Pacific Journal of Mathematics

ON THE ABSOLUTE CONTINUITY OF p-HARMONIC MEASURE AND SURFACE MEASURE IN REIFENBERG FLAT DOMAINS

MURAT AKMAN

Volume 286 No. 1 January 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.286.25

ON THE ABSOLUTE CONTINUITY OF p-HARMONIC MEASURE AND SURFACE MEASURE IN REIFENBERG FLAT DOMAINS

MURAT AKMAN

We study the set of absolute continuity of p-harmonic measure μ associated to a positive weak solution to the p-Laplace equation with continuous zero boundary values and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on locally flat domains in space. We prove that when $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2-\eta < p < 2$ for some $\eta > 0$ there exist locally flat domains $\Omega \subset \mathbb{R}^n$ with locally finite perimeter and Borel sets $E \subset \partial\Omega$ such that $\mu(E) > 0 = \mathcal{H}^{n-1}(E)$.

1. Introduction and statement of main results

A well-known result of F. and M. Riesz says that if Ω is a simply connected domain whose boundary has finite length in the plane then harmonic measure and arclength are mutually absolutely continuous. Makarov [1985] gives a sharp description of the support of harmonic measure and shows that the function λ given below is the proper function to measure the size of the support of ω. In particular, if $\Omega \subset \mathbb{R}^2$ is a simply connected domain in the plane, then $\omega \ll \mathcal{H}^{\lambda}$, where

$$\lambda(r) := r \exp\left\{ C\sqrt{\log\frac{1}{r}\log\log\log\frac{1}{r}} \right\}$$

for sufficiently large C. Here "$\ll$" stands for absolute continuity of the measures, we use "$\perp$" to denote measures are singular, and $\mathcal{H}^{\lambda}$ to denote the Hausdorff measure with respect to the function λ (see (1.4) for definition of $\mathcal{H}^{\lambda}$). In [Makarov 1985], it is also shown that this result is sharp in the following sense; there is an example of a simply connected domain for which $\omega \perp \mathcal{H}^{\lambda}$ whenever C is sufficiently small in the definition of λ. In higher dimensions, due to examples of Ziemer [1974] and Wu [1986], neither $\mathcal{H}^n|_{\partial\Omega} \ll \omega$ nor $\omega \ll \mathcal{H}^n|_{\partial\Omega}$ are true in general without imposing extra topological or nontopological conditions on $\partial\Omega$. David and Jerison [1990] prove

MSC2010: 28A75, 28A78, 35J25, 37F35, 35J92, 31A15.
Keywords: Hausdorff dimension of p-harmonic measure, harmonic measure, p-harmonic measure, nonlinear elliptic PDEs, Hausdorff measure, Hausdorff dimension, singular sets for p-harmonic measure, Reifenberg flat domains, NTA domains.

that if Ω is a nontangentially accessible (NTA for short; see Definition 2.1) domain and $\partial\Omega$ is Ahlfors–David regular (ADR for short; see Definition 2.5) then harmonic measure is mutually absolutely continuous on $\partial\Omega$ with respect to surface measure, and in fact they are A_∞-equivalent (see [Azzam et al. 2014]). Badger [2012] considers the same problem by relaxing the ADR property by $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and proves that $\mathcal{H}^{n-1} \ll \omega$ on $\partial\Omega$. He also shows that $\omega \ll \mathcal{H}^{n-1} \ll \omega$ on the set $A \subset \partial\Omega$, where

$$A = \left\{ x \in \partial\Omega : \liminf_{r\to 0} \frac{\mathcal{H}^{n-1}(\Delta(x,r))}{r^{n-1}} < \infty \right\}.$$

Here $\Delta(x,r) = B(x,r) \cap \partial\Omega$. Badger also conjectures that when Ω is an NTA domain then the same result holds not only on $A \subset \partial\Omega$ but on the whole $\partial\Omega$ (see Conjecture 1.3 in [Badger 2012]). However, it turns out that this is not true in general. In fact, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2016] construct an example of a Reifenberg flat domain (see Definition 2.3) Ω in $\mathbb{R}^n$, $n \geq 3$, with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and a Borel set $E \subset \partial\Omega$ such that

$$\omega(E) > 0 = \mathcal{H}^{n-1}(E).$$

One can consider the same problem for the p-harmonic measure associated with a positive weak solution to the p-Laplace equation for $1 < p \neq 2 < \infty$. To define p-harmonic measure and the p-Laplace equation, we let $\Omega \subset \mathbb{R}^n$ be a domain and let N be a neighborhood of $\partial\Omega$. Fix p, $1 < p < \infty$, and suppose that $\hat{u}$ is a positive weak solution to the p-Laplace equation in $\Omega \cap N$. That is, $\hat{u} \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \, \mathrm{d}x = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Equivalently, we say that $\hat{u}$ is p-harmonic in $\Omega \cap N$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in $\Omega \cap N$ then

$$\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$$

in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that $\hat{u}$ has zero boundary values on $\partial\Omega \cap N$ in the Sobolev sense. More specifically, if $\zeta \in C_0^\infty(\Omega \cap N)$, then $\hat{u}\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend $\hat{u}$ to N by putting $\hat{u} \equiv 0$ on $N \setminus \Omega$. Then $\hat{u} \in W^{1,p}(N)$ and it follows from (1.1), as in [Heinonen et al. 1993, Chapter 21], that there exists a finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$ with support contained in $\partial\Omega \cap N$ satisfying

