Pacific Journal of Mathematics

LOCAL SYMMETRIC SQUARE L-FACTORS OF REPRESENTATIONS OF GENERAL LINEAR GROUPS

SHUNSUKE YAMANA

Volume 286 No. 1 January 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 1, 2017
dx.doi.org/10.2140/pjm.2017.286.215

LOCAL SYMMETRIC SQUARE L-FACTORS OF REPRESENTATIONS OF GENERAL LINEAR GROUPS

SHUNSUKE YAMANA

This paper develops a theory of local symmetric square L-factors of representations of general linear groups. We will prove a certain characterization of a pole of symmetric square L-factors of square-integrable representations, the uniqueness of certain trilinear forms and the nonexistence of Whittaker models of higher exceptional representations.

Introduction

The purpose of this paper is to elaborate on the Rankin–Selberg construction of the twisted symmetric square L-functions of general linear groups, developed in [Bump and Ginzburg 1992; Takeda 2014]. We will mainly focus on the local aspects here.

Fix an integer $n \geq 2$. The setup involves an exceptional representation θ of an appropriate double cover $\overline{G}$ of a general linear group $G = G_n = \mathrm{GL}_n(F)$ over a nonarchimedean local field F of characteristic zero. This rather mysterious representation, which is the smallest genuine representation of this covering group in many senses, was first constructed in generality by Kazhdan and Patterson [1984].

We can associate to each representation φ of the Weil–Deligne group WD_F of F the local L-factor $L(s, \varphi)$ of Artin type. Let sym^2 and Λ^2 be the symmetric and exterior square representations of $\mathrm{GL}_n(\mathbb{C})$. Given an irreducible admissible representation π of G, we can define its local symmetric and exterior square L-factors as $L(s, \mathrm{sym}^2 \circ \phi(\pi))$ and $L(s, \Lambda^2 \circ \phi(\pi))$, where ϕ stands for the local Langlands correspondence between irreducible admissible representations of G and n-dimensional representations of WD_F.

MSC2010: primary 11F66; secondary 11F70.
Keywords: symmetric square L-factors, exceptional representations, distinguished representations.

The factorization

$$L(s,\phi(\pi)\otimes\phi(\pi))=L(s,\Lambda^2\circ\phi(\pi))L(s,\mathrm{sym}^2\circ\phi(\pi)) \tag{0-1}$$

is an easy consequence of the Langlands formalism. Assume that π is an irreducible square-integrable self-dual representation of G. Then $L(s,\phi(\pi)\otimes\phi(\pi))$ has a simple pole at $s=0$, and hence exactly one of the symmetric or exterior square L-factors of π has a pole at $s=0$.

It is known that $L(s,\Lambda^2\circ\phi(\pi))$ has a pole at $s=0$ only if n is even. Let ψ be a nontrivial additive character of F. When n is even, the L-factor $L(s,\Lambda^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if π admits a nonzero linear form λ on π which satisfies

$$\lambda\left(\pi\left(\begin{bmatrix}h & hX\\ 0 & h\end{bmatrix}\right)v\right)=\psi(\mathrm{tr}(X))\lambda(v)$$

for all $v\in\pi$, $h\in G_{n/2}$ and $X\in \mathrm{M}_{n/2}(F)$ (see [Kewat and Raghunathan 2012; Kewat 2011; Lapid and Mao 2017]). A linear form with this property is called a Shalika functional. As is well known, the space of Shalika functionals on any irreducible admissible representation is at most one-dimensional (see [Jacquet and Rallis 1996]).

We will prove analogous results for symmetric square L-factors. We call π distinguished if there is a nonzero G-invariant linear functional on $\pi\otimes\theta\otimes\theta^\vee$. The following theorem, which is a special case of Theorem 3.19, indicates that this notion of distinction is closely connected with the symmetric square L-factor.

Theorem A. *Let π be an irreducible admissible square-integrable representation of G with central character ω_π. Then $L(s,\mathrm{sym}^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if $\omega_\pi^2=1$ and $\pi\otimes\omega_\pi$ is distinguished.*

It should be noted that if n is even, π is distinguished and $\chi^2=1$, then $\omega_\pi^2=1$ and $\pi\otimes\chi$ is distinguished (see Lemma 1.12). Thus in the case of even n the L-factor $L(s,\mathrm{sym}^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if π is distinguished. Notice that $L(s,\mathrm{sym}^2\circ\phi(\pi\otimes\chi))=L(s,\mathrm{sym}^2\circ\phi(\pi))$.

As with many L-factors, the symmetric square L-factor may currently be defined not only by the local Langlands correspondence, but also via integral representations or through analysis of Fourier coefficients of Eisenstein series. Henniart [2010] has shown that the first and third definitions agree. We will define the symmetric square L-factor of irreducible admissible generic representations via the integral representation (see Definitions 3.10 and 3.12) and show that this approach gives the same L-factor at least for square-integrable representations (see Theorem 3.18).

Now the following corollary can trivially be deduced from Theorem A and the relevant result for $L(s,\Lambda^2\circ\phi(\pi))$, alluded to above.

Corollary A. *Let π be an irreducible square-integrable representation of G with central character ω_π.*

(1) *Assume that n is odd. Then π is distinguished if and only if ω_π is trivial and π is self-dual.*

(2) *Assume that n is even and ω_π is nontrivial. Then π is distinguished if and only if π is self-dual.*

(3) *Assume that n is even and ω_π is trivial. Then π is self-dual if and only if either a nonzero G-invariant linear functional on $\pi \otimes \theta \otimes \theta^\vee$ or a nonzero Shalika functional on π exists. Moreover, if one of the two functionals exists, then the other does not.*

The following theorem is included in Theorem 2.14.

Theorem B. *If π is an irreducible admissible unitary representation of G, then the space of G-invariant linear functionals on $\pi \otimes \theta \otimes \theta^\vee$ is at most one-dimensional.*

The unitarity assumption is expected to be unnecessary. Sun [2012] proved uniqueness of another trilinear form. Our proof of Theorem B is a refinement of the proof of the generic uniqueness in [Kable 2001, Theorem 6.1], combined with the same idea as in the proof of [Matringe 2014, Proposition 2.3]. Though the hypothesis is essential to this method, we can prove a somewhat stronger uniqueness, which is entirely analogous to the well-known theorem of Bernstein [1984] and its twisted analogue [Ok 1997] (cf. Remark 2.15(1) and [Anandavardhanan et al. 2004; Matringe 2014]).

Since $\bar{G}$ has a subgroup N, which is isomorphic to the group of upper unitriangular matrices of G, we can consider Jacquet modules, Whittaker models and derivatives of representations of $\bar{G}$.

Theorem C. *If $n \geq 3$, then the exceptional representations of $\bar{G}$ carry no Whittaker functionals.*

This result has been proved by Kazhdan and Patterson for nonarchimedean local fields of odd residual characteristic (see Theorem I.3.5 of [Kazhdan and Patterson 1984]). When $n = 3$, this is Lemma 6 of [Flicker et al. 1990]. We will give a different proof which covers the dyadic case. Eyal Kaplan indicated another proof, which uses Lemma 6 of [Flicker et al. 1990] together with induction. It is noteworthy that our proof covers the twisted case as well.

Theorem C completes the computation of derivatives of the exceptional representations. For all nonarchimedean local fields of characteristic zero, the first derivative has been computed by Kable [2001], and the second derivative has been considered by Bump and Ginzburg [1992]. Theorem C combined with the periodicity (see Theorem 5.1 of [Kable 2001]) implies that the third and higher derivatives of the exceptional representations are zero.

Our proof of Theorem A uses a local functional equation, which is a direct consequence of the generic uniqueness, and a stronger uniqueness result given in Theorem 2.14(2). The proofs of these uniqueness results rely upon the knowledge of derivatives of the exceptional representations. The local functional equation and uniqueness principle have not been previously discussed in the dyadic case because of a gap in this knowledge for the exceptional representations over dyadic fields. One of the contributions of this paper is to remove this restriction.

Takeda [2014] has recently constructed twisted exceptional representations and generalized the Rankin–Selberg integral to represent the twisted symmetric square L-functions. In the case of even n the results described so far except for Corollary A will be proved for twisted symmetric square L-factors and twisted exceptional representations (cf. Remark 3.20). When n is odd, we will discuss the symmetric square L-factors without twisting. In order to deal with the twisted case, we only have to analyze the representation of $\overline{G}$ induced from a twisted exceptional representation of $\overline{G}_{n-1}$. Though this analysis is not very difficult, if somewhat involved, we think that our formulation keeps our exposition a reasonable length and sufficient for future applications (cf. Theorem 3.19).

1. Exceptional representations

In this section we aim to review those properties of the exceptional representations that will be required below. Since the proper home for the exceptional representation is not really $\mathrm{GL}_r(F)$, but rather its covering group, we begin this section by recalling some relevant facts from the theory of the covering groups.

1A. ***Notation.*** The notation introduced here will be used constantly in later sections. Throughout, F will be a local field of characteristic 0. We write $|x|$ for the normalized absolute value of an element x of F. There is a quadratic Hilbert symbol $(\ ,\ )$ on $F^\times \times F^\times$ which takes values in $\mu_2 = \{\pm 1\}$. This symbol is symmetric and bimultiplicative, and its left kernel is the subgroup $F^{\times 2}$ of squares in $F^\times$. In the nonarchimedean case the symbols $\mathfrak{o}$ and q will denote, respectively, the ring of integers of F and the cardinality of the residue field of F.

By a character of a locally compact group H we mean any continuous homomorphism of H into $\mathbb{C}^\times$.

Definition 1.1. A character χ of $F^\times$ is said to be unitary (resp. quadratic, even, odd) if $\chi(a)$ is a complex number of modulus 1 for every $a \in F^\times$ (resp. $\chi^2 = 1$, $\chi(-1) = 1$, $\chi(-1) = -1$). When $a \in F^\times$, we define a quadratic character χ_a of $F^\times$ by $\chi_a(b) = (a, b)$ for $b \in F^\times$.

For each positive integer r, we denote by $G_r = \mathrm{GL}_r(F)$ the group of invertible matrices of size r, by T_r its subgroup of diagonal matrices, by B_r its subgroup of upper triangular matrices, by N_r its subgroup of upper triangular matrices with unit diagonal, by Z_r its subgroup of scalar matrices, by $\mathscr{P}_r$ its subgroup consisting of matrices whose last row is $(0, 0, \ldots, 0, 1) \in F^r$ and by $\mathscr{Y}_r$ the unipotent radical of $\mathscr{P}_r$. Put $\mathcal{P}_r = Z_r \cdot \mathscr{P}_r$. We denote the group of permutation matrices in G_r by W_r and identify it with the Weyl group of G_r. For a representation π of G_r we will denote its central character, if it exists, by ω_π unless otherwise mentioned.

We fix a maximal compact subgroup K_r of G_r. Let $K_r = \mathrm{GL}_r(\mathfrak{o})$ in the p-adic case. When $m < r$, we shall systematically regard G_m as a subgroup of G_r via the embedding into the upper left corner. We here allow the specific case $m = 0$ so that G_0 is the identity group. For a parabolic subgroup P of G_r we denote by δ_P the modulus function of P and extend it to the right K_r-invariant function on G_r.

By a standard parabolic subgroup of G_r we shall mean a parabolic subgroup of G_r which contains B_r. A composition of r is an ordered partition of r. To such a composition $\boldsymbol{r} = (r_1, \ldots, r_k)$ of r, we associate the standard parabolic subgroup $P_{\boldsymbol{r}} = M_{\boldsymbol{r}} U_{\boldsymbol{r}}$ of G_r, where $U_{\boldsymbol{r}}$ is the unipotent radical of $P_{\boldsymbol{r}}$ and the group $M_{\boldsymbol{r}} = G_{r_1} \times \cdots \times G_{r_k}$, regarded as embedded in the natural way as a block-diagonal subgroup of G_r, is a Levi subgroup of $P_{\boldsymbol{r}}$.

We define the subgroup $G_r^{\square}$ of G_r by

$$G_r^{\square} = \{g \in G_r \mid \det g \in F^{\times 2}\}.$$

Further we define the subgroup $M_{\boldsymbol{r}}^{\square}$ of $M_{\boldsymbol{r}}$ by

$$M_{\boldsymbol{r}}^{\square} = \left\{\mathrm{diag}[m_1, \ldots, m_k] \in M_{\boldsymbol{r}} \mid m_i \in G_{r_i}^{\square} \text{ for } i = 1, 2, \ldots, k\right\}.$$

Put

$$\mathscr{Z}_r = \{z^{e(r)} \mid z \in Z_r\},$$

where $e(r)$ is 1 or 2 according to whether r is odd or even. Set

$$\mathscr{T}_r = \left\{t \in T_r \mid t_{r-2i+1} t_{r-2i+2}^{-1} \in F^{\times 2} \text{ for } i = 1, 2, \ldots, \left[\tfrac{r}{2}\right]\right\},$$

writing a diagonal matrix $t \in T_r$ in the form $\mathrm{diag}[t_1, t_2, \ldots, t_r]$. We define the two compositions of r by

$$\boldsymbol{e}(r) = (2, 2, 2, \ldots, 2, 2), \quad \boldsymbol{o}(r) = (1, 2, 2, \ldots, 2, 1)$$

if r is even, and by

$$\boldsymbol{e}(r) = (1, 2, 2, \ldots, 2, 2), \quad \boldsymbol{o}(r) = (2, 2, 2, \ldots, 2, 1)$$

if r is odd. Lastly, we define the subgroup $\mathscr{M}_r$ of $M_{\boldsymbol{e}(r)}$ by $\mathscr{M}_r = Z_r \cdot M_{\boldsymbol{e}(r)}^{\square}$.

1B. ***The double covers of general linear groups.*** A central double covering $p_r : \overline{G}_r \to G_r$ corresponds in the usual way to a class in the cohomology group $H^2(G_r, \mu_2)$, where G_r acts trivially on the coefficients μ_2, and choosing a cocycle to represent this class is equivalent to choosing a section $\boldsymbol{s}_r : G_r \to \overline{G}_r$ of the map p_r. We shall choose $\boldsymbol{s}_r$ in such a way that the resulting cocycle σ_r agrees with the one constructed by Banks, Levy and Sepanski in [Banks et al. 1999, Section 3]. Let μ_2 inject into the center of $\overline{G}_r$. Then we can write typical elements of $\overline{G}_r$ uniquely in the form $\zeta \boldsymbol{s}_r(g)$ for $g \in G_r$ and $\zeta \in \mu_2$. The composition rule is given by

$$\zeta \boldsymbol{s}_r(g) \cdot \zeta' \boldsymbol{s}_r(g') = \zeta\zeta' \sigma_r(g, g') \boldsymbol{s}_r(gg') \quad (g, g' \in G_r,\ \zeta, \zeta' \in \mu_2).$$

The 2-cocycles $\{\sigma_r\}_{r=1}^{\infty}$ are well behaved with respect to restriction and satisfy a nice block formula on all standard Levi subgroups, i.e., if $r = r_1 + \cdots + r_k$ and $g_i, g_i' \in G_{r_i}$ for $i = 1, 2, \dots, k$, then

$$\sigma_r\left[\begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{bmatrix}, \begin{bmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{bmatrix}\right] = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g_i') \prod_{j<l} (\det g_j, \det g_l').$$

The 2-cocycle σ_1 is trivial and σ_2 is the Kubota 2-cocycle on G_2.

For any subgroup H of G_r we write $\widetilde{H}$ for its preimage $p_r^{-1}(H)$. An irreducible admissible representation of $\widetilde{H}$ is said to be genuine if it does not descend to a representation of H. Since the restriction of σ_r to any copy of G_{r_i} embedded along the diagonal in G_r agrees with the 2-cocycle σ_{r_i}, we can naturally identify $\widetilde{G}_{r_i}$ with $\overline{G}_{r_i}$. The block-compatibility of σ_r guarantees that the map

$$(\zeta_1 \boldsymbol{s}_{r_1}(g_1), \dots, \zeta_k \boldsymbol{s}_{r_k}(g_k)) \mapsto (\zeta_1 \cdots \zeta_k) \boldsymbol{s}_r(\operatorname{diag}[g_1, \dots, g_k])$$

is a surjective group homomorphism $\widetilde{G}_{r_1}^{\square} \times \cdots \times \widetilde{G}_{r_k}^{\square} \to \widetilde{M}_{\boldsymbol{r}}^{\square}$, which gives the decomposition

$$\text{(1-1)} \quad \widetilde{M}_{\boldsymbol{r}}^{\square} \simeq \widetilde{G}_{r_1}^{\square} \times \widetilde{G}_{r_2}^{\square} \times \cdots \times \widetilde{G}_{r_k}^{\square} \big/ \big\{(\zeta_1, \zeta_2, \dots, \zeta_k) \mid \zeta_i \in \mu_2,\ \zeta_1\zeta_2\cdots\zeta_k = 1\big\}.$$

Remark 1.2. (1) The center of $\overline{G}_r$ is $\widetilde{\mathscr{Z}}_r$.

(2) The center of $\widetilde{T}_r$ is $\widetilde{\mathscr{Z}}_r \widetilde{T}_r^{\square}$.

(3) The preimage $\widetilde{\mathscr{T}}_r$ is a maximal abelian subgroup of $\widetilde{T}_r$.

(4) It is known that

$$\sigma_r(ugu', g'u'') = \sigma_r(g, u'g') \quad (g, g' \in G_r,\ u, u', u'' \in N_r).$$

In particular, the restriction of $\boldsymbol{s}_r$ to N_r is a group homomorphism, by which we view subgroups of N_r as those of $\overline{G}_r$. If P is a standard parabolic subgroup of G_r with unipotent radical U, then

$$\tilde{p} \boldsymbol{s}_r(u) \tilde{p}^{-1} = \boldsymbol{s}_r(p_r(\tilde{p}) u p_r(\tilde{p})^{-1}) \quad (u \in U,\ \tilde{p} \in \widetilde{P}).$$

If F is nonarchimedean, then there are an open subgroup $\mathcal{K}_r$ of K_r and a map $\kappa_r : K_r \to \mu_2$ such that $k \mapsto \kappa_r(k)\boldsymbol{s}_r(k)$ is a group homomorphism from $\mathcal{K}_r$ to $\overline{G}_r$ by Proposition 0.1.2 of [Kazhdan and Patterson 1984]. The topology of $\overline{G}_r$ as a locally compact group is determined by this embedding. If the residual characteristic of F is odd, then we can take $\mathcal{K}_r = K_r$. The splitting $\mathcal{K}_r \to \overline{G}_r$ is not unique. We shall fix what Kazhdan and Patterson refer to as the canonical lift of K_r to $\overline{G}_r$ (see [Kazhdan and Patterson 1984, Proposition 0.I.3]).

1C. *Lifts of the main involution.* When φ is an automorphism of G_r, a lift of φ to $\overline{G}_r$ is an automorphism $\tilde{\varphi}$ of $\overline{G}_r$ such that $\tilde{\varphi}(\zeta) = \zeta$ and $p_r(\tilde{\varphi}(\tilde{g})) = \varphi(p_r(\tilde{g}))$ for all $\zeta \in \mu_2$ and $\tilde{g} \in \overline{G}_r$. The lift of any topological automorphism of G_r to $\overline{G}_r$ is a topological automorphism by Corollary 1 of [Kable 1999]. We consider a lift of the automorphism $g \mapsto {}^{\iota}g$ of G_r defined by ${}^{\iota}g = w_0^{(r)}\,{}^{t}g^{-1}w_0^{(r)}$, where ${}^{t}g$ is the transpose of the matrix g and $w_0^{(r)} \in W_r$ is the longest element.

