpha}{}_{\bar\beta} + 2i f^{\alpha} f_{\bar\beta}\big). \tag{2-6}$$

See [Lee 1986] for details.

Folland–Stein spaces and subelliptic estimates. Roughly speaking, a *normal coordinate system* of a pseudohermitian manifold (M,θ) of dimension $2n+1$ is a local approximation by the standard pseudohermitian structure on the Heisenberg group $(\mathcal{H}^n,\theta_0)$. For $p\in M$, let $W_1,\dots,W_n$ be a local frame defined on a neighborhood V of p such that the coefficients of the Levi form for θ are given by $g_{\alpha\bar\beta}=\delta_{\alpha\bar\beta}$. Such a frame is called a *unitary frame*. We denote by T the characteristic vector field for θ. Let (z,t) be the standard coordinates of $\mathcal{H}^n$ and let $|(z,t)|=(|z|^4+t^2)^{1/4}$ be the Heisenberg group norm. We define Z_α and θ_0 on $\mathcal{H}^n$ as (2-1) and (2-2).

Theorem 2.3 [Folland and Stein 1974]. *There is a neighborhood of the diagonal $\Omega\subset V\times V$ and a C^∞-smooth mapping $\Theta:\Omega\to\mathcal{H}^n$ satisfying*:

(a) *We have $\Theta(\xi,\eta)=-\Theta(\eta,\xi)=\Theta(\eta,\xi)^{-1}$. (In particular, $\Theta(\xi,\xi)=0$.)*

(b) *Let $\Theta_\xi(\eta)=\Theta(\xi,\eta)$. Then Θ_ξ is a diffeomorphism of a neighborhood Ω_ξ of ξ onto a neighborhood of the origin in $\mathcal{H}^n$. Denote by $y=(z,t)=\Theta(\xi,\eta)$ the coordinates of $\mathcal{H}^n$. Denote by O^k $(k=1,2,\dots)$ a C^∞ function f of ξ and y such that for each compact set $K\subset V$, there is a constant C_K with $f(\xi,y)\le C_K|y|^k$ (Heisenberg norm) for $\xi\in K$. Then we have the following approximation formula*:

$$(\Theta_\xi^{-1})^*\theta = \theta_0 + O^1 dt + \sum_{\alpha=1}^{n}(O^2 dz^\alpha + O^2 dz^{\bar\alpha}),$$

$$(\Theta_\xi^{-1})^*(\theta\wedge d\theta^n) = (1+O^1)\,\theta_0\wedge d\theta_0^n,$$

$$\Theta_{\xi *}W_\alpha = Z_\alpha + O^1\mathcal{E}(\partial_z) + O^2\mathcal{E}(\partial_t),$$

$$\Theta_{\xi *}T = \partial/\partial t + O^1\mathcal{E}(\partial_z,\partial_t),$$

$$\Theta_{\xi *}\Delta_\theta = \Delta_{\theta_0} + \mathcal{E}(\partial_z) + O^1\mathcal{E}(\partial_t,\partial_z^2) + O^2\mathcal{E}(\partial_z\partial_t) + O^3\mathcal{E}(\partial_t^2).$$

Here $O^k\mathcal{E}$ indicates an operator involving linear combinations of the indicated derivatives with smooth coefficients in O^k, and we have used ∂_z to denote any of the derivatives $\partial/\partial z^\alpha$, $\partial/\partial z^{\bar\alpha}$.

The smooth map Θ_ξ is called the *Folland–Stein normal coordinates* centered at ξ with respect to the frame $\{W_\alpha\}$. (This coordinate system depends on the choice of local unitary frame. Another construction of pseudohermitian normal coordinates which does not depend on local frames is given in [Jerison and Lee 1989].) Here and in the sequel, we use the term *frame constants* to mean bounds on finitely many derivatives of the coefficients in the $O^k\mathcal{E}$ terms in Theorem 2.3.

Let V be an open neighborhood of a point $p \in M$ with a fixed local unitary frame $W_1, \dots, W_n$ and let U be a relatively compact open neighborhood of p in V such that Ω_ξ in Theorem 2.3 contains $\overline{U}$ for every $\xi \in \overline{U}$. Let $X_\alpha = \operatorname{Re} W_\alpha$ and $X_{\alpha+n} = \operatorname{Im} W_\alpha$ for $\alpha = 1, \dots, n$. For a multi-index $A = (\alpha_1, \dots, \alpha_k)$, with $1 \le \alpha_j \le 2n$, $j = 1, \dots, k$, we denote k by $\ell(A)$ and write $X^A f = X_{\alpha_1} \cdots X_{\alpha_k} f$ for a smooth function f on U. The $S_k^p(U)$-norm of a smooth function f on U is

$$\|f\|_{S_k^p(U)} = \sup_{\ell(A) \le k} \|X^A f\|_{L^p(U)},$$

where $\|g\|_{L^p(U)} = \left(\int_U |g|^p \, \theta \wedge d\theta^n\right)^{1/p}$ is the L^p-norm of g on U with respect to the volume element induced by θ. The completion of $C_0^\infty(U)$ with respect to $\|\cdot\|_{S_k^p(U)}$ is denoted by $S_k^p(U)$.

Hölder type spaces suited to Δ_θ are defined as follows. For $x, y \in U$, let $\rho(x, y) = |\Theta(x, y)|$ (Heisenberg norm). For a positive real number $0 < s < 1$,

$$\Gamma_s(U) = \{f \in C^0(\overline{U}) : |f(x) - f(y)| \le C\rho(x, y)^s \text{ for some constant } C > 0\}.$$

If s is a positive nonintegral real number such that $k < s < k+1$ for some integer $k \ge 1$, then

$$\Gamma_s(U) = \{f \in C^0(\overline{U}) : X^A f \in \Gamma_{s-k}(U),\ \ell(A) \le k\}.$$

Then the $\Gamma_s(U)$-norm for $f \in \Gamma_s(U)$ is defined by

$$\|f\|_{\Gamma_s(U)} = \sup_{x \in U} |f(x)| + \sup\left\{\frac{|X^A f(x) - X^A f(y)|}{\rho(x, y)^{s-k}} : x, y \in U,\ x \ne y,\ \ell(A) \le k\right\}.$$

The function spaces $S_k^p(U)$ and $\Gamma_s(U)$ are called the *Folland–Stein spaces* on U. We denote by $\Lambda_s(U)$ the Euclidean Hölder space when we regard U as a subset of $\mathbb{R}^{2n+1}$.

Theorem 2.4 [Folland and Stein 1974]. *For each positive real number s which is not an integer, each $1 < r < \infty$ and each integer $k \ge 1$, there exists a constant $C > 0$ such that for every $f \in C_0^\infty(U)$,*

(a) $\|f\|_{\Gamma_s(U)} \le C\|f\|_{S_k^r(U)}$, *where* $1/r = (k - s)/(2n + 2)$,

(b) $\|f\|_{\Lambda_{s/2}(U)} \le C\|f\|_{\Gamma_s(U)}$,

(c) $\|f\|_{S_2^r(U)} \le C(\|\Delta_\theta f\|_{L^r(U)} + \|f\|_{L^r(U)})$,

(d) $\|f\|_{\Gamma_{s+2}(U)} \le C(\|\Delta_\theta f\|_{\Gamma_s(U)} + \|f\|_{\Gamma_s(U)})$.

Moreover the constant C depends only on frame constants.

One should notice that the constants C in the theorem above depend on frame constants rather than the pseudohermitian structure itself. Therefore, if $\mathcal{U}$ is a small neighborhood (J_0, θ_0) in the C^∞-topology, then we can choose constants C in Theorem 2.4 which are independent of the choice of $(J, \theta) \in \mathcal{U}$.

If M is compact, we can choose a finite open covering $U_1, \dots, U_m$, each of which is contained in a normal coordinate. Let $\phi_1, \dots, \phi_m$ be a partition of unity subordinate to this covering. Then the spaces of $S_k^p(M)$ and $\Gamma_s(M)$ are defined as spaces of a function u such that $\phi_j u \in S_k^p(U_j)$ or $\phi_j u \in \Gamma_s(U_j)$, respectively, for every $j = 1, \dots, m$.

3. Proof of Theorem 1.3

The proof of Theorem 1.3 is based on the following fundamental fact about the semicontinuity property of compact group actions proved by Ebin [1970] for Theorem 1.1. We denote by Diff(M) the group of C^∞-smooth diffeomorphisms. Recall that the C^∞-topology on Diff(M) is metrizable. We denote a metric inducing the C^∞-topology by d.

