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ELLIPTIC CURVES, RANDOM MATRICES AND ORBITAL INTEGRALS

JEFFREY D. ACHTER AND JULIA GORDON

WITH AN APPENDIX BY S. ALI ALTUĞ

An isogeny class of elliptic curves over a finite field is determined by a quadratic Weil polynomial. Gekeler has given a product formula, in terms of congruence considerations involving that polynomial, for the size of such an isogeny class (over a finite prime field). In this paper we give a new transparent proof of this formula; it turns out that this product actually computes an adelic orbital integral which visibly counts the desired cardinality. This answers a question posed by N. Katz and extends Gekeler's work to ordinary elliptic curves over arbitrary finite fields.

1. Introduction

The isogeny class of an elliptic curve over a finite field $\mathbb{F}_p$ of p elements is determined by its trace of Frobenius; calculating the size of such an isogeny class is a classical problem. Fix a number a with $|a| \le 2\sqrt{p}$, and let $I(a, p)$ be the set of all elliptic curves over $\mathbb{F}_p$ with trace of Frobenius a. Further suppose that $p \nmid a$, so that the isogeny class is ordinary.

Gekeler [2003] proposed a random matrix model to compute the size of $I(a, p)$ (see also [Katz 2009]). For each rational prime $\ell \neq p$, let

(1-1)
$$\nu_\ell(a, p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) : \mathrm{tr}(\gamma) \equiv a \bmod \ell^n, \det(\gamma) \equiv p \bmod \ell^n\}}{\#\,\mathrm{SL}_2(\mathbb{Z}/\ell^n)/\ell^n}.$$

For $\ell = p$, let

(1-2)
$$\nu_p(a, p) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{M}_2(\mathbb{Z}/p^n) : \mathrm{tr}(\gamma) \equiv a \bmod p^n, \det(\gamma) \equiv p \bmod p^n\}}{\#\,\mathrm{SL}_2(\mathbb{Z}/p^n)/p^n}.$$

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On average, the number of elements of $\mathrm{GL}_2(\mathbb{Z}/\ell^n)$ with a given characteristic polynomial is $\#\,\mathrm{GL}_2(\mathbb{Z}/\ell^n)/(\#(\mathbb{Z}/\ell^n)^\times \cdot \ell^n)$. Thus, $\nu_\ell(a, p)$ measures the departure of the frequency of the event that a random matrix γ satisfies $f_\gamma(T) = T^2 - aT + p$ from the average (over all possible characteristic polynomials).

It turns out [Gekeler 2003, Theorem 5.5] that

$$\widetilde{\#} I(a, p) = \frac{1}{2}\sqrt{p}\,\nu_\infty(a, p) \prod_\ell \nu_\ell(a, p), \tag{1-3}$$

where

$$\nu_\infty(a, p) = \frac{2}{\pi}\sqrt{1 - \frac{a^2}{4p}},$$

$\widetilde{\#} I(a, p)$ is a count weighted by automorphisms (2-1), and we note that the term $H^*(a, p)$ of [Gekeler 2003] actually computes $2\widetilde{\#} I(a, p)$ (see [Gekeler 2003, (2.10) and (2.13); Katz 2009, Theorem 8.5, p. 451]). This equation is almost miraculous. An equidistribution assumption about Frobenius elements, which is so strong that it can't possibly be true, leads one to the correct conclusion.

In contrast to the heuristic, the proof of (1-3) is somewhat pedestrian. Let $\Delta_{a,p} = a^2 - 4p$, let $K_{a,p} = \mathbb{Q}(\sqrt{\Delta_{a,p}})$, and let $\chi_{a,p}$ be the associated quadratic character. Classically, the size of the isogeny class $I(a, \mathbb{F}_p)$ is given by the Kronecker class number $H(\Delta_{a,p})$. Direct calculation [Gekeler 2003] shows that, at least for unramified primes ℓ,

$$\nu_\ell(a, p) = \frac{1}{1 - \chi_{a,p}(\ell)/\ell}$$

is the term at ℓ in the Euler product expansion of $L(1, \chi_{a,p})$. More generally, a term by term comparison shows that the right-hand side of (1-3) computes $H(\Delta_{a,p})$.

Even though (1-3) is striking and unconditional, one might still want a pure thought derivation of it. (We are not alone in this desire; Katz calls attention to this question in [Katz 2009, Remark 8.7].) Our goal in the present paper is to provide a conceptual explanation of (1-3). We will show that Gekeler's random matrix model (i.e., the right-hand side of (1-3)) directly calculates $\widetilde{\#} I(a, p)$, *without* appeal to class numbers. A further payoff of our method is that we extend Gekeler's results to the case of ordinary elliptic curves over an arbitrary finite field $\mathbb{F}_q$.

Our method relies on the description, due to Langlands (for modular curves) and Kottwitz (in general), of the points on a Shimura variety over a finite field. A consequence of their study is that one can calculate the cardinality of an ordinary isogeny class of elliptic curves over $\mathbb{F}_q$ using orbital integrals on the finite adelic points of GL_2 (Proposition 2.1). Our main observation is that one can, without explicit calculation, relate each local factor $\nu_\ell(a, q)$ to an orbital integral

$$\int_{G_{\gamma_\ell}(\mathbb{Q}_\ell)\backslash \mathrm{GL}_2(\mathbb{Q}_\ell)} \mathbb{1}_{\mathrm{GL}_2(\mathbb{Z}_\ell)}(x^{-1}\gamma_\ell x)\, dx, \tag{1-4}$$

where γ_ℓ is an element of $\mathrm{GL}_2(\mathbb{Q}_\ell)$ of trace a and determinant q, G_{γ_ℓ} is its centralizer in $\mathrm{GL}_2(\mathbb{Q}_\ell)$, and $\mathbb{1}_{\mathrm{GL}_2(\mathbb{Z}_\ell)}$ is the characteristic function of the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_\ell)$. Here the choice of the invariant measure dx on the orbit is crucial. On one hand, the measure that is naturally related to Gekeler's numbers is the so-called *geometric measure* (see [Frenkel et al. 2010]), which we review in Section 3A3. On the other hand, this measure is inconvenient for computing the global volume term that appears in the formula of Langlands and Kottwitz. The main technical difficulty is the comparison, which should be well-known but is hard to find in the literature, between the geometric measure and the so-called *canonical measure*.

We start in Section 2 by establishing notation and reviewing the Langlands–Kottwitz formula. We define the relevant natural measures in Section 3, and study the comparison factor between them in Section 4. Finally, in Section 5, we complete the global calculation.

It is perhaps not surprising that one can use a similar method to give an analogous product formula for the size of an isogeny class of simple ordinary principally polarized abelian varieties over a finite field. (The fact that the group controlling the moduli problem is GSp_{2g} rather than GL_2 means that, for example, conjugacy and stable conjugacy no longer coincide, the explicit invocation of the fundamental lemma is more involved, the comparison of measures (Proposition 4.5) is more difficult, the global volume calculation is less immediate, etc.) We take up this challenge in a companion work.

It turns out that [Frenkel et al. 2010, §3] has much of the information one needs for the crucial comparison of measures. This is explained in the Appendix by S. Ali Altuğ.

As we were finishing this paper, the authors of [David et al. 2016] shared their work with us, which takes Gekeler's random matrix model as its starting point; we invite the interested reader to consult that work.

Notation. Throughout, $\mathbb{F}_q$ is a finite field of characteristic p and cardinality $q = p^e$. Let $\mathbb{Q}_q$ be the unique unramified extension of $\mathbb{Q}_p$ of degree e, and let $\mathbb{Z}_q \subset \mathbb{Q}_q$ be its ring of integers. We use σ to denote both the canonical generator of $\mathrm{Gal}(\mathbb{F}_q/\mathbb{F}_p)$ and its lift to $\mathrm{Gal}(\mathbb{Q}_q/\mathbb{Q}_p)$.

Typically, G will denote the algebraic group GL_2. While many of our results admit immediate generalization to other reductive groups, as a rule we resist this temptation unless the statement and its proof require no additional notation.

Shortly, we will fix a regular semisimple element $\gamma_0 \in G(\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})$; its centralizer will variously be denoted G_{γ_0} and T.

Conjugacy in an abstract group is denoted by $\sim$.

2. Preliminaries

Here we collect notation concerning isogeny classes (Section 2A) as well as basic information on Gekeler's ratios (Section 2C) and the Langlands–Kottwitz formula (Section 2D).

2A. *Isogeny classes of elliptic curves.* If $E/\mathbb{F}_q$ is an elliptic curve, then its characteristic polynomial of Frobenius has the form $f_{E/\mathbb{F}_q}(T) = T^2 - a_{E/\mathbb{F}_q}T + q$, where $|a_{E/\mathbb{F}_q}| \le 2\sqrt{q}$. Moreover, E_1 and E_2 are $\mathbb{F}_q$-isogenous if and only if $a_{E_1/\mathbb{F}_q} = a_{E_2/\mathbb{F}_q}$. In particular, for a given integer a with $|a| \le 2\sqrt{q}$, the set

$$I(a,q) = \{E/\mathbb{F}_q : a_{E/\mathbb{F}_q} = a\}$$

is a single isogeny class of elliptic curves over $\mathbb{F}_q$. Its weighted cardinality is

$$\widetilde{\#} I(a,q) := \sum_{E \in I(a,q)} \frac{1}{\#\,\mathrm{Aut}(E)}. \tag{2-1}$$

A member of this isogeny class is ordinary if and only if $p \nmid a$; henceforth, we assume this is the case.

Fix an element $\gamma_0 \in G(\mathbb{Q})$ with characteristic polynomial

$$f_0(T) = f_{a,q}(T) := T^2 - aT + q.$$

Newton polygon considerations show that exactly one root of $f_{a,q}(T)$ is a p-adic unit, and in particular $f_{a,q}(T)$ has distinct roots. Therefore, γ_0 is regular semisimple. Moreover, any other element of $G(\mathbb{Q})$ with the same characteristic polynomial is conjugate to γ_0. (Here and elsewhere, we use the fact that in a general linear group, two elements are conjugate if and only if they are stably conjugate.)

Let $K = K_{a,q} = \mathbb{Q}[T]/f(T)$; it is a quadratic imaginary field. If $E \in I(a,q)$, then its endomorphism algebra is $\mathrm{End}(E) \otimes \mathbb{Q} \cong K$. The centralizer G_{γ_0} of γ_0 in G is the restriction of scalars torus $G_{\gamma_0} \cong \mathrm{R}_{K/\mathbb{Q}}\mathbb{G}_m$.

If α is an invariant of an isogeny class, we will variously denote it as $\alpha(a,q)$, $\alpha(f_0)$, or $\alpha(\gamma_0)$, depending on the desired emphasis.

2B. *The Steinberg quotient.* We review the general definition of the Steinberg quotient. Let G be a split, reductive group of rank r, with simply connected derived group G^{der} and Lie algebra $\mathfrak{g}$; further assume that $G/G^{\mathrm{der}} \cong \mathbb{G}_m$. (In the case of interest for this paper, $G = \mathrm{GL}_2$, $r = 2$, and $G^{\mathrm{der}} = \mathrm{SL}_2$.)

Let T be a split maximal torus in G, $T^{\mathrm{der}} = T \cap G^{\mathrm{der}}$ (note that T^{der} is *not* the derived group of T), and W be the Weyl group of G relative to T. Let $A^{\mathrm{der}} = T^{\mathrm{der}}/W$ be the Steinberg quotient for the semisimple group G^{der}. It is isomorphic to the affine space of dimension $r-1$.

Let $A = A^{\mathrm{der}} \times \mathbb{G}_m$ be the analogue of the Steinberg quotient for the reductive group G, see [Frenkel et al. 2010]. We think of A as the space of "characteristic polynomials". There is a canonical map

$$G \xrightarrow{\mathfrak{c}} A. \tag{2-2}$$

Since $G/G^{\mathrm{der}} \cong \mathbb{G}_m$, we have

$$A \cong \mathbb{A}^{r-1} \times \mathbb{G}_m \subset \mathbb{A}^r.$$

2C. *Gekeler numbers.* We resume our earlier discussion of elliptic curves, and let $G = \mathrm{GL}_2$. As in Section 2A, fix data (a, q) defining an ordinary isogeny class over $\mathbb{F}_q$. Recall that, to each finite prime ℓ, Gekeler has assigned a local probability $\nu_\ell(a, q)$, see (1-1) and (1-2). We give a geometric interpretation of this ratio, as follows.

Since G is a group scheme over $\mathbb{Z}$, for any finite prime ℓ we have a well-defined group $G(\mathbb{Z}_\ell)$, which is a (hyperspecial) maximal compact subgroup of $G(\mathbb{Q}_\ell)$, as well as the "truncated" groups $G(\mathbb{Z}_\ell/\ell^n)$ for every integer $n \geq 0$.

