Pacific Journal of Mathematics

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Volume 286 No. 1 January 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.286.25

ON THE ABSOLUTE CONTINUITY OF p-HARMONIC MEASURE AND SURFACE MEASURE IN REIFENBERG FLAT DOMAINS

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We study the set of absolute continuity of p-harmonic measure μ associated to a positive weak solution to the p-Laplace equation with continuous zero boundary values and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on locally flat domains in space. We prove that when $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2 - \eta < p < 2$ for some $\eta > 0$ there exist locally flat domains $\Omega \subset \mathbb{R}^n$ with locally finite perimeter and Borel sets $E \subset \partial\Omega$ such that $\mu(E) > 0 = \mathcal{H}^{n-1}(E)$.

1. Introduction and statement of main results

A well-known result of F. and M. Riesz says that if Ω is a simply connected domain whose boundary has finite length in the plane then harmonic measure and arclength are mutually absolutely continuous. Makarov [1985] gives a sharp description of the support of harmonic measure and shows that the function λ given below is the proper function to measure the size of the support of ω. In particular, if $\Omega \subset \mathbb{R}^2$ is a simply connected domain in the plane, then $\omega \ll \mathcal{H}^\lambda$, where

$$\lambda(r) := r \exp\left\{ C \sqrt{\log \frac{1}{r} \log\log\log \frac{1}{r}} \right\}$$

for sufficiently large C. Here "$\ll$" stands for absolute continuity of the measures, we use "$\perp$" to denote measures are singular, and $\mathcal{H}^\lambda$ to denote the Hausdorff measure with respect to the function λ (see (1.4) for definition of $\mathcal{H}^\lambda$). In [Makarov 1985], it is also shown that this result is sharp in the following sense; there is an example of a simply connected domain for which $\omega \perp \mathcal{H}^\lambda$ whenever C is sufficiently small in the definition of λ. In higher dimensions, due to examples of Ziemer [1974] and Wu [1986], neither $\mathcal{H}^n|_{\partial\Omega} \ll \omega$ nor $\omega \ll \mathcal{H}^n|_{\partial\Omega}$ are true in general without imposing extra topological or nontopological conditions on $\partial\Omega$. David and Jerison [1990] prove

MSC2010: 28A75, 28A78, 35J25, 37F35, 35J92, 31A15.

Keywords: Hausdorff dimension of p-harmonic measure, harmonic measure, p-harmonic measure, nonlinear elliptic PDEs, Hausdorff measure, Hausdorff dimension, singular sets for p-harmonic measure, Reifenberg flat domains, NTA domains.

that if Ω is a nontangentially accessible (NTA for short; see Definition 2.1) domain and $\partial\Omega$ is Ahlfors–David regular (ADR for short; see Definition 2.5) then harmonic measure is mutually absolutely continuous on $\partial\Omega$ with respect to surface measure, and in fact they are A_∞-equivalent (see [Azzam et al. 2014]). Badger [2012] considers the same problem by relaxing the ADR property by $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and proves that $\mathcal{H}^{n-1} \ll \omega$ on $\partial\Omega$. He also shows that $\omega \ll \mathcal{H}^{n-1} \ll \omega$ on the set $A \subset \partial\Omega$, where

$$A = \left\{ x \in \partial\Omega : \liminf_{r\to 0} \frac{\mathcal{H}^{n-1}(\Delta(x,r))}{r^{n-1}} < \infty \right\}.$$

Here $\Delta(x,r) = B(x,r) \cap \partial\Omega$. Badger also conjectures that when Ω is an NTA domain then the same result holds not only on $A \subset \partial\Omega$ but on the whole $\partial\Omega$ (see Conjecture 1.3 in [Badger 2012]). However, it turns out that this is not true in general. In fact, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2016] construct an example of a Reifenberg flat domain (see Definition 2.3) Ω in $\mathbb{R}^n$, $n \geq 3$, with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and a Borel set $E \subset \partial\Omega$ such that

$$\omega(E) > 0 = \mathcal{H}^{n-1}(E).$$

One can consider the same problem for the p-harmonic measure associated with a positive weak solution to the p-Laplace equation for $1 < p \neq 2 < \infty$. To define p-harmonic measure and the p-Laplace equation, we let $\Omega \subset \mathbb{R}^n$ be a domain and let N be a neighborhood of $\partial\Omega$. Fix p, $1 < p < \infty$, and suppose that $\hat{u}$ is a positive weak solution to the p-Laplace equation in $\Omega \cap N$. That is, $\hat{u} \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \, \mathrm{d}x = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Equivalently, we say that $\hat{u}$ is p-harmonic in $\Omega \cap N$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in $\Omega \cap N$ then

$$\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$$

in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that $\hat{u}$ has zero boundary values on $\partial\Omega \cap N$ in the Sobolev sense. More specifically, if $\zeta \in C_0^\infty(\Omega \cap N)$, then $\hat{u}\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend $\hat{u}$ to N by putting $\hat{u} \equiv 0$ on $N \setminus \Omega$. Then $\hat{u} \in W^{1,p}(N)$ and it follows from (1.1), as in [Heinonen et al. 1993, Chapter 21], that there exists a finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$ with support contained in $\partial\Omega \cap N$ satisfying

