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We give a concrete description of the two-fold cover of a simply connected, split real reductive group and its maximal compact subgroup as Chevalley groups. We study the representations of the maximal compact subgroups called pseudospherical representations, which appear with multiplicity one in the principal series representation. We introduce a family of canonically defined intertwining operators and compute their action on pseudospherical *K*-types, obtaining explicit formulas of the Harish-Chandra *c*-function.

1. Introduction

Assume that $\underline{G}$ is the split real form of a simply connected complex algebraic group. It turns out that $\underline{G}$ admits a unique nontrivial two-fold cover (or double cover) G, which is the nonlinear group we wish to study. Such coverings are well-studied. There are several general results about coverings of algebraic groups in [Steinberg 1968]. We are interested in *pseudospherical principal series representations*, that is, principal series representations that contain a pseudospherical K-type. These representations are defined for G and are related to a conjectural Shimura correspondence for split real groups; see [Adams et al. 2007]. Pseudospherical representation can refer to three definitions: Let $G = PK$ be an Iwasawa decomposition with $P = MAN$ a minimal parabolic subgroup. We have pseudospherical representations of M, pseudospherical representations of K and pseudospherical representations of G; see the definition at the beginning of Section 3.

The intertwining operators between two principal series representations, when considered as integral operators, reveal many properties of the principal series representations, such as reducibility points. The intertwining operators play an important role in the general Plancherel formula for semisimple Lie groups developed by Harish-Chandra. They are also related to the theory of Eisenstein series. A nice discussion of the formalism can be found in [Schiffmann 1971]. In this paper, we normalize the intertwining operators between two pseudospherical principal

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series, in a way that it is independent of the choice of representative in $N_K(A)$ of $w \in W = N_K(A)/Z_K(A)$, and obtain a canonical definition. We are interested in the action of intertwining operators on pseudospherical K-types. We compute explicitly the *Harish-Chandra c-function* associated to this action, which is our main result (Theorem 6.5). There is an analogous result in the p-adic case obtained by H. Y. Loke and G. Savin [2010].

The structure of this paper is arranged as follows: In Section 2, we recall some basic facts on Chevalley groups and their covering groups. We define the maximal compact subgroup K of the covering group G using Steinberg symbols. We calculate the structure of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the nontrivial two-fold cover of $\mathrm{SL}(2, \mathbb{R})$, making a comparison between Kubota cocycles and Steinberg symbols and writing down the exponential map from the Lie algebra to the cover. In Section 3, we define the pseudospherical representation following [Adams et al. 2007] and list some properties regarding the action of W on it. In Section 4, we define a family of canonical intertwining operators among pseudospherical principal series. In Section 5 we compute the intertwining operators of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, which are important for the general groups. Finally, we calculate the action of intertwining operators on pseudospherical K-types and obtain our main result in Section 6.

2. Chevalley groups and their covering groups

In Section 2A, we recall the well-known construction of the Chevalley groups. In Section 2B, we state a number of results for the covering group of a Chevalley group that we will need in later sections. In particular, we give the generators and relations of the double cover in terms of the Hilbert symbol. We define the minimal parabolic subgroup $P = NAM$ and the maximal compact subgroup K in terms of the Steinberg symbol; see Proposition 2.6. In Section 2C, we specialize our discussion in Section 2B to the case $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and make a comparison between the definition based on Kubota symbols and the definition based on Steinberg symbols; see Proposition 2.8. We also compute explicitly an exponential map from the Lie algebra to the cover; see Proposition 2.11.

2A. ***Construction of a Chevalley group.*** In this section, we recall the construction of Chevalley groups following [Steinberg 1968]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Φ the corresponding root system. We use $\alpha, \beta, \gamma, \ldots$ to denote the roots. Let B be the Killing form on $\mathfrak{g}$. Since it is nondegenerate, there exists $H'_\alpha \in \mathfrak{h}$ such that $B(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. Define $(\alpha, \beta) = B(H'_\alpha, H'_\beta)$ for all $\alpha, \beta \in \Phi$. The Cartan integer $\langle \alpha, \beta \rangle$ is defined to be $2(\alpha, \beta)/(\beta, \beta)$. The root system Φ is invariant under all reflections w_α $(\alpha \in \Phi)$, where w_α is the reflection across the hyperplane orthogonal to α. These reflections generate the Weyl group W.

For each α, define $H_\alpha = 2H'_\alpha/(\alpha,\alpha)$ and $H_i = H_{\alpha_i}$, where $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots. By [Steinberg 1968], one can choose $X_\alpha \in \mathfrak{g}_\alpha$ such that

- $[H_\beta, X_\alpha] = \langle \alpha, \beta \rangle X_\alpha$,
- $[X_\alpha, X_{-\alpha}] = H_\alpha$ is an integer linear combination of the H_i, and
- $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$, where $N_{\alpha\beta}$ is an integer which is 0 if $\alpha+\beta$ is not a root.

The collection of H_i and X_α is called a *Chevalley basis* of the complex semisimple Lie algebra $\mathfrak{g}$. It is important that the integer span, $\mathfrak{g}_\mathbb{Z}$, of the basis elements is stable under the Lie bracket.

Let L_0 be the root lattice, i.e, the integer span of all roots in Φ, and let L_1 be the weight lattice, which is the set of all $\mu \in \mathfrak{h}^*$ such that $\mu(H_\alpha) \in \mathbb{Z}$ for all roots α. Assume $(\mathfrak{g}, V)$ is a complex finite-dimensional representation of $\mathfrak{g}$. One can show that its weight lattice L_V is contained between L_0 and L_1. To construct the Chevalley group based on the representation $(\mathfrak{g}, V)$, choose a full-rank lattice M in V which is invariant under the set

$$\{X_\alpha^n/n! : n \in \mathbb{Z}_{\geq 0},\ \alpha \in \Phi\},$$

where we are thinking of $X_\alpha^n/n!$ as a member of $\mathrm{End}(V)$. One can show (see [Steinberg 1968]) that such a lattice exists. For any field k, set V^k to be the vector space $M \otimes_\mathbb{Z} k$ on which $X_\alpha^n/n!$ acts in a natural way. Since the representation V has a finite number of weights, there is some n for each α such that $X_\alpha^n \in \mathrm{End}(V^k)$ is zero. Therefore, for $t \in k$ and $\alpha \in \Phi$,

$$x_\alpha(t) = \exp(tX_\alpha) = 1 + tX_\alpha + \frac{(tX_\alpha)^2}{2!} + \frac{(tX_\alpha)^3}{3!} + \cdots \in \mathrm{GL}(V^k)$$

is a finite sum and hence is well-defined.

Define the *Chevalley group* to be the subgroup $G(k)$ of $\mathrm{GL}(V^k)$ generated by $x_\alpha(t)$, with $t \in k$, $\alpha \in \Phi$. We say G is *simply connected* if $L_V = L_1$. Note that this definition is different from simply-connectedness in the topological sense. We assume all Chevalley groups are simply connected for the rest of this paper.

Define

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \quad \text{and} \quad h_\alpha(t) = w_\alpha(t)w_\alpha(-1) \quad \text{for } t \in k^\times.$$

Let T (the Cartan subgroup, or maximal torus) be the subgroup of G generated by $h_\alpha(t)$, with $t \in k^\times$, $\alpha \in \Phi$. By [Steinberg 1968, Lemma 28], $h_\alpha(t)$ is multiplicative as a function of t, and simply-connectedness implies that any element of T can be written uniquely as $h_1(t_1)h_2(t_2)\cdots h_l(t_l)$ for some $t_1, \dots, t_l \in k^\times$, where $h_i(t_i) = h_{\alpha_i}(t_i)$.

Now let us describe the generators and relations of a simply connected Chevalley group G over k:

(A) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$,

(B) $(x_\alpha(t), x_\beta(u)) = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(c_{ij}t^i u^j)$,

(B′) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$,

(C) $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$.

Here the c_{ij} are integers depending on α, β and the chosen ordering, but not on t or u. By [Steinberg 1968, Theorem 8], if Φ is not of type A_1, then (A), (B), (C) form a complete set of relations for G constructed from Φ and k; if Φ is of type A_1, then (A), (B′), (C) form a complete set of relations. By [Steinberg 1968, Lemma 37], (B′) is also true when Φ is not of type A_1, and it implies that

$$w_\alpha(t) = w_{-\alpha}(-t^{-1}), \quad w_\alpha(1)h_\alpha(t)w_\alpha(-1) = h_\alpha(t^{-1}),$$

which we will use later.

2B. ***Covering groups.*** To study the covering group of a simply connected Chevalley group, we need some preparations. First, a central extension of a group G is a couple (π, G'), where G' is a group, and π is a homomorphism of G' onto G such that $\operatorname{Ker}\pi$ is a subset of the center of G'. A central extension (π, E) of a group G is *universal* if for any central extension (π', E') of G there exists a unique homomorphism $\phi: E \to E'$ such that $\pi' \circ \phi = \pi$. It is easy to see that if a universal central extension exists, it is unique up to isomorphism.

Theorem 2.1 [Steinberg 1968, Theorem 10]. *Let Φ be an irreducible root system and k a field such that $|k| > 4$ and if* $\operatorname{rank}\Phi = 1$, *then $|k| > 9$. Let G be the corresponding simply connected Chevalley group abstractly defined by the relations* (A), (B), (B′), (C), *let E be the group defined by the relations* (A), (B), (B′) (*we use* (B′) *only if* $\operatorname{rank}\Phi = 1$), *and let π be the natural homomorphism from E to G. Then (π, E) is a universal central extension of G.*

From now on, we use $x_\alpha(t)$, $w_\alpha(t)$, $h_\alpha(t)$ to denote the elements in the central extension of G, and $\underline{x}_\alpha(t)$, $\underline{w}_\alpha(t)$, $\underline{h}_\alpha(t)$ to denote the elements in G.

