A STRONG MULTIPLICITY ONE THEOREM FOR $SL_2$

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It is known that the multiplicity one property holds for SL$_2$ while the strong multiplicity one property fails. However, in this paper we show that if we require further that a pair of cuspidal representations $\pi$ and $\pi'$ of SL$_2$ have the same local components at the archimedean places and the places above 2, and they are generic with respect to the same additive character, then they also satisfy the strong multiplicity one property. The proof is based on a local converse theorem for SL$_2$.

1. Introduction

Let $F$ be a number field and $\mathbb{A} = \mathbb{A}_F$ be its ring of adeles. Let $G$ be a linear reductive algebraic group defined over $F$. The study of the space of automorphic forms $L^2(G(F)\backslash G(\mathbb{A}))$ has been a central topic in the Langlands program and representation theory. Let $L^2_0(G(F)\backslash G(\mathbb{A}))$ be the subspace of cuspidal representations. Suppose $\pi$ is an irreducible automorphic representation of $G(\mathbb{A})$. It is known that $\pi$ occurs discretely with finite multiplicity $m_{\pi}$ in $L^2_0(G(F)\backslash G(\mathbb{A}))$.

The multiplicities $m_{\pi}$ are important in the study of automorphic forms and number theory. By [Jacquet and Shalika 1981; Badulescu 2008] and the work of Piatetski-Shapiro, the group $G = \text{GL}_n$ and its inner forms have the property of multiplicity one, that is, $m_{\pi} \leq 1$ for any $\pi$. This is also true for SL$_2$ by the famous work of D. Ramakrishnan [2000]. But in general the multiplicity one property fails, for example [Blasius 1994; Gan et al. 2002; Li 1997; Labesse and Langlands 1979] to list a few.

In the case of GL$_n$ a stronger theorem, called the strong multiplicity one, holds. It says that for two cuspidal representations $\pi_1$ and $\pi_2$, if they have isomorphic local components almost everywhere, then they coincide in the space of cusp forms (not only isomorphic). It follows from the results in [Labesse and Langlands 1979] that SL$_2$ does not have this strong multiplicity one property. The multiplicity one property is already rare and the strong multiplicity one is even rarer. To the authors’ knowledge the examples other than GL$_n$ in this direction are the strong multiplicity one theorems for U$(2,1)$ [Gelbart and Piatetski-Shapiro 1984; Baruch 1997] and

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345
The main purpose of this paper is to prove a weaker version of the strong multiplicity one result for $\text{Sp}_{2n} = \text{SL}_{2n}$. Although we know strong multiplicity one does not hold in general for a pair of cuspidal representations $\pi_1$ and $\pi_2$ of $\text{SL}_2(\mathbb{A})$, if we require that both $\pi_1$ and $\pi_2$ are generic with respect to the same additive character $\psi$ of $\mathbb{A}$, then we can show that they also satisfy the strong multiplicity one property.

The reason for the failure of the strong multiplicity one for $\text{SL}_2$ is the existence of $L$-packets. According to the local conjecture of Gan–Gross–Prasad [2012, Conjecture 17.3] there is at most one $\psi$-generic representation in each $L$-packet. For $\text{SL}_2$, the result is known by the local discussion in [Labesse and Langlands 1979]. In this paper, we prove a local converse theorem for $\text{SL}_2(F)$ when $F$ is a $p$-adic field such that its residue characteristic is not 2, which will reprove the result of Labesse and Langlands [1979] and confirm a local converse conjecture of Jiang, see [Jiang 2006, Conjecture 3.7] and [Jiang and Nien 2013, Conjecture 6.3]. This also implies our version of strong multiplicity one easily.

We now give some details of our results. Gelbart and Piatetski-Shapiro [1987] constructed some Rankin–Selberg integrals to study $L$-functions on the group $G_n \times \text{GL}(n)$, for $G_n = \text{Sp}(n)$ and $\text{U}(n, n)$. In particular, in Method C in that paper, if $\pi$ is a globally generic cuspidal representation of $\text{Sp}_{2n}(\mathbb{A})$, then $\tau$ is a cuspidal representation of $\text{GL}_n(\mathbb{A})$. Consider the global Shimura type zeta-integral

$$I(s, \phi, E) = \int_{\text{Sp}_{2n}(F) \backslash \text{Sp}_{2n}(\mathbb{A})} \phi(g) \theta(g) E(g, s) dg,$$

where $\phi$ belongs to the space of $\pi$, $E(g, s)$ is a genuine Eisenstein series on $\tilde{\text{Sp}}_{2n}(\mathbb{A})$ built from the representation induced from the representation $\tau$ of $\text{GL}_n(\mathbb{A})$ twisted by $|\det|^s$ and $\theta(g)$ is some theta series on $\tilde{\text{Sp}}_{2n}(\mathbb{A})$. Note that the product $\theta(g) E(g, s)$ is well-defined on $\text{Sp}_{2n}$. The global integral is shown to be Eulerian. The local functional equations and unramified calculations were also carried out by Gelbart and Piatetski-Shapiro [1987]. Although we will only consider the easiest case when $n = 1$ of Gelbart and Piatetski-Shapiro’s construction, we remark here that Ginzburg, Rallis and Soudry [1997; 1998] generalized the above construction to $\text{Sp}_{2n} \times \text{GL}_k$, for any $k$.

We study more details of Gelbart and Piatetski-Shapiro’s local integral

$$\Psi(W_v, \phi_v, f_{s,v}) = \int_{N(F_v) \backslash \text{SL}_2(F_v)} W_v(h)(\omega_{\psi_v^{-1}}(h)\phi_v)(1) f_{s,v}(h) dh$$

(for the unexplained notations, see sections below) when $v$ is finite. These local zeta-integrals satisfy certain functional equations, which come from the intertwining
operators on induced representations and certain uniqueness statements. These functional equations can then be used to define local gamma factors $\gamma(s, \pi_v, \eta_v, \psi_v)$, where $\pi_v$ is a generic representation of $\text{SL}_2(F_v)$, $\eta_v$ is a character of $F_v^\times$ and $\psi_v$ is a nontrivial additive character. The main local result of this paper can be formulated as follows.

**Theorem 3.10** (Local converse theorem and stability of $\gamma$). Suppose that the residue characteristic of the $p$-adic field $F$ is not 2 and $\psi$ is a nontrivial additive character of $F$. Let $(\pi, V_\pi)$ and $(\pi', V_{\pi'})$ be two $\psi$-generic representations of $\text{SL}_2(F)$ with the same central character.

1. If $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasicharacters $\eta$ of $F^\times$, then $\pi \cong \pi'$.
2. There is an integer $l = l(\pi, \pi')$ such that if $\eta$ is a quasicharacter of $F^\times$ with conductor $\text{cond}(\eta) > l$, then
   \[ \gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi). \]

The proof of this result follows closely [Baruch 1995; 1997; Zhang 2015] and Howe vectors play an important role. With the help of this result, combined with a nonvanishing result on archimedean local integrals proved in Lemma 4.9, we follow the argument in [Baruch 1997, Theorem 7.2.13], or in [Casselman 1973, Theorem 2], to prove the main global result of this paper.

**Theorem 4.8** (Strong multiplicity one for $\text{SL}_2$). Let $\psi$ be a nontrivial additive character of $F \setminus \mathbb{A}$ and let $\pi = \otimes_v \pi_v$ and $\pi' = \otimes_v \pi'_v$ be two irreducible cuspidal automorphic representations of $\text{SL}_2(\mathbb{A})$ with the same central character. Suppose that $\pi$ and $\pi'$ are both $\psi$-generic. Let $S$ be a finite set of finite places such that no place in $S$ is above 2. If $\pi_v \cong \pi'_v$ for all $v \not\in S$, then $\pi = \pi'$.

**Remark.** The restriction on residue characteristic comes from Lemma 3.3. It is expected that this restriction can be removed.

Besides the above, we also in this paper include a discussion of relations between global genericity and local genericity. An irreducible cuspidal automorphic representation $(\pi, V_\pi)$ is called globally $\psi$-generic if for some $\phi \in V_\pi$, the integral
\[ \int_{N(F) \backslash N(\mathbb{A})} \phi(ug)\psi^{-1}(u)du \neq 0 \]
for some $g \in \text{SL}_2(\mathbb{A})$. The representation $\pi$ is called locally $\psi$-generic if each of its local component is generic for the corresponding local components of $\psi$. It is easy to see that if $\pi$ is globally $\psi$-generic, then $\pi$ is also locally $\psi$-generic. It is a conjecture that on a reductive algebraic group $G$, the converse is also true. This conjecture is closely related to the Ramanujan conjecture. See [Shahidi 2011] for more detailed discussions. We confirm this conjecture for $\text{SL}_2$. 
Theorem 4.3. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\text{SL}_2(\mathbb{A})$ and $\psi = \otimes_v \psi_v$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then $\pi$ is $\psi$-generic if and only if each $\pi_v$ is $\psi_v$-generic.

Gelbart, Rogawski and Soudry [1997, Proposition 2.5] proved similar results for $\text{U}(1, 1)$ and for endoscopic cuspidal automorphic representations of $\text{U}(2, 1)$. From the discussions given in [Gelbart et al. 1997] Theorem 4.3 follows directly from the results of Labesse and Langlands [1979]. Here, we include this result because we adopt a local argument (see Proposition 2.1) which is different from that given in [Labesse and Langlands 1979]. Hopefully, this local argument can be extended to more general groups.

As explained above, there is essentially nothing new in this paper. All the results and proofs should be known to the experts. Our task here is simply to try to write down the details and to check everything works out as expected.

This paper is organized as follows. In Section 2 we collect basic results about the local zeta-integrals which will be needed. In Section 3 we study the Howe vectors and use them to prove the local converse theorem and stability of local gamma factors. In Section 4 we prove the main global results.

1A. Notations. Let $F$ be a field. In $\text{SL}_2(F)$, we consider the following subgroups. Let $B$ be the upper triangular subgroup. Let $B = TN$ be the Levi-decomposition, where $T$ is the diagonal torus and $N$ is the upper triangular unipotent. Denote

$$t(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in T, \quad \text{for } a \in F^\times, \quad \text{and} \quad n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \in N, \quad \text{for } b \in F.$$

Let $\widetilde{N}$ be the lower triangular unipotent and denote

$$\tilde{n}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix} \quad \text{and} \quad w = \begin{pmatrix} 1 \\ -1 & 1 \end{pmatrix}.$$

Denote by $St$ the natural inclusion $\text{SO}_3(\mathbb{C}) \to \text{GL}_3(\mathbb{C})$ and view it as the “standard” representation of $^L\text{SL}_2 = \text{SO}_3(\mathbb{C})$.

2. The local zeta-integral

2A. The genericity of representations of $\text{SL}_2(F)$. In this section let $F$ be a local field and $\psi$ be a nontrivial additive character of $F$, which is also viewed as a character of $N(F)$. For $\kappa \in F^\times$ and $g \in \text{SL}_2(F)$ we define

$$g^\kappa = \begin{pmatrix} \kappa \\ 1 \end{pmatrix} g \begin{pmatrix} \kappa^{-1} \\ 1 \end{pmatrix}.$$
Explicitly
\[
\begin{pmatrix} x & y \\ z & w \end{pmatrix}^\kappa = \begin{pmatrix} x & \kappa y \\ \kappa^{-1}z & w \end{pmatrix}.
\]

Note that if \( \kappa \in F^\times \), say \( \kappa = a^2 \), then \( g^\kappa = t(a)g t(a)^{-1} \), i.e., \( g \mapsto g^\kappa \) is an inner automorphism on \( SL_2(F) \). Let \( (\pi, V_\pi) \) be an infinite dimensional irreducible smooth representation of \( SL_2(F) \). We consider the representation \( (\pi^\kappa, V_{\pi^\kappa}) \) defined by
\[
V_{\pi^\kappa} = V_\pi \quad \text{and} \quad \pi^\kappa(g) = \pi(g^\kappa).
\]

Let \( \psi_\kappa \) be the character of \( F \) defined by \( \psi_\kappa(b) = \psi(\kappa b) \). If \( (\pi, V_\pi) \) is \( \psi \)-generic with a nonzero \( \psi \) Whittaker functional \( \Lambda : V_\pi \rightarrow \mathbb{C} \), one verifies that
\[
\Lambda(\pi^\kappa(n)v) = \Lambda(\pi(n^\kappa)v) = \psi_\kappa(n)\Lambda(v)
\]
for all \( n \in N(F) \) and all \( v \in V_{\pi^\kappa} = V_\pi \). Hence \( (\pi^\kappa, V_{\pi^\kappa}) \) is \( \psi_\kappa \)-generic.

