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ON THE FOURIER–JACOBI MODEL FOR SOME ENDOSCOPIC ARTHUR PACKET OF $U(3) \times U(3)$: THE NONGENERIC CASE

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For a generic L-parameter of $U(n) \times U(n)$, it is conjectured that there is a unique representation in their associated relevant Vogan L-packet which produces the unique Fourier–Jacobi model. We investigated this conjecture for some nongeneric L-parameters of $U(3) \times U(3)$ and discovered that it is true for some nongeneric L-parameters and false for some nongeneric L-parameters. In the case when it holds, we specified such representation under the local Langlands correspondence for unitary groups.

1. Introduction

The local Gan–Gross–Prasad conjecture deals with certain restriction problems between p-adic groups. In this paper, we shall investigate it for some nongeneric case not treated before.

Let E/F be a quadratic extension of number fields and $G = U(3)$ be the quasisplit unitary group of rank 3 relative to E/F. Then $H = U(2) \times U(1)$ is the unique elliptic endoscopic group for G. Rogawski [1990] has defined a certain enlarged class of L-packets, or A-packets, of G using endoscopic transfer of one-dimensional characters of H to G. In more detail, let $\varrho = \otimes_v \varrho_v$ be a one-dimensional automorphic character of H. The A-packet $\Pi(\varrho) \simeq \otimes \Pi(\varrho_v)$ is the transfer of ϱ with respect to functoriality for an embedding of L-groups $\xi : {}^L H \to {}^L G$. Then for all places v of F, the packet $\Pi(\varrho_v)$ contains a certain nontempered representation $\pi^n(\varrho_v)$ and it contains an additional supercuspidal representation $\pi^s(\varrho_v)$ precisely when v remains prime in E. Gelbart and Rogawski [1991] showed that the representations in this A-packet arise in the Weil representation of G. Our goal is to study the branching rule of the representations in this A-packet.

For the branching problem, there is a fascinating conjecture, the so-called Gan–Gross–Prasad (GGP) conjecture, which was first proposed by Gross and Prasad [1992] for orthogonal groups and later they, together with Gan, extended it

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to all classical groups in [Gan et al. 2012]. Since it concerns our main theorem, we shall give a brief review on the GGP conjecture, especially for unitary groups.

Let E/F be a quadratic extension of nonarchimedean local fields of characteristic zero. Let V_{n+1} be a hermitian space of dimension $n+1$ over E and W_n a skew-hermitian space of dimension n over E. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1, so that if we set

$$G_n = U(V_n) \times U(V_{n+1}) \quad \text{or} \quad U(W_n) \times U(W_n)$$

and

$$H_n = U(V_n) \quad \text{or} \quad U(W_n),$$

then we have a diagonal embedding

$$\Delta : H_n \hookrightarrow G_n.$$

Let π be an irreducible smooth representation of G_n. In the hermitian case, one is interested in computing

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n}(\pi, \mathbb{C}).$$

We shall call this the *Bessel* case (B) of the GGP conjecture. For the GGP conjecture in the skew-hermitian case, we need to introduce a certain Weil representation ω_{ψ,χ,W_n} of H_n, where ψ is a nontrivial additive character of F and χ is a character of $E^\times$ whose restriction to $F^\times$ is the quadratic character $\omega_{E/F}$ associated to E/F by local class field theory. (For the exact definition of ω_{ψ,χ,W_n}, please see page 77.) In this case, one is interested in computing

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n}(\pi, \omega_{\psi,\chi,W_n}).$$

We shall call this the *Fourier–Jacobi* case (FJ) of the GGP conjecture. To treat both cases using one notation, we shall let ν equal $\mathbb{C}$ or ω_{ψ,χ,W_n} in the respective cases.

By the results of [Aizenbud et al. 2010; Sun 2012], it is known that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n}(\pi, \nu) \leq 1,$$

and so the next step is to specify irreducible smooth representations π such that

$$\operatorname{Hom}_{\Delta H_n}(\pi, \nu) = 1.$$

(A nonzero element of $\operatorname{Hom}_{\Delta H_n}(\pi, \nu)$ is called a *Bessel* (*Fourier–Jacobi*) model of π in the hermitian (skew-hermitian) case.)

Gan, Gross and Prasad [2012] brought this problem into a more general setting using the notion of relevant pure inner forms of G_n and Vogan L-packets. A pure inner form of G_n is a group of the form

$$G_n' = U(V_n') \times U(V_{n+1}') \quad \text{or} \quad U(W_n') \times U(W_n''),$$

where $V_n' \subset V_{n+1}'$ are n and $n+1$ dimensional hermitian spaces over E, and W_n', W_n'' are n-dimensional skew-hermitian spaces over E.

Furthermore, if

$$V_{n+1}'/V_n' \cong V_{n+1}/V_n \quad \text{or} \quad W_n' = W_n'',$$

we say that G_n' is a relevant pure inner form.
(Indeed, there are four pure inner forms of G_n and among them, only two are relevant.)

If G_n' is relevant, we set

$$H_n' = U(V_n') \quad \text{or} \quad U(W_n'),$$

so that we have a diagonal embedding

$$\Delta : H_n' \hookrightarrow G_n'.$$

Now suppose that ϕ is an L-parameter for the group G_n. Then the (relevant) Vogan L-packet Π_ϕ associated to ϕ consists of certain irreducible smooth representations of G_n and its (relevant) pure inner forms G_n' whose L-parameter is ϕ. We denote the relevant Vogan L-packet of ϕ by Π_ϕ^R.

With these notions, we can loosely state the result of Beuzart-Plessis [2014; 2012; 2015] for the *Bessel* case and Gan and Ichino [2016] for the *Fourier–Jacobi* case as follows:

Theorem 1.1. *For a tempered L-parameter ϕ of G_n, the following hold*:

(i) $\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n'}(\pi', \nu) = 1$.

(ii) *Using the local Langlands correspondence for unitary groups, we can pinpoint the unique $\pi' \in \Pi_\phi^R$ such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{\Delta H_n'}(\pi', \nu) = 1.$$

To emphasize its dependence on the number n, we denote the *Fourier–Jacobi* and *Bessel* cases of Theorem 1.1 as $(\mathrm{FJ})_n$ and $(\mathrm{B})_n$ respectively, and later we shall elaborate more on this notation. The GGP conjecture predicts that this theorem also holds for a generic L-parameter ϕ of G_n.