The next theorem gives a complete description of $C = \operatorname{Ker}\pi$:

Theorem 2.2 [Steinberg 1968, Theorem 12]. *Keep the assumptions in the previous theorem. $C = \operatorname{Ker}\pi$ is isomorphic to the abstract group A generated by the symbols $f(t, u)$ $(t, u \in k^*)$ subject to the relations*

(a) $f(t,u)f(tu,v) = f(t,uv)f(u,v), \quad f(1,u) = f(u,1) = 1$,

(b) $f(t,u)f(t,-u^{-1}) = f(t,-1)$,

(c) $f(t,u)=f(u^{-1},t)$,

(d) $f(t,u)=f(t,-tu)$,

(e) $f(t,u)=f(t,(1-t)u)$.

In the case when Φ *is not of type* C_n $(n\geq 1)$ *the relations above may be replaced by*

(ab′) $f(t,u)f(t',u)=f(tt',u)$, $f(t,u)f(t,u')=f(t,uu')$,

(c′) $f(t,u)=f(u,t)^{-1}$,

(d′) $f(t,-t)=1$,

(e′) $f(t,1-t)=1$.

The isomorphism is given by

$$\phi: f(t,u)\mapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1},$$

where α *is a fixed long root. One can write*

$$h_\alpha(t)h_\alpha(u)=f(t,u)h_\alpha(tu)$$

if we identify $C=\operatorname{Ker}\pi$ *with* A *via* ϕ.

Remark. Because all long roots are conjugate by W, the isomorphism ϕ does not depend on the choice of a long root α.

Remark. These relations are satisfied by the norm residue symbol in class field theory.

For the application to real groups, we specialize our result to the case when $k=\mathbb{R}$, and consider the double cover. First, recall the real Hilbert quadratic symbol $(\,,\,)_{\mathbb{R}}$. It is a map from $\mathbb{R}^*\times\mathbb{R}^*$ to $\mu_2=\{\pm 1\}$. For $t,u\in\mathbb{R}^*$, $(t,u)=1$ if and only if $x^2-ty^2-uz^2$ has a nontrivial solution $(x,y,z)\in\mathbb{R}^3$. It is easy to see that $(t,u)=1$ unless both of t and u are negative. Assume G' is a *double cover* of G, more precisely, a central extension (p,G') of G such that $\operatorname{Ker} p$ is of order 2 and such that it *does not split*, i.e, there is no homomorphism $i:G\to G'$ such that $p\circ i=\mathrm{id}_G$. Since (π,E) is the universal central extension of G, there exists a homomorphism $q:E\to G'$ such that $p\circ q=\pi$. Any such q maps C onto $\operatorname{Ker} p$, that is, $\operatorname{Ker} p$ is a quotient of $C\cong A$. Passing to quotient, we use $\bar f(t,u)\in\mu_2$ to denote the image of $f(t,u)\in A$. Since the $\bar f(t,u)$ satisfy (a), (b), (c), (d), (e), G' is unique up to isomorphism. On the other hand, the Hilbert symbol (t,u) satisfies the relations that $\bar f(t,u)$ satisfies, hence $\bar f(t,u)=(t,u)$. Thus we have:

Corollary 2.3. *Assume* G *is a simply connected Chevalley group over* $\mathbb{R}$. *Then there exists a unique* (*up to isomorphism*) *double cover* (p,G') *of* G. *Moreover, an isomorphism* $\phi:\mu_2\to\operatorname{Ker} p$ *is given by*

$$(t,u)\mapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1},$$

where α is a fixed long root and (t,u) is the real Hilbert quadratic symbol. One can write

$$h_\alpha(t)h_\alpha(u)=(t,u)h_\alpha(tu)$$

by identifying Ker p *and* μ_2 *via* ϕ. *Combining with* (A), (B), (B′), *we get a complete set of relations for* G'.

In the universal cover E, let T be the subgroup generated by $h_\alpha(t)$, $\alpha\in\Phi$, $t\in k^\times$. It is called the *metaplectic torus* of E. We also refer to the image of T in any cover of G as the metaplectic torus. The proposition below lists some relations in T.

Proposition 2.4. *Keep the assumptions in Theorems 2.1 and 2.2. Assume furthermore that Φ is not of type C_n. Then*

$$\begin{aligned}
h_\alpha(t)h_\alpha(u)&=f(t,u)h_\alpha(tu) &&\text{if } \alpha \text{ is long};\\
h_\alpha(t)h_\alpha(u)&=f(t,u)^{n_\Phi}h_\alpha(tu) &&\text{if } \alpha \text{ is short};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\alpha,\beta\rangle} &&\text{if } \alpha,\beta \text{ are long};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\alpha,\beta\rangle} &&\text{if } \alpha \text{ is long}, \beta \text{ is short};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\beta,\alpha\rangle} &&\text{if } \alpha \text{ is short}, \beta \text{ is long};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{n_\Phi\cdot\langle\alpha,\beta\rangle} &&\text{if } \alpha,\beta \text{ are short.}
\end{aligned}$$

Here $n_\Phi=\max_{\alpha,\beta\in\Phi}(\alpha,\alpha)/(\beta,\beta)$ *and we identify* $f(t,u)$ *with its image in* C *via* ϕ.

Proof. By [Steinberg 1968, Lemma 37],

$$(h_\alpha(t),h_\beta(u))=h_\beta(t^{\langle\beta,\alpha\rangle}u)h_\beta(t^{\langle\beta,\alpha\rangle})^{-1}h_\beta(u)^{-1}$$

for any α,β. If β is long, the right-hand side is $f(u,t^{\langle\beta,\alpha\rangle})^{-1}$, which is equal to $f(t,u)^{\langle\beta,\alpha\rangle}$ since Φ is not of type C_n. Taking the inverse on both sides, we get $(h_\beta(u),h_\alpha(t))=f(u,t)^{\langle\beta,\alpha\rangle}$. Now assume β is short, α is long. Then

$$\begin{aligned}
h_\beta(u)h_\beta(t^{\langle\beta,\alpha\rangle})h_\beta(t^{\langle\beta,\alpha\rangle}u)^{-1}&=(h_\beta(u),h_\alpha(t))=f(u,t)^{\langle\alpha,\beta\rangle}\\
&=f(u,t^{\langle\alpha,\beta\rangle})=f(u,t^{\langle\beta,\alpha\rangle})^{\frac{(\alpha,\alpha)}{(\beta,\beta)}}=f(u,t^{\langle\beta,\alpha\rangle})^{n_\Phi}.
\end{aligned}$$

Because $\langle\beta,\alpha\rangle=\pm1$, $t^{\langle\beta,\alpha\rangle}$ runs through all the elements in $k^\times$. Finally, if both of α,β are short,

$$\begin{aligned}
(h_\alpha(t),h_\beta(u))&=(h_\beta(u)h_\beta(t^{\langle\beta,\alpha\rangle})h_\beta(t^{\langle\beta,\alpha\rangle}u)^{-1})^{-1}\\
&=f(u,t^{\langle\beta,\alpha\rangle})^{-n_\Phi}=f(t,u)^{n_\Phi\cdot\langle\beta,\alpha\rangle}.
\end{aligned}$$

□

Remark. Assume G is a real group and G' is its double cover. The relations above are still true if we replace $f(t,u)$ by (t,u). Because the Hilbert symbol (t,u) is bimultiplicative, by the proof of Proposition 2.4, one can remove the assumption that Φ is not of type C_n.

Proposition 2.5 [Steinberg 1968, Lemma 37]. *Let $c = c(\alpha, \beta) = \pm 1$ be independent of t and u. Then*

$$h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{\langle\beta,\alpha\rangle} u),$$
$$w_\alpha(1) h_\beta(t) w_\alpha(-1) = h_{w_\alpha\beta}(ct) h_{w_\alpha\beta}(c)^{-1}.$$

The next proposition gives a description of maximal compact subgroups in the setting of Chevalley groups:

Proposition 2.6. *Assume $k = \mathbb{C}$ or $\mathbb{R}$. Then there exists an automorphism σ of E such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for any $\alpha \in \Phi$, and an automorphism $\underline{\sigma}$ of G such that $\underline{\sigma}\, \underline{x}_\alpha(t) = \underline{x}_{-\alpha}(-t)$ for any $\alpha \in \Phi$. We have $\sigma h_\alpha(t) = h_\alpha(t^{-1})$, $\underline{\sigma}\, \underline{h}_\alpha(t) = \underline{h}_\alpha(t^{-1})$. Moreover, the group K of fixed points of σ is a subgroup of E containing $C =$ $\operatorname{Ker}\pi$, the group $\underline{K}$ of fixed points of $\underline{\sigma}$ is a maximal compact subgroup of G, and $K = \pi^{-1}(\underline{K})$.*