Recall that, given the fixed data (a, q), we have chosen an element $\gamma_0 \in G(\mathbb{Q})$. Since the conjugacy class of a semisimple element of a general linear group is determined by its characteristic polynomial, γ_0 is well-defined up to conjugacy.

Let ℓ be any finite prime (we allow the possibility $\ell = p$); using the inclusion $\mathbb{Q} \hookrightarrow \mathbb{Q}_\ell$ we identify γ_0 with an element of $G(\mathbb{Q}_\ell)$. In fact, if $\ell \neq p$, then γ_0 is a regular semisimple element of $G(\mathbb{Z}_\ell)$.

For a fixed positive integer n, the average value of $\#\mathfrak{c}^{-1}(a)$, as a ranges over $A(\mathbb{Z}_\ell/\ell^n)$, is

$$\#G(\mathbb{Z}_\ell/\ell^n)/\#A(\mathbb{Z}_\ell/\ell^n).$$

Consequently, we set

$$\nu_{\ell,n}(a, q) = \nu_{\ell,n}(\gamma_0) = \frac{\#\{\gamma \in G(\mathbb{Z}_\ell/\ell^n) : \gamma \sim (\gamma_0 \bmod \ell^n)\}}{\#G(\mathbb{Z}_\ell/\ell^n)/\#A(\mathbb{Z}_\ell/\ell^n)}, \tag{2-3}$$

and rewrite (1-3) (and extend it to the case of $\mathbb{F}_q$) as

$$\nu_\ell(a, q) = \lim_{n \to \infty} \nu_{\ell,n}(a, q). \tag{2-4}$$

Again, we have exploited the fact that two semisimple elements of GL_2 are conjugate if and only if their characteristic polynomials are the same. Note that the denominator of (2-4) coincides with that of Gekeler's definition [2003, (3.7)]. Indeed,

$$\frac{\#G(\mathbb{Z}/\ell^n)}{\#A(\mathbb{Z}/\ell^n)} = \frac{\ell(\ell-1)(\ell^2-1)\ell^{4n-4}}{(\ell-1)\ell^{n-1}\ell^n} = (\ell^2-1)\ell^{2n-2}. \tag{2-5}$$

For $\ell = p$, γ_0 lies in $\mathrm{GL}_2(\mathbb{Q}_p) \cap \mathrm{Mat}_2(\mathbb{Z}_p)$. We make the apparently ad hoc definition

$$\nu_p(a,q) = \lim_{n\to\infty} \frac{\#\{\gamma \in \mathrm{Mat}_2(\mathbb{Z}_p/p^n) : \gamma \sim (\gamma_0 \bmod p^n)\}}{\#G(\mathbb{Z}_p/p^n)/\#A(\mathbb{Z}_p/p^n)}, \tag{2-6}$$

where we have briefly used $\sim$ to denote similarity of matrices under the action of $\mathrm{GL}_2(\mathbb{Z}_p/p^n)$. In the case where $q = p$, this recovers Gekeler's definition (1-2).

Finally, we follow [Gekeler 2003, (3.3)] and, inspired by the Sato–Tate measure, define an archimedean term

$$\nu_\infty(a,q) = \frac{2}{\pi}\sqrt{1-\frac{a^2}{4q}}. \tag{2-7}$$

2D. ***The Langlands and Kottwitz approach.*** For Shimura varieties of PEL type, Kottwitz [1992] proved Langlands's conjectural expression of the zeta function of that Shimura variety in terms of automorphic L-functions on the associated group. A key, albeit elementary, tool in this proof is the fact that the isogeny class of a (structured) abelian variety can be expressed in terms of an orbital integral. The special case where the Shimura variety in question is a modular curve, so that the abelian varieties are simply elliptic curves, has enjoyed several detailed presentations in the literature (e.g., [Clozel 1993; Scholze 2011] and, to a lesser extent, [Achter and Cunningham 2002]), and so we content ourselves here with the relevant statement.

As in Section 2A, fix data (a,q) which determines an isogeny class of ordinary elliptic curves over $\mathbb{F}_q$, and let $\gamma_0 \in G(\mathbb{Q})$ be a suitable choice. If $E \in I(a,q)$, then for each ℓ not dividing q there is an isomorphism

$$H^1(E_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\oplus 2}$$

which takes the Frobenius endomorphism of E to γ_0.

There is an additive operator F on $H^1_{\mathrm{cris}}(E, \mathbb{Q}_q)$. It is σ-linear, in the sense that if $a \in \mathbb{Q}_q$ and $x \in H^1_{\mathrm{cris}}(E, \mathbb{Q}_q)$, then $F(ax) = a^\sigma F(x)$. To F corresponds some $\delta_0 \in G(\mathbb{Q}_q)$, well-defined up to σ-conjugacy. (Recall that δ and δ' are σ-conjugate if there exists some $h \in G(\mathbb{Q}_q)$ such that $h^{-1}\delta h^\sigma = \delta'$.) The two elements are related by $\mathrm{N}_{\mathbb{Q}_q/\mathbb{Q}_p}(\delta_0) \sim \gamma_0$.

Let G_{γ_0} be the centralizer of γ_0 in G. Let $G_{\delta_0\sigma}$ be the twisted centralizer of δ_0 in $G_{\mathbb{Q}_q}$; it is an algebraic group over $\mathbb{Q}_p$.

Finally, let $\mathbb{A}_f^p$ denote the prime-to-p finite adeles, and let $\hat{\mathbb{Z}}_f^p \subset \mathbb{A}_f^p$ be the subring of everywhere-integral elements. With these notational preparations, we have:

Proposition 2.1. *The weighted cardinality of an ordinary isogeny class of elliptic curves is given by*

$$\widetilde{\#}I(a,q) = \mathrm{vol}(G_{\gamma_0}(\mathbb{Q})\backslash G_{\gamma_0}(\mathbb{A}_f)) \cdot \int_{G_{\gamma_0}(\mathbb{A}_f^p)\backslash G(\mathbb{A}_f^p)} \mathbb{1}_{G(\hat{\mathbb{Z}}_f^p)}(g^{-1}\gamma_0 g)\,dg \cdot \int_{G_{\delta_0\sigma}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_q)} \mathbb{1}_{G(\mathbb{Z}_q)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)G(\mathbb{Z}_q)}(h^{-1}\delta_0 h^\sigma)\,dh. \tag{2-8}$$

Here, each group $G(\mathbb{Q}_\ell)$ has been given the Haar measure which assigns volume one to $G(\mathbb{Z}_\ell)$ (this is the so-called *canonical measure*, see Section 3A2). The choice of nonzero Haar measure on the centralizer $G_\gamma(\mathbb{Q}_\ell)$ is irrelevant, as long as the same choice is made for the global volume computation. Similarly, in the second, twisted orbital integral, $G(\mathbb{Q}_q)$ is given the Haar measure which assigns volume one to $G(\mathbb{Z}_q)$. Since we shall need to say something about the volume term later, we need to fix the measures on $G_{\gamma_0}(\mathbb{Q}_\ell)$ for every ℓ. We choose the canonical measures μ^{can} on both G and G_{γ_0} at every place. These measures are defined below in Section 3A2.

The idea behind Proposition 2.1 is straightforward. (We defer to [Clozel 1993] for details.) Fix an $E \in I(a,q)$ and $H^1(E_{\bar{\mathbb{F}}_q}, \mathbb{Q}_\ell) \cong \mathbb{Q}_\ell^{\oplus 2}$ as above. This singles out an integral structure

$$H^1(E_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell) \subseteq \mathbb{Q}_\ell^{\oplus 2}.$$

If E' is any other member of $I(a,q)$, then the prime-to-p part of an $\mathbb{F}_q$-rational isogeny $E \to E'$ gives a new integral structure $H^1(E'_{\bar{\mathbb{F}}_q}, \mathbb{Z}_\ell)$ on $\mathbb{Q}_\ell^{\oplus 2}$. Similarly, p-power isogenies give rise to new integral structures on the crystalline cohomology $H^1_{\mathrm{cris}}(E, \mathbb{Q}_q)$. In this way, $I(a,q)$ is identified with $K^\times\backslash Y^p \times Y_p$, where Y^p ranges among γ_0-stable lattices in $Y^1(E_{\bar{\mathbb{F}}_q}, \mathbb{A}^p)$, and Y_p ranges among lattices in $H^1_{\mathrm{cris}}(E, \mathbb{Q}_q)$ stable under δ_0 and $p\delta_0^{-1}$. It is now straight forward to use an orbital integral to calculate the automorphism-weighted, or groupoid, cardinality of the quotient set $K^\times\backslash Y^p \times Y_p$ (e.g., [Hales 2012, §6]).

We remark that most expositions of Proposition 2.1 refer to a geometric context in which $\mathbb{1}_{G(\hat{\mathbb{Z}}_f^p)}$ is replaced with the characteristic function of an open compact subgroup which is sufficiently small that objects have trivial automorphism groups, so that the corresponding Shimura variety is a smooth and quasiprojective fine moduli space. However, this assumption is not necessary for the counting argument underlying (2-8); see, for instance, [Clozel 1993, Section 3(b)].

3. Comparison of Gekeler numbers with orbital integrals

The calculation is based on the interplay between several G-invariant measures on the adjoint orbits in G. We start by carefully reviewing the definitions and the normalizations of all Haar measures involved.

3A. ***Measures on groups and orbits.*** Let $\pi_n : \mathbb{Z}_\ell \to \mathbb{Z}_\ell/\ell^n$ be the truncation map. For any $\mathbb{Z}_\ell$-scheme $\mathcal{X}$, we denote by $\pi_n^{\mathcal{X}}$ the corresponding map

$$\pi_n^{\mathcal{X}} : \mathcal{X}(\mathbb{Z}_\ell) \to \mathcal{X}(\mathbb{Z}_\ell/\ell^n)$$

induced by π_n.

Once and for all, fix the Haar measure on $\mathbb{A}^1(\mathbb{Q}_\ell)$ such that the volume of $\mathbb{Z}_\ell$ is 1. We will denote this measure by dx. For our calculations the key observation is that, with this normalization, the fibers of the standard projection $\pi_n^{\mathbb{A}^d} : \mathbb{A}^d(\mathbb{Z}_\ell) \to \mathbb{A}^d(\mathbb{Z}/\ell^n\mathbb{Z})$ have volume ℓ^{-nd}.

There are two fundamental approaches to normalizing a Haar measure on the set of $\mathbb{Q}_\ell$-points of an arbitrary algebraic group G. One can either fix a maximal compact subgroup and assign volume 1 to it, or one can fix a volume form ω_G on G with coefficients in $\mathbb{Z}$, and thus get the measure $|\omega_G|_\ell$ on each $G(\mathbb{Q}_\ell)$.

For the $\mathbb{Q}_\ell$-points of a general variety, one also has the Serre–Oesterlé measure; it is this measure which naturally arises in studying Gekeler-type ratios. In the case of GL_2, this measure comes from the volume form which Gross calls *canonical*.

We now review these constructions and the relations between them.

3A1. *Serre–Oesterlé measure.* Let $\mathcal{X}$ be a smooth scheme over $\mathbb{Z}_\ell$. Then there is the so-called Serre–Oesterlé measure on X, which we will denote by μ_X^{SO}. It is defined in [Serre 1981, §3.3], see also [Veys 1992] for an attractive equivalent definition. For a smooth scheme that has a nonvanishing gauge form this definition coincides with the definition of A. Weil [1982], and by Theorem 2.2.5 of that paper (extended by Batyrev [1999, Theorem 2.7]), this measure has the property that $\mathrm{vol}_{\mu_X^{\mathrm{SO}}}(\mathcal{X}(\mathbb{Z}_\ell)) = \#\mathcal{X}(\mathbb{F}_\ell)\ell^{-d}$, where d is the dimension of the generic fiber of $\mathcal{X}$. In particular, $\mu_{\mathbb{A}^1}^{\mathrm{SO}}$ is the Haar measure on the affine line such that $\mathrm{vol}_{\mu_{\mathbb{A}^1}^{\mathrm{SO}}}(\mathbb{A}^1(\mathbb{Z}_\ell)) = \ell\ell^{-1} = 1$, i.e., $\mu_{\mathbb{A}^1(\mathbb{Q}_\ell)}^{\mathrm{SO}}$ coincides with $|dx|_\ell$. Similarly, on any d-dimensional affine space $\mathbb{A}^d$, the Serre–Oesterlé measure gives $\mathbb{A}^d(\mathbb{Z}_\ell)$ volume 1.