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \, \mathrm{d}x = -\int \phi \, \mathrm{d}\hat{\mu} \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. Existence of $\hat{\mu}$ follows from the maximum principle, basic Caccioppoli inequalities for $\hat{u}$ and the Riesz representation theorem for a positive linear functional. We note that if $\partial\Omega$ is smooth enough and $\nabla u \neq 0$ in Ω, then

$$\mathrm{d}\hat{\mu} = |\nabla \hat{u}|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega\cap N}.$$

Remark 1.3. When $p=2$ in (1.1), we have the usual Laplace's equation. Moreover, if u is the Green's function for Laplace's equation with pole at, say $z_0 \in \Omega$, then the measure in (1.2) corresponding to this harmonic function u is harmonic measure, ω, relative to z_0. Note also that the p-Laplace equation in (1.1) is degenerate when $p > 2$ and is singular when $1 < p < 2$. The nonlinear structure of this PDE makes it difficult to work with.

We next introduce the notion of the *Hausdorff dimension of a measure*. To this end, let $\hat{r}_0 > 0$ be given, and let $0 < \delta < \hat{r}_0$ be fixed. Let $\lambda : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\lambda(0) = 0$. Let $d(\cdot)$ denote the diameter of a set. For a given Borel set $E \subset \mathbb{R}^n$, we define (δ, λ)-*Hausdorff content* of E in the usual way:

$$\mathcal{H}^{\lambda}_{\delta}(E) := \inf\left\{\sum_i \lambda(d(U_i)) : E \subset \bigcup U_i,\ \text{each } U_i \text{ is open with } d(U_i) < \delta\right\}.$$

Then the *Hausdorff measure* of E is defined by

$$\mathcal{H}^{\lambda}(E) := \lim_{\delta\to 0} \mathcal{H}^{\lambda}_{\delta}(E). \tag{1.4}$$

In case $\lambda(r) = r^{\alpha}$ we write $\mathcal{H}^{\alpha}$ for $\mathcal{H}^{\lambda}$. The Hausdorff dimension of $\hat{\mu}$, denoted by $\mathcal{H}-\dim\hat{\mu}$, is defined by

$$\mathcal{H}-\dim\hat{\mu} := \inf\big\{\alpha : \text{there exists Borel} \quad E \subset \partial\Omega \quad \text{such that} \quad \mathcal{H}^{\alpha}(E) = 0 \quad \text{and} \quad \hat{\mu}(\mathbb{R}^n \setminus E) = 0\big\}.$$

We return to our study of singular sets of p-harmonic measure with respect to $\mathcal{H}^{n-1}$ measure. For arbitrary $p,\ 1 < p \neq 2 < \infty$, Bennewitz and Lewis [2005] observed that the natural candidates, i.e., snowflake-type domains, which give sharpness in the harmonic case shown by Makarov, do not provide sharpness. In the same paper it was also shown that if $\partial\Omega$ is the von Koch snowflake in the plane and $2 < p < \infty$ then $\mathcal{H}-\dim\mu < 1$. In [Lewis et al. 2011], a weaker version of Makarov's result was obtained under the p-harmonic setting for $1 < p \neq 2 < \infty$. Finally, Lewis [2015] proved a p-harmonic analogue of Makarov's result; let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain and let μ be the p-harmonic measure described above. Let $\lambda(r)$ be as in Makarov's result. Then the following are true.

(a) If $1 < p < 2$, there is $A = A(p) \geq 1$ such that $\mu \ll \mathcal{H}^{\lambda}$.

(b) If $2 < p < \infty$, there is $A = A(p) \leq -1$ such that μ is concentrated on a set of σ-finite $\mathcal{H}^1$ measure.