Theorem 3.1 ([Ebin 1970]; cf. [Greene et al. 2011; 2013; Grove and Karcher 1973; Kim 1987]). *Let M be a compact C^∞-smooth manifold and let G_k $(k = 1, 2, \dots)$ and G_0 be compact subgroups of* Diff(M). *Suppose $G_j \to G_0$ in the C^∞-topology as $j \to \infty$; that is, for every $\epsilon > 0$, there exists an integer N such that $d(f, G_0) := \inf_{g \in G_0} d(f, g) < \epsilon$ for every $f \in G_j$, whenever $j > N$. Then G_j is isomorphic to a subgroup of G_0 for every sufficiently large j. Moreover, the isomorphism can be obtained by the conjugation by a diffeomorphism ϕ_j of M which converges to the identity map in the C^∞ sense.*

Therefore, it suffices to prove the following proposition for the conclusion of Theorem 1.3.

Proposition 3.2. *Let $\{J_k : k = 1, 2, \dots\}$ be a sequence of strongly pseudoconvex CR structures on a compact manifold M which tends to a strongly pseudoconvex CR structure J_0 as in Theorem 1.3. Suppose that* $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ *is compact. Then* $\mathrm{Aut}_{\mathrm{CR}}(M, J_k)$ *is also compact for every sufficiently large k. Furthermore, every sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k) : k = 1, 2, \dots\}$ admits a subsequence converging to an element $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ in the C^∞ sense.*

We will make use of the solutions of the CR Yamabe problem for the proof of Proposition 3.2. According to the variational approach introduced by Jerison and Lee [1987; 1989], it is very natural to consider the sign of the CR Yamabe invariant defined as follows: Let (M, θ) be a compact pseudohermitian manifold. For a C^∞-smooth real-valued function u, let

$$A(\theta; u) := \int_M u\, L_\theta u\, \theta \wedge d\theta^n = \int_M (b_n |du|_\theta^2 + R u^2)\, \theta \wedge d\theta^n$$

and

$$B(\theta; u) := \int_M |u|^p \, \theta \wedge d\theta^n.$$

Then the *CR Yamabe invariant* $Y(M)$ is defined by

$$Y(M) := \inf\{A(\theta; u) : u \in C^\infty(M),\ B(\theta; u) = 1\}.$$

It is well known that $Y(M)$ does not depend on the choice of contact form θ. Let J_k be a sequence of strongly pseudoconvex CR structures on M tending to a strongly pseudoconvex CR structure J_0 as $k \to \infty$. We denote by Y_k the CR Yamabe invariant of (M, J_k). For the proof, we may assume either that $Y_k \le 0$ for every k or that $Y_k > 0$ for every k.

Case $\boldsymbol{Y_k \le 0}$**.** In this case, we use the minimal solution of the Yamabe problem.

Theorem 3.3 [Jerison and Lee 1987]. *Let M be a compact strongly pseudoconvex CR manifold of dimension* $2n+1$.

(i) $Y(S^{2n+1}) > 0$, *where $Y(S^{2n+1})$ is the CR Yamabe invariant for the sphere S^{2n+1} with the standard structure.*

(ii) $Y(M) \le Y(S^{2n+1})$.

(iii) *If $Y(M) < Y(S^{2n+1})$, then there exists a positive C^∞-smooth function u which satisfies $B(\theta; u) = 1$ and $A(\theta; u) = Y(M)$ for a given pseudohermitian structure θ. This function u satisfies*

$$L_\theta u = Y(M) u^{p-1}.$$

That is, the pseudohermitian structure $\tilde{\theta} = u^{p-2}\theta$ has a constant Webster scalar curvature $\widetilde{R} = Y(M)$.

It is known from [Jerison and Lee 1989] that $Y(M) < Y(S^{2n+1})$ if M is not locally spherical and $2n+1 \ge 5$. The cases that $2n+1 = 3$ or that M is spherical are dealt with in [Gamara 2001; Gamara and Yacoub 2001].

Proposition 3.4 [Jerison and Lee 1987, Theorem 7.1]. *If $Y(M) \le 0$, then a pseudohermitian structure with constant Webster scalar curvature is unique up to constant multiples. As a consequence, there is a unique pseudohermitian structure with constant Webster scalar curvature under the unit volume condition, if $Y(M) \le 0$.*

Proposition 3.5 [Jerison and Lee 1987, Theorem 5.15]. *Let M be a compact strongly pseudoconvex CR manifold of dimension $2n+1$ and let θ be a pseudohermitian structure. Suppose that $f, g \in C^\infty(M)$, $u \ge 0$, $u \in L^r$ for some $r > p = 2 + 2/n$ and*

$$\Delta_\theta u + gu = f u^{q-1}$$

in the distribution sense for some $2 \le q \le p$. *Then* $u \in C^\infty(M)$, $u > 0$. *Furthermore,* $\|u\|_{C^k}$ *depends only on* $\|u\|_{L^r}$, $\|f\|_{C^k}$, $\|g\|_{C^k}$ *and frame constants, but not on* q.

Indeed, a local version of the above lemma is stated in [Jerison and Lee 1987]. But it is obvious it holds globally by taking a partition of unity subordinate to a chart of normal coordinates.

Proposition 3.6 [Jerison and Lee 1987, Proposition 5.5, case $k = 1$, $r = 2$ and $s = p$]. *For a compact pseudohermitian manifold* (M, θ) *of dimension* $2n+1$, *there exists a constant* $C > 0$ *such that*

$$\int_M |v|^p \,\theta \wedge d\theta^n \le C \int_M (|dv|_\theta^2 + |v|^2)\,\theta \wedge d\theta^n$$

for every C^∞*-smooth function* v *on* M.

Since we are considering CR structures converging to the target structure J_0, we can choose also a sequence $\{\theta_k\}$ of contact forms which tends to a target pseudohermitian structure θ_0 in the C^∞ sense. Without loss of generality, we always assume that $\int_M \theta_k \wedge d\theta_k^n = 1$ for every k.

Lemma 3.7. *Suppose that* $Y_k \le 0$ *for every* k. *Let* $u_k > 0$ *be the (unique) solution as in Theorem 3.3(iii) with respect to* (J_k, θ_k). *Then for each nonnegative integer* l, *there exists a constant* C *such that* $\|u_k\|_{C^l} \le C$ *for every* k.

Proof. Since u_k satisfies

$$b_n \Delta_{\theta_k} u_k + R_k u_k = Y_k u_k^{p-1}, \tag{3-1}$$

where R_k is the Webster scalar curvature for θ_k, we have

$$\int_M \tfrac{1}{2}(p-1)\, b_n u_k^{p-2} |du_k|_{\theta_k}\, \theta_k \wedge d\theta_k^n \le \int_M |R_k u_k^p|\, \theta_k \wedge d\theta_k^n$$

by integrating after multiplying by u_k^{p-1} on both sides of (3-1), since $Y_k \le 0$. Therefore, the function $w_k := u_k^{p/2}$ satisfies

$$\int_M |dw_k|_{\theta_k}^2\, \theta_k \wedge d\theta_k^n \le C \int_M w_k^2\, \theta_k \wedge d\theta_k^n = C \int_M u_k^p\, \theta_k \wedge d\theta_k^n = C,$$

since R_k is bounded uniformly for k. Moreover since $(J_k, \theta_k) \to (J_0, \theta_0)$ in the C^∞ sense, Proposition 3.6 implies that there exists a constant $C > 0$ independent of k such that

$$\int_M w_k^p\, \theta_k \wedge d\theta_k^n \le C \int_M (|dw_k|_{\theta_k}^2 + w_k^2)\, \theta_k \wedge d\theta_k^n,$$

which is uniformly bounded for every k. This implies that $\|u_k\|_{L^r}$ is uniformly bounded as $(J_k, \theta_k) \to (J_0, \theta_0)$, where $r = \frac{1}{2}p^2 > p$. Then the conclusion follows from Proposition 3.5, since frame constants for (J_k, θ_k) are also uniformly bounded as $(J_k, \theta_k) \to (J_0, \theta_0)$ in the C^∞ sense. □

If $Y_k \leq 0$ for every $k \geq 1$, then by taking a subsequence, we may assume the sequence $\{u_k\}$ of solutions of the Yamabe problem with respect to (J_k, θ_k) converges to u_0, the solution of the Yamabe problem with respect to (J_0, θ_0) in the C^∞ sense by Lemma 3.7. Replacing θ_k by $u_k^{p-2}\theta_k$, then we may assume the Webster scalar curvature of θ_k is a nonpositive constant for every k. In this case, it is known that the CR automorphism group of (M, J_k) coincides with the pseudohermitian automorphism group for (M, J_k, θ_k). Let g_k be the Riemannian metric on M defined by

$$g_k = \theta_k \otimes \theta_k + d\theta_k(\,\cdot\,, J_k\,\cdot\,)$$

for each k. Then we see that $g_k \to g_0 = \theta_0 \otimes \theta_0 + d\theta_0(\,\cdot\,, J_0\,\cdot\,)$ in the C^∞ sense, and the CR automorphism groups $\mathrm{Aut}_{\mathrm{CR}}(M, J_k)$ and $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ are subgroups of the isometry groups of g_k and g_0, respectively. Then the conclusion follows from the proof of Theorem 1.1.