The algebraic group GL_2 is a smooth group scheme defined over $\mathbb{Z}$. In particular, for every ℓ, $\mathrm{GL}_2 \times_{\mathbb{Z}} \mathbb{Z}_\ell$ is a smooth scheme over $\mathbb{Z}_\ell$, so μ^{SO} gives $\mathrm{GL}_2(\mathbb{Z}_\ell)$ volume

$$\mathrm{vol}_{\mu_{\mathrm{GL}_2}^{\mathrm{SO}}}(\mathrm{GL}_2(\mathbb{Z}_\ell)) = \frac{\#\,\mathrm{GL}_2(\mathbb{F}_\ell)}{\ell^d} = \frac{\ell(\ell-1)(\ell^2-1)}{\ell^4}.$$

3A2. *The canonical measures.* Let G be a reductive group over $\mathbb{Q}_\ell$; Gross [1997, Section 4] defines a canonical integral model $\underline{G}/\mathbb{Z}_\ell$. If G is unramified and connected, then $\underline{G}(\mathbb{Z}_\ell)$ is a hyperspecial maximal compact subgroup of $G(\mathbb{Q}_\ell)$. If T is a (possibly ramified) torus, then $\underline{T}$ is the identity component $\mathcal{T}^\circ$ of the weak Néron model $\mathcal{T}$ of T (discussed in more detail in Section 4A).

The measure most commonly used in orbital integrals, μ^{can}, is the Haar measure which assigns volume 1 to $\underline{G}(\mathbb{Z}_\ell)$.

In fact, Gross uses $\underline{G}$ to define a canonical volume form ω_G, which does not vanish on the special fiber $\underline{G}^\kappa$ of $\underline{G}$. If G is unramified over $\mathbb{Q}_\ell$, then ω_G recovers the Serre–Oesterlé measure, insofar as

$$\int_{\underline{G}(\mathbb{Z}_\ell)} |\omega_G|_\ell = \frac{\#\underline{G}^\kappa(\mathbb{F}_\ell)}{\ell^{\dim G}}$$

[Gross 1997, Proposition 4.7].

3A3. *The geometric measure.* We will use a certain quotient measure μ^{geom} on the orbits, which is called the geometric measure in [Frenkel et al. 2010]. This measure is defined using the Steinberg map $\mathfrak{c}$, (2-2). We return to the setting of Section 2B.

For a general reductive group G and $\gamma \in G(\mathbb{Q}_\ell)$ regular semisimple, the fiber over $\mathfrak{c}(\gamma)$ is the stable orbit of γ, which is a finite union of rational orbits. In our setting with $G = \mathrm{GL}_2$, the fiber $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$ is a single rational orbit, which substantially simplifies the situation. From here onwards, we work only with $G = \mathrm{GL}_2$.

Consider the measure given by the form ω_G on G, and the measure on $A = \mathbb{A}^1 \times \mathbb{G}_m$ which is the product of the measures associated with the form dt on $\mathbb{A}^1$ and ds/s on $\mathbb{G}_m$, where we denote the coordinates on A by (t, s). We will denote this measure by $|d\omega_A|$.

The form ω_G is a generator of the top exterior power of the cotangent bundle of G. For each orbit $\mathfrak{c}^{-1}(t, s)$ (note that such an orbit is a variety) there is a unique generator $\omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)}$ of the top exterior power of the cotangent bundle on the orbit $\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))$ such that

$$\omega_G = \omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)} \wedge \omega_A.$$

Then for any $\phi \in C_c^\infty(G(\mathbb{Q}_\ell))$,

$$\int_{G(\mathbb{Q}_\ell)} \phi(g)\,|d\omega_G| = \int_{A(\mathbb{Q}_\ell)} \int_{\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))} \phi(g)\,|d\omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)}|\,|d\omega_A(t, s)|.$$

This measure also appears in [Frenkel et al. 2010], and it is discussed in detail in Section 4 below.

3A4. *Orbital integrals.* There are two kinds of orbital integrals that will be relevant for us; they differ only in the normalization of measures on the orbits. Let γ be a regular semisimple element of $G(\mathbb{Q}_\ell)$, and let ϕ be a locally constant compactly supported function on $G(\mathbb{Q}_\ell)$. Let T be the centralizer G_γ of γ. Since γ is regular (i.e., the roots of the characteristic polynomial of γ are distinct) and semisimple, T is a maximal torus in G.

First, we consider the orbital integral with respect to the geometric measure.

Definition 3.1. Define $O_\gamma^{\mathrm{geom}}(\phi)$ by

$$O_\gamma^{\mathrm{geom}}(\phi) := \int_{T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \phi(g^{-1}\gamma g)\, d\mu_\gamma^{\mathrm{geom}},$$

where $\mu_\gamma^{\mathrm{geom}}$ is the measure on the orbit of γ associated with the corresponding differential form $\omega^{\mathrm{geom}}_{\mathfrak{c}^{-1}(\mathfrak{c}(\gamma))}$ as in Section 3A3 above.

Second, there is the canonical orbital integral over the orbit of γ, defined as follows. The orbit of γ can be identified with the quotient $T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)$. Both $T(\mathbb{Q}_\ell)$ and $G(\mathbb{Q}_\ell)$ are endowed with canonical measures, as above in Section 3A2. Then there is a unique quotient measure on $T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)$, which will be denoted $\mu_\gamma^{\mathrm{can}}$. The canonical orbital integral will be the integral with respect to this measure on the orbit (also considered as a distribution on the space of locally constant compactly supported functions on $G(\mathbb{Q}_\ell)$).

Definition 3.2. Define $O_\gamma^{\mathrm{can}}(\phi)$ by

$$O_\gamma^{\mathrm{can}}(\phi) := \int_{T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \phi(g^{-1}\gamma g)\, d\mu_\gamma^{\mathrm{can}}.$$

By definition, the distributions O_γ^{geom} and O_γ^{can} differ by a multiple that is a function of γ. This ratio (which we feel should probably be well-known but was hard to find in the literature, see also [Frenkel et al. 2010] and the Appendix) is computed in Section 4 below.

We will first relate Gekeler's ratios to orbital integrals with respect to the geometric measure, in a natural way, and from there will get the relationship with the canonical orbital integrals, which are more convenient to use for the purposes of computing the global volume term appearing in the formula of Langlands and Kottwitz.

3B. ***Gekeler numbers and volumes, for ℓ not equal to p.*** From now on, $G = \mathrm{GL}_2$, $\gamma_0 = \gamma_{a,q}$, and ℓ is a fixed prime distinct from p. Our first goal is to relate the Gekeler number $\nu_\ell(a,q)$, (2-4), to an orbital integral $O_{\gamma_0}^{\mathrm{geom}}(\phi_0)$ of a suitable test function ϕ_0 with respect to $|d\omega^{\mathrm{geom}}_{\mathfrak{c}(\gamma)}|$. (Recall that γ_0 is the element of $G(\mathbb{Q}_\ell)$ determined by E, and in this case since $\ell \neq p$, it lies in $G(\mathbb{Z}_\ell)$.) In order to do this we define natural subsets of $G(\mathbb{Q}_\ell)$ whose volumes are responsible for this relationship.

Recall (2-3), the definition of $\nu_{\ell,n}(\gamma_0)$. For each positive integer n, consider the subset V_n of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ defined as

$$\begin{aligned} V_n = V_n(\gamma_0) &:= \{\gamma \in \mathrm{GL}_2(\mathbb{Z}_\ell) \mid f_\gamma(T) \equiv f_0(T) \bmod \ell^n\} \\ &= \{\gamma \in \mathrm{GL}_2(\mathbb{Z}_\ell) \mid \pi_n^A(\mathfrak{c}(\gamma)) = \pi_n^A(\mathfrak{c}(\gamma_0))\}, \end{aligned} \tag{3-1}$$

and set

$$V(\gamma_0) := \bigcap_{n\geq 1} V_n(\gamma_0).$$

We define an auxiliary ratio

$$v_n(\gamma_0) := \frac{\mathrm{vol}_{\mu^{\mathrm{so}}_{GL_2}}(V_n(\gamma_0))}{\ell^{-2n}}. \tag{3-2}$$

Now we would like to relate the limit of these ratios $v_n(\gamma_0)$ both to the limit of Gekeler ratios $\nu_{\ell,n}(\gamma_0)$ and to an orbital integral.

Let $\phi_0 = \mathbb{1}_{\mathrm{GL}_2(\mathbb{Z}_\ell)}$ be the characteristic function of the maximal compact subgroup $\mathrm{GL}_2(\mathbb{Z}_\ell)$ in $\mathrm{GL}_2(\mathbb{Q}_\ell)$.

Proposition 3.3. *We have*

$$\lim_{n\to\infty} v_n(\gamma_0) = O^{\mathrm{geom}}_{\gamma_0}(\phi_0).$$

Proof. Because equality of characteristic polynomials is equivalent to conjugacy in $\mathrm{GL}_2(\mathbb{Q}_\ell)$, $V(\gamma_0)$ is the intersection of $\mathrm{GL}_2(\mathbb{Z}_\ell)$ with the orbit $\mathcal{O}(\gamma_0)$ of γ_0 in $G = \mathrm{GL}_2(\mathbb{Q}_\ell)$. Then the orbital integral $O^{\mathrm{geom}}_{\gamma_0}(\phi_0)$ is nothing but the volume of the set $V(\gamma_0)$, as a subset of $\mathcal{O}(\gamma_0)$, with respect to the measure $d\mu^{\mathrm{geom}}_{\gamma_0}$.

Let $a_0 = \mathfrak{c}(\gamma_0) = (a, q) \in \mathbb{A}^1 \times \mathbb{G}_m(\mathbb{Q}_\ell)$, and let $U_n(a_0)$ be its $\ell^{-n} \times \ell^{-n}$-neighborhood. Its Serre–Oesterlé volume is $\mathrm{vol}_{\mu^{\mathrm{so}}_A}(U_n(\gamma_0)) = \ell^{-2n}$.

Moreover, $V_n(\gamma_0) = \mathfrak{c}^{-1}(U_n(\gamma_0)) \cap \mathrm{GL}_2(\mathbb{Z}_\ell)$. Consequently,

$$\begin{aligned} \lim_{n\to\infty} v_n(\gamma_0) &= \lim_{n\to\infty} \frac{\mathrm{vol}_{\mu^{\mathrm{so}}_{\mathrm{GL}_2}}\big(\mathfrak{c}^{-1}(U_n(\gamma_0)) \cap \mathrm{GL}_2(\mathbb{Z}_\ell)\big)}{\mathrm{vol}_{\mu^{\mathrm{so}}_A}(U_n(\gamma_0))} \\ &= \lim_{n\to\infty} \frac{\mathrm{vol}_{|d\omega_G|}\big(\mathfrak{c}^{-1}(U_n(\gamma_0)) \cap \mathrm{GL}_2(\mathbb{Z}_\ell)\big)}{\mathrm{vol}_{|d\omega_A|}(U_n(\gamma_0))} \\ &= \mathrm{vol}_{\mu^{\mathrm{geom}}_{\gamma_0}}(V(\gamma_0)), \end{aligned} \tag{3-3}$$

by definition of the geometric measure. □

Next, let us relate the ratios v_n to the Gekeler ratios.

Proposition 3.4. *The ratios $v_n(\gamma_0)$ (and thus, also $\nu_{\ell,n}(\gamma_0)$) stabilize, in the sense that when n is large enough, $v_n(\gamma_0) = \lim_{n\to\infty} v_n(\gamma_0)$, and we have*

$$\begin{aligned} \lim_{n\to\infty} v_n(\gamma_0) &= \frac{\#\,\mathrm{SL}_2(\mathbb{F}_\ell)}{\ell^3} \cdot \lim_{n\to\infty} \nu_{\ell,n}(\gamma_0) \\ &= \frac{\ell^2-1}{\ell^2} \cdot \nu_\ell(a, q). \end{aligned}$$

Remark 3.5. We do not need the claim that Gekeler's ratios $\nu_{\ell,n}$ stabilize for large n in order to relate them to the orbital integrals. However, we have included this claim in order to point out that this behavior (also proved by Gekeler by direct computation) is a special case of a very general phenomenon (which can be thought of as a multivariable version of Hensel's lemma) that has appeared in the work of Igusa, Serre, and later Veys, Denef, and others, and was at the foundation of the theory of motivic integration (see [Veys 2006] for related results), but does not appear to be widely known. We provide more specific references in the proof of the proposition.

Proof. Let $\pi_n = \pi_n^{\mathrm{GL}_2} : \mathrm{GL}_2(\mathbb{Z}_\ell) \to \mathrm{GL}_2(\mathbb{Z}/\ell^n)$. To ease notation slightly, let $V_n = V_n(\gamma_0)$. Let $S_n \subset \mathrm{GL}_2(\mathbb{Z}/\ell^n\mathbb{Z})$ be the set that appears in the numerator of (2-3),

$$S_n := \{\gamma \in \mathrm{GL}_2(\mathbb{Z}/\ell^n) \mid f_\gamma(T) \equiv f_0(T) \bmod \ell^n\}.$$

First, observe that for all $n \geq 1$, we have $V_n = \pi_n^{-1}(S_n)$. Indeed, taking characteristic polynomials commutes with reduction $\bmod \ell^n$, since the coefficients of the characteristic polynomial are themselves polynomial in the matrix entries of γ, and reduction $\bmod \ell^n$ is a ring homomorphism. We claim that, for large enough n (with the restriction depending on the discriminant of f), the following hold:

(i) $\pi_n|_{V_n} : V_n \to S_n$ is surjective.