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \, \mathrm{d}x = -\int \phi \, \mathrm{d}\hat{\mu} \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. Existence of $\hat{\mu}$ follows from the maximum principle, basic Caccioppoli inequalities for $\hat{u}$ and the Riesz representation theorem for a positive linear functional. We note that if $\partial\Omega$ is smooth enough and $\nabla u \neq 0$ in Ω, then

$$\mathrm{d}\hat{\mu} = |\nabla \hat{u}|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega\cap N}.$$

Remark 1.3. When $p = 2$ in (1.1), we have the usual Laplace's equation. Moreover, if u is the Green's function for Laplace's equation with pole at, say $z_0 \in \Omega$, then the measure in (1.2) corresponding to this harmonic function u is harmonic measure, ω, relative to z_0. Note also that the p-Laplace equation in (1.1) is degenerate when $p > 2$ and is singular when $1 < p < 2$. The nonlinear structure of this PDE makes it difficult to work with.

We next introduce the notion of the *Hausdorff dimension of a measure*. To this end, let $\hat{r}_0 > 0$ be given, and let $0 < \delta < \hat{r}_0$ be fixed. Let $\lambda : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\lambda(0) = 0$. Let $d(\cdot)$ denote the diameter of a set. For a given Borel set $E \subset \mathbb{R}^n$, we define (δ, λ)-*Hausdorff content* of E in the usual way:

$$\mathcal{H}^{\lambda}_{\delta}(E) := \inf\left\{ \sum_i \lambda(d(U_i)) : E \subset \bigcup U_i, \text{ each } U_i \text{ is open with } d(U_i) < \delta \right\}.$$

Then the *Hausdorff measure* of E is defined by

$$\mathcal{H}^{\lambda}(E) := \lim_{\delta\to 0} \mathcal{H}^{\lambda}_{\delta}(E). \tag{1.4}$$

In case $\lambda(r) = r^{\alpha}$ we write $\mathcal{H}^{\alpha}$ for $\mathcal{H}^{\lambda}$. The Hausdorff dimension of $\hat{\mu}$, denoted by $\mathcal{H}-\dim\hat{\mu}$, is defined by

$$\mathcal{H}-\dim\hat{\mu} := \inf\big\{\alpha : \text{there exists Borel } \; E \subset \partial\Omega \text{ such that } \; \mathcal{H}^{\alpha}(E) = 0 \;\text{ and }\; \hat{\mu}(\mathbb{R}^n \setminus E) = 0\big\}.$$

We return to our study of singular sets of p-harmonic measure with respect to $\mathcal{H}^{n-1}$ measure. For arbitrary p, $1 < p \neq 2 < \infty$, Bennewitz and Lewis [2005] observed that the natural candidates, i.e., snowflake-type domains, which give sharpness in the harmonic case shown by Makarov, do not provide sharpness. In the same paper it was also shown that if $\partial\Omega$ is the von Koch snowflake in the plane and $2 < p < \infty$ then $\mathcal{H}-\dim\mu < 1$. In [Lewis et al. 2011], a weaker version of Makarov's result was obtained under the p-harmonic setting for $1 < p \neq 2 < \infty$. Finally, Lewis [2015] proved a p-harmonic analogue of Makarov's result; let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain and let μ be the p-harmonic measure described above. Let $\lambda(r)$ be as in Makarov's result. Then the following are true.

(a) If $1 < p < 2$, there is $A = A(p) \geq 1$ such that $\mu \ll \mathcal{H}^{\lambda}$.

(b) If $2 < p < \infty$, there is $A = A(p) \leq -1$ such that μ is concentrated on a set of σ-finite $\mathcal{H}^1$ measure.

The nonlinearity and degeneracy of the p-Laplace equation makes it difficult to study the Hausdorff dimension of this measure in $\mathbb{R}^n$, $n \geq 3$. The tools developed by Lewis, Nyström, and Vogel [Lewis et al. 2013] for p-harmonic functions were used to obtain that:

(1) If $\partial\Omega$ is sufficiently flat in the sense of Reifenberg and $p \geq n \geq 3$, then μ is concentrated on a set of σ-finite $\mathcal{H}^{n-1}$ measure.

(2) If $n \geq 3$ and $2 < p < n$, there exist Wolff snowflakes such that $\mathcal{H}-\dim \mu < n-1$, while if $1 < p < 2$, there exist Wolff snowflakes such that $\mathcal{H}-\dim \mu > n-1$.

(3) All examples produced by Wolff's snowflake method have $\mathcal{H}-\dim \mu < n-1$ when $p \geq n$.

