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THE SU(2) CASSON–LIN INVARIANT OF THE HOPF LINK

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We compute the SU(2) Casson–Lin invariant for the Hopf link and determine the sign in the formula of Harper and Saveliev relating this invariant to the linking number.

The Casson–Lin invariant $h(K)$ was defined for knots K by X.-S. Lin [1992] as a signed count of conjugacy classes of irreducible SU(2) representations of the knot group $G_K = \pi_1(S^3 \setminus K)$ with traceless meridional image, and Corollary 2.10 of the same paper shows that $h(K)$ is equal to $\frac{1}{2}\operatorname{sign}(K)$, one half the knot signature. E. Harper and N. Saveliev [2010] introduced the Casson–Lin invariant $h_2(L)$ of 2-component links, which they defined as a signed count of certain projective SU(2) representations of the link group $G_L = \pi_1(S^3 \setminus L)$. They showed that $h_2(L)$ equals the linking number of $L = \ell_1 \cup \ell_2$, up to an overall sign: $h_2(L) = \pm \operatorname{lk}(\ell_1, \ell_2)$. Harper and Saveliev [2012] also show that $h_2(L)$ can be regarded as an Euler characteristic associated to a certain SU(2) instanton Floer homology theory, defined by Kronheimer and Mrowka [2011].

The purpose of this note is to determine the sign in the formula of Harper and Saveliev, establishing the following.

Theorem 1. *If $L = \ell_1 \cup \ell_2$ is an oriented 2-component link in S^3, then its Casson–Lin invariant satisfies $h_2(L) = -\operatorname{lk}(\ell_1, \ell_2)$.*

We remark that the braid approach in [Harper and Saveliev 2010] is close in spirit to Lin's original definition, and it shows that $h_2(L)$ is an invariant of *oriented* links, because the Alexander and Markov theorems hold for oriented links; see Theorems 2.3 and 2.8 of [Kassel and Turaev 2008]. The sign of the invariant $h_2(L)$ depends not only on the choice of orientation on the braid, but also on the more subtle choice of identification of geometric braids with elements in the abstract braid group B_n, viewed as a subgroup of $\operatorname{Aut}(F_n)$. Here we follow Conventions 1.13 of [Kassel and Turaev 2008] in making this choice.

Note that extensions of the Casson–Lin invariants to SU(N) and to oriented links L in S^3 with at least two components are presented in [Boden and Harper

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2016], where, as before, they are defined by counting certain projective $\mathrm{SU}(N)$ representations of the link group G_L.

The rest of this paper is devoted to proving Theorem 1.

Proof. The proof of Proposition 5.7 in [Harper and Saveliev 2010] shows that the sign of $\mathrm{lk}(\ell_1, \ell_2)$ in our theorem is independent of L. (See also the proof of their Theorem 2 and their general discussion in Section 5.) Thus Theorem 1 will follow from a single computation.

To that end, we will determine the Casson–Lin invariant for the right-handed Hopf link. Since there is just one irreducible projective SU(2) representation of the link group, up to conjugation, it suffices to determine the sign associated to this one point.

We identify

$$\mathrm{SU}(2) = \{x + yi + zj + wk \mid |x|^2 + |y|^2 + |z|^2 + |w|^2 = 1\}$$

with the group of unit quaternions and consider the conjugacy class

$$C_i = \{yi + zj + wk \mid |y|^2 + |z|^2 + |w|^2 = 1\} \subset \mathrm{SU}(2)$$

of purely imaginary unit quaternions. Notice that C_i is diffeomorphic to S^2 and coincides with the set of SU(2) matrices of trace zero.

Let L be an oriented link in S^3, represented as the closure of an n-strand braid $\sigma \in B_n$. We follow Conventions 1.13 on page 17 of [Kassel and Turaev 2008] for writing geometric braids σ as words in the standard generators $\sigma_1, \dots, \sigma_{n-1}$. In particular, braids are oriented from top to bottom and σ_i denotes a right-handed crossing in which the $(i+1)$-st strand crosses over the i-th strand. The braid group B_n gives a faithful right action on the free group F_n on n generators, and here we follow the conventions in [Boden and Harper 2016] for associating an automorphism of F_n to a given braid $\sigma \in B_n$, which we write as $x_i \mapsto x_i^\sigma$ for $i = 1, \dots, n$. To be precise, to each braid group generator σ_i we associate the map $\sigma_i : F_n \to F_n$ given by

$$x_i \mapsto x_{i+1}, \quad x_{i+1} \mapsto (x_{i+1})^{-1} x_i x_{i+1}, \quad x_j \mapsto x_j \ (j \neq i,\ i+1),$$

and this is a right action, i.e., if $\sigma, \sigma' \in B_n$ are two braids, then $(x_i)^{\sigma\sigma'} = (x_i^\sigma)^{\sigma'}$ for all $1 \le i \le n$. Note that each braid $\sigma \in B_n$ fixes the product $x_1 \cdots x_n$.