**Proposition 2.1.** If \( \pi \) is both \( \psi \)- and \( \psi_\kappa \)-generic, then \( \pi \cong \pi^\kappa \).

**Proof.** If \( F \) is nonarchimedean, the proof is similar to the \( U(1, 1) \) case as in [Zhang 2015].

If \( F \) is archimedean the case \( F = \mathbb{C} \) is easy, as every \( \kappa \) has a square root in \( \mathbb{C} \). Now consider \( F = \mathbb{R} \). We will work with the category of smooth representations of moderate growth of finite length. The Whittaker functional is an exact functor from this category to the category of vector spaces by [Casselman et al. 2000, Theorem 8.2].

We first consider the case when \( I(\chi) = \text{Ind}_G^H(\chi) \) for some quasicharacter \( \chi \) of \( F^\times \). For \( f \in I(\chi) \), consider the function \( f^\chi \) on \( SL_2(F) \) defined by \( f^\kappa(g) = f(g^{\kappa^{-1}}) \). It is clear that \( f^\chi \in I(\chi)^\kappa \) and the map \( f \mapsto f^\chi \) defines an isomorphism \( I(\chi) \rightarrow I(\chi)^\kappa \).

By results in [Vogan 1981, Chapter 2], if \( \pi \) is not a fully induced representation then it can be embedded into a principal series \( I(\chi) \). This \( I(\chi) \) has two irreducible infinite dimensional subrepresentations, use \( \pi' \) to denote the other one. The quotient of \( I(\chi) \) by the sum of \( \pi \) and \( \pi' \), denoted by \( \pi'' \), is finite dimensional, i.e., we have a short exact sequence
\[
0 \rightarrow \pi \oplus \pi' \rightarrow I(\chi) \rightarrow \pi'' \rightarrow 0.
\]

First, by [Casselman et al. 2000, Theorem 6.1], we know that the Whittaker functionals on \( I(\chi) \) are one dimensional for either \( \psi \) or \( \psi_\kappa \). Note that \( \pi'' \) cannot be generic as it is finite dimensional. Since the Whittaker functor is exact, it follows that the dimension of Whittaker functionals on \( \pi \oplus \pi' \) is also one for either \( \psi \) or \( \psi_\kappa \). By the assumption \( \pi \) is both \( \psi \)- and \( \psi_\kappa \)-generic, thus \( \pi' \) is neither \( \psi \)- nor \( \psi_\kappa \)-generic.
Now since \( \pi \) is \( \psi \)-generic, \( \pi^\kappa \) is \( \psi_\kappa \)-generic. Hence the image of \( \pi \) under the isomorphism \( I(\chi) \to I(\chi)^\kappa \) given by \( f \mapsto f^\kappa \) is again \( \psi_\kappa \)-generic and hence it has to be \( \psi \)-generic and isomorphic to \( \pi \), which finishes the proof. \( \square \)

2B. Weil representations of \( \widetilde{\text{SL}}_2 \). Let \( \widetilde{\text{SL}}_2 \) be the metaplectic double cover of \( \text{SL}_2 \). Then we have an exact sequence

\[
0 \to \mu_2 \to \widetilde{\text{SL}}_2 \to \text{SL}_2 \to 0,
\]

where \( \mu_2 = \{ \pm 1 \} \).

The product on \( \widetilde{\text{SL}}_2(F) \) is given by

\[
(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1\xi_2c(g_1, g_2)),
\]

where \( c : \text{SL}_2(F) \times \text{SL}_2(F) \to \{ \pm 1 \} \) is defined by Hilbert symbols as

\[
c(g_1, g_2) = (x(g_1), x(g_2))_F (x(-g_1)x(g_2), x(g_1g_2)),
\]

where

\[
x \left( \begin{array}{cc} a & b \\ c & d \end{array} \right) = \begin{cases} c & c \neq 0, \\ d & c = 0, \end{cases}
\]

and \( (,)_F \) is the Hilbert symbol. For these formulas for the Kubota cocycle see [Kubota 1969, Section 3].

For a subgroup \( A \) of \( \text{SL}_2(F) \), we denote by \( \widetilde{A} \) the preimage of \( A \) in \( \widetilde{\text{SL}}_2(F) \), which is a subgroup of \( \widetilde{\text{SL}}_2(F) \). For an element \( g \in \text{SL}_2(F) \), we sometimes abuse notation by writing \( (g, 1) \in \widetilde{\text{SL}}_2(F) \) as \( g \).

A representation \( \pi \) of \( \widetilde{\text{SL}}_2(F) \) is called genuine if \( \pi(\zeta g) = \zeta \pi(g) \) for all \( g \in \widetilde{\text{SL}}_2(F) \) and \( \zeta \in \mu_2 \). Let \( \psi \) be an additive character of \( F \). Then there is a Weil representation \( \omega_\psi \) of \( \widetilde{\text{SL}}_2(F) \) on the space \( \mathcal{S}(F) \) of Schwartz–Bruhat functions on \( F \). For \( f \in \mathcal{S}(F) \), we have the well-known formulas:

\[
\left( \omega_\psi \left( \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) \right) f(x) = \gamma(\psi) \hat{f}(x),
\]

\[
\left( \omega_\psi \left( \begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \right) f(x) = \psi(bx^2) f(x), \quad b \in F
\]

\[
\left( \omega_\psi \left( \begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array} \right) \right) f(x) = |a|^{1/2} \frac{\gamma(\psi)}{\gamma(\psi_a)} f(ax), \quad a \in F^\times.
\]

\[
\omega_\psi(\zeta) f(x) = \zeta f(x), \quad \zeta \in \mu_2.
\]

Here \( \hat{f}(x) = \int_F f(y) \psi(2xy) dy \), where \( dy \) is normalized so that \( \hat{f}(x) = f(-x) \), \( \gamma(\psi) \) is the Weil index and \( \psi_a(x) = \psi(ax) \).
Let \( \tilde{T} \) be the inverse image of \( T = \{ t(a) := \left( \begin{smallmatrix} a & 1 \\ -1 & a \end{smallmatrix} \right), a \in F^\times \} \subset SL_2(F) \) in \( \tilde{SL}_2(F) \). The product in \( \tilde{T} \) is given by the Hilbert symbol, i.e.,

\[
(t(a), \xi_1)(t(b), \xi_2) = (t(ab), \xi_1 \xi_2(a, b)_F).
\]

The function

\[
\mu_\psi(a) = \frac{\gamma(\psi(a))}{\gamma(\psi_a)}
\]

satisfies

\[
\mu_\psi(a)\mu_\psi(b) = \mu_\psi(ab)(a, b)_F,
\]

and thus extends to a genuine character of \( \tilde{T} \).

The representation \( \omega_\psi \) is not irreducible and we have \( \omega_\psi = \omega_\psi^+ \oplus \omega_\psi^- \), where \( \omega_\psi^+ \) and \( \omega_\psi^- \) are the subrepresentations on even and odd functions in \( \mathcal{F}(F) \), respectively. All the above facts can be found in [Gelbart and Piatetski-Shapiro 1980, Section 1].

**2C. The local zeta-integral.** Let \( \mu_\psi(a) = \gamma(\psi)/\gamma(\psi_a) \) be as above, which is viewed as a character of \( \tilde{T} \). Let \( \eta \) be a quasicharacter of \( F^\times \) and \( \eta_s \) denote the character \( \eta|\cdot|^s \) of \( F^\times \). Consider the induced representation \( I(s, \eta, \psi) = \text{Ind}_{B(F)}^{SL_2(F)}(\eta_{s-1/2} \mu_\psi) \).

Let \((\pi, V)\) be a \( \psi \)-generic representation of \( SL_2(F) \) with its Whittaker model \( \mathcal{W}(\pi, \psi) \). Choose \( W \in \mathcal{W}(\pi, \psi) \), \( \phi \in \mathcal{F}(F) \) and \( f_s \in I(s, \eta, \psi^{-1}) \). Note that \((\omega_\psi^{-1}(h)\phi)(1)f_s(h) \) is well-defined as a function on \( SL_2(F) \) and consider the integral

\[
\Psi(W, \phi, f_s) = \int_{N(F)\backslash SL_2(F)} W(h)(\omega_\psi^{-1}(h)\phi)(1)f_s(h)dh.
\]

By results in [Gelbart et al. 1987, Sections 5 and 12], the above integral is absolutely convergent when \( \text{Re}(s) \) is large enough and has a meromorphic continuation to the whole plane.

**Remark.** Gelbart and Piatetski-Shapiro [1987, Method C] constructed a global zeta-integral for \( \text{Sp}_{2n} \times \text{GL}_n \) which showed that it is Eulerian. They also sketched a proof of the local functional equation. The above integral is the simplest case of the Gelbart and Piatetski-Shapiro integral, namely when \( n = 1 \).

**2D. Local functional equation.** The trilinear form \((W, \phi, f_s) \mapsto \Psi(W, \phi, f_s) \) defines an element in

\[
\text{Hom}_{SL_2}(\pi \otimes \omega_\psi^{-1} \otimes I(s, \eta, \psi^{-1}), \mathbb{C}),
\]

which has dimension at most one. The proof of this fact is given in [Gelbart et al. 1987, §11] and also can be deduced by the uniqueness of the Fourier–Jacobi model.
for SL₂, see [Sun 2012]. Let
\[ M_s : I(s, \eta, \psi^{-1}) \to I(1-s, \eta^{-1}, \psi^{-1}) \]
be the standard intertwining operator, i.e.,
\[ M_s(f_s)(g) = \int_N f_s(wn g)dn. \]
By the one dimensionality of the above Hom space we get the following:

**Proposition 2.2.** There is a meromorphic function \( \gamma(s, \pi, \eta, \psi) \) such that
\[ \Psi(W, \phi, M_s(f_s)) = \gamma(s, \pi, \eta, \psi)\Psi(W, \phi, f_s), \]
for all \( W \in \mathcal{W}(\pi, \psi), \phi \in \mathcal{F}(F) \) and \( f_s \in I(s, \eta, \psi^{-1}) \).

**2E. Unramified calculation.** The unramified calculation of Method C is in fact not included in [Gelbart et al. 1987], but it can be simply done in the SL₂-case.