(4) There is a Wolff snowflake for which the sign of $(n-1)-(\mathcal{H}-\dim \mu)$ equals the sign of $(n-1)-(\mathcal{H}-\dim \omega)$, where μ is the p-harmonic measure for p in an open interval containing 2 and ω is the harmonic measure with pole at infinity.

Lewis, Vogel, and the author [Akman et al. 2015] improved these results by proving the following: let $O \subset \mathbb{R}^n$ be any open set, $\hat{z} \in \partial O$, and let $\rho > 0$. Let u be a positive weak solution to (1.1) in $O \cap B(\hat{z}, \rho)$. Assume also that u has continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$. Extend u to all $B(\hat{z}, \rho)$ by defining 0 in $B(\hat{z}, \rho) \setminus O$. Let μ be the measure associated to u as in (1.2). If $p > n$ then μ is concentrated on the set

$$\mathcal{P} = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{r \to 0} \frac{\mu(B(w, r))}{r^{n-1}} > 0 \right\}.$$

This set $\mathcal{P}$ has σ-finite $\mathcal{H}^{n-1}$ measure. The same result holds when $p = n$, provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of n-capacity. Therefore, $\mathcal{H}-\dim \mu_p \leq n-1$ when $p \geq n$.

On the other hand, the result of David and Jerison described above for harmonic measure is extended to the p-harmonic setting for $1 < p \neq 2 < \infty$ by Lewis and Nyström [2012]. To state this result, we let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with constants M, r_0 whose boundary is ADR. Let u be p-harmonic in $\Omega \cap B(w, 4r)$, $w \in \partial\Omega$, $0 < r < r_0$, and continuous in $\bar{\Omega} \cap B(w, 4r)$ with $u \equiv 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and let μ be the p-harmonic measure as in (1.2) associated with u. Then it is shown in [Lewis and Nyström 2012, Proposition 3.4] that $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on $\partial\Omega$; in fact they are A_∞-equivalent. It also is proven in the same paper that Badger's result holds under the p-harmonic setting; if Ω is an NTA domain then $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on the set $A' \subset \Delta(w, 4r) \subset \partial\Omega$, where

$$A' = \left\{ x \in \Delta(w, 4r) : \liminf_{\rho \to 0} \frac{\mathcal{H}^{n-1}(\Delta(x, \rho))}{\rho^{n-1}} < \infty \right\}.$$

The main result proved in this paper is that there are examples of domains for which absolute continuity of p-harmonic measure and $(n-1)$-dimensional Hausdorff measure does not hold on the whole boundary when the domain is even locally flat in the sense of Reifenberg.

Theorem 1.5. *When $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2 - \eta < p < 2$ for some $\eta > 0$, there exist domains $\Omega \subset \mathbb{R}^n$ and Borel sets $E \subset \partial\Omega$ such that*:

(1) *Ω is a $(\hat{\delta}, \infty)$-Reifenberg flat domain.*

(2) *$\sigma = \mathcal{H}^{n-1}$ is Radon.*

(3) *$\mu_p(E) > 0 = \sigma(E)$, where μ_p is the p-harmonic measure associated to a positive p-harmonic function in Ω with continuous zero boundary values on $\partial\Omega$.*

As the plan of this paper, we first state the definition of nontangentially accessible domains, Reifenberg flatness, and Ahlfors–David regularity, and we give some lemmas concerning the regularity of p-harmonic function in NTA domains in Section 2. We give the construction of Wolff snowflakes in Section 3. Following [Azzam et al. 2016], we construct "an enlarged domain Ω_ϵ^+" from a certain domain Ω and, using some results from [Lewis et al. 2013] concerning the dimension of p-harmonic measure, we give a proof of Theorem 1.5 in Section 4.

2. Definitions and preparatory lemmas

To proceed, some notation and definitions are in order. In the sequel, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n, unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1 which may depend only on $p, n, a_1, \dots, a_n$, not necessarily the same at each occurrence.

Let $x = (x_1, \dots, x_n)$ denote points in $\mathbb{R}^n$ and let $\bar{E} = \mathrm{cl}(E)$, $\mathrm{int}\, E$, ∂E, and E^{c} be the closure, interior, boundary, and the complement of the set $E \subset \mathbb{R}^n$, respectively. Let $\mathrm{diam}(E)$ be the diameter of a set E. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$. Let $d(E, F)$ denote the usual distance between the sets E and F and let $d_{\mathcal{H}}(E, F)$ denote the Hausdorff distance between the sets E and F, which is defined by

$$d_{\mathcal{H}}(E, F) := \max(\sup\{d(E, y) : y \in F\}, \sup\{d(x, F) : x \in E\}).$$

Let $B(x, r)$ be the usual open ball centered at x with radius $r > 0$ in $\mathbb{R}^n$ and let $\mathrm{d}x$ denote the Lebesgue n-measure in $\mathbb{R}^n$. Let $\Delta(w, r) = \partial\Omega \cap B(w, r)$. For a given number $t > 0$ and a cube Q, let $l(Q)$ be the side length of Q and let tQ denote the cube whose side length is $tl(Q)$ with the same center as Q.

We state the notion of nontangentially accessible domain which was initially introduced by Jerison and Kenig [1982].