Our main theorem is to investigate $(\mathrm{FJ})_3$ for some L-parameter of G_3 involving a nongeneric L-parameter of $U(W_3)$. More precisely, we have:

Main Theorem. *For an irreducible smooth representation π_2 of $U(W_3)$, let $\pi = \pi^n(\varrho) \otimes \pi_2$ as a representation of G_3. Then*

(i) $\operatorname{Hom}_{\Delta H_3}(\pi, \omega_{\psi,\chi,W_3}) = 0$ *if π_2 is not a theta lift from $U(V_2)$.*

(ii) *Assume that π_2 is the theta lift from $U(V_1')$ and let $\phi = \phi^n \otimes \phi_2$ be the L-parameter of π. Then*

$$\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_3'}(\pi', \omega_{\psi,\chi,W_3}) = 1.$$

(iii) *Using the local Langlands correspondence for unitary groups, we can explicitly describe the representation $\pi' \in \Pi_\phi^R$ appearing in (ii) such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_3'}(\pi', \omega_{\psi,\chi,W_3}) = 1.$$

Remark 1.2. As we shall see in Theorem 3.2, the L-parameter of $\pi^n(\varrho_v)$ is not only nontempered but also nongeneric. Thus if we choose the L-parameters of ϕ_2 in ϕ apart from those obtained by the theta lift from $U(V_2)$ to $U(W_3)$, then the first part of the Main Theorem tells us that the GGP conjecture may not be true for nongeneric L-parameters of G_n.

The proof of the Main Theorem is based on the following see-saw diagram:

$$\begin{tikzcd} U(W_3)\times U(W_3) \arrow[d, dash] \arrow[dr, dash] & U(V_2) \arrow[d, dash] \arrow[dl, dash] \\ U(W_3) & U(V_1)\times U(V_1) \end{tikzcd}$$

Since all elements in the A-packet $\Pi(\varrho)$ can be obtained by theta lift from $U(V_1)$, we can write $\pi^n(\varrho) = \Theta_{\psi,\chi,W_3,V_1}(\sigma)$ where σ is an irreducible smooth character of $U(V_1)$, and ψ,χ are some characters, which are needed to fix a relevant Weil representation. Then by the see-saw identity, we have

$$\mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \simeq \mathrm{Hom}_{U(V_1)}(\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee), \sigma).$$

From this, we see that for having $\mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \neq 0$, it should be preceded by $\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0$. This accounts for (i) in the Main Theorem because

$$\begin{aligned} &\mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \\ &\qquad \simeq \mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \pi_2, \omega_{\psi,\chi,W_3}). \end{aligned}$$

If $\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0$, then by the local theta correspondence, $\pi_2^\vee$ should be $\Theta_{\psi,\chi,W_3,V_2}(\pi_0)$, where π_0 is an irreducible representation of $U(V_2)$. By applying $(\mathrm{B})_1$, we can pinpoint π_0 and σ in the framework of local Langlands correspondence such that $\mathrm{Hom}_{U(V_1)}(\pi_0, \sigma) \neq 0$. Next we shall use the precise local theta correspondences for $(U(V_1), U(W_3))$ and $(U(V_1), U(W_1))$ in order to transfer the recipe for $(\mathrm{B})_1$ to $(\mathrm{FJ})_3$.

The rest of the paper is organized as follows: In Section 2, we shall give a brief sketch of the local Langlands correspondence for unitary groups. In Section 3, we collect some results on the local theta correspondence for unitary groups which we will use in the proof of our main results. In Section 4, we shall prove our Main Theorem.

Notations. We fix some notations we shall use throughout this paper:

- E/F is a quadratic extension of nonarchimedean local fields of characteristic zero.
- c is the nontrivial element of $\mathrm{Gal}(E/F)$.
- Fr_E is a Frobenius element of $\mathrm{Gal}(\bar{E}/E)$.
- Denote by $\mathrm{Tr}_{E/F}$ and $\mathrm{N}_{E/F}$ the trace and norm maps from E to F.
- δ is an element of E such that $\mathrm{Tr}_{E/F}(\delta) = 0$.
- Let ψ be an additive character of F and define
$$\psi^E(x) := \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big) \quad \text{and} \quad \psi_2^E(x) := \psi(\mathrm{Tr}_{E/F}(\delta x)).$$
- Let χ be a character of $E^\times$ whose restriction to $F^\times$ is $\omega_{E/F}$, which is the quadratic character associated to E/F by local class field theory.
- For a linear algebraic group G, its F-points will be denoted by $G(F)$ or simply by G.

2. Local Langlands correspondence for unitary groups

By the recent work of Mok [2015] and Kaletha, Mínguez, Shin and White [2014], the local Langlands correspondence is now known for unitary groups conditional on the stabilization of the twisted trace formula and weighted fundamental lemma. The twisted trace formula has now been stabilized in [Waldspurger 2014a; Waldspurger 2014b] and [Moeglin and Waldspurger 2014a; Moeglin and Waldspurger 2014b] and the proof of the weighted fundamental lemma is an ongoing project of Chaudouard and Laumon. Since our main results are expressed using the local Langlands correspondence, we shall assume the local Langlands correspondence for unitary groups. In this section, we list some of its properties which are used in this paper. Indeed, much of this section is excerpts from Section 2 in [Gan and Ichino 2016].

Hermitian and skew-hermitian spaces. For $\varepsilon = \pm 1$, let V be a finite n-dimensional vector space over E equipped with a nondegenerate ε-hermitian c-sesquilinear form $\langle\,\cdot\,,\cdot\,\rangle_V : V \times V \to E$. That means for $v, w \in V$ and $a, b \in E$,
$$\langle av, bw\rangle_V = ab^c\langle v, w\rangle_V, \qquad \langle w, v\rangle_V = \varepsilon\cdot\langle v, w\rangle_V^c.$$

We define $\operatorname{disc} V = (-1)^{(n-1)n/2} \cdot \det V$, so that

$$\operatorname{disc} V \in \begin{cases} F^\times / \mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = +1, \\ \delta^n \cdot F^\times / \mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = -1, \end{cases}$$

and we can define $\epsilon(V) = \pm 1$ by

$$\epsilon(V) = \begin{cases} \omega_{E/F}(\operatorname{disc} V) & \text{if } \varepsilon = +1, \\ \omega_{E/F}(\delta^{-n} \cdot \operatorname{disc} V) & \text{if } \varepsilon = -1. \end{cases} \tag{2-1}$$

By a theorem of Landherr, for a given positive integer n, there are exactly two isomorphism classes of ε-hermitian spaces of dimension n and they are distinguished from each other by $\epsilon(V)$. Let $U(V)$ be the unitary group of V defined by

$$U(V) = \{ g \in \mathrm{GL}(V) \mid \langle gv, gw \rangle_V = \langle v, w \rangle_V \text{ for } v, w \in V \}.$$

Then $U(V)$ turns out to be the connected reductive algebraic group defined over F.