A standard application of the Seifert–van Kampen theorem shows that the link complement $S^3 \setminus L$ has fundamental group

$$\pi_1(S^3 \setminus L) = \langle x_1, \dots, x_n \mid x_i^\sigma = x_i,\ i = 1, \dots, n \rangle.$$

We can therefore identify representations in $\mathrm{Hom}(\pi_1(S^3 \setminus L), \mathrm{SU}(2))$ with fixed points in $\mathrm{Hom}(F_n, \mathrm{SU}(2))$ under the induced action of the braid σ. We further identify $\mathrm{Hom}(F_n, \mathrm{SU}(2))$ with $\mathrm{SU}(2)^n$ by associating to a homomorphism ϱ the

n-tuple $(X_1, \ldots, X_n) = (\varrho(x_1), \ldots, \varrho(x_n))$. Note that $\sigma : \mathrm{SU}(2)^n \to \mathrm{SU}(2)^n$ is equivariant with respect to conjugation, so that fixed points come in whole orbits.

Every projective SU(2) representation can be identified with a fixed point in $\mathrm{Hom}(F_n, \mathrm{SU}(2))$ under the action of $\varepsilon\sigma$ for some n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ such that $\varepsilon_1 \cdots \varepsilon_n = 1$. Notice that the action of $\varepsilon\sigma$ on $(X_1, \ldots, X_n) \in \mathrm{SU}(2)^n$ preserves the product $X_1 \cdots X_n$ and is equivariant with respect to conjugation. The Casson–Lin invariant $h_2(L)$ is then defined as a signed count of orbits of fixed points of $\varepsilon\sigma$ for a suitably chosen n-tuple $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. The choice is made so that the resulting projective representations ϱ all have $w_2(\mathrm{Ad}\,\varrho) \neq 0$, meaning that the representations $\mathrm{Ad}\,\varrho$ do not lift to SU(2) representations. It has the consequence that for all fixed points ϱ of $\varepsilon\sigma$, each $\varrho(x_i)$ is a traceless SU(2) element.

We therefore restrict our attention to the subset of traceless representations, which are elements $\varrho \in \mathrm{Hom}(F_n, \mathrm{SU}(2))$ with $\varrho(x_j) \in C_i$ for $j = 1, \ldots, n$. Define $f : C_i^n \times C_i^n \to \mathrm{SU}(2)$ by setting

$$f(X_1, \ldots, X_n, Y_1, \ldots, Y_n) = (X_1 \cdots X_n)(Y_1 \cdots Y_n)^{-1}.$$

We obtain an orientation on $f^{-1}(1)$ by applying the base-fiber rule, using the product orientation on $C_i^n \times C_i^n$ and the standard orientation on the codomain of f. The quotient $f^{-1}(1)/\mathrm{conj}$ is then oriented by another application of the base-fiber rule, using the standard orientation on SU(2). This step uses the fact that, if $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$ is chosen so that the associated SO(3) representation $\mathrm{Ad}\,\varrho$ has nontrivial second Stiefel–Whitney class $w_2 \neq 0$, then every fixed point of $\varepsilon\sigma$ in $\mathrm{Hom}(F_n, \mathrm{SU}(2))$ is necessarily irreducible.

We view conjugacy classes of fixed points of $\varepsilon\sigma$ as points in the intersection $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$, where $\widehat{\Delta} = \Delta/\mathrm{conj}$ is the quotient of the diagonal $\Delta \subset C_i^n \times C_i^n$, and where $\widehat{\Gamma}_{\varepsilon\sigma} = \Gamma_{\varepsilon\sigma}/\mathrm{conj}$ is the quotient of the graph $\Gamma_{\varepsilon\sigma}$ of $\varepsilon\sigma : C_i^n \to C_i^n$.