Proposition 1.3 [Kable 1999]. *There exists a lift $\tilde{g} \mapsto {}^{\iota}\tilde{g}$ of the automorphism $g \mapsto {}^{\iota}g$ to $\overline{G}_r$ satisfying*

$$ {}^{\iota}\boldsymbol{s}_r(t) = \boldsymbol{s}_r({}^{\iota}t)\prod_{i>j}(t_i, t_j), \quad {}^{\iota}\tilde{z} = \tilde{z}^{-1}, \quad {}^{\iota}({}^{\iota}\tilde{g}) = \tilde{g}, \quad {}^{\iota}\boldsymbol{s}_r(u) = \boldsymbol{s}_r({}^{\iota}u) $$

for all $t = \mathrm{diag}[t_1, \dots, t_r] \in T_r$, $\tilde{z} \in \widetilde{\mathscr{Z}}_r$, $\tilde{g} \in \overline{G}_r$ and $u \in N_r$. All lifts are of the form $\tilde{g} \mapsto \varrho(\det p_r(\tilde{g}))\,{}^{\iota}\tilde{g}$, where ϱ is an arbitrary quadratic character of $F^{\times}$. Moreover, if the residual characteristic of F is odd and $f : K_r \to \overline{G}_r$ is a homomorphism, then $f({}^{\iota}k) = {}^{\iota}f(k)$ for all $k \in K_r$.

Proof. Kable has determined the lifts of the main involution and proved their basic properties. However, we need to keep track of his computations, using the cocycle defined in [Banks et al. 1999]. To that end, we recall how our cocycle σ_r is constructed. Put $\mathbb{G}_k = \mathrm{SL}_k(F)$ and define the embedding of G_r into $\mathbb{G}_{r+1}$ by $\jmath_r(g) = \mathrm{diag}[g, (\det g)^{-1}]$. There is a double cover $\bar{\mathbb{G}}_k$ of $\mathbb{G}_k$ by a theorem of Matsumoto [1969]. Banks, Levy and Sepanski [Banks et al. 1999] defined an explicit cocycle τ_k that represents the cohomology class of this cover and defined σ_r by

$$ \sigma_r(g, g') = \tau_{r+1}(\jmath_r(g), \jmath_r(g'))(\det g, \det g'). \tag{1-2} $$

The cocycle τ_{r+1} satisfies

$$ \tau_{r+1}(u, u') = (\det t, \det t')\prod_{1\le i<j\le r}(t_i, t'_j) = \prod_{r\ge i\ge j\ge 1}(t_i, t'_j) = \prod_{r+1\ge i\ge j\ge 1}(u_i, u'_j) $$

for $t = \mathrm{diag}[t_1, \dots, t_r]$ and $t' = \mathrm{diag}[t'_1, \dots, t'_r]$ by the block-compatibility of σ_r. Here we write

$$ u = \jmath_r(t) = \mathrm{diag}[u_1, \dots, u_{r+1}], $$

and similarly for $u' = \jmath_r(t')$. Kable chooses a cocycle on G_{r+1} which agrees with τ_{r+1} on the torus (see [Kable 1999, (3)]) and defines his cocycle on G_r by the relation [Kable 1999, (4)]. When $m = 0$ and $A = \mu_2$, it is the same as (1-2). Since he does not impose any other condition on his cocycle, we can apply all of his results to our cocycle.

Finally, we prove the last statement. We can define a quadratic character $\varrho_0 : K_r \to \mu_2$ by $\varrho_0(k) = {}^{\iota}f(k)f({}^{\iota}k)^{-1}$ for $k \in K_r$. Since $\mathrm{SL}_r(\mathfrak{o})$ is a perfect group, there is a quadratic character $\varrho_1 : \mathfrak{o}^\times \to \mu_2$ such that $\varrho_0(k) = \varrho_1(\det k)$ for all $k \in K_r$. Similarly, there is a quadratic character $\varrho_2 : \mathfrak{o}^\times \to \mu_2$ such that $f(k) = \varrho_2(\det k)\kappa_r(k)\boldsymbol{s}_r(k)$ for all $k \in K_r$. If $k \in K_r \cap T_r$, then $\kappa_r(k) = 1$ by (1.6) of [Takeda 2014], and

$$\varrho_1(\det k) = \varrho_0(k) = {}^{\iota}(\varrho_2(\det k)\boldsymbol{s}_r(k))(\varrho_2(\det {}^{\iota}k)\boldsymbol{s}_r({}^{\iota}k))^{-1} = {}^{\iota}\boldsymbol{s}_r(k)\boldsymbol{s}_r({}^{\iota}k)^{-1} = 1.$$

Therefore ϱ_1 must be trivial and hence ϱ_0 is trivial. □

Let π be a representation of $\widetilde{H}$. Taking a preimage $\tilde{g}$ of $g \in G_r$ in $\overline{G}_r$, we define the representation ${}^g\pi$ of ${}^g\widetilde{H} = \tilde{g}\widetilde{H}\tilde{g}^{-1}$ by ${}^g\pi(\tilde{h}) = \pi(\tilde{g}^{-1}\tilde{h}\tilde{g})$ for $\tilde{h} \in {}^g\widetilde{H}$, where conjugation is independent of the choice of $\tilde{g}$. We define a subgroup ${}^{\iota}\widetilde{H}$ of $\overline{G}_r$ by ${}^{\iota}\widetilde{H} = \{{}^{\iota}\tilde{h} \mid \tilde{h} \in \widetilde{H}\}$ and define a representation ${}^{\iota}\pi$ of ${}^{\iota}\widetilde{H}$ on the same space by ${}^{\iota}\pi(\tilde{h}) = \pi({}^{\iota}\tilde{h})$. If f is a function on $\widetilde{H}$, then we define a function ${}^{\iota}f$ on ${}^{\iota}\widetilde{H}$ by ${}^{\iota}f(\tilde{h}) = f({}^{\iota}\tilde{h})$ for $\tilde{h} \in {}^{\iota}\overline{H}$. If H is a subgroup of $M_{\boldsymbol{r}}$ containing $M_{\boldsymbol{r}}^{\square}$, where $\boldsymbol{r}$ is a composition of r, then H normalizes $U_{\boldsymbol{r}}$ in view of Remark 1.2(4) and we can construct, out of its pull-back to $\widetilde{H}U_{\boldsymbol{r}}$, the induced representation $\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi$. Here the induction is normalized in order that $\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi$ is unitarizable whenever π is unitarizable. Observe that ${}^{\iota}\delta_{P_{\boldsymbol{r}}} = \delta_{{}^{\iota}P_{\boldsymbol{r}}}$ and ${}^{\iota}P_{\boldsymbol{r}} = P_{\overleftarrow{\boldsymbol{r}}}$, where $\overleftarrow{\boldsymbol{r}} = (r_k, r_{k-1}, \dots, r_1)$. Note that $f \mapsto {}^{\iota}f$ gives a $\overline{G}_r$-equivariant isomorphism

$$\text{(1-3)} \qquad {}^{\iota}\big(\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi\big) \simeq \mathrm{Ind}_{{}^{\iota}\widetilde{H}\,{}^{\iota}U_{\boldsymbol{r}}}^{\overline{G}_r}{}^{\iota}\pi.$$

1D. ***The Weil representations of $\widetilde{G}_2^{\square}$.*** The Weil representation of $\overline{G}_2$ can be identified as the original example of the exceptional representation. Fix a nontrivial additive character ψ of F. Put $\mu_\psi(a) = \gamma(\psi_a)/\gamma(\psi)$ for $a \in F^\times$, where $\psi_a(x) = \psi(ax)$ and $\gamma(\psi)$ is the Weil constant associated to ψ. Recall that

$$\mu_\psi(ab) = \mu_\psi(a)\mu_\psi(b)(a,b), \quad \mu_\psi(ab^2) = \mu_\psi(a)$$

for $a, b \in F^\times$.

We will denote the space of Schwartz functions in k variables by $\mathscr{S}(F^k)$. For $x \in F^k$ we define the $\mathbb{C}$-linear functional e_x on $\mathscr{S}(F^k)$ by $e_x(\Phi) = \Phi(x)$. The Weil representation Ω^ψ associated to ψ is a genuine representation of the metaplectic double cover of $\mathrm{SL}_2(F)$ realized on the space $\mathscr{S}(F)$. The explicit action of the

Borel subgroup of $\mathrm{SL}_2(F)$ is given by

$$\Omega^\psi\left[\boldsymbol{s}_2\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right]\right]\Phi(x)=\mu_\psi(a)|a|^{1/2}\Phi(xa), \tag{1-4}$$

$$\Omega^\psi\left[\boldsymbol{s}_2\left[\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\right]\right]\Phi(x)=\psi(bx^2)\Phi(x) \tag{1-5}$$

for $\Phi\in\mathscr{S}(F)$, $a\in F^\times$ and $b,x\in F$. It is well known that Ω^ψ is reducible and written as the direct sum $\Omega^\psi=\Omega^\psi_1\oplus\Omega^\psi_{-1}$, where Ω^ψ_1 (resp. Ω^ψ_{-1}) is an irreducible representation realized in the space of even (resp. odd) Schwartz functions in one variable. For a character ϱ of $F^\times$ one can extend $\Omega^\psi_{\varrho(-1)}$ to an irreducible representation Ω^ψ_ϱ of $\widetilde{G}^\square_2$ by setting

$$\Omega^\psi_\varrho(\boldsymbol{s}_2(a\mathbf{1}_2))=\varrho(a)\mu_\psi(a) \tag{1-6}$$

for $a\in F^\times$. When ϱ is trivial, we will sometimes write $\Omega^\psi_+=\Omega^\psi_\varrho$. For $a\in F^\times$ we put $d(a)=\mathrm{diag}[a,1]\in G_2$.

Proposition 1.4. *Let ϱ be a character of $F^\times$.*

1. *If $a\in F^\times$, then ${}^{d(a^{-1})}\Omega^\psi_\varrho\simeq\Omega^{\psi_a}_\varrho$.*
2. *The representation $\mathrm{Ind}^{\widetilde{G}_2}_{\widetilde{G}^\square_2}\Omega^\psi_\varrho$ is irreducible and its equivalence class is independent of ψ.*
3. *If $\varPhi\in\mathrm{Ind}^{\widetilde{G}_2}_{\widetilde{G}^\square_2}\Omega^\psi_\varrho$ and $e_1(\varPhi(\tilde p))=0$ for all $\tilde p\in\widetilde{\mathscr{P}}_2$, then $\varPhi=0$.*

Proof. We will prove only the last part, for the other results are recalled or derived in Section 2.2 of [Takeda 2014]. By (1-4), (1-6) and the assumption on $\varPhi$,

$$\begin{aligned}0&=e_1\big(\varPhi(\boldsymbol{s}_2(d(a^2))\tilde p)\big)\\&=e_1\big(\Omega^\psi_\varrho(\boldsymbol{s}_2(d(a^2)))\varPhi(\tilde p)\big)\\&=(a,-1)\varrho(a)\mu_\psi(a)^2|a|^{1/2}e_a(\varPhi(\tilde p))\end{aligned}$$

for all $a\in F^\times$ and $\tilde p\in\widetilde{\mathscr{P}}_2$. Therefore $e_a(\varPhi(\tilde p))=0$ for all $a\in F^\times$, and so in view of continuity, $e_a(\varPhi(\tilde p))=0$ for all $a\in F$. Bear in mind that $\varPhi$ is a $\mathscr{S}(F)$-valued function on $\widetilde{G}_2$. We conclude that $\varPhi(\tilde p)=0$ for all $\tilde p\in\widetilde{\mathscr{P}}_2$. Since $G_2=G^\square_2\cdot\mathscr{P}_2$, we conclude that $\varPhi=0$. ☐

1E. ***Exceptional representations.*** We can define a genuine character ξ^ψ_r of $\widetilde{\mathscr{T}}_r$ by

$$\xi^\psi_r(\boldsymbol{s}_r(t))=\prod_{i=0}^{[r/2]-1}\mu_\psi(t_{r-2i})^{-1}.$$

The exceptional representation θ_r^ψ is the unique irreducible subrepresentation of

$$\mathscr{I}_r^\psi = \mathrm{Ind}_{\widetilde{\mathscr{T}}_r N_r}^{\bar{G}_r} \xi_r^\psi \otimes \delta_{B_r}^{-1/4}$$

(see Theorem I.2.9 of [Kazhdan and Patterson 1984]). Next we recall Takeda's construction [2014] of the twisted exceptional representations.

Definition 1.5. Fix a positive integer r and a character χ of $F^\times$. In light of (1-1) and Remark 1.2(1) we can define the genuine representation $\Upsilon_{r,\chi}^\psi$ of $\widetilde{\mathscr{M}}_r$ to be the tensor product

$$\Upsilon_{r,\chi}^\psi = (\xi_r^\psi|_{\widetilde{Z}_r}) \boxtimes \Omega_+^{\psi^{-1}} \boxtimes \cdots \boxtimes \Omega_+^{\psi^{-1}} \quad \text{or} \quad \Upsilon_{r,\chi}^\psi = \Omega_\chi^{\psi^{-1}} \boxtimes \cdots \boxtimes \Omega_\chi^{\psi^{-1}}$$

according to whether r is odd or even. Put

$$I_{r,\chi}^\psi = \mathrm{Ind}_{\widetilde{\mathscr{M}}_r U_{e(r)}}^{\bar{G}_r} \Upsilon_{r,\chi}^\psi \otimes \delta_{P_{e(r)}}^{-1/4}.$$

By the Langlands theorem [Ban and Jantzen 2013] the representation $I_{r,\chi}^\psi$ has a unique irreducible subrepresentation, which we denote by $\theta_{r,\chi}^\psi$. Exceptional representations of $\bar{G}_r$ are twists of these representations $\theta_{r,\chi}^\psi$ by characters of $F^\times$.

Remark 1.6. Proposition 1.4(2) implies that the equivalence class of $\theta_{r,\chi}^\psi$ is independent of ψ whenever r is even. We will sometimes suppress the superscript ψ and write $\theta_{r,\chi} = \theta_{r,\chi}^\psi$ when r is even.

Remark 1.7. Whenever r is odd, the representation $\theta_{r,\chi}^\psi$ is defined independently of χ contrary to what one might guess from the notation. If χ is trivial, then by (1-4), (1-5), (1-6) and the invariant distribution theorem, the map $\Phi \mapsto e_0 \circ \Phi$ gives a $\bar{G}_r$-intertwining embedding $I_{r,\chi}^\psi \hookrightarrow \mathscr{I}_r^\psi$ and hence $\theta_r^\psi \simeq \theta_{r,\chi}^\psi$. We may therefore omit the subscript χ from the notation either if r is odd or if χ is trivial. In view of Remark 1.6 we may write θ_r when r is even and χ is trivial. We trust this will cause no confusion.

A little more generally, we assume that χ is even. Then we can define a character ϱ of $F^{\times 2}$ by $\varrho(a^2) = \chi(a)$ for $a \in F^\times$. We extend ϱ to a character of $F^\times$ and denote it also by ϱ. If r is even, then since the map $\Phi \mapsto e_0 \circ \Phi$ gives a $\bar{G}_r$-intertwining embedding $I_{r,\chi}^\psi \hookrightarrow \mathscr{I}_r^\psi \otimes \varrho$, we conclude $\theta_r \simeq \theta_{r,\chi} \otimes \varrho^{-1}$.

The notion of principal series representations of $\bar{G}_r$ is introduced in Section 1.1 of [Kazhdan and Patterson 1984]. The following result is an easy consequence of an analogue of the Stone–von Neumann theorem, which states that the genuine irreducible representations of the two-step nilpotent group $\widetilde{T}_r$ are parametrized by the genuine characters of its center $\widetilde{\mathscr{Z}}_r \widetilde{T}_r^\square$ (cf. [Kazhdan and Patterson 1984; Bump and Ginzburg 1992, Proposition 1.1]).

Lemma 1.8. *Let $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathcal{T}}_2$ be maximal abelian subgroups of $\widetilde{T}_r$. Let ξ_i be a genuine character of $\widetilde{\mathcal{T}}_i$. If the restrictions of ξ_1 and ξ_2 to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ coincide, then* $\operatorname{Ind}_{\widetilde{\mathcal{T}}_1 N_r}^{\overline{G}_r}\xi_1 \simeq \operatorname{Ind}_{\widetilde{\mathcal{T}}_2 N_r}^{\overline{G}_r}\xi_2$.

1F. ***Distinction by pairs of exceptional representations.***

Lemma 1.9. *Let χ and μ be characters of $F^\times$.*

(1) $(\theta_r^{\psi})^\vee \simeq \theta_r^{\psi^{-1}}$.

(2) *If r is odd and $a \in F^\times$, then* $\theta_r^{\psi_a} \simeq \theta_r^{\psi} \otimes \chi_a^{(r-1)/2}$.

(3) *If r is even, then* $\theta_{r,\chi}\otimes\mu \simeq \theta_{r,\chi\mu^2}$.

(4) ${}^{\iota}\theta_{r,\chi}^{\psi} \simeq \theta_{r,\chi^{-1}}^{\psi^{-1}}$.

Proof. We have $(\xi_r^{\psi})^{-1} = \xi_r^{\psi^{-1}}$ simply because $\mu_{\psi^{-1}} = \mu_{\psi}^{-1}$. Assertion (1) therefore follows from Theorem 5.1(5) of [Kable 2001].

Note that $\mu_{\psi_a} = \chi_a\cdot\mu_{\psi}$. The restrictions of $\xi_r^{\psi_a}$ and $\xi_r^{\psi}\cdot(\chi_a\circ\det)^{(r-1)/2}$ to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ agree when r is odd. Assertion (2) follows from Lemma 1.8.

Since $\Omega_\chi^{\psi}\otimes\mu \simeq \Omega_{\chi\mu^2}^{\psi}$ by definition, assertion (3) readily follows.

Finally, we will prove (4). First assume that χ is trivial. Since the restrictions of ${}^{\iota}\xi_r^{\psi}$ and $\xi_r^{\psi^{-1}}$ to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ coincide, we see that

$$ {}^{\iota}\mathscr{I}_r^{\psi} \simeq \operatorname{Ind}_{{}^{\iota}\widetilde{\mathscr{T}}_r N_r}^{\overline{G}_r} {}^{\iota}\xi_r^{\psi}\otimes\delta_{B_r}^{-1/4} \simeq \mathscr{I}_r^{\psi^{-1}} $$

by (1-3) and Lemma 1.8, from which assertion (4) follows.