L-parameters and component groups. Let I_F be the inertia subgroup of $\mathrm{Gal}(\bar{F}/F)$. Let $W_F = I_F \rtimes \langle \mathrm{Fr}_F \rangle$ be the Weil group of F and $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of F. For a homomorphism $\phi : \mathrm{WD}_F \to \mathrm{GL}_n(\mathbb{C})$, we say that it is a representation of WD_F if

(i) ϕ is continuous and $\phi(\mathrm{Fr}_F)$ is semisimple,

(ii) the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is induced by a morphism of algebraic groups $\mathrm{SL}_2 \to \mathrm{GL}_n$.

If, moreover, the image of W_F is bounded then we say that ϕ is tempered. Define $\phi^\vee$ by $\phi^\vee(w) = {}^t\phi(w)^{-1}$ and call this the contragredient representation of ϕ. If E/F is a quadratic extension of local fields and ϕ is a representation of WD_E, fix $s \in W_F \setminus W_E$ and define a representation ϕ^c of WD_E by $\phi^c(w) = \phi(sws^{-1})$. The equivalence class of ϕ^c is independent of the choice of s. We say that ϕ is conjugate self-dual if there is an isomorphism $b : \phi \mapsto (\phi^\vee)^c$. Note that there is a natural isomorphism $(((\phi^\vee)^c)^\vee)^c \simeq \phi$ so that $(b^\vee)^c$ can be considered as an isomorphism from ϕ onto $(\phi^\vee)^c$. For $\varepsilon = \pm 1$, we say that ϕ is conjugate self-dual with sign ε if there exists such a b satisfying the extra condition $(b^\vee)^c = \varepsilon \cdot b$. Define $\mathrm{As}(\phi) : \mathrm{WD}_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ by tensor induction of ϕ as follows:

$$\mathrm{As}(\phi)(w) = \begin{cases} \phi(w) \otimes \phi(s^{-1}ws) & \text{if } w \in \mathrm{WD}_E, \\ \iota \circ (\phi(s^{-1}w) \otimes \phi(ws)) & \text{if } w \in \mathrm{WD}_F \setminus \mathrm{WD}_E, \end{cases}$$

where ι is the linear isomorphism of $\mathbb{C}^n \otimes \mathbb{C}^n$ given by $\iota(x \otimes y) = y \otimes x$.

Then the equivalence class of $\mathrm{As}(\phi)$ is also independent of the choice of s. We set $\mathrm{As}^+(\phi) = \mathrm{As}(\phi)$ and $\mathrm{As}^-(\phi) = \mathrm{As}(\chi \otimes \phi)$.

Let V be an n-dimensional ε-hermitian space over E. An L-parameter for the unitary group $U(V)$ is a $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of admissible homomorphisms

$$\phi : \mathrm{WD}_F \to {}^L U(V) = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F),$$

such that the composite of ϕ with the projection onto $\mathrm{Gal}(E/F)$ is the natural projection of WD_F to $\mathrm{Gal}(E/F)$.

The following proposition from [Gan et al. 2012, Section 8] enables us to remove the cumbersome $\mathrm{Gal}(E/F)$-factor in the definition of L-parameters of $U(V)$.

Proposition 2.1. *Restriction to the Weil–Deligne group* WD_E *gives a bijection between the set of L-parameters for* $U(V)$ *and the set of equivalence classes of conjugate self-dual representations*

$$\phi : \mathrm{WD}_E \to \mathrm{GL}_n(\mathbb{C})$$

of sign $(-1)^{n-1}$.

With this proposition, by an L-parameter for $U(V)$, we mean an n-dimensional conjugate self-dual representation ϕ of WD_E of sign $(-1)^{n-1}$.

Given an L-parameter ϕ of $U(V)$, we say that ϕ is generic if its adjoint L-function $L(s, \mathrm{Ad}\circ\phi) = L(s, \mathrm{As}^{(-1)^{n-1}}(\phi))$ is holomorphic at $s = 1$. Write ϕ as a direct sum

$$\phi = \bigoplus_i m_i \phi_i$$

of pairwise inequivalent irreducible representations ϕ_i of WD_E with multiplicities m_i. We say that ϕ is square-integrable if it has no multiplicity (i.e., $m_i = 1$ for all i) and ϕ_i is conjugate self-dual with sign $(-1)^{n-1}$ for all i. Furthermore, we can associate its component group S_ϕ to ϕ. As explained in [Gan et al. 2012, Section 8], S_ϕ is a finite 2-abelian group which can be described as

$$S_\phi = \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j$$

with a canonical basis $\{a_j\}$, where the product ranges over all j such that ϕ_j is conjugate self-dual with sign $(-1)^{n-1}$. If we denote the image of $-1 \in \mathrm{GL}_n(\mathbb{C})$ in S_ϕ by z_ϕ, we have

$$z_\phi = (m_j a_j) \in \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j.$$

Local Langlands correspondence for unitary groups. The aim of the local Langlands correspondence for unitary groups is to classify the irreducible smooth representations of unitary groups. To state it, we first introduce some notations.

- Let V^+ and V^- be the n-dimensional ε-hermitian spaces with $\epsilon(V^+) = +1$, $\epsilon(V^-) = -1$ respectively.

- Let $\mathrm{Irr}(U(V^{\pm}))$ be the set of irreducible smooth representations of $U(V^{\pm})$.

Then a form of the local Langlands correspondence enhanced by Vogan [1993], says that for an L-parameter ϕ of $U(V^{\pm})$, there is the so-called Vogan L-packet Π_ϕ, a finite set consisting of irreducible smooth representations of $U(V^{\pm})$, such that

$$\mathrm{Irr}(U(V^{+})) \sqcup \mathrm{Irr}(U(V^{-})) = \bigsqcup_{\phi} \Pi_\phi,$$

where ϕ on the right-hand side runs over all equivalence classes of L-parameters for $U(V^{\pm})$. Then under the local Langlands correspondence, we may also decompose Π_ϕ as

$$\Pi_\phi = \Pi_\phi^{+} \sqcup \Pi_\phi^{-},$$

where for $\epsilon = \pm 1$, the L-packet Π_ϕ^{ϵ} consists of the representations of $U(V^{\epsilon})$ in Π_ϕ.