Let \( F \) be a nonarchimedian local field with odd residue characteristic. Suppose everything is unramified. Then the character \( \mu_\psi \) is unramified, [Szpruch 2009, p. 2188]. Suppose the representation \( (\pi, V) \) has Satake parameter \( a \), which means that \( \pi \) is the unramified component \( \text{Ind}_{B(F)}^{\text{SL}_2(F)}(v) \) for an unramified character \( v \) and \( a = v(p_F) \), where \( p_F \) is some prime element of \( F \). Let
\[ b_k = t(p_F^k) = \text{diag}(p_F^k, p_F^{-k}), \]
and \( W \) be the spherical Whittaker functional normalized by \( W(e) = 1 \). Then \( W(b_k) = 0 \) for \( k < 0 \) and
\[ W(b_k) = \frac{q^{-k}}{a-1}(a^{k+1} - a^{-k}), \]
by the general Casselman–Shalika formula [1980, Theorem 5.4]. For \( k \geq 0 \) we have
\[ (\omega_{\psi^{-1}}(b_k)\phi)(1) = \mu_{\psi^{-1}}(p_F^k)|p_F^k|^{1/2}, \]
where \( \phi \) is the characteristic function of the ring of integers \( \mathcal{O}_F \). On the other hand, let \( f_s \) be the standard spherical section of \( I(s, \eta, \psi^{-1}) \) normalized by \( f_s(1) = 1 \). Then we have
\[ f_s(b_k) = \eta(p_F^k)|p_F^k|^{s+1/2}\mu_{\psi^{-1}}(p_F^k). \]
A STRONG MULTIPLICITY ONE THEOREM FOR SL

Since \( \mu_{\psi^{-1}}(p_F^k) \mu_{\psi^{-1}}(p_F^k) = (p_F^k, p_F^k)_F = (p_F, -1)^k_F \), we have

\[
\Psi(W, \phi, f_s) = \int_{F^\times} \int_K W(t(a)k) \omega_{\psi^{-1}}(t(ak)\phi)(1) f_s(t(a)k) |a|^{-2} \, dk \, da
\]

\[
= \int_{F^\times} W(t(a)) \omega_{\psi^{-1}}(t(a)) \phi(1) f_s(t(a)) |a|^{-2} \, da
\]

\[
= \sum_{k \geq 0} W(b_k) (\omega_{\psi^{-1}}(b_k) \phi)(1) f_s(b_k) |p_F|^k |^{-2}
\]

\[
= \frac{1}{a - 1} \sum_{k \geq 0} (a^{k+1} - a^{-k})(p_F, -1)^k \eta(p_F)^k q_F^{-ks}
\]

\[
= \frac{1 + c}{(1 - ac)(1 - a^{-1}c)} = \frac{1 - c^2}{((1 - ac)(1 - c)(1 - a^{-1}c)}
\]

\[
= \frac{L(s, \pi, St \otimes \eta \chi)}{L(2s, \eta^2)},
\]

where \( c = (p_F, -1) \eta(p_F) q_F^{-s} \), and \( \chi(a) = (a, -1)_F \). Recall that \( St \) is the standard representation of \( L \) \( SL_2 = SO_3(\mathbb{C}) \).

**Remark.** From the calculation of the \( \mu_{\psi} \) given in [Szpruch 2009, Lemmas 1.5 and 1.10], one can check that

\[
M_s(f_s) = \frac{L(2s - 1, \eta^2)}{L(2s, \eta^2)} f_{1-s},
\]

where \( f_s \) and \( f_{1-s} \) are the standard spherical sections in, respectively, \( I(s, \eta, \psi^{-1}) \) and \( I(1-s, \eta^{-1}, \psi^{-1}) \). Thus the factor \( L(2s, \eta^2) \) appearing in the above unramified calculation will play the role of the normalizing factor of a global intertwining operator or Eisenstein series.

### 3. Howe vectors and the local converse theorem

In this section, we assume \( F \) is a \( p \)-adic field with odd residue characteristic. We will follow Baruch’s method [1995; 1997] to give a proof of the local converse theorem for generic representations of \( SL_2(F) \).

**3A. Howe vectors.** Let \( \psi \) be an unramified character. For a positive integer \( m \), let \( K_m = (1 + M_{2 \times 2}(\mathbb{O}_F^{m^*})) \cap SL_2(F) \) where \( \mathbb{O}_F = (p_F) \) denotes the maximal ideal in \( \mathbb{O}_F \). Define a character \( \tau_m \) of \( K_m \) by

\[
\tau_m(k) = \psi(p_F^{-2m}k_{12})
\]

for \( k = (k_{ij}) \in K_m \). It is easy to see that \( \tau_m \) is indeed a character on \( K_m \).
Let $d_m = t(p_F^{-m})$. Consider the subgroup $J_m = d_m K_m d_m^{-1}$. Then

$$J_m = \begin{pmatrix} 1 + \mathfrak{p}_F^m & \mathfrak{p}_F^{-m} \\ \mathfrak{p}_F^{-3m} & 1 + \mathfrak{p}_F^m \end{pmatrix} \cap \text{SL}_2(F).$$

Define $\psi_m(j) = \tau_m(d_m^{-1} j d_m)$ for $j \in J_m$. For a subgroup $H \subset \text{SL}_2(F)$, denote $H_m = H \cap J_m$. It is easy to check that $\psi_m|_{N_m} = \psi|_{N_m}$.

Let $\pi$ be an irreducible smooth $\psi$-generic representation of $\text{SL}_2(F)$ and let $v \in V_\pi$ be a vector such that $W_v(1) = 1$. For $m \geq 1$, as in [Baruch 1995; 1997] we consider

$$v_m = \frac{1}{\text{Vol}(N_m)} \int_{N_m} \psi(n)^{-1} \pi(n)v dn.$$

Let $L \geq 1$ be an integer such that $v$ is fixed by $K_L$. Following E. M. Baruch, we call $v_m, m \geq L$ Howe vectors.

**Lemma 3.1.** We have:

1. $W_{v_m}(1) = 1$.
2. If $m \geq L$ then $\pi(j)v_m = \psi_m(j)v_m$ for all $j \in J_m$.
3. If $k \leq m$ then

$$v_m = \frac{1}{\text{Vol}(N_m)} \int_{N_m} \psi(u)^{-1} \pi(u)v_k du.$$

The proof of this lemma is the same as the proof in the $U(2, 1)$ case, which is given in [Baruch 1997, Lemma 5.2].

**Lemma 3.2.** Let $m \geq L$ and $t = t(a)$ for $a \in F^\times$:

1. If $W_{v_m}(t) \neq 0$, we have

$$a^2 \in 1 + \mathfrak{p}_F^m.$$

2. If $W_{v_m}(tw) \neq 0$, we have

$$a^2 \in \mathfrak{p}_F^{-3m}.$$

**Proof.**

1. Take $x \in \mathfrak{p}_F^{-m}$. We then have $n(x) \in N_m \subset J_m$. From the relation

$$tn(x) = n(a^2x)t$$

and (2) of Lemma 3.1 we have

$$\psi(x)W_{v_m}(t) = \psi(a^2x)W_{v_m}(t).$$

If $W_{v_m}(t) \neq 0$ we get $\psi(x) = \psi(a^2x)$ for all $x \in \mathfrak{p}_F^{-m}$. Since $\psi$ is unramified we get $a^2 \in 1 + \mathfrak{p}_F^m$. 

3. Let $d_m = t(p_F^{-m})$. Consider the subgroup $J_m = d_m K_m d_m^{-1}$. Then

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1. Take $x \in \mathfrak{p}_F^{-m}$. We then have $n(x) \in N_m \subset J_m$. From the relation

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Let $L \geq 1$ be an integer such that $v$ is fixed by $K_L$. Following E. M. Baruch, we call $v_m, m \geq L$ Howe vectors.

**Lemma 3.1.** We have:

1. $W_{v_m}(1) = 1$.
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The proof of this lemma is the same as the proof in the $U(2, 1)$ case, which is given in [Baruch 1997, Lemma 5.2].

**Lemma 3.2.** Let $m \geq L$ and $t = t(a)$ for $a \in F^\times$:

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**Proof.**

1. Take $x \in \mathfrak{p}_F^{-m}$. We then have $n(x) \in N_m \subset J_m$. From the relation

$$tn(x) = n(a^2x)t$$

and (2) of Lemma 3.1 we have

$$\psi(x)W_{v_m}(t) = \psi(a^2x)W_{v_m}(t).$$

If $W_{v_m}(t) \neq 0$ we get $\psi(x) = \psi(a^2x)$ for all $x \in \mathfrak{p}_F^{-m}$. Since $\psi$ is unramified we get $a^2 \in 1 + \mathfrak{p}_F^m$. 

(2) For \( x \in \mathbb{P}^{3m} \) we have \( \tilde{n}(x) \in \mathbb{N}_m \). From the relation \( tw\tilde{n}(x) = n(-a^2x)tw \) and Lemma 3.1 (2) we get
\[
W_{v_m}(tw) = \psi(-a^2x)W_{v_m}(tw).
\]
Thus if \( W_{v_m}(tw) \neq 0 \) we get \( \psi(-a^2x) = 1 \) for all \( x \in \mathbb{P}^{3m} \). Thus \( a^2 \in \mathbb{P}^{-3m} \). \( \square \)

**Lemma 3.3.** For \( m \geq 1 \) the squaring map from \( 1 + \mathbb{P}^m \rightarrow 1 + \mathbb{P}^m \), sending \( a \mapsto a^2 \), is well-defined and surjective.

This lemma requires that the residue field of \( F \) is not of characteristic 2 which we assume throughout this section.

**Proof.** For \( x \in \mathbb{P}^m \), it is clear that \( (1 + x)^2 = 1 + 2x + x^2 \in 1 + \mathbb{P}^m \). Thus the square map is well-defined. On the other hand, we take \( u \in 1 + \mathbb{P}^m \) and consider the equation \( f(X) := X^2 - u = 0 \). We have \( f'(X) = 2X \). Since \( q^{-m} = |1 - u| = |f(1)| < |f'(1)|^2 = 2 |1| = 1 \) by Newton’s Lemma, see for example [Lang 1994, Proposition 2, Chapter II], there is a root \( a \in \mathbb{O}_F \) of \( f(X) \) such that
\[
|a - 1| \leq \frac{|f(1)|}{|f'(1)|^2} = |1 - u| = q^{-m}.
\]
Thus we get a root \( a \in 1 + \mathbb{P}^m \) of \( f(X) \). This completes the proof. \( \square \)

Let \( Z = \{ \pm 1 \} \) and identify \( Z \) with the center of \( \text{SL}_2(F) \). Denote by \( \omega_\pi \) the central character of \( \pi \).

**Corollary 3.4.** Let \( m \geq L \). Then we have
\[
W_{v_m}(t(a)) = \begin{cases} 
\omega_\pi(z) & \text{if } a = za' \text{ for some } z \in Z \text{ and } a' \in 1 + \mathbb{P}^m, \\
0 & \text{otherwise}.
\end{cases}
\]

**Proof.** Suppose that \( W_{v_m}(t(a)) \neq 0 \). Then by Lemma 3.2 we have \( a^2 \in 1 + \mathbb{P}^m \).

By Lemma 3.3 there exists an \( a' \in 1 + \mathbb{P}^m \) such that \( a^2 = (a')^2 \). Thus \( a = za' \) for some \( z \in Z \). Since \( a' \in 1 + \mathbb{P}^m \) we get \( t(a') \in J_m \). The assertion follows from Lemma 3.1. \( \square \)

From now on, we fix two \( \psi \)-generic representations \((\pi, V_\pi)\) and \((\pi', V_{\pi'})\) with the same central characters. Fix \( v \) and \( v' \) such that \( W_v(1) = 1 = W_{v'}(1) \). Let \( L \) be an integer such that both \( v \) and \( v' \) are fixed by \( K_L \). For \( m \geq 1 \) consider the Howe vectors \( v_m \) and \( v'_m \).

By Corollary 3.4 and the fact that \( \omega_\pi = \omega_{\pi'} \) we get the following:

**Corollary 3.5.** For \( m \geq L \) we have \( W_{v_m}(g) = W_{v'_m}(g) \) for all \( g \in B \).

**Lemma 3.6** (Baruch). If \( m \geq 4L \) and \( n \in N - N_m \) we have
\[
W_{v_m}(twn) = W_{v'_m}(twn),
\]
for all \( t \in T \).
Proof. This is a special case of [Baruch 1995, Lemma 6.2.2]. A similar result for U(2, 1) is given in [Baruch 1997, Proposition 5.7]. We just remark that the proof of this lemma depends on Corollary 3.5, and hence requires that the residue characteristic of \( F \) is not 2. \( \square \)

3B. Induced representations. Note that \( \overline{N}(F) \) and \( N(F) \) split in \( \text{SL}_2(F) \). Moreover, for \( g_1 \in N \) and \( g \in \overline{N} \) we have \( c(g_1, g_2) = 1 \). In fact if \( g_1 = n(y) \) and \( g_2 = \tilde{n}(x) \) with \( x \neq 0 \) we have \( x(g_1) = 1 \) and \( x(g_2) = x \). Thus

\[
c(g_1, g_2) = (1, x)_F (-x, x)_F = 1.
\]

This shows that \( N(F) \cdot \overline{N}(F) \subset \text{SL}_2(F) \), where \( \text{SL}_2(F) \) denotes the subset of \( \widehat{\text{SL}}_2(F) \) which consists of elements of the form \((g, 1)\) for \( g \in \text{SL}_2(F) \).