The nonlinearity and degeneracy of the p-Laplace equation makes it difficult to study the Hausdorff dimension of this measure in $\mathbb{R}^n$, $n \geq 3$. The tools developed by Lewis, Nyström, and Vogel [Lewis et al. 2013] for p-harmonic functions were used to obtain that:

(1) If $\partial\Omega$ is sufficiently flat in the sense of Reifenberg and $p \geq n \geq 3$, then μ is concentrated on a set of σ-finite $\mathcal{H}^{n-1}$ measure.

(2) If $n \geq 3$ and $2 < p < n$, there exist Wolff snowflakes such that $\mathcal{H}-\dim\mu < n-1$, while if $1 < p < 2$, there exist Wolff snowflakes such that $\mathcal{H}-\dim\mu > n-1$.

(3) All examples produced by Wolff's snowflake method have $\mathcal{H}-\dim\mu < n-1$ when $p \geq n$.

(4) There is a Wolff snowflake for which the sign of $(n-1)-(\mathcal{H}-\dim\mu)$ equals the sign of $(n-1)-(\mathcal{H}-\dim\omega)$, where μ is the p-harmonic measure for p in an open interval containing 2 and ω is the harmonic measure with pole at infinity.

Lewis, Vogel, and the author [Akman et al. 2015] improved these results by proving the following: let $O \subset \mathbb{R}^n$ be any open set, $\hat{z} \in \partial O$, and let $\rho > 0$. Let u be a positive weak solution to (1.1) in $O \cap B(\hat{z}, \rho)$. Assume also that u has continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$. Extend u to all $B(\hat{z}, \rho)$ by defining 0 in $B(\hat{z}, \rho) \setminus O$. Let μ be the measure associated to u as in (1.2). If $p > n$ then μ is concentrated on the set

$$\mathcal{P} = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{r \to 0} \frac{\mu(B(w, r))}{r^{n-1}} > 0 \right\}.$$

This set $\mathcal{P}$ has σ-finite $\mathcal{H}^{n-1}$ measure. The same result holds when $p = n$, provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of n-capacity. Therefore, $\mathcal{H}-\dim\mu_p \leq n-1$ when $p \geq n$.

On the other hand, the result of David and Jerison described above for harmonic measure is extended to the p-harmonic setting for $1 < p \neq 2 < \infty$ by Lewis and Nyström [2012]. To state this result, we let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with constants M, r_0 whose boundary is ADR. Let u be p-harmonic in $\Omega \cap B(w, 4r)$, $w \in \partial\Omega$, $0 < r < r_0$, and continuous in $\bar{\Omega} \cap B(w, 4r)$ with $u \equiv 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and let μ be the p-harmonic measure as in (1.2) associated with u. Then it is shown in [Lewis and Nyström 2012, Proposition 3.4] that $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on $\partial\Omega$; in fact they are A_∞-equivalent. It also is proven in the same paper that Badger's result holds under the p-harmonic setting; if Ω is an NTA domain then $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on the set $A' \subset \Delta(w, 4r) \subset \partial\Omega$, where

$$A' = \left\{ x \in \Delta(w, 4r) : \liminf_{\rho \to 0} \frac{\mathcal{H}^{n-1}(\Delta(x, \rho))}{\rho^{n-1}} < \infty \right\}.$$

The main result proved in this paper is that there are examples of domains for which absolute continuity of p-harmonic measure and $(n-1)$-dimensional Hausdorff measure does not hold on the whole boundary when the domain is even locally flat in the sense of Reifenberg.

Theorem 1.5. *When $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2 - \eta < p < 2$ for some $\eta > 0$, there exist domains $\Omega \subset \mathbb{R}^n$ and Borel sets $E \subset \partial\Omega$ such that*:

(1) *Ω is a $(\hat{\delta}, \infty)$-Reifenberg flat domain.*

(2) *$\sigma = \mathcal{H}^{n-1}$ is Radon.*

(3) *$\mu_p(E) > 0 = \sigma(E)$, where μ_p is the p-harmonic measure associated to a positive p-harmonic function in Ω with continuous zero boundary values on $\partial\Omega$.*

As the plan of this paper, we first state the definition of nontangentially accessible domains, Reifenberg flatness, and Ahlfors–David regularity, and we give some lemmas concerning the regularity of p-harmonic function in NTA domains in Section 2. We give the construction of Wolff snowflakes in Section 3. Following [Azzam et al. 2016], we construct "an enlarged domain Ω_ϵ^+" from a certain domain Ω and, using some results from [Lewis et al. 2013] concerning the dimension of p-harmonic measure, we give a proof of Theorem 1.5 in Section 4.

2. Definitions and preparatory lemmas

To proceed, some notation and definitions are in order. In the sequel, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n, unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1 which may depend only on $p, n, a_1, \dots, a_n$, not necessarily the same at each occurrence.