Next assume that r is even. Since

$$ {}^{\iota}g = \begin{pmatrix}1&0\\0&-1\end{pmatrix} g \begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1} $$

for $g\in \mathrm{SL}_2(F)$, Proposition 1.3 shows that

$$ {}^{\iota}\tilde g = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \tilde g \begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1} $$

for all elements $\tilde g\in\overline{G}_2$ such that $\det p_2(\tilde g)=1$. Proposition 1.4(1) tells us that ${}^{\iota}\Omega^{\psi^{-1}} \simeq \Omega^{\psi}$ and so ${}^{\iota}\Omega_\chi^{\psi^{-1}} \simeq \Omega_{\chi^{-1}}^{\psi}$. If $g=\operatorname{diag}[g_1,\dots,g_{r/2}]\in M_{\boldsymbol{e}(r)}^{\square}$, then ${}^{\iota}g = w_r^{-1}\operatorname{diag}[{}^{\iota}g_1,\dots,{}^{\iota}g_{r/2}]w_r$, where the matrix w_r is defined in (2-1), and hence

$$ {}^{\iota}\Upsilon_{r,\chi}^{\psi} \simeq {}^{w_r}\Upsilon_{r,\chi^{-1}}^{\psi^{-1}} \simeq \Upsilon_{r,\chi^{-1}}^{\psi^{-1}} $$

(cf. Proposition 2.9 of [Takeda 2015]). □

We define the notion of distinction in our current setup. No subgroup of G_r appears, but the exceptional representations play the role of “restriction to the subgroup”.

Definition 1.10. We assume χ to be trivial whenever r is odd. Let π be an admissible representation of G_r. We say that π is χ-distinguished if there is a nonzero G_r-invariant linear form on $\pi \otimes \theta^{\psi}_{r,\chi} \otimes \theta^{\psi^{-1}}_r$. We say that π is distinguished if there is a nonzero G_r-invariant linear form on $\pi \otimes \theta^{\psi}_r \otimes \theta^{\psi^{-1}}_r$.

Remark 1.11. This notion of distinction is independent of the choice of ψ on account of Lemma 1.9(1)–(2) and Remark 1.6.

Lemma 1.12. *Let π be an irreducible admissible representation of G_r. Let χ be a character of $F^\times$.*

(1) *If r is odd and π is distinguished, then the central character ω_π of π is trivial and $\pi^\vee$ is distinguished.*

(2) *If r is even and π is χ-distinguished, then $\omega_\pi^2\chi^r$ is trivial, $\pi^\vee$ is χ^{-1}-distinguished and $\pi \otimes \mu$ is $\chi\mu^{-2}$-distinguished for all characters μ of $F^\times$.*

Remark 1.13. By Theorem 3.19, if π is square-integrable and χ-distinguished, then $\pi \simeq \pi^\vee \otimes \chi^{-1}$. It is expected that all irreducible admissible χ-distinguished representations π satisfy $\pi \simeq \pi^\vee \otimes \chi^{-1}$ (cf. [Flicker 1991, Proposition 12; Jacquet and Rallis 1996, Theorem 1.1, Proposition 6.1]).

Proof. For π to be χ-distinguished, the product of the three central characters must be trivial on $F^{\times e(r)}$ as $\widetilde{\mathscr{Z}}_r$ is the center of $\overline{G}_r$. This gives the stated conditions on ω_π (see Lemma 1.9(1) and (1-6)). We can easily deduce the remaining parts from the relevant properties of exceptional representations stated in Lemma 1.9(3)–(4). $\square$

1G. ***The intertwining operator.*** We will fix, once and for all, a positive integer $n \geq 2$ and write $G = G_n$ and $G' = G_{n-1}$. Put $\ell = \left[\frac{n}{2}\right]$. We embed G' into G via the map $h \mapsto \begin{pmatrix} h & \\ & 1 \end{pmatrix}$. We omit the subscript n and adapt the same notation adding a prime $'$ for G'; that is,

$$\mathscr{P} = \mathscr{P}_n, \qquad \mathscr{T} = \mathscr{T}_n, \qquad \mathscr{Z} = \mathscr{Z}_n, \qquad N = N_n, \qquad B' = B_{n-1},$$
$$T' = T_{n-1}, \qquad \mathscr{Z}' = \mathscr{Z}_{n-1}, \qquad \xi^\psi = \xi^\psi_n, \qquad \theta^\psi_\chi = \theta^\psi_{n,\chi}, \qquad \theta^\psi = \theta^\psi_n,$$

and so on.

For each character ϱ of $F^\times$ we define a genuine character ζ^ψ_ϱ of $\widetilde{\mathscr{Z}}$ by

$$\zeta^\psi_\varrho(\boldsymbol{s}(z\mathbf{1}_n)) = \varrho(z)^{-1}\mu_\psi(z)^\ell$$

for $z \in F^{\times e(n)}$. Then we can extend θ^ψ_{n-1} to the representation $\theta^\psi_{n-1} \boxtimes \zeta^\psi_\varrho$ of the semidirect product $(\overline{G}' \times \widetilde{\mathscr{Z}}) \ltimes \mathscr{Y}$ by letting $\widetilde{\mathscr{Z}}$ act by ζ^ψ_ϱ and letting $\mathscr{Y}$ act trivially. For $s \in \mathbb{C}$ we consider the induced representation

$$I_\psi(s, \varrho) = \mathrm{Ind}^{\overline{G}}_{\widetilde{\mathscr{Z}}\widetilde{\mathscr{P}}}(\theta^\psi_{n-1} \boxtimes \zeta^\psi_\varrho) \otimes \delta^{s/4}_{\mathcal{P}}.$$

We define the intertwining operator

$$M(s,\varrho): I_\psi(s,\varrho) \to J_\psi(-s,\varrho)$$

for $\Re s \gg 0$ by the integrals

$$M(s,\varrho) f^{(s)}(\tilde{g}) = \int_{\iota\mathscr{Y}} f^{(s)}(\boldsymbol{s}(\delta)^{-1} y \tilde{g})\, \mathrm{d}y$$

and by meromorphic continuation otherwise, where

$$J_\psi(s,\varrho) = \mathrm{Ind}_{\widetilde{\mathscr{Z}}\iota\widetilde{\mathscr{P}}}^{\overline{G}} ({}^{\delta}\theta_{n-1}^{\psi} \boxtimes \zeta_\varrho^\psi) \otimes \delta_{\iota\mathcal{P}}^{s/4}, \quad \delta = \begin{pmatrix} & 1 \\ \mathbf{1}_{n-1} & \end{pmatrix}. \tag{1-7}$$

The operator $M(s,\varrho)$ is holomorphic at $s = 1$ due to the analysis in Sections 4.5 and 4.6 of [Takeda 2015] (see Lemma 3.2).

Lemma 1.14. *If $\widetilde{\mathcal{T}}'$ is a maximal abelian subgroup of $\widetilde{T}'$, then $\widetilde{\mathcal{T}}'\widetilde{\mathscr{Z}}$ is a maximal abelian subgroup of $\widetilde{T}$.*

Proof. Suppose that $\tilde{t} \in \widetilde{T}$ commutes with all elements in $\widetilde{\mathscr{Z}}\widetilde{\mathcal{T}}'$. We can write $\tilde{t} = \boldsymbol{s}(z\mathbf{1}_n)\cdot\tilde{t}'$ ($z \in F^\times$, $\tilde{t}' \in \widetilde{T}'$). If n is odd, then $\widetilde{\mathscr{Z}} = \widetilde{Z}$ and hence $\tilde{t}'$ commutes with all elements in $\widetilde{\mathcal{T}}'$, so that $\tilde{t}' \in \widetilde{\mathcal{T}}'$. If n is even, then since $\widetilde{\mathcal{T}}'$ contains $\widetilde{\mathscr{Z}}' = \widetilde{Z}'$, we have

$$(z,z')^{(n-2)/2} = \sigma\left[z\mathbf{1}_n, \begin{bmatrix} z'\mathbf{1}_{n-1} & \\ & 1 \end{bmatrix}\right] = \sigma\left[\begin{bmatrix} z'\mathbf{1}_{n-1} & \\ & 1 \end{bmatrix}, z\mathbf{1}_n\right] = (z,z')^{n/2}$$

for all $z' \in F^\times$, so that z must be a square, and hence $\tilde{t}' \in \widetilde{\mathcal{T}}'$. □

Lemma 1.15. *Let ϱ be a quadratic character of $F^\times$. The representation $I_\psi(1,\varrho)$ has a unique irreducible quotient, which is isomorphic to $\theta^{\psi^{-1}} \otimes \varrho$. Moreover, the quotient map*

$$I_\psi(1,\varrho) \to \theta^{\psi^{-1}} \otimes \varrho$$

is realized as the intertwining operator $M(1,\varrho)$.

Proof. Let W and W' denote the Weyl groups of G and G', respectively. Let $w_0 \in W$ and $w_0' \in W'$ be the longest elements. Since θ_{n-1}^ψ is a quotient of the principal series representation

$$\mathrm{Ind}_{w_0'\widetilde{\mathcal{T}}'N'}^{\overline{G}'} {}^{w_0'}\xi_{n-1}^{\psi} \otimes \delta_{B'}^{1/4}$$

by Theorem I.2.9 of [Kazhdan and Patterson 1984], the representation $I_\psi(1,\varrho)$ is a quotient of

$$\mathrm{Ind}_{w_0'\widetilde{\mathcal{T}}'\widetilde{\mathscr{Z}}N}^{\overline{G}} ({}^{w_0'}\xi_{n-1}^{\psi} \boxtimes \zeta_\varrho^\psi) \otimes \delta_B^{1/4} \simeq (\mathrm{Ind}_{w_0\widetilde{\mathcal{T}}N}^{\overline{G}} {}^{w_0}\xi^{\psi^{-1}} \otimes \delta_B^{1/4}) \otimes \varrho,$$

where we use Lemma 1.8 and the assumption on ϱ, observing that the inducing characters agree on $\widetilde{\mathscr{Z}}\widetilde{T}^{\square}$. Therefore the first part follows. Similarly, $J_\psi(-1,\varrho)$ is a submodule of $\mathscr{I}^{\psi^{-1}} \otimes \varrho$, and hence $\theta^{\psi^{-1}} \otimes \varrho$ is a submodule of $J_\psi(-1, \varrho)$. We have an injective $\mathbb{C}$-linear map from $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$ to

$$\mathrm{Hom}_{\overline{G}}\big((\mathrm{Ind}_{w_0 \tilde{\mathscr{T}} N}^{\overline{G}} {}^{w_0}\xi^{\psi^{-1}} \otimes \delta_B^{1/4}) \otimes \varrho, \mathscr{I}^{\psi^{-1}} \otimes \varrho\big).$$

Since the latter space is one-dimensional by Proposition I.2.2 of [Kazhdan and Patterson 1984],

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho)) \leq 1.$$

Since $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho)$ is a subspace of $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$ and since $\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho) \geq 1$, these spaces are equal. Because $M(1, \varrho)$ gives a nonzero element in $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$, it is proportional to the basis vector in $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho)$. □

2. Derivatives of exceptional representations

Throughout this section we suppose that F is a nonarchimedean local field of characteristic 0.

2A. ***Whittaker models of exceptional representations.*** For an l-group $\mathcal{G}$, its closed subgroup H and a smooth representation ρ of H we define $\mathrm{ind}_H^{\mathcal{G}} \rho$ to be the space of all functions $f : \mathcal{G} \to \rho$ such that $f(hg) = \rho(h) f(g)$ for all $h \in H$ and $g \in \mathcal{G}$ and such that f is right invariant under some compact open subgroup of $\mathcal{G}$. Define $\text{c-ind}_H^{\mathcal{G}} \rho$ to be the subspace of $\mathrm{ind}_H^{\mathcal{G}} \rho$ which consists of functions with compact support modulo H. The group $\mathcal{G}$ acts on both of these by right translation.

Definition 2.1. If U is a closed subgroup of $\mathcal{G}$, Ψ a character of U and π a smooth representation of $\mathcal{G}$, then we call the quotient space $\pi_{U,\Psi} = \pi/\pi(U, \Psi)$ the Jacquet module of π with respect to U and Ψ, where $\pi(U, \Psi)$ is the space spanned by the vectors of the form $\pi(u)v - \Psi(u)v$ for $v \in \pi$ and $u \in U$. When $\mathcal{G} = \overline{G}_r$ and $U = N_r$, a Ψ-Whittaker functional on π is a complex linear functional λ on π which satisfies $\lambda(\pi(u)v) = \Psi(u)\lambda(v)$ for all $v \in \pi$ and $u \in N_r$. The space of Ψ-Whittaker functionals on π can be identified with the space of complex linear functionals on $\pi_{N_r,\Psi}$.

We say that a character Ψ of N_r is generic if it is nontrivial on $U_{\boldsymbol{r}}$ for all compositions $\boldsymbol{r}$ of r. We define, as usual, a generic character ψ_r of N_r by

$$\psi_r(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r-1,r}), \quad u \in N_r.$$

Remark 2.2. (1) The space $\mathrm{Ind}_{N_r}^{\overline{G}_r} \Psi$ consists of all smooth functions W on $\overline{G}_r$ satisfying $W(u\tilde{g}) = \Psi(u) W(\tilde{g})$ for all $u \in N_r$ and $\tilde{g} \in \overline{G}_r$. The group $\overline{G}_r$ acts on this space by right translation, and a nontrivial intertwining map $\pi \to \mathrm{Ind}_{N_r}^{\overline{G}_r} \Psi$ is called the Ψ-Whittaker model of π. Note that π has a nonzero Ψ-Whittaker functional λ if and only if π has a Ψ-Whittaker model Λ. To obtain a model from a functional, set $\Lambda(\tilde{g}, v) = \lambda(\pi(\tilde{g})v)$, and to obtain a functional from a model, set $\lambda(v) = \Lambda(\tilde{e}, v)$, where $\tilde{e}$ denotes the identity element of $\overline{G}_r$.

(2) The group $\widetilde{T}_r$ acts transitively on the set of generic characters of N_r thanks to Remark 1.2(4). For $\tilde{t} \in \widetilde{T}_r$ the $\mathbb{C}$-linear map $v \mapsto \pi(\tilde{t})v$ is an isomorphism of $\pi_{N_r,\Psi}$ and $\pi_{N_r,{}^{\tilde{t}}\Psi}$.

(3) The vector space $\pi_{N_r,\Psi}$ can be identified with ${}^{\iota}\pi_{N_r,{}^{\iota}\Psi}$.

(4) For $a \in F^\times$ we define a character ψ_a of N_2 by $\psi_a\left[\left(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\right)\right] = \psi(ax)$. Recall that

$$\dim_{\mathbb{C}}(\Omega^{\psi}_{\chi})_{N_2,\psi_a} = \begin{cases} 1 & \text{if } a \in F^{\times 2}, \\ 0 & \text{if } a \in F^\times \setminus F^{\times 2} \end{cases}$$

(Proposition 2.16 of [Takeda 2014]). When $a \in F^\times$, the complex linear maps on $(\Omega^{\psi}_{\chi})_{N_2,\psi_{a^2}}$ are scalar multiples of e_a in view of (1-5).

We define a matrix $w_r \in G_r$ by

$$w_r = \begin{pmatrix} & & & \mathbf{1}_2 \\ & & \mathbf{1}_2 & \\ & \cdot^{\cdot^{\cdot}} & & \\ \mathbf{1}_2 & & & \end{pmatrix} \quad \text{or} \quad w_r = \begin{pmatrix} & & & & \mathbf{1}_2 \\ & & & \mathbf{1}_2 & \\ & & \cdot^{\cdot^{\cdot}} & & \\ & \mathbf{1}_2 & & & \\ 1 & & & & \end{pmatrix} \tag{2-1}$$

according to whether r is even or odd. Put $k = \left[\frac{r}{2}\right]$ and

$$J^{\psi}_{r,\chi} = \mathrm{Ind}^{\overline{G}_r}_{{}^{\iota}\widetilde{\mathscr{M}}_r U_{\overleftarrow{e(r)}}} {}^{w_r}\Upsilon^{\psi}_{r,\chi} \otimes \delta^{1/4}_{P_{\overleftarrow{e(r)}}}, \quad \mathcal{J}^{\psi}_{r,\chi} = \mathrm{Ind}^{\overline{G}_r}_{\widetilde{\mathscr{M}}_r U_{e(r)}} \Upsilon^{\psi^{-1}}_{r,\chi} \otimes \delta^{1/4}_{P_{e(r)}}.$$

Lemma 2.3. *If Ψ is generic, then the space $(J^{\psi}_{r,\chi})_{N_r,\Psi}$ is one-dimensional.*

Remark 2.4. Kazhdan and Patterson [1984] studied Whittaker functionals on the principal series representations of $\overline{G}_r$. Its space of Whittaker functionals is not one-dimensional:

$$\dim_{\mathbb{C}}(\mathscr{I}^{\psi}_r)_{N_r,\Psi} = [F^\times : F^{\times 2}]^k$$

(see Lemma I.3.2 of that paper).

Proof. From Remark 2.2(2) we may assume that $\Psi = \psi_r$. We will apply Theorem 5.2 of [Bernstein and Zelevinsky 1977] to $J^{\psi}_{r,\chi}$ with

$$G = \overline{G}_r, \quad M = {}^{\iota}\widetilde{\mathscr{M}}_r, \quad U = U_{\overleftarrow{e(r)}}, \quad \theta = 1, \quad N = \{e\}, \quad V = N_r.$$

If we set

$$P = MU = {}^{\iota}\widetilde{\mathscr{M}}_r U_{\overleftarrow{e(r)}}, \quad Q = NV = N_r, \quad V' = M \cap {}^{w^{-1}}V, \quad \psi' = {}^{w^{-1}}\psi_r|_{V'}$$

for $w \in G$, then the space $(J^{\psi}_{r,\chi})_{N_r,\Psi}$ is glued from ${}^{w}(({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'})$, where wP runs through the Q-orbits on G/P such that Ψ is trivial on ${}^{w}U \cap V$. Fix a set Σ of representatives of $F^{\times 2}\backslash F^{\times}$. The Q-orbits satisfying this condition are of the form $s_r(\iota_r(a)w_r^{-1})P$ for $a = (a_1, \dots, a_k) \in \Sigma^{\oplus k}$, where

$$\iota_r(a) = \mathrm{diag}[d(a_1), \dots, d(a_k)] \quad \text{or} \quad \iota_r(a) = \mathrm{diag}[d(a_1), \dots, d(a_k), 1]$$

according to whether r is even or odd. If $w = \iota_r(a)w_r^{-1}$, then

$$^{w_r^{-1}}V' = M_{e(r)} \cap N_r \simeq N_2^{\oplus k}, \quad ^{w_r^{-1}}\psi' = \psi_{a_1} \oplus \psi_{a_2} \oplus \cdots \oplus \psi_{a_k}.$$

In light of Remark 2.2(4) the space $({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'}$ is zero unless $-a_i \in F^{\times 2}$ for all $i = 1, 2, \dots, k$, and when this is the case, $({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'}$ is one-dimensional. □

Lemma 2.5. *Fix a preimage $\tilde{w}_r$ of w_r in $\overline{G}_r$. The integral*

$$\lambda^{\Psi}_x(\Phi) = \int_{\overleftarrow{U_{e(r)}}} e_x(\Phi(\tilde{w}_r^{-1}u))\overline{\Psi(u)}\,\mathrm{d}u$$

converges absolutely for all $\Phi \in \mathcal{J}^{\psi}_{r,\chi}$, $x \in F^k$ and characters Ψ of N_r.