Let \( X \) be an open compact subgroup of \( N(F) \). For \( x \in X \) and \( i > 0 \) consider the set \( A(x, i) = \{ \tilde{n} \in \overline{N}(F) : \tilde{n}x \in B \cdot \overline{N}_i \} \).

Lemma 3.7. (1) For any positive integer \( c \) there exists an integer \( i_1 = i_1(X, c) \) such that for all \( i \geq i_1 \), \( x \in X \) and \( \tilde{n} \in A(x, i) \) we have

\[
\tilde{n}x = nt(a)\tilde{n}_0,
\]

with \( n \in N \), \( \tilde{n}_0 \in \overline{N}_i \) and \( a \in 1 + \mathcal{O}_c \).

(2) There exists an integer \( i_0 = i_0(X) \) such that for all \( i \geq i_0 \) we have \( A(x, i) = \overline{N}_i \).

Proof. By abuse of notation, for \( x \in X \) we write \( x = n(x) \). Since \( X \) is compact there is a constant \( C \) such that \( |x| < C \) for all \( n(x) \in X \subset N \).

For \( n(x) \in X \) and \( \tilde{n}(y) \in A(x, i) \) we have \( \tilde{n}(y)n(x) \in B \cdot \overline{N}_i \). Thus we can assume that

\[
\tilde{n}(y)n(x) = \begin{pmatrix} a & b \\ a^{-1} \\ \end{pmatrix} \tilde{n}(\tilde{y})
\]

for \( a \in F^\times \), \( b \in F \) and \( \tilde{y} \in \mathcal{O}_3^i \). Rewrite the above expression as

\[
\tilde{n}(-y) \begin{pmatrix} a & b \\ a^{-1} \\ \end{pmatrix} = n(x)\tilde{n}(-\tilde{y}),
\]

or

\[
\begin{pmatrix} a & b \\ -ay & a^{-1} - by \end{pmatrix} = \begin{pmatrix} 1 - x\tilde{y} & x \\ -\tilde{y} & 1 \end{pmatrix}.
\]

Thus we get

\[
a = 1 - x\tilde{y} \text{ and } ay = \tilde{y}.
\]

Since \( |x| < C \) and \( \tilde{y} \in \mathcal{O}_3^i \) it is clear that for any positive integer \( c \) we can choose \( i_1(X, c) \) such that \( a = 1 - x\tilde{y} \in 1 + \mathcal{O}_c \) for all \( n(x) \in X \) and \( \tilde{n}(y) \in A(x, i) \). This proves (1).
If we take \( i_0(X) = i_1(X, 1) \) we get \( a \in 1 + \mathcal{P} \subset \mathcal{O}_F^\times \) for \( i \geq i_0 \). From \( ay = \tilde{y} \) we get \( y \in \mathcal{P}^3i \). Thus for \( i \geq i_0(X) \) we have that \( \tilde{n}(y) \in \tilde{N}_i \), i.e., \( A(x, i) \subset \tilde{N}_i \).

The other direction can be checked similarly if \( i \) is large. We omit the details. \( \square \)

Given a positive integer \( i \) and a complex number \( s \in \mathbb{C} \) we consider the following function \( f_s^i \) on \( \tilde{SL}_2(F) \):

\[
    f_s^i(\tilde{g}) = \begin{cases} 
        \zeta \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) \quad & \text{if } \tilde{g} = \left( \begin{pmatrix} a & b \\ a^{-1} & \zeta \end{pmatrix}, \zeta \right) \tilde{n}(x), \\
        0 \quad & \text{otherwise.}
    \end{cases}
\]

\( \zeta \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) \) with \( a \in F^\times, b \in F, \zeta \in \mu_2, x \in \mathcal{P}^3i \), \( \tilde{n}(x) \) for all \( x \in X \), where \( \tilde{f}_s^i = M_s(f_s^i) \) and \( w = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

**Lemma 3.8.** (1) There exists an integer \( i_2(\eta) \) such that for all \( i \geq i_2(\eta) \), \( f_s^i \) defines a section in \( I(s, \eta, \psi^{-1}) \).

(2) Let \( X \) be an open compact subset of \( N \). There exists an integer \( I(X, \eta) \geq i_2(\eta) \) such that for all \( i \geq I(X, \eta) \) we have

\[
    \tilde{f}_s^i(wx) = \vol(\tilde{N}_i) = q_F^{-3i}
\]

for all \( x \in X \), where \( \tilde{f}_s^i = M_s(f_s^i) \) and \( w = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \).

**Proof.** (1) From the definition it is clear that

\[
    f_s^i\left( \left( \begin{pmatrix} a & b \\ a^{-1} & \zeta \end{pmatrix}, \zeta \right) \tilde{g} \right) = \zeta \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) f_s^i(\tilde{g}),
\]

for \( a \in F^\times, b \in F, \zeta \in \mu_2 \) and \( \tilde{g} \in \tilde{SL}_2(F) \). It suffices to show that for \( i \) large there is an open compact subgroup \( \tilde{H}_i \subset \tilde{SL}_2(F) \) such that \( f_s^i(\tilde{g}h) = f_s^i(\tilde{g}) \) for all \( \tilde{g} \in \tilde{SL}_2(F) \) and \( \tilde{h} \in \tilde{H}_i \).

If \( \psi \) is unramified and the residue characteristic is not 2 as we assumed then the character \( \mu_{\psi^{-1}} \) is trivial on \( \mathcal{O}_F^\times \), see for example [Szpruch 2009, p. 2188].

Let \( c \) be a positive integer such that \( \eta \) is trivial on \( 1 + \mathcal{P}^c \). Let \( i_2(\eta) = \max\{c, i_0(N \cap K_c), i_1(N \cap K_c, c)\} \). For \( i \geq i_2(\eta) \) we take \( \tilde{H}_i = K_{4i} = 1 + M_2(\mathcal{P}^{4i}) \). Note that \( K_{4i} \) splits and thus can be viewed as a subgroup of \( \tilde{SL}_2 \). We now check that for \( i \geq i_2(\eta) \) we have \( f_s^i(\tilde{g}h) = f_s^i(\tilde{g}) \) for all \( \tilde{g} \in \tilde{SL}_2 \) and \( h \in K_{4i} \). We have the decomposition \( K_{4i} = (N \cap K_{4i})(T \cap K_{4i})(\tilde{N} \cap K_{4i}) \). For \( h = \tilde{n} \in \tilde{N} \cap K_{4i} \subset \tilde{N}_i \) we have \( f_s^i(\tilde{g}h) = f_s^i(\tilde{g}) \) by the definition of \( f_s^i \). Now we take \( h \in T \cap K_{4i} \). Write \( h = t(a_0) \) with \( a_0 \in 1 + \mathcal{P}^{4i} \). We have \( \tilde{n}(x)h = h\tilde{n}(a_0^{-2}x) \). It is clear that \( x \in \mathcal{P}^{3i} \) if and only if \( a_0^{-2}x \in \mathcal{P}^{3i} \). On the other hand, for any \( a = F^\times \) and \( b \in F \) we have

\[
    c\left( \begin{pmatrix} a & b \\ a^{-1} & \zeta \end{pmatrix}, t(a_0) \right) = (a, a_0) = 1.
\]
since $a_0 \in 1 + \mathcal{P}^{4i}_F \subset F^\times$ by Lemma 3.3. Thus we get
\[
\left(\begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \zeta \right) \tilde{n}(x)h = \left(\begin{pmatrix} a_0 & ba_0^{-1} \\ a^{-1}a_0^{-1} \end{pmatrix}, \zeta \right) \tilde{n}(a_0^{-2}x).
\]
By the definition of $f^i_s$, if $x \in \mathcal{P}^{3i}$ for $g = \left(\begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \zeta \right) \tilde{n}(x)$ then we get
\[
f^i_s(gh) = \mu_{\psi^{-1}}(a_0a)\eta_{s+1/2}(aa_0) = \mu_{\psi^{-1}}(a)\eta_{s+1/2}(a) = f^i_s(g)
\]
by the assumption on $i$.

Finally, we consider $h \in N \cap K_{4i} \subset N \cap K_c$. By the assumption on $i$ we get
\[
A(h, i) = A(h^{-1}, i) = \tilde{N}_i.
\]
In particular, for $\tilde{n} \in \tilde{N}_i$ we have $\tilde{n}h \in B \cdot \tilde{N}_i$ and $\tilde{n}h^{-1} \in B \cdot \tilde{N}_i$. Now it is clear that $\tilde{g} \in \tilde{B} \cdot \tilde{N}_i$ if and only if $\tilde{gh} \in \tilde{B} \cdot \tilde{N}_i$. Thus $f^i_s(\tilde{g}) = 0$ if and only if $f^i_s(\tilde{gh}) = 0$. Moreover, for $\tilde{n} \in \tilde{N}_i$, we have
\[
\tilde{n}h = \begin{pmatrix} a_0 & b_0 \\ a_0^{-1} \end{pmatrix} \tilde{n}_0
\]
for $a_0 \in 1 + \mathcal{P}^c$, $b_0 \in F$ and $\tilde{n}_0 \in \tilde{N}_i$. Thus for $\tilde{g} = \left(\begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \zeta \right) \tilde{n}$ with $\tilde{n} \in \tilde{N}_i$ we get
\[
\tilde{gh} = \left(\begin{pmatrix} aa_0 & ab_0 + a_0^{-1}b \\ a^{-1}a_0^{-1} \end{pmatrix}, \zeta \right) \tilde{n}_0.
\]
Here we used the fact that $a_0 \in 1 + \mathcal{P}^c$ is a square and thus
\[
c\left(\begin{pmatrix} a & b \\ a^{-1} \end{pmatrix}, \begin{pmatrix} a_0 & b_0 \\ a_0^{-1} \end{pmatrix} \right) = 1.
\]
Since $\mu_{\psi^{-1}}(a_0) = 1$, $(a, a_0) = 1$ and $\eta_{s+1/2}(a_0) = 1$ we get
\[
f^i_s(\tilde{gh}) = f^i_s(g).
\]
This finishes the proof of (1).

(2) As in the proof of (1) let $c$ be a positive integer such that $\eta$ is trivial on $1 + \mathcal{P}^c$. Take $I(X, \eta) = \max\{i_1(X, c), i_0(X)\}$. We have
\[
\tilde{f}^i_s(wx) = \int_N f^i_s(w^{-1}nw)dn.
\]
By the definition of $f^i_s$, $f^i_s(w^{-1}nw) \neq 0$ if and only if $w^{-1}nw \in B\tilde{N}_i$ if and only if $w^{-1}nw \in A(x, i) = \tilde{N}_i$ for all $i \geq I(X)$ and $x \in X$. On the other hand, if
w^{-1}nw \in A(x, i), we have

\begin{align*}
w^{-1}nwx &= \begin{pmatrix} a & b \\ \frac{1}{a} & 0 \end{pmatrix} \tilde{n}0
\end{align*}

with \( a \in 1 + \mathfrak{P}^c_F \). Thus

\[ f^i_s (w^{-1}nwx) = \eta_{s+1/2}(a) \mu_{\psi^{-1}}(a) = 1. \]

Now the assertion is clear. \( \square \)

3C. The local converse theorem.