Furthermore, as explained in [Gan et al. 2012, Section 12], there is a bijection

$$J^{\psi}(\phi) : \Pi_\phi \to \mathrm{Irr}(S_\phi),$$

which is canonical when n is odd and depends on the choice of an additive character of $\psi : F^{\times} \to \mathbb{C}$ when n is even. More precisely, such bijection is determined by the $\mathrm{N}_{E/F}(E^{\times})$-orbit of nontrivial additive characters

$$\begin{cases} \psi^{E} : E/F \to \mathbb{C}^{\times} & \text{if } \varepsilon = +1, \\ \psi : F \to \mathbb{C}^{\times} & \text{if } \varepsilon = -1. \end{cases}$$

According to this choice, when n is even, we write

$$J^{\psi} = \begin{cases} J_{\psi^{E}} & \text{if } \varepsilon = +1, \\ J_{\psi} & \text{if } \varepsilon = -1, \end{cases}$$

and even when n is odd, we retain the same notation $J^{\psi}(\phi)$ for the canonical bijection. Hereafter, if a nontrivial additive character $\psi : F \to \mathbb{C}^{\times}$ is fixed, we define $\psi^{E} : E/F \to \mathbb{C}^{\times}$ by

$$\psi^{E}(x) := \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big),$$

and using these two characters, we fix once and for all a bijection

$$J^{\psi}(\phi) : \Pi_\phi \to \mathrm{Irr}(S_\phi),$$

as above.

With these fixed bijections, we can label all irreducible smooth representations of $U(V^{\pm})$ as $\pi(\phi, \eta)$ for some unique L-parameter ϕ of $U(V^{\pm})$ and $\eta \in \mathrm{Irr}(S_\phi)$.

Properties of the local Langlands correspondence. We briefly list some properties of the local Langlands correspondence for unitary groups, which we will use in this paper:

- $\pi(\phi, \eta)$ is a representation of $U(V^\epsilon)$ if and only if $\eta(z_\phi) = \epsilon$.
- $\pi(\phi, \eta)$ is tempered if and only if ϕ is tempered.
- $\pi(\phi, \eta)$ is square-integrable if and only if ϕ is square-integrable.
- The component groups S_ϕ and $S_{\phi^\vee}$ are canonically identified. Under this canonical identification, if $\pi = \pi(\phi, \eta)$, then its contragredient representation $\pi^\vee$ is $\pi(\phi^\vee, \eta \cdot \nu)$ where
$$\nu(a_j) = \begin{cases} \omega_{E/F}(-1)^{\dim \phi_j} & \text{if } \dim_{\mathbb{C}} \phi \text{ is even,} \\ 1 & \text{if } \dim_{\mathbb{C}} \phi \text{ is odd.} \end{cases}$$
(This property follows from a result of Kaletha [2013, Theorem 4.9].)

3. Local theta correspondence

In this section, we state the local theta correspondence of unitary groups for two low rank cases. From now on, for $\epsilon = \pm 1$, we shall denote by V_n^ϵ the n-dimensional hermitian space with $\epsilon(V_n^\epsilon) = \epsilon$ and by W_n^ϵ the n-dimensional skew-hermitian space with $\epsilon(W_n^\epsilon) = \epsilon$, so that $W_n^\epsilon = \delta \cdot V_n^\epsilon$.

The Weil representation for unitary groups. In this subsection, we introduce the Weil representation.

Let E/F be a quadratic extension of local fields and let $(V_m, \langle\, , \rangle_{V_m})$ be an m-dimensional hermitian space and $(W_n, \langle\, , \rangle_{W_n})$ an n-dimensional skew-hermitian space over E. Define the symplectic space
$$\mathbb{W}_{V_m, W_n} := \mathrm{Res}_{E/F}\, V_m \otimes_E W_n$$
with the symplectic form
$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}_{V_m, W_n}} := \tfrac{1}{2} \mathrm{Tr}_{E/F}(\langle v, v' \rangle_{V_m} \langle w, w' \rangle_{W_n}).$$
We also consider the associated symplectic group $\mathrm{Sp}(\mathbb{W}_{V_m, W_n})$ which preserves $\langle\cdot, \cdot\rangle_{\mathbb{W}_{V_m, W_n}}$ and the metaplectic group $\widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m, W_n})$ which sits in a short exact sequence:
$$1 \to \mathbb{C}^\times \to \widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m, W_n}) \to \mathrm{Sp}(\mathbb{W}_{V_m, W_m}) \to 1.$$
Let $\mathbb{X}_{V_m, W_n}$ be a Lagrangian subspace of $\mathbb{W}_{V_m, W_n}$ and fix an additive character $\psi : F \to \mathbb{C}^\times$. Then we have a Schrödinger model of the Weil representation ω_ψ of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ on $\mathcal{S}(\mathbb{X}_{V_m, W_n})$, where $\mathcal{S}$ is the Schwartz–Bruhat function space.

If we set
$$\chi_{V_m} := \chi^m \quad \text{and} \quad \chi_{W_n} := \chi^n,$$

where χ is a character of $E^\times$ whose restriction to $F^\times$ is $\omega_{E/F}$, which is the quadratic character associated to E/F by local class field theory, then (χ_{V_m}, χ_{W_n}) gives a splitting homomorphism

$$\iota_{\chi_{V_m},\chi_{W_n}} : U(V_m) \times U(W_n) \to \widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m,W_n}),$$

and so by composing this with ω_ψ, we have a Weil representation $\omega_\psi \circ \iota_{\chi_{V_m},\chi_{W_n}}$ of $U(V_m) \times U(W_n)$ on $\mathbb{S}(\mathbb{X}_{V_m,W_n})$.

When the choice of ψ and (χ_{V_m}, χ_{W_n}) is fixed as above, we simply write

$$\omega_{\psi,W_n,V_m} := \omega_\psi \circ \iota_{\chi_{V_m},\chi_{W_n}}.$$

Throughout the rest of the paper, we shall denote the Weil representation of $U(V_m) \times U(W_n)$ by ω_{ψ,W_n,V_m} with the choice of characters (χ_{V_m}, χ_{W_n}) understood as above.

Remark 3.1. When $m = 1$, the image of $U(V_1)$ in $\widetilde{\mathrm{Sp}}(\mathbb{W}_{V_1,W_n})$ coincides with the image of the center of $U(W_n)$, and so we regard the Weil representation of $U(V_1) \times U(W_n)$ as a representation of $U(W_n)$. In this case, we denote the Weil representation of $U(W_n)$ as ω_{ψ,W_n}. Furthermore, we can also use $\chi_{V_1} = \chi^{-1}$ for the choice of splitting homomorphism $\iota_{\chi_{V_1},\chi_{W_n}}$ instead of $\chi_{V_1} = \chi$. In this case, the Weil representation of $U(W_n)$ is $\omega^\vee_{\psi,W_n}$.