Case $Y_k > 0$. We will show that if a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$ is divergent, then it generates a single "bubble" which is CR equivalent to (M, J_0). This case should be excluded by proving the CR structure of the bubble is the same as that of the standard sphere, which contradicts the hypothesis that $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact. An essential ingredient for analyzing the bubbling phenomenon is the reparametrization of the pseudohermitian structure by the Green function of the CR Laplacian. The existence of the Green function is guaranteed by the hypothesis $Y_k > 0$ (see, for instance, [Cheng et al. 2014; Gamara 2001]). We discuss the bubbling after the following fundamental lemma on the convergence of CR automorphisms.

Lemma 3.8. *Suppose for a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$, $F_k \to F$ and $F_k^{-1} \to G$ in the C^0 sense for some continuous mappings F and G. Then $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$, $G = F^{-1}$ and $F_k \to F$ in the C^∞ sense.*

Proof. This lemma is a sequential version of Proposition 1.1′ in [Schoen 1995]. Let θ_k and θ_0 be pseudohermitian structures for J_k and J_0, respectively, and suppose $\theta_k \to \theta_0$ in the C^∞ sense. For a given point $p \in M$, let $q_k = F_k(p)$ and $q = F(p)$. Let $q \in \widetilde{U} \Subset \widetilde{V} \Subset \widetilde{W}$ be relatively compact neighborhoods of q. Since $q_k \to q$, we can assume that $q_k \in \widetilde{U}$ for every k. The fact that $Y_k > 0$ implies that the principal eigenvalue of L_{θ_k} on M, and hence the Dirichlet principal eigenvalue of L_{θ_k} on $\widetilde{W}$, is also positive for every k. Then by the local scalar flattening argument of Fischer-Colbrie and Schoen [1980; 1995], we have a positive C^∞-smooth function u_k on $\widetilde{W}$ such that $L_{\theta_k} u_k = 0$ on $\widetilde{W}$ for every k. Multiplying by a positive constant, we may assume that $u_k(q) = 1$ for every k. Then the subelliptic theory in Theorem 2.3 for the sublaplacian and the Harnack principle (cf. Proposition 5.12 in [Jerison and Lee 1987]) imply that $\{u_k\}$ has a convergent subsequence which tends to a positive function u_0 on the closure of $\widetilde{V}$ in the C^∞ sense. We denote the convergent subsequence by $\{u_k\}$ again. Then $\tilde{\theta}_k = u_k^{p-2}\theta_k$ and $\tilde{\theta}_0 = u_0^{p-2}\theta_0$ have the trivial

Webster scalar curvatures on $\widetilde{V}$. From the equicontinuity of the sequence $\{F_k\}$, we can choose a neighborhood W of p such that $F_k(W) \Subset \widetilde{U}$ for every k. Let v_k be a positive smooth function on V defined by $F_k^*\tilde{\theta}_k = v_k^{p-2}\theta_k$. Then for every k, we have

$$L_{\theta_k} v_k = 0 \quad \text{on } W. \tag{3-2}$$

We denote by $\mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U})$ the volume of $\widetilde{U}$ with respect to the volume form $\tilde{\theta}_k \wedge d\tilde{\theta}_k^n$. Since $\tilde{\theta}_k$ converges to $\tilde{\theta}_0$ in the C^∞ sense in $\widetilde{V}$, there exists a uniform bound C of $\mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U})$. Therefore, it turns out that

$$\int_W v_k^p\, \theta_k \wedge d\theta_k^n = \int_W F_k^*(\tilde{\theta}_k \wedge d\tilde{\theta}_k) = \mathrm{Vol}_{\tilde{\theta}_k}(F_k(W)) \le \mathrm{Vol}_{\tilde{\theta}_k}(\widetilde{U}) \le C$$

for every k. Fix a neighborhood $V \Subset W$ of p. Then the subelliptic mean-value inequality for (3-2) implies that there exists a constant C such that $v_k(x) \le C$ for every $x \in V$. We can also choose this C independently on k by the convergence of structures. Then for a given neighborhood $U \Subset V$ of p and for each positive integer l, there exists a constant C_l which is independent of k such that

$$\|v_k\|_{C^l(U)} \le C_l$$

for every k, by Theorem 2.3. Since each F_k is pseudoconformal, the C^l-norm of F_k on U is completely determined by that of v_k and is uniformly bounded on U. This yields that every subsequence of $\{F_k\}$ contains a subsequence converging in the C^l sense, for every positive integer l. Since F_k converges to F in the C^0 sense on M and since M is compact, we conclude that F_k converges to F in the C^∞ sense. By the same reasoning, $F_k^{-1} \to G$ in the C^∞ sense. It follows immediately that $F \in \mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ and $G = F^{-1}$. □

For a CR diffeomorphism $F : (M, \theta) \to (\widetilde{M}, \tilde{\theta})$ between two pseudohermitian manifolds, we denote by $|F'|_{\theta,\tilde{\theta}}$ the pseudoconformal factor of F, that is, $F^*\tilde{\theta} = |F'|_{\theta,\tilde{\theta}}\,\theta$. We abbreviate it to $|F'|_\theta$ in case $(M, \theta) = (\widetilde{M}, \tilde{\theta})$.

Lemma 3.9. *Let (M, θ) and $(\widetilde{M}, \tilde{\theta})$ be pseudohermitian manifolds of the same dimension. Let K be a relatively compact subset of M and suppose that the Webster scalar curvature for $\tilde{\theta}$ vanishes on $\widetilde{M}$. Then there exist constants $r_0 > 0$ and $C > 0$ such that for every CR diffeomorphism F on a Carnot–Carathéodory ball $B_\theta(x, r)$ into M,*

$$B_{\tilde{\theta}}(F(x), C^{-1}\lambda r) \subset F(B_\theta(x, r)) \subset B_{\tilde{\theta}}(F(x), C\lambda r)$$

whenever $x \in K$ and $r \le \frac{1}{2}r_0$, where $\lambda = |F'|_{\theta,\tilde{\theta}}(x)$. The constant C depends only on r_0, K and uniform bounds of finite-order derivatives of the CR and pseudohermitian structures of (M, θ).

This lemma is a restatement of Proposition 2.1′(i) in [Schoen 1995], which is a consequence of the subelliptic Harnack principle.

To prove Proposition 3.2, assume the contrary. Then there exists a sequence $\{F_k \in \mathrm{Aut}_{\mathrm{CR}}(M, J_k)\}$ such that $\sup_{x\in M} |F_k'|_{\theta_k}(x) \to \infty$ as $k \to \infty$, thanks to Lemma 3.8. Let $x_k \in M$ be a point of M with $|F_k'|_{\theta_k}(x_k) = \sup_{x\in M} |F_k'|_{\theta_k}(x)$. Extracting a subsequence, we assume that $x_k \to x_0 \in M$ and $F_k(x_k) \to z_0$ as $k \to \infty$. Choose $r > 0$ small enough that the Carnot–Carathéodory balls satisfy $B_{\theta_k}(x_k, r) \Subset B_{\theta_k}(x_k, 2r) \Subset U$ for each k, where U is a relatively compact neighborhood of x_0 in M, and $2r < r_0$ for r_0 given in Lemma 3.9.

Lemma 3.10. *There exists a subsequence $\{F_{k_j} : j = 1, 2, \dots\}$ of $\{F_k : k = 1, 2, \dots\}$ which admits a point $y_0 \in M$ such that for every compact subset K in $M \setminus \{y_0\}$, there exists $N > 0$ such that $K \subset F_{k_j}(B_{\theta_{k_j}}(x_{k_j}, 2r))$ if $k_j > N$. Moreover, for the subsequence, one can choose the point y_0 independently of $r > 0$ as $r \to 0$.*

Proof. Suppose for every $r > 0$, there exists no sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ such that $d(y_k, F_k(x_k)) > \epsilon$ for any given $\epsilon > 0$, where d is the sub-Riemannian distance induced from θ_0. Then it turns out every sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ converges to z_0. In this case, we just need to put $y_0 = z_0$.