Lemma 3.9. Let \( \phi^m \) be the characteristic function of \( 1 + \mathfrak{P}^m \). Then

1. for \( n \in N_m \) we have \( \omega_{\psi^{-1}}(n)\phi^m = \psi^{-1}(n)\phi^m \), and
2. for \( \tilde{n} \in \tilde{N}_m \) we have \( \omega_{\psi^{-1}}(\tilde{n})\phi^m = \phi^m \).

Proof.

1. For \( n = n(b) \in N_m \) we have \( b \in \mathfrak{P}^m \). For \( x \in 1 + \mathfrak{P}^m \) we have \( bx^2 - b \in \mathcal{O}_F \). Thus

\[ \omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = \psi^{-1}(b)\phi^m(x). \]

For \( x \notin 1 + \mathfrak{P}^m \) we have \( \omega_{\psi^{-1}}(n)\phi^m(x) = \psi^{-1}(bx^2)\phi^m(x) = 0 \). The first assertion follows.

2. For \( \tilde{n} \in \tilde{N}_m \) we can write \( \tilde{n} = w^{-1}n(b)w \) with \( b \in \mathfrak{P}^{3m} \). Let \( \phi' = \omega_{\psi^{-1}}(w)\phi^m \). We have

\[ \phi'(x) = \gamma(\psi^{-1}) \int_{\mathcal{O}} \phi^m(y)\psi^{-1}(2xy)dy \]

\[ = \gamma(\psi^{-1})\psi^{-1}(2x) \int_{\mathfrak{P}^m} \psi^{-1}(2xz)dz \]

\[ = \gamma(\psi^{-1})\psi^{-1}(2x) \text{vol}(\mathfrak{P}^m) \text{Char}(\mathfrak{P}^{-m})(x), \]

where \( \text{Char}(\mathfrak{P}^{-m}) \) denotes the characteristic function of the set \( \mathfrak{P}^{-m} \). It is clear that \( \omega_{\psi^{-1}}(n(b))\phi' = \phi' \). Thus we have

\[ \omega_{\psi^{-1}}(\tilde{n})\phi^m = \omega_{\psi^{-1}}(w^{-1}n(b))\phi' = \omega_{\psi^{-1}}(w^{-1})\phi' = \omega_{\psi^{-1}}(w^{-1})\omega_{\psi^{-1}}(w)\phi^m = \phi^m. \]

This completes the proof. \( \square \)

Given a quasicharacter \( \eta \) of \( F^x \) recall that we have defined a local gamma factor \( \gamma(s, \pi, \eta, \psi) \) in Proposition 2.2.

Theorem 3.10. Suppose that the residue characteristic of \( F \) is not 2 and \( \psi \) is a nontrivial additive character of \( F \). Let \( (\pi, V_\pi) \) and \( (\pi', V_{\pi'}) \) be two \( \psi \)-generic representations of \( \text{SL}_2(F) \) with the same central character.
(1) If \( \gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi) \) for all quasicharacters \( \eta \) of \( F^\times \), then \( \pi \cong \pi' \).

(2) There is an integer \( l = l(\pi, \pi') \) such that if \( \eta \) is quasicharacter of \( F^\times \) with conductor \( \text{cond}(\eta) > l \), then

\[
\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).
\]

**Remark.** Theorem 3.10 can be viewed as one example of a general local converse conjecture for classical groups, see [Jiang 2006, Conjecture 3.7] or [Jiang and Nien 2013, Conjecture 6.3].

**Proof.** We will first treat the case where \( \psi \) is unramified and prove the general case at the end. We fix the notations \( v \in V_{\pi}, v' \in V_{\pi'} \) and \( L \) as before.

Let \( \eta \) be a quasicharacter of \( F^\times \). We take an integer \( m \geq \max\{6L, \text{cond}(\eta)\} \) and consider the Howe vectors \( v_m \) and \( v'_m \). Additionally, we take an integer \( i \geq \max\{i_2(\eta), I(N_m, \eta), m\} \). In particular we have a section \( f_s^i \in I(s, \eta, \psi) \) as in Section 3C. Let \( W_m = W_{v_m} \) or \( W_{v'_m} \). We compute the integral of \( \Psi(W_m, \phi^m, f_s^i) \) on the open dense subset \( T\tilde{N}(F) = N(F) \setminus N(F)T\tilde{N}(F) \) of \( N(F) \setminus \text{SL}_2(F) \).

For \( g = nt(a)\tilde{n} \) we can take the quotient measure as \( dg = |a|^{-2}d\tilde{n}da \). By the definition of \( f_s^i \) we get

\[
\Psi(W_m, \phi^m, f_s^i) = \int_{T \times \tilde{N}(F)} W_m(t(a)\tilde{n})(\omega_{\psi^{-1}}(t(a)\tilde{n})\phi^m)(1)f_s^i(t(a)\tilde{n})|a|^{-2}d\tilde{n}da
\]

\[
= \int_{T \times \tilde{N}_i} W_m(t(a)\tilde{n})\mu_{\psi^{-1}}(a)|a|^{1/2}\omega_{\psi^{-1}}(\tilde{n})
\]

\[
\cdot \phi^m(a)\mu_{\psi^{-1}}(a)\eta_{s+1/2}(a)|a|^{-2}d\tilde{n}da
\]

\[
= \int_{T \times \tilde{N}_i} W_m(t(a)\tilde{n})\omega_{\psi^{-1}}(\tilde{n})\phi^m(a)\chi(a)\eta_{s-1}(a)d\tilde{n}da,
\]

where \( \chi(a) = \mu_{\psi^{-1}}(a)\mu_{\psi^{-1}}(a) = (a, -1)_F \). Since \( i \geq m \) we get \( \tilde{N}_i \subset \tilde{N}_m \). By Lemmas 3.1 and 3.9 we get \( W_m(t(a)\tilde{n}) = W_m(t(a)) \) and \( \omega_{\psi^{-1}}(\tilde{n})\phi^m = \phi^m \). Thus we get

\[
\Psi(W_m, \phi^m, f_s^i) = q^{-3i} \int_{F^\times} W_m(t(a))\phi^m(a)\chi(a)\eta_{s-1}(a)da.
\]

Since \( \phi^m = \text{Char}(1 + \mathcal{P}^m) \) and, for \( a \in 1 + \mathcal{P}^m \), we have \( W_m(t(a)) = 1 \). By Lemma 3.1 we get

\[
\Psi(W_m, \phi^m, f_s^i) = q^{-3i} \int_{1 + \mathcal{P}^m} \chi(a)\eta(a)da.
\]

Since \( \chi(a) = 1 \) for \( a \in 1 + \mathcal{P}^m \) and \( m \geq \text{cond}(\eta) \) by assumption we get

\[
\Psi(W_m, \phi^m, f_s^i) = q^{-3i-m}.
\]
The above calculation works for both \( W_{v_m} \) and \( W_{v'_m} \). Thus we have

\[
\Psi(W_{v_m}, \phi^m, f_s^i) = \Psi(W_{v'_m}, \phi^m, f_s^i) = q^{-3i-m}.
\]

Next we compute the other side of the local functional equation, \( \Psi(W_m, \phi^m, f_s^i) \), on the open dense subset \( N(F) \setminus N(F)TwN(F) \subset N(F) \setminus SL_2(F) \), where \( f_s^i = M_s(f_s^i) \).

We have

\[
\Psi(W_m, \phi^m, f_s^i) = \int_{T \times N(F)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)f_s^i(t(a)wn)|a|^{-2}dnda
\]

\[
= \int_{T \times N_m} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)f_s^i(t(a)wn)|a|^{-2}dnda
\]

\[
+ \int_{T \times (N(F) - N_m)} W_m(t(a)wn)(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)f_s^i(t(a)wn)|a|^{-2}dnda.
\]

By Lemma 3.6 we get \( W_{v_m}(t(a)wn) = W_{v'_m}(t(a)wn) \) for all \( n \in N(F) - N_m \). Thus

\[
\Psi(W_{v_m}, \phi^m, f_s^i) - \Psi(W_{v'_m}, \phi^m, f_s^i)
\]

\[
= \int_{T \times N_m} (W_{v_m}(t(a)wn) - W_{v'_m}(t(a)wn))(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1)
\]

\[
\cdot f_s^i(t(a)wn)|a|^{-2}dnda.
\]

Since \( i \geq I(N_m, \eta) \) we get

\[
f_s^i(t(a)wn) = \mu_{\psi^{-1}}(a)\eta^{-1}(a)q_F^{-3i}
\]

by Lemma 3.8. On the other hand, by Lemma 3.1 and Lemma 3.9, for \( n \in N_m \) we get

\[
W_m(t(a)wn) = \psi(n)W_m(t(a)w),
\]

\[
(\omega_{\psi^{-1}}(t(a)wn)\phi^m)(1) = \psi^{-1}(n)(\omega_{\psi^{-1}}(t(a)w)\phi^m)(1).
\]

Thus

\[
\Psi(W_{v_m}, \phi^m, f_s^i) - \Psi(W_{v'_m}, \phi^m, f_s^i)
\]

\[
= q_F^{-3i+m} \int_{T} (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)
\]

\[
\cdot \chi(a)\eta^{-1}(a)|a|^{-s}da.
\]

By (3-2), (3-3) and the local functional equation we get

\[
q^{-2m}(\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi))
\]

\[
= \int_{F \times} (W_{v_m}(t(a)w) - W_{v'_m}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da.
\]
Let $k = 4L$. Since $m \geq 6L > k$, by Lemmas 3.1 and 3.6, we get
\[
W_{v_m}(t(a)w) - W_{v'_m}(t(a)w) = \frac{1}{\text{vol}(N_m)} \int_{N_m} (W_{v_k}(t(a)wn) - W_{v'_k}(t(a)wn))\psi^{-1}(n)dn
\]
\[
= \frac{1}{\text{vol}(N_m)} \int_{N_k} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(n)dn
\]
\[
= \text{vol}(N_k) \text{vol}(N_m)^{-1} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))
\]
\[
= q^{k-m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w)).
\]

Now we can rewrite (3-4) as
\[
(3-5) \quad q^{-m-k}(\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi))
\]
\[
= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))(\omega_{\psi^{-1}}(w)\phi^m)(a)\chi(a)\eta^{-1}(a)|a|^{-s}da.
\]
By Lemma 3.2, if $a \notin \mathcal{P}^{-6L}$, i.e., $a^2 \notin \mathcal{P}^{-3k}$, we get $W_{v_k}(t(a)w) = W_{v'_k}(t(a)w)$. Thus the integral on the right side of formula (3-5) can be taken over $\mathcal{P}^{-6L}$. For $a \in \mathcal{P}^{-6L}$ and $m \geq 6L$ (as we assumed), by the calculation given in the proof of Lemma 3.9, we have
\[
(\omega_{\psi^{-1}}(w)\phi^m)(a) = \gamma(\psi^{-1})\psi^{-1}(2a) \text{vol}(\mathcal{P}^m) \text{Char}(\mathcal{P}^{-m})(a)
\]
\[
= \gamma(\psi^{-1})\psi^{-1}(2a)q^{-m}.
\]
Plugging this into (3-5) we get
\[
(3-6) \quad q^{-k}\gamma(\psi^{-1})^{-1}(\gamma(s, \pi, \eta, \psi) - \gamma(s, \pi', \eta, \psi))
\]
\[
= \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da.
\]
Now we can prove our theorem. We consider (1) first. Suppose $\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi)$ for all quasicharacters $\eta$ of $F^\times$. Then we get
\[
\int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da = 0
\]
for all quasicharacters $\eta$.