Definition 2.1 (NTA domain). A domain Ω is called a *nontangentially accessible* (NTA) domain if there exist $M \geq 2$ and r_0 such that the following are fulfilled.

(i) *Corkscrew condition*: for any $w \in \partial\Omega,\ 0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying
$$M^{-1}r < |a_r(w) - w| < r \quad \text{and} \quad M^{-1}r < d(a_r(w), \partial\Omega).$$

(ii) $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew condition.

(iii) *Uniform condition*: if $w \in \partial\Omega,\ 0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$ then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1$ and $\gamma(1) = w_2$ such that

(a) $\mathcal{H}^1(\gamma) \leq M|w_1 - w_2|$,

(b) $\min\{\mathcal{H}^1(\gamma([0, t])), \mathcal{H}^1([t, 1]))\} \leq Md(\gamma(t), \partial\Omega)$.

Remark 2.2. We use the definition of this notion given in [Lewis and Nyström 2012]. Note that (iii) of Definition 2.1 is different but equivalent to the Harnack chain condition given in [Jerison and Kenig 1982].

Next we give the definition of Reifenberg flatness from [Azzam et al. 2016].

Definition 2.3 ((δ, r_0)-Reifenberg flat domain). Let Ω be a domain and $r_0, \delta > 0$ with $0 < \delta < \frac{1}{2}$. Then Ω is said to be (δ, r_0)-*Reifenberg flat* provided that the following two conditions hold.

(i) For every $w \in \partial\Omega$ and every $0 < r < r_0$ there exists a hyperplane $\mathcal{P}(w, r)$ containing w such that
$$d_{\mathcal{H}}(\Delta(w, r), \mathcal{P}(w, r) \cap B(w, r)) \leq \delta r.$$

(ii) For every $x \in \partial\Omega$, one of the connected components of
$$B(x, r_0) \cap \{x \in \mathbb{R}^n;\ \mathrm{d}(x, \mathcal{P}(x, r_0)) \geq 2\delta r_0\}$$
is contained in Ω and the other is contained in $\mathbb{R}^n \setminus \Omega$.

We say that Ω is (δ, ∞)-Reifenberg flat if it is (δ, r_0)-Reifenberg flat for every $r_0 > 0$.

Remark 2.4. An equivalent definition of Reifenberg flatness is given in [Lewis and Nyström 2012], and it is remarked that these two definitions are equivalent (see observation after their Definition 1.2).

Definition 2.5 (Ahlfors–David regular set). We say that $\partial\Omega$ is n-dimensional *Ahlfors–David regular* (ADR) if there is some uniform constant C such that
$$C^{-1}r^n \leq \mathcal{H}^n(\Delta(x, r)) \leq Cr^n \quad \text{for all } r \in (0, \mathrm{diam}(\Omega)),\ x \in \partial\Omega.$$

We next give some estimates from when $n \geq 3$ [Lewis et al. 2013, Lemmas 3.2–3.6] and when $n = 2$ given under the p-harmonic settings [Bennewitz and Lewis 2005, Lemmas 2.6, 2.7, 2.13, 2.14]. For Lemmas 2.6–2.8, let p be fixed with $1 < p \neq 2 < \infty$.

Lemma 2.6. *Let u be a positive p-harmonic function in $B(w, 2r) \subset \mathbb{R}^n$, $n \geq 3$. Then*

$$r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \, \mathrm{d}x \leq c\Big(\max_{B(w,r)} u\Big)^p$$

and

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Moreover, there exists $\beta = \beta(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$ then

$$|u(x) - u(y)| \leq c\left(\frac{|x-y|}{r}\right)^{\beta} \max_{B(w,2r)} u.$$

For Lemmas 2.7 and 2.8 let Ω be an NTA domain in $\mathbb{R}^n$ and let $w \in \partial\Omega$, $0 < r < r_0$.

Lemma 2.7. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular, there exists $\sigma = \sigma(p, n) \in (0, 1]$ such that if $x, y \in B\big(\hat{w}, \frac{1}{2}\hat{r}\big)$, where $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$ then*

$$\frac{1}{c}|\nabla u(x) - \nabla u(y)| \leq \left(\frac{|x-y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},\hat{r})} |\nabla u| \leq \frac{c}{\hat{r}}\left(\frac{|x-y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},2\hat{r})} u.$$

If $\nabla u(\hat{w}) \neq 0$ then u is real analytic in a neighborhood of $\hat{w}$.

The next lemma gives a relation between a p-harmonic function and its corresponding measure.

Lemma 2.8. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. As in* (1.2), *there exists a unique locally finite positive Borel measure μ on $\mathbb{R}^n$ with support in $\Delta(w, 2r)$ such that*

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla\theta \rangle \, \mathrm{d}x = -\int \theta \, \mathrm{d}\mu$$

whenever $\theta \in C_0^{\infty}(B(w, 2r))$. Moreover, there exists $c = c(p, n, M) \in [1, \infty)$ such that if $\tilde{r} = r/c$ then

$$c^{-1} r^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c r^{p-n} \mu\big(\Delta\big(w, \tfrac{1}{2}\tilde{r}\big)\big),$$

where $a_{\tilde{r}}(w)$ is as in Definition 2.1.