Proof. This is basically [Steinberg 1968, Theorem 16], which proves the existence of $\underline{\sigma}$ and $\underline{K}$ for G. In particular, $x_\alpha(t) \mapsto x_{-\alpha}(-t)$, for all $\alpha \in \Phi$, preserves the relations (A) and (B). Hence $\underline{\sigma}$ can be lifted to an automorphism of E, which we denote by σ, such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for any $\alpha \in \Phi$. By the definition of $w_\alpha(t)$, $\sigma w_\alpha(t) = w_{-\alpha}(-t)$. So

$$\sigma h_\alpha(t) = \sigma w_\alpha(t) w_\alpha(-1) = \sigma w_\alpha(t) \sigma w_\alpha(-1) = w_{-\alpha}(-t) w_{-\alpha}(1).$$

Since $w_\alpha(t) = w_{-\alpha}(-t^{-1})$ for any $\alpha \in \Phi$, $t \in k^\times$, the last term is $w_\alpha(t^{-1}) w_\alpha(-1) = h_\alpha(t^{-1})$. Thus $\sigma h_\alpha(t) = h_\alpha(t^{-1})$ as in the linear case. Next, with the notation of Theorem 2.2, C is generated by $f(t, u)$, $t, u \in k^\times$, if we identify the groups A, C via ϕ. We have $h_\alpha(t) h_\alpha(u) = f(t, u) h_\alpha(tu)$. Let σ act on both sides. Then one has $h_\alpha(t^{-1}) h_\alpha(u^{-1}) = \sigma f(t, u) h_\alpha(t^{-1}u^{-1})$, which implies that $\sigma f(t, u) = f(t^{-1}, u^{-1})$. By relation (c) in Theorem 2.2, $f(t^{-1}, u^{-1}) = f(u, t^{-1}) = f(t, u)$ and hence σ fixes C. □

For the rest of this paper, we use $\underline{G}$ to denote a simply connected Chevalley group over $\mathbb{R}$, and G to denote the *double cover* of $\underline{G}$. For any subgroup H of G, let $\underline{H}$ be the image of H under the covering projection $p : G \to \underline{G}$. Define the real metaplectic torus T to be the subgroup of G generated by $h_\alpha(t)$, with $\alpha \in \Phi$ and $t \in \mathbb{R}^*$. Let $A \cong (\mathbb{R}^+)^l$ be the subgroup of T generated by $h_\alpha(t)$, with $\alpha \in \Phi$, $t > 0$. Here l is the rank of Φ. By the remark on page 196, $p|_A : A \to \underline{A}$ is an isomorphism, and hence for simplicity we just use A to denote this group. Let M be the subgroup of T generated by $h_\alpha(-1)$, with $\alpha \in \Phi$. It is easy to see that A is in the center of T, and T is the direct product of A and M. Note that M is a central extension of $\underline{M} \cong (\mathbb{Z}/2\mathbb{Z})^l$ by $\mu_2 = \{\pm 1\}$. Let Δ be a set of simple roots, and let Φ^+ be the corresponding set of positive roots. Let N be the group generated by $x_\alpha(t)$, with $\alpha \in \Phi^+$, $t \in \mathbb{R}$. Then $p|_N : N \to \underline{N}$ is an isomorphism, and hence for simplicity we just use N to denote this group. Define P to be subgroup of G generated by N and T, which we call a minimal parabolic subgroup (or Borel subgroup). We have the Langlands decomposition $P = NAM$. By Proposition 2.6, there exists an automorphism σ of G such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for all $\alpha \in \Phi$. Similarly for $\underline{G}$. The group K of fixed points of σ is a maximal compact subgroup of G which is the double cover of $\underline{K}$. It is easy to see that M (resp. $\underline{M}$) is a subgroup of K (resp. $\underline{K}$). One has $Z_{\underline{K}}(A) = \underline{M}$, which implies that $Z_K(A) = M$. Define the Weyl group W to be $N_K(A)/Z_K(A) = N_K(A)/M$. Then W is isomorphic to $N_{\underline{K}}(A)/\underline{M}$.

Lemma 2.7. *The $w_\alpha(1)$ lie in $N_K(A)$, for any $\alpha \in \Phi$, and their images in $N_K(A)/M$ generate W.*

Proof. Since $\sigma w_\alpha(1) = w_{-\alpha}(-1) = w_\alpha(1)$, we have $w_\alpha(1) \in K$. Also, by the second relation in Proposition 2.5, $w_\alpha(1)$ normalizes A. Each $w_\alpha(1)$ corresponds to the reflection s_α through the hyperplane determined by α, which gives an isomorphism between $W = N_K(A)/Z_K(A)$ and the Weyl group $\widehat{W}$ defined in the abstract root system setting. In particular, the $w_\alpha(1)$, for $\alpha \in \Phi$, generate W. □

2C. *The group* SL(2, $\mathbb{R}$) *and its double cover* $\widetilde{\mathrm{SL}}$(2, $\mathbb{R}$). In this section, we recall some basic facts about SL(2, $\mathbb{R}$) and its double cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, which are important for the study of representation theory of general covering groups.

$\underline{G} = \mathrm{SL}(2, \mathbb{R})$ may be described in Steinberg symbols: Let $X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, $Y = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, $H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ be the $\mathfrak{sl}_2$ triple. For $t \in \mathbb{R}$, define

$$\underline{x}(t) = \exp(tX) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \underline{w}(t) = \underline{x}(t)\underline{y}(-t^{-1})\underline{x}(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix},$$
$$\underline{y}(t) = \exp(tY) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \underline{h}(t) = \underline{w}(t)\underline{w}(-1) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let N be the subgroup generated by $\underline{x}(t)$, $t \in \mathbb{R}$, and let A be the subgroup generated by $\underline{h}(t)$, $t > 0$. Then $\underline{K} = \mathrm{SO}(2)$ consists of $r_\phi = \begin{pmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{pmatrix}$, $\phi \in \mathbb{R}$, and $\underline{G} = NA\underline{K}$. Let $\underline{M} = \{\underline{h}(\pm 1)\} \in \underline{K}$. Then the subgroup $\underline{P}$ of upper-triangular matrices has the Langlands decomposition $\underline{P} = NA\underline{M}$.

By Corollary 2.3, there exists a unique nontrivial double cover $G = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ of $\underline{G} = \mathrm{SL}(2, \mathbb{R})$, that is, a central extension of $\underline{G}$ by $\mu_2 = \{\pm 1\}$. We use p to denote the covering map. It is generated by the symbols $x(t)$, $y(t)$ satisfying the same relations as that of $\underline{G}$, except that $h(t)h(u) = (t, u)h(tu)$, where $(\,,\,)$ is the real Hilbert quadratic symbol. The map $\phi : N \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$, $\underline{x}(t) \mapsto x(t)$, $t \in \mathbb{R}$, is a group homomorphism; $\psi : A \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$, $\underline{h}(t) \mapsto h(t)$, $t > 0$, is also a group homomorphism. Moreover, ϕ is the only homomorphism from N to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ satisfying $p \circ \phi = \mathrm{Id}_N$. Assume ϕ' is another one. Consider $f : N \to \mu_2$, $n \mapsto \phi(n)\phi'(n)^{-1}$. Then we have $f(\underline{x}(t)) = f(\underline{x}(t/2)^2) = f(\underline{x}(t/2))^2 = 1$. So f is trivial, whence $\phi = \phi'$. A similar fact holds for ψ. We still denote the images of ϕ, ψ by N, A. Let K be the subgroup fixed by the automorphism σ of G, where σ sends $x(t)$ to $y(-t)$, $y(t)$ to $x(-t)$. Then K is a double cover of SO(2). We have the Iwasawa decomposition $G = NAK$. Let $M = \{\pm h(\pm 1)\} \subset K$. It is isomorphic to C_4, the cyclic group of order four, and it is an extension of $\underline{M}$ by μ_2. The group $P = NAM$ is an extension of $\underline{P}$ by μ_2.

We may also describe the group structure of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ using Kubota cocycles. The only reason we introduce this is that Kubota cocycles make some calculations involving K more explicit. They will be used in Section 5. As a set, $\widetilde{\mathrm{SL}}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) \times \mu_2$. The group law is given by

$$(g, \varepsilon)(g', \varepsilon') = (gg', \varepsilon\varepsilon' c(g, g')).$$

Here c, called the *Kubota cocycle*, is given by the formula

$$c(g, g') = (x(g), x(g'))(-x(g)x(g'), x(gg')),$$

where

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0, \end{cases}$$

and (,) is the quadratic Hilbert symbol. The map $x(t) \mapsto (\underline{x}(t), 1)$, $y(t) \mapsto (\underline{y}(t), 1)$ gives an isomorphism between the two definitions. Direct calculation using the Kubota cocycle shows that $w(t) \mapsto (\underline{w}(t), 1)$ and $h(t) \mapsto (\underline{h}(t), \mathrm{sgn}(t))$. Thus:

Proposition 2.8. *We may write*

$$x(t) = (\underline{x}(t), 1), \quad y(t) = (\underline{y}(t), 1), \quad w(t) = (\underline{w}(t), 1), \quad h(t) = (\underline{h}(t), \mathrm{sgn}(t)).$$

The exponential map

$$\underline{\exp} : \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$$

is given by the exponents of matrices. In particular,

$$\underline{\exp}(tX) = \underline{x}(t), \quad \underline{\exp}(tH) = \underline{h}(e^t), \quad \underline{\exp}(-tZ) = r_t,$$

where $Z = X - Y$.