Local theta correspondence. Given a Weil representation ω_{ψ,W_n,V_m} of $U(V_m) \times U(W_n)$ and an irreducible smooth representation π of $U(W_n)$, the maximal π-isotypic quotient of ω_{ψ,V_m,W_n} is of the form

$$\Theta_{\psi,V_m,W_n}(\pi) \boxtimes \pi$$

for some smooth representation $\Theta_{\psi,V_m,W_n}(\pi)$ of $U(V_m)$ of finite length. By Howe duality[1], the maximal semisimple quotient $\theta_{\psi,V_m,W_n}(\pi)$ of $\Theta_{\psi,V_m,W_n}(\pi)$ is either zero or irreducible.

In this paper, we consider two kinds of theta correspondences for $(U(1) \times U(3))$ and $(U(2) \times U(3))$:

***Case* 1.** First we shall consider the theta correspondence for $U(V_1^\epsilon) \times U(W_3^{\epsilon'})$. The following is a compound of Theorem 3.4 and Theorem 3.5 in [Haan 2016].

Theorem 3.2. *Let ϕ be an L-parameter of $U(V_1^\pm)$. Then*:

(i) *For any $\epsilon, \epsilon' = \pm 1$ and any $\pi \in \Pi_\phi^{\epsilon'}$, we have $\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi)$ is nonzero and irreducible.*

[1]It was first proved by Waldspurger [1990] for all residual characteristics except $p = 2$. Recently, Gan and Takeda [2014; 2016] have made it available for all residual characteristics.

(ii) $\Theta_{\psi,W_3^{\epsilon},V_1^{\epsilon'}}(\pi) =$

$$\begin{cases} \textit{a nontempered representation} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon', \\ \textit{a supercuspidal representation} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon', \end{cases}$$

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

(iii) *The L-parameter $\theta(\phi)$ of $\Theta_{\psi,W_3^{\epsilon},V_1^{\epsilon'}}(\pi)$ has the following form*:

(3-1) $\theta(\phi) =$

$$\begin{cases} \theta^n(\phi) = \chi|\cdot|_E^{\frac{1}{2}} \oplus \phi\cdot\chi^{-2} \oplus \chi|\cdot|_E^{-\frac{1}{2}} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon', \\ \theta^s(\phi) = \phi\cdot\chi^{-2} \oplus \chi\boxtimes \boldsymbol{S}_2 & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon', \end{cases}$$

where $\boldsymbol{S}_2$ is the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

(iv) *For ϵ, ϵ' such that $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi,W_3^{\epsilon},V_1^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi_{\theta^n(\phi)}.$$

(v) *For ϵ, ϵ' such that $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi,W_3^{\epsilon},V_1^{\epsilon'}}(\pi)$ gives an injection*

$$\Pi_\phi \hookrightarrow \Pi_{\theta^s(\phi)}.$$

Write

- $S_\phi = (\mathbb{Z}/2\mathbb{Z})a_1$,
- $S_{\theta^n(\phi)} = (\mathbb{Z}/2\mathbb{Z})a_1$ *if* $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon'$,
- $S_{\theta^s(\phi)} = (\mathbb{Z}/2\mathbb{Z})a_1 \times (\mathbb{Z}/2\mathbb{Z})a_2$ *if* $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon'$,

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

(*Note that $\theta^s(\phi)$ is a square-integrable L-parameter of $U(W_3^{\epsilon})$ and the summand $(\mathbb{Z}/2\mathbb{Z})a_2$ of $S_{\theta^s(\phi)}$ arises from the summand $\chi\boxtimes\boldsymbol{S}_2$ in $\theta^s(\phi)$.*)

Since we are only dealing with odd dimensional spaces, *there are three canonical bijections*:

- $J^\psi(\phi) : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi)$,
- $J^\psi(\theta^n(\phi)) : \Pi_{\theta^n(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta^n(\phi)})$,
- $J^\psi(\theta^s(\phi)) : \Pi_{\theta^s(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta^s(\phi)})$.

Using these maps, the following bijections and inclusions

$$\begin{gathered}\mathrm{Irr}(S_\phi) \longleftrightarrow \mathrm{Irr}(S_{\theta^n(\phi)})\\ \eta \mapsto \theta^n(\eta),\\ \mathrm{Irr}(S_\phi) \hookrightarrow \mathrm{Irr}(S_{\theta^s(\phi)})\\ \eta \mapsto \theta^s(\eta),\end{gathered}$$

induced by the theta correspondence can be explicated as follows:

$$\theta^n(\eta)(a_1) = \eta(a_1)\cdot\epsilon\big(\tfrac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\big), \tag{3-2}$$

$$\theta^s(\eta)(a_1) = \eta(a_1)\cdot\epsilon\big(\tfrac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\big), \qquad \theta_2(\eta)(a_2) = -1. \tag{3-3}$$

Remark 3.3. Note that $\theta^n(\phi)$ is a nongeneric L-parameter. Gan and Ichino [2016, Proposition B.1 in Appendix] proved that an L-parameter is generic if and only if its associated L-packet Π_ϕ contains a generic representation (i.e., one possessing a Whittaker model). Together with Corollary 4.2.3 in [Gelbart and Rogawski 1990], which asserts that all elements in $\Pi_{\theta^n(\phi)}$ have no Whittaker models, we see that $\theta^n(\phi)$ is a nongeneric L-parameter.

***Case* 2.** Now we shall consider the theta correspondence for $U(V_2^{\epsilon'}) \times U(W_3^\epsilon)$. The following summarizes some results of Gan and Ichino [2014; 2016], which are specialized to this case.

Theorem 3.4. *Let ϕ be an L-parameter for $U(V_2^\pm)$. Then*:

(i) *Suppose that ϕ does not contain χ^3.*

  (a) *For any $\pi \in \Pi_\phi^{\epsilon'}$ and any $\epsilon \in \{\pm 1\}$, we have $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is nonzero and $\theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi\otimes\chi^{-1}) \oplus \chi^2.$$

  (b) *For each $\epsilon = \pm 1$, the theta correspondence $\pi \mapsto \theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi^\epsilon_{\theta(\phi)}.$$

(ii) *Suppose that ϕ contains χ^3.*

  (a) *For any $\pi \in \Pi_\phi^{\epsilon'}$, exactly one of $\Theta_{\psi, W_3^+, V_2^{\epsilon'}}(\pi)$ or $\Theta_{\psi, W_3^-, V_2^{\epsilon'}}(\pi)$ is nonzero.*

  (b) *If $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is nonzero, then $\theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi\otimes\chi^{-1}) \oplus \chi^2.$$

  (c) *The theta correspondence $\pi \mapsto \theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi_{\theta(\phi)}.$$

(iii) *We have fixed a bijection*

$$J_{\psi^E}(\phi) : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi),$$

where

$$\psi^E(x) = \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big)$$

and there is the bijection

$$J^{\psi}(\theta(\phi)) : \Pi_{\theta(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}).$$