3. Construction of Wolff snowflakes

In this section, following [Lewis et al. 2013] when $n \geq 3$ and [Bennewitz and Lewis 2005] when $n = 2$, we describe the construction of Wolff snowflakes in $\mathbb{R}^n$ which was originally introduced in [Wolff 1995]. To this end, let

$$\Omega_0 = \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\} \subset \mathbb{R}^n.$$

Set

$$Q(r) = \left\{x' \in \mathbb{R}^{n-1} : -\tfrac{1}{2}r \leq |x_i| \leq \tfrac{1}{2}r \ \text{ for }\ 1 \leq i \leq n-1\right\}.$$

Then $Q(r)$ is an $(n-1)$-dimensional cube with side length r and centered at 0. Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a piecewise linear function with support contained in $\left\{x' : |x'| < \frac{1}{2}\right\}$ satisfying

$$\|\nabla\phi\|_\infty \leq \theta_0. \tag{3.1}$$

For fixed large N, define $\psi(x') = N^{-1}\phi(Nx')$. Let $b > 0$ be a small constant and let Q be an $(n-1)$-dimensional cube with center a_Q and length $l(Q)$ contained in some hyperplane. Let $\operatorname{cch}(E)$ denote the closed convex hull. Let e be a unit normal to Q and define

$$P_Q = \operatorname{cch}(Q \cup \{a_Q + bl(Q)e\}) \quad \text{and} \quad \tilde{P}_Q = \operatorname{int}\operatorname{cch}(Q \cup \{a_Q - bl(Q)e\}).$$

We set $e = -e_n$ for $Q(1)$. We also define

$$\Lambda := \{x \in P_{Q(1)} \cup \tilde{P}_{Q(1)},\ x_n \geq \psi(x)\} \quad \text{and} \quad \partial := \{x \in \mathbb{R}^n,\ x' \in Q(1),\ x_n = \psi(x')\}.$$

We assume that $N = N(b, M)$ is so large that

$$d(\partial \setminus \partial\Omega_0, \partial[P_{Q(1)} \cup \tilde{P}_{Q(1)}]) \geq \tfrac{b}{100}.$$

From the construction, it can be easily seen that $\partial \subset Q(1) \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ consists of a finite number of $(n-1)$-dimensional faces. We fix a Whitney decomposition of each face; we divide each face of ∂ into an $(n-1)$-dimensional cube Q, with side lengths 8^{-k}, $k = 1, 2, \ldots$, and $8^{-k} \approx$ to their distance from the edges of the face they lie on. We next choose a distinguished $(n-2)$-dimensional "side" for each $(n-1)$-dimensional cube.

Suppose Ω is a domain and $Q \subset \partial\Omega$ is an $(n-1)$-dimensional cube with distinguished side γ. Let e be a unit normal to $\partial\Omega$ on Q and assume that $P_Q \cap \Omega = \varnothing$ and $\tilde{P} \subset \Omega$. We form a new domain $\tilde{\Omega}$ as follows. Let $\mathcal{T}$ be the conformal affine map, i.e., composition of a translation, dilation, and rotation, with $\mathcal{T}(Q(1)) = Q$ which fixes dilation, $\mathcal{T}(0) = a_Q$ which fixes translation, $\mathcal{T}\left(\left\{x \in \partial Q(1) : x_1 = \frac{1}{2}\right\}\right)$ and $\mathcal{T}(-e_n)$ in the direction of e which fixes rotation. Let $\Lambda_Q = \mathcal{T}(\Lambda)$ and $\partial_Q = \mathcal{T}(\partial)$.

Then we define $\tilde{\Omega}$ through the relations

$$\tilde{\Omega} \cap (P_Q \cup \tilde{P}_Q) \quad \text{and} \quad \tilde{\Omega} \setminus (P_Q \cup \tilde{P}_Q) = \Omega \setminus (P_Q \cup \tilde{P}_Q).$$

Note that ∂_Q inherits from ∂ a natural subdivision into Whitney cubes with distinguished sides. This process is called "adding a blip to Ω along Q".

To use the process of "adding a blip" to construct a Wolff snowflake Ω_∞, starting from Ω_0, we first add a blip to Ω_0 along $Q(1)$ obtaining a new domain Ω_1. We then inherit a subdivision of $\partial\Omega_1 \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)})$ into Whitney cubes with distinguished sides, together with a finite set of edges E_1 (the edges of the faces of the graph are not in the Whitney cubes). Let G_1 be the set of all Whitney cubes in the subdivision. Then Ω_2 is obtained from Ω_1 by adding a blip along each $Q \in G_1$. From this process, we inherit a family of cubes $G_2 \subset \partial\Omega_2$ (each with a distinguished side) and a set of edges $E_2 \subset \partial\Omega_2$ of σ-finite $\mathcal{H}^{n-2}$ measure. Continuing by induction we get $(\Omega_m)_{m=n-1}^\infty$, $(G_m)_{m=n-1}^\infty$, and $(E_m)_{m=n-1}^\infty$, where

$$\partial\Omega_m \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)}) = E_m \cup \bigcup_{Q \in G_m} Q \quad \text{for } m \geq n-1.$$

If $N = N(b, M)$ is large enough, then $\Omega_m \to \Omega_\infty$ in the Hausdorff distance sense. We call Ω_∞ a *Wolff snowflake*. We state a result which says that Wolff snowflakes are locally flat in the sense of Reifenberg.