Proof. We may assume χ to be unitary. Define a function f_0 on G_r by

$$\begin{aligned} f_0(g) &= \delta_{P_{e(r)}}(g)^{3/4} \prod_{i=1}^{k} \left| \frac{t_{r-2i+1}}{t_{r-2i+2}} \right|^{1/4} \\ &= \delta_{B_r}(g)^{1/2} \prod_{i=1}^{k} |t_{r-2i+1}|^{\alpha_i} |t_{r-2i+2}|^{\beta_i} \times \begin{cases} 1 & \text{if } r \text{ is even,} \\ |t_1|^{(r-1)/4} & \text{if } r \text{ is odd,} \end{cases} \end{aligned}$$

writing g in the form utk with $t = \mathrm{diag}[t_1, \dots, t_r] \in T_r$, $u \in N_r$ and $k \in K_r$, where $\alpha_i = i - \frac{1}{4}(r+3)$ and $\beta_i = i - \frac{1}{4}(r+1)$. In view of (1-4) we can find a positive constant c such that $|e_x(\Phi(\tilde{g}))| \le c f_0(p_r(\tilde{g}))$ for all $\tilde{g} \in \overline{G}_r$. Since

$$\tfrac{1}{4}(r-1) \ge \beta_k > \alpha_k > \beta_{k-1} > \alpha_{k-1} > \cdots > \beta_1 > \alpha_1,$$

the integral

$$\int_{U_{\overleftarrow{e(r)}}} f_0(w_r^{-1}u)\,\mathrm{d}u$$

is convergent by applying Proposition IV.2.1 of [Waldspurger 2003] with $P = B_r$ and $P' = w_r^{-1}B_r w_r$. □

Lemma 2.6. *If $\Phi \in \mathcal{J}^{\psi}_{r,\chi}$, $b \in (F^{\times})^{\oplus k}$, Ψ is a generic character of N_r and $\lambda^{\Psi}_b(\mathcal{J}^{\psi}_{r,\chi}(\tilde{p})\Phi) = 0$ for all $\tilde{p} \in \widetilde{\mathscr{P}}_r$, then $\Phi = 0$.*

Proof. The proof proceeds as in that of Proposition 3.2 of [Jacquet and Shalika 1983], where an analogous result was proved for standard modules of general linear groups. There is no harm in assuming that $\Psi = \psi_r$ in view of Remark 2.2(3).

The case $r=1$ is trivial. Proposition 1.4(3) proves the case $r=2$. We suppose that $r>2$, assuming the result up to $r-2$. Take a preimage $\tilde{w}_{r-2}$ of w_{r-2} in $\overline{G}_r$. Put $\tilde{w}=\tilde{w}_{r-2}\tilde{w}_r^{-1}$ and $b'=(b_1,\dots,b_{k-1})\in(F^\times)^{\oplus k-1}$. We define the $\mathbb{C}$-linear map $e^*_{b'}:\mathscr{S}(F^k)\to\mathscr{S}(F)$ by the relation

$$e_x(e^*_{b'}(\Phi))=\Phi(b_1,\dots,b_{k-1},x)$$

for $x\in F$. For each $\tilde{g}\in\overline{G}_r$ we define the map on $\mathcal{J}^{\psi}_{r,\chi}$ by

$$\Phi\mapsto W^*(\tilde{g},\Phi)=\int_{U_{\overleftarrow{e(r-2)}}}e^*_{b'}(\Phi(\tilde{w}_{r-2}^{-1}u\tilde{g}))\overline{\psi_r(u)}\,\mathrm{d}u\in\mathscr{S}(F).$$

Observe that

$$\lambda^{\Psi}_b(\mathcal{J}^{\psi}_{r,\chi}(\tilde{g})\Phi)=\int_{U_{(2,r-2)}}e_{b_k}(W^*(\tilde{w}u\tilde{g},\Phi))\overline{\psi_r(u)}\,\mathrm{d}u.$$

Hence the integrals are absolutely convergent in view of Lemma 2.5.

Suppose that $\lambda^{\Psi}_b(\mathcal{J}^{\psi}_{r,\chi}(\tilde{p})\Phi)=0$ for all $\tilde{p}\in\widetilde{\mathscr{P}}_r$. If we replace $\tilde{p}$ by $\boldsymbol{s}_2(g)\tilde{p}$, then a simple computation yields

$$\int_{M_{2,r-2}(F)}e_{b_k}\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx))\,\mathrm{d}x=0$$

for all $g\in G_2^{\square}$, where

$$\varepsilon=\begin{pmatrix}0&0&\cdots&0&0\\1&0&\cdots&0&0\end{pmatrix}\in\mathrm{M}_{2,r-2}(F).$$

Replacing g by $\mathrm{diag}[b_k^{-2}a^2,1]g$, we obtain

$$\int_{M_{2,r-2}(F)}e_{a}\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx))\,\mathrm{d}x=0$$

for all $a\in F^\times$, and so by continuity, this holds for all $a\in F$.

For $x\in M_{2,r-2}(F)$ we define $\mathcal{F}_x\in\mathscr{S}(F)$ by

$$\mathcal{F}_x(y)=e_y\left(W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx)),\quad y\in F.$$

Since the integral

$$\int_{M_{2,r-2}(F)}\left|e_y\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))\mathcal{F}_x\right)\right|\mathrm{d}x$$

is convergent uniformly in y,

$$\begin{aligned} 0 &= \int_F \int_{M_{2,r-2}(F)} e_y\big(\Omega_\chi^{\psi^{-1}}(\boldsymbol{s}_2(g))\mathcal{F}_x\big)\overline{\Phi(y)}\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{M_{2,r-2}(F)} \int_F e_y\big(\Omega_\chi^{\psi^{-1}}(\boldsymbol{s}_2(g))\mathcal{F}_x\big)\overline{\Phi(y)}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_{M_{2,r-2}(F)} \int_F \mathcal{F}_x(y)\overline{e_y\big(\Omega_{\bar\chi^{-1}}^{\psi^{-1}}(\boldsymbol{s}_2(g))^{-1}\Phi\big)}\,\mathrm{d}y\,\mathrm{d}x \\ &= \int_F \overline{e_y\big(\Omega_{\bar\chi^{-1}}^{\psi^{-1}}(\boldsymbol{s}_2(g))^{-1}\Phi\big)} \int_{M_{2,r-2}(F)} \mathcal{F}_x(y)\,\mathrm{d}x\,\mathrm{d}y \end{aligned}$$

for all $\Phi \in \mathscr{S}(F)$, where $\bar\chi$ is defined by $\bar\chi(a) = \overline{\chi(a)}$ for $a \in F^\times$. We get

$$\int_{M_{2,r-2}(F)} \mathcal{F}_x(y)\,\mathrm{d}x = 0$$

for all $g \in G_2^\square$, $\tilde p \in \widetilde{\mathscr{P}}_r$ and $y \in F$. Since this integral is absolutely convergent, we may apply the Fourier inversion to conclude that for all $\tilde p \in \widetilde{\mathscr{P}}_r$

$$\int_{M_{2,r-3}(F)} e_y\left(W^*\left[\tilde w \boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & 0 & x \\ & & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde p, \varPhi\right]\mathrm{d}x\right) = 0.$$

We can prove that for any j with $1 \le j < r-2$ the relation

$$\int_{M_{2,r-2-j}(F)} e_y\left(W^*\left[\tilde w \boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & 0 & x \\ & & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde p, \varPhi\right]\mathrm{d}x\right) = 0$$

for all $\tilde p \in \widetilde{\mathscr{P}}_r$ implies the same relation with j replaced by $j+1$ by arguing exactly as on p. 118 of [Jacquet and Shalika 1983]. We ultimately get $W^*(\tilde w \tilde p, \varPhi) = 0$ for all $\tilde p \in \widetilde{\mathscr{P}}_r$.

Substituting $\boldsymbol{s}_r(\mathrm{diag}[\mathbf{1}_2, p'])\tilde p$ for $\tilde p$, we see that $W^*(\boldsymbol{s}_r(p')\tilde w \tilde p, \varPhi) = 0$ for all $p' \in \mathscr{P}_{r-2}$ and $\tilde p \in \widetilde{\mathscr{P}}_r$. The induction hypothesis applied to $\mathcal{J}_{r-2,\chi}^{\psi}$ gives $W^*(\boldsymbol{s}_r(g')\tilde w \tilde p, \varPhi) = 0$ for all $g' \in G_{r-2}$ and $\tilde p \in \widetilde{\mathscr{P}}_r$. But then $W^*(u \tilde w \tilde p, \varPhi) = 0$ for all $u \in U_{(r-2,2)}$ and $\tilde p \in \widetilde{P}_{(2,r-2)}$, and so by continuity, $W^*(\tilde g, \varPhi) = 0$ for all $\tilde g \in \overline{G}_r$. We obtain $\varPhi = 0$ by induction on r. ☐

Lemma 2.7. *When $r > 2$, the representation $J_{r,\chi}^{\psi}$ is reducible.*

Proof. The periodicity of $\theta_{r,\chi}$ stated in [Kazhdan and Patterson 1984, Theorem I.2.9(e)] or [Takeda 2014, Proposition 2.36] shows that $(\theta_{r,\chi}^{\psi})_{U_{e(r)},1} \not\simeq (J_{r,\chi}^{\psi})_{U_{e(r)},1}$, which completes our proof. ☐

Proposition 2.8. *If $r > 2$, then $(\theta_{r,\chi}^{\psi})_{N_r,\psi_r} = 0$.*

Proof. Take a subrepresentation V_0 of $J_{r,\chi}^{\psi}$ such that $\theta_{r,\chi}^{\psi} = J_{r,\chi}^{\psi}/V_0$. There are $b \in (F^\times)^{\oplus k}$ and a generic character Ψ such that $\lambda_0(\varPhi) = \lambda_b^{\Psi}({}^\iota\varPhi)$ gives a ψ_r-Whittaker functional on $J_{r,\chi}^{\psi}$. Suppose that $\theta_{r,\chi}^{\psi}$ admits a nonzero ψ_r-Whittaker functional λ. We can view λ as a linear form on $J^{\psi}_{r,\chi}$ which vanishes on V_0. Since λ is a scalar multiple of λ_0 by the uniqueness of the Whittaker model of $J^{\psi}_{r,\chi}$ (see Lemma 2.3), if $\Phi \in V_0$, then $\lambda_0(J^{\psi}_{r,\chi}(\tilde{g})\Phi) = 0$ for all $\tilde{g} \in \overline{G}_r$, and hence $\Phi = 0$ by Lemma 2.6. Thus $V_0 = 0$, which contradicts Lemma 2.7. □

2B. ***The restriction to the group*** $\widetilde{\mathscr{P}}$**.** Define the character ν_r of $\overline{G}_r$ by $\nu_r(\tilde{g}) = |\det p_r(\tilde{g})|$ for $\tilde{g} \in \overline{G}_r$. We denote its restriction to $\widetilde{\mathscr{P}}_r$ by the same symbol. The five functors $\Phi^{\pm}$, $\Psi^{\pm}$ and $\hat{\Phi}^{+}$ play an important role in the theory of representations of $\widetilde{\mathscr{P}}_r$. These functors are the exact analogues of the functors described in [Zelevinsky 1980]. Although the theory is stated for G_r, the same principle works in the setting of the double covers $\overline{G}_r$ (see [Bump and Ginzburg 1992; Kable 2001]). Given a smooth representation π of $\overline{G}_r$ we write $\Psi^{+}\pi$ for the representation of $\widetilde{\mathscr{P}}_{r+1}$ on the same space such that $\mathscr{Y}_{r+1}$ acts trivially and $\overline{G}_r$ acts by $\pi \otimes \nu_r^{1/2}$. For a smooth representation τ of $\widetilde{\mathscr{P}}_r$ put

$$\Phi^{+}(\tau) = \text{c-ind}_{\widetilde{\mathscr{P}}_r \mathscr{Y}_{r+1}}^{\widetilde{\mathscr{P}}_{r+1}} \tau \otimes \nu_r^{1/2} \boxtimes (\psi_{r+1}|_{\mathscr{Y}_{r+1}}), \quad \Phi^{-}(\tau) = \tau_{\mathscr{Y}_r, \psi_r|_{\mathscr{Y}_r}},$$

$$\hat{\Phi}^{+}(\tau) = \text{ind}_{\widetilde{\mathscr{P}}_r \mathscr{Y}_{r+1}}^{\widetilde{\mathscr{P}}_{r+1}} \tau \otimes \nu_r^{1/2} \boxtimes (\psi_{r+1}|_{\mathscr{Y}_{r+1}}), \qquad \Psi^{-}(\tau) = \tau_{\mathscr{Y}_r, 1}.$$

The actions of the groups $\widetilde{\mathscr{P}}_{r-1}$ and $\overline{G}_{r-1}$ on $\Phi^{-}(\tau)$ and $\Psi^{-}(\tau)$ are normalized respectively in order that the following results hold (see Propositions 4.2 and 4.3 of [Kable 2001]):

Lemma 2.9. *If ρ, τ and κ are smooth representations of $\overline{G}_{r-1}$, $\widetilde{\mathscr{P}}_r$ and $\widetilde{\mathscr{P}}_{r-1}$, respectively, then*

$$\mathrm{Hom}_{\widetilde{\mathscr{P}}_r}(\tau, \Psi^{+}(\rho)) = \mathrm{Hom}_{\overline{G}_{r-1}}(\Psi^{-}(\tau), \rho), \quad \Psi^{+}(\rho)^{\vee} \simeq \nu_r^{-1} \otimes \Psi^{+}(\rho^{\vee}),$$

$$\mathrm{Hom}_{\widetilde{\mathscr{P}}_r}(\Phi^{+}(\kappa), \tau) = \mathrm{Hom}_{\widetilde{\mathscr{P}}_{r-1}}(\kappa, \Phi^{-}(\tau)), \quad \Phi^{+}(\kappa)^{\vee} \simeq \nu_r^{-1} \otimes \hat{\Phi}^{+}(\nu_{r-1} \otimes \kappa^{\vee}),$$

$$\mathrm{Hom}_{\widetilde{\mathscr{P}}_r}(\tau, \hat{\Phi}^{+}(\kappa)) = \mathrm{Hom}_{\widetilde{\mathscr{P}}_{r-1}}(\Phi^{-}(\tau), \kappa), \quad \Phi^{-}(\tau)^{\vee} \simeq \Phi^{-}(\tau^{\vee}).$$

Definition 2.10. Let π be an admissible representation of $\overline{G}_r$. For $i = 1, 2, \dots, r$ the i-th derivative of a smooth representation π of $\overline{G}_r$ is a representation of $\overline{G}_{r-i}$ defined by $\pi^{(i)} = \Psi^{-}(\Phi^{-})^{i-1}(\pi|_{\widetilde{\mathscr{P}}_r})$. If $\pi^{(h)} \neq 0$ and $\pi^{(j)} = 0$ for all $j > h$, then we call the number h the depth of π and call $\pi^{(h)}$ the highest derivative of π. It is convenient to introduce the shifted derivatives $\pi^{[i]} = \pi^{(i)} \otimes \nu_{r-i}^{1/2}$.

If π is irreducible, then so is its highest derivative by Theorem 8.1 of [Zelevinsky 1980].

We identify the multiplicative group $F^{\times}$ with the center Z_r of the group G_r for $r > 0$. When π is an irreducible admissible representation of G_r, its central exponent is the real number $e(\pi)$ defined by $|\omega_\pi(z)| = |z|^{e(\pi)}$ for $z \in F^{\times}$. In the next subsection we will use the following consequence of the unitarizability criterion given in Section 7.3 of [Bernstein 1984].

Proposition 2.11 (Bernstein). *Let π be an irreducible unitary representation of G_r of depth h. Then $\pi^{[h]}$ is an irreducible unitary representation of G_{r-h} and all the central exponents of irreducible subquotients of $\pi^{[k]}$ are strictly positive for all $k = 1, 2, \dots, h-1$.*

Thanks to Proposition 2.8, we have the following generalization of Theorem 5.4 of [Kable 2001] to the dyadic and twisted cases. The exceptional representations are very small in the following sense:

Theorem 2.12. *If $3 \le k \le r$, then the k-th derivatives of the exceptional representations of $\overline{G}_r$ are zero.*

2C. *Uniqueness of invariant trilinear forms.*

Proposition 2.13 (Kable). (1) $(\theta_{r,\chi}^{\psi^{-1}})^{[2]} \simeq \theta_{r-2,\chi}^{\psi^{-1}}$.

(2) *If r is odd, then* $(\theta_r^{\psi})^{(1)} \otimes \nu_{r-1}^{1/4} \simeq \theta_{r-1}$.

(3) *If r is even, then*

$$\theta_r^{(1)} \otimes \nu_{r-1}^{1/4} \simeq \bigoplus_{a \in F^{\times 2} \backslash F^{\times}} (\theta_{r-1}^{\psi} \otimes \chi_a).$$

(4) *If r is even and χ is odd, then* $\theta_{r,\chi}^{(1)} = 0$.

Proof. After Bump and Ginzburg [1992] showed that the second derivative of an exceptional representation must again be exceptional, Kable identified it precisely [2001, Theorem 5.3]. Although they discussed only the case when χ is trivial, one can similarly prove the twisted case. The second and third assertions are Theorem 5.2 of [Kable 2001]. The last assertion is obvious as Ω_{χ}^{ψ} is supercuspidal if χ is odd. □

Here and throughout the rest of this paper we will retain the notation from Section 1G.

Theorem 2.14. *Let ϱ be a character of $F^{\times}$, π an irreducible admissible representation of G and ϑ an exceptional representation of $\overline{G}$.*

(1) *For all but finitely many values of q^{-s} we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \vartheta \otimes I_{\psi}(s, \varrho), \mathbb{C}) \le \dim_{\mathbb{C}} \pi^{(n)}.$$

(2) *Assume that χ is trivial if n is odd. If π and χ are unitary, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \theta_{\chi}^{\psi} \otimes \theta^{\psi^{-1}}, \mathbb{C}) \le 1,$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \theta_{\chi}^{\psi} \otimes I_{\psi}(1, \varrho), \mathbb{C}) \le 1.$$

Remark 2.15. (1) One can view the second inequality of (2) as an analogue of Bernstein's theorem that $\dim_{\mathbb{C}} \operatorname{Hom}_{\mathscr{P}}(\pi \otimes \pi^\vee, \mathbb{C}) = 1$ for all irreducible admissible representations π of G, in view of

$$\begin{aligned}\operatorname{Hom}_G(\pi \otimes \pi^\vee \otimes \operatorname{Ind}_{\mathcal{P}}^G \delta_{\mathcal{P}}^{1/2}, \mathbb{C}) &\simeq \operatorname{Hom}_G(\pi \otimes \pi^\vee, \operatorname{Ind}_{\mathcal{P}}^G \delta_{\mathcal{P}}^{-1/2}) \\ &\simeq \operatorname{Hom}_{\mathcal{P}}(\pi \otimes \pi^\vee, \mathbb{C}) \\ &\simeq \operatorname{Hom}_{\mathscr{P}}(\pi \otimes \pi^\vee, \mathbb{C}).\end{aligned}$$

(2) Matringe [2014, Proposition 2.3] proved that if E is a quadratic extension of F and if π is an irreducible admissible unitary representation of $\mathrm{GL}_n(E)$, then the space of $\mathscr{P}$-invariant linear functionals on π is at most one-dimensional (cf. Theorem 1.1 of [Anandavardhanan et al. 2004]). This is an analogue of the second part in the context of Asai L-factors.