If the link L is the closure of a 2-strand braid, as it is for the Hopf link, then $\varepsilon = (-1, -1)$ is the only choice whose associated SO(3) bundle has $w_2 \neq 0$. Furthermore, in this case the intersection $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ takes place in the pillowcase $f^{-1}(1)/\mathrm{conj}$, which is defined as the quotient

$$P = \{(a, b, c, d) \in C_i^4 \mid ab = cd\}/\mathrm{conj}. \tag{1}$$

It is well known that P is homeomorphic to S^2. To see this, first conjugate so that $a = i$, then conjugate by elements of the form $e^{i\theta}$ to arrange that b lies in the (i, j)-circle. A straightforward calculation using the equation $ab = cd$ shows that d must also lie on the (i, j)-circle. Clearly c is determined by a, b, d. We thus obtain an embedded 2-torus of elements of C_i^4 satisfying $ab = cd$, parametrized by

$$g(\theta_1, \theta_2) = (i, e^{k\theta_1} i, e^{k(\theta_2-\theta_1)} i, e^{k\theta_2} i)$$

for $\theta_1, \theta_2 \in [0, 2\pi)$, which maps onto P. It is easy to verify that this is a two-to-one submersion, except when $\theta_1, \theta_2 \in \{0, \pi\}$. This realizes P as a quotient of the torus by the hyperelliptic involution. In particular, this involution is orientation-preserving, and away from the four singular points of P, we can lift all orientation questions up to the torus.

Let L be the right-handed Hopf link, which we view as the closure of the braid $\sigma = \sigma_1^2 \in B_2$, and suppose $\varepsilon = (-1, -1)$. The intersection $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ consists of only one point, the conjugacy class of $g\left(\frac{\pi}{2}, \frac{\pi}{2}\right)$, that is, the point $[(i, j, i, j)] \in P$. (This is easily verified using the action of σ_1^2 on $F_2 = \langle x, y\rangle$; see Figure 1.) Thus, in order to pin down the sign of the Casson–Lin invariant $h_2(L)$, we must determine the orientations of $\widehat{\Delta}$, $\widehat{\Gamma}_{\varepsilon\sigma}$, and P at this point.

Notice that

$$\frac{\partial}{\partial\theta_1} g(\theta_1, \theta_2) = (0, e^{k\theta_1} j, -e^{k(\theta_2-\theta_1)} j, 0),$$
$$\frac{\partial}{\partial\theta_2} g(\theta_1, \theta_2) = (0, 0, e^{k(\theta_2-\theta_1)} j, e^{k\theta_2} j).$$

Evaluating at $\theta_1 = \theta_2 = \frac{\pi}{2}$ gives two tangent vectors $u_1 := (0, -i, -j, 0)$ and $u_2 := (0, 0, j, -i)$ to C_i^4 which span a complementary subspace in $\ker df$ to the orbit tangent space. Therefore, an ordering of these vectors determines an orientation on $P = f^{-1}(1)/\text{conj}$.

The orbit tangent space is spanned by the three tangent vectors

$$v_1 := \frac{\partial}{\partial t}\Big|_{t=0} e^{it}(i, j, i, j)e^{-it} = (0, 2k, 0, 2k),$$
$$v_2 := \frac{\partial}{\partial t}\Big|_{t=0} e^{jt}(i, j, i, j)e^{-jt} = (-2k, 0, -2k, 0),$$
$$v_3 := \frac{\partial}{\partial t}\Big|_{t=0} e^{kt}(i, j, i, j)e^{-kt} = (2j, -2i, 2j, -2i).$$

Then $\{u_1, u_2, v_1, v_2, v_3\}$ is a basis for $\ker(df|_{(i,j,i,j)}) = T_{(i,j,i,j)} f^{-1}(1)$. We choose vectors $w_1 = (k, 0, 0, 0)$, $w_2 = (0, k, 0, 0)$, $w_3 = (j, 0, 0, 0)$ to extend this to a basis for $T_{(i,j,i,j)} C_i^4$.

The orientation conventions in the definition of $h_2(L)$ (see Section 5d of [Harper and Saveliev 2010]) involve pulling back the orientation from $\mathfrak{su}(2) = T_1\,\mathrm{SU}(2)$ by df to obtain a coorientation for $\ker(df|_{(i,j,i,j)})$. With that in mind, we compute the action of df on $\{w_1, w_2, w_3\}$, namely, $df(w_1) = -j$, $df(w_2) = i$, $df(w_3) = k$.

Notice that the ordered triple $\{df(w_1), df(w_2), df(w_3)\} = \{-j, i, k\}$ gives the same orientation as the standard basis for $\mathfrak{su}(2)$. Thus, the base-fiber rule gives the coorientation $\{w_1, w_2, w_3\}$ on $\ker df$, so we choose the orientation $\mathcal{O}_{\ker df}$ on $\ker df$ such that $\mathcal{O}_{\{w_1,w_2,w_3\}} \oplus \mathcal{O}_{\ker df}$ agrees with the product orientation on $C_i^2 \times C_i^2$.