(ii) We have the equality

$$\mathrm{vol}_{\mu^{\mathrm{SO}}_{\mathrm{GL}_2}}(V_n) = \ell^{-4n} \# S_n. \tag{3-4}$$

(iii) The number $\ell^{2n}\,\mathrm{vol}_{\mu^{\mathrm{SO}}_{\mathrm{GL}_2}}(V_n)$ does not depend on n.

We only need the second and third claims to establish the Proposition; we have singled out the first claim since it is key to the proof of claims (ii) and (iii). First, let us finish the proof of the Proposition assuming (ii) holds. Handling the denominator of Gekeler's ratio as in (2-5) above, we get

$$v_n(\gamma_0) = \frac{\ell^{-4n}\# S_n}{\ell^{-2n}} = \frac{\# S_n}{\ell^{2n}} = \frac{\# S_n \# \mathrm{SL}_2(\mathbb{F}_\ell)}{\# \mathrm{SL}_2(\mathbb{F}_\ell)\ell^{3(n-1)}\ell^{-n}\ell^3} = \frac{\# \mathrm{SL}_2(\mathbb{F}_\ell)}{\ell^3}\nu_{\ell,n}(\gamma_0), \tag{3-5}$$

as required.

Thus, it remains to address the three claims. The set $V(\gamma_0)$ is the subset of $\mathbb{A}^4(\mathbb{Z}_\ell)$ cut out by the algebraic equations $\mathrm{tr}(\gamma) = \mathrm{tr}(\gamma_0)$ and $\det(\gamma) = \det(\gamma_0)$. Since γ_0 is a regular semisimple element, these equations define a 2-dimensional ℓ-adic analytic submanifold of $\mathbb{A}^4$ (namely, the orbit of γ_0). For such submanifolds, all three claims were proved by J.-P. Serre in [Serre 1981] (see Theorem 9 in §3.3 and the remarks following it; see also [Veys 1992, Proposition 0.1], and the discussion before Corollary 1.8.2 in the survey [Denef 2000]). We note that (i) is key, and the other two claims follow easily. Indeed, since GL_2 is smooth over the residue field $\mathbb{F}_\ell$, all fibers of π_n have volume equal to ℓ^{-4n}. The set V_n is a disjoint union of fibers of π_n, and by (i), the number of these fibers is $\#\pi_n(V_n) = \#S_n$. Thus, the volume of V_n is exactly ℓ^{-4n} times the number of points in the image of the set in the numerator under this projection. Claim (iii) follows in a similar fashion by considering $\pi_{n+1}(V_n) = S_{n+1}$ as a fibration over S_n. □

Combining Propositions 3.3 and 3.4, we immediately obtain:

Corollary 3.6. *The Gekeler numbers relate to orbital integrals via*

$$\nu_\ell(a,q) = \frac{\ell^3}{\#\,\mathrm{SL}_2(\mathbb{F}_\ell)} O^{\mathrm{geom}}_{\gamma_0}(\phi_0).$$

3C. *$\ell = p$ revisited.* We now consider $\nu_p(a,q)$ in a similar light. Since $\det(\gamma_0) = q$, γ_0 lies in $\mathrm{Mat}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p)$ but *not* in $\mathrm{GL}_2(\mathbb{Z}_p)$, and we must consequently modify the argument of Section 3B.

For integers m and n, let $\lambda_{m,n} = \left(\begin{smallmatrix} p^m & 0 \\ 0 & p^n \end{smallmatrix}\right)$, and let $C_{m,n} = \mathrm{GL}_2(\mathbb{Z}_p)\lambda_{m,n}\,\mathrm{GL}_2(\mathbb{Z}_p)$. The Cartan decomposition for GL_2 asserts that $\mathrm{GL}_2(\mathbb{Q}_p)$ is the disjoint union

$$\mathrm{GL}_2(\mathbb{Q}_p) = \bigcup_{m \ge n} C_{m,n},$$

so that

$$\mathrm{Mat}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p) = \bigcup_{0 \le n \le m} C_{m,n}.$$

We now express $\nu_p(a,q)$ as an orbital integral. Recall that $q = p^e$. Since we consider an ordinary isogeny class, the element $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q}_p)$ actually can be chosen to have the form $\gamma_0 = \left(\begin{smallmatrix} u_1 p^e & 0 \\ 0 & u_2 \end{smallmatrix}\right)$, where $u_1, u_2 \in \mathbb{Z}_p$ are units and thus, in particular, $\gamma_0 \in C_{e,0}$.

Lemma 3.7. *Let ϕ_q be the characteristic function of $C_{e,0} = \mathrm{GL}_2(\mathbb{Z}_p)\left(\begin{smallmatrix} q & 0 \\ 0 & 1 \end{smallmatrix}\right)\mathrm{GL}_2(\mathbb{Z}_p)$. Then*

$$\nu_p(a,q) = \frac{p^3}{\#\,\mathrm{SL}_2(\mathbb{F}_p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_q).$$

Proof. The proof is similar to the case $\ell \neq p$, with one key modification. There, we use the reduction $\bmod \ell^n$ map π_n defined on $G(\mathbb{Z}_\ell)$. Here, we need to extend the map π_n to a set that contains γ_0.

Let $\pi_n^{\mathrm{M}} : \mathrm{Mat}_2(\mathbb{Z}_p) \to \mathrm{Mat}_2(\mathbb{Z}_p/p^n)$ be the projection map, and let $\mathfrak{c} : \mathrm{GL}_2(\mathbb{Q}_p) \to A(\mathbb{Q}_p)$ be the characteristic polynomial map. As in Section 3B, we define the sets

$$\begin{aligned} U_n &:= \{a = (a_0, a_1) \in A(\mathbb{Z}_p) \mid a_i \equiv a_i(\gamma_0) \bmod p^n, i = 0, 1\}, \\ S_n &:= \{\gamma \in \mathrm{Mat}_2(\mathbb{Z}_p/p^n) : \gamma \sim \pi_n^{\mathrm{M}}(\gamma_0)\}, \\ V_n &:= (\pi_n^{\mathrm{M}})^{-1}(S_n) \subset \mathrm{Mat}_2(\mathbb{Z}_p) \cap \mathrm{GL}_2(\mathbb{Q}_p). \end{aligned}$$

As before, informally, we think of U_n as a neighborhood of the point given by the coefficients of the characteristic polynomial of γ_0 in the Steinberg–Hitchin base, and we think of V_n as the intersection of the corresponding neighborhood of the orbit of γ_0 in $\mathrm{GL}_2(\mathbb{Q}_p)$ with $\mathrm{Mat}_2(\mathbb{Z}_p)$. In the case $\ell \neq p$ we had $\mathrm{GL}_2(\mathbb{Z}_\ell)$ in the place of $\mathrm{Mat}_2(\mathbb{Z}_p)$ in this description, and so it was clear that the evaluation of the volume of V_n would lead to the orbital integral of ϕ_0, the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_\ell)$. Here, we need to make the connection between the set V_n and our function ϕ_q.

We claim that if $n > e$, then $V_n \subset C_{e,0}$. Indeed, suppose $\gamma \in V_n$. Then, since the characteristic polynomial of γ is congruent to that of γ_0, the trace of γ is a p-adic unit. Then γ cannot lie in any double coset $C_{m,n}$ with both m, n positive, because if it did, its trace would have been divisible by $p^{\min(m,n)}$. Then γ has to lie in a double coset of the form $C_{e+m,-m}$ for some $m \geq 0$, but if $m > 0$, then such a double coset has empty intersection with $\mathrm{Mat}_2(\mathbb{Z}_p)$, so $m = 0$ and the claim is proved.

As in the proof of Proposition 3.4 (iii), the volume of the set V_n equals $p^{-4n}\#S_n$. The rest of the proof repeats the proofs of Proposition 3.4 and Corollary 3.6. We again set $V(\gamma_0) = \bigcap_{n\geq 1} V_n \subset C_{e,0}$. Since π_n^{M} is surjective, $V(\gamma_0) = O(\gamma_0) \cap C_{e,0}$. By (3-3),

$$O_{\gamma_0}^{\mathrm{geom}}(\phi_q) = \lim_{n\to\infty} \frac{\mathrm{vol}_{\mu^{\mathrm{SO}}_{\mathrm{GL}_2}} V_n(\gamma_0)}{\mathrm{vol}_{\mu^{\mathrm{SO}}_A}(U_n)} = \lim_{n\to\infty} \frac{\#S_n(\gamma_0)p^{-4n}}{p^{-2n}},$$

and the statement follows by (3-5), which does not require any modification. $\square$

Recall that, in terms of the data (a, q), we have also computed a representative δ_0 for a σ-conjugacy class in $\mathrm{GL}_2(\mathbb{Q}_q)$. It is characterized by the fact that, possibly after adjusting γ_0 in its conjugacy class, we have $\mathrm{N}_{\mathbb{Q}_q/\mathbb{Q}_p}(\delta_0) = \gamma_0$. (Here we exploit the fact that, in a general linear group, conjugacy and stable conjugacy coincide.)

The twisted centralizer $G_{\delta_0\sigma}$ of δ_0 is an inner form of the centralizer G_{γ_0} [Kottwitz 1982, Lemma 5.8]; since γ_0 is regular semisimple, G_{γ_0} is a torus, and thus $G_{\delta_0\sigma}$ is isomorphic to G_{γ_0}. Using this, any choice of Haar measure on $G_{\delta_0\sigma}(\mathbb{Q}_p)$ induces one on $G_{\gamma_0}(\mathbb{Q}_p)$.

If ϕ is a function on $G(\mathbb{Q}_q)$, denote its twisted (canonical) orbital integral along the orbit of δ_0 by

$$TO_{\delta_0}^{\mathrm{can}}(\phi) = \int_{G_{\delta_0\sigma}(\mathbb{Q}_p)\backslash G(\mathbb{Q}_q)} \phi(h^{-1}\delta_0 h^{\sigma})\, d\mu^{\mathrm{can}}.$$

Lemma 3.8. *Let $\phi_{p,q}$ be the characteristic function of* $\mathrm{GL}_2(\mathbb{Z}_q)\lambda_{0,1}\,\mathrm{GL}_2(\mathbb{Z}_q)$. *Then*

$$TO_{\delta_0}^{\mathrm{can}}(\phi_{p,q}) = O_{\gamma_0}^{\mathrm{can}}(\phi_q).$$

Proof. The asserted matching of twisted orbital integrals on $\mathrm{GL}_2(\mathbb{Q}_q)$ with orbital integrals on $\mathrm{GL}_2(\mathbb{Q}_p)$ is one of the earliest known instances of the fundamental lemma ([Langlands 1980]; see also [Laumon 1996, Section 4; Getz and Goresky 2012, (E.4.9)] or even [Achter and Cunningham 2002, Section 2.1]]). Indeed, the base change homomorphism of the Hecke algebras matches the characteristic function of $\mathrm{GL}_2(\mathbb{Z}_q)\lambda_{1,0}\,\mathrm{GL}_2(\mathbb{Z}_q)$ with $\phi_q+\phi$, where ϕ is a linear combination of the characteristic functions of $C_{a,b}$ with $a+b=e$ and $a, b>0$. As shown in the proof of the previous lemma, the orbit of γ_0 does not intersect the double cosets $C_{a,b}$ with $a, b>0$, and thus the only nonzero term on the right-hand side is $O^{\mathrm{can}}_{\gamma_0}(\phi_q)$. □

4. Canonical measure versus geometric measure

Finally, we need to relate the orbital integral with respect to the geometric measure as above to the canonical orbital integrals. A very similar calculation is discussed in [Frenkel et al. 2010] (and as the authors point out, surprisingly, it seemed impossible to find in earlier literature). Since our normalization of local measures seems to differ by an interesting constant from that of [Frenkel et al. 2010] at ramified finite primes, we carry out this calculation in our special case.