Proposition 2.9. *Let $\underline{e} : \mathbb{R} \to \mathrm{SO}(2)$ be the homomorphism sending ϕ to r_ϕ. Then there exists a unique homomorphism $e : \mathbb{R} \to K$ such that $p \circ e = \underline{e}$. It is given by $e(\phi) = (r_\phi, \epsilon(\phi/2))$, where $\epsilon : \mathbb{R}/2\pi\mathbb{Z} \to \pm 1$ is defined by $\epsilon(\theta) = \mathrm{sgn}(\sin\theta \sin 2\theta)$ when $\theta \neq 0, \pi/2, \pi, 3\pi/2$, $\epsilon(0) = 1$, $\epsilon(\pi/2) = -1$, $\epsilon(\pi) = -1$, $\epsilon(3\pi/2) = 1$.*

Proof. It is clear that e is of the form appearing in the proposition for some $\epsilon : \mathbb{R}/2\pi\mathbb{Z} \to \pm 1$. By working out $\exp(\theta)\exp(\theta) = \exp(2\theta)$, one sees that

$$\epsilon(\theta) = (x(r_\theta), x(r_\theta))(-1, x(r_{2\theta})) = (-1, x(r_\theta)x(r_{2\theta})) = \mathrm{sgn}(x(r_\theta)x(r_{2\theta})).$$

Direct calculations show that $x(r_\theta)x(r_{2\theta}) = \sin\theta \sin 2\theta$ when $\theta \neq 0, \pi/2, \pi, 3\pi/2$, $0 \leq \theta < 2\pi$, and $\epsilon(0) = 1,\ \epsilon(\pi/2) = -1,\ \epsilon(\pi) = -1,\ \epsilon(3\pi/2) = 1$. □

Corollary 2.10. *For any integer n, the map* $\sigma_{n/2} : K \to S^1,\ (r_\phi, \epsilon(\phi/2)) \mapsto e^{in\phi/2}$ *is a character of K. In particular,* $\sigma = \sigma_{n/2}|_M$ *is a character of M satisfying* $\sigma(I, 1) = 1,\ \sigma(-I, -1) = i^n,\ \sigma(I, -1) = (-1)^n,\ \sigma(-I, 1) = (-i)^n$.

There exists a unique exponential map

$$\exp : \mathfrak{sl}(2, \mathbb{R}) \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$$

such that $p \circ \exp = \underline{\exp}$. For any $X \in \mathfrak{sl}(2, \mathbb{R})$, let $\gamma(t)$ be the unique one-parameter subgroup of $\underline{G}$ whose tangent vector at the identity is equal to X. Since p is a covering map, $t \mapsto \gamma(t)$ can be partially lifted to a continuous map $\tilde{\gamma} : I \to G$ such that it pushes forward to $\gamma|_I$ for some neighborhood $I \subset \mathbb{R}$ around 0 and $\tilde{\gamma}(0) = 1 \in G$. Since γ is a continuous homomorphism, one can extend $\tilde{\gamma}$ to a homomorphism from $\mathbb{R}$ to G which lifts γ. We define $\exp(X)$ to be $\tilde{\gamma}(X)$.

Proposition 2.11. *We have*

$$\exp(tX) = (\underline{x}(t), 1), \quad \exp(tH) = (\underline{h}(e^t), 1), \quad \exp(-tZ) = (r_t, \epsilon(t/2)).$$

Proof. The first two are obvious and the third follows from Proposition 2.9. □

2D. *Connections between $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and general covering groups.* Let G be the unique nontrivial two-fold cover of a split real group $\underline{G}$. For each root α,

$$\Phi_\alpha : \widetilde{\mathrm{SL}}(2, \mathbb{R}) \to G$$

is defined to be the homomorphism sending $x(t)$ to $x_\alpha(t)$, $y(t)$ to $x_{-\alpha}(t)$, and $h(t)$ to $h_\alpha(t)$.

We now state a definition from [Adams et al. 2007], which will be used later:

Definition. A root α is said to be *metaplectic* if Φ_α does not factor through $\mathrm{SL}(2, \mathbb{R})$.

The next proposition follows from the first two equations in Proposition 2.4:

Proposition 2.12. *If G is not of type* G_2*, then* α *is metaplectic if and only if it is long. If G is of type* G_2*, then all roots are metaplectic.*

3. Pseudospherical Representations

For each $\alpha \in \Phi$, let $m_\alpha = h_\alpha(-1) \in G$ and $Z_\alpha = X_\alpha - X_{-\alpha} \in \mathfrak{g}$. Then we have $\exp(-\pi Z_\alpha) = m_\alpha$ by Propositions 2.8 and 2.11. The following definition is from Definition 4.9 and Lemma 4.11 of [Adams et al. 2007]:

Definition (pseudospherical representations). An irreducible representation σ of M is pseudospherical if the eigenvalues of $\sigma(m_\alpha)$ belong to $\{\pm i\}$ when α is a metaplectic root, and $\{1\}$ otherwise. An irreducible representation μ of K is pseudospherical if the eigenvalues of $d\mu(iZ_\alpha)$ belong to $\left\{\pm\frac{1}{2}\right\}$ when α is a metaplectic root, and $\{0\}$ otherwise. A representation of G is pseudospherical if it contains a pseudospherical K-type.

Remark. When $G = \widetilde{\mathrm{SL}}(n, \mathbb{R})$, the double cover of $\mathrm{SL}(n, \mathbb{R})$, the *Spinor representation* of $K = \mathrm{Spin}(n)$ is pseudospherical.

Remark. If G is simply laced or of type G_2, then every irreducible genuine representation of M is pseudospherical. In fact, all roots are metaplectic in this case, and so $m_\alpha^2 = h_\alpha(-1)h_\alpha(-1) = -1 \in \mu_2 \subset Z(G)$. So $\sigma(m_\alpha)^2 = \sigma(-1) = -I$ and hence its eigenvalues are $\pm i$ with multiplicities.

Also notice that eigenvalues in the pseudospherical conditions for M and K are compatible: $\log(1) = 2\pi i\mathbb{Z}$, $\log(\pm i) = \pm i\pi/2 + 2\pi i\mathbb{Z}$.

Example. In the case $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the representation $\mu = \sigma_{1/2}$ is a pseudospherical representation of $K = \widetilde{\mathrm{SO}}(2)$ whose restriction $\sigma = \sigma_{1/2}|_M$ to M is a pseudospherical representation of M. In fact,

$$d\mu(Z_\alpha) = \lim_{t\to 0} \frac{\mu(\exp(tZ_\alpha)) - 1}{t}.$$

But $\mu(\exp(tZ_\alpha)) = \sigma_{1/2}(r_{-t}, \epsilon(-t/2)) = e^{-it/2}$, so the limit is

$$\lim_{t\to 0} \frac{e^{-it/2} - 1}{t} = -\frac{i}{2}.$$

Thus $d\mu(iZ_\alpha) = \frac{1}{2}$ and

$$\sigma(m_\alpha) = \sigma(h_\alpha(-1)) = \sigma(r_\pi, \epsilon(\pi/2)) = e^{i\pi/2} = i.$$

Below is a fundamental fact on pseudospherical representations:

Theorem 3.1 [Adams et al. 2007, Proposition 5.2]. *Let σ be a pseudospherical representation of M. There is a unique pseudospherical representation μ_σ of K such that $\mu_\sigma|_M = \sigma$ and this defines a bijection between pseudospherical representations of M and K.*

Now we want to define an action of W on irreducible representations (σ, V) of M that do not factor through $\underline{M}$ (or equivalently, $\sigma(-1) \neq 1$). We call such representations *genuine representations*. We use $\Pi_g(M)$ to denote the set of isomorphism classes of genuine representations of M. We will show that W fixes every isomorphism class of irreducible genuine pseudospherical representations of M. This is proved in [Adams et al. 2007, Lemma 4.11(3)]. We repeat the argument below for completeness.

Proposition 3.2. *Let $Z(M) \supset \mu_2$ be the center of M, and let $\Pi_g(Z(M))$ be the set of genuine characters of $Z(M)$, that is, those characters χ satisfying $\chi(-1) = -1$.*

For every $\chi \in \Pi_g(Z(M))$, there is a unique representation $\sigma(\chi)$ of M such that $\sigma(\chi)|_{Z(M)} = \chi \cdot I$. The map $\chi \mapsto \sigma(\chi)$ defines a bijection $\Pi_g(Z(M)) \to \Pi_g(M)$. The dimension of $\sigma(\chi)$ is $|M/Z(M)|^{1/2}$ and $\mathrm{Ind}_{Z(M)}^M(\chi) \cong |M/Z(M)|^{1/2}\sigma(\chi)$.

Proof. The key point of the proof in [Adams et al. 2007] is that if σ is a genuine representation of M, then the character $\mathrm{tr}\,\sigma$ is supported on $Z(M)$. Suppose m does not belong to $Z(M)$. Choose $h \in M$ such that $hmh^{-1} \neq m$. Since $\underline{M}$ is abelian, $p(hmh^{-1}) = p(h)p(m)p(h)^{-1} = p(m)$, so $hmh^{-1} = -m$. Taking the trace on both sides, we have $\mathrm{tr}\,\sigma(m) = \chi(-1)\,\mathrm{tr}\,\sigma(m)$. Since χ is genuine, $\chi(-1) = -1$, so $\mathrm{tr}\,\sigma(m) = 0$.