- *If ϕ does not contain χ^3, we have*

  $$S_{\theta(\phi)} = S_\phi \times (\mathbb{Z}/2\mathbb{Z})b_1,$$

  where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ of $S_{\theta(\phi)}$ arises from the summand χ^2 in $\theta(\phi)$. Then for each ϵ, using the above bijections J and J_{ψ^E}, one has a canonical bijection

  $$\begin{aligned} \mathrm{Irr}(S_\phi) &\longleftrightarrow \mathrm{Irr}^{\epsilon}(S_{\theta(\phi)}), \\ \eta &\longleftrightarrow \theta(\eta), \end{aligned}$$

  induced by the theta correspondence, where $\mathrm{Irr}^{\epsilon}(S_{\theta(\phi)})$ is the set of irreducible characters η' of $S_{\theta(\phi)}$ such that $\eta'(z_{\theta(\phi)}) = \epsilon$ and the bijection is determined by

  $$\theta(\eta)|_{S_\phi} = \eta.$$

- *If ϕ contains χ^3, then $\phi \otimes \chi^{-1}$ contains χ^2, and so*

  $$S_{\theta(\phi)} = S_\phi.$$

  Thus, one has a canonical bijection

  $$\begin{aligned} \mathrm{Irr}(S_\phi) &\longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}), \\ \eta &\longleftrightarrow \theta(\eta), \end{aligned}$$

  induced by the theta correspondence and it is given by

  $$\theta(\eta) = \eta.$$

(iv) *If π is tempered and $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi)$ is nonzero, then $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi)$ is irreducible and tempered.*

4. Main Theorem

In this section, we prove our Main Theorem. To prove it, we first state the precise result of Beuzart-Plessis which we shall use in the proof of Theorem 4.1[2].

Theorem (B)$_n$. *Let $\phi = \phi^{(n+1)} \times \phi^{(n)}$ be a tempered L-parameter of $U(V_{n+1}^{\pm}) \times U(V_n^{\pm})$ and write $S_{\phi^{(n+1)}} = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i$ and $S_{\phi^{(n)}} = \prod_j (\mathbb{Z}/2\mathbb{Z})b_j$. Let*

$$\Delta : U(V_n^{\pm}) \hookrightarrow U(V_{n+1}^{\pm}) \times U(V_n^{\pm})$$

be the diagonal map. Then for $\pi(\eta) \in \Pi_{\phi}^{R,\,\pm} = \Pi_{\phi^{(n+1)}}^{\pm} \times \Pi_{\phi^{(n)}}^{\pm}$, where $\eta \in \mathrm{Irr}(S_\phi) = \mathrm{Irr}(S_{\phi^{(n+1)}}) \times \mathrm{Irr}(S_{\phi^{(n)}})$,

$$\mathrm{Hom}_{\Delta(U(V_n^{\pm}))}(\pi(\eta), \mathbb{C}) = 1 \Leftrightarrow \eta = \eta^{\ddagger},$$

where

$$\begin{cases} \eta^{\ddagger}(a_i) = \epsilon\big(\tfrac{1}{2}, \phi_i^{(n+1)} \otimes \phi^{(n)}, \psi_{-2}^{E}\big), \\ \eta^{\ddagger}(b_j) = \epsilon\big(\tfrac{1}{2}, \phi^{(n+1)} \otimes \phi_j^{(n)}, \psi_{-2}^{E}\big), \end{cases}$$

where $\psi_{-2}^{E}(x) = \psi(-\mathrm{Tr}_{E/F}(\delta x))$.

Theorem 4.1. *Let $\phi^{(1)}, \phi^{(2)}$ be tempered L-parameters of $U(V_1^{\pm})$ and $U(V_2^{\pm})$, respectively and suppose that $\phi^{(2)}$ does not contain χ^{-3}. Let*

$$\theta^n(\phi^{(1)}) = \chi|\cdot|_E^{\frac{1}{2}} \oplus \phi^{(1)} \cdot \chi^{-2} \oplus \chi|\cdot|_E^{-\frac{1}{2}},$$
$$\theta^s(\phi^{(1)}) = \phi^{(1)} \cdot \chi^{-2} \oplus \chi \boxtimes \mathbf{S}_2$$

be the two L-parameters of $U(W_3^{\pm})$ appearing in (3-1) and let

$$\theta(\phi^{(2)}) = \phi^{(2)} \otimes \chi \oplus \chi^{-2}$$

be the L-parameters of $U(W_3^{\pm})$ appearing in Theorem 3.4 (ii), in which χ is replaced by χ^{-1}.

Write

- $S_{\phi^{(1)}} = S_{\theta^n(\phi^{(1)})} = (\mathbb{Z}/2\mathbb{Z})a_1$,
- $S_{\theta^s(\phi^{(1)})} = S_{\phi^{(1)}} \times (\mathbb{Z}/2\mathbb{Z})a_2$,
- $S_{\phi^{(2)}} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})b_1 & \text{if } \phi^{(2)} \text{ is irreducible,} \\ (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})b_2 & \text{if } \phi^{(2)} = \phi_1^{(2)} \oplus \phi_2^{(2)} \text{ is reducible,} \end{cases}$
- $S_{\theta(\phi^{(2)})} = S_{\phi^{(2)}} \times (\mathbb{Z}/2\mathbb{Z})c_1$,

where c_1 comes from the component χ^{-2} of $\theta(\phi^{(2)})$. We use the fixed character ψ to fix the local Langlands correspondence for $\Pi_{\phi^{(2)}} \leftrightarrow \mathrm{Irr}(S_{\phi^{(2)}})$.

[2]Recently, Gan and Ichino [2016] extended Beuzart-Plessis's work to the generic case relating it to the (FJ) case.

For $x = n, s$, let

$$\theta^x(\phi^{(1)}, \phi^{(2)}) = \theta^x(\phi^{(1)}) \times \theta(\phi^{(2)})$$

be an L-parameter of $G_3^{\pm} = U(W_3^{\pm}) \times U(W_3^{\pm})$ and $\pi^x(\eta) \in \Pi^{R,\,\epsilon}_{\theta^x(\phi^{(1)},\phi^{(2)})}$ be a representation of a relevant pure inner form of G_3. Then,

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^x(\eta), \omega_{\psi, W_3^{\epsilon}}) \neq 0 \Longleftrightarrow \eta = \eta_x^{\dagger},$$

where $\eta_x^{\dagger} \in \mathrm{Irr}(S_{\theta^x(\phi^{(1)},\phi^{(2)})}) = \mathrm{Irr}(S_{\theta^x(\phi^{(1)})}) \times \mathrm{Irr}(S_{\theta(\phi^{(2)})})$ is specified as follows:

(i) When $\phi^{(2)}$ is irreducible,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(c_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big), \end{cases}$$

$$\begin{cases} \eta_s^{\dagger}(a_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(a_2) = -1, \\ \eta_s^{\dagger}(b_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(c_1) = -\epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

(ii) When $\phi^{(2)} = \phi_1^{(2)} \oplus \phi_2^{(2)}$ is reducible,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi_1^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_2) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi_2^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(c_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big), \end{cases}$$

$$\begin{cases} \eta_s^{\dagger}(a_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(a_2) = -1, \\ \eta_s^{\dagger}(b_1) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi_1^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(b_2) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi_2^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(c_1) = -\epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

Furthermore,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^x(\eta_x^{\dagger}), \omega_{\psi, W_3^{\epsilon}}) = 1.$$

Remark 4.2. When $x = n$ or s, we have $\eta_x^{\dagger}(z_{\theta^x(\phi^{(1)})}) = \eta_x^{\dagger}(z_{\theta(\phi^{(2)})})$ so that $\eta_x^{\dagger}$ always corresponds to a representation $\pi^x(\eta_x^{\dagger})$ of a relevant pure inner form of G_3.

Proof. For each $x = n, s$, we first prove the existence of some $\epsilon_x \in \{\pm 1\}$ and $\pi^x(\eta) \in \Pi^{R,\,\epsilon_x}_{\theta^x(\phi^{(1)},\phi^{(2)})}$ such that

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon_x})}(\pi^x(\eta), \omega_{\psi, W_3^{\epsilon_x}}) \neq 0.$$

For $a \in F^\times$, let L_a be the 1-dimensional hermitian space with form $a \cdot \mathrm{N}_{E/F}$. Then

$$V_2^+/V_1^+ \simeq V_2^-/V_1^- \simeq L_{-1}.$$

We consider the following see-saw diagram (ϵ, ϵ' will be determined soon):

$$\begin{tikzcd} U(W_3^\epsilon) \times U(W_3^\epsilon) \arrow[d, dash] \arrow[drr, dash] & & U(V_2^{\epsilon'}) \arrow[d, dash] \arrow[dll, dash] \\ U(W_3^\epsilon) & & U(V_1^{\epsilon'}) \times U(L_{-1}) \end{tikzcd} \tag{4-1}$$

In this diagram, we shall use three theta correspondences:

(i) $U(V_2^{\epsilon'}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ^2, χ^3),

(ii) $U(V_1^{\epsilon'}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ, χ^3),

(iii) $U(L_{-1}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ^{-1}, χ^3).

By $(\mathrm{B})_1$, there is a unique $\epsilon' \in \{\pm 1\}$ and a unique pair of component characters

$$(\eta_2, \eta_1) \in \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(2)})^\vee}) \times \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(1)})^\vee}),$$

such that

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2) \otimes \pi(\eta_1), \mathbb{C}) \neq 0.$$

Moreover, $\epsilon' = \eta_1(a_1) = \epsilon\big(\frac{1}{2}, (\phi^{(1)})^\vee \otimes (\phi^{(2)})^\vee, \psi_{-2}^E\big) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big)$.

By Theorem 3.4 (i), (iv) and [Atobe and Gan 2016, Theorem 4.1],

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is nonzero for any $\epsilon \in \{\pm 1\}$. Since

$$\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2)) \boxtimes \Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is the maximal $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))$-isotypic quotient of ω_{ψ, V_2, W_3} and we have that $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2)) \boxtimes \pi(\eta_2)$ is a $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))$-isotypic quotient of ω_{ψ, V_2, W_3}, the representation $\pi(\eta_2)$ should be a quotient of $\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$. By Proposition 5.4 in [Atobe and Gan 2016],

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is irreducible and thus we have $\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big) = \pi(\eta_2)$.

Since

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2), \pi^\vee(\eta_1)) \neq 0,$$

by the see-saw identity and Remark 3.1, we have

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}\big(\Theta_{\psi, W_3^\epsilon, V_1^{\epsilon'}}(\pi^\vee(\eta_1)) \otimes \omega_{\psi, W_3^\epsilon}^\vee, \Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big) \neq 0,$$

and since $\Theta_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2))$ and $\omega^\vee_{\psi,W_3^\epsilon}$ are admissible, we have

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}\big(\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^\vee(\eta_1))\otimes\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2)),\omega_{\psi,W_3^\epsilon}\big)\neq 0.$$

By Theorem 3.2 (i) and Theorem 3.4 (i), the L-parameter of $\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^\vee(\eta_1))\otimes\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2))$ is

$$\begin{cases}\theta^n(\phi^{(1)},\phi^{(2)}) & \text{if } \epsilon=\epsilon\big(\frac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big)\cdot\epsilon\big(\frac12,\phi^{(1)}\otimes\phi^{(2)},\psi_2^E\big),\\ \theta^s(\phi^{(1)},\phi^{(2)}) & \text{if } \epsilon=-\epsilon\big(\frac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big)\cdot\epsilon\big(\frac12,\phi^{(1)}\otimes\phi^{(2)},\psi_2^E\big),\end{cases}$$

and by Theorem 3.2 (v) and Theorem 3.4 (iii), we see that their associated component characters are $\eta_n^\dagger$ and $\eta_s^\dagger$ in each case.

Next we shall prove that these are the unique representations which yield *Fourier–Jacobi* models in each of the L-packets, $\Pi_{\theta^n(\phi^{(1)},\phi^{(2)})}$ and $\Pi_{\theta^s(\phi^{(1)},\phi^{(2)})}$.

Since $\theta^s(\phi^{(1)},\phi^{(2)})$ is a tempered L-parameter, the uniqueness easily follows from $(\mathrm{FJ})_3$ in this case. Therefore, we shall only consider the nontempered L-parameter $\theta^n(\phi^{(1)},\phi^{(2)})$. Let $\pi_2\otimes\pi_1\in\Pi^{R,\,\epsilon}_{\theta^n(\phi^{(1)},\phi^{(2)})}=\Pi^\epsilon_{\theta^n(\phi^{(1)})}\times\Pi^\epsilon_{\theta(\phi^{(2)})}$ be a representation satisfying

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\pi_1,\omega_{\psi,W_3^\epsilon})\neq 0,$$

and in turn

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\omega^\vee_{\psi,W_3^\epsilon},\pi_1^\vee)\neq 0.$$

(The existence of such $\pi_2\otimes\pi_1$ was insured by the previous step.) Then by Theorem 3.2 (iv), we can write $\pi_2=\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^{(1)})$ for some $\pi^{(1)}\in\Pi^{\epsilon'}_{\theta^{(1)}}$ where

$$\epsilon'=\epsilon\cdot\epsilon\big(\tfrac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big).$$