4A. ***Canonical measure and L-functions.*** Here we briefly review the facts that go back to the work of Weil, Langlands, Ono, Gross, and many others, that show the relationship between convergence factors that can be used for Tamagawa measures and various Artin L-functions. Our goal is to introduce the Artin L-factors that naturally appear in the computation of the canonical measures. To any reductive group G over $\mathbb{Q}_\ell$, Gross [1997] attaches a motive $M=M_G$; following his notation, we consider $M^\vee(1)$ — the Tate twist of the dual of M. For any motive M we let $L_\ell(s, M)$ be the associated local Artin L-function. We will write $L_\ell(M)$ for the value of $L_\ell(s, M)$ at $s=0$. The value $L_\ell(M^\vee(1))$ is always a positive rational number, related to the canonical measure reviewed in Section 3A2. In particular, if G is quasisplit over $\mathbb{Q}_\ell$, then

$$\mu_G^{\mathrm{can}} = L_\ell(M^\vee(1))|\omega_G|_\ell \tag{4-1}$$

[Gross 1997, Proposition 4.7 and (5.1)].

We shall also need a similar relation between volumes and Artin L-functions in the case when $G=T$ is an algebraic torus which is not necessarily anisotropic. Here we follow [Bitan 2011]. Suppose that T splits over a finite Galois extension L of $\mathbb{Q}_\ell$; let κ_L be the residue field of L, and let I be the inertia subgroup of the Galois group $\mathrm{Gal}(L/\mathbb{Q}_\ell)$. Let $X^*(T)$ be the group of rational characters of T. Let $\mathcal{T}$ be the Néron model of T over $\mathbb{Z}_\ell$, with the connected component of the identity denoted by $\mathcal{T}^\circ$. This is the canonical model for T referred to in Section 3A2.

Let Fr_L be the Frobenius element of $\mathrm{Gal}(\kappa_L/\mathbb{F}_\ell)$. The Galois group of the maximal unramified subextension of L, which is isomorphic to $\mathrm{Gal}(\kappa_L/\mathbb{F}_\ell)$, acts naturally on the I-invariants $X^*(T)^I$, giving rise to a representation which we will denote by ξ_T (and which is denoted by h in [Bitan 2011]),

$$\xi_T : \mathrm{Gal}(\kappa_L/\mathbb{F}_\ell) \to \mathrm{Aut}(X^*(T)^I) \simeq \mathrm{GL}_{d_I}(\mathbb{Z}),$$

where $d_I = \mathrm{rank}(X^*(T)^I)$. Then the associated local Artin L-factor is defined as

$$L_\ell(s, \xi_T) := \det\Big(1_{d_I} - \frac{\xi_T(\mathrm{Fr}_L)}{\ell^s}\Big)^{-1}.$$

Proposition 4.1 [Bitan 2011, Proposition 2.14].

$$L_\ell(1, \xi_T)^{-1} = \#\mathcal{T}^\circ(\mathbb{F}_\ell)\ell^{-\dim(T)} = \int_{\mathcal{T}^\circ(Z_\ell)} |\omega_T|_\ell.$$

We observe that by definition [Gross 1997, § 4.3], since $G = T$ is an algebraic torus, the canonical parahoric $\underline{T}^\circ$ is $\mathcal{T}^\circ$; the canonical volume form ω_T is the same as the volume form denoted by $\omega_{\mathfrak{p}}$ in [Bitan 2011].

We also note that the motive of the torus T is the Artin motive $M = X^*(T) \otimes \mathbb{Q}$. If T is anisotropic over $\mathbb{Q}_\ell$, by the formula (6.6) (see also (6.11)) in [Gross 1997], we have

$$L_\ell(M^\vee(1)) = L_\ell(1, \xi_T).$$

As in the first paragraph of Section 3A3, let G be a reductive group over $\mathbb{Q}_\ell$ with simply connected derived group G^{der} and connected center Z, and assume that $G/G^{\mathrm{der}} \cong \mathbb{G}_m$.

Lemma 4.2. *Let $T \subset G$ be a maximal torus and let $T^{\mathrm{der}} = T \cap G^{\mathrm{der}}$. Then*

$$\frac{L_\ell(M_G^\vee(1))}{L_\ell(1, \xi_T)} = \frac{L_\ell(M_{G^{\mathrm{der}}}^\vee(1))}{L_\ell(1, \xi_{T^{\mathrm{der}}})}. \tag{4-2}$$

Proof. The motive M_H of a reductive group H, and thus $L_\ell(M_H^\vee(1))$, depends on H only up to isogeny [Gross 1997, Lemma 2.1]. Since G is isogenous to $Z \times G^{\mathrm{der}}$,

$$L_\ell(M_G^\vee(1)) = L_\ell(M_Z^\vee(1)) L_\ell(M_{G^{\mathrm{der}}}^\vee(1)).$$

Because $G^{\mathrm{der}} \cap Z$ is finite [Frenkel et al. 2010, (3.1)], so is $T^{\mathrm{der}} \cap Z$. Therefore, the natural map $T^{\mathrm{der}} \to T/Z$ is an isogeny onto its image. For dimension reasons it is an actual isogeny, and induces an isomorphism $X^*(T^{\mathrm{der}}) \otimes \mathbb{Q} \cong X^*(T/Z) \otimes \mathbb{Q}$ of $\mathrm{Gal}(\mathbb{Q}_\ell)$-modules. Therefore, $L(s, \xi_{T^{\mathrm{der}}}) = L(s, \xi_{T/Z})$, and thus

$$L(s, \xi_T) = L(s, \xi_{T/Z}) L(s, \xi_Z) = L(s, \xi_{T^{\mathrm{der}}}) L(s, \xi_Z).$$

Identity (4-2) is now immediate. $\square$

4B. ***Weyl discriminants and measures.*** Our next immediate goal is to find an explicit constant $d(\gamma)$ such that $\mu_\gamma^{\mathrm{can}} = d(\gamma)\mu_\gamma^{\mathrm{geom}}$. We note that a similar calculation is carried out in [Frenkel et al. 2010]. However, the notation there is slightly different, and the key proof in [Frenkel et al. 2010] only appears for the field of complex numbers; hence, we decided to include this calculation here.

Let G be a split reductive group over $\mathbb{Q}_\ell$. Choose a split maximal torus and associated root system R and set of positive roots R^+.

Definition 4.3. Let $\gamma \in G(\mathbb{Q}_\ell)$, let T be the centralizer of γ, and let $\mathfrak{t}$ be the Lie algebra of T. Then the discriminant of γ is

$$D(\gamma) = \prod_{\alpha \in R}(1-\alpha(\gamma)) = \det(I - \mathrm{Ad}(\gamma^{-1})|_{\mathfrak{g}/\mathfrak{t}}).$$

4B1. *Weyl integration formula, revisited.* As pointed out in [Frenkel et al. 2010, the paragraph above equation (3.28)], since both $\mu_\gamma^{\mathrm{can}}$ and $\mu_\gamma^{\mathrm{geom}}$ are invariant under the center, it suffices to consider the case $G = G^{\mathrm{der}}$. So for the moment, let us assume that the group G is semisimple and simply connected. Let $\phi \in C_c^\infty(\mathbb{Q}_\ell)$.

On one hand, the Weyl integration formula (we write a group-theoretic version of the formulation for the Lie algebra in [Kottwitz 2005, §7.7]) asserts that

(4-3)
$$\int_{G(\mathbb{Q}_\ell)} \phi(g)|d\omega_G| = \sum_T \frac{1}{|W_T|}\int_{T(\mathbb{Q}_\ell)} |D(\gamma)| \int_{T(\mathbb{Q}_\ell)\backslash G(\mathbb{Q}_\ell)} \phi(g^{-1}\gamma g)|d\omega_{T\backslash G}||d\omega_T|,$$

by our definition of the measure $|d\omega_{T\backslash G}|$. (Here, the sum ranges over a set of representatives for $G(\mathbb{Q}_\ell)$-conjugacy classes of maximal $\mathbb{Q}_\ell$-rational tori in G, and W_T is the finite group $W_T = N_G(T)(\mathbb{Q}_\ell)/T(\mathbb{Q}_\ell)$.)

On the other hand we have, by definition of the geometric measure,

$$\int_{G(\mathbb{Q}_\ell)} \phi(g)|d\omega_G| = \int_{A(\mathbb{Q}_\ell)} \int_{\mathfrak{c}^{-1}(a)} \phi(g)|d\omega_\gamma^{\mathrm{geom}}(g)||d\omega_A|.$$

To compare the two measures, we need to match the integration over $A(\mathbb{Q}_\ell)$ with the sum of integrals over tori.

Up to a set of measure zero, $A(\mathbb{Q}_\ell)$ is a disjoint union of images of $T(\mathbb{Q}_\ell)$, as T ranges over the same set as in (4-3); and for each such T, the restriction of $\mathfrak{c}$ to T is $|W_T|$-to-one.

It remains to compute the Jacobian of this map for a given T. Over the algebraic closure of $\mathbb{Q}_\ell$ this calculation is done, for example, in [Kottwitz 2005, § 14]; over $\mathbb{Q}_\ell$, this only applies to the split torus T^{spl}. The answer over the algebraic closure is $c_T \prod_{\alpha>0}(\alpha(x)-1)$, where $c_T \in \bar{F}^\times$ is a constant (which depends on the torus T). We compute $|c_T|_\ell$ in the special case where T comes from a restriction of scalars in GL_2.

Lemma 4.4. *Let T be a torus in $\mathrm{GL}_2(\mathbb{Q}_\ell)$, and let c_T be the constant defined above. Then $|c_T|_\ell = 1$ if T is split or splits over an unramified extension, and $|c_T| = \ell^{-1/2}$ if T splits over a ramified quadratic extension. In particular, if $\gamma_0 \in \mathrm{GL}_2(\mathbb{Q})$ and $T = \mathrm{R}_{K/\mathbb{Q}}\mathbb{G}_m$ is the centralizer of γ_0 as in Section 2A, then $|c_T| = |\Delta_K|_\ell^{-1/2}$.*

Proof. We prove the lemma by direct calculation for GL_2. First, let us compute $|c_T|$ for the split torus. Here we can just compute the Jacobian of the map $T^{\mathrm{der}} \to T^{\mathrm{der}}/W$ by hand. Since we are working with invariant differential forms, we can just do the Jacobian calculation on the Lie algebra; it suffices to compute the Jacobian of the map from $\mathfrak{t}$ to $\mathfrak{t}/W$. Choose coordinates on the split torus in $\mathrm{SL}_2 = \mathrm{GL}_2^{\mathrm{der}}$ so that elements of $\mathfrak{t}$ are diagonal matrices with entries $(t, -t)$, then the canonical measure on $\mathfrak{t}$ is nothing but dt. Now, the coordinate on $\mathfrak{t}/W$ is $y = -t^2$ and the form $\omega_{\mathbb{A}^1}$ is dx. The Jacobian of the change of variables from $\mathfrak{t}/W$ to $\mathbb{A}^1$ is $-2t$. Thus, for the split torus $c = -1$. Note that $2t$ is the product of positive roots (on the Lie algebra). Thus, $|c_T| = 1$.

Now, consider a general maximal torus T in GL_2. Let T^{spl} be a split maximal torus; we have shown that $|c_{T^{\mathrm{spl}}}| = 1$. The torus T is conjugate to T^{spl} over a quadratic field extension L. Let us briefly denote this conjugation map by ψ. Then the map $\mathfrak{c}|T$ can be thought of as the conjugation $\psi : T \to T^{\mathrm{spl}}$ (defined over L) followed by the map $\mathfrak{c}|T^{\mathrm{spl}}$. Then

$$c_T = c_{T^{\mathrm{spl}}} \frac{\omega_T}{\psi^*(\omega_{T^{\mathrm{spl}}})},$$

where $\psi^*(\omega_{T^{\mathrm{spl}}})$ is the pullback of the canonical volume form on T^{spl} under ψ and the ratio $\omega_T/(\psi^*(\omega_{T^{\mathrm{spl}}}))$ is a constant in L. We thus have

$$c_T = \left| \frac{\omega_T}{\psi^*(\omega_{T^{\mathrm{spl}}})} \right|_L, \tag{4-4}$$

where $|\cdot|_L$ is the unique extension of the absolute value on $\mathbb{Q}_\ell$ to L.

At this point this is just a question about two tori, no longer requiring Steinberg section, and so we pass back to working with the group GL_2 rather than SL_2. Now T is obtained by restriction of scalars from $\mathbb{G}_m$, and so we can compute $\psi^*(\omega_{T^{\mathrm{spl}}})$ by hand. By definition, $T = \mathrm{R}_{L/\mathbb{Q}_\ell} \mathrm{G}_m$ and $T^{\mathrm{spl}} = \mathbb{G}_m \times \mathbb{G}_m$. The form $\omega_{T^{\mathrm{spl}}}$ is

$$\omega_{T^{\mathrm{spl}}} = \frac{du}{u} \wedge \frac{dv}{v},$$

where we denote the coordinates on $\mathbb{G}_m \times \mathbb{G}_m$ by (u, v). Let $L = \mathbb{Q}_\ell(\sqrt{\epsilon})$, where ϵ is nonsquare in $\mathbb{Q}_\ell$ (assume for the moment that $\ell \neq 2$). Then every element of T is conjugate in $\mathrm{GL}_2(\mathbb{Q}_\ell)$ to $\left(\begin{smallmatrix} x & \epsilon y \\ y & x \end{smallmatrix}\right)$, and using (x, y) as the coordinates on T, the map ψ can be written as $\psi(x, y) = (x + \sqrt{\epsilon} y, x - \sqrt{\epsilon} y)$. Then one can simply compute

$$\psi^*\left(\frac{du}{u} \wedge \frac{dv}{v}\right) = 2\sqrt{\epsilon} \frac{dx \wedge dy}{x^2 - \epsilon y^2} = 2\sqrt{\epsilon}\,\omega_T.$$

Thus we get (for $\ell \neq 2$),

$$|c_T|_\ell = |2\sqrt{\epsilon}|_L = \begin{cases} 1 & \text{if } L \text{ is unramified,} \\ \sqrt{\ell} & \text{if } L \text{ is ramified,} \end{cases}$$

which completes the proof of the lemma in the case $\ell \neq 2$.