Let $x = (x_1, \dots, x_n)$ denote points in $\mathbb{R}^n$ and let $\bar{E} = \mathrm{cl}(E)$, $\operatorname{int} E$, ∂E, and E^{c} be the closure, interior, boundary, and the complement of the set $E \subset \mathbb{R}^n$, respectively. Let $\operatorname{diam}(E)$ be the diameter of a set E. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$. Let $d(E, F)$ denote the usual distance between the sets E and F and let $d_{\mathcal{H}}(E, F)$ denote the Hausdorff distance between the sets E and F, which is defined by

$$d_{\mathcal{H}}(E, F) := \max(\sup\{d(E, y) : y \in F\}, \sup\{d(x, F) : x \in E\}).$$

Let $B(x, r)$ be the usual open ball centered at x with radius $r > 0$ in $\mathbb{R}^n$ and let $\mathrm{d}x$ denote the Lebesgue n-measure in $\mathbb{R}^n$. Let $\Delta(w, r) = \partial\Omega \cap B(w, r)$. For a given number $t > 0$ and a cube Q, let $l(Q)$ be the side length of Q and let tQ denote the cube whose side length is $tl(Q)$ with the same center as Q.

We state the notion of nontangentially accessible domain which was initially introduced by Jerison and Kenig [1982].

Definition 2.1 (NTA domain). A domain Ω is called a *nontangentially accessible* (NTA) domain if there exist $M \geq 2$ and r_0 such that the following are fulfilled.

(i) *Corkscrew condition*: for any $w \in \partial\Omega$, $0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying

$$M^{-1}r < |a_r(w) - w| < r \quad \text{and} \quad M^{-1}r < d(a_r(w), \partial\Omega).$$

(ii) $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew condition.

(iii) *Uniform condition*: if $w \in \partial\Omega$, $0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$ then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1$ and $\gamma(1) = w_2$ such that

(a) $\mathcal{H}^1(\gamma) \leq M|w_1 - w_2|$,

(b) $\min\{\mathcal{H}^1(\gamma([0, t])), \mathcal{H}^1([t, 1]))\} \leq Md(\gamma(t), \partial\Omega)$.

Remark 2.2. We use the definition of this notion given in [Lewis and Nyström 2012]. Note that (iii) of Definition 2.1 is different but equivalent to the Harnack chain condition given in [Jerison and Kenig 1982].

Next we give the definition of Reifenberg flatness from [Azzam et al. 2016].

Definition 2.3 ((δ, r_0)-Reifenberg flat domain). Let Ω be a domain and $r_0, \delta > 0$ with $0 < \delta < \frac{1}{2}$. Then Ω is said to be (δ, r_0)-*Reifenberg flat* provided that the following two conditions hold.

(i) For every $w \in \partial\Omega$ and every $0 < r < r_0$ there exists a hyperplane $\mathcal{P}(w, r)$ containing w such that

$$d_{\mathcal{H}}(\Delta(w, r), \mathcal{P}(w, r) \cap B(w, r)) \leq \delta r.$$

(ii) For every $x \in \partial\Omega$, one of the connected components of

$$B(x, r_0) \cap \{x \in \mathbb{R}^n;\ \mathrm{d}(x, \mathcal{P}(x, r_0)) \geq 2\delta r_0\}$$

is contained in Ω and the other is contained in $\mathbb{R}^n \setminus \Omega$.

We say that Ω is (δ, ∞)-Reifenberg flat if it is (δ, r_0)-Reifenberg flat for every $r_0 > 0$.

Remark 2.4. An equivalent definition of Reifenberg flatness is given in [Lewis and Nyström 2012], and it is remarked that these two definitions are equivalent (see observation after their Definition 1.2).

Definition 2.5 (Ahlfors–David regular set). We say that $\partial\Omega$ is n-dimensional *Ahlfors–David regular* (ADR) if there is some uniform constant C such that

$$C^{-1}r^n \leq \mathcal{H}^n(\Delta(x, r)) \leq Cr^n \quad \text{for all } r \in (0, \mathrm{diam}(\Omega)),\ x \in \partial\Omega.$$

We next give some estimates from when $n \geq 3$ [Lewis et al. 2013, Lemmas 3.2–3.6] and when $n = 2$ given under the p-harmonic settings [Bennewitz and Lewis 2005, Lemmas 2.6, 2.7, 2.13, 2.14]. For Lemmas 2.6–2.8, let p be fixed with $1 < p \neq 2 < \infty$.