Therefore, every irreducible genuine representation of M is uniquely determined by its central character. Fix $\chi \in \Pi_g(Z(M))$. Let $I(\chi) = \mathrm{Ind}_{Z(M)}^M(\chi)$. This has central character χ, so it is a multiple of the irreducible representation $\sigma(\chi)$ of M with central character χ. Put $I(\chi) = n\sigma(\chi)$. By Frobenius reciprocity,

$$\mathrm{Hom}_M(I(\chi), I(\chi)) = \mathrm{Hom}_{Z(M)}(I(\chi)|_{Z(M)}, \chi),$$

which has dimension $|M/Z(M)|$. On the other hand, by Schur's lemma,

$$\mathrm{Hom}_M(I(\chi), I(\chi)) = \mathrm{Hom}_M(n\sigma(\chi), n\sigma(\chi)),$$

which has dimension n^2. Therefore $n = |M/Z(M)|^{1/2}$ and the dimension of $\sigma(\chi)$ is $|M/Z(M)|^{1/2}$. ☐

Since $N_K(A)$ acts on M by conjugation, it also acts on its center $Z(M)$, which factors down to $W = N_K(A)/M$. Thus we have an action of W on $\Pi_g(Z(M))$. By the proposition above, this gives rise to an action of W on $\Pi_g(M)$. More precisely, pick a representative $\hat{w} \in N_K(A)$ of $w \in W$. Then $\hat{w}\sigma(m) = \sigma(\hat{w}^{-1}m\hat{w})$ is a representation of M. Up to isomorphism, it is independent of the choice of a representative of w because different representatives $\hat{w}$ give the same central character, hence isomorphic representations. Therefore one can denote this representation by $w\sigma$, as an isomorphism class in $\Pi_g(M)$.

Proposition 3.3. *The action of the Weyl group W on the isomorphism classes of irreducible genuine representations of M fixes each isomorphism class of pseudospherical representations.*

Proof. Assume (σ, V) is a genuine representation of M. For all $w \in W$, choose a representative $\hat{w}$ of w in $N_K(A)$. By Theorem 3.1, there is a unique pseudospherical representation (μ_σ, V) of K such that $\mu_\sigma|_M = \sigma$. Let $\phi : V \to V,\ v \mapsto \mu_\sigma(\hat{w}^{-1})v$. Then

$$\phi(\mu_\sigma(k)v) = \mu_\sigma(\hat{w}^{-1})\mu_\sigma(k)v = \mu_\sigma(\hat{w}^{-1}k\hat{w})\mu_\sigma(\hat{w}^{-1})v = (\hat{w}\mu_\sigma)(k)\phi(v)$$

for any $k \in K$. Thus ϕ is a K-isomorphism, and restricting it to M, we get

$$\sigma \cong (\hat{w}\mu_\sigma)|_M = \hat{w}\sigma.$$

☐

4. Principal series representations and intertwining operators

In this section, let G be the double cover of a split real group. We define the principal series representation of G and the intertwining operator. Most of the results are well-known in the linear group case; see [Schiffmann 1971]. The discussion for covering groups is almost identical to the linear case. The highlight is that the intertwining maps can be defined in a canonical way using the theory of pseudospherical representations.

Let χ be a character of A, and let δ be the modular character of A such that

$$\int_N f(a^{-1}na)\,dn = \delta(a)\int_N f(n)\,dn$$

for any $a \in A$ and any compact supported function f on N. Here we fix a Haar measure on N, which is topologically isomorphic to $\mathbb{R}^{|\Phi^+|}$. Since δ depends on N, we will write δ_N instead of δ when needed. The character δ is equal to the product of the roots in Φ^+, considered as multiplicative characters of A. Let (σ, V) be a pseudospherical representation of M.

Definition. Let $I(P, \sigma, \chi)$, the space of principal series, be the space of smooth functions $f : G \to V$ such that

$$f(namx) = \delta(a)^{1/2}\chi(a)\sigma(m)f(x)$$

for all $n \in N$, $a \in A$, $m \in M$, and $x \in G$. Then G acts on $I(P, \sigma, \chi)$ by right translation: $\rho(g)f(x) = f(xg)$. This defines a representation of G called the *principal series representation, or induced representation*, of G. For simplicity, we denote this representation by $I(\sigma, \chi)$ or $I(\chi)$ when there is no confusion.

Assume χ is a character of A. For all $w \in N_K(A)$, $w\chi(a) = \chi(w^{-1}aw)$ is another character of A. This action factors down to W. Note that $w_1(w_2\chi) = (w_1w_2)\chi$. In other words, we have an action of the Weyl group W on $\Pi(A) =$ the set of characters of A.

By Theorem 3.1, there is an irreducible representation (μ_σ, V) of K such that $\mu_\sigma|_M = \sigma$. For any $f \in I(P, \sigma, \chi)$ and any $w \in W$, pick a representative $\hat{w} \in N_K(A)$ of w, and define a function

$$M(w, \sigma, \chi)f(x) = \mu_\sigma(\hat{w})\int_{N\cap\hat{w}N\hat{w}^{-1}\backslash N} f(\hat{w}^{-1}nx)\,dn.$$

Note that $n \to f(\hat{w}^{-1}nx)$ is left $(N \cap \hat{w}N\hat{w}^{-1})$-invariant, so the integral makes sense. Also it is well-defined, i.e, independent of the choice of a representative of w in $N_K(A)$ due to the normalizing factor μ_σ. For simplicity, we write w in place of $\hat{w}$ when there is no confusion. Let us remark that N_w, which is equal to

$N \cap wNw^{-1} \setminus N$, corresponds to those positive roots that are sent to negative by w^{-1}, and it has one-to-one correspondence with $B \setminus Bw^{-1}N$.

Let $S(w)$ be the set of χ such that the above integral is absolutely convergent for any $x \in G$, $f \in I(P, \sigma, \chi)$. We are going to show that $M(w, \sigma, \chi)$ maps $I(P, \sigma, \chi)$ into $I(P, \sigma, w\chi)$ for $\chi \in S(w)$. This map is called the *intertwining map*. For simplicity, we sometimes denote this map by $M(w, \chi)$ or $M(w)$.

Lemma 4.1. *Let w be an element in W, and let δ_w be a character of A such that*

$$\int_{N_w} f(a^{-1}na)\,dn = \delta_w(a) \int_{N_w} f(n)\,dn$$

for any $a \in A$ and any integrable function f on N. Then $(w\delta)^{1/2}\delta_w = \delta^{1/2}$.

Proof. For a simple reflection w, take $Q = P \cup Pw^{-1}P$ and let L, U be its Levi factor and unipotent radical. Note that $U = N \cap wNw^{-1}$. We have $\delta_N = \delta_U \delta_{N/U}$ and $\delta_{wNw^{-1}} = \delta_U \delta_{wNw^{-1}/U}$. But $\delta_{wNw^{-1}} = w\delta_N$ and $\delta_{wNw^{-1}/U} = \delta_{N/U}^{-1}$. The conclusion now follows from simple algebraic manipulations. □

Proposition 4.2. *If $\chi \in S(w)$, then $M(w, \sigma, \chi)$ maps $I(P, \sigma, \chi)$ into $I(P, \sigma, w\chi)$.*

Proof. $M(w, \sigma, \chi) f(nx) = M(w, \sigma, \chi) f(x)$ is obvious. Next we have

$$\begin{aligned} M(w, \sigma, \chi) f(ax) &= \mu_\sigma(w) \int_{N_w} f(w^{-1}nax)\,dn \\ &= \mu_\sigma(w) \int_{N_w} f((w^{-1}aw)w^{-1}(a^{-1}na)x)\,dn \\ &= \mu_\sigma(w) w\delta(a)^{1/2} w\chi(a) \int_{N_w} f(w^{-1}(a^{-1}na)x)\,dn \\ &= \mu_\sigma(w) w\delta(a)^{1/2} w\chi(a) \delta_w(a) \int_{N_w} f(w^{-1}nx)\,dn \\ &= \delta(a)^{1/2} w\chi(a) M(w, \sigma, \chi) f(x). \end{aligned}$$

Similarly,

$$\begin{aligned} M(w, \sigma, \chi) f(mx) &= \mu_\sigma(w) \int_{N_w} f(w^{-1}nmx)\,dn \\ &= \mu_\sigma(w) \int_{N_w} f((w^{-1}mw)w^{-1}(m^{-1}nm)x)\,dn \\ &= \mu_\sigma(w) \sigma(w^{-1}mw) \int_{N_w} f(w^{-1}(m^{-1}nm)x)\,dn \\ &= \sigma(m) \mu_\sigma(w) \int_{N_w} f(w^{-1}nx)\,dn \\ &= \sigma(m) M(w, \sigma, \chi) f(x). \end{aligned}$$

□

Assume that the Haar measures on the N_w are normalized so that, when $l(w_1 w_2) = l(w_1) + l(w_2)$,

$$\int_{N_{w_1 w_2}} f(n)\, dn = \int_{N_{w_1} \times N_{w_2}} f(w_1 n_2 w_1^{-1} n_1)\, dn_1\, dn_2$$

for any integrable function f on $N_{w_1 w_2}$. Under this assumption, the following proposition holds:

Proposition 4.3. *Assume $w_1, w_2 \in W$ such that $l(w_1 w_2) = l(w_1) + l(w_2)$. Then*

$$S(w_1 w_2) = S(w_2) \cap w_2^{-1} S(w_1),$$

and for $\chi \in S(w)$ regular (only fixed by the trivial element in W),

$$M(w_1, \sigma, w_2 \chi) \circ M(w_2, \sigma, \chi) = M(w_1 w_2, \sigma, \chi).$$

Proof. Since χ is regular, the dimension of $\mathrm{Hom}_G(I(\chi), I(w\chi))$ is one for any $w \in W$, by Frobenius reciprocity. So it suffices to show that $(M(w_1) \circ M(w_2) f)(1) = M(w_1 w_2) f(1)$:

$$\begin{aligned}
(M(w_1) \circ M(w_2) f)(1) &= \mu_\sigma(w_1) \int_{N_{w_1}} (M(w_2) f)(w_1^{-1} n_1)\, dn_1 \\
&= \mu_\sigma(w_1) \mu_\sigma(w_2) \int_{N_{w_1}} dn_1 \int_{N_{w_2}} f(w_2^{-1} n_2 w_1^{-1} n_1)\, dn_2 \\
&= \mu_\sigma(w_1 w_2) \int_{N_{w_1}} dn_1 \int_{N_{w_2}} f(w_2^{-1} w_1^{-1} \cdot w_1 n_2 w_1^{-1} n_1)\, dn_2.
\end{aligned}$$

By the assumption on the Haar measures, the last expression is equal to

$$\mu_\sigma(w_1 w_2) \int_{N_{w_1 w_2}} f(w_2^{-1} w_1^{-1} n)\, dn = M(w_1 w_2) f(1). \qquad \square$$

5. Representations of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$

We carry out the detailed study of principal series and intertwining maps in the SL_2 case first, since it is the fundamental building block of the general case. The results in Section 5A are well-known, but we list them here for the purpose of making a comparison with the nonlinear case.