Now suppose for some $r > 0$, there exists a sequence $\{y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))\}$ such that $d(y_k, F_k(x_k)) > \epsilon$ for infinitely many k for some $\epsilon > 0$. Extracting a subsequence, we may assume that $y_k \to y_0 \in M$ and $d(y_k, F_k(x_k)) > \epsilon$ for every k so that the sequence $\{F_k(x_k)\}$ is relatively compact in $M \setminus \{y_0\}$. Let G_k be the Green function for L_{θ_k} with pole at y_k. We normalize G_k by the condition $\min_{M\setminus\{y_k\}} G_k = 1$. Since each $G_k > 0$ and $L_{\theta_k} G_k = 0$ on $M \setminus \{y_k\}$, we may assume $\{G_k : k = 1, 2, \dots\}$ converges to a positive function G_0 on $M \setminus \{y_0\}$ in the local C^∞ sense, by extracting a subsequence if necessary. Let $\tilde{\theta}_k = G_k^{p-2}\theta_k$. Then $\tilde{\theta}_k$ is a pseudohermitian structure on $M \setminus \{y_k\}$ which is Webster scalar flat. Therefore, if we denote $\lambda_k = |F_k'|_{\theta_k, \tilde{\theta}_k}(x_k)$, then Lemma 3.9 implies that there exists a constant C independent of k such that

$$B_{\tilde{\theta}_k}(F_k(x_k), C^{-1}\lambda_k r) \subset F_k(B_{\theta_k}(x_k, r)) \subset B_{\tilde{\theta}_k}(F_k(x_k), C\lambda_k r).$$

Since $G_k \geq 1$ and $|F_k'|_{\theta_k}(x_k) \to \infty$, λ_k also tends to infinity as $k \to \infty$. Therefore, a relatively compact subset K in $M \setminus \{y_0\}$ should be included in $F_k(B_{\theta_k}(x_k, r))$ for every sufficiently large k, since $F_k(x_k)$ lies on a fixed relatively compact subset of $M \setminus \{y_0\}$ and $\tilde{\theta}_k \to \tilde{\theta}_0 = G_0^{p-2}\theta_0$ in the local C^∞-smooth sense on $M \setminus \{y_0\}$. Note that the choice of the sequence $\{y_k\}$ and y_0 still works for every $r' \leq r$. This yields the independence of y_0 on r as $r \to 0$. □

As a consequence of Lemma 3.10, it turns out that $M \setminus \{y_0\}$ is simply connected and complete with respect to the sub-Riemannian distance induced by $\tilde{\theta}_0$. In fact, any loop in $M \setminus \{y_0\}$ is contained in $F_k(B_{\theta_k}(x_k, 2r))$ for some large k by Lemma 3.10. Since $B_{\theta_k}(x_k, 2r)$ is simply connected if $r > 0$ is small enough and since F_k is a diffeomorphism, $F_k(B_{\theta_k}(x_k, 2r))$ is simply connected as well. Therefore, the given loop should be contractible. This shows that $M \setminus \{y_0\}$ is simply connected.

Extracting a subsequence, we assume that Lemma 3.10 holds for the entire sequence $\{F_k\}$. Choose $y_k \in M \setminus F_k(B_{\theta_k}(x_k, 2r))$ which tends to y_0. Let v_k and f_k be real-valued functions on $B_{\theta_k}(x_k, 2r)$ defined by

$$v_k^{p-2} = |F_k'|_{\theta_k, \tilde{\theta}_k} = e^{2f_k},$$

where G_k is the normalized Green function for L_{θ_k} with pole at y_k which converges to a positive function G_0 in the local C^∞-smooth sense on $M \setminus \{y_0\}$ as $k \to \infty$, and $\tilde{\theta}_k = G_k^{p-2}\theta_k$. Since $L_{\theta_k} v_k = 0$, we see that there exists a constant C independent of k such that $|F_k'|_{\theta_k, \tilde{\theta}_k} \geq C\lambda_k$ on $B_{\theta_k}(x_k, r)$ by the Harnack principle, where $\lambda_k = |F_k'|_{\theta_k, \tilde{\theta}_k}(x_k)$. Let $\{Z_k \in \Gamma(H_k^{1,0})\}$ be a sequence of vector fields on U which tends to $Z_0 \in \Gamma(H_0^{1,0})$ as $k \to \infty$, where $H_k^{1,0}$ represents the $(1, 0)$-bundle with respect to J_k. Since $f_k = (1/n) \log v_k$, we have

$$Z_k f_k = \frac{Z_k v_k}{n v_k}$$

for every k. Since $L_{\theta_k} v_k = 0$ on $B_{\theta_k}(x_k, 2r)$, the subelliptic estimates in Theorem 2.4 imply that $Z_k f_k$ is uniformly bounded on $B_{\theta_k}(x_k, r)$ for every k. So is $\bar{Z}_k f_k$, and if W_k is another sequence of vector fields, then $Z_k W_k f_k$ and $Z_k \overline{W}_k f_k$ are all uniformly bounded on $B_{\theta_k}(x_k, r)$ as $k \to \infty$. Therefore, if we denote by $\boldsymbol{R}_k$ and $\widetilde{\boldsymbol{R}}_k$ the Webster curvature tensors for θ_k and $\tilde{\theta}_k$, respectively, then (2-3) implies that

$$|\widetilde{\boldsymbol{R}}_k|^2_{\tilde{\theta}_k}(F_k(x)) \leq C\lambda_k^{-2}\bigl\{|\boldsymbol{R}_k|^2_{\theta_k}(x) + A_k|\boldsymbol{R}_k|_{\theta_k}(x) + B_k\bigr\}$$

for every $x \in B_{\theta_k}(x_k, r)$, where A_k and B_k are some functions of the first and second covariant derivatives of f_k with respect to the pseudohermitian structure θ_k which are uniformly bounded on $B_{\theta_k}(x_k, r)$ as $k \to \infty$. Since $\lambda_k \to \infty$ and $|\boldsymbol{R}_k|_{\theta_k}$ is uniformly bounded on $B_{\theta_k}(x_k, r)$ for every k, it turns out that $|\widetilde{\boldsymbol{R}}_k|_{\tilde{\theta}_k} \to 0$ uniformly on every compact subset of $M \setminus \{y_0\}$ by Lemma 3.10. Therefore, we see that the pseudohermitian manifold $(M \setminus \{y_0\}, \tilde{\theta}_0)$ has trivial Webster curvature. A similar argument using (2-6) implies that the torsion tensor of $\tilde{\theta}_0$ is also trivial. Therefore, we can conclude that $(M \setminus \{y_0\}, \tilde{\theta}_0)$ is equivalent to the standard pseudohermitian structure of the Heisenberg group and therefore, (M, J_0) is CR equivalent to the sphere by the removable singularity theorem. This contradicts the hypothesis that $\mathrm{Aut}_{\mathrm{CR}}(M, J_0)$ is compact and hence yields the conclusion of Proposition 3.2.

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GRADIENT ESTIMATES FOR A NONLINEAR LICHNEROWICZ EQUATION UNDER GENERAL GEOMETRIC FLOW ON COMPLETE NONCOMPACT MANIFOLDS

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We study gradient estimates for positive solutions to the nonlinear parabolic equation

$$\frac{\partial u}{\partial t} = \Delta u + c u^{-\alpha}$$

under general geometric flow on complete noncompact manifolds, where α, c are two real constants and $\alpha > 0$. As an application, we give the corresponding Harnack inequality.

1. Introduction

Recently, there has been active interest in the study of gradient estimates for partial differential equations on noncompact manifolds. Wu [2010] gave a local Li–Yau type gradient estimate for positive solutions to a general nonlinear parabolic equation

$$u_t = \Delta u - \nabla\varphi \nabla u - au \log u - qu$$

in $M \times [0, \tau]$, where $a \in R$, φ is a C^2-smooth function and $q = q(x, t)$ is a function, which generalizes many previous well-known gradient estimates. Zhu [2011] investigated the fast diffusion equation

$$u_t = \Delta u^{\alpha} \quad (0 < \alpha < 1). \tag{1-1}$$

Theorem 1.1 [Zhu 2011]. *Let M be a Riemannian manifold of dimension $n \geq 2$ with $\operatorname{Ric} M \geq -k$ for some $k \geq 0$. Suppose that $v = -(\alpha/(\alpha-1))u^{\alpha-1}$ is any positive solution to (1-1) in $Q_{R,T} \equiv B(x_0, R) \times [t_0 - T, t_0] \subset M \times (-\infty, \infty)$. Suppose also that $v \leq \widetilde{M}$ in $Q_{R,T}$. Then there exists a constant $C = C(\alpha, M)$ such that*

$$\frac{|\nabla v|}{v^{1/2}} \leq C \widetilde{M}^{1/2} \left(\frac{1}{R} + \frac{1}{\sqrt{T}} + \sqrt{k} \right)$$

in $Q_{\frac{R}{2}, \frac{T}{2}}$.