Lemma 2.6. *Let u be a positive p-harmonic function in $B(w, 2r) \subset \mathbb{R}^n$, $n \geq 3$. Then*

$$r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \, \mathrm{d}x \leq c\Big(\max_{B(w,r)} u\Big)^p$$

and

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Moreover, there exists $\beta = \beta(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$ then

$$|u(x) - u(y)| \leq c\left(\frac{|x-y|}{r}\right)^{\beta} \max_{B(w,2r)} u.$$

For Lemmas 2.7 and 2.8 let Ω be an NTA domain in $\mathbb{R}^n$ and let $w \in \partial\Omega$, $0 < r < r_0$.

Lemma 2.7. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular, there exists $\sigma = \sigma(p, n) \in (0, 1]$ such that if $x, y \in B\big(\hat{w}, \frac{1}{2}\hat{r}\big)$, where $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$ then*

$$\frac{1}{c}|\nabla u(x) - \nabla u(y)| \leq \left(\frac{|x-y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},\hat{r})} |\nabla u| \leq \frac{c}{\hat{r}}\left(\frac{|x-y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},2\hat{r})} u.$$

If $\nabla u(\hat{w}) \neq 0$ then u is real analytic in a neighborhood of $\hat{w}$.

The next lemma gives a relation between a p-harmonic function and its corresponding measure.

Lemma 2.8. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. As in (1.2), there exists a unique locally finite positive Borel measure μ on $\mathbb{R}^n$ with support in $\Delta(w, 2r)$ such that*

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \, \mathrm{d}x = -\int \theta \, \mathrm{d}\mu$$

whenever $\theta \in C_0^{\infty}(B(w, 2r))$. Moreover, there exists $c = c(p, n, M) \in [1, \infty)$ such that if $\tilde{r} = r/c$ then

$$c^{-1} r^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c r^{p-n} \mu\big(\Delta\big(w, \tfrac{1}{2}\tilde{r}\big)\big),$$

where $a_{\tilde{r}}(w)$ is as in Definition 2.1.

3. Construction of Wolff snowflakes

In this section, following [Lewis et al. 2013] when $n \geq 3$ and [Bennewitz and Lewis 2005] when $n = 2$, we describe the construction of Wolff snowflakes in $\mathbb{R}^n$ which was originally introduced in [Wolff 1995]. To this end, let

$$\Omega_0 = \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\} \subset \mathbb{R}^n.$$

Set

$$Q(r) = \left\{x' \in \mathbb{R}^{n-1} : -\tfrac{1}{2}r \leq |x_i| \leq \tfrac{1}{2}r \text{ for } 1 \leq i \leq n-1\right\}.$$

Then $Q(r)$ is an $(n-1)$-dimensional cube with side length r and centered at 0. Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a piecewise linear function with support contained in $\left\{x' : |x'| < \frac{1}{2}\right\}$ satisfying

$$\|\nabla\phi\|_\infty \leq \theta_0. \tag{3.1}$$

For fixed large N, define $\psi(x') = N^{-1}\phi(Nx')$. Let $b > 0$ be a small constant and let Q be an $(n-1)$-dimensional cube with center a_Q and length $l(Q)$ contained in some hyperplane. Let $\operatorname{cch}(E)$ denote the closed convex hull. Let e be a unit normal to Q and define

$$P_Q = \operatorname{cch}(Q \cup \{a_Q + bl(Q)e\}) \quad \text{and} \quad \tilde{P}_Q = \operatorname{int}\operatorname{cch}(Q \cup \{a_Q - bl(Q)e\}).$$

We set $e = -e_n$ for $Q(1)$. We also define

$$\Lambda := \{x \in P_{Q(1)} \cup \tilde{P}_{Q(1)},\ x_n \geq \psi(x)\} \quad \text{and} \quad \partial := \{x \in \mathbb{R}^n,\ x' \in Q(1),\ x_n = \psi(x')\}.$$

We assume that $N = N(b, M)$ is so large that

$$d(\partial \setminus \partial\Omega_0, \partial[P_{Q(1)} \cup \tilde{P}_{Q(1)}]) \geq \tfrac{b}{100}.$$

From the construction, it can be easily seen that $\partial \subset Q(1) \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ consists of a finite number of $(n-1)$-dimensional faces. We fix a Whitney decomposition of each face; we divide each face of ∂ into an $(n-1)$-dimensional cube Q, with side lengths 8^{-k}, $k = 1, 2, \ldots$, and $8^{-k} \approx$ to their distance from the edges of the face they lie on. We next choose a distinguished $(n-2)$-dimensional "side" for each $(n-1)$-dimensional cube.