(3) When χ is trivial and F is not dyadic, Kable [2001, Theorem 6.1] proved the first part by modifying the proof of [Bump and Ginzburg 1992, Theorem 5.1], and moreover, if π is generic and unitary, then his result implies the second part. Actually, our proof combines his argument and the idea of [Matringe 2014]. Since the restriction to nondyadic F entered only through the lack of Theorem 2.12, his computation is now applicable to the dyadic case, and even to the twisted case.

Proof. Since $\tilde{\mathscr{Z}}$ is the center of $\bar{G}$, the space $\operatorname{Hom}_G(\pi \otimes \vartheta \otimes I_\psi(4s, \varrho), \mathbb{C})$ is zero unless the product of the three central characters is trivial on $F^{\times e(n)}$. Assume that this is the case. Then the space is isomorphic to

$$\begin{aligned}\operatorname{Hom}_{\bar{G}}(\pi \otimes \vartheta, I_{\psi^{-1}}(-4s, \varrho^{-1})) &\simeq \operatorname{Hom}_{\tilde{\mathscr{P}}}(\pi|_{\mathscr{P}} \otimes \vartheta|_{\tilde{\mathscr{P}}}, \Psi^+\theta_{n-1}^{\psi^{-1}} \otimes \nu^{-s}) \\ &\simeq \operatorname{Hom}_{\mathscr{P}}(\pi|_{\mathscr{P}} \otimes \vartheta|_{\tilde{\mathscr{P}}} \otimes \Psi^+\theta_{n-1}^{\psi}, \nu^{1-s})\end{aligned}$$

by the Frobenius reciprocity and Lemma 2.9. Recall that

$$(\theta_{n-1}^{\psi^{-1}})^\vee \simeq \theta_{n-1}^{\psi}.$$

For $1 \le k \le n$ and exceptional representations θ of $\bar{G}_k$ and θ' of $\bar{G}_{k-1}$ we shall consider the space

$$\mathcal{H}_{k,\theta,\theta'}(\pi, s) = \operatorname{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}}) \otimes \theta|_{\tilde{\mathscr{P}}_k} \otimes \Psi^+\theta', \nu_k^{1-s}\big).$$

Assume that $k \ge 2$. Since there is a short exact sequence

$$0 \to \Phi^+\Phi^-(\theta|_{\tilde{\mathscr{P}}_k}) \to \theta|_{\tilde{\mathscr{P}}_k} \to \Psi^+\Psi^-(\theta|_{\tilde{\mathscr{P}}_k}) \to 0$$

as recorded in Section 3 of [Bernstein and Zelevinsky 1977], we have an exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\to \mathcal{H}_{k,\theta,\theta'}(\pi,s)\\ &\to \mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Phi^+\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big).\end{aligned}$$

Lemma 2.9 shows that

$$\begin{aligned}\text{(2-2)}\quad &\mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta',\Psi^+(\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee)\otimes\nu_k^{-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{G}_{k-1}}\big(\Psi^-\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta'\big),\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1}^{-s}\big)\\ &\simeq \mathrm{Hom}_{G_{k-1}}(\pi^{[n-k+1]}\otimes\theta'\otimes\theta^{(1)},\nu_{k-1}^{-s}).\end{aligned}$$

Lemma 2.9 again shows that

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Phi^+\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta',\hat{\Phi}^+(\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1})\otimes\nu_k^{-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_{k-1}}\big(\Phi^-\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta'\big),\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1}^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\mathscr{P}_{k-1}}\big((\Phi^-)^{n-k+1}(\pi|_{\mathscr{P}})\otimes(\theta'|_{\widetilde{\mathscr{P}}_{k-1}}\otimes\nu_{k-1}^{1/2})\otimes\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k}),\nu_{k-1}^{1-s}\big).\end{aligned}$$

Now we use Theorem 2.12. It implies that $\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\simeq\Psi^+\theta^{(2)}$ (see [Kable 2001, (6.8)]). The last space is isomorphic to $\mathcal{H}_{k-1,\theta',\theta^{[2]}}(\pi,s)$ and

$$\begin{aligned}&\dim_{\mathbb{C}}\mathcal{H}_{k,\theta,\theta'}(\pi,s)\\ &\le \dim_{\mathbb{C}}\mathcal{H}_{k-1,\theta',\theta^{[2]}}(\pi,s)+\dim_{\mathbb{C}}\mathrm{Hom}_{G_{k-1}}\big(\pi^{[n-k+1]}\otimes\theta'\otimes\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k}),\nu_{k-1}^{-s}\big).\end{aligned}$$

We can see by comparing the central characters that the latter dimension must be zero except for finitely many q^{-s}. From this point onwards the exceptional representations with respect to which the spaces $\mathcal{H}_{k,\theta,\theta'}(\pi,s)$ are formed will not play a significant role and we shall allow ourselves to omit them from the notation. Then

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{Hom}_G(\pi\otimes\vartheta\otimes I_\psi(4s,\varrho),\mathbb{C}) &\le \dim_{\mathbb{C}}\mathcal{H}_n(\pi,s)\\ &\le\cdots\\ &\le \dim_{\mathbb{C}}\mathcal{H}_1(\pi,s)=\dim_{\mathbb{C}}\pi^{(n)}\end{aligned}$$

for all but finitely many q^{-s} by descending induction.

Next we prove (2). Since we obtain the injective map

$$\text{(2-3)}\qquad \mathrm{Hom}_G(\pi\otimes\varrho\otimes\theta_\chi^\psi\otimes\theta^{\psi^{-1}},\mathbb{C})\hookrightarrow\mathrm{Hom}_G(\pi\otimes\theta_\chi^\psi\otimes I_\psi(1,\varrho),\mathbb{C})$$

by composition with the quotient map in Lemma 1.15, we get the first inequality from the second. The proof of the second inequality is a variation on the proof of Proposition 2.2 of [Matringe 2014]. Let h denote the depth of π. Note that $(\Phi^-)^h(\pi|_{\mathscr{P}}) = 0$ and hence $\mathcal{H}_{n-h}(\pi, s) = 0$. If θ is a unitary exceptional representation, then $\theta^{(1)} \otimes \nu_{k-1}^{1/4}$ is zero or a unitary exceptional representation or a sum of such by Proposition 2.13. Thus the space (2-2) must vanish at $s = \frac{1}{4}$ for $k = n, n-1, \dots, n-h+2$ as the central characters do not match by Proposition 2.11. We conclude that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \vartheta \otimes I_\psi(1, \varrho), \mathbb{C}) \le \dim_{\mathbb{C}} \mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes (\theta')^{(1)}, \nu_{n-h}^{-1/4}\big)$$

for some unitary exceptional representations θ of $\overline{G}_{n-h}$ and θ' of $\overline{G}_{n-h+1}$.

Our task is to prove that the right-hand side is at most one. Without loss of generality we may suppose that $\theta' = \theta^{\psi}_{n-h+1}$ by replacing θ by $\theta \otimes \eta$ for some unitary character η of $F^\times$ in view of Proposition 2.13(4) and Remark 1.7. Then the space is zero by Proposition 2.13 unless the product of the central characters of $\pi^{[h]}$ and ϑ is quadratic. By comparing the central characters, one can find a nonzero element a_0 in F such that

$$\mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes (\theta')^{(1)}, \nu_{n-h}^{-1/4}\big) \simeq \mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes \theta^{\psi}_{n-h}, \chi_{a_0}\big).$$

Notice that the central characters of $\theta^{\psi}_{n-h} \otimes \chi_a$ ($a \in F^{\times 2} \backslash F^\times$) are mutually different if $n-h$ is odd. Now our proof is complete by induction. □

3. Twisted symmetric square L-factors

One of the most significant uses of exceptional representations in number theory so far is as an ingredient in the Rankin–Selberg integral for the symmetric square L-function of an irreducible cuspidal automorphic representation of a general linear group found by Bump and Ginzburg [1992]. Let F for the moment be a local field of characteristic zero.

3A. *A normalization of the intertwining operator.*

Definition 3.1. A normalized intertwining operator is defined by

$$N(s, \varrho) = \frac{b(-s, \varrho^{-1})}{a(s, \varrho)} M(s, \varrho),$$

where

$$a(s, \varrho) = L\big(\tfrac{1}{2}n(s-1) + 1, \varrho^2\big), \quad b(s, \varrho) = L\big(\tfrac{1}{2}n(s+1), \varrho^2\big).$$

Lemma 3.2. *The operator $M^*(s, \varrho) = a(s, \varrho)^{-1} M(s, \varrho)$ is entire.*

Proof. This is proved in Sections 4.5 and 4.6 of [Takeda 2015]. □

Lemma 3.3. *If we put $\mathcal{M} = M_{(n-1,1)}$, then*

$$^{\iota}\big(\mathrm{Ind}^{\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\widetilde{G}'}\,\theta^{\psi}_{n-1}\boxtimes\zeta^{\psi}_{\varrho}\big)\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\,{}^{\delta}\theta^{\psi^{-1}}_{n-1}\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}},$$

where the matrix δ is defined in (1-7).

Proof. Recall the longest element w_0' of the Weyl group of G'. The automorphism $g\mapsto w_0'\,{}^tg^{-1}w_0'$ of G stabilizes the subgroup G'. Its restriction to G' is the main involution ι' of G'. Since $\tilde{g}\mapsto\delta^{-1}\,{}^{\iota}\tilde{g}\delta$ is a lift of this automorphism, its restriction to $\overline{G}'$ differs from the lift of ι' only by twisting by a quadratic character η of $F^{\times}$ on account of Proposition 1.3. It follows from Lemma 1.9(4) that

$$\theta^{\psi}_{n-1}\simeq{}^{\iota'}\theta^{\psi^{-1}}_{n-1}\simeq{}^{\iota}({}^{\delta}\theta^{\psi^{-1}}_{n-1})\otimes\eta.$$

Thus ${}^{\iota}\theta^{\psi}_{n-1}\simeq{}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta$. Since ${}^{\iota}\zeta^{\psi}_{\varrho}=\zeta^{\psi^{-1}}_{\varrho^{-1}}$, we obtain

$$^{\iota}\big(\mathrm{Ind}^{\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\widetilde{G}'}\,\theta^{\psi}_{n-1}\boxtimes\zeta^{\psi}_{\varrho}\big)\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\big({}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta\big)\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

by (1-3). If n is odd, then ${}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta\simeq{}^{\delta}\theta^{\psi^{-1}}_{n-1}$ by Lemma 1.9(3).

Suppose that n is even. Take a genuine character ξ' of ${}^{\iota}\widetilde{\mathscr{T}}'$ in such a way that

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\,{}^{\delta}\theta^{\psi^{-1}}_{n-1}\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

is the unique irreducible subrepresentation of

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

(cf. Lemma 1.14). Since the restrictions of ξ' and $\xi'\cdot(\eta\circ\det)$ to $\widetilde{\mathscr{Z}}\widetilde{T}^{\square}=\widetilde{T}^{\square}$ coincide,

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\cdot(\eta\circ\det)\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

by Lemma 1.8, which concludes our proof. □

Lemma 3.3 gives an important isomorphism,

$$\jmath^{\psi}_{s,\varrho}:{}^{\iota}J_{\psi}(s,\varrho)\simeq I_{\psi^{-1}}(s,\varrho^{-1}).$$

The isomorphism depends on s in a fairly trivial way.

Definition 3.4. We call a right $\widetilde{K}$-finite function $(s,\tilde{g})\mapsto f^{(s)}(\tilde{g})$ on $\mathbb{C}\times\overline{G}$ a holomorphic section of $I_{\psi}(s,\varrho)$ if $f^{(s)}(\tilde{g})$ is holomorphic in s for each $\tilde{g}\in\overline{G}$ and $f^{(s)}\in I_{\psi}(s,\varrho)$ for each $s\in\mathbb{C}$. A holomorphic section $f^{(s)}$ is a standard section if its restriction to $\mathbb{C}\times\widetilde{K}$ does not depend on s. We call a function $f^{(s)}$ on $\mathbb{C}\times\overline{G}$ a meromorphic section of $I_{\psi}(s,\varrho)$ if there is a nonzero entire function β such that $\beta(s)f^{(s)}$ is a holomorphic section.

We call $h^{(s)}$ a meromorphic section of $J_\psi(s,\varrho)$ if $j^\psi_{s,\varrho}({}^\iota h^{(s)})$ is. We define a $\mathbb{C}$-linear map

$$\hat{N}(s,\varrho)=\hat{N}_\psi(s,\varrho):I_\psi(s,\varrho)\to I_{\psi^{-1}}(-s,\varrho^{-1})$$

by $\hat{N}(s,\varrho)f^{(s)}=j^\psi_{-s,\varrho}({}^\iota N(s,\varrho)f^{(s)})$, where

$$[{}^\iota N(s,\varrho)f^{(s)}](\tilde{g})=[N(s,\varrho)f^{(s)}]({}^\iota\tilde{g}).$$

We can define a meromorphic function $\alpha_\psi(s,\varrho)$ by

$$\hat{N}_{\psi^{-1}}(-s,\varrho^{-1})\hat{N}_\psi(s,\varrho)=\alpha_\psi(s,\varrho)\cdot\mathrm{Id}.$$

Lemma 3.5. *The function $\alpha_\psi(s,\varrho)$ has neither pole nor zero.*

Proof. We can view $I_\psi(s,\varrho)$ as a subrepresentation of $\mathrm{Ind}^{\overline{G}}_{\widetilde{\mathscr{T}}N}\,\mu_s$, where μ_s is an extension to $\widetilde{\mathscr{T}}$ of the genuine character of $\widetilde{\mathscr{Z}}\widetilde{T}^\square$ defined by

$$\mu_s(\boldsymbol{s}(t))=\varrho(t_n)^{-1}|t_n|^{-(n-1)s/4}\prod_{i=0}^{l-1}\mu_\psi(t_{n-2i})\prod_{j=1}^{n-1}|t_j|^{(2j-n+s)/4}$$

for $t=\mathrm{diag}[t_1,\dots,t_n]\in\mathscr{Z}T^\square$. Theorem I.2.6 of [Kazhdan and Patterson 1984] shows that

$$\begin{aligned}&\frac{a(s,\varrho)a(-s,\varrho^{-1})}{b(s,\varrho)b(-s,\varrho^{-1})}\alpha_\psi(s,\varrho)\\&\quad\approx\prod_{j=1}^{n-1}\frac{L\big(j+\frac{1}{2}n(s-1),\varrho^2\big)L\big(j+\frac{1}{2}n(-1-s),\varrho^{-2}\big)}{L\big(j+\frac{1}{2}n(s-1)+1,\varrho^2\big)L\big(j+\frac{1}{2}n(-1-s)+1,\varrho^{-2}\big)}\\&\quad=\frac{L\big(1+\frac{1}{2}n(s-1),\varrho^2\big)L\big(1+\frac{1}{2}n(-1-s),\varrho^{-2}\big)}{L\big(\frac{1}{2}n(s+1),\varrho^2\big)L\big(\frac{1}{2}n(1-s),\varrho^{-2}\big)}\\&\quad=\frac{a(s,\varrho)a(-s,\varrho^{-1})}{b(s,\varrho)b(-s,\varrho^{-1})},\end{aligned}$$

where $\approx$ denotes equality up to multiplication by invertible functions. □

3B. ***Semi-Whittaker functions.*** When $r>2$, the exceptional representations of $\overline{G}_r$ fail to possess Whittaker models with respect to generic characters of N_r, but they have models with respect to certain degenerate characters of N_r. We define the degenerate characters of N_r by

$$\begin{aligned}\psi_{\boldsymbol{e},r}(u)&=\psi(u_{1,2}+u_{3,4}+\cdots+u_{r-1,r}),\\\psi_{\boldsymbol{o},r}(u)&=\psi(u_{2,3}+u_{4,5}+\cdots+u_{r-2,r-1})\end{aligned}$$

when r is even. When r is odd, we define the degenerate characters by

$$\psi_{\boldsymbol{e},r}(u) = \psi(u_{2,3} + u_{4,5} + \cdots + u_{r-1,r}),$$
$$\psi_{\boldsymbol{o},r}(u) = \psi(u_{1,2} + u_{3,4} + \cdots + u_{r-2,r-1}).$$

It is important to note that $\psi_r = \psi_{\boldsymbol{e},r} \cdot \psi_{\boldsymbol{o},r}$ and $\psi_r^{-1} = {}^{\iota}\psi_{\boldsymbol{e},r} \cdot {}^{\iota}\psi_{\boldsymbol{o},r}$.

Recall that χ is assumed to be trivial whenever r is odd. We define the $\mathbb{C}$-linear functional ϵ_j on $\mathscr{S}(F^j)$ by

$$\epsilon_j(\Phi) = \Phi(1, 1, \ldots, 1)$$

for $\Phi \in \mathscr{S}(F^j)$. The functional $\Phi \mapsto \epsilon_k(\Phi(\tilde{e}))$ gives a $\overline{\psi_{\boldsymbol{e},r}}$-Whittaker functional on $I^{\psi}_{r,\chi}$ by (1-5), where $k = \left[\frac{r}{2}\right]$. The $\overline{\psi_{\boldsymbol{e},r}}$-Whittaker functional corresponds to a $\bar{G}_r$-intertwining map

$$Q = Q^{\psi}_{r,\chi} : I^{\psi}_{r,\chi} \to \operatorname{Ind}^{\bar{G}_r}_{N_r} \overline{\psi_{\boldsymbol{e},r}}$$

(see Remark 2.2(1)). One can see from the proof of Proposition 1.4(3) that Q is injective. Note that

$$Q(\boldsymbol{s}_r(zu)\tilde{g}, \Theta) = \overline{\psi_{\boldsymbol{e},r}(u)} \frac{\chi(z)^k}{\mu_{\psi}(z)^k} Q(\tilde{g}, \Theta)$$

with $z \in F^{\times e(r)}$, $u \in N_r$, $\tilde{g} \in \bar{G}_r$ and $\Theta \in I^{\psi}_{r,\chi}$. When $r = n$, we will suppress the subscript r.

For $f \in I_{\psi}(s, \varrho)$ we define a $\overline{\psi_{\boldsymbol{o}}}$-Whittaker function $R(f) = R^{\psi}_{s,\varrho}(f)$ by $R(\tilde{g}, f) = \epsilon_{\ell'}(f(\tilde{g}))$ for $\tilde{g} \in \bar{G}$, where $\ell' = \left[\frac{n-1}{2}\right]$. Note that

$$R(\boldsymbol{s}(zu)\tilde{g}, f) = \varrho(z)^{-1}\mu_{\psi}(z)^{\ell}\overline{\psi_{\boldsymbol{o}}(u)}R(\tilde{g}, f) \quad (z \in F^{\times e(n)},\ u \in N,\ \tilde{g} \in \bar{G}).$$

Lemma 3.6.