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Later, Huang and Li [2014] studied the generalized equation

$$u_t = \Delta_f u^\alpha \quad (\alpha > 0)$$

on Riemannian manifolds and got some interesting gradient estimates, where f is a smooth function and Δ_f is defined by

$$\Delta_f = \Delta - \nabla f \cdot \nabla.$$

For the elliptic case, Zhang and Ma [2011] considered the equation

$$\Delta_f u + cu^{-\alpha} = 0 \quad (\alpha > 0) \tag{1-2}$$

on complete noncompact manifolds when the constant N is finite and the N-Bakry–Émery Ricci tensor is bounded from below, obtaining the following gradient estimate.

Theorem 1.2 [Zhang and Ma 2011]. *Suppose (M, g) is a complete noncompact n-dimensional Riemannian manifold with N-Bakry–Émery Ricci tensor bounded from below by the constant $-K =: -K(2R)$, where $R > 0$ and $K(2R) > 0$ in the metric ball $B_{2R}(p)$ around $p \in M$. Let u be a positive solution of (1-2). Then*

(1) *if $c > 0$, we have*

$$\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \le \frac{(N+n)(N+n+2)c_1^2}{R^2} + \frac{(N+n)\big((N+n-1)c_1 + c_2\big)}{R^2} + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + 2(N+n)K,$$

(2) *if $c < 0$, we have*

$$\begin{aligned}\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} \le \big(A+\sqrt{A}\big)|c|\Big(\inf_{B_p(2R)} u\Big)^{-\alpha-1} + \frac{(N+n)\big((N+n-1)c_1+c_2\big)}{R^2} \\ + \frac{(N+n)c_1^2}{R^2}\left(n+N+2+\frac{n+N}{2\sqrt{A}}\right) \\ + \frac{(N+n)\sqrt{(N+n)K}c_1}{R} + \left(2+\frac{1}{\sqrt{A}}\right)(n+N)K,\end{aligned}$$

where $A = (N+n)(\alpha+1)(\alpha+2)$ and c_1, c_2 are absolute positive constants.

For interesting gradient estimates on manifolds with fixed metric, see [Chen and Chen 2009; 2010; Li 2005; Ma 2006; 2010; Zhao 2013; 2014].

However, in the above works, the authors considered gradient estimates for positive solutions to nonlinear equations on complete noncompact manifolds with fixed metric, so it is natural to ask how gradient estimates vary if the metric on a manifold evolves with time. In Perelman's breakthrough work [2002] on the

Poincaré conjecture, the author showed the gradient estimate for the fundamental solution of the conjugate heat equation

$$\Delta u - Ru + \partial_t u = 0$$

under Ricci flow on a closed Riemannian manifold M, where R is the scalar curvature. Since then, a large amount work has been done to study gradient estimates along geometric flow for the solution of the nonlinear equation. Kuang and Zhang [2008] established the corresponding pointwise gradient estimate. For the heat equation under Ricci flow, Liu [2009] got first-order gradient estimates for its positive solutions and derived Harnack inequalities and second-order gradient estimates. Later, Sun [2011] extended it to general geometric flow.

Since gradient estimates often lead to Liouville type theorems and Harnack inequalities, which played an important role in the proof of the Poincaré conjecture, for nonlinear heat equations on manifolds, to get good control of suitable Harnack quantities (depending on nonlinear terms), one may need the key lower bound assumption about Ricci curvature. The results of Theorem 1.2 are about gradient estimates for the elliptic equation (1-2). In this paper, we will extend these results to the parabolic variant of the problem. Thus, we consider the equation

$$\frac{\partial u}{\partial t} = \Delta u + cu^{-\alpha} \tag{1-3}$$

on complete noncompact manifolds M with evolving metric, where α, c are two real constants and $\alpha > 0$. The motivation for this paper is that (1-3) can be viewed as a simple parabolic Lichnerowicz equation. It is well known that the Lichnerowicz equation arises from the Hamiltonian constraint equation for the Einstein-scalar field. Since (1-3) contains a negative power nonlinearity, it is interesting to discuss gradient estimates for it.

We state our main results about (1-3) as follows.

Theorem 1.3. *Let $(M, g(t))$ be a smooth one-parameter family of complete Riemannian manifolds evolving by*

$$\frac{\partial}{\partial t} g = 2h \tag{1-4}$$

for t in some time interval $[0, T]$. Let M be complete under the initial metric $g(0)$. Given $x_0 \in M$ and $R > 0$, let u be a positive solution to the nonlinear equation

$$\frac{\partial u}{\partial t} = \Delta u + cu^{-\alpha}$$

in the cube $Q_{2R,T} := \{(x, t) \mid d(x, x_0, t) \le 2R,\ 0 \le t \le T\}$. Suppose that there exist constants $K_1, K_2, K_3, K_4 \ge 0$ such that

$$\mathrm{Ric} \ge -K_1 g, \quad -K_2 g \le h \le K_3 g, \quad |\nabla h| \le K_4$$

on $Q_{2R,T}$. Then for $(x,t) \in Q_{R,T}$ and positive constants c_1, c_2,

(1) *if $c<0$ and for a positive constant $\widetilde{M}$, $u^{-(\alpha+1)} \le \widetilde{M}$ for all $(x,t) \in M \times [0,T]$, we have*

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \le H_1 + H_2 + \frac{n}{\beta}\frac{1}{t},$$

where

$$H_1 = \frac{n}{\beta}\left(\frac{(n-1)\left(1+\sqrt{K_1}R\right)c_1^2+c_2+2c_1^2}{R^2} + \sqrt{c_3}K_2 - c(\alpha+1)\widetilde{M} + \frac{nc_1^2}{2\beta(1-\beta)R^2}\right),$$

$$H_2 = \left(\frac{n^2}{4\beta^2(1-\beta)^2}\left(2\beta K_1 + 2(1-\beta)K_3 - c(\beta+\alpha)(\alpha+1)\widetilde{M} + \tfrac{3}{2}K_4\right)^2 + \frac{n^2}{\beta}\left((K_2+K_3)^2 + \tfrac{3}{2}K_4\right)\right)^{\frac{1}{2}},$$

(2) *if $c>0$, we have*

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \le \widetilde{H}_1 + \widetilde{H}_2 + \frac{n}{\beta}\frac{1}{t},$$

where

$$\widetilde{H}_1 = \frac{n}{\beta}\left(\frac{(n-1)\left(1+\sqrt{K_1}R\right)c_1^2+c_2+2c_1^2}{R^2} + \sqrt{c_3}K_2 + \frac{nc^2}{2\beta(1-\beta)R^2}\right),$$

$$\widetilde{H}_2 = \left(\frac{n^2}{4\beta^2(1-\beta)^2}\left(2\beta K_1 + 2(1-\beta)K_3 + \tfrac{3}{2}K_4\right)^2 + \frac{n^2}{\beta}\left((K_2+K_3)^2 + \tfrac{3}{2}K_4\right)\right)^{\frac{1}{2}}.$$

Here $0<\beta<1$, c_1, c_2, c_3 are positive constants.

Remark. In fact, our result is the parabolic version of Theorem 1.2 under the evolving metric.

Letting $R \to \infty$, we can get the following global gradient estimate for the nonlinear parabolic equation (1-3).

Corollary 1.4. *Let $(M, g(0))$ be a complete noncompact Riemannian manifold without boundary, and suppose $g(t)$ evolves by (1-4) for $t \in [0,T]$ and satisfies*

$$\mathrm{Ric} \ge -K_1 g, \quad -K_2 g \le h \le K_3 g, \quad |\nabla h| \le K_4.$$

If u is a positive solution to (1-3), then for $(x,t) \in M \times [0,T]$,

(1) *if $c<0$ and $u^{-(\alpha+1)} \le \widetilde{M}$ for all $(x,t) \in M \times [0,T]$, we have*

$$\beta \frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \le \overline{H}_1 + H_2 + \frac{n}{\beta}\frac{1}{t},$$

where

$$\overline{H}_1 = \sqrt{c_3}K_2 - \frac{n}{\beta}c(\alpha+1)\widetilde{M},$$

(2) *if* $c > 0$, *we have*

$$\beta\frac{|\nabla u|^2}{u^2} + cu^{-(\alpha+1)} - \frac{u_t}{u} \le \widehat{H}_1 + \widetilde{H}_2 + \frac{n}{\beta}\frac{1}{t},$$

where $\widehat{H}_1 = \sqrt{c_3}K_2$ *and* H_2, $\widetilde{H}_2$ *are the same as in Theorem 1.3.*

As an application, we get the following Harnack inequality.