5A. ***Linear case.*** Let $\underline{G} = \mathrm{SL}(2, \mathbb{R})$, and let $\underline{P}$ be the standard parabolic subgroup with Langlands decomposition $\underline{P} = NA\underline{M}$. Then $\underline{M}$ has only two characters. Let σ be any of them. For any complex number s, define a character χ of A by $\chi(a) = a^s$, where $a = \mathrm{diag}(a, a^{-1})$. The modular character $\delta(a)$ of A is a^2. So the space

$I(\underline{P}, \sigma, \chi)$ of principal series in this case is the collection of functions f such that

$$f(namx) = a^{s+1}\sigma(m) f(x).$$

For simplicity, we denote it by $I(\sigma, s)$. Let $\underline{K} = \mathrm{SO}(2)$. For any $n \in \mathbb{Z}$, $\tau_n(r_\phi) = e^{in\phi}$ is a character of $\underline{K}$. Define f_s^n such that

$$f_s^n(nak) = a^{s+1}\tau_n(k).$$

Then $f_s^n \in I(\tau_n|_{\underline{M}}, s)$. When n is even, $\tau_n|_{\underline{M}}$ is trivial, denoted by σ_0. When n is odd, $\tau_n|_{\underline{M}}$ is nontrivial, denoted by σ_1. We say f_s^n is of $\underline{K}$-type n.

The Weyl group W is of order two. Let w be its nontrivial element. Now we define the intertwining map $M(\sigma, s) : I(\sigma, s) \to I(\sigma, -s)$. For $f \in I(\sigma, s)$,

$$M(\sigma, s) f(x) = \int_N f(wnx)\, dn \qquad \text{when } \sigma = \sigma_0,$$

$$M(\sigma, s) f(x) = \tau_1(w)^{-1} \int_N f(wnx)\, dn \quad \text{when } \sigma = \sigma_1.$$

It does not depend on the choice of a representative element of w in $N_K(A)$. We have $M(\sigma, s) f_s^n = c_n(s) f_{-s}^n$ for some constant $c_n(s) = (M(\sigma, s) f_s^n)(1)$. The following proposition is well-known. We will give a proof of a more general proposition in the next subsection.

Proposition 5.1.
$$c_n(s) = \sqrt{\pi}\, \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+n+1}{2}\right)\Gamma\left(\frac{s-n+1}{2}\right)}.$$

5B. ***Nonlinear case.*** Let $G = \widetilde{\mathrm{SL}}(2, \mathbb{R})$, and let $P = NAM$ be its standard parabolic subgroup which is the double cover of $\underline{P}$. Let K be the double cover of $\underline{K} = \mathrm{SO}(2)$. We are going to study the principal series of G and calculate the intertwining map using the Kubota cocycle. Let σ be a character of M, and define a character χ of A by

$$\chi(a) = a^s,$$

where $s \in \mathbb{C}$, $a = (\mathrm{diag}(a, a^{-1}), 1) \in A$. Since $\delta_N(a) = a^2$ and $\delta_{\bar{N}}(a) = a^{-2}$, $I(P, \sigma, \chi)$, which we denote by $I(\sigma, s)$ for simplicity, consists of functions f such that

$$f(namx) = a^{s+1}\sigma(m) f(x).$$

$I(\bar{P}, \sigma, \chi)$, which we denote by $\bar{I}(\sigma, s)$ for simplicity, consists of functions f such that

$$f(\bar{n}amx) = a^{s-1}\sigma(m) f(x).$$

For any $n \in \mathbb{Z}$, define $f_s^{n/2}$ such that

$$f_s^{n/2}(nak) = a^{s+1}\sigma_{n/2}(k),$$

where $\sigma_{n/2}$ is a character of K defined in Proposition 2.9. Then $f_s^{n/2} \in I(\sigma_{n/2}|_M, s)$. We say $f_s^{n/2}$ is of K-type $n/2$. Similarly, define $\bar{f}_s^{n/2}$ such that

$$\bar{f}_s^{n/2}(\bar{n}ak) = a^{s-1}\sigma_{n/2}(k).$$

Then $\bar{f}_s^{n/2} \in \bar{I}(\sigma_{n/2}|_M, s)$.

There are four different characters σ of M: $\sigma_0|_M, \sigma_{1/2}|_M, \sigma_1|_M, \sigma_{3/2}|_M$. Define the intertwining map $M(\sigma, s) : I(\sigma, s) \to I(\sigma, -s)$ by

$$M(\sigma, s)f(x) = \sigma_{i/2}(w)^{-1}\int_N f(wnx)\,dn, \quad \sigma = \sigma_{i/2}|_M,\ i = 0, 1, 2, 3.$$

This definition is canonical. Define $T : \bar{I}(\sigma, s) \to I(\sigma, -s)$ by

$$Tf(x) = \sigma_{i/2}(w)^{-1} f(wx), \quad \sigma = \sigma_{i/2}|_M,\ i = 0, 1, 2, 3.$$

Also define intertwining maps $A(\sigma, s) : I(\sigma, s) \to \bar{I}(\sigma, s)$ such that

$$A(\sigma, s)f(x) = \int_{\bar{N}} f(\bar{n}x)\,d\bar{n}$$

and $\bar{A}(\sigma, s) : \bar{I}(\sigma, s) \to I(\sigma, s)$ such that

$$\bar{A}(\sigma, s)f(x) = \int_N f(nx)\,dn.$$

Then we have

$$M(\sigma, s) = T \circ A(\sigma, s).$$

$M(\sigma, s)$ sends $f_s^{n/2}$ to $c_{n/2}(s) f_{-s}^{n/2}$ for some constant $c_{n/2}(s)$. It is sometimes called the *Harish-Chandra c-function*. It is easy to see that $c_{n/2}(s) = A(\sigma, s) f_s^{n/2}(1)$. For simplicity, we sometimes use $I(s)$ in place of $I(\sigma, s)$ and $A(s)$ in place of $A(\sigma, s)$.

Define a pairing $(\,,\,) : I(s) \times I(-\bar{s}) \to \mathbb{C}$ by

$$(f, g) = \int_K f(k)\overline{g(k)}\,dk.$$

There is also a pairing $(\,,\,) : \bar{I}(s) \times \bar{I}(-\bar{s}) \to \mathbb{C}$ defined using the same formula. The following lemma follows from formal calculations:

Lemma 5.2. *For $f \in I(s)$ and $g \in \bar{I}(-\bar{s})$, we have $(A(s)f, g) = (f, \bar{A}(-\bar{s})g)$.*

Proposition 5.3. *For those $s \in i\mathbb{R}$ such that $I(s)$ is irreducible, $\bar{A}(s) \circ A(s)$ is a nonnegative constant.*

Proof. By Schur's lemma, $\bar{A}(s) \circ A(s)$ is a constant, say $\lambda(s)$. When $s \in i\mathbb{R}$, $s = -\bar{s}$, so by Lemma 5.2 $(A(s)f, g) = (f, \bar{A}(s)g)$. Taking $g = A(s)f$, we get $(A(s)f, A(s)f) = (f, \bar{A}(s) \circ A(s)f) = \bar{\lambda}(s)(f, f)$, hence $\lambda(s)$ is nonnegative. $\square$

Below is a nice property of the c-function:

Proposition 5.4. $c_{n/2}(s) = c_{-n/2}(s)$.

Proof. Let $d = \mathrm{diag}(1, -1) \in \mathrm{GL}_2$. The conjugation action of d on SL_2 satisfies $d r_\phi d^{-1} = r_{-\phi}$. This action lifts to the covering group and it gives an inverse on K. Hence the conjugation of functions by d gives an intertwining map $I(\sigma, s) \to I(\sigma^{-1}, s)$ which we denote by $d(s)$. We have

$$M(\sigma^{-1}, s) \circ d(s) = d(-s) \circ M(\sigma, s).$$

In fact, for any $f \in I(\sigma, s)$,

$$\begin{aligned} M(\sigma^{-1}, s) \circ d(s) f(x) &= \sigma_{-n/2}(w)^{-1} \int_N (d(s) f)(wnx)\, dn \\ &= \sigma_{-n/2}(w)^{-1} \int_N f(dwnxd^{-1})\, dn \\ &= \sigma_{n/2}(w) \int_N f(w^{-1} dnxd^{-1})\, dn. \end{aligned}$$

Since $w^{-1} = mw$ for some $m \in M$, the last expression is

$$\sigma_{n/2}(w)\sigma(m) \int_N f(wdnxd^{-1})\, dn = \sigma_{n/2}(w^{-1}) \int_N f(wdnxd^{-1})\, dn.$$

On the other hand,

$$\begin{aligned} d(-s) \circ M(\sigma, s) f(x) &= \sigma_{n/2}(w)^{-1} \int_N f(wndxd^{-1})\, dn \\ &= \sigma_{n/2}(w)^{-1} \int_N f(wdnxd^{-1})\, dn, \end{aligned}$$

hence the two operators are equal.