We rewrite the equality as
\[
0 = \int_{F^\times} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)|a|^{-s}da
\]
\[
= \sum_{m=-\infty}^{\infty} \int_{|a|=q^m} (W_{v_k}(t(a)w) - W_{v'_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)da q^{-ms}.
\]
It follows that all the coefficients in the above Laurent series in $q^s$ have to be zero. So

$$\int_{|a|=q^m} (W_{v_k}(t(a)w) - W'_{v_k}(t(a)w))\psi^{-1}(2a)\chi(a)\eta^{-1}(a)da = 0$$

for all quasicharacters $\eta$.

Since the set $\{a \in F^\times : |a| = q^m\}$ is compact open in $F^\times$, the left side of equation (3-7) can be viewed as Mellin transform of a compactly supported function on $F^\times$. By the inverse Mellin transform we get

$$(W_{v_k}(t(a)w) - W'_{v_k}(t(a)w))\psi^{-1}(2a) = 0,$$

or

$$W_{v_k}(t(a)w) = W'_{v_k}(t(a)w).$$

By Lemmas 3.1 and 3.6, Corollary 3.5 and the Bruhat decomposition $SL_2(F) = B \cup BwB$ we get

$$W_{v_k}(g) = W'_{v_k}(g)$$

for all $g \in SL_2(F)$. By the uniqueness of Whittaker model we get $\pi \cong \pi'$. This proves (1).

Next we consider (2). Let $l = l(\pi, \pi')$ be an integer such that $l \geq 6L$, then

$$W_{v_k}(t(a_0a)w) = W_{v_k}(t(a)w) \quad \text{and} \quad W'_{v_k}(t(a_0a)w) = W'_{v_k}(t(a)w)$$

for all $a_0 \in 1 + \mathcal{P}^l$ and all $a \in \mathcal{P}^{-6L}$. Such an $l$ exists because the functions $a \mapsto W_{v_k}(t(a)w)$ and $a \mapsto W'_{v_k}(t(a)w)$ on $\mathcal{P}^{-6L} \subset F^\times$ are continuous. Note that $k = 4L$ and $L$ only depends on the choices of $v$ and $v'$. On the other hand, for $a \in \mathcal{P}^{-6L}$, it is easy to see that

$$\psi^{-1}(2a_0a) = \psi^{-1}(2a) \quad \text{for all} \ a_0 \in 1 + \mathcal{P}^l,$$

since $l \geq 6L$. It is also clear that $\chi(a_0a) = \chi(a)$ for all $a_0 \in 1 + \mathcal{P}^l$, since the character $\chi$ is unramified. As we noted before, the integrand of the right side integral of (3-6) has support in $\mathcal{P}^{-6L}$. Let $\eta$ be a quasicharacter of $F^\times$ with cond($\eta$) $> l$. Then it is clear that the integral of the right side of (3-6) vanishes. Thus we get

$$\gamma(s, \pi, \eta, \psi) = \gamma(s, \pi', \eta, \psi).$$

This finishes the proof when $\psi$ is unramified.

Now let us consider the general case when $\psi$ is ramified. The proof is essentially the same as the unramified case. We will indicate the necessary changes in the above proof. If $\psi$ has conductor $c$, i.e., $\psi(\mathcal{P}_F^c) = 1$ but $\psi(\mathcal{P}_F^{c-1}) \neq 1$, we define
\[ d_m = \text{diag}(p_F^{-2m+c}, 1) \in \text{GL}_2(F) \] and \[ J_m = d_m K_m a_m^{-1}. \] Then

\[ J_m = \begin{pmatrix} 1 + \mathfrak{p}^m_F & \mathfrak{p}^{-m+c}_F \\ \mathfrak{p}^{3m-c}_F & 1 + \mathfrak{p}^m_F \end{pmatrix} \cap \text{SL}_2(F). \]

For \( j = (j_{il})_{1 \leq i, l \leq 2} \in J_m \) we define \( \psi_m(j) = \psi(j_{12}) \). It is clear that \( \psi_m \) defines a character of \( J_m \). Given a \( \psi \)-generic representation \((\pi, V)\) of \( \text{SL}_2(F) \) and a vector \( v \in V \) we define \( v_m \) in the same way as before, i.e., by \( (3-1) \). In this case, we fix an integer \( L \) such that \( L \geq c \) and \( v \) is fixed by \( K_L \). We call \( \{v_m\}_{m \geq L} \) the Howe vectors. We note that in the proof of Lemma 3.8, we used that \( \psi \) is unramified to make sure \( \mu_{\psi^{-1}} \) is trivial on \( \mathbb{O}_F^\times \). If \( \psi \) is ramified, by continuity, \( \mu_{\psi^{-1}} \) is trivial on \( 1 + \mathfrak{p}_F^i \) for \( i \) large. This is all what we need in the proof of Lemma 3.8 to extend it to the ramified case. Now one can check easily that all of the above proofs go through and we get the theorem in general. \( \square \)

### 4. A strong multiplicity one theorem

Let \( F \) be a number field and \( \mathbb{A} \) be its adele ring.

**4A. Global genericity.** In this subsection we discuss the relation between global genericity and local genericity. Let \( \varphi \) be a cusp form on \( \text{SL}_2(F) \setminus \text{SL}_2(\mathbb{A}) \). Since the group \( N(F) \setminus N(\mathbb{A}) \) is compact and abelian we have the Fourier expansion

\[ \varphi(g) = \sum_{\psi \in \hat{N}(F) \setminus \hat{N}(\mathbb{A})} W_{\psi}(g), \]

where

\[ W_{\psi}(g) = \int_{N(F) \setminus N(\mathbb{A})} \varphi(ng) \psi^{-1}(n) dg. \]

Since \( \varphi \) is a cusp form we get \( W_{\psi_0} \equiv 0 \), where \( \psi_0 \) is the trivial character of \( F \setminus \mathbb{A} \). Thus we get

\[ \varphi(g) = \sum_{\psi \in \hat{N}(F) \setminus \hat{N}(\mathbb{A})} W_{\psi}(g). \]

Fix a nontrivial additive character \( \psi \) of \( N(F) \setminus N(\mathbb{A}) \). Then

\[ (N(F) \setminus \hat{N}(\mathbb{A})) \setminus \{\psi_0\} = \{\psi_\kappa : \kappa \in F^\times\}, \]

where \( \psi_\kappa(a) = \psi(\kappa a) \) and \( a \in \mathbb{A} \). If \( \kappa \in F^{\times,2} \), say \( \kappa = a^2 \), we have

\[ W_{\psi_\kappa}(g) = W_{\psi}(t(a)g). \]

Thus we get

\[ \varphi(g) = \sum_{\kappa \in F^{\times}/F^{\times,2}} \sum_{a \in F^\times} W_{\psi_\kappa}(t(a)g). \]
Corollary 4.1. If $\varphi$ is a nonzero cusp form, there exists $\kappa \in F^\times$ such that

$$W_{\varphi}^{\psi^\kappa} \neq 0.$$ 

Let $(\pi, V_\pi)$ be a cuspidal automorphic representation of $\text{SL}_2(F) \setminus \text{SL}_2(\mathbb{A})$. We say $\pi$ is $\psi_\kappa$-generic if there exists $\varphi \in V_\pi$ such that

$$W_{\varphi}^{\psi^\kappa} \neq 0.$$ 

Corollary 4.2. Let $\pi$ be a cuspidal automorphic representation of $\text{SL}_2(F) \setminus \text{SL}_2(\mathbb{A})$ and $\psi$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then there exists $\kappa \in F^\times$ such that $\pi$ is $\psi_\kappa$-generic.

Theorem 4.3. Let $\pi = \otimes_v \pi_v$ be an irreducible cuspidal automorphic representation of $\text{SL}_2(\mathbb{A})$ and $\psi = \otimes \psi_v$ be a nontrivial additive character of $F \setminus \mathbb{A}$. Then $\pi$ is $\psi$-generic if and only if each $\pi_v$ is $\psi_v$-generic.

Proof. A similar result is proved for $U(1, 1)$ by Gelbart, Rogawski and Soudry [1997, Proposition 2.5].

It is clear that global genericity implies local genericity. Now we consider the other direction. We assume each $\pi_v$ is $\psi_v$-generic.

We assume $\pi$ is $\psi_\kappa$-generic for some $\kappa \in F^\times$, i.e., there exists $\varphi \in V_\pi$ such that

$$W_{\varphi}^{\psi^\kappa}(g) = \int_{N(F) \setminus N(\mathbb{A})} \varphi(ng) \psi^{-1}_\kappa(n) dn \neq 0.$$ 

Then $\pi_v$ is also $\psi_{\kappa,v}$-generic, where $\psi_{\kappa,v}(a) = \psi_v(\kappa a)$. By Proposition 2.1 we get $\pi_v \simeq \pi^\kappa_v$.

For $\varphi \in V_\pi$ consider the function $\varphi^\kappa(g) = \varphi(g^\kappa)$, where $g^\kappa$ is defined by

$$g^\kappa = \text{diag}(\kappa, 1) g \text{ diag}(\kappa^{-1}, 1).$$

Then

$$\int_{N(F) \setminus N(\mathbb{A})} \varphi^\kappa(ng) dn = \int_{N(F) \setminus N(\mathbb{A})} \varphi((ng)^\kappa) dn = \int_{N(F) \setminus N(\mathbb{A})} \varphi(n^\kappa g^\kappa) dn$$

$$= \int_{N(F) \setminus N(\mathbb{A})} \varphi(ng^\kappa) dn = 0,$$

hence $\varphi^\kappa$ is also a cusp form. Let $V^\kappa_\pi$ be the space which consists of functions of the form $\varphi^\kappa$ for all $\varphi \in V_\pi$. Let $\pi^\kappa$ denote the cuspidal automorphic representation of $\text{SL}_2(\mathbb{A})$ on $V^\kappa_\pi$.

Claim. $(\pi^\kappa)_v = \pi^\kappa_v$.

Proof. Let $\Lambda : V_\pi \to \mathbb{C}$ be a nonzero $\psi_\kappa$-Whittaker functional for $\pi$ and let $\Lambda_v$ be a nonzero $(\psi_\kappa)_v$-Whittaker functional on $V_\pi_v$ satisfying that if $\varphi = \otimes_v \varphi_v$ is a pure
tensor, then
\[ \Lambda(\pi(g)\varphi) = \prod_v \Lambda_v(\pi_v(g_v)\varphi_v). \]

Note that \( \Lambda \) is in fact given by
\[ \Lambda(\varphi) = \int_{N(F) \setminus N(\mathbb{A})} \varphi(n)\psi_\kappa^{-1}(n)dn. \]

Then the \( \psi_\kappa^2 \)-Whittaker functional of \( \pi^\kappa \) is given by
\[ \int_{N(F) \setminus N(\mathbb{A})} \varphi^\kappa(n)\psi_\kappa^{-1}^2(n)dn. \]

This means that if \( W_\varphi(g) \) is a \( \psi_\kappa \)-Whittaker function of \( \pi \), then \( W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa) \) is a \( \psi_\kappa^2 \)-Whittaker function of \( \pi^\kappa \).

Hence, with \( \varphi = \otimes_v \varphi_v \) a pure tensor, we have \( W_\varphi(g) = \prod_v W_{\varphi_v}(g_v) \) and \( \{W_{\varphi_v}(g_v)\} \) is the Whittaker model of \( \pi_v \), while \( W_{\varphi^\kappa}(g) = W_\varphi(g^\kappa) = \prod_v W_{\varphi_v}(g_v^\kappa) \) and \( \{W_{\varphi_v}(g_v^\kappa)\} \) is the Whittaker model of \( (\pi^\kappa)_v \). Now \( W_{\varphi_v}(g_v) \rightarrow W_{\varphi^\kappa}(g_v^\kappa) \) gives an isomorphism between \( \pi^\kappa_v \) and \( (\pi^\kappa)_v \), which proves the claim.