There is, however, a better argument, which also covers the case $\ell = 2$. Namely, to find the ratio $|\omega_T/\psi^*(\omega_{T^{\mathrm{spl}}})|_L$ of (4-4), we just need to find the ratio of the volume of $\mathcal{T}^\circ(\mathbb{Z}_\ell)$ with respect to the measure $|d\omega_T|$ to its volume with respect to $|d\psi^*(\omega_{T^{\mathrm{spl}}})|$. This is, in fact, the same calculation as the one carried out in [Weil 1982, p. 22 (before Theorem 2.3.2)], and the answer is that the convergence factors for the pull-back of the form $\omega_{T^{\mathrm{spl}}}$ to the restriction of scalars is $(\sqrt{|\Delta_K|_\ell})^{\dim(\mathbb{G}_m)}$, in this case. $\square$

Finally, summarizing the above discussion, we obtain

Proposition 4.5. *Let $\gamma \in \mathrm{GL}_2(\mathbb{Q})$ be a regular element. Let T be the centralizer of γ, and let K be as in Section 2A. Abusing notation, we also denote by γ the image of γ in $\mathrm{GL}_2(\mathbb{Q}_\ell)$ for every finite prime ℓ. Then for every finite prime ℓ,*

$$\mu_\gamma^{\mathrm{geom}} = \frac{L_\ell(1, \xi_T)}{L_\ell(M_G^\vee(1))} |\Delta_K|_\ell^{-1/2} |D(\gamma)|_\ell^{1/2} \mu_\gamma^{\mathrm{can}}$$

as measures on the orbit of γ.

5. The global calculation

In this section, we put all the above local comparisons together, and thus show that Gekeler's formula reduces to a special case of the formula of Langlands and Kottwitz. In the process we will need a formula for the global volume term that arises in that formula. We are now in a position to give a new proof of Gekeler's theorem, and of its generalization to arbitrary finite fields.

Theorem 5.1. *Let q be a prime power, and let a be an integer with $|a| \leq 2\sqrt{q}$ and $\gcd(a, p) = 1$. The number of elliptic curves over $\mathbb{F}_q$ with trace of Frobenius a is*

$$\widetilde{\#}I(a, q) = \frac{\sqrt{q}}{2} \nu_\infty(a, q) \prod_\ell \nu_\ell(a, q). \tag{5-1}$$

Here, $\nu_\ell(a, q)$ (for $\ell \neq p$), $\nu_p(a, q)$, and $\nu_\infty(a, q)$ are defined, respectively, in (2-4), (2-6), and (2-7), and the weighted count $\widetilde{\#}I(a, q)$ is defined in (2-1).

Proof. Recall the notation surrounding γ_0 and δ_0 established in Section 2A. Given Proposition 2.1, it suffices to show that the right-hand side of (5-1) calculates the right-hand side of (2-8).

Let $G = \mathrm{GL}_2$. First, let

$$\phi^p = \otimes_{\ell \neq p} \mathbb{1}_{G(\mathbb{Z}_\ell)}$$

be the characteristic function of $G(\hat{\mathbb{Z}}_f^p)$ in $G(\mathbb{A}_f^p)$. The first integral appearing in (2-8) is equal to

$$O_{\gamma_0}(\phi^p) = \int_{G(\mathbb{A}^p)} \phi^p |d\omega_G| = \prod_{\ell \neq p} O^{\mathrm{can}}(\mathbb{1}_{G(\mathbb{Z}_\ell)}).$$

Combining Corollary 3.6, relation (4-2), and Proposition 4.5 we get, for $\ell \neq p$,

$$\begin{aligned}
\nu_\ell(a,q) &= \frac{\ell^3}{\#G^{\mathrm{der}}(\mathbb{F}_\ell)} O^{\mathrm{geom}}_{\gamma_0}(\mathbb{1}_{G(\mathbb{Z}_\ell)}) \\
&= \frac{\ell^3}{\#G^{\mathrm{der}}(\mathbb{F}_\ell)} \frac{L_\ell(1,\xi_{T^{\mathrm{der}}})}{L_\ell(M^\vee_{G^{\mathrm{der}}}(1))} |\Delta_K|_\ell^{-1/2} |D(\gamma_0)|_\ell^{1/2} O^{\mathrm{can}}_{\gamma_0}(\mathbb{1}_{G(\mathbb{Z}_\ell)}) \\
&= L_\ell(1,\xi_{T^{\mathrm{der}}}) |D(\gamma_0)|_\ell^{1/2} |\Delta_K|_\ell^{-1/2} O^{\mathrm{can}}_{\gamma_0}(\mathbb{1}_{G(\mathbb{Z}_\ell)}).
\end{aligned}$$

Second, let ϕ_q be the characteristic function of $G(\mathbb{Z}_p)\left(\begin{smallmatrix}1&0\\0&q\end{smallmatrix}\right)G(\mathbb{Z}_p)$ in $G(\mathbb{Q}_p)$, and let $\phi_{p,q}$ be the characteristic function of $G(\mathbb{Z}_q)\left(\begin{smallmatrix}1&0\\0&p\end{smallmatrix}\right)G(\mathbb{Z}_q)$ in $G(\mathbb{Q}_q)$. Using Lemmas 3.7 and 3.8, we find that

$$\begin{aligned}
\nu_p(a,q) &= \frac{p^3}{\#G^{\mathrm{der}}(\mathbb{F}_p)} O^{\mathrm{geom}}_{\gamma_0}(\phi_q) \\
&= \frac{p^3}{\#G^{\mathrm{der}}(\mathbb{F}_p)} \frac{L_p(1,\xi_{T^{\mathrm{der}}})}{L_p(M^\vee_{G^{\mathrm{der}}}(1))} |\Delta_K|_p^{-1/2} |D(\gamma_0)|_p^{1/2} O^{\mathrm{can}}_{\gamma_0}(\phi_q) \\
&= \frac{p^3}{\#G^{\mathrm{der}}(\mathbb{F}_p)} \frac{L_p(1,\xi_{T^{\mathrm{der}}})}{L_p(M^\vee_{G^{\mathrm{der}}}(1))} |\Delta_K|_p^{-1/2} |D(\gamma_0)|_p^{1/2} T\, O^{\mathrm{can}}_{\delta_0}(\phi_{p,q}).
\end{aligned}$$

Taking a product over all finite primes, we obtain

$$\prod_{\ell<\infty} \nu_\ell(a,q) = L(1,\xi_{T^{\mathrm{der}}}) \sqrt{\frac{|\Delta_K|}{|D(\gamma_0)|}}\, T O^{\mathrm{can}}_{\delta_0\sigma}(\phi_{p,q}) O^{\mathrm{can}}_{\gamma_0}(\phi^p). \tag{5-2}$$

Recall that $f_0(T)$, the characteristic polynomial of γ_0, is $f_0(T) = T^2 - aT + q$. The (polynomial) discriminant of $f_0(T)$ and the (Weyl) discriminant of γ_0 are related by $|D(\gamma_0)\det(\gamma_0)| = |\mathrm{disc}(f_0)| = 4q - a^2$. Consequently,

$$\sqrt{q}\,\nu_\infty(a,q) = \frac{1}{\pi}\sqrt{|D(\gamma_0)|}.$$

Since $L(1,\xi_{T^{\mathrm{der}}}) = L(1,\xi_{T/Z})$ (Lemma 4.2), to deduce (5-1) from (5-2) it suffices to show that

$$\frac{\sqrt{|\Delta_K|}}{2\pi} L(1,\xi_{T/Z}) = \mathrm{vol}(T(\mathbb{Q})\backslash T(\mathbb{A}_f)). \tag{5-3}$$

On one hand $L(s,\xi_{T/Z})$ coincides with $L(s,K/\mathbb{Q})$, the Dirichlet L-function attached to the quadratic character of K. Therefore, the analytic class number formula implies that the left-hand side of (5-3) is h_K/w_K, the ratio of the class number of K to the number of roots of unity in K. On the other hand, the right-hand side of (5-3) is also well-known to be h_K/w_K (e.g., [Weil 1973, Proposition VII.6.12]); we defer to Lemma A.4 in the Appendix for details. $\square$

Appendix: Orbital integers and measure conversions

by S. Ali Altuğ

In this appendix, we explain how to deduce the comparison factor of Proposition 4.5 from [Frenkel et al. 2010] and certain computations in [Langlands 2013] as well as calculate the volume factor that goes into the proof of Theorem 5.1. We also remark that the same measure comparison also appears in [Altuğ 2015] (although implicitly) in the passage from equation (2) to (3).

Comparison of measures. Let $G = \mathrm{GL}_2$. Let ω_G be the same volume form as in Section 3A2. For a torus $T \subset G$, let ω_T be as in Proposition 4.1. Recall that $\mathcal{T}^\circ$ is the connected component of the identity in the Néron model of T.

Lemma A.1. *Let ℓ be a finite prime, let $\gamma \in G(\mathbb{Q}_\ell)$ be regular semisimple, and let $T = G_\gamma$ be its centralizer. Let μ_G and μ_T be nonzero Haar measures on $G(\mathbb{Q}_\ell)$ and $T(\mathbb{Q}_\ell)$, respectively. Then*

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = \sqrt{|D(\gamma)|_\ell}\,\frac{\mathrm{vol}(|\omega_G|_\ell)\,\mathrm{vol}(\mu_{T,\ell})}{\mathrm{vol}(|\omega_T|_\ell)\,\mathrm{vol}(\mu_{G,\ell})}\,\bar{\mu}_{T\backslash G,\ell},$$

where $|D(\gamma)| = |\mathrm{tr}(\gamma)^2 - 4\det(\gamma)|$ and $\bar{\mu}_\ell = \mu^{\mathrm{can}}_{\mathrm{GL}_2,\ell}/\mu_{T\ell}$.

Proof. By equation (3.30) of [Frenkel et al. 2010], we have

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = \sqrt{|D(\gamma)|_\ell}\,|\omega_{T\backslash G}|_\ell,$$

where we note that the left hand side of (3.30) of loc. cit. is what we denoted by $\mu^{\mathrm{geom}}_\gamma$. Since the Haar measure is unique up to a constant we have $|\omega_G|_\ell = c_\ell(G)d\mu_{G,\ell}$ and $|\omega_T|_\ell = c_\ell(T)d\mu_{T,\ell}$. The constants can be calculated easily by comparing the volumes of the integral points

$$c_\ell(G) = \frac{\mathrm{vol}_{|\omega_G|}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{\mu_{G,\ell}}(G(\mathbb{Z}_\ell))} \quad \text{and} \quad c_\ell(T) = \frac{\mathrm{vol}_{|\omega_T|}(\mathcal{T}^\circ(\mathbb{Z}_\ell))}{\mathrm{vol}_{\mu_{T,\ell}}(\mathcal{T}^\circ(\mathbb{Z}_\ell))}.$$

Therefore, the quotient measures $\bar{\mu}_{T\backslash G,\ell}$ and $|\omega_{T\backslash G}|_\ell$ are related by

$$|\omega_{T\backslash G}|_\ell = \frac{c_\ell(G)}{c_\ell(T)}\bar{\mu}_{T\backslash G,\ell}.$$

The lemma follows. $\square$

As an immediate corollary to Lemma A.1 we get:

Corollary A.2. *Let $\mu^{\mathrm{can}}_{G,\ell}$ and $\mu^{\mathrm{can}}_{T,\ell}$ be normalized to give measure* 1 *to $G(\mathbb{Z}_\ell)$ and $\mathcal{T}^\circ(\mathbb{Z}_\ell)$ respectively, and let the rest of the notation be as in Lemma A.1. Then*

$$\mu^{\mathrm{geom}}_{\gamma,\ell} = \sqrt{|D(\gamma)|_\ell}\,\frac{\mathrm{vol}_{|\omega_G|_\ell}(G(\mathbb{Z}_\ell))}{\mathrm{vol}_{|\omega_T|_\ell}(\mathcal{T}^\circ(\mathbb{Z}_\ell))}\bar{\mu}_{T\backslash G,\ell}.$$

We now quote a result of [Langlands 2013]. Let $\zeta_\ell(s) = 1/(1-\ell^{-s})$.