The orientation on the pillowcase P is then obtained by applying the base-fiber rule a second time to the quotient (1), using $\mathcal{O}_{\ker df}$ to orient $f^{-1}(1)$ and giving the

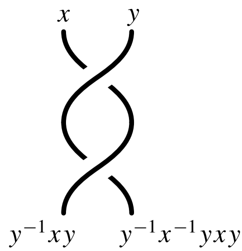

Figure 1. The action of $\sigma = \sigma_1^2$ on $F_2 = \langle x, y \rangle$.

orbit tangent space the orientation induced from that on SU(2) as well. We claim that the basis $\{u_1, u_2\}$ for the tangent space to the pillowcase has the opposite orientation. To see this, we note that $\{v_1, v_2, v_3\}$ is the fiber orientation for $SO(3) \to f^{-1}(1) \to P$ and compare $S = \{w_1, w_2, w_3, u_1, u_2, v_1, v_2, v_3\}$ to the product orientation on $C_i^2 \times C_i^2$. Using the basis $\{(j,0), (k,0), (0,k), (0,i)\}$ for $T_{(i,j)}(C_i^2)$, we see that

$$\beta = \{(j,0,0,0), (k,0,0,0), (0,k,0,0), (0,i,0,0),\\ (0,0,j,0), (0,0,k,0), (0,0,0,k), (0,0,0,i)\}$$

is an oriented basis for $T_{(i,j,i,j)} C_i^4 = T_{(i,j)} C_i^2 \times T_{(i,j)} C_i^2$ with the product orientation.[1]

Let M be the matrix expressing the vectors in S in terms of the basis β. Since

$$M = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 \\ 1 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -2 \\ 0 & 0 & 0 & -1 & 1 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & -2 \end{bmatrix},$$

one easily computes that $\det M = -8$, confirming our claim that $\{u_2, u_1\}$ is a positively oriented basis for the pillowcase tangent space.

Recall that L is the right-handed Hopf link, which we represent as the closure of the braid $\sigma = \sigma_1^2 \in B_2$. For $\varepsilon = (-1, -1)$, as in Figure 1, one can verify that

$$\varepsilon\sigma(X, Y) = (-Y^{-1}XY, -Y^{-1}X^{-1}YXY).$$

Consider the curve $\alpha(\theta) = (i, e^{k\theta} i)$, passing through the point $(i, j) \in C_i^2$ when $\theta = \frac{\pi}{2}$, which is transverse to the orbit $[(i, j)]$. Then $(\alpha(\theta), \alpha(\theta))$ and $(\alpha(\theta), \varepsilon\sigma \circ \alpha(\theta))$ are curves in Δ and $\Gamma_{\varepsilon\sigma}$, respectively, and both are necessarily transverse to the orbit in

[1] As explained in Section 5d of [Harper and Saveliev 2010], the invariant $h_2(L)$ is independent of the choice of orientation on C_i. In fact, C_i^2 can be oriented arbitrarily provided one uses the *product* orientation on $C_i^2 \times C_i^2$.

C_i^4/conj. Thus, we can compare the orientations induced by the parametrizations $[(\alpha(\theta), \alpha(\theta))]$ and $[(\alpha(\theta), \varepsilon\sigma \circ \alpha(\theta))]$ of $\widehat{\Delta}$ and $\widehat{\Gamma}_{\varepsilon\sigma}$ to the pillowcase orientation determined above, namely $\{u_2, u_1\}$. The velocity vectors for the paths $(\alpha(\theta), \alpha(\theta)) = (i, e^{k\theta} i, i, e^{k\theta} i)$ and $(\alpha(\theta), \varepsilon\sigma \circ \alpha(\theta)) = (i, e^{k\theta} i, -e^{2k\theta} i, -e^{3k\theta} i)$ at $\theta = \frac{\pi}{2}$ are given by $(0, -i, 0, -i) = u_1 + u_2$ and $(0, -i, 2j, -3i) = u_1 + 3u_2$, respectively.

The Casson–Lin invariant is defined as the intersection number $h_2(L) = \langle \widehat{\Delta}, \widehat{\Gamma}_{\varepsilon\sigma} \rangle$, and in our case the sign of the unique intersection point in $\widehat{\Delta} \cap \widehat{\Gamma}_{\varepsilon\sigma}$ is determined by comparing the orientation of $\{u_1 + u_2, u_1 + 3u_2\}$ with $\{u_2, u_1\}$. Since the change of basis matrix $\left[\begin{smallmatrix} 1 & 3 \\ 1 & 1 \end{smallmatrix}\right]$ has negative determinant, it follows that $h_2(L) = -1$. □

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