Now take $f = f_s^{n/2}$. Then

$$\begin{aligned} M(\sigma^{-1}, s) \circ d(s) f_s^{n/2} &= c_{-n/2}(s) f_{-s}^{-n/2}, \\ d(-s) \circ M(\sigma, s) f_s^{n/2} &= c_{n/2}(s) f_{-s}^{-n/2}. \end{aligned}$$

Thus $c_{n/2}(s) = c_{-n/2}(s)$. $\square$

Now we calculate $c_{n/2}(s)$:

Proposition 5.5.
$$c_{n/2}(s) = \sqrt{\pi} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2} + \frac{n}{4}\right)\Gamma\left(\frac{s+1}{2} - \frac{n}{4}\right)}.$$

Proof. We have $c_{n/2}(s)=\int_{\bar N} f_s^{n/2}(\bar n)\,d\bar n$. For $\bar n=\left(\begin{smallmatrix}1&0\\t&1\end{smallmatrix}\right)\in\bar N$,

$$\begin{aligned}\bar n&=\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}y^{1/2}&0\\0&y^{-1/2}\end{pmatrix}(r_\phi,1)\\&=\begin{pmatrix}y^{1/2}\cos\phi+xy^{-1/2}\sin\phi&-y^{1/2}\sin\phi+xy^{-1/2}\cos\phi\\y^{-1/2}\sin\phi&y^{-1/2}\cos\phi\end{pmatrix}.\end{aligned}$$

Then

$$\bar n\cdot i=\frac{i}{ti+1}=\frac{t}{t^2+1}+\frac{1}{t^2+1}i=x+yi,$$

so $a(t)=y^{1/2}=1/\sqrt{t^2+1},\ \tan\phi=t,\ \phi=\phi(t)=\arctan t$. Then

$$c_{n/2}(s)=\int_{\mathbb R} f_s^{n/2}\left(\begin{pmatrix}1&0\\t&1\end{pmatrix}\right)dt=\int_{\mathbb R}\frac{1}{(t^2+1)^{(s+1)/2}}\sigma_{n/2}(r_{\phi(t)},1)\,dt.$$

Since $\phi(t)\in(-\pi/2,\pi/2)$, we have $\epsilon(\phi(t)/2)=\operatorname{sgn}(\sin(\phi(t)/2)\sin(\phi(t)))=1$, hence $\sigma_{n/2}(r_{\phi(t)},1)=e^{in\phi(t)/2}$. Finally, by substituting those expressions into the last integral, we get

$$c_{n/2}(s)=\int_{\mathbb R}\frac{1}{(t^2+1)^{(s+1)/2}}e^{in(\arctan t)/2}=\int_{\mathbb R}\frac{1}{(t^2+1)^{(s+1)/2}}\left(\frac{1-it}{\sqrt{t^2+1}}\right)^{-n/2}dt.$$

Now the proposition follows from the lemma below. □

Lemma 5.6. *For any $n\in\mathbb Z$,*

$$\int_{\mathbb R}\frac{1}{(t^2+1)^{(s+1)/2}}\left(\frac{1-it}{\sqrt{t^2+1}}\right)^{-n/2}dt=\sqrt\pi\,\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\frac{n}{4}\right)\Gamma\left(\frac{s+1}{2}-\frac{n}{4}\right)}.$$

Proof. The integral is absolutely convergent for $\operatorname{Re}(s)>0$. The integrand is equal to

$$(1+it)^{(-2s-n-2)/4}(1-it)^{(-2s+n-2)/4},$$

which we denote by $f(t)$. By Lebesgue's dominated convergence theorem,

$$\lim_{y\to0}\int_{\mathbb R}f(t)e^{-ity}\,dt=\int_{\mathbb R}f(t)\,dt.$$

Let $2u=\frac14(2s+n+2),\ 2v=\frac14(2s-n+2)$. By [Erdélyi et al. 1954],

$$\hat f(y)=\int_{\mathbb R}f(t)e^{-ity}\,dt=2\pi2^{-u-v}\Gamma(2v)^{-1}y^{u+v-1}W_{v-u,1/2-v-u}(2y)$$

for $y>0$. Here W is the Whittaker function

$$W_{\rho,\sigma}(z)=\frac{\Gamma(-2\sigma)}{\Gamma\left(\frac12-\sigma-\rho\right)}M_{\rho,\sigma}(z)+\frac{\Gamma(2\sigma)}{\Gamma\left(\frac12+\sigma-\rho\right)}M_{\rho,-\sigma}(z),$$

where

$$M_{\rho,\sigma}(z) = z^{1/2+\sigma} e^{-z/2} F\big(\tfrac{1}{2}+\sigma-\rho, 2\sigma+1, z\big),$$

$$F(a,b,z) = 1 + \sum_{k\geq 1} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)} \frac{z^k}{k!}.$$

So

$$W_{v-u,1/2-v-u}(2y) = \frac{\Gamma(-1+2u+2v)}{\Gamma(2u)}(2y)^{1-u-v}e^{-y}F(1-2v, 2-2u-2v, 2y) + \frac{\Gamma(1-2u-2v)}{\Gamma(1-2v)}(2y)^{u+v}e^{-y}F(2u, 2u+2v, 2y).$$

Thus

$$y^{u+v-1}W_{v-u,1/2-v-u}(2y) \to \frac{\Gamma(-1+2u+2v)}{\Gamma(2u)}2^{1-u-v}$$

as $y \to 0$. It follows that

$$\int_{\mathbb{R}} f(t)\,dt = \lim_{y\to 0} \hat{f}(y) = 2\pi 2^{-u-v}\Gamma(2v)^{-1}\frac{\Gamma(-1+2u+2v)}{\Gamma(2u)}2^{1-u-v} = \pi 2^{1-s}\frac{\Gamma(s)}{\Gamma\big(\frac{s+1}{2}+\frac{n}{4}\big)\Gamma\big(\frac{s+1}{2}-\frac{n}{4}\big)}.$$

By the double formula,

$$\Gamma(s) = \frac{1}{\sqrt{\pi}}2^{s-1}\Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big),$$

hence

$$\int_{\mathbb{R}} f(t)\,dt = \sqrt{\pi}\frac{\Gamma\big(\frac{s}{2}\big)\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{s+1}{2}+\frac{n}{4}\big)\Gamma\big(\frac{s+1}{2}-\frac{n}{4}\big)}. \qquad \square$$

Now we consider a slightly more general situation, which will be used in the next section. Let (σ, V) be a finite-dimensional representation of M which is the restriction of a representation (μ, V) of K. Let $I(\sigma, s)$ be the space of functions $f : G \to V$ such that $f(namg) = a^{s+1}\sigma(m)f(g)$. For $f \in I(\sigma, s)$, define

$$M(s)f(x) = \mu(w)^{-1}\int_N f(wnx)\,dn.$$

By Proposition 4.2, $M(s)$ maps $I(\sigma, s)$ into $I(\sigma, -s)$. For $v \in V$, define

$$f_s^v(nak) = a^{s+1}\mu(k)v.$$

Then $v \mapsto f_s^v$ is an embedding of (μ, V) into $I(\sigma, s)$, as a K-subrepresentation.

Proposition 5.7. *Assume (μ, V) is a direct sum of $\sigma_{\pm n/2}$ for a fixed integer n. Then $(M(s)f_s^v)(1) = c_{n/2}(s)v$.*

Proof. If v belongs to one of those summands, then by the definition of $M(s)$ and Proposition 5.5, $(M(s)f_s^v)(1) = c_{n/2}(s)v$. Because $c_{n/2}(s) = c_{-n/2}(s)$, this is valid for any $v \in V$. □

6. Action of intertwining operators on pseudospherical K-types

This section contains the main result of this paper. Let G be the unique nontrivial two-fold cover of a split real group $\underline{G}$. Assume σ is a pseudospherical representation of M and μ_σ is the pseudospherical representation of K corresponding to σ. We note that the multiplicity of μ_σ in $I(P, \sigma, \chi)$ is one and then calculate the action of the intertwining operator on it, obtaining explicit formulas of the Harish-Chandra c-function.

The following lemma is fairly simple; see Definition 5.5 of [Adams et al. 2007].

Lemma 6.1. *As a K-representation, the multiplicity of μ_σ in $I(P, \sigma, \chi)$ is* 1.

Proof. It is easy to see that, as a K-representation, $I(P, \sigma, \chi)$ is isomorphic to $\mathrm{Ind}_M^K(\sigma)$. By Frobenius reciprocity, $\mathrm{Hom}_K(\mu_\sigma, \mathrm{Ind}_M^K(\sigma)) = \mathrm{Hom}_M(\sigma, \sigma)$, which is isomorphic to $\mathbb{C}$ by Schur's lemma. □

Let ϕ be the unique element in $\mathrm{Hom}_K(\mu_\sigma, I(P, \sigma, \chi))$ such that $(\phi v)(1) = v$ for all $v \in V$, and let ψ be the unique element in $\mathrm{Hom}_K(\mu_\sigma, I(P, \sigma, w\chi))$ such that $(\psi v)(1) = v$ for all $v \in V$. Then $M(w, \sigma, \chi)(\phi v) = c \cdot (\psi v)$ for some nonzero constant $c \in \mathbb{C}$ which does not depend on v.