Lemma A.3. *We have*

$$\mathrm{vol}(|\omega_G|_\ell) = \zeta_\ell(1)^{-1}\zeta_\ell^{-1}(2),$$
$$\mathrm{vol}(|\omega_T|_\ell) = \sqrt{|\Delta_K|_\ell}\begin{cases}\zeta_\ell(1)^{-2} & \text{if } K/\mathbb{Q} \text{ is split at } \ell,\\ \zeta_\ell(2)^{-1} & \text{if } K/\mathbb{Q} \text{ is unramified at } \ell,\\ \zeta_\ell(1)^{-1} & \text{if } K/\mathbb{Q} \text{ is ramified at } \ell,\end{cases}$$

where $K/\mathbb{Q}$ is the quadratic extension which splits T and Δ_K is the discriminant of K.

Proof. The result for odd primes ℓ is given on pages 41 and 42 of [Langlands 2013]. The case for $\ell = 2$ follows the same lines. The only point to keep in mind is the extra factor of 2 that appears in the calculation of the differential form on page 42 of [Langlands 2013]; we leave the details to the reader. □

Corollary A.2 and Lemma A.3 then give the conversion factor between the two measures.

***Calculation of* $\mathbf{vol}(K^\times\backslash\mathbb{A}_K^{\times,\mathrm{fin}})$.** Let (a,p) be such that $a^2-4p<0$. Let $d\mu^{\mathrm{can}}_{T,\ell}$ be the Haar measure normalized to give measure 1 to $T(\mathbb{Z}_\ell)$ and set

$$d\mu^{\mathrm{can}}_{T,\mathrm{fin}} := \otimes_{l\neq\infty} d\mu^{\mathrm{can}}_{T,\ell}.$$

Lemma A.4. *We have*

$$\mu^{\mathrm{can}}_{T,\mathrm{fin}}(T(\mathbb{Q})\backslash T(\mathbb{A}^{\mathrm{fin}})) = \frac{h_K}{\omega_K},$$

where $K/\mathbb{Q}$ is the quadratic extension which splits T, ω_K is the number of roots of unity in K, and h_K is its class number.

Proof. By identifying $T = G_\gamma$ with $\mathbb{G}_m$ over the quadratic extension K we have

$$\mathrm{M}^{\mathrm{can}}_{T,\mathrm{fin}}(T(\mathbb{Q})\backslash T(\mathbb{A}^{\mathrm{fin}})) = \mu^{\mathrm{can}}_{K,\mathrm{fin}}(K^\times\backslash\mathbb{A}_K^{\times,\mathrm{fin}}),$$

where the measure on the right is such that $\mu^{\mathrm{can}}_{K,\mathrm{fin}}(O_v^\times) = 1$ for each place v. Let $\hat{\mathcal{O}}_K^\times = \prod_v \mathcal{O}_v^\times$. Recall that

$$1 \to (K^\times \cap \hat{\mathcal{O}}_K^\times)\backslash\hat{\mathcal{O}}_K^\times \to K^\times\backslash\mathbb{A}_K^{\times,\mathrm{fin}} \to \mathrm{Cl}(K) \to 1,$$

which implies that $\mu(K^\times\backslash\mathbb{A}_K^{\times,\mathrm{fin}}) = h_K\mu(K^\times\cap\hat{\mathcal{O}}_K^\times)\backslash\hat{\mathcal{O}}_K^\times) = h_K/\omega_K$. □

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ON THE ABSOLUTE CONTINUITY OF p-HARMONIC MEASURE AND SURFACE MEASURE IN REIFENBERG FLAT DOMAINS

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We study the set of absolute continuity of p-harmonic measure μ associated to a positive weak solution to the p-Laplace equation with continuous zero boundary values and $(n-1)$-dimensional Hausdorff measure $\mathcal{H}^{n-1}$ on locally flat domains in space. We prove that when $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2-\eta < p < 2$ for some $\eta > 0$ there exist locally flat domains $\Omega \subset \mathbb{R}^n$ with locally finite perimeter and Borel sets $E \subset \partial\Omega$ such that $\mu(E) > 0 = \mathcal{H}^{n-1}(E)$.

1. Introduction and statement of main results

A well-known result of F. and M. Riesz says that if Ω is a simply connected domain whose boundary has finite length in the plane then harmonic measure and arclength are mutually absolutely continuous. Makarov [1985] gives a sharp description of the support of harmonic measure and shows that the function λ given below is the proper function to measure the size of the support of ω. In particular, if $\Omega \subset \mathbb{R}^2$ is a simply connected domain in the plane, then $\omega \ll \mathcal{H}^\lambda$, where

$$\lambda(r) := r \exp\left\{ C\sqrt{\log\frac{1}{r}\log\log\log\frac{1}{r}} \right\}$$

for sufficiently large C. Here "$\ll$" stands for absolute continuity of the measures, we use "$\perp$" to denote measures are singular, and $\mathcal{H}^\lambda$ to denote the Hausdorff measure with respect to the function λ (see (1.4) for definition of $\mathcal{H}^\lambda$). In [Makarov 1985], it is also shown that this result is sharp in the following sense; there is an example of a simply connected domain for which $\omega \perp \mathcal{H}^\lambda$ whenever C is sufficiently small in the definition of λ. In higher dimensions, due to examples of Ziemer [1974] and Wu [1986], neither $\mathcal{H}^n|_{\partial\Omega} \ll \omega$ nor $\omega \ll \mathcal{H}^n|_{\partial\Omega}$ are true in general without imposing extra topological or nontopological conditions on $\partial\Omega$. David and Jerison [1990] prove

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that if Ω is a nontangentially accessible (NTA for short; see Definition 2.1) domain and $\partial\Omega$ is Ahlfors–David regular (ADR for short; see Definition 2.5) then harmonic measure is mutually absolutely continuous on $\partial\Omega$ with respect to surface measure, and in fact they are A_∞-equivalent (see [Azzam et al. 2014]). Badger [2012] considers the same problem by relaxing the ADR property by $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and proves that $\mathcal{H}^{n-1} \ll \omega$ on $\partial\Omega$. He also shows that $\omega \ll \mathcal{H}^{n-1} \ll \omega$ on the set $A \subset \partial\Omega$, where

$$A = \left\{ x \in \partial\Omega : \liminf_{r\to 0} \frac{\mathcal{H}^{n-1}(\Delta(x,r))}{r^{n-1}} < \infty \right\}.$$

Here $\Delta(x,r) = B(x,r) \cap \partial\Omega$. Badger also conjectures that when Ω is an NTA domain then the same result holds not only on $A \subset \partial\Omega$ but on the whole $\partial\Omega$ (see Conjecture 1.3 in [Badger 2012]). However, it turns out that this is not true in general. In fact, Azzam, Mourgoglou, and Tolsa [Azzam et al. 2016] construct an example of a Reifenberg flat domain (see Definition 2.3) Ω in $\mathbb{R}^n$, $n \geq 3$, with $\mathcal{H}^{n-1}(\partial\Omega) < \infty$ and a Borel set $E \subset \partial\Omega$ such that

$$\omega(E) > 0 = \mathcal{H}^{n-1}(E).$$

One can consider the same problem for the p-harmonic measure associated with a positive weak solution to the p-Laplace equation for $1 < p \neq 2 < \infty$. To define p-harmonic measure and the p-Laplace equation, we let $\Omega \subset \mathbb{R}^n$ be a domain and let N be a neighborhood of $\partial\Omega$. Fix p, $1 < p < \infty$, and suppose that $\hat{u}$ is a positive weak solution to the p-Laplace equation in $\Omega \cap N$. That is, $\hat{u} \in W^{1,p}(\Omega \cap N)$ and

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \theta \rangle \,\mathrm{d}x = 0 \tag{1.1}$$

whenever $\theta \in W_0^{1,p}(\Omega \cap N)$. Equivalently, we say that $\hat{u}$ is p-harmonic in $\Omega \cap N$. Observe that if $\hat{u}$ is smooth and $\nabla \hat{u} \neq 0$ in $\Omega \cap N$ then

$$\nabla \cdot (|\nabla \hat{u}|^{p-2} \nabla \hat{u}) \equiv 0$$

in the classical sense, where $\nabla \cdot$ denotes divergence. We assume that $\hat{u}$ has zero boundary values on $\partial\Omega \cap N$ in the Sobolev sense. More specifically, if $\zeta \in C_0^\infty(\Omega \cap N)$, then $\hat{u}\zeta \in W_0^{1,p}(\Omega \cap N)$. Extend $\hat{u}$ to N by putting $\hat{u} \equiv 0$ on $N \setminus \Omega$. Then $\hat{u} \in W^{1,p}(N)$ and it follows from (1.1), as in [Heinonen et al. 1993, Chapter 21], that there exists a finite positive Borel measure $\hat{\mu}$ on $\mathbb{R}^n$ with support contained in $\partial\Omega \cap N$ satisfying

$$\int |\nabla \hat{u}|^{p-2} \langle \nabla \hat{u}, \nabla \phi \rangle \,\mathrm{d}x = -\int \phi \,\mathrm{d}\hat{\mu} \tag{1.2}$$

whenever $\phi \in C_0^\infty(N)$. Existence of $\hat{\mu}$ follows from the maximum principle, basic Caccioppoli inequalities for $\hat{u}$ and the Riesz representation theorem for a positive linear functional. We note that if $\partial\Omega$ is smooth enough and $\nabla u \neq 0$ in Ω, then

$$\mathrm{d}\hat{\mu} = |\nabla \hat{u}|^{p-1}\, \mathrm{d}\mathcal{H}^{n-1}|_{\partial\Omega\cap N}.$$

Remark 1.3. When $p=2$ in (1.1), we have the usual Laplace's equation. Moreover, if u is the Green's function for Laplace's equation with pole at, say $z_0 \in \Omega$, then the measure in (1.2) corresponding to this harmonic function u is harmonic measure, ω, relative to z_0. Note also that the p-Laplace equation in (1.1) is degenerate when $p > 2$ and is singular when $1 < p < 2$. The nonlinear structure of this PDE makes it difficult to work with.

We next introduce the notion of the *Hausdorff dimension of a measure*. To this end, let $\hat{r}_0 > 0$ be given, and let $0 < \delta < \hat{r}_0$ be fixed. Let $\lambda : [0, \infty) \to [0, \infty)$ be a nondecreasing function with $\lambda(0) = 0$. Let $d(\cdot)$ denote the diameter of a set. For a given Borel set $E \subset \mathbb{R}^n$, we define (δ, λ)-*Hausdorff content* of E in the usual way:

$$\mathcal{H}^{\lambda}_{\delta}(E) := \inf\left\{\sum_i \lambda(d(U_i)) : E \subset \bigcup U_i,\ \text{each } U_i \text{ is open with } d(U_i) < \delta\right\}.$$

Then the *Hausdorff measure* of E is defined by

$$\mathcal{H}^{\lambda}(E) := \lim_{\delta\to 0} \mathcal{H}^{\lambda}_{\delta}(E). \tag{1.4}$$

In case $\lambda(r) = r^{\alpha}$ we write $\mathcal{H}^{\alpha}$ for $\mathcal{H}^{\lambda}$. The Hausdorff dimension of $\hat{\mu}$, denoted by $\mathcal{H}-\dim\hat{\mu}$, is defined by

$$\mathcal{H}-\dim\hat{\mu} := \inf\big\{\alpha : \text{there exists Borel } \quad E \subset \partial\Omega \\ \text{such that} \quad \mathcal{H}^{\alpha}(E) = 0 \quad \text{and} \quad \hat{\mu}(\mathbb{R}^n \setminus E) = 0\big\}.$$

We return to our study of singular sets of p-harmonic measure with respect to $\mathcal{H}^{n-1}$ measure. For arbitrary $p,\ 1 < p \neq 2 < \infty$, Bennewitz and Lewis [2005] observed that the natural candidates, i.e., snowflake-type domains, which give sharpness in the harmonic case shown by Makarov, do not provide sharpness. In the same paper it was also shown that if $\partial\Omega$ is the von Koch snowflake in the plane and $2 < p < \infty$ then $\mathcal{H}-\dim\mu < 1$. In [Lewis et al. 2011], a weaker version of Makarov's result was obtained under the p-harmonic setting for $1 < p \neq 2 < \infty$. Finally, Lewis [2015] proved a p-harmonic analogue of Makarov's result; let $\Omega \subset \mathbb{R}^2$ be any bounded simply connected domain and let μ be the p-harmonic measure described above. Let $\lambda(r)$ be as in Makarov's result. Then the following are true.