Theorem 1.5. *Let* $(M, g(0))$ *be a complete noncompact Riemannian manifold without boundary, and suppose* $g(t)$ *evolves by (1-4) for* $t \in [0, T]$ *and satisfies*

$$\mathrm{Ric} \ge -K_1 g, \quad -K_2 g \le h \le K_3 g, \quad |\nabla h| \le K_4.$$

Let u *be a positive solution to (1-3) with* $u^{-(\alpha+1)} \le \widetilde{M}$ *for all* $(x, t) \in M \times (0, T]$. *Then for any points* (x_1, t_1) *and* (x_2, t_2) *on* $M \times (0, T]$ *with* $0 < t_1 < t_2$, *we have the following Harnack inequality*:

(1) *if* $c < 0$, *we have*

$$u(x_1, t_1) \le u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^{n/\beta} e^{\varphi(x_1,x_2,t_1,t_2)+(\overline{H}_1+H_2)(t_2-t_1)},$$

(2) *if* $c > 0$, *we have*

$$u(x_1, t_1) \le u(x_2, t_2)\left(\frac{t_2}{t_1}\right)^{n/\beta} e^{\varphi(x_1,x_2,t_1,t_2)+(c\widetilde{M}+\widehat{H}_1+\widetilde{H}_2)(t_2-t_1)},$$

where $\varphi(x_1, x_2, t_1, t_2) = \inf_\gamma \int_{t_1}^{t_2} \frac{1}{4\beta}|\dot\gamma|^2\,dt$ *and* γ *is any spacetime path joining* (x_1, t_2) *and* (x_2, t_2).

2. Proof of Theorem 1.3

Let u be a positive solution to (1-3). Set $w = \log u$; then w satisfies

$$w_t = \Delta w + |\nabla w|^2 + ce^{-w(\alpha+1)}. \tag{2-1}$$

Lemma 2.1 [Sun 2011]. *Suppose the metric evolves by (1-4). Then, for any smooth function* w, *we have*

$$\frac{\partial}{\partial t}|\nabla w|^2 = -2h(\nabla w, \nabla w) + 2\nabla w \nabla w_t$$

and

$$\frac{\partial}{\partial t}\Delta w = \Delta\frac{\partial}{\partial t}w - 2h\nabla^2 w - 2\nabla w\big(\mathrm{div}\, h - \tfrac{1}{2}\nabla(\mathrm{tr}_g\, h)\big),$$

where $\mathrm{div}\, h$ *is the divergence of* h.

Lemma 2.2. *Assume $(M, g(t))$ satisfies the hypotheses of Theorem 1.3. We have*

$$\left(\Delta-\frac{\partial}{\partial t}\right)F \geq -2\nabla w\nabla F+t\left(\frac{\beta}{n}\big(|\nabla w|^2+c(\alpha+1)e^{-w(\alpha+1)}-w_t\big)^2 + \big((\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}+2(\beta-1)K_3-2\beta K_1-\tfrac{3}{2}K_4\big)|\nabla w|^2 - n\left(\frac{1}{\beta}(K_2+K_3)^2+\tfrac{3}{2}K_4\right)\right) + c(\alpha+1)e^{-w(\alpha+1)}F-\frac{F}{t},$$

where $F = t\big(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t\big)$ and $0 < \beta < 1$.

Proof. Define $F = t\big(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t\big)$. It is well known that for the Ricci tensor, we have the Bochner formula:

$$\Delta|\nabla w|^2 \geq 2|\nabla^2 w|^2 + 2\nabla w\nabla(\Delta w) - 2K_1|\nabla w|^2.$$

Noting that

$$\begin{aligned}\Delta w_t &= (\Delta w)_t+2h\nabla^2 w+2\nabla w\big(\operatorname{div} h-\tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\\ &= -(|\nabla w|^2)_t+c(\alpha+1)e^{-w(\alpha+1)}w_t+w_{tt}+2h\nabla^2 w+2\nabla w\big(\operatorname{div} h-\tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\\ &= 2h(\nabla w,\nabla w)-2\nabla w\nabla w_t+c(\alpha+1)e^{-w(\alpha+1)}w_t\\ &\qquad +w_{tt}+2h\nabla^2 w+2\nabla w\big(\operatorname{div} h-\tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big),\end{aligned}$$

$$\Delta w = -|\nabla w|^2-ce^{-w(\alpha+1)}+w_t = \left(1-\frac{1}{\beta}\right)(-ce^{-w(\alpha+1)}+w_t)-\frac{F}{\beta t},$$

we have

$$\begin{aligned}\Delta F &= t\big(\beta\Delta|\nabla w|^2 + c\Delta e^{-w(\alpha+1)} - \Delta w_t\big)\\ &= t(\beta\Delta|\nabla w|^2) + tc\big((\alpha+1)^2e^{-w(\alpha+1)}|\nabla w|^2 - (\alpha+1)e^{-w(\alpha+1)}\Delta w\big) - t\Delta w_t\\ &\geq t\Bigg(2\beta|\nabla^2 w|^2 + 2\beta\nabla w\nabla(\Delta w) - 2K_1\beta|\nabla w|^2 + c(\alpha+1)^2e^{-w(\alpha+1)}|\nabla w|^2\\ &\qquad - c(\alpha+1)e^{-w(\alpha+1)}\left(\left(1-\frac{1}{\beta}\right)(-ce^{-w(\alpha+1)}+w_t)-\frac{F}{\beta t}\right) + (|\nabla w|^2)_t\\ &\qquad -c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt} - 2h\nabla^2 w - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\Bigg)\\ &= t\Bigg(2\beta|\nabla^2 w|^2 - \frac{2}{t}\nabla w\nabla F + 2\beta\nabla w\nabla w_t - 2h(\nabla w,\nabla w)\\ &\qquad + \big((2\beta+\alpha-1)c(\alpha+1)e^{-w(\alpha+1)} - 2K_1\beta\big)|\nabla w|^2\\ &\qquad + c^2(\alpha+1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} + c\left(\frac{1}{\beta}-2\right)(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt}\\ &\qquad - 2h\nabla^2 w - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big) + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta t}\Bigg),\end{aligned}$$

and by Lemma 2.1, we get

$$
\begin{aligned}
F_t &= \big(\beta|\nabla w|^2 + ce^{-w(\alpha+1)} - w_t\big) + t\big(\beta(|\nabla w|^2)_t - c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt}\big) \\
&= \frac{F}{t} + t\big(2\beta\nabla w\nabla w_t - 2\beta h(\nabla w,\nabla w) - c(\alpha+1)e^{-w(\alpha+1)}w_t - w_{tt}\big).
\end{aligned}
$$

Therefore, it follows that

$$
\begin{aligned}
&\left(\Delta - \frac{\partial}{\partial t}\right)F \\
&\quad\geq -2\nabla w\nabla F + t\bigg(2\beta|\nabla^2 w|^2 + \big((2\beta+\alpha-1)c(\alpha+1)e^{-w(\alpha+1)}\big)|\nabla w|^2 \\
&\qquad\qquad + c^2(\alpha+1)\frac{\beta-1}{\beta}e^{-2w(\alpha+1)} - \frac{\beta-1}{\beta}c(\alpha+1)e^{-w(\alpha+1)}w_t \\
&\qquad\qquad - 2h\nabla^2 w + 2(\beta-1)K_3|\nabla w|^2 \\
&\qquad\qquad - 2K_1\beta|\nabla w|^2 - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\bigg) \\
&\qquad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&\quad= -2\nabla w\nabla F + t\bigg(2\beta|\nabla^2 w|^2 \\
&\qquad\qquad + \big((\beta-1)c(\alpha+1)e^{-w(\alpha+1)}\big)\Big(|\nabla w|^2 + \frac{ce^{-w(\alpha+1)}}{\beta} - \frac{1}{\beta}w_t\Big) \\
&\qquad\qquad + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 + 2(\beta-1)K_3|\nabla w|^2 \\
&\qquad\qquad - 2K_1\beta|\nabla w|^2 - 2h\nabla^2 w - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\bigg) \\
&\qquad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&\quad= -2\nabla w\nabla F + t\bigg(2\beta|\nabla^2 w|^2 + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 \\
&\qquad\qquad + (\beta-1)c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta t} + 2(\beta-1)K_3|\nabla w|^2 \\
&\qquad\qquad - 2K_1\beta|\nabla w|^2 - 2h\nabla^2 w - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\bigg) \\
&\qquad + c(\alpha+1)e^{-w(\alpha+1)}\frac{F}{\beta} - \frac{F}{t} \\
&\quad= -2\nabla w\nabla F + t\bigg(2\beta|\nabla^2 w|^2 + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 \\
&\qquad\qquad + 2(\beta-1)K_3|\nabla w|^2 - 2K_1\beta|\nabla w|^2 \\
&\qquad\qquad - 2h\nabla^2 w - 2\nabla w\big(\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)\big)\bigg) \\
&\qquad + c(\alpha+1)e^{-w(\alpha+1)}F - \frac{F}{t}.
\end{aligned}
$$