Now let us continue the proof of the theorem. By the claim we have \( \pi_v \cong (\pi^\kappa)_v \) or \( \pi \cong \pi^\kappa \). By the multiplicity one theorem for \( SL_2 \) of Ramakrishnan [2000] we get \( \pi = \pi^\kappa \). Since \( \pi \) is \( \psi_\kappa \)-generic we get that \( \pi^\kappa \) is \( \psi_\kappa^2 \)-generic and hence \( \psi \)-generic. Since \( \pi = \pi^\kappa \) the theorem follows. \( \square \)

4B. Eisenstein series on \( \widetilde{SL}_2(\mathbb{A}) \). Let \( \widetilde{SL}_2(\mathbb{A}) \) be the double cover of \( SL_2(\mathbb{A}) \). It is well-known that \( SL_2(F) \) splits over the projection \( \tilde{SL}_2(\mathbb{A}) \rightarrow SL_2(\mathbb{A}) \). Let \( \mu_\psi \) be the genuine character of \( T(F) \setminus \tilde{T}(\mathbb{A}) \) whose local components are \( \mu_\psi \) as given in §2.

Let \( \eta \) be a quasicharacter of \( F^\times \setminus \mathbb{A}^\times \) and \( s \in \mathbb{C} \). We consider the induced representation
\[ I(s, \chi, \psi) = \text{Ind}_{B(\mathbb{A})}^{\tilde{SL}_2(\mathbb{A})}(\mu_\psi \eta s^{-1/2}). \]

For \( f_s \in I(s, \eta, \psi) \) we consider the Eisenstein series \( E(g, f_s) \) on \( \widetilde{SL}_2(\mathbb{A}) \):
\[ E(g, f_s) = \sum_{B(F) \setminus SL_2(F)} f_s(\gamma g), \ g \in \widetilde{SL}_2(\mathbb{A}). \]

The above sum is absolutely convergent when \( \text{Re}(s) \gg 0 \) and can be meromorphically continued to the whole \( s \)-plane.

There is an intertwining operator \( M_s = M_s(\eta) : I(s, \eta, \psi) \rightarrow I(1-s, \eta^{-1}, \psi) \) with
\[ M_s(f_s)(g) = \int_{N(F) \setminus N(\mathbb{A})} f_s(wng)dn. \]
The above integral is absolutely convergent for \( \Re(s) \gg 0 \) and defines a meromorphic function of \( s \in \mathbb{C} \).

**Proposition 4.4.** (1) If \( \eta^2 \neq 1 \), then the Eisenstein series \( E(g, f_s) \) is holomorphic for all \( s \). If \( \eta^2 = 1 \), the only possible poles of \( E(g, f_s) \) are at \( s = 0 \) and \( s = 1 \). Moreover, the order of the poles are at most 1.

(2) We have the functional equation

\[
E(g, f_s) = E(g, M_s(f_s)) \quad \text{and} \quad M_s(\eta) \circ M_{1-s}(\eta^{-1}) = 1.
\]


**4C. The global zeta-integral.** Let \( \psi \) be a nontrivial additive character of \( F \setminus A \). Then there is a global Weil representation \( \omega_{\psi} \) of \( \widetilde{SL}_2(A) \) on \( \mathcal{F}(A) \). For \( \phi \in \mathcal{F}(A) \) we consider the theta series

\[
\theta_{\psi}(\phi)(g) = \sum_{x \in F} (\omega_{\psi}(g)\phi)(x).
\]

It is well-known that \( \theta_{\psi} \) defines an automorphic form on \( \widetilde{SL}_2(A) \).

Let \((\pi, V_\pi)\) be a \( \psi \)-generic cuspidal automorphic representation of \( SL_2(A) \). For \( \varphi \in V_\pi, \phi \in \mathcal{F}(A) \) and \( f_s \in I(s, \eta, \psi^{-1}) \) consider the integral

\[
(4-1) \quad Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \int_{SL_2(F) \setminus SL_2(A)} \varphi(g)\theta_{\psi^{-1}}(\phi)(g)E(g, f_s)dg.
\]

**Proposition 4.5** [Gelbart et al. 1987, Theorem 4.C]. For \( \Re(s) \gg 0 \), the integral \( Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) \) is absolutely convergent and

\[
Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \int_{N(A) \setminus SL_2(A)} W^\psi_\varphi(g)(\omega_{\psi^{-1}}(g))\phi(1)f_s(g)dg,
\]

where \( W^\psi_\varphi(g) = \int_{N(F) \setminus N(A)} \varphi(ng)\psi^{-1}(n)dn \) is the \( \psi \)-th Whittaker coefficient of \( \varphi \).

**Corollary 4.6.** We take \( \varphi = \otimes \varphi_v, \phi = \otimes_v \phi_v \) and \( f_s = \otimes f_{s,v} \) to be pure tensors. Let \( S \) be a finite set of places such that for all \( v \notin S, v \) is finite and \( \pi_v, \psi_v, f_{s,v} \) are unramified. Then for \( \Re(s) \gg 0 \) we have

\[
Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = \prod_{v \in S} \Psi(W_{\varphi_v}, \phi_v, f_{s,v}) \frac{L^S(s, \pi, St \otimes (\chi \eta))}{L^S(2s, \eta^2)},
\]

where \( \chi \) is the character of \( F^\times \setminus A^\times \) defined by

\[
\chi((a_v)) = \prod_v (a_v, -1)_{F_v}, \quad (a_v)_v \in A^\times.
\]

Moreover, we have the following functional equation

\[
Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, f_s)) = Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, M_s(f_s))).
\]
This follows directly from Proposition 2.2, the unramified calculation, and the functional equation of Eisenstein series in Proposition 4.4.

**Corollary 4.7.** (1) The partial L-function $L^S(s, \pi, St \otimes \chi \eta)$ can be extended to a meromorphic function of $s$.

(2) If $\eta^2 \neq 1$, then $L^S(s, \pi, St \otimes \chi \eta)$ is holomorphic for $\text{Re}(s) > 1/2$.

(3) If $\eta^2 = 1$, then, on the region $\text{Re}(s) > 1/2$, the only possible pole of the function $L^S(s, \pi, St \otimes \chi \eta)$ is at $s = 1$. Moreover, the order of the pole of $L^S(s, \pi, St \otimes (\chi \eta))$ at $s = 1$ is at most 1.

(4) Let $S_\infty$ be the set of infinity places of $F$, then we can find data $\varphi_v \in V_{\pi_v}$, $\phi_v \in \mathcal{F}(F_v)$ and $f_{s,v} \in I(s, \eta_v, \psi_v)$ for $v \in S_\infty$ such that

$$L^S(s, \pi, St \otimes (\chi \eta))$$

$$= \prod_{v \in S_\infty} \frac{\Psi(W_{\varphi_v}, \phi_v, M_s(f_{s,v}))}{\Psi(W_{\varphi_v}, \phi_v, f_{s,v})} \cdot \prod_{v \in S-S_\infty} \gamma(s, \pi_v, \eta_v, \psi_v) \cdot \frac{L^S(2s - 1, \eta^2)}{L^S(2 - 2s, \eta^{-2})},$$

where $S$ is a large enough finite set of places which contains $S_\infty$, all finite places $v$ such that $v|2$ and all finite places such that our data is ramified. Here $\gamma(s, \pi_v, \eta_v, \psi_v)$ is the local gamma factors defined in Proposition 2.2.

**Proof.** By Proposition 4.4 and Corollary 4.6 to prove (1)-(3) it suffices to show that, for each place $v$ and for any fixed point $s \in \mathbb{C}$, we can choose the data $(W_v, \phi_v, f_{s,v})$ such that $\Psi(W_{\varphi_v}, \phi_v, f_{s,v}) \neq 0$. If $v$ is nonarchimedean this is shown in the proof of Theorem 3.10, see equation (3-2). We will prove the general case later, see Lemma 4.9. We now consider (4). For $s \in \mathbb{C}$ with $\text{Re}(s) \gg 0$ we choose data $\varphi = \otimes \varphi_v$, $\phi = \otimes \phi_v$ and $f_s = \otimes f_{s,v}$ such that $\Psi(W_{\varphi_v}, \phi_v, f_{s,v}) \neq 0$ for each $v \in S$ and $\varphi_v$, $\phi_v$, $f_{s,v}$ and $\psi_v$ are unramified for $v \notin S$. By the Remark at the end of §2, for $v \notin S$, we have

$$M_s(f_{s,v}) = \frac{L(2s - 1, \eta_v^2)}{L(2s, \eta_v^2)} f_{1-s,v}.$$

Thus, by Corollary 4.6, for $\text{Re}(s) \ll 0$ we have

$$Z(\varphi, \theta_{\psi^{-1}}(\phi), E(\cdot, M_s(f_s)))$$

$$= \prod_{v \in S} \Psi(W_{\varphi_v}, \phi_v, M_s(f_{s,v})) \frac{L^S(1-s, \pi, St \otimes (\chi \eta^{-1}))}{L^S(2-2s, \eta^{-2})} \cdot \frac{L^S(2s - 1, \eta^2)}{L^S(2s, \eta^2)}.$$

Note that the above equation also holds after meromorphic continuation. Now (4) follows from Corollary 4.6 and Proposition 2.2 directly. \qed
4D. A strong multiplicity one theorem. With the above preparation, we are now ready to prove the main global result of this paper.

**Theorem 4.8.** Let $\pi = \otimes \pi_v$ and $\pi' = \otimes \pi'_v$ be two irreducible cuspidal automorphic representations of $\text{SL}_2(\mathbb{A})$ with the same central character. Suppose that $\pi$ and $\pi'$ are both $\psi$-generic. Let $S$ be a finite set of finite places such that no place in $S$ is above 2. If $\pi_v \cong \pi'_v$ for all $v \notin S$, then $\pi = \pi'$.

**Proof.** The following argument follows from the proof of [Casselman 1973, Theorem 2, p. 307].

Let $S_1$ be a large finite set of places which contains $S_\infty \cup S$. Since $\pi_v \cong \pi'_v$ for all $v \notin S$, we have $L^{S_1}(s, \pi, St \otimes (\chi \eta)) = L^{S_1}(s, \pi', St \otimes (\chi \eta))$ and $L^{S_1}(1 - s, \pi, St \otimes (\chi \eta^{-1})) = L^{S_1}(1 - s, \pi', St \otimes (\chi \eta^{-1}))$. Thus, by Corollary 4.7 (4), for each quasicharacter $\eta$, we can find data $\varphi_v \in V_{\pi_v}$, $\phi_v \in \mathcal{F}(F_v)$ and $f_{s,v}$ for $v \in S_\infty$ such that

$$\prod_{v \in S_\infty} \frac{\Psi(W_{\varphi_v}, \phi_v, M_s(f_{s,v}))}{\Psi(W_{\varphi'_v}, \phi_v, f_{s,v})} \cdot \prod_{v \in S_1 - S_\infty} \gamma(s, \pi_v, \eta_v, \psi_v)$$

$$= \prod_{v \in S_\infty} \frac{\Psi(W_{\varphi'_v}, \phi_v, M_s(f_{s,v}))}{\Psi(W_{\varphi'_v}, \phi_v, f_{s,v})} \cdot \prod_{v \in S_1 - S_\infty} \gamma(s, \pi'_v, \eta_v, \psi_v),$$

where $\varphi'_v$ is the image of $\varphi_v$ under a fixed isomorphism $\pi_v \cong \pi'_v$ for $v \in S_\infty$. Since $\pi_v \cong \pi'_v$ for $v \in S_1 - S$, we get

$$\prod_{v \in S} \gamma(s, \pi_v, \eta_v, \psi_v) = \prod_{v \in S} \gamma(s, \pi'_v, \eta_v, \psi_v).$$

Fix $v_0 \in S$. By [Jacquet and Langlands 1970, Lemma 12.5], given an arbitrary character $\eta_{v_0}$, we can find a character $\eta$ of $\mathbb{A}^\times$ which restricted to $v_0$ is $\eta_{v_0}$ and has arbitrarily high conductor at the other places of $S$. By Theorem 3.10 (2) we conclude that

$$\gamma(s, \pi_{v_0}, \eta_{v_0}, \psi_{v_0}) = \gamma(s, \pi'_{v_0}, \eta_{v_0}, \psi_{v_0})$$

for all characters $\eta_{v_0}$. Thus, by Theorem 3.10 (1), we conclude that $\pi_{v_0} \cong \pi'_{v_0}$. This applies also to the other places of $S$. Thus we proved that $\pi_v \cong \pi'_v$ for all places $v$. Now the theorem follows from the multiplicity one theorem for $\text{SL}_2$ of [Ramakrishnan 2000]. □

**Remark.** We expect that the restriction about residue characteristics on the finite set $S$ in Theorem 4.8 can be removed.