(a) If $1 < p < 2$, there is $A = A(p) \geq 1$ such that $\mu \ll \mathcal{H}^{\lambda}$.

(b) If $2 < p < \infty$, there is $A = A(p) \leq -1$ such that μ is concentrated on a set of σ-finite $\mathcal{H}^1$ measure.

The nonlinearity and degeneracy of the p-Laplace equation makes it difficult to study the Hausdorff dimension of this measure in $\mathbb{R}^n$, $n \geq 3$. The tools developed by Lewis, Nyström, and Vogel [Lewis et al. 2013] for p-harmonic functions were used to obtain that:

(1) If $\partial\Omega$ is sufficiently flat in the sense of Reifenberg and $p \geq n \geq 3$, then μ is concentrated on a set of σ-finite $\mathcal{H}^{n-1}$ measure.

(2) If $n \geq 3$ and $2 < p < n$, there exist Wolff snowflakes such that $\mathcal{H}-\dim\mu < n-1$, while if $1 < p < 2$, there exist Wolff snowflakes such that $\mathcal{H}-\dim\mu > n-1$.

(3) All examples produced by Wolff's snowflake method have $\mathcal{H}-\dim\mu < n-1$ when $p \geq n$.

(4) There is a Wolff snowflake for which the sign of $(n-1)-(\mathcal{H}-\dim\mu)$ equals the sign of $(n-1)-(\mathcal{H}-\dim\omega)$, where μ is the p-harmonic measure for p in an open interval containing 2 and ω is the harmonic measure with pole at infinity.

Lewis, Vogel, and the author [Akman et al. 2015] improved these results by proving the following: let $O \subset \mathbb{R}^n$ be any open set, $\hat{z} \in \partial O$, and let $\rho > 0$. Let u be a positive weak solution to (1.1) in $O \cap B(\hat{z}, \rho)$. Assume also that u has continuous zero boundary values on $\partial O \cap B(\hat{z}, \rho)$. Extend u to all $B(\hat{z}, \rho)$ by defining 0 in $B(\hat{z}, \rho) \setminus O$. Let μ be the measure associated to u as in (1.2). If $p > n$ then μ is concentrated on the set

$$\mathcal{P} = \left\{ x \in \partial O \cap B(\hat{z}, \rho) : \limsup_{r \to 0} \frac{\mu(B(w, r))}{r^{n-1}} > 0 \right\}.$$

This set $\mathcal{P}$ has σ-finite $\mathcal{H}^{n-1}$ measure. The same result holds when $p = n$, provided that $\partial O \cap B(\hat{z}, \rho)$ is locally uniformly fat in the sense of n-capacity. Therefore, $\mathcal{H}-\dim\mu_p \leq n-1$ when $p \geq n$.

On the other hand, the result of David and Jerison described above for harmonic measure is extended to the p-harmonic setting for $1 < p \neq 2 < \infty$ by Lewis and Nyström [2012]. To state this result, we let $\Omega \subset \mathbb{R}^n$ be a bounded NTA domain with constants M, r_0 whose boundary is ADR. Let u be p-harmonic in $\Omega \cap B(w, 4r)$, $w \in \partial\Omega$, $0 < r < r_0$, and continuous in $\bar{\Omega} \cap B(w, 4r)$ with $u \equiv 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$ and let μ be the p-harmonic measure as in (1.2) associated with u. Then it is shown in [Lewis and Nyström 2012, Proposition 3.4] that $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on $\partial\Omega$; in fact they are A_∞-equivalent. It also is proven in the same paper that Badger's result holds under the p-harmonic setting; if Ω is an NTA domain then $\mu \ll \mathcal{H}^{n-1} \ll \mu$ on the set $A' \subset \Delta(w, 4r) \subset \partial\Omega$, where

$$A' = \left\{ x \in \Delta(w, 4r) : \liminf_{\rho \to 0} \frac{\mathcal{H}^{n-1}(\Delta(x, \rho))}{\rho^{n-1}} < \infty \right\}.$$

The main result proved in this paper is that there are examples of domains for which absolute continuity of p-harmonic measure and $(n-1)$-dimensional Hausdorff measure does not hold on the whole boundary when the domain is even locally flat in the sense of Reifenberg.

Theorem 1.5. *When $n \geq 2$ and $2 < p < \infty$ and when $n \geq 3$ and $2 - \eta < p < 2$ for some $\eta > 0$, there exist domains $\Omega \subset \mathbb{R}^n$ and Borel sets $E \subset \partial\Omega$ such that*:

(1) *Ω is a $(\hat{\delta}, \infty)$-Reifenberg flat domain.*

(2) *$\sigma = \mathcal{H}^{n-1}$ is Radon.*

(3) *$\mu_p(E) > 0 = \sigma(E)$, where μ_p is the p-harmonic measure associated to a positive p-harmonic function in Ω with continuous zero boundary values on $\partial\Omega$.*

As the plan of this paper, we first state the definition of nontangentially accessible domains, Reifenberg flatness, and Ahlfors–David regularity, and we give some lemmas concerning the regularity of p-harmonic function in NTA domains in Section 2. We give the construction of Wolff snowflakes in Section 3. Following [Azzam et al. 2016], we construct "an enlarged domain Ω_ϵ^+" from a certain domain Ω and, using some results from [Lewis et al. 2013] concerning the dimension of p-harmonic measure, we give a proof of Theorem 1.5 in Section 4.

2. Definitions and preparatory lemmas

To proceed, some notation and definitions are in order. In the sequel, c will denote a positive constant ≥ 1 (not necessarily the same at each occurrence), which may depend only on p, n, unless otherwise stated. In general, $c(a_1, \dots, a_n)$ denotes a positive constant ≥ 1 which may depend only on $p, n, a_1, \dots, a_n$, not necessarily the same at each occurrence.

Let $x = (x_1, \dots, x_n)$ denote points in $\mathbb{R}^n$ and let $\bar{E} = \mathrm{cl}(E)$, $\mathrm{int}\, E$, ∂E, and E^{c} be the closure, interior, boundary, and the complement of the set $E \subset \mathbb{R}^n$, respectively. Let $\mathrm{diam}(E)$ be the diameter of a set E. Let $\langle \cdot, \cdot \rangle$ be the usual inner product in $\mathbb{R}^n$. Let $d(E, F)$ denote the usual distance between the sets E and F and let $d_{\mathcal{H}}(E, F)$ denote the Hausdorff distance between the sets E and F, which is defined by

$$d_{\mathcal{H}}(E, F) := \max(\sup\{d(E, y) : y \in F\}, \sup\{d(x, F) : x \in E\}).$$

Let $B(x, r)$ be the usual open ball centered at x with radius $r > 0$ in $\mathbb{R}^n$ and let $\mathrm{d}x$ denote the Lebesgue n-measure in $\mathbb{R}^n$. Let $\Delta(w, r) = \partial\Omega \cap B(w, r)$. For a given number $t > 0$ and a cube Q, let $l(Q)$ be the side length of Q and let tQ denote the cube whose side length is $tl(Q)$ with the same center as Q.

We state the notion of nontangentially accessible domain which was initially introduced by Jerison and Kenig [1982].

Definition 2.1 (NTA domain). A domain Ω is called a *nontangentially accessible* (NTA) domain if there exist $M \geq 2$ and r_0 such that the following are fulfilled.

(i) *Corkscrew condition*: for any $w \in \partial\Omega$, $0 < r < r_0$, there exists $a_r(w) \in \Omega$ satisfying

$$M^{-1}r < |a_r(w) - w| < r \quad \text{and} \quad M^{-1}r < d(a_r(w), \partial\Omega).$$

(ii) $\mathbb{R}^n \setminus \overline{\Omega}$ satisfies the corkscrew condition.

(iii) *Uniform condition*: if $w \in \partial\Omega$, $0 < r < r_0$, and $w_1, w_2 \in B(w, r) \cap \Omega$ then there exists a rectifiable curve $\gamma : [0, 1] \to \Omega$ with $\gamma(0) = w_1$ and $\gamma(1) = w_2$ such that

(a) $\mathcal{H}^1(\gamma) \leq M|w_1 - w_2|$,

(b) $\min\{\mathcal{H}^1(\gamma([0, t])), \mathcal{H}^1([t, 1]))\} \leq Md(\gamma(t), \partial\Omega)$.

Remark 2.2. We use the definition of this notion given in [Lewis and Nyström 2012]. Note that (iii) of Definition 2.1 is different but equivalent to the Harnack chain condition given in [Jerison and Kenig 1982].

Next we give the definition of Reifenberg flatness from [Azzam et al. 2016].

Definition 2.3 ((δ, r_0)-Reifenberg flat domain). Let Ω be a domain and $r_0, \delta > 0$ with $0 < \delta < \frac{1}{2}$. Then Ω is said to be (δ, r_0)-*Reifenberg flat* provided that the following two conditions hold.

(i) For every $w \in \partial\Omega$ and every $0 < r < r_0$ there exists a hyperplane $\mathcal{P}(w, r)$ containing w such that

$$d_{\mathcal{H}}(\Delta(w, r), \mathcal{P}(w, r) \cap B(w, r)) \leq \delta r.$$

(ii) For every $x \in \partial\Omega$, one of the connected components of

$$B(x, r_0) \cap \{x \in \mathbb{R}^n;\ \mathrm{d}(x, \mathcal{P}(x, r_0)) \geq 2\delta r_0\}$$

is contained in Ω and the other is contained in $\mathbb{R}^n \setminus \Omega$.

We say that Ω is (δ, ∞)-Reifenberg flat if it is (δ, r_0)-Reifenberg flat for every $r_0 > 0$.

Remark 2.4. An equivalent definition of Reifenberg flatness is given in [Lewis and Nyström 2012], and it is remarked that these two definitions are equivalent (see observation after their Definition 1.2).

Definition 2.5 (Ahlfors–David regular set). We say that $\partial\Omega$ is n-dimensional *Ahlfors–David regular* (ADR) if there is some uniform constant C such that

$$C^{-1}r^n \leq \mathcal{H}^n(\Delta(x, r)) \leq Cr^n \quad \text{for all } r \in (0, \mathrm{diam}(\Omega)),\ x \in \partial\Omega.$$

We next give some estimates from when $n \geq 3$ [Lewis et al. 2013, Lemmas 3.2–3.6] and when $n = 2$ given under the p-harmonic settings [Bennewitz and Lewis 2005, Lemmas 2.6, 2.7, 2.13, 2.14]. For Lemmas 2.6–2.8, let p be fixed with $1 < p \neq 2 < \infty$.

Lemma 2.6. *Let u be a positive p-harmonic function in $B(w, 2r) \subset \mathbb{R}^n$, $n \geq 3$. Then*

$$r^{p-n} \int_{B(w,r/2)} |\nabla u|^p \,\mathrm{d}x \leq c\Big(\max_{B(w,r)} u\Big)^p$$

and

$$\max_{B(w,r)} u \leq c \min_{B(w,r)} u.$$

Moreover, there exists $\beta = \beta(p, n) \in (0, 1)$ such that if $x, y \in B(w, r)$ then

$$|u(x) - u(y)| \leq c\left(\frac{|x - y|}{r}\right)^{\beta} \max_{B(w,2r)} u.$$

For Lemmas 2.7 and 2.8 let Ω be an NTA domain in $\mathbb{R}^n$ and let $w \in \partial\Omega$, $0 < r < r_0$.

Lemma 2.7. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 4r)$ and $u = 0$ on $\Delta(w, 4r)$. Extend u to $B(w, 4r)$ by defining $u \equiv 0$ on $B(w, 4r) \setminus \Omega$. Then u has a representative in $W^{1,p}(B(w, 4r))$ with Hölder continuous partial derivatives in $\Omega \cap B(w, 4r)$. In particular, there exists $\sigma = \sigma(p, n) \in (0, 1]$ such that if $x, y \in B\big(\hat{w}, \frac{1}{2}\hat{r}\big)$, where $B(\hat{w}, 4\hat{r}) \subset \Omega \cap B(w, 4r)$ then*

$$\frac{1}{c}|\nabla u(x) - \nabla u(y)| \leq \left(\frac{|x - y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},\hat{r})} |\nabla u| \leq \frac{c}{\hat{r}}\left(\frac{|x - y|}{\hat{r}}\right)^{\sigma} \max_{B(\hat{w},2\hat{r})} u.$$

If $\nabla u(\hat{w}) \neq 0$ then u is real analytic in a neighborhood of $\hat{w}$.

The next lemma gives a relation between a p-harmonic function and its corresponding measure.

Lemma 2.8. *Suppose that u is a nonnegative continuous p-harmonic function in $\bar{\Omega} \cap B(w, 2r)$ and $u = 0$ on $\Delta(w, 2r)$. Extend u to $B(w, 2r)$ by defining $u \equiv 0$ on $B(w, 2r) \setminus