By our assumption, we have

$$-(K_2+K_3)g \le h \le (K_2+K_3)g,$$

which implies that

$$|h|^2 \le (K_2+K_3)^2|g|^2 = n(K_2+K_3)^2.$$

Applying those bounds and Young's inequality yields

$$|h\nabla^2 w| \le \frac{\beta}{2}|\nabla^2 w|^2 + \frac{1}{2\beta}|h|^2 \le \frac{\beta}{2}|\nabla^2 w|^2 + \frac{n}{2\beta}(K_2+K_3)^2.$$

On the other hand,

$$|\operatorname{div} h - \tfrac{1}{2}\nabla(\operatorname{tr}_g h)| = |g^{ij}\nabla_i h_{jl} - \tfrac{1}{2}g^{ij}\nabla_l h_{ij}| \le \tfrac{3}{2}|g||\nabla h| \le \tfrac{3}{2}\sqrt{n}K_4.$$

Finally, with the help of the inequality

$$|\nabla^2 w|^2 \ge \frac{1}{n}(\operatorname{tr}\nabla^2 w)^2 = \frac{1}{n}(\Delta w)^2 = \frac{1}{n}\big(|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t\big)^2,$$

we get

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)F &\\ \ge -2\nabla w\nabla F + t\Big(&\frac{\beta}{n}|\Delta w|^2 + (\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}|\nabla w|^2 + 2(\beta-1)K_3|\nabla w|^2 \\ &- 2K_1\beta|\nabla w|^2 - \frac{n}{\beta}(K_2+K_3)^2 - 3\sqrt{n}K_4|\nabla w|\Big) \\ &+ c(\alpha+1)e^{-w(\alpha+1)}F - \frac{F}{t}.\end{aligned}$$

Since

$$3\sqrt{n}K_4|\nabla w| \le 3K_4\big(\tfrac{1}{2}n + \tfrac{1}{2}|\nabla w|^2\big),$$

we have

$$\begin{aligned}\Big(\Delta - \frac{\partial}{\partial t}\Big)F &\\ \ge -2\nabla w\nabla F + t\Big(&\frac{\beta}{n}\big(|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t\big)^2 \\ &+ \big((\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)} + 2(\beta-1)K_3 - 2\beta K_1 - \tfrac{3}{2}K_4\big)|\nabla w|^2 \\ &- n\Big(\frac{1}{\beta}(K_2+K_3)^2 + \tfrac{3}{2}K_4\Big)\Big) \\ &+ c(\alpha+1)e^{-w(\alpha+1)}F - \frac{F}{t}.\end{aligned}$$

This completes the proof of Lemma 2.2. □

Remark. If the general geometric flow is Ricci flow, that is, if $h = \mathrm{const} \cdot \mathrm{Ric}\, g$, the term $\operatorname{div} h - \frac{1}{2}\nabla(\mathrm{tr}_g\, h)$ in the above proof will vanish.

We take a C^2 cutoff function $\tilde{\varphi}$ defined on $[0, \infty)$ such that $\tilde{\varphi}(r) = 1$ for $r \in [0, 1]$, $\tilde{\varphi}(r) = 0$ for $r \in [2, \infty)$, and $0 \le \tilde{\varphi}(r) \le 1$. Furthermore $\tilde{\varphi}$ satisfies

$$-\frac{\tilde{\varphi}'(r)}{\tilde{\varphi}^{1/2}(r)} \le c_1$$

and

$$\tilde{\varphi}''(r) \ge -c_2$$

for two constants $c_1, c_2 > 0$. Set

$$\varphi(x,t) = \tilde{\varphi}\left(\frac{r(x,t)}{R}\right),$$

where $r(x,t) = d(x, x_0, t)$. Using an argument of Calabi [1958], we can assume $\varphi(x,t) \in C^2(M)$ with support in $Q_{2R,T}$. Direct calculation shows that on $Q_{2R,T}$

$$\frac{|\nabla\varphi|^2}{\varphi} \le \frac{c_1^2}{R^2}. \tag{2-2}$$

By the Laplacian comparison theorem in [Aubin 1982],

$$\Delta\varphi \ge -\frac{(n-1)\bigl(1+\sqrt{K_1}\,R\bigr)c_1^2 + c_2}{R^2}. \tag{2-3}$$

For any $0 < T_1 < T$, let (x_0, t_0) be a point in the cube Q_{2R,T_1} at which φF attains its maximum value. We can assume that this value is positive (otherwise the proof is trivial). At the point (x_0, t_0), we have

$$\nabla(\varphi F) = 0, \quad \Delta(\varphi F) \le 0, \quad (\varphi F)_t \ge 0.$$

It follows that

$$0 \ge \left(\Delta - \frac{\partial}{\partial t}\right)(\varphi F) = (\Delta\varphi)F - \varphi_t F + \varphi\left(\Delta - \frac{\partial}{\partial t}\right)F + 2\nabla\varphi\nabla F.$$

By [Sun 2011, p. 494], we know there exists a positive constant c_3 such that

$$-\varphi_t F \ge -\sqrt{c_3}\,K_2 F.$$

So we obtain

$$\varphi\left(\Delta - \frac{\partial}{\partial t}\right)F + F\Delta\varphi - \varphi_t F - 2F\varphi^{-1}|\nabla\varphi|^2 \le 0.$$

This inequality, together with the inequalities (2-2) and (2-3), yields

$$\varphi\left(\Delta - \frac{\partial}{\partial t}\right)F \le AF,$$

where

$$A=\frac{\big((n-1)\big(1+\sqrt{K_1}R\big)\big)c_1^2+c_2+2c_1^2}{R^2}+\sqrt{c_3}K_2.$$

At (x_0,t_0), by Lemma 2.2, we have

$$\begin{aligned}
0&\ge\varphi\Big(\Delta-\frac{\partial}{\partial t}\Big)F-AF\\
&\ge -AF+\varphi\Big(-\frac{F}{t_0}+c(\alpha+1)e^{-w(\alpha+1)}F+\frac{\beta t_0}{n}\big(|\nabla w|^2+c(\alpha+1)e^{-w(\alpha+1)}-w_t\big)^2\\
&\qquad -2\nabla w\nabla F\\
&\qquad +\big((\beta+\alpha)c(\alpha+1)e^{-w(\alpha+1)}+2(\beta-1)K_3-2\beta K_1-\tfrac{3}{2}K_4\big)|\nabla w|^2t_0\\
&\qquad -n\Big(\frac{1}{\beta}(K_2+K_3)^2+\tfrac{3}{2}K_4\Big)t_0\Big).
\end{aligned}$$

(1) If $c<0$ and $u^{-(\alpha+1)}\le\widetilde{M}$ for all $(x,t)\in M\times[0,T]$, we have

$$\begin{aligned}
0&\ge -AF-\varphi\frac{F}{t_0}+\varphi\frac{\beta t_0}{n}\big(|\nabla w|^2+c(\alpha+1)e^{-w(\alpha+1)}-w_t\big)^2-2\varphi\nabla w\nabla F\\
&\quad +c\widetilde{M}(\alpha+1)\varphi F+\big((\beta+\alpha)c(\alpha+1)\widetilde{M}+2(\beta-1)K_3-2\beta K_1-\tfrac{3}{2}K_4\big)|\nabla w|^2\varphi t_0\\
&\quad -\Big(\frac{1}{\beta}(K_2+K_3)^2+\tfrac{3}{2}K_4\Big)n\varphi t_0.
\end{aligned}$$

Set

$$\widetilde{C}_1=-(\beta+\alpha)c(\alpha+1)\widetilde{M}+2(1-\beta)K_3+2\beta K_1+\tfrac{3}{2}K_4$$

and

$$\widetilde{C}_2=\frac{1}{\beta}(K_2+K_3)^2+\tfrac{3}{2}K_4.$$

Multiplying by φt_0 on both sides of the above inequality, we get

$$\begin{aligned}
0&\ge -A\varphi t_0F-\varphi F+2t_0F\varphi\nabla w\nabla\varphi+c\widetilde{M}(\alpha+1)\varphi Ft_0\\
&\quad -\widetilde{C}_1|\nabla w|^2\varphi^2t_0^2-\widetilde{C}_2n\varphi^2t_0^2+\varphi^2t_0^2\frac{\beta t_0}{n}\big(|\nabla w|^2+c(\alpha+1)e^{-w(\alpha+1)}-w_t\big)^2\\
&\ge\varphi F\big(-At_0-1+c\widetilde{M}(\alpha+1)t_0\big)-\frac{2c_1}{R}t_0F\varphi^{3/2}|\nabla w|\\
&\quad +\frac{\beta t_0^2}{n}\Big(\varphi^2\big(|\nabla w|^2+c(\alpha+1)e^{-w(\alpha+1)}-w_t\big)^2-\frac{n}{\beta}\widetilde{C}_1\varphi^2|\nabla w|^2\Big)-\widetilde{C}_2nt_0^2,
\end{aligned}$$

where the last inequality used

$$-2\varphi\nabla w\nabla F=2F\nabla w\nabla\varphi\ge-2F|\nabla w||\nabla\varphi|\ge-\frac{2c_1}{R}\varphi^{1/2}F|\nabla w|.$$