Lemma 3.2 [Lewis et al. 2013, Lemma 7.1]. *If θ_0, N^{-1} are small enough, depending only n, then the Wolff snowflake domain Ω_∞ is $(c\theta_0, \infty)$-Reifenberg flat, where $c = c(n)$.*

4. Proof of Theorem 1.5

In this section we give a proof of Theorem 1.5 using some results from [Lewis et al. 2013; Azzam et al. 2016]. To this end, let Ω_∞ be a Wolff snowflake with constants θ_0, N as described in Section 3. For fixed p, $1 < p \neq 2 < \infty$, let u_∞ be the unique positive p-harmonic function in Ω_∞ with continuous boundary value zero on $\partial\Omega_\infty$ and $|x_n - u_\infty(x)| \to 0$ uniformly as $|x| \to \infty$. Let μ_∞ be the p-harmonic measure associated with u_∞ as in (1.2). A proof of existence and uniqueness of u_∞ can be found in [Lewis et al. 2013, Lemma 6.1]. Let Ω'_∞ be the restriction of Ω_∞ to $Q(1) \times [-1, 1]$ and let μ'_∞ be the restriction of μ_∞ to $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$.

The following lemma can be easily deduced by combining Lemma 7.4 and Proposition 7.6 from [Lewis et al. 2013] when $n \geq 3$ and combining Lemma 3.23 and Theorem 1 from [Bennewitz and Lewis 2005] when $n = 2$. Moreover, when $n \geq 3$ and $2-\eta < p < 2$ it follows from Theorem 4 in [Lewis et al. 2013]. We first state a lemma.

Lemma 4.1. *When $n \geq 3$ let p be fixed, $2 < p < \infty$, and when $n \geq 2$ let p be fixed with $2 - \eta < p < 2$ for some $\eta > 0$. Let Ω'_∞ and μ'_∞ be described as above. Then* *for some $d > 0$ we have*

$$\lim_{r\to 0} \frac{\log \mu'_\infty(\Delta(x,r))}{\log r} \le d < n-1 \quad \textit{for all } x \in \partial\Omega'_\infty \setminus \Lambda,$$

where $\Lambda \subset \partial\Omega'_\infty$ *with* $\mu'_\infty(\Lambda) = 0$. *Moreover*, $\mathcal{H} - \dim \mu'_\infty \le d < n-1$.

Proof. We first show that there exist Wolff snowflakes for which $\mathcal{H} - \dim \mu < n-1$ in all cases stated in Lemma 4.1. To this end, as we discussed in Section 1, when $n \ge 3$ and $2 < p < \infty$ there exist Wolff snowflakes such that $\mathcal{H} - \dim \mu < n-1$ (see Theorems 2 and 3 in [Lewis et al. 2013]). When $n = 2$, it follows from [Bennewitz and Lewis 2005, Theorem 1] that there is a Wolff snowflake for which $\mathcal{H} - \dim \mu < 1$ whenever p is fixed with $2 < p < \infty$. Next, there exist Wolff snowflakes for which $\mathcal{H} - \dim \omega < n-1$, which is a well-known result of Wolff [1995] when $n \ge 3$. On the other hand, it is observed in [Lewis et al. 2013, Proposition 6.4] that there exists a Wolff snowflake such that the sign of $(n-1) - (\mathcal{H} - \dim \omega)$ equals the sign of $(n-1) - (\mathcal{H} - \dim \mu)$ for $p \in (2-\eta, 2)$. Therefore, combining these two results, we first conclude that there exists a Wolff snowflake for which $\mathcal{H} - \dim \omega < n-1$ when $2 - \eta < p < 2$ for some $\eta > 0$. Using these observations and Lemma 7.4 in [Lewis et al. 2013] we finish the proof of lemma. □

We are now ready to prove Theorem 1.5. Under the p-harmonic setting, we closely follow the arguments given in [Azzam et al. 2016] after Theorem 4.3. We first observe from Lemma 4.1, more specifically from the fact $\mathcal{H} - \dim \mu'_\infty \le d < n-1$, and the definition of Hausdorff dimension of p-harmonic measure, that there is a Borel set $E \subset \partial\Omega'_\infty$ such that $\mu'_\infty(\mathbb{R}^n \setminus E) = 0$ and $\mathcal{H}^d(E) = 0$. From this observation and once again from Lemma 4.1 we also have

$$\lim_{r\to 0} \frac{\log \mu'_\infty(B(x,r))}{\log r} \le d < n-1 \quad \text{for all } x \in E. \tag{4.2}$$