Then by applying the see-saw duality in the see-saw diagram in (4-1), one has

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\omega^\vee_{\psi,W_3^\epsilon},\pi_1^\vee)\simeq\mathrm{Hom}_{U(V_1^{\epsilon'})}(\pi^{(2)},\pi^{(1)})\neq 0,$$

where

$$\pi^{(2)}=\Theta_{\psi,V_2^{\epsilon'},W_3^\epsilon}(\pi_1^\vee).$$

Note that $\pi^{(2)}\neq 0$ and so it has the tempered L-parameter $(\phi^{(2)})^\vee$. Then by $(\mathrm{B})_1$, $(\pi^{(2)},\pi^{(1)})$ is the unique pair in the L-packet $\Pi_{(\phi^{(2)})^\vee}\times\Pi_{\phi^{(1)}}$ such that

$$\mathrm{Hom}_{U(V_1^{\epsilon'})}(\pi^{(2)},\pi^{(1)})\neq 0,$$

and so (π_2,π_1) should be $(\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^{(1)}),\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi^{(2)}))$. This settles the uniqueness issue. □

Remark 4.3. When the L-parameter $\phi^{(2)}$ of $U(V_2^{\pm})$ contains χ^{-3}, we can write $\phi^{(2)} = \phi_0 \oplus \chi^{-3}$ for an L-parameter ϕ_0 of $U(V_1^{\pm})$. Then, we set

$$\theta(\phi^{(2)}) = \begin{cases} 3 \cdot \chi^{-2} & \text{if } \phi_0 = \chi^{-3}, \\ \phi_0 \cdot \chi \oplus 2 \cdot \chi^{-2} & \text{if } \phi_0 \neq \chi^{-3}, \end{cases}$$

and

$$S_{\theta(\phi^{(2)})} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})b_1 & \text{if } \phi_0 = \chi^{-3}, \\ (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})c_1 & \text{if } \phi_0 \neq \chi^{-3}. \end{cases}$$

If one develops a similar argument in this case, one could also have a recipe as in Theorem 4.4 for the nontempered case. However, we need some assumption on the irreducibility of the theta lifts because we cannot apply Proposition 5.4 in [Atobe and Gan 2016].

Theorem 4.4. *Let the notations be as in* Theorem 4.1 *and assume this time that* $\phi^{(2)}$ *contains* χ^{-3}. *Assume that for* $\pi \in \Pi^{\epsilon}_{\theta((\phi^{(2)})^{\vee})}$, *if* $\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}(\pi)$ *is nonzero, it is irreducible.*

Then,

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^n(\eta), \omega_{\psi, W_3^{\epsilon}}) \neq 0 \Longleftrightarrow \eta = \eta_n^{\dagger},$$

where $\eta_n^{\dagger} \in \mathrm{Irr}(S_{\theta^n(\phi^{(1)}, \phi^{(2)})}) = \mathrm{Irr}(S_{\theta^n(\phi^{(1)})}) \times \mathrm{Irr}(S_{\theta(\phi^{(2)})})$ *is specified as follows:*

- *When* $\phi_0 = \chi^{-3}$,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

- *When* $\phi_0 \neq \chi^{-3}$,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big), \\ \eta_n^{\dagger}(c_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

Proof. We first prove the existence of some $\epsilon \in \{\pm 1\}$ and $\pi^n(\eta) \in \Pi^{R,\epsilon}_{\theta^n(\phi^{(1)}, \phi^{(2)})}$ such that

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^n(\eta), \omega_{\psi, W_3^{\epsilon}}) \neq 0.$$

By $(\mathrm{B})_1$, there is a unique $\epsilon' \in \{\pm 1\}$ and a unique pair of component characters

$$(\eta_2, \eta_1) \in \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(2)})^{\vee}}) \times \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(1)})^{\vee}}),$$

such that

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2) \otimes \pi(\eta_1), \mathbb{C}) \neq 0.$$

Moreover, $\eta_2(b_1) = \epsilon((\phi^{(1)})^{\vee} \otimes (\phi_0)^{\vee}, \psi^E_{-2}) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big)$ and

$$\epsilon' = \eta_1(a_1) = \epsilon\big(\tfrac{1}{2}, (\phi^{(1)})^{\vee} \otimes (\phi^{(2)})^{\vee}, \psi^E_{-2}\big) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big).$$

By Theorem 3.4 (ii) and (iv),

$$\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$$

is a nonzero irreducible representation of $U(W_3^{\epsilon})$ for some $\epsilon \in \{\pm 1\}$ and by Theorem 4.1 in [Atobe and Gan 2016], $\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}\big(\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big)$ is nonzero. So by our assumption, it is irreducible and

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}\big(\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big) = \pi(\eta_2).$$

Since

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2), \pi^{\vee}(\eta_1)) \neq 0,$$

by the see-saw identity and Remark 3.1, we have

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}\big(\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \omega^{\vee}_{\psi, W_3^{\epsilon}}, \Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big) \neq 0,$$

and since $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$ and $\omega^{\vee}_{\psi, W_3^{\epsilon}}$ are admissible, we have

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}\big(\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2)), \omega_{\psi, W_3^{\epsilon}}\big) \neq 0.$$

Let $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2)) = \pi(\eta_3)$ for some $\eta_3 \in \prod_{\theta((\phi^{(2)})^{\vee})}$. If $\phi_0 = \chi^{-3}$, then $\eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(3 \cdot b_1) = \eta_3(b_1)$ and if $\phi_0 \neq \chi^{-3}$, then $\eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(b_1)$. Thus in both cases, we have

$$\epsilon = \eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(b_1) = \eta_2(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big).$$

Since

$$\epsilon' = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big),$$

we have $\epsilon \cdot \epsilon' = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big)$ and so by Theorem 3.2 (iii) and Theorem 3.4 (i), the L-parameter of $\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$ is $\theta^n(\phi^{(1)}, \phi^{(2)})$.

Furthermore, by applying Theorem 3.2 (v) and Theorem 3.4 (iii), we see that the associated component character of

$$\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$$

is exactly $\eta_n^{\dagger}$ in each case and it proves the existence part. The proof of the uniqueness part is essentially the same as the one in Theorem 4.1. □

Remark 4.5. It is remarkable that for the supercuspidal L-parameter $\theta^s(\phi^{(1)}, \phi^{(2)})$ with $\theta(\phi^{(2)})$ as above, the recipe, which is suggested in $(\mathrm{FJ})_3$, does not occur with the theta lift from $U(V_1^{\pm})$ and $U(V_2^{\pm})$. This is quite similar to Proposition 4.6 in [Haan 2016], which concerns the nongeneric aspect of *Bessel* cases of the GGP conjecture.

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