Finally, we prove a nonvanishing result about the archimedean local zeta-integrals which is used in the above proof. We formulate and prove the result both for the $p$-adic and the archimedean cases simultaneously.
Lemma 4.9. Let \( F \) be a local field, \( \psi \) be a nontrivial additive character of \( F \), \( \eta \) be a quasicharacter of \( F^\times \) and \( \pi \) be a \( \psi \)-generic representation of \( \text{SL}_2(F) \). Then there exists \( W \in \mathcal{W}(\pi, \psi) \), \( \phi \in \mathcal{S}(F) \) and \( f_s \in \text{Ind}_B^{\text{SL}_2(F)}(\eta_{s-1/2} \mu \psi) \) such that

\[
\Psi(W, \phi, f_s) = \int_{N(F)\setminus \text{SL}_2(F)} W(h)(\omega_{\psi^{-1}}\phi)(h)f_s(h) \neq 0.
\]

Proof. We note that the Bruhat cell \( \Omega = N(F)T w N(F) \) is open and dense in \( \text{SL}_2(F) \). Thus the above integral is reduced to

\[
\Psi(W, \phi, f_s) = \int_{TN(F)} W(wt(a)n(u))(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(1)f_s(wt(a)n(u))\Delta(a) dadu,
\]

where \( \Delta(a) = |a|^{-2} \).

Using the formulas for the Weil representation \( \omega_{\psi^{-1}} \) we find

\[
(\omega_{\psi^{-1}}(wt(a)n(u))\phi)(x) = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \int_F \psi(ua^2y^2)\phi(ay)\psi(2xy)dy = |a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \Phi_{a,u}(x),
\]

where \( \Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax) \) which is again a Schwartz function on \( F \) and depends continuously on \( a \) and \( u \).

We next explain that the set \( \{(g, 1) : g \in N(F)T w N(F)\} \), still denoted as \( \Omega \), is open in \( \widetilde{\text{SL}}_2(F) \). Note that there is a double covering map \( p : \widetilde{\text{SL}}_2(F) \to \text{SL}_2(F) \). For any \( (g, 1) \in \Omega \) its projection under \( p \) is \( g \). As \( p \) is a covering map there exists an open neighborhood \( U_g \) of \( g \) contained in \( N(F)T w N(F) \) such that \( p^{-1}(U_g) \) is a disjoint union of two open subsets of \( \widetilde{\text{SL}}_2(F) \), each is homeomorphic to \( U_g \) by \( p \). Then one component of \( p^{-1}(U_g) \) is an open neighborhood of \( (g, 1) \) in \( \Omega \), which shows that \( \Omega \) is open in \( \text{SL}_2(F) \).

Now define \( f_s \in I(s, \eta, \psi^{-1}) \) on the set \( \{(g, 1) : g \in \text{SL}_2(F)\} \) by

\[
f_s(g) = \begin{cases} \delta(b)^{1/2}(\eta_{s-1/2} \mu \psi^{-1})(b)f_2(u) & \text{if } g = bun(u) \in \Omega, \\ 0 & \text{otherwise}, \end{cases}
\]

where \( b \in B(F) = TN(F) \), \( u \in F \) and \( f_2 \) is a compactly supported function to be determined later. Then we extend the definition of \( f_2 \) to the set \( \{(g, 1) : g \in \text{SL}_2(F)\} \) to make it genuine, i.e., \( f_s(g, -1) = -1f_s(g, 1). \)

Then the integral \( \Psi \) can be reduced further to

\[
(4-2) \quad \Psi(W, \phi, f_s) = \int_{TN(F)} W(wau)|a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi_a^{-1})} \Phi_{a,u}(1)\delta(a)^{1/2}(\eta_{s-1/2} \mu \psi^{-1})(a) 
\cdot f_2(u)\Delta(a) dadu.
\]
Case 1 ($F$ is $p$-adic). Consider the Howe vector $W_{\psi_m}$. By Corollary 3.4, taking $m$ large enough, $W_{\psi_m}$ can have arbitrarily small compact open support around 1 when restricted to $T$. Then $W_{w_\psi}(t(a^{-1})w)$ has small compact open support around $a = 1$.

First choose $\phi$ so that $\hat{\Phi}_{a,u}(1) \neq 0$ when $a = 1$, $u = 0$. Then choose $m$ so that $W_{w_\psi}(wt(a)) = W_{w_\psi}(t(a^{-1})w)$ has small compact support around 1 and all the other data involving $a$ in the integral (*) are nonzero constants. For this $W_{w_\psi}$, consider $W_{w_\psi}(wt(a)u)$ with $u \in N$. When $u$ is close to 1 enough, we have $W_{w_\psi}(wt(a)u) = W_{w_\psi}(wt(a))$ for all $a$ in that small compact support around 1. Then take $f_2$ with support $u$ close to 1 satisfying the above. With these choices of $W_{w_\psi}(g)$, $f_2$, $\phi$, the integral (4-2) is nonzero.

Case 2 ($F$ is archimedean). We will concentrate on the case $F = \mathbb{R}$. The case $F = \mathbb{C}$ is similar as we have the same formulas for the Weil representation by [Jacquet and Langlands 1970, Proposition 1.3]. We begin with the formulas

\[ \Psi(W, \phi, f_s) = \int_{TN(F)} W(wau)|a|^{1/2} \frac{\gamma(\psi^{-1})}{\gamma(\psi a^{-1})} \hat{\Phi}_{a,u}(1) \delta(a)^{1/2}(\eta_{s-1/2}\mu_{\psi^{-1}})(a) \cdot f_2(u) \Delta(a) dadu, \]

where $\Phi_{a,u}(x) = \psi(ua^2x^2)\phi(ax)$ is again a Schwartz function, as is $\phi$, and it depends on $a$ and $u$ continuously. Since the Fourier transform is an isometry of the Schwartz space we can choose $\phi$ so that the Fourier transform $\hat{\Phi}_{a,u}(1) > 0$ when $a = 1$ and $u = 0$, and it depends on $a$ and $u$ continuously.

Now let $(\pi, V)$ be an irreducible generic smooth representation of $SL_2(\mathbb{R})$ of moderate growth. Realize $\pi$ as a quotient of a smooth principal series $I(\chi, s)$, i.e.,

\[ 0 \to V' \to I(\chi, s) \to V \to 0. \]

Let $\lambda : V \to \mathbb{C}$ be the unique nonzero continuous Whittaker functional on $V$. Then the composition

\[ \Lambda : I(\chi, s) \longrightarrow V \xrightarrow{\lambda} \mathbb{C} \]

gives the unique nonzero continuous Whittaker functional on $I(\chi, s)$ up to a scalar. It follows that the two spaces $\{\lambda(\pi(g)v) : g \in SL_2(F), v \in V\}$ and $\{\Lambda(R(g).f) : g \in SL_2(F), f \in I(\chi, s)\}$ are the same, although the first is the Whittaker model of $\pi$ while the later may not be a Whittaker model of $I(\chi, s)$.

The Whittaker functional on $I(\chi, s)$ is given by the following

\[ \Lambda(f) = \int_{N(F)} f(wu)\psi^{-1}(u) du, \]

when $s$ is in some right half plane and its continuation gives Whittaker functionals for all $I(\chi, s)$. Also when $f$ has support inside $\Omega = N(F)T\omega N(F)$ the above integral always converges for any $s$ and gives the Whittaker functional.
Now for such \( f \) one computes that, for \( a = t(a) \in T \),
\[
\Lambda(I(a), f) = \int_{N(F)} f(wua)\psi^{-1}(u)du = \chi'(a) \int_{F} f(wu)\psi^{-1}(a^2u)du
\]
\[= \chi'(a) \int_{F} f_1(u)\psi^{-1}(a^2u)du = \chi'(a)\hat{f}_1(a^2),\]
where \( f_1 \) is the restriction of \( f \) to \( wN \) which can be chosen to be a Schwartz function, \( \hat{f}_1 \) is its Fourier transform and \( \chi' \) is a certain character. Again, as the Fourier transform gives an isometry of Schwartz functions, we can always choose \( f \) so that its Whittaker function \( W_f(a) \) has arbitrarily small compact support around 1.

By a right translation by \( w \) we show that one can always choose \( f \) so that \( \hat{W}_w f(a) \) has small compact support around 1.

In order to prove the proposition note that we have chosen \( \Phi \). Let
\[
R(a, u) = |a|^{1/2} \gamma(\psi^{-1}) \Phi_{a, u}(1)\delta(a)^{1/2}(\eta_{s-1/2}^2)(\mu_{\psi^{-1}})(a) \Delta(a).
\]
Then \( R(a, u) \) is a continuous function of \( a \) and \( u \), and \( R(1, 0) \neq 0 \). This means that there exist neighborhoods \( U_1 \) of \( a = 1 \) and \( U_2 \) of \( u = 0 \), such that \( R(a, u) > R(1, 0)/2 > 0 \) for all \( a \in U_1 \) and \( u \in U_2 \).

Now choose \( f \) so that \( W_{w, f}(aw) \) has small compact support in a neighborhood \( V_1 \) of 1 with \( V_1 \subset U_1 \) and \( W_{w, f}(w) > 0 \). For this Whittaker function, since \( W_{w, f}(awu) \) is continuous on \( u \), we can choose \( f_2 \) so that it is positively supported in a neighborhood \( V_2 \) of 0 such that:

1. \( V_2 \subset U_2 \).
2. \( W_{w, f}(awu) > W_{w, f}(w)/2 > 0 \) for all \( u \in V_2 \).

Then (4-3) becomes
\[
\int W_{w, f}(awu)R(a, u)f_2(u)dadu > \frac{W_{w, f}(w)}{2} \frac{R(1, 0)}{2} \int_{V_1} \int_{V_2} f_2(u)dadu > 0,
\]
which proves the nonvanishing.

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The SU($N$) Casson–Lin invariants for links 257
HANS U. BODEN and ERIC HARPER

The SU(2) Casson–Lin invariant of the Hopf link 283
HANS U. BODEN and CHRISTOPHER M. HERALD

Commensurations and metric properties of Houghton’s groups 289
JOSÉ BURILLO, SEAN CLEARY, ARMANDO MARTINO and CLAAS E. RÖVER

Conformal holonomy equals ambient holonomy 303
ANDREAS ČAP, A. ROD GOVER, C. ROBIN GRAHAM and MATTHIAS HAMMERL

Nonorientable Lagrangian cobordisms between Legendrian knots 319
ORSOLA CAPOVILLA-SEARLE and LISA TRAYNOR

A strong multiplicity one theorem for SL$_2$ 345
JINGSONG CHAI and QING ZHANG

The Yamabe problem on noncompact CR manifolds 375
PAK TUNG HO and SEONGTAG KIM

Isometry types of frame bundles 393
WOUTER VAN LIMBEEK

Bundles of spectra and algebraic K-theory 427
JOHN A. LIND

Hidden symmetries and commensurability of 2-bridge link complements 453
CHRISTIAN MILLICHAP and WILLIAM WORDEN

On seaweed subalgebras and meander graphs in type $C$ 485
DMITRI I. PAN'YUSHEV and OKSANA S. YAKIMOVA

The genus filtration in the smooth concordance group 501
SHIDA WANG