Suppose Ω is a domain and $Q \subset \partial\Omega$ is an $(n-1)$-dimensional cube with distinguished side γ. Let e be a unit normal to $\partial\Omega$ on Q and assume that $P_Q \cap \Omega = \varnothing$ and $\tilde{P} \subset \Omega$. We form a new domain $\tilde{\Omega}$ as follows. Let $\mathcal{T}$ be the conformal affine map, i.e., composition of a translation, dilation, and rotation, with $\mathcal{T}(Q(1)) = Q$ which fixes dilation, $\mathcal{T}(0) = a_Q$ which fixes translation, $\mathcal{T}\left(\left\{x \in \partial Q(1) : x_1 = \frac{1}{2}\right\}\right)$ and $\mathcal{T}(-e_n)$ in the direction of e which fixes rotation. Let $\Lambda_Q = \mathcal{T}(\Lambda)$ and $\partial_Q = \mathcal{T}(\partial)$.

Then we define $\tilde{\Omega}$ through the relations

$$\tilde{\Omega} \cap (P_Q \cup \tilde{P}_Q) \quad \text{and} \quad \tilde{\Omega} \setminus (P_Q \cup \tilde{P}_Q) = \Omega \setminus (P_Q \cup \tilde{P}_Q).$$

Note that ∂_Q inherits from ∂ a natural subdivision into Whitney cubes with distinguished sides. This process is called "adding a blip to Ω along Q".

To use the process of "adding a blip" to construct a Wolff snowflake Ω_∞, starting from Ω_0, we first add a blip to Ω_0 along $Q(1)$ obtaining a new domain Ω_1. We then inherit a subdivision of $\partial\Omega_1 \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)})$ into Whitney cubes with distinguished sides, together with a finite set of edges E_1 (the edges of the faces of the graph are not in the Whitney cubes). Let G_1 be the set of all Whitney cubes in the subdivision. Then Ω_2 is obtained from Ω_1 by adding a blip along each $Q \in G_1$. From this process, we inherit a family of cubes $G_2 \subset \partial\Omega_2$ (each with a distinguished side) and a set of edges $E_2 \subset \partial\Omega_2$ of σ-finite $\mathcal{H}^{n-2}$ measure. Continuing by induction we get $(\Omega_m)_{m=n-1}^\infty$, $(G_m)_{m=n-1}^\infty$, and $(E_m)_{m=n-1}^\infty$, where

$$\partial\Omega_m \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)}) = E_m \cup \bigcup_{Q \in G_m} Q \quad \text{for } m \geq n-1.$$

If $N = N(b, M)$ is large enough, then $\Omega_m \to \Omega_\infty$ in the Hausdorff distance sense. We call Ω_∞ a *Wolff snowflake*. We state a result which says that Wolff snowflakes are locally flat in the sense of Reifenberg.

Lemma 3.2 [Lewis et al. 2013, Lemma 7.1]. *If θ_0, N^{-1} are small enough, depending only n, then the Wolff snowflake domain Ω_∞ is $(c\theta_0, \infty)$-Reifenberg flat, where $c = c(n)$.*

4. Proof of Theorem 1.5

In this section we give a proof of Theorem 1.5 using some results from [Lewis et al. 2013; Azzam et al. 2016]. To this end, let Ω_∞ be a Wolff snowflake with constants θ_0, N as described in Section 3. For fixed p, $1 < p \neq 2 < \infty$, let u_∞ be the unique positive p-harmonic function in Ω_∞ with continuous boundary value zero on $\partial\Omega_\infty$ and $|x_n - u_\infty(x)| \to 0$ uniformly as $|x| \to \infty$. Let μ_∞ be the p-harmonic measure associated with u_∞ as in (1.2). A proof of existence and uniqueness of u_∞ can be found in [Lewis et al. 2013, Lemma 6.1]. Let Ω'_∞ be the restriction of Ω_∞ to $Q(1) \times [-1, 1]$ and let μ'_∞ be the restriction of μ_∞ to $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$.

The following lemma can be easily deduced by combining Lemma 7.4 and Proposition 7.6 from [Lewis et al. 2013] when $n \geq 3$ and combining Lemma 3.23 and Theorem 1 from [Bennewitz and Lewis 2005] when $n = 2$. Moreover, when $n \geq 3$ and $2-\eta < p < 2$ it follows from Theorem 4 in [Lewis et al. 2013]. We first state a lemma.