Assume that $y=\varphi|\nabla w|^2$ and $z=\varphi(-ce^{-w(\alpha+1)}+w_t)$. We have

$$0\geq \varphi F\big(-At_0-1+c\widetilde{M}(\alpha+1)t_0\big) \\ +\frac{\beta t_0^2}{n}\left((y-z)^2-\frac{n}{\beta}\widetilde{C}_1 y-2\frac{nc_1}{R}y^{1/2}\left(y-\frac{z}{\beta}\right)\right)-\widetilde{C}_2 n t_0^2.$$

Using the inequality $ax^2-bx\geq -b^2/(4a)$, valid for $a,b>0$, one obtains

$$\begin{aligned}
&\frac{\beta t_0^2}{n}\left((y-z)^2-\frac{n}{\beta}\widetilde{C}_1 y-2\frac{nc_1}{R}y^{1/2}\left(y-\frac{z}{\beta}\right)\right)\\
&=\frac{\beta t_0^2}{n}\left(\beta^2\left(y-\frac{z}{\beta}\right)^2+(1-\beta)^2y^2-\frac{n}{\beta}\widetilde{C}_1 y+\left(2(\beta-\beta^2)y-2\frac{nc_1}{R}y^{1/2}\right)\left(y-\frac{z}{\beta}\right)\right)\\
&\geq \frac{\beta t_0^2}{n}\left(\beta^2\left(y-\frac{z}{\beta}\right)^2-\frac{n^2\widetilde{C}_1^2}{4\beta^2(1-\beta)^2}-\frac{n^2c_1^2}{2R^2(\beta-\beta^2)}\left(y-\frac{z}{\beta}\right)\right)\\
&=\frac{\beta}{n}(\varphi F)^2-\frac{n\widetilde{C}_1^2t_0^2}{4\beta(1-\beta)^2}-\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}(\varphi F).
\end{aligned}$$

Hence,

$$\frac{\beta}{n}(\varphi F)^2+\left(-At_0-1+c\widetilde{M}(\alpha+1)t_0-\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\right)(\varphi F) \\ -\frac{n\widetilde{C}_1^2t_0^2}{4\beta(1-\beta)^2}-\widetilde{C}_2nt_0^2\leq 0.$$

From the inequality $Ax^2-2Bx\leq C$, we have $x\leq 2B/A+\sqrt{C/A}$. We can get

$$\varphi F\leq \frac{n}{\beta}\left(At_0+1-c\widetilde{M}(\alpha+1)t_0+\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\right)+\left(\frac{n}{\beta}\left(\frac{n\widetilde{C}_1^2}{4\beta(1-\beta)^2}+\widetilde{C}_2n\right)\right)^{\frac{1}{2}}t_0.$$

If $d(x,x_0,T_1)<R$, we have $\varphi(x,T_1)=1$. Then

$$\begin{aligned}
&F(x,T_1)\\
&\quad =T_1\big(\beta|\nabla w|^2+ce^{-w(\alpha+1)}-w_t\big)\\
&\quad \leq \varphi F(x_0,t_0)\\
&\quad \leq \frac{n}{\beta}\left(At_0+1-c\widetilde{M}(\alpha+1)t_0+\frac{nc_1^2t_0}{2R^2(\beta-\beta^2)}\right)+\left(\frac{n}{\beta}\left(\frac{n\widetilde{C}_1^2}{4\beta(1-\beta)^2}+\widetilde{C}_2n\right)\right)^{\frac{1}{2}}t_0.
\end{aligned}$$

As T_1 is arbitrary, we can get case (1) of Theorem 1.3.

(2) If $c > 0$, we have

$$0 \ge -AF + \varphi\Big(-\frac{F}{t_0} + \frac{\beta t_0}{n}\big(|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t\big)^2 - 2\nabla w \nabla F + \big(2(\beta-1)K_3 - 2\beta K_1 - \tfrac{3}{2}K_4\big)|\nabla w|^2 t_0 - n\Big(\frac{1}{\beta}(K_2+K_3)^2 + \tfrac{3}{2}K_4\Big)t_0\Big).$$

Therefore,

$$0 \ge -AF - \varphi\frac{F}{t_0} + \varphi\frac{\beta t_0}{n}\big(|\nabla w|^2 + c(\alpha+1)e^{-w(\alpha+1)} - w_t\big)^2 - 2\varphi\nabla w \nabla F + \big(2(\beta-1)K_3 - 2\beta K_1 - \tfrac{3}{2}K_4\big)|\nabla w|^2 \varphi t_0 - \Big(\frac{1}{\beta}(K_2+K_3)^2 + \tfrac{3}{2}K_4\Big)n\varphi t_0.$$

Similarly, we can get case (2) of Theorem 1.3. This completes the proof of Theorem 1.3. □

Proof of Theorem 1.5. For any points (x_1, t_1) and (x_2, t_2) on $M \times (0, T]$ with $0 < t_1 < t_2$, we take a curve $\gamma(t)$ parametrized with $\gamma(t_1) = x_1$ and $\gamma(t_2) = x_2$. One gets from Corollary 1.4:

(1) If $c < 0$, we have

$$\begin{aligned}
\log u(x_2,t_2) - \log u(x_1,t_1) &= \int_{t_1}^{t_2}\big((\log u)_t + \langle \nabla \log u, \dot\gamma\rangle\big)\,dt \\
&\ge \int_{t_1}^{t_2}\Big(\beta|\nabla \log u|^2 - \frac{n}{\beta t} - cu^{-(\alpha+1)} - \overline{H}_1 - H_2 - |\nabla \log u||\dot\gamma|\Big)\,dt \\
&\ge -\int_{t_1}^{t_2}\Big(\frac{1}{4\beta}|\dot\gamma|^2 + \frac{n}{\beta t} + \overline{H}_1 + H_2\Big)\,dt \\
&= -\Big(\int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\,dt + \log\Big(\frac{t_2}{t_1}\Big)^{n/\beta} + (\overline{H}_1 + H_2)(t_2 - t_1)\Big),
\end{aligned}$$

which means that

$$\log\frac{u(x_1,t_1)}{u(x_2,t_2)} \le \int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\,dt + \log\Big(\frac{t_2}{t_1}\Big)^{n/\beta} + (\overline{H}_1 + H_2)(t_2 - t_1).$$

Therefore,

$$u(x_1,t_1) \le u(x_2,t_2)\Big(\frac{t_2}{t_1}\Big)^{n/\beta} e^{\varphi(x_1,x_2,t_1,t_2) + (\overline{H}_1 + H_2)(t_2 - t_1)},$$

where $\varphi(x_1,x_2,t_1,t_2) = \inf_\gamma \int_{t_1}^{t_2}\frac{1}{4\beta}|\dot\gamma|^2\,dt$.

(2) If $c > 0$, we have

$$\begin{aligned}
\log u(x_2,t_2)-\log u(x_1,t_1)
&= \int_{t_1}^{t_2} \big((\log u)_t + \langle \nabla \log u, \dot{\gamma}\rangle\big)\, dt \\
&\geq \int_{t_1}^{t_2} \left(\beta|\nabla \log u|^2 - \frac{n}{\beta t} - cu^{-(\alpha+1)} - \widehat{H}_1 - \widetilde{H}_2 - |\nabla \log u||\dot{\gamma}|\right) dt \\
&\geq -\int_{t_1}^{t_2} \left(\frac{1}{4\beta}|\dot{\gamma}|^2 + \frac{n}{\beta t} + c\widetilde{M} + \widehat{H}_1 + \widetilde{H}_2\right) dt \\
&= -\left(\int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2\, dt + \log\left(\frac{t_2}{t_1}\right)^{n/\beta} + (c\widetilde{M} + \widehat{H}_1 + \widetilde{H}_2)(t_2-t_1)\right),
\end{aligned}$$

which means that

$$\log \frac{u(x_1,t_1)}{u(x_2,t_2)} \leq \int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2\, dt + \log\left(\frac{t_2}{t_1}\right)^{n/\beta} + (c\widetilde{M} + \widehat{H}_1 + \widetilde{H}_2)(t_2-t_1).$$

Therefore,

$$u(x_1,t_1) \leq u(x_2,t_2)\left(\frac{t_2}{t_1}\right)^{n/\beta} e^{\varphi(x_1,x_2,t_1,t_2)+(c\widetilde{M}+\widehat{H}_1+\widetilde{H}_2)(t_2-t_1)},$$

where $\varphi(x_1,x_2,t_1,t_2) = \inf_\gamma \int_{t_1}^{t_2} \frac{1}{4\beta}|\dot{\gamma}|^2\, dt$. □

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