Note that Ω'_∞ is the restriction of Ω_∞ to $Q(1) \times [-1, 1]$; therefore,

$$\partial\Omega_\infty \setminus \{(x', x_n) \in \mathbb{R}^n : x_n = 0\} \subset \partial\Omega'_\infty.$$

For ease of notation we let

$$\mathfrak{R}^{n-1} := \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ and } x_n = 0\}.$$

From (4.2) it follows that for α, $0 < \alpha < n-1-d$, one can find small enough ρ such that $\mu'_\infty(E_1) > 0$, where

$$E_1 = \left\{ x \in (E \cap \partial\Omega_\infty) \setminus \mathfrak{R}^{n-1} : \frac{\log \mu'_\infty(B(x,r))}{\log r} < n-1-\alpha \text{ for all } r \in (0, \rho] \right\}.$$

We next fix a point $\zeta_0 \in E_1$. By the regularity of p-harmonic measure we can find $\rho_0 \in (0, \rho]$ and a compact set $K \subset E_1 \cap B(\zeta_0, \rho_0)$ such that for all $x \in K$ and

$r \in (0, \rho_0)$ the following property holds:

$$\mu'_\infty(K) > 0 \quad \text{and} \quad \mu'_\infty(B(x,r)) > r^{n-1-\alpha}.$$

The construction yields that $K \subset \partial\Omega'_\infty \cap \partial\Omega_\infty$ and $\mathrm{cl}(\Omega'_\infty) \subset \mathrm{cl}(\Omega_\infty)$. Then using the fact that the support of μ'_∞ is contained in $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$ we have

$$\mu_\infty(K) > 0 \quad \text{and} \quad \mu_\infty(B(x,r) \cap \partial\Omega_\infty) > r^{n-1-\alpha} \tag{4.3}$$

for all $x \in K$ and $r \in (0, \rho_0)$.

For a given number t, $4 \le t$, and given open set $O \subset \mathbb{R}^{n-1}$ we use $\mathcal{W}_t(O)$ to denote the set of maximal dyadic cubes $Q \subset O$ satisfying $tQ \cap K^{\mathrm{c}} = \varnothing$. Let $0 < \epsilon < \frac{1}{100}$ and let $\mathcal{I}$ be the family of cubes $Q \in \mathcal{W}_{\epsilon^{-2}}(K^{\mathrm{c}})$ such that

$$Q \cap (Q(1) \times [-1, 1]) \cap \partial\Omega_\infty \neq \varnothing.$$

Note that

$$l(Q) \approx \epsilon^2 \,\mathrm{dist}(Q, K) \quad \text{for all } Q \in \mathcal{I} \quad \text{and} \quad \partial\Omega'_\infty \setminus K \subset \bigcup_{Q \in \mathcal{I}} Q.$$

For each $Q \in \mathcal{I}$, fix some point $z_Q \in Q \cap \partial\Omega'_\infty$. We then define a new domain Ω^+_ϵ by

$$\Omega^+_\epsilon := \Omega'_\infty \cup \Big(\bigcup_{Q \in \mathcal{I}} B_Q\Big), \quad \text{where } B_Q = B(z_Q, \epsilon \,\mathrm{dist}(z_Q, K)).$$

It is observed in [Azzam et al. 2016, Lemma 2.2] that if θ, ϵ in the construction of the Wolff snowflake in Section 3 are small enough then Ω^+_ϵ is $(c\epsilon^{1/2}, r_0)$-Reifenberg flat and $K \subset \partial\Omega^+_\epsilon$, provided that the original domain Ω_∞ is (δ, r_0)-Reifenberg flat. Note that from Lemma 3.2 we have that Wolff snowflake domain Ω_∞ is $(c\theta_0, r_0)$-Reifenberg flat, where $r_0 = \infty$. Therefore if we choose θ and ϵ small enough and use Lemma 2.2 from [Azzam et al. 2016] then Ω^+_ϵ is a $(c\epsilon^{1/2}, \infty)$-Reifenberg flat domain satisfying

$$K \subset \partial\Omega'_\infty \cap \partial\Omega^+_\epsilon \quad \text{and} \quad \mathrm{cl}(\Omega_\infty) \subset \mathrm{cl}(\Omega^+_\epsilon). \tag{4.4}$$

Let u^+_ϵ be a positive p-harmonic function in Ω^+_ϵ with continuous boundary value zero on $\partial\Omega^+_\epsilon$. Let μ^+_ϵ be the p-harmonic measure associated with u^+_ϵ as in (1.2). From the construction of Ω^+_ϵ we have $u^+_\epsilon \ge u'_\infty$ on $\partial\Omega'_\infty$. Then it follows from the maximum principle for positive p-harmonic functions and (4.4) that $u^+_\epsilon \ge u'_\infty$ in Ω'_∞. This observation, Lemmas 2.6–2.8 and (4.3) yield

$$\mu^+_\epsilon(K) > 0 \quad \text{and} \quad \mu^+_\epsilon(B(x,r)) > r^{n-1-\alpha} \quad \text{for all } x \in K,\ r \in (0, \rho_0). \tag{4.5}$$