Lemma 4.1. *When $n \geq 3$ let p be fixed, $2 < p < \infty$, and when $n \geq 2$ let p be fixed with $2 - \eta < p < 2$ for some $\eta > 0$. Let Ω'_∞ and μ'_∞ be described as above. Then* *for some $d > 0$ we have*

$$\lim_{r\to 0} \frac{\log \mu'_\infty(\Delta(x,r))}{\log r} \le d < n-1 \quad \textit{for all } x \in \partial\Omega'_\infty \setminus \Lambda,$$

where $\Lambda \subset \partial\Omega'_\infty$ with $\mu'_\infty(\Lambda) = 0$. Moreover, $\mathcal{H}-\dim \mu'_\infty \le d < n-1$.

Proof. We first show that there exist Wolff snowflakes for which $\mathcal{H}-\dim\mu < n-1$ in all cases stated in Lemma 4.1. To this end, as we discussed in Section 1, when $n \ge 3$ and $2 < p < \infty$ there exist Wolff snowflakes such that $\mathcal{H}-\dim\mu < n-1$ (see Theorems 2 and 3 in [Lewis et al. 2013]). When $n = 2$, it follows from [Bennewitz and Lewis 2005, Theorem 1] that there is a Wolff snowflake for which $\mathcal{H}-\dim\mu < 1$ whenever p is fixed with $2 < p < \infty$. Next, there exist Wolff snowflakes for which $\mathcal{H}-\dim\omega < n-1$, which is a well-known result of Wolff [1995] when $n \ge 3$. On the other hand, it is observed in [Lewis et al. 2013, Proposition 6.4] that there exists a Wolff snowflake such that the sign of $(n-1)-(\mathcal{H}-\dim\omega)$ equals the sign of $(n-1)-(\mathcal{H}-\dim\mu)$ for $p \in (2-\eta, 2)$. Therefore, combining these two results, we first conclude that there exists a Wolff snowflake for which $\mathcal{H}-\dim\omega < n-1$ when $2-\eta < p < 2$ for some $\eta > 0$. Using these observations and Lemma 7.4 in [Lewis et al. 2013] we finish the proof of lemma. □

We are now ready to prove Theorem 1.5. Under the p-harmonic setting, we closely follow the arguments given in [Azzam et al. 2016] after Theorem 4.3. We first observe from Lemma 4.1, more specifically from the fact $\mathcal{H}-\dim\mu'_\infty \le d < n-1$, and the definition of Hausdorff dimension of p-harmonic measure, that there is a Borel set $E \subset \partial\Omega'_\infty$ such that $\mu'_\infty(\mathbb{R}^n \setminus E) = 0$ and $\mathcal{H}^d(E) = 0$. From this observation and once again from Lemma 4.1 we also have

$$\lim_{r\to 0} \frac{\log \mu'_\infty(B(x,r))}{\log r} \le d < n-1 \quad \text{for all } x \in E. \tag{4.2}$$

Note that Ω'_∞ is the restriction of Ω_∞ to $Q(1)\times[-1,1]$; therefore,

$$\partial\Omega_\infty \setminus \{(x', x_n) \in \mathbb{R}^n : x_n = 0\} \subset \partial\Omega'_\infty.$$

For ease of notation we let

$$\mathfrak{R}^{n-1} := \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ and } x_n = 0\}.$$

From (4.2) it follows that for α, $0 < \alpha < n-1-d$, one can find small enough ρ such that $\mu'_\infty(E_1) > 0$, where

$$E_1 = \left\{ x \in (E \cap \partial\Omega_\infty) \setminus \mathfrak{R}^{n-1} : \frac{\log \mu'_\infty(B(x,r))}{\log r} < n-1-\alpha \text{ for all } r \in (0,\rho] \right\}.$$

We next fix a point $\zeta_0 \in E_1$. By the regularity of p-harmonic measure we can find $\rho_0 \in (0,\rho]$ and a compact set $K \subset E_1 \cap B(\zeta_0, \rho_0)$ such that for all $x \in K$ and

$r \in (0, \rho_0)$ the following property holds:

$$\mu'_\infty(K) > 0 \quad \text{and} \quad \mu'_\infty(B(x,r)) > r^{n-1-\alpha}.$$

The construction yields that $K \subset \partial\Omega'_\infty \cap \partial\Omega_\infty$ and $\mathrm{cl}(\Omega'_\infty) \subset \mathrm{cl}(\Omega_\infty)$. Then using the fact that the support of μ'_∞ is contained in $(Q(1) \times [-1,1]) \cap \partial\Omega_\infty$ we have

$$\mu_\infty(K) > 0 \quad \text{and} \quad \mu_\infty(B(x,r) \cap \partial\Omega_\infty) > r^{n-1-\alpha} \tag{4.3}$$

for all $x \in K$ and $r \in (0, \rho_0)$.