(1) *There is a $\bar{G}_r$-intertwining embedding $\hat{Q} = \hat{Q}^{\psi}_{r,\chi} : \theta^{\psi}_{r,\chi} \to \operatorname{Ind}^{\bar{G}_r}_{N_r} {}^{\iota}\psi_{\boldsymbol{e},r}$.*

(2) *There is a $\bar{G}$-intertwining embedding $\hat{R} = \hat{R}^{\psi}_{s,\varrho} : J_{\psi}(s, \varrho) \to \operatorname{Ind}^{\bar{G}}_{N} {}^{\iota}\psi_{\boldsymbol{o},r}$.*

Proof. Lemma 1.9(4) gives an isomorphism $\iota^{\psi}_{r,\chi} : {}^{\iota}\theta^{\psi}_{r,\chi} \simeq \theta^{\psi^{-1}}_{r,\chi^{-1}}$. We obtain a ${}^{\iota}\psi_{\boldsymbol{e},r}$-Whittaker model of $\theta^{\psi}_{r,\chi}$ and ${}^{\iota}\psi_{\boldsymbol{o},r}$-Whittaker model of $J_{\psi}(s, \varrho)$ by setting

$$\hat{Q}^{\psi}_{r,\chi}(\tilde{g}, \Theta) = Q^{\psi^{-1}}_{r,\chi^{-1}}({}^{\iota}\tilde{g}, \iota^{\psi}_{r,\chi}(\Theta)), \quad \hat{R}^{\psi}_{s,\varrho}(\tilde{g}, h) = R^{\psi^{-1}}_{s,\varrho^{-1}}({}^{\iota}\tilde{g}, \jmath^{\psi}_{s,\varrho}({}^{\iota}h))$$

for $\tilde{g} \in \bar{G}_r$, $\Theta \in \theta^{\psi}_{r,\chi}$ and $h \in J_{\psi}(s, \varrho)$. □

3C. ***The local zeta integrals.*** Let π be an irreducible admissible generic representation of G and $\mathcal{W}^{\psi}(\pi)$ its ψ_n-Whittaker model. For $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_\chi^{\psi}$ and a meromorphic section $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$ the integral

$$Z(W, \Theta, f^{(s)}) = \int_{\mathscr{Z} N \backslash G} W(g) Q(g, \Theta) R(g, f^{(s)})\, \mathrm{d}g$$

makes sense at least formally. For a meromorphic section $h^{(s)}$ of $J_\psi(s, \chi^\ell \omega_\pi)$ we define the integral $Z(W, \Theta, h^{(s)})$ by

$$Z(W, \Theta, h^{(s)}) = \int_{\mathscr{Z} N \backslash G} W(g) \hat{Q}(g, \Theta) \hat{R}(g, h^{(s)})\, \mathrm{d}g.$$

We will use the following estimate for Whittaker functions.

Proposition 3.7 [Jacquet and Shalika 1990, Proposition 3, p. 177]. *If π is an irreducible admissible unitary generic representation of G, then for each $1 \le j \le n-1$ there is a finite set C_j of characters of $F^\times$ with positive real parts, and for each $\chi \in C_j$, an integer n_χ with the following property: Let X_j be the set of functions of the form $\chi(a)(\log|a|)^k$ with $0 \le k \le n_\chi$ and X the functions on $(F^\times)^{\oplus n-1}$ which are products of functions in the X_j. Then for each $W \in W^{\psi}(\pi)$ there are Schwartz functions $\phi_\xi \in \mathcal{S}(F^{n-1} \times K)$ such that for $g = tk$*

$$W(g) = \delta_{B'}(t)^{1/2} \sum_{\xi \in X} \phi_\xi\left(\frac{t_1}{t_2}, \dots, \frac{t_{n-1}}{t_n}, k\right) \xi\left(\frac{t_1}{t_2}, \dots, \frac{t_{n-1}}{t_n}\right).$$

In the following proposition by "local Euler factor" in the p-adic case we mean a function of the form $P(q^{-s})^{-1}$, where P is a polynomial satisfying $P(0) = 1$, and in the archimedean case we mean a product of functions of the form $\pi^{-s/2}\Gamma\left(\frac{1}{2}(s+b)\right)$ for constants $b \in \mathbb{C}$.

Proposition 3.8 (cf. [Bump and Ginzburg 1992; Takeda 2014]). *Let F be a (not necessarily nonarchimedean) local field of characteristic zero. Let π be an irreducible admissible generic representation of G. We assume χ to be trivial if n is odd.*

(1) *There is $\beta \in \mathbb{R}$ such that the integrals $Z(W, \Theta, f^{(s)})$ converge absolutely in the right half-plane $\Re s > \beta$ for all $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_\chi^{\psi}$ and holomorphic sections $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$.*

(2) *$Z(W, \Theta, f^{(s)})$ possesses a meromorphic continuation to $\mathbb{C}$. If F is nonarchimedean and $f^{(s)}$ is a standard section, then it represents a rational function of $q^{-s/4}$.*

(3) *There is a local Euler factor $L(s)$ such that $Z(W, \Theta, f^{(2s-1)})/L\left(\frac{s}{2}\right)$ is entire for all $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_\chi^{\psi}$ and holomorphic sections $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$.*

(4) *For each point* $s_0 \in \mathbb{C}$ *there are* $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_{\chi}^{\psi}$ *and a holomorphic section* $f^{(s)}$ *of* $I_{\psi}(s, \chi^{\ell}\omega_{\pi})$ *such that* $Z(W, \Theta, f^{(s)})$ *does not have a zero at* $s = s_0$.

(5) *If* π *and* χ *are unitary, then* $Z(W, \Theta, f^{(s)})$ *converges absolutely for the closed right half-plane* $\Re s \geq 1$.

(6) *Suppose that* F *is nonarchimedean*, χ *is unitary and* π *is square-integrable. Then* $Z(W, \Theta, f^{(s)})$ *converges absolutely for* $\Re s \geq -1$.

Proof. The paper [Bump and Ginzburg 1992] deals with some basic local theory, and Proposition 5.5 of [Takeda 2014] discusses the twisted case. Strictly speaking, our zeta integrals are slightly different from those treated in [Bump and Ginzburg 1992] and [Takeda 2014] when n is even. However, the arguments can easily be modified to deal with our integrals.

Assertions (2) and (4) are in Proposition 5.2 and Theorem 7.2 of [Bump and Ginzburg 1992], respectively. It is easy to see from the proof of [Bump and Ginzburg 1992, Proposition 5.2] that the integral $Z(W, \Theta, f^{(s)})$ is a finite sum of products of entire functions and Tate integrals. The exponents of the quasicharacters occurring in the Tate integrals are finite in number and are independent of the choice of W, Q and $f^{(s)}$, which verifies (1) and (3).

Finally, we assume χ to be unitary and prove (5) and (6). Since Z^2 and $T'^{\square}$ have finite indices in Z and T', it suffices to prove the convergence of the integral

$$\int_{T'^{\square}} \left| W(t't) Q(t't, \Theta) R(t't, f^{(s)}) \right| \delta_B(t')^{-1} \, \mathrm{d}t'$$

for $\Re s \geq -1$ and all $t \in T$. We may assume that $t = 1$, taking Proposition 1.4(1) into account. From (1-4) there are positive constants c and c' such that

$$|Q(\tilde{t}', \Theta)| \leq c\delta_B^{1/4}(\tilde{t}'), \quad |R(\tilde{t}', f^{(s)}))| \leq c'\delta_B^{1/4}(\tilde{t}')\delta_{\mathcal{P}}(\tilde{t}')^{(\Re s+1)/4}$$

for all $\tilde{t}' \in \widetilde{T}'^{\square}$. Therefore all that is required is to show that if π is unitary generic or square-integrable, then the integral

$$\int_{T'^{\square}} |W(t')| \delta_B(t')^{-1/2} \delta_{\mathcal{P}}(t')^{(\Re s+1)/4} \, \mathrm{d}t$$

is convergent for $\Re s \geq 1$ or $\Re s \geq -1$, respectively. Note that $\delta_B(t') = \delta_{B'}(t')\delta_{\mathcal{P}}(t')$ for $t' \in T'$. Since the integrals

$$\int_{F^{\times}} |a|^{\delta} \left|\log|a|\right|^{k} |\Phi(a)| \, \mathrm{d}a$$

are convergent for all $0 < \delta \in \mathbb{R}$, $0 \leq k \in \mathbb{Z}$ and $\Phi \in \mathscr{S}(F)$, Proposition 3.7 proves (5). The proof of (6) proceeds exactly as in that of Lemma 2 of [Kable 2004]. □

Corollary 3.9. *Assume that F is nonarchimedean. Let π be an irreducible generic unitary representation of G and χ a unitary character of $F^\times$. Assume that χ is trivial if n is odd. Put $\varrho = \chi^\ell \omega_\pi$. If $\varrho^2 = 1$, then the following conditions are equivalent*:

(a) *$\pi \otimes \varrho$ is χ-distinguished*;

(b) $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho, \mathbb{C}) = \mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho), \mathbb{C})$;

(c) *the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ factors through the quotient*

$$\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho) \to \pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho.$$

Proof. Proposition 3.8(4)–(5) combined with Theorem 2.14(2) shows that the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ gives a basis vector in the one-dimensional vector space $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho), \mathbb{C})$. Since $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho, \mathbb{C})$ is its subspace, the equivalence of the three conditions is evident. □

3D. *Good sections.*

Definition 3.10. Assume that ϱ is unitary. Let $s_0 \in \mathbb{C}$ and $f^{(s)}$ be a meromorphic section of $I_\psi(s, \varrho)$. When $\Re s_0 > -1$, we say that $f^{(s)}$ is good at $s = s_0$ if it is holomorphic at $s = s_0$. When $\Re s_0 < 0$, we say that $f^{(s)}$ is good at $s = s_0$ if $\hat{N}(s, \varrho) f^{(s)}$ is holomorphic at $s = s_0$. We call $f^{(s)}$ a good section if it is good at every point $s_0 \in \mathbb{C}$.

The following result can be proved in the same way as in the proof of Proposition 3.1 of [Yamana 2014] by utilizing Lemmas 3.2 and 3.5.

Proposition 3.11. (1) *Holomorphic sections are good sections.*

(2) *If $f^{(s)}$ is a good section of $I_\psi(s, \varrho)$, then $b(s, \varrho)^{-1} f^{(s)}$ is a holomorphic section.*

(3) *If $f^{(s)}$ is a meromorphic section which is good at $s = s_0$, then there is a good section $F^{(s)}$ such that $f^{(s)} - F^{(s)}$ has a zero of any prescribed order at $s = s_0$.*

(4) *Given a meromorphic section $f^{(s)}$ of $I_\psi(s, \varrho)$ the following conditions are equivalent*:

- *$f^{(s)}$ is a good section of $I_\psi(s, \varrho)$*;
- *$h^{(s)} = \hat{N}(-s, \varrho) f^{(-s)}$ is a good section of $I_{\psi^{-1}}(s, \varrho^{-1})$*;
- *there exist holomorphic sections $f_1^{(s)}$ of $I_\psi(s, \varrho)$ and $f_2^{(-s)}$ of $I_{\psi^{-1}}(-s, \varrho^{-1})$ such that*

$$f^{(s)} = f_1^{(s)} + \hat{N}_{\psi^{-1}}(-s, \varrho^{-1}) f_2^{(-s)}.$$

Definition 3.10 coincides in the strip $-1 < \Re s_0 < 0$ by Proposition 3.11(2).

3E. ***The twisted symmetric square L-factors.*** In Sections 3E–3G we will assume F to be nonarchimedean. Let π be an irreducible admissible generic representation of G. Suppose that χ is trivial if n is odd. Proposition 3.8(2) tells us that if $f^{(s)}$ is a standard section of $I_\psi(s,\chi^\ell\omega_\pi)$ multiplied by an element of $\mathbb{C}[q^{-s/4},q^{s/4}]$ or a section obtained by applying the normalized intertwining operator to such a section of $I_{\psi^{-1}}(-s,\chi^{-\ell}\omega_\pi^{-1})$, then $Z(W,\Theta,f^{(2s-1)})$ is a rational function of $q^{-s/2}$. Let $\mathcal{I}(\pi,\chi)$ be the subspace of $\mathbb{C}(q^{-s/2})$ spanned by these local integrals. One can see from Propositions 3.8(2) and 3.11(2) that each such rational function can be written with a common denominator. That is, $\mathcal{I}(\pi,\chi)$ is a fractional $\mathbb{C}[q^{-s/2},q^{s/2}]$-ideal. Proposition 3.8(4) shows that it contains 1. It is not difficult to see that $\mathcal{I}(\pi,\chi)$ is independent of the choice of ψ. With these properties of $\mathcal{I}(\pi,\chi)$ in hand, we can now define the twisted symmetric square L-factor.

Definition 3.12. The ideal $\mathcal{I}(\pi,\chi)$ has a unique generator of the form $Q_{\pi,\chi}(q^{-s/2})^{-1}$, where the polynomial $Q_{\pi,\chi}$ satisfies $Q_{\pi,\chi}(0)=1$. We will define the twisted symmetric L-factor by $L(s,\pi,\mathrm{sym}^2\otimes\chi)=Q_{\pi,\chi}(q^{-s/2})^{-1}$.

We expect that $Q_{\pi,\chi}(q^{-s/2})$ is a polynomial of q^{-s}. It may be worth noting the simple fact that $\delta_{\mathcal{P}}(t)^s$ is a power of q^{-2s} for $t\in\mathscr{T}$.

In other words, $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ is the minimal factor such that the ratios $Z(W,\Theta,f^{(2s-1)})/L(s,\pi,\mathrm{sym}^2\otimes\chi)$ are entire for all $W\in\mathcal{W}^\psi(\pi)$, $\Theta\in\theta_\chi^\psi$ and good sections $f^{(s)}$ of $I_\psi(s,\chi^\ell\omega_\pi)$, simply because any holomorphic section can be expressed as a linear combination of standard sections with coefficients entire functions of s.

Remark 3.13. Recall that ν_k is the character of G_k defined by $\nu_k(g)=|\det g|$. Since $I_\psi(s,\varrho)\otimes\nu^y\simeq I_\psi(s+4y,\varrho\nu_1^{-ny})$,

$$L(s,\pi\otimes\nu^y,\mathrm{sym}^2\otimes\chi)=L(s+2y,\pi,\mathrm{sym}^2\otimes\chi)$$

for all $y\in\mathbb{C}$. If n is even and μ is a character of $F^\times$, then Lemma 1.9(3) implies

$$L(s,\pi\otimes\mu,\mathrm{sym}^2\otimes\chi)=L(s,\pi,\mathrm{sym}^2\otimes\chi\mu^2).$$

3F. ***Local functional equations.*** The need for normalizing $M(s,\chi^\ell\omega_\pi)$ and the need for including sections of the second type are clear from the following result:

Proposition 3.14. *Suppose that F is nonarchimedean. Let π be an irreducible admissible generic representation of G. We assume χ to be trivial if n is odd. Then there is a nowhere-vanishing entire function $\mathcal{E}(s,\pi,\chi,\psi)$ such that*

$$\frac{Z(W,\Theta,N(s,\chi^\ell\omega_\pi)f^{(s)})}{L\big(\frac{1}{2}(1-s),\pi^\vee,\mathrm{sym}^2\otimes\chi^{-1}\big)}=\mathcal{E}\big(\tfrac{1}{2}(1+s),\pi,\chi,\psi\big)\frac{Z(W,\Theta,f^{(s)})}{L\big(\frac{1}{2}(1+s),\pi,\mathrm{sym}^2\otimes\chi\big)}$$

for $W\in\mathcal{W}^\psi(\pi)$, $\Theta\in\theta_\chi^\psi$ and meromorphic sections $f^{(s)}$ of $I_\psi(s,\chi^\ell\omega_\pi)$.

Proof. The generic uniqueness in Theorem 2.14(1) produces the functional equation above. It is well known that the contragredient representation $\pi^\vee$ of π is isomorphic to ${}^\iota\pi$, and we shall allow ourselves to confuse the two. The image of $\mathcal{W}^\psi(\pi)$ under the map $W \mapsto {}^\iota W$ is precisely the space $\mathcal{W}^{\psi^{-1}}(\pi^\vee)$. If $h^{(s)}$ is a meromorphic section of $J_\psi(s, \chi^\ell\omega_\pi)$, then

$$\begin{aligned} Z(W, \Theta, h^{(s)}) &= \int_{\mathscr{Z}N\backslash G} W({}^\iota g)\hat{Q}^\psi_\chi({}^\iota g, \Theta)\hat{R}^\psi_{s,\chi^\ell\omega_\pi}({}^\iota g, h^{(s)})\,\mathrm{d}g \\ &= \int_{\mathscr{Z}N\backslash G} {}^\iota W(g)Q^{\psi^{-1}}_{\chi^{-1}}(g, \iota^\psi_\chi(\Theta))R^{\psi^{-1}}_{s,\chi^{-\ell}\omega_\pi^{-1}}(g, j^\psi_{s,\chi^\ell\omega_\pi}({}^\iota h^{(s)}))\,\mathrm{d}g \\ &= Z({}^\iota W, \iota^\psi_\chi(\Theta), j^\psi_{s,\chi^\ell\omega_\pi}({}^\iota h^{(s)})) \end{aligned}$$

by the proof of Lemma 3.6. This combined with Proposition 3.11(4) shows that the ratios on both sides of the functional equation are holomorphic and nonzero everywhere on $\mathbb{C}$, and hence so is its factor of proportionality $\mathcal{E}\big(\frac{1}{2}(1+s), \pi, \chi, \psi\big)$. □

3G. ***Poles of the symmetric square L-factor and distinction.*** We will continue to assume F to be nonarchimedean.