As μ^+_ϵ is a Radon measure which follows from Lemma 2.8 and satisfies (4.5) and Ω^+_ϵ is $(\hat\delta, \hat r_0)$-Reifenberg flat domain, it follows from [Azzam et al. 2016, Lemma 3.1] that $\mathcal{H}^{n-1}|_{\partial\Omega^+_\epsilon}$ is locally finite. Let $\Omega := \Omega^+_\epsilon$ be the $(\hat\delta, \hat r_0)$-Reifenberg flat domain with locally finite surface measure and let $\mu := \mu_\epsilon^+$ be the p-harmonic measure as above. From (4.5) we conclude that Theorem 1.5 is true. $\square$

Acknowledgments

The author would like to thank John Lewis for his suggestions and fruitful discussions to improve the case when $p \in (2-\eta, 2)$. He also warmly thanks the referee for their careful reading of the manuscript and useful suggestions. The author has been supported in part by ICMAT Severo Ochoa project SEV-2011-0087. He acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013)/ ERC agreement no. 615112 HAPDEGMT.

References

[Akman et al. 2015] M. Akman, J. Lewis, and A. Vogel, "Hausdorff dimension and σ finiteness of p harmonic measures in space when $p \geq n$", *Nonlinear Anal.* **129** (2015), 198–216. MR Zbl

[Azzam et al. 2014] J. Azzam, S. Hofmann, J. M. Martell, K. Nyström, and T. Toro, "A new characterization of chord-arc domains", 2014. To appear in *J. Eur. Math. Soc.* arXiv

[Azzam et al. 2016] J. Azzam, M. Mourgoglou, and X. Tolsa, "Singular sets for harmonic measure on locally flat domains with locally finite surface measure", *Int. Math. Res. Not.* (online publication June 2016).

[Badger 2012] M. Badger, "Null sets of harmonic measure on NTA domains: Lipschitz approximation revisited", *Math. Z.* **270**:1-2 (2012), 241–262. MR Zbl

[Bennewitz and Lewis 2005] B. Bennewitz and J. L. Lewis, "On the dimension of p-harmonic measure", *Ann. Acad. Sci. Fenn. Math.* **30**:2 (2005), 459–505. MR Zbl

[David and Jerison 1990] G. David and D. Jerison, "Lipschitz approximation to hypersurfaces, harmonic measure, and singular integrals", *Indiana Univ. Math. J.* **39**:3 (1990), 831–845. MR Zbl

[Heinonen et al. 1993] J. Heinonen, T. Kilpeläinen, and O. Martio, *Nonlinear potential theory of degenerate elliptic equations*, Oxford University Press, 1993. MR Zbl

[Jerison and Kenig 1982] D. S. Jerison and C. E. Kenig, "Boundary behavior of harmonic functions in nontangentially accessible domains", *Adv. in Math.* **46**:1 (1982), 80–147. MR Zbl

[Lewis 2015] J. L. Lewis, "p harmonic measure in simply connected domains revisited", *Trans. Amer. Math. Soc.* **367**:3 (2015), 1543–1583. MR Zbl

[Lewis and Nyström 2012] J. L. Lewis and K. Nyström, "Regularity and free boundary regularity for the p-Laplace operator in Reifenberg flat and Ahlfors regular domains", *J. Amer. Math. Soc.* **25**:3 (2012), 827–862. MR Zbl

[Lewis et al. 2011] J. L. Lewis, K. Nyström, and P. Poggi-Corradini, "p harmonic measure in simply connected domains", *Ann. Inst. Fourier (Grenoble)* **61**:2 (2011), 689–715. MR Zbl

[Lewis et al. 2013] J. Lewis, K. Nyström, and A. Vogel, "On the dimension of p-harmonic measure in space", *J. Eur. Math. Soc.* **15**:6 (2013), 2197–2256. MR Zbl

[Makarov 1985] N. G. Makarov, "On the distortion of boundary sets under conformal mappings", *Proc. London Math. Soc.* (3) **51**:2 (1985), 369–384. MR Zbl

[Wolff 1995] T. H. Wolff, "Counterexamples with harmonic gradients in $\mathbb{R}^3$", pp. 321–384 in *Essays on Fourier analysis in honor of Elias M. Stein* (Princeton, NJ, 1991), edited by C. Fefferman et al., Princeton Math. Ser. **42**, Princeton Univ. Press, 1995. MR Zbl

[Wu 1986] J.-M. Wu, "On singularity of harmonic measure in space", *Pacific J. Math.* **121**:2 (1986), 485–496. MR Zbl

[Ziemer 1974] W. P. Ziemer, "Some remarks on harmonic measure in space", *Pacific J. Math.* **55** (1974), 629–637. MR Zbl

Received November 5, 2015. Revised August 16, 2016.

MURAT AKMAN
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CONNECTICUT
STORRS, CT 06268-3009
UNITED STATES
murat.akman@uconn.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

PUBLISHED BY
mathematical sciences publishers
nonprofit scientific publishing
http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 1 January 2017

0030-8730(201701)286:1;1-Z