For a given number t, $4 \le t$, and given open set $O \subset \mathbb{R}^{n-1}$ we use $\mathcal{W}_t(O)$ to denote the set of maximal dyadic cubes $Q \subset O$ satisfying $tQ \cap K^{\mathrm{c}} = \varnothing$. Let $0 < \epsilon < \frac{1}{100}$ and let $\mathcal{I}$ be the family of cubes $Q \in \mathcal{W}_{\epsilon^{-2}}(K^{\mathrm{c}})$ such that

$$Q \cap (Q(1) \times [-1,1]) \cap \partial\Omega_\infty \neq \varnothing.$$

Note that

$$l(Q) \approx \epsilon^2 \operatorname{dist}(Q, K) \quad \text{for all } Q \in \mathcal{I} \quad \text{and} \quad \partial\Omega'_\infty \setminus K \subset \bigcup_{Q \in \mathcal{I}} Q.$$

For each $Q \in \mathcal{I}$, fix some point $z_Q \in Q \cap \partial\Omega'_\infty$. We then define a new domain Ω^+_ϵ by

$$\Omega^+_\epsilon := \Omega'_\infty \cup \Big(\bigcup_{Q \in \mathcal{I}} B_Q\Big), \qquad \text{where } B_Q = B(z_Q, \epsilon \operatorname{dist}(z_Q, K)).$$

It is observed in [Azzam et al. 2016, Lemma 2.2] that if θ, ϵ in the construction of the Wolff snowflake in Section 3 are small enough then Ω^+_ϵ is $(c\epsilon^{1/2}, r_0)$-Reifenberg flat and $K \subset \partial\Omega^+_\epsilon$, provided that the original domain Ω_∞ is (δ, r_0)-Reifenberg flat. Note that from Lemma 3.2 we have that Wolff snowflake domain Ω_∞ is $(c\theta_0, r_0)$-Reifenberg flat, where $r_0 = \infty$. Therefore if we choose θ and ϵ small enough and use Lemma 2.2 from [Azzam et al. 2016] then Ω^+_ϵ is a $(c\epsilon^{1/2}, \infty)$-Reifenberg flat domain satisfying

$$K \subset \partial\Omega'_\infty \cap \partial\Omega^+_\epsilon \quad \text{and} \quad \mathrm{cl}(\Omega_\infty) \subset \mathrm{cl}(\Omega^+_\epsilon). \tag{4.4}$$

Let u^+_ϵ be a positive p-harmonic function in Ω^+_ϵ with continuous boundary value zero on $\partial\Omega^+_\epsilon$. Let μ^+_ϵ be the p-harmonic measure associated with u^+_ϵ as in (1.2). From the construction of Ω^+_ϵ we have $u^+_\epsilon \ge u'_\infty$ on $\partial\Omega'_\infty$. Then it follows from the maximum principle for positive p-harmonic functions and (4.4) that $u^+_\epsilon \ge u'_\infty$ in Ω'_∞. This observation, Lemmas 2.6–2.8 and (4.3) yield

$$\mu^+_\epsilon(K) > 0 \quad \text{and} \quad \mu^+_\epsilon(B(x,r)) > r^{n-1-\alpha} \quad \text{for all } x \in K,\ r \in (0, \rho_0). \tag{4.5}$$

As μ^+_ϵ is a Radon measure which follows from Lemma 2.8 and satisfies (4.5) and Ω^+_ϵ is $(\hat\delta, \hat r_0)$-Reifenberg flat domain, it follows from [Azzam et al. 2016, Lemma 3.1] that $\mathcal{H}^{n-1}|_{\partial\Omega^+_\epsilon}$ is locally finite. Let $\Omega := \Omega^+_\epsilon$ be the $(\hat\delta, \hat r_0)$-Reifenberg flat domain with locally finite surface measure and let $\mu := \mu_\epsilon^+$ be the p-harmonic measure as above. From (4.5) we conclude that Theorem 1.5 is true. $\square$

Acknowledgments

The author would like to thank John Lewis for his suggestions and fruitful discussions to improve the case when $p \in (2-\eta, 2)$. He also warmly thanks the referee for their careful reading of the manuscript and useful suggestions. The author has been supported in part by ICMAT Severo Ochoa project SEV-2011-0087. He acknowledges that the research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007–2013)/ ERC agreement no. 615112 HAPDEGMT.

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Received November 5, 2015. Revised August 16, 2016.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 1 January 2017