Lemma 3.15. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$. Assume that χ is trivial when n is odd.*

(1) *$L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ is holomorphic for $\Re s > 0$.*

(2) *If $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ has a pole at $s = 0$, then $\chi^n\omega_\pi^2$ is trivial.*

(3) *$L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ has at most a simple pole on $\Re s = 0$.*

Proof. Recall that $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ has the same poles as the family of local integrals $Z(W, \Theta, f^{(2s-1)})$ for good sections. Therefore the poles of $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ in $\Re s \geq 0$ are contained in the poles of good sections of $I_\psi(2s-1, \chi^\ell\omega_\pi)$ with multiplicity by Proposition 3.8(6). Our assertions now amount to the relevant analytic properties of $b(2s-1, \chi^\ell\omega_\pi) = L(ns, \chi^n\omega_\pi^2)$ in view of Proposition 3.11(2). □

Lemma 3.16. (We keep the notation of Lemma 3.15.) *Assume that $\chi^n\omega_\pi^2 = 1$. Then there are $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta^\psi_\chi$ and a good section $f^{(s)}$ of $I_\psi(s, \chi^\ell\omega_\pi)$ such that*

$$M_\psi(1, \chi^\ell\omega_\pi)f^{(1)} = 0, \quad \lim_{s\to 1} Z(W, \Theta, N(s, \chi^\ell\omega_\pi)f^{(s)}) \neq 0.$$

Proof. Proposition 3.8(4) enables us to choose $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta^\psi_\chi$ and a holomorphic section $h^{(s)}$ of $I_{\psi^{-1}}(s, \chi^\ell\omega_\pi)$ so that $Z({}^\iota W, \iota^\psi_\chi(\Theta), h^{(-1)}) \neq 0$. Put $f^{(-s)} = \hat{N}_{\psi^{-1}}(s, \chi^\ell\omega_\pi)h^{(s)}$. Then $f^{(s)}$ is a good section in view of Proposition 3.11(1) and (4). Lemma 3.5 shows that

$$\begin{aligned}\lim_{s\to 1} M_\psi(s,\chi^\ell\omega_\pi)f^{(s)} &= \lim_{s\to -1} M_\psi(-s,\chi^\ell\omega_\pi)\hat{N}_{\psi^{-1}}(s,\chi^\ell\omega_\pi)h^{(s)}\\ &= \lim_{s\to -1}\alpha_{\psi^{-1}}(s,\chi^\ell\omega_\pi)\frac{a(-s,\chi^\ell\omega_\pi)}{b(s,\chi^\ell\omega_\pi)}\,{}^\iota\big((J^\psi_{s,\chi^\ell\omega_\pi})^{-1}(h^{(s)})\big)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\lim_{s\to 1} Z(W,\Theta,N_\psi(s,\chi^\ell\omega_\pi)f^{(s)})\\ &\quad= \lim_{s\to -1} Z\big(W,\Theta,N_\psi(-s,\chi^\ell\omega_\pi)\hat{N}_{\psi^{-1}}(s,\chi^\ell\omega_\pi)h^{(s)}\big)\\ &\quad= \lim_{s\to -1}\alpha_{\psi^{-1}}(s,\chi^\ell\omega_\pi)Z\big(W,\Theta,{}^\iota\big((J^\psi_{s,\chi^\ell\omega_\pi})^{-1}(h^{(s)})\big)\big)\\ &\quad= \alpha_{\psi^{-1}}(-1,\chi^\ell\omega_\pi)Z({}^\iota W,\iota^\psi_\chi(\Theta),h^{(-1)})\\ &\quad\neq 0\end{aligned}$$

(see the proof of Proposition 3.14). □

Theorem 3.17. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$.*

(1) *Assume that n is even. Then $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has a pole at $s=0$ if and only if π is χ-distinguished.*

(2) *Assume that n is odd. Then $L(s,\pi,\mathrm{sym}^2)$ has a pole at $s=0$ if and only if ω_π is quadratic and $\pi\otimes\omega_\pi$ is distinguished.*

Proof. First we shall prove the "only if" part, which, in view of Lemma 1.12, is equivalent to showing that π is χ-distinguished if $L(s,\pi^\vee,\mathrm{sym}^2\otimes\chi^{-1})$ has a pole at $s=0$. Then $\chi^n\omega_\pi^2$ is trivial by Lemma 3.15(2). In the case of odd n we may assume that ω_π is trivial at the cost of replacing π by $\pi\otimes\omega_\pi$ if necessary. If n is even, then $\theta^\psi_\chi\otimes\chi^\ell\omega_\pi\simeq\theta^\psi_\chi$ by Lemma 1.9(3). We get

$$Z(W,\Theta,M_\psi(1,\chi^\ell\omega_\pi)f^{(1)}) = cZ(W,\Theta,f^{(1)})$$

by evaluating the functional equation stated in Proposition 3.14 at $s=1$, where

$$c = 2\frac{a(1,\chi^\ell\omega_\pi)\mathcal{E}(1,\pi,\chi,\psi)\,\mathrm{Res}_{s=0}L(s,\pi^\vee,\mathrm{sym}^2\otimes\chi^{-1})}{L(1,\pi,\mathrm{sym}^2\otimes\chi)\,\mathrm{Res}_{s=-1}b(s,\chi^\ell\omega_\pi)}\neq 0.$$

Since the zeta integral is convergent by Proposition 3.8(6), the functional

$$W\otimes\Theta\otimes f\mapsto Z(W,\Theta,M_\psi(1,\chi^\ell\omega_\pi)f)$$

factors through the quotient

$$\pi\otimes\theta^\psi_\chi\otimes I_\psi(1,\chi^\ell\omega_\pi)\to\pi\otimes\theta^\psi_\chi\otimes\theta^{\psi^{-1}}$$

by Lemma 1.15, and hence so does $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$. Therefore π is χ-distinguished by Corollary 3.9.

Next suppose that $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ is holomorphic at $s = 0$ and that $\chi^n \omega_\pi^2$ is trivial. If we take $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta_\chi^\psi$ and a good section $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$ as in Lemma 3.16, then the functional equation in Proposition 3.14 shows that $Z(W, \Theta, f^{(1)}) \neq 0$. Thus the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ fails to factor through the quotient, and hence π cannot be χ-distinguished by Corollary 3.9. □

3H. ***Shahidi's symmetric square L-factor.*** Let π be an irreducible admissible generic representation of G and χ a character of $F^\times$. We can define the twisted symmetric square L-factor by the Langlands–Shahidi method. We refer to [Shahidi 1990] for its precise definition. Henniart [2010] showed that this L-factor coincides with the Artin L-factor $L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)$, where ϕ denotes the local Langlands correspondence.

If F is a nonarchimedean local field of odd residual characteristic, π and χ are unramified and the order of ψ is 0, then there are a K-fixed Whittaker function $W^0 \in \mathcal{W}^\psi(\pi)$, a K-fixed semi-Whittaker function $\Theta^0 \in \theta_\chi^\psi$ and a K-fixed good section $f_0^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$ such that

$$Z(W^0, \Theta^0, f_0^{(2s-1)}) = L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi), \tag{3-1}$$
$$Z\big(W^0, \Theta^0, N(2s-1, \chi^\ell \omega_\pi) f_0^{(2s-1)}\big) = L(1-s, \mathrm{sym}^2 \circ \phi(\pi^\vee) \otimes \chi^{-1})$$

by Theorem 4.1 and Proposition 5.6 of [Bump and Ginzburg 1992] (cf. [Takeda 2014]). Though our zeta integral is slightly different if n is even, one can easily see that the unramified computation of our integral is reduced to their computation.

Thus $L(s, \pi, \mathrm{sym}^2 \otimes \chi)^{-1}$ is divisible by $L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)^{-1}$ if π and χ are unramified. However, the coincidence of the two L-factors still remains open even in the unramified case. Nevertheless, we can prove that the two L-factors agree in the square-integrable case.

Theorem 3.18. *Suppose that F is nonarchimedean. Let π be an irreducible square-integrable representation of G and χ a character of $F^\times$. Suppose that χ is the trivial character if n is odd. Then*

$$L(s, \pi, \mathrm{sym}^2 \otimes \chi) = L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi).$$

Proof. We may assume that χ is unitary, taking Remark 3.13 into account. The proof is similar to those of [Kewat and Raghunathan 2012, Theorem 1.1] and [Kable 2004, Theorem 6]. Although the statement is purely local, its proof uses the global functional equations for both Shahidi's L-function and the Rankin–Selberg integrals.

Let p_0 be the residual characteristic of F and q the cardinality of the residue field of F. We can find a number field $\mathbb{F}$ which has a unique place v_0 lying over p_0

and such that the completion $\mathbb{F}_{v_0}$ of $\mathbb{F}$ at v_0 is isomorphic to F. By Lemma 6.5 of Chapter 1 of [Arthur and Clozel 1989] there is an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ such that the local component Π_{v_0} of Π at v_0 is isomorphic to π, where $\mathbb{A}$ denotes the adèle ring of $\mathbb{F}$. Take a nontrivial additive character $\Psi : \mathbb{F}\backslash\mathbb{A} \to \mathbb{C}^\times$ and a Hecke character $\mathcal{X}$ of $\mathbb{A}^\times$ such that $\Psi_{v_0} = \psi$ and $\mathcal{X}_{v_0} = \chi$. We define the completed twisted symmetric square L-function by the infinite product

$$L(s, \Pi, \mathcal{X}, \mathrm{sym}^2) = \prod_v L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v).$$

The L-function $L(s, \Pi, \mathcal{X}, \mathrm{sym}^2)$ admits a meromorphic continuation to the entire complex plane and satisfies a functional equation

$$L(s, \Pi, \mathcal{X}, \mathrm{sym}^2) = \varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2) L(1-s, \Pi^\vee, \mathcal{X}^{-1}, \mathrm{sym}^2)$$

by Theorem 7.7 of [Shahidi 1990], where the function $\varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2)$ is entire and nonvanishing. The double cover $\overline{G}_{\mathbb{A}}$ of $G(\mathbb{A})$ and its global exceptional representation $\theta^\Psi_{\mathcal{X}}$ are constructed in [Kazhdan and Patterson 1984; Takeda 2014]. Note that $\overline{G}_{\mathbb{A}}$ is split over $G(F)$ and $\theta^\Psi_{\mathcal{X}}$ is an automorphic representation of $\overline{G}_{\mathbb{A}}$, which is isomorphic to the restricted tensor product $\bigotimes'_v \theta^{\Psi_v}_{\mathcal{X}_v}$. Let S_∞ be the set of archimedean places of $\mathbb{F}$ and S_r the set of finite places v for which Π_v or Ψ_v or $\theta^{\Psi_v}_{\mathcal{X}_v}$ is ramified. We set $S = S_\infty \cup S_r$.

We form the global induced representation and global intertwining operator. They have decompositions

$$I_\Psi(s, \mathcal{X}^\ell \omega_\Pi) \simeq \textstyle\bigotimes'_v I_{\Psi_v}(s, \mathcal{X}^\ell_v \omega_{\Pi_v}), \quad M(s, \mathcal{X}^\ell \omega_\Pi) = \textstyle\bigotimes'_v M(s, \mathcal{X}^\ell_v \omega_{\Pi_v}).$$

The global functional equation of the completed Hecke L-function yields

$$M(s, \mathcal{X}^\ell \omega_\Pi) = \varepsilon\bigl(\tfrac{1}{2}n(s-1) + 1, \mathcal{X}^n \omega^2_\Pi\bigr) \textstyle\bigotimes'_v N(s, \mathcal{X}^\ell_v \omega_{\Pi_v}). \tag{3-2}$$

For any holomorphic section $f^{(s)}$ of $I_\Psi(s, \mathcal{X}^\ell \omega_\Pi)$ we form the associated Eisenstein series $E(f^{(s)})$ on $G(F)\backslash\overline{G}_{\mathbb{A}}$ by

$$E(\tilde{g}, f^{(s)}) = \sum_{\gamma \in \mathcal{P}(F)\backslash G(F)} \sum_{\delta \in \mathscr{Z}\backslash Z(F)} f^{(s)}(\delta\gamma\tilde{g}),$$

where $\mathscr{Z} = \{z^{e(n)} \mid z \in Z(F)\}$. The series converges absolutely for $\Re s$ sufficiently large. By the theory of Eisenstein series, it can be continued to a meromorphic function on all of $\mathbb{C}$ satisfying the functional equation

$$E(f^{(s)}) = E(M(s, \mathcal{X}^\ell \omega_\Pi) f^{(s)}).$$

For $\varphi \in \Pi$, $\mathbf{\Theta} \in \theta_{\mathcal{X}}^{\Psi}$ and a meromorphic section $\boldsymbol{f}^{(s)}$ of $I_{\Psi}(s, \mathcal{X}^{\ell}\omega_{\Pi})$ we can consider the global zeta integral defined by

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = \int_{\mathscr{Z}_{\mathbb{A}} G(F)\backslash G(\mathbb{A})} \varphi(g)\mathbf{\Theta}(g)E(g, \boldsymbol{f}^{(s)})\,\mathrm{d}g,$$

where $\mathscr{Z}_{\mathbb{A}} = \{z^{e(n)} \mid z \in Z(\mathbb{A})\}$. This integral converges absolutely for all s away from the poles of the Eisenstein series and defines a meromorphic function in s satisfying

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = Z(\varphi, \mathbf{\Theta}, M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}).$$

The ψ_n-Whittaker coefficient of φ and the semi-Whittaker coefficients of $\mathbf{\Theta}$ and $\boldsymbol{f}^{(s)}$ are defined by

$$W^{\psi}(g, \varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\,\mathrm{d}u,$$

$$Q^{\psi}(\tilde{g}, \mathbf{\Theta}) = \int_{N(F)\backslash N(\mathbb{A})} \mathbf{\Theta}(\boldsymbol{s}(u)\tilde{g})\psi_{\boldsymbol{e}}(u)\,\mathrm{d}u,$$

$$R^{\psi}(\tilde{g}, \boldsymbol{f}^{(s)}) = \int_{N(F)\backslash N(\mathbb{A})} \boldsymbol{f}^{(s)}(\boldsymbol{s}(u)\tilde{g})\psi_{\boldsymbol{o}}(u)\,\mathrm{d}u.$$

In the case of even n the Rankin–Selberg integral differs slightly from those considered by Bump and Ginzburg [1992] or by Takeda [2014], but it can be unfolded to an adelic integral of the product of $W^{\psi}(\varphi)$, $Q^{\psi}(\mathbf{\Theta})$ and $R^{\psi}(\boldsymbol{f}^{(s)})$ in the same manner as in [Bump and Ginzburg 1992]. If $W^{\psi}(\varphi) = \bigotimes_v W_v$, $\mathbf{\Theta} = \bigotimes_v \Theta_v$ and $\boldsymbol{f}^{(s)} = \bigotimes_v f_v^{(s)}$ are factorizable, then

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = \prod_v Z(W_v, \Theta_v, f_v^{(s)}),$$

$$Z(\varphi, \mathbf{\Theta}, M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}) = \prod_v Z(W_v, \Theta_v, M(s, \mathcal{X}_v^{\ell}\omega_{\Pi_v})f_v^{(s)}).$$

The first factorization was proved by the author and Eyal Kaplan [Kaplan and Yamana 2016]. We here prove the second one. Put $\boldsymbol{h}^{(-s)} = M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}$. Unfolding the Eisenstein series, we have

$$\begin{aligned} Z(\varphi, \mathbf{\Theta}, \boldsymbol{h}^{(-s)}) &= \int_{\mathscr{Z}_{\mathbb{A}}{}^{\iota}\mathscr{P}(F)\backslash G(\mathbb{A})} \varphi(g)\Theta(g)\boldsymbol{h}^{(-s)}(g)\,\mathrm{d}g \\ &= \int_{\mathscr{Z}_{\mathbb{A}}\mathscr{P}(F)\backslash G(\mathbb{A})} \varphi({}^{\iota}g)\Theta({}^{\iota}g)\boldsymbol{h}^{(-s)}({}^{\iota}g)\,\mathrm{d}g. \end{aligned}$$

Substituting the Fourier expansion

$$\varphi({}^{\iota}g) = {}^{\iota}\varphi(g) = \sum_{\gamma \in N(F)\backslash \mathscr{P}(F)} W^{\psi^{-1}}(\gamma g, {}^{\iota}\varphi) = \sum_{\gamma \in N(F)\backslash {}^{\iota}\mathscr{P}(F)} W^{\psi}(\gamma\, {}^{\iota}g, \varphi),$$

we get

$$Z(\varphi, \boldsymbol{\Theta}, \boldsymbol{h}^{(-s)}) = \int_{\mathscr{Z}_{\mathbb{A}} N(F)\backslash G(\mathbb{A})} W^{\psi}({}^{\iota}g, \varphi)\Theta({}^{\iota}g)\boldsymbol{h}^{(-s)}({}^{\iota}g)\,\mathrm{d}g,$$

where our formal manipulations can be justified by the absolute convergence of this integral for $\Re s \ll 0$, which can be checked by a gauge estimate. For $i = 1, 2, \dots, n-1$ we put

$$\mathscr{U}^{(i)} = \left\{ \begin{pmatrix} \mathbf{1}_{n-i} & b \\ 0 & u \end{pmatrix} \,\middle|\, b \in \mathrm{M}_{n-i,i},\ u \in N_i \right\}.$$

Proposition 1.3 enables us to lift the main involution of $G(\mathbb{A})$ to $\bar{G}_{\mathbb{A}}$. For $i = 1, 2, \dots, n-1$ we define

$$Q_i(\tilde{g}) = \int_{\mathscr{U}^{(i)}(F)\backslash \mathscr{U}^{(i)}(\mathbb{A})} {}^{\iota}\Theta(\boldsymbol{s}(u)\tilde{g})\overline{\psi_{\boldsymbol{e}}(u)}\,\mathrm{d}u,$$

$$R_i(\tilde{g}, -s) = \int_{\mathscr{U}^{(i)}(F)\backslash \mathscr{U}^{(i)}(\mathbb{A})} {}^{\iota}\boldsymbol{h}^{(-s)}(\boldsymbol{s}(u)\tilde{g})\overline{\psi_{\boldsymbol{o}}(u)}\,\mathrm{d}u,$$

$$Z_i(\varphi, \Theta, \boldsymbol{h}^{-s}) = \int_{\mathscr{Z}_{\mathbb{A}} N(F)\mathscr{U}^{(i)}(\mathbb{A})\backslash G(\mathbb{A})} {}^{\iota}W^{\psi}(g, \varphi)Q_i(g)R_i(g, -s)\,\mathrm{d}g.$$

Let $\mathscr{N}_i$ be the subgroup of N consisting of matrices whose only nonzero off-diagonal elements are in the $(n-i)$-th column. When i is odd, Propositions 2.4 and 2.5 of [Bump and Ginzburg 1992] and Lemma 3.11 of [Takeda 2014] state that $Q_i(\boldsymbol{s}(u)\tilde{g})$ is independent of $u \in \mathscr{N}_i(\mathbb{A})$ and equal to $Q_{i+1}(\tilde{g})$, and hence

$$\begin{aligned} Z_i(\varphi, \Theta, \boldsymbol{h}^{(-s)}) &= \int_{\mathscr{Z}_{\mathbb{A}} N(F)(\mathbb{A})\mathscr{U}^{(i+1)}(\mathbb{A})\backslash G(\mathbb{A})} W^{\psi}(g, \varphi)Q_{i+1}(g) \\ &\qquad \times \int_{\mathscr{N}_i(F)\backslash \mathscr{N}_i(\mathbb{A})} R_i(\boldsymbol{s}(u)g, -s)\overline{\psi(u)}\,\mathrm{d}u\,\mathrm{d}g \\ &= Z_{i+1}(\varphi, \Theta, \boldsymbol{h}^{(-s)}). \end{aligned}$$

When i is even, Propositions 2.4 and 2.5 of [Bump and Ginzburg 1992] and Lemma 3.11 of [Takeda 2014] again imply that $Z_i(\varphi, \Theta, h^{(-s)}) = Z_{i+1}(\varphi, \Theta, h^{(-s)})$. Consequently,

$$\begin{aligned} Z(\varphi, \Theta, \boldsymbol{h}^{(-s)}) &= Z_1(\varphi, \Theta, \boldsymbol{h}^{(-s)}) = \cdots = Z_{n-1}(\varphi, \Theta, \boldsymbol{h}^{(-s)}) \\ &= \int_{\mathscr{Z}_{\mathbb{A}} N(\mathbb{A})\backslash G(\mathbb{A})} W^{\psi}(g, \varphi)Q^{\psi^{-1}}({}^{\iota}g, {}^{\iota}\Theta)R^{\psi^{-1}}({}^{\iota}g, {}^{\iota}\boldsymbol{h}^{(-s)})\,\mathrm{d}g. \end{aligned}$$

Since the semi-Whittaker function of Θ is the Whittaker function of the $M_{\boldsymbol{e}(n)}$-part of the constant term of Θ along $P_{\boldsymbol{e}(n)}$, one can verify that it is factorizable, and similarly for $\boldsymbol{h}^{(-s)}$, which gives rise to the factorization we want.