Let $s = (s_1, \ldots, s_l) \in \mathbb{C}^l$ and take $\chi = \chi_s$ to be the character of A such that

$$\chi_s(h_1(t_1) \cdots h_l(t_l)) = t_1^{s_1} \cdots t_l^{s_l}, \quad t_i > 0.$$

We write $I(P, \sigma, s)$ instead of $I(P, \sigma, \chi)$. Let $ws \in \mathbb{C}^l$ be such that $w\chi_s = \chi_{ws}$. We write $M(w, s)$ for the intertwining map instead of $M(w, \sigma, \chi_s)$.

Lemma 6.2. *Define a function $f_{P,\mu_\sigma,s}^v : G \to V$ such that*

$$f_{P,\mu_\sigma,s}^v(nak) = \chi_s(a)\delta_N(a)^{1/2}\mu_\sigma(k)v.$$

Then $f_{P,\mu_\sigma,s}^v$ is well-defined and lies in $I(P, \sigma, s)$.

Proof. For simplicity, we write f_s^v in place of $f_{P,\mu_\sigma,s}^v$ when there is no confusion. Since the Iwasawa decomposition is unique (this is not true in the p-adic case), f_s^v is well-defined. It is evident that $f_s^v(nx) = f_s^v(x)$. For any $a \in A$, we have $f_s^v(ax) = f_s^v(an(x)a(x)k(x))$. Since T normalizes N, it is equal to $\chi_s(a)\delta_N(a)^{1/2} f_s^v(x)$. Finally, since T normalizes N and A is contained in the center of T,

$$\begin{aligned} f_s^v(mx) &= f_s^v(mn(x)a(x)k(x)) = f_s^v(n'(x)ma(x)k(x)) \\ &= f_s^v(n'(x)a(x)mk(x)) = \sigma(m) f_s^v(x). \end{aligned}$$

Thus $f_s^v \in I(P, \sigma, s)$. □

Lemma 6.3. *Define $\phi : \mu_\sigma \to I(P, \sigma, s)$, $v \mapsto f_s^v$. Then ϕ is a K-intertwining map.*

Proof. We need to show $\phi(\sigma(k)v) = R(k)\phi(v)$. For $x \in G$, let $x = n(x)a(x)k(x)$ be the Iwasawa decomposition of x. Then

$$\phi(\sigma(k)v)(x) = f_s^{\sigma(k)v}(x) = \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x))\sigma(k)v.$$

On the other hand,

$$\begin{aligned} R(k)\phi(v)(x) = f_s^v(xk) &= \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x)k)v \\ &= \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x))\sigma(k)v, \end{aligned}$$

which proves the identity. □

Proposition 6.4. *Assume σ is a genuine pseudospherical representation of M. Then $\mu_\sigma|_{K_\alpha} = m\sigma_{1/2} \oplus m'\sigma_{-1/2}$ for some integers m, m' when α is metaplectic, and $\mu_\sigma|_{K_\alpha} = m \cdot 1$ for some integer m when α is not metaplectic. Here $K_\alpha = \Phi_\alpha(\widetilde{\mathrm{SO}}(2))$.*

Proof. For each α, K_α is generated by $\exp(tZ_\alpha)$, $t \in \mathbb{R}$. By the definition at the beginning of Section 3, the eigenvalues of $\mu(\exp(tZ_\alpha))$ are $e^{\pm it/2}$ with multiplicities for α metaplectic, and 1 otherwise. On the other hand, for each $n \in \mathbb{Z}$, $\sigma_{n/2} : K_\alpha \to S^1$, $\exp(tZ_\alpha) \mapsto e^{-int/2}$ is a character of K_α, and $K_\alpha \cong S^1$ has no other characters. Thus $\mu_\sigma|_{K_\alpha}$ is a direct sum of $\sigma_{\pm 1/2}$ when α is metaplectic and 1 otherwise. □

Let $G_\alpha = \Phi_\alpha(\widetilde{\mathrm{SL}}(2, \mathbb{R})) \subset G$. Then $G_\alpha \cong \widetilde{\mathrm{SL}}(2, \mathbb{R})$ when α is metaplectic, and $G_\alpha \cong \mathrm{SL}(2, \mathbb{R})$ when α is not metaplectic. Let T_α be the image of the metaplectic torus of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and let N_α be the image of the unipotent radical of the standard parabolic subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Consider $Q = P \cup P w_\alpha P$, where $P = NT = NAM$ is a minimal parabolic subgroup of G. Then $U = N \cap w_\alpha N w_\alpha^{-1}$ is the unipotent radical of Q. We have $\delta_N(t) = \delta_U(t)\delta_{N/U}(t)$ for $t \in T$. In particular, taking $t \in T_\alpha$, we get $\delta_U(t) = 1$, hence $\delta_N(t) = \delta_{N/U}(t) = \delta_{N_\alpha}(t)$. Thus $\delta_N(t) = \delta_{N_\alpha}(t)$ for $t \in T_\alpha$.

Now we get to the main result of this paper; a similar result on double covers of p-adic groups can be found in [Loke and Savin 2010].

Theorem 6.5 (action of intertwining operators on pseudospherical K-types). *Let $M(w, s) : I(P, \sigma, s) \to I(P, \sigma, ws)$ be the intertwining map. Then $M(w, s) f_s^v = c(w, s) f_{ws}^v$ for some constant $c(w, s)$. Moreover, let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be the set of simple roots, and let $w_i = w_{\alpha_i}$. Then in the case when Φ is simply laced or of type G_2,*

$$c(w_i, s) = c_{1/2}(s_i) \quad \textit{for all } i.$$

Otherwise,

$$\begin{aligned} c(w_i, s) &= c_0(s_i) && \textit{when } \alpha_i \textit{ is short}, \\ c(w_i, s) &= c_{1/2}(s_i) && \textit{when } \alpha_i \textit{ is long}. \end{aligned}$$

Here, for $\nu \in \mathbb{C}$,

$$c_0(\nu) := \sqrt{\pi}\,\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}, \quad c_{1/2}(\nu) := \sqrt{\pi}\,\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}\right)}.$$

Proof. The idea is reduction to the SL_2 case.

The multiplicities of (μ_σ, V) in $I(P,\sigma,s)$ and $I(P,\sigma,ws)$ are both 1, hence $M(w,s)f_s^v = c(w,s)f_{ws}^v$ for some constant $c(w,s)$. Evaluating at $g=1$ on both sides, we get $M(w,s)f_s^v(1)=c(w,s)v$. For $w=w_i$, there is a map from $I(P,\sigma,s)$ to $I(\sigma,s_i)$ given by restricting functions on G to G_{α_i}, where $I(\sigma,s_i)$ is the space of functions $f: G_{\alpha_i} \to V$ such that $f(namx) = a^{s_i+1}\sigma(m)f(x)$ (here a stands for $h_{\alpha_i}(a)$). Since $N_{w_i} = N \cap w_i N w_i^{-1} \setminus N = N_{\alpha_i}$, $M(w_i,s)$ induces a map from $I(\sigma,s_i)$ to $I(\sigma,-s_i)$, and $f_s^v|_{G_{\alpha_i}}$ satisfies $f_s^v(nak) = a^{s_i+1}\mu_\sigma(k)v$ for $n \in N_{\alpha_i}$, $a \in A_{\alpha_i}$, $k \in K_{\alpha_i}$.

By Proposition 2.12, when Φ is simply laced or of type G_2, all roots are metaplectic By Proposition 6.4,

$$\mu_\sigma|_{K_{\alpha_i}} = m\sigma_{1/2} \oplus m'\sigma_{-1/2}$$

for some positive integers m, m'. Applying Proposition 5.7, we see that $c(w_i,s) = c_{1/2}(s_i)$.

Now assume Φ is of type B_n, C_n, or F_4. If α_i is long, then it is metaplectic, by the same argument as the paragraph above, and we have $c(w_i,s)=c_{1/2}(s_i)$; if α_i is short, then it is not metaplectic by Proposition 2.12. Hence by Proposition 6.4,

$$\mu_\sigma|_{K_{\alpha_i}} = m \cdot 1$$

for some positive integer m. Applying Proposition 5.7 again, $c(w_i,s)=c_0(s_i)$. □

Remark. We may write any $w \in W$ as a reduced product of simple reflections: $w = w_1 w_2 \cdots w_n$. Then by Proposition 4.3,

$$M(w,s) = M(w_1, w_2\cdots w_n s)M(w_2, w_3\cdots w_n s)\cdots M(w_{n-1}, w_n s)M(w_n,s),$$

which implies

$$c(w,s) = c(w_1, w_2\cdots w_n s)c(w_2, w_3\cdots w_n s)\cdots c(w_{n-1}, w_n s)c(w_n,s).$$

Define

$$\overline{M}(w,s) = \frac{M(w,s)}{c(w,s)}.$$

Then

$$\overline{M}(ww',s) = \overline{M}(w,w's)\circ\overline{M}(w',s)$$

for any $w, w' \in W$. These are called *normalized intertwining operators* and their composition law behaves like the Weyl group.

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