There are $W_i \in \mathcal{W}^{\psi}(\pi)$, $\Theta_i \in \theta_{\chi}^{\psi}$ and good sections $f_i^{(s)}$ such that

$$\sum_i Z(W_i, \Theta_i, f_i^{(2s-1)}) = L(s, \pi, \mathrm{sym}^2 \otimes \chi).$$

On substituting each of these triplets into the functional equation in Proposition 3.14 and summing the results, we find that

$$\sum_i Z\big(W_i, \Theta_i, N(2s-1, \chi^{\ell}\omega_\pi) f_i^{(2s-1)}\big) = \mathcal{E}(s, \pi, \chi, \psi) L(1-s, \pi^\vee, \mathrm{sym}^2 \otimes \chi^{-1}). \tag{3-3}$$

For $v \in S \setminus \{v_0\}$ we choose $W_v \in \mathcal{W}^{\Psi_v}(\Pi_v)$, $\Theta_v \in \theta_{\mathcal{X}_v}^{\Psi_v}$ and standard sections $f_v^{(s)}$ such that $Z(W_v, \Theta_v, f_v^{(s)})$ is not identically zero. Put

$$\boldsymbol{W}_i = W_i \otimes \Big(\bigotimes_{v \in S\setminus\{v_0\}} W_v\Big) \otimes \Big(\bigotimes_{v \notin S} W_v^0\Big),$$

$$\boldsymbol{\Theta}_i = \Theta_i \otimes \Big(\bigotimes_{v \in S\setminus\{v_0\}} \Theta_v\Big) \otimes \Big(\bigotimes_{v \notin S} \Theta_v^0\Big),$$

$$\boldsymbol{f}_i^{(s)} = f_i^{(s)} \otimes \Big(\bigotimes_{v \in S\setminus\{v_0\}} f_v^{(s)}\Big) \otimes \Big(\bigotimes_{v \notin S} f_{v,0}^{(s)}\Big).$$

Further set

$$A(s) = L(s, \Pi, \mathcal{X}, \mathrm{sym}^2)^{-1} \sum_i Z(\boldsymbol{W}_i, \boldsymbol{\Theta}_i, \boldsymbol{f}_i^{(2s-1)}) = a(s)\alpha(s)a(s, \pi, \chi),$$

where

$$a(s, \pi, \chi) = L(s, \pi, \mathrm{sym}^2 \otimes \chi)/L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)$$

and

$$a(s) = \prod_{v \in S_r\setminus\{v_0\}} \frac{Z(W_v, \Theta_v, f_v^{(2s-1)})}{L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v)},$$

$$\alpha(s) = \prod_{v \in S_\infty} \frac{Z(W_v, \Theta_v, f_v^{(2s-1)})}{L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v)}.$$

Similarly, we put

$$B(s) = L(s, \Pi^\vee, \mathcal{X}^{-1}, \mathrm{sym}^2)^{-1} \sum_i Z\big(\boldsymbol{W}_i, \boldsymbol{\Theta}_i, M(1-2s, \mathcal{X}^{\ell}\omega_\Pi) \boldsymbol{f}_i^{(1-2s)}\big).$$

Note that

$$B(s) = \varepsilon(1-ns, \mathcal{X}^n \omega_\Pi^2) \mathcal{E}(1-s, \pi, \chi, \psi) b(s)\beta(s) a(s, \pi^\vee, \chi^{-1})$$

by (3-2) and (3-3), where $b(s)$ (resp. $\beta(s)$) is a product of the ratios

$$Z\big(W_v, \Theta_v, N(1-2s, \mathcal{X}_v^{\ell}\omega_{\Pi_v}) f_v^{(1-2s)}\big)/L\big(s, \mathrm{sym}^2 \circ \phi(\Pi_v^\vee) \otimes \mathcal{X}_v^{-1}\big)$$

over $v \in S_r \setminus \{v_0\}$ (resp. $v \in S_\infty$). Plugging the functional equation of Shahidi's L-function into the functional equation of the global zeta integral, we are led to

$$\varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2)A(s) = B(1-s),$$

that is,

$$\begin{aligned}(3\text{-}4)\quad & \varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2)a(s)\alpha(s)a(s, \pi, \chi) \\ &= \mathcal{E}(s, \pi, \chi, \psi)\varepsilon(n(s-1)+1, \mathcal{X}^n\omega_\Pi^2)b(1-s)\beta(1-s)a(1-s, \pi^\vee, \chi^{-1}).\end{aligned}$$

To prove Theorem 3.18, it is enough to prove that $a(s, \pi, \chi)$ is entire and nowhere vanishing. First suppose that $a(s, \pi, \chi)$ has a zero at $s = s_0$. This means that $a(s, \pi, \chi)$ has zeros at $s_0 + k(2\pi\sqrt{-1})/\log q$ for all $k \in \mathbb{Z}$. We claim that all but finitely many of these zeros must also be zeros of $A(s)$. This fails to happen only if all but finitely many zeros are canceled by the poles of $a(s)\alpha(s)$. The function $\alpha(s)$ can contribute only finitely many poles on any line with real part constant by Proposition 3.8(3), and this set of poles is independent of the choice of W_v, Θ_v and $f_v^{(s)}$ at the archimedean places. Hence $a(s)$ must have infinitely many poles of this form. Since the poles of $a(s)$ are of the form $s_j + m(4\pi\sqrt{-1})/\log q_v$ for $m \in \mathbb{Z}$ with $v \in S_r \setminus \{v_0\}$, there are a place v and $s_j \in \mathbb{C}$ and two integers $m_1 \neq m_2$ such that

$$s_0 + k_1\frac{2\pi\sqrt{-1}}{\log q} = s_j + m_1\frac{4\pi\sqrt{-1}}{\log q_v}, \quad s_0 + k_2\frac{2\pi\sqrt{-1}}{\log q} = s_j + m_2\frac{4\pi\sqrt{-1}}{\log q_v}$$

for some $k_1, k_2 \in \mathbb{Z}$ (in fact, there are infinitely many distinct integers with this property). Then $\log q_v/\log q$ is rational, which contradicts $(q_v, q) = 1$. Thus the points $s_0 + k(2\pi\sqrt{-1})/\log q$ are zeros of $A(s)$ for all but finitely many k.

Since $L(s, \mathrm{sym}\circ\phi(\pi)\otimes\chi)$ is holomorphic in the region $\Re s > 0$ by Proposition 7.2 of [Shahidi 1990], the function $a(s, \pi, \chi)$ is nonvanishing in the region $\Re s > 0$. Thus $\Re s_0 \leq 0$. From (3-4) we see that all but finitely many of the points $1 - s_0 + k(2\pi\sqrt{-1})/\log q$ are zeros of the function $B(s)$. Since $a(s, \pi^\vee, \chi^{-1})$ is nonzero for $\Re s > 0$, these zeros have to be the zeros of $b(s)\beta(s)$. Arguing as above, these cannot be zeros of $b(s)$ for infinitely many k. Since the poles of

$$\prod_{v\in S_\infty} L(s, \mathrm{sym}\circ\phi(\Pi_v^\vee)\otimes\mathcal{X}_v^{-1})$$

lie along horizontal lines, this product can contribute only finitely many poles on any vertical line. Thus these must be common zeros of functions

$$\prod_{v\in S_\infty} Z\big(W_v, \Theta_v, N(1-2s, \mathcal{X}_v^\ell\omega_{\Pi_v})f_v^{(1-2s)}\big)$$

for all W_v, Θ_v and $f_v^{(s)}$. This contradicts Proposition 3.8(4) in view of the proof of Proposition 3.14.

Suppose that $a(s,\pi,\chi)$ has a pole at $s=s_0$. Since $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ is holomorphic in the region $\Re s>0$ by Lemma 3.15(1), we obtain $\Re s_0\leq 0$. By Proposition 3.8(5) the product $a(s)\alpha(s)$ is holomorphic in $\Re s\geq 1$ and the function $b(1-s)\beta(1-s)$ is holomorphic in $\Re s\leq 0$. Therefore $A(s)$ is holomorphic in $\Re s\geq 1$ and $\Re s\leq 0$ by (3-4), so that the pole of $a(s,\pi,\chi)$ must be canceled by the zeros of $a(s)\alpha(s)$. Arguing as above, we can see that $s_0+k(4\pi\sqrt{-1})/\log q$ cannot be zeros of $a(s)$ for infinitely many integers k. Since the poles of

$$\prod_{v\in S_\infty} L(s,\mathrm{sym}^2\circ\phi(\Pi_v)\otimes\mathcal{X}_v)$$

lie along horizontal lines, this product can contribute only finitely many poles on any vertical line. Thus these must be common zeros of functions

$$\prod_{v\in S_\infty} Z(W_v,\Theta_v,f_v^{(2s-1)})$$

for all W_v, Θ_v and $f_v^{(s)}$, which contradicts Proposition 3.8(4). ☐

3I. *Proof of Theorem A and Corollary A.*

Theorem 3.19. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$.*

(1) *Assume that n is even. Then $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$ if and only if π is χ-distinguished.*

(2) *Assume that n is odd. Then $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$ if and only if $\omega_\pi^2=\chi^{-n}$ and $\pi\otimes(\omega_\pi^{-1}\chi^{-(n-1)/2})$ is distinguished.*

Proof. Theorems 3.17 and 3.18 prove the first part. The factorization (0-1) is extended to the twisted case as follows:

$$L(s,\phi(\pi)\otimes\phi(\pi)\otimes\chi)=L(s,\Lambda^2\circ\phi(\pi)\otimes\chi)L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi).$$

It is a consequence of Proposition 8.1 and Theorem 8.2 of [Jacquet et al. 1983] that $L(s,\phi(\pi)\otimes\phi(\pi)\otimes\chi)$ has a simple pole at $s=0$ exactly when $\pi\simeq\pi^\vee\otimes\chi^{-1}$.

Suppose that n is odd. If $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$, then $\pi\simeq\pi^\vee\otimes\chi^{-1}$ and hence $\omega_\pi^2=\chi^{-n}$. Put $\mu=\omega_\pi\chi^{(n-1)/2}$ and $\pi'=\pi\otimes\mu^{-1}$. If $\omega_\pi^2=\chi^{-n}$, then $\mu^2=\chi^{-1}$, $\omega_{\pi'}=\omega_\pi\mu^{-n}=\omega_\pi\chi^{(n-1)/2}\mu^{-1}=1$ and

$$L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)=L(s,\mathrm{sym}^2\circ\phi(\pi'))=L(s,\pi',\mathrm{sym}^2).$$

The equivalence now amounts to a combination of Theorems 3.17 and 3.18. ☐

When $\pi\simeq\pi^\vee$, one of the L-factors on the right-hand side of the factorization (0-1) must have a pole at $s=0$, and the other does not.

If n is odd or ω is nontrivial, then $L(s, \Lambda^2 \circ \phi(\pi))$ cannot have a pole at $s = 0$ by Theorems 4.3 and 6.1 of [Kewat and Raghunathan 2012], so that $\pi \simeq \pi^\vee$ if and only if $L(s, \mathrm{sym}^2 \circ \phi(\pi))$ has a pole at $s = 0$. Thus Lemma 1.12(1) and Theorem 3.17(2) prove Corollary A(1). Theorem 3.19(1) proves Corollary A(2).

Assume that n is even and ω is trivial. Then $L(s, \Lambda^2 \circ \phi(\pi))$ has a pole at $s = 0$ if and only if π admits a nontrivial Shalika model by Proposition 3.4 of [Lapid and Mao 2017]. This combined with Theorem 3.19(1) proves Corollary A(3).

Remark 3.20. In the proof of Corollary A we limit ourselves to the nontwisted case even when n is even, because of the lack of knowledge of suitable generalizations of the results for the twisted exterior square L-factors.

Acknowledgements

The research leading to these results has received funding from the European Research Council under the European Community's Seventh Framework Programme (FP7/2007-2013)/ERC Grant agreement n° 290766 (AAMOT). The author is partially supported by JSPS Grant-in-Aid for Young Scientists (B) 26800017. Finally, we would like to thank the referee for constructive criticism and a number of useful comments.

References

[Anandavardhanan et al. 2004] U. K. Anandavardhanan, A. C. Kable, and R. Tandon, "Distinguished representations and poles of twisted tensor L-functions", *Proc. Amer. Math. Soc.* **132**:10 (2004), 2875–2883. MR Zbl

[Arthur and Clozel 1989] J. Arthur and L. Clozel, *Simple algebras, base change, and the advanced theory of the trace formula*, Annals of Mathematics Studies **120**, Princeton University Press, 1989. MR Zbl

[Ban and Jantzen 2013] D. Ban and C. Jantzen, "The Langlands quotient theorem for finite central extensions of p-adic groups", *Glas. Mat. Ser. III* **48(68)**:2 (2013), 313–334. MR Zbl

[Banks et al. 1999] W. D. Banks, J. Levy, and M. R. Sepanski, "Block-compatible metaplectic cocycles", *J. Reine Angew. Math.* **507** (1999), 131–163. MR Zbl

[Bernstein 1984] J. N. Bernstein, "P-invariant distributions on $\mathrm{GL}(N)$ and the classification of unitary representations of $\mathrm{GL}(N)$ (non-Archimedean case)", pp. 50–102 in *Lie group representations, II* (College Park, MD, 1982–1983), edited by R. Herb et al., Lecture Notes in Math. **1041**, Springer, Berlin, 1984. MR Zbl

[Bernstein and Zelevinsky 1977] I. N. Bernstein and A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, I", *Ann. Sci. École Norm. Sup.* (4) **10**:4 (1977), 441–472. MR Zbl

[Bump and Ginzburg 1992] D. Bump and D. Ginzburg, "Symmetric square L-functions on $\mathrm{GL}(r)$", *Ann. of Math.* (2) **136**:1 (1992), 137–205. MR Zbl

[Flicker 1991] Y. Z. Flicker, "On distinguished representations", *J. Reine Angew. Math.* **418** (1991), 139–172. MR Zbl

[Flicker et al. 1990] Y. Flicker, D. Kazhdan, and G. Savin, "Explicit realization of a metaplectic representation", *J. Analyse Math.* **55** (1990), 17–39. MR Zbl

[Henniart 2010] G. Henniart, "Correspondance de Langlands et fonctions L des carrés extérieur et symétrique", *Int. Math. Res. Not.* **2010**:4 (2010), 633–673. MR Zbl

[Jacquet and Rallis 1996] H. Jacquet and S. Rallis, "Uniqueness of linear periods", *Compositio Math.* **102**:1 (1996), 65–123. MR Zbl

[Jacquet and Shalika 1983] H. Jacquet and J. Shalika, "The Whittaker models of induced representations", *Pacific J. Math.* **109**:1 (1983), 107–120. MR Zbl

[Jacquet and Shalika 1990] H. Jacquet and J. Shalika, "Exterior square L-functions", pp. 143–226 in *Automorphic forms, Shimura varieties, and L-functions* (Ann Arbor, MI, 1988), vol. 2, edited by L. Clozel and J. S. Milne, Perspect. Math. **11**, Academic Press, Boston, 1990. MR Zbl

[Jacquet et al. 1983] H. Jacquet, I. I. Piatetskii-Shapiro, and J. A. Shalika, "Rankin–Selberg convolutions", *Amer. J. Math.* **105**:2 (1983), 367–464. MR Zbl

[Kable 1999] A. C. Kable, "The main involutions of the metaplectic group", *Proc. Amer. Math. Soc.* **127**:4 (1999), 955–962. MR Zbl

[Kable 2001] A. C. Kable, "The tensor product of exceptional representations on the general linear group", *Ann. Sci. École Norm. Sup.* (4) **34**:5 (2001), 741–769. MR Zbl

[Kable 2004] A. C. Kable, "Asai L-functions and Jacquet's conjecture", *Amer. J. Math.* **126**:4 (2004), 789–820. MR Zbl

[Kaplan and Yamana 2016] E. Kaplan and S. Yamana, "Twisted symmetric square L-functions for $\mathrm{GL}(n)$ and invariant trilinear forms", *Math. Z.* (online publication September 2016).

[Kazhdan and Patterson 1984] D. A. Kazhdan and S. J. Patterson, "Metaplectic forms", *Inst. Hautes Études Sci. Publ. Math.* 59 (1984), 35–142. MR Zbl

[Kewat 2011] P. K. Kewat, "The local exterior square L-function: holomorphy, non-vanishing and Shalika functionals", *J. Algebra* **347** (2011), 153–172. MR Zbl

[Kewat and Raghunathan 2012] P. K. Kewat and R. Raghunathan, "On the local and global exterior square L-functions of GL_n", *Math. Res. Lett.* **19**:4 (2012), 785–804. MR Zbl

[Lapid and Mao 2017] E. Lapid and Z. Mao, "Whittaker–Fourier coefficients of cusp forms on $\widetilde{\mathrm{Sp}_n}$: reduction to a local statement", *Amer. J. Math.* **139**:1 (2017). arXiv

[Matringe 2014] N. Matringe, "Unitary representations of $\mathrm{GL}(n, K)$ distinguished by a Galois involution for a p-adic field K", *Pacific J. Math.* **271**:2 (2014), 445–460. MR Zbl

[Matsumoto 1969] H. Matsumoto, "Sur les sous-groupes arithmétiques des groupes semi-simples déployés", *Ann. Sci. École Norm. Sup.* (4) **2** (1969), 1–62. MR Zbl

[Ok 1997] Y. Ok, *Distinction and gamma factors at 1/2: supercuspidal case*, Ph.D. thesis, Columbia University, 1997, available at http://search.proquest.com/docview/304339470. MR

[Shahidi 1990] F. Shahidi, "A proof of Langlands' conjecture on Plancherel measures; complementary series for p-adic groups", *Ann. of Math.* (2) **132**:2 (1990), 273–330. MR Zbl

[Sun 2012] B. Sun, "Multiplicity one theorems for Fourier–Jacobi models", *Amer. J. Math.* **134**:6 (2012), 1655–1678. MR Zbl

[Takeda 2014] S. Takeda, "The twisted symmetric square L-function of $\mathrm{GL}(r)$", *Duke Math. J.* **163**:1 (2014), 175–266. MR Zbl

[Takeda 2015] S. Takeda, "On a certain metaplectic Eisenstein series and the twisted symmetric square L-function", *Math. Z.* **281**:1-2 (2015), 103–157. MR Zbl

[Waldspurger 2003] J.-L. Waldspurger, "La formule de Plancherel pour les groupes p-adiques (d'après Harish-Chandra)", *J. Inst. Math. Jussieu* **2**:2 (2003), 235–333. MR Zbl

[Yamana 2014] S. Yamana, "L-functions and theta correspondence for classical groups", *Invent. Math.* **196**:3 (2014), 651–732. MR Zbl

[Zelevinsky 1980] A. V. Zelevinsky, "Induced representations of reductive $\mathfrak{p}$-adic groups, II: On irreducible representations of $\mathrm{GL}(n)$", *Ann. Sci. École Norm. Sup.* (4) **13**:2 (1980), 165–210. MR Zbl

Received April 30, 2015. Revised September 3, 2015.

SHUNSUKE YAMANA
DEPARTMENT OF MATHEMATICS
KYOTO UNIVERSITY
KITASHIRAKAWA OIWAKE-CHO
SAKYO-KU
KYOTO 606-8502
JAPAN

and

HAKUBI CENTER
KYOTO UNIVERSITY
YOSHIDA-HONMACHI
SAKYO-KU
KYOTO 606-8501
JAPAN

yamana07@math.kyoto-u.ac.jp

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US $450/year for the electronic version, and $625/year for print and electronic.
Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

mathematical sciences publishers

nonprofit scientific publishing

http://msp.org/

PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 1 January 2017

0030-8730(201701)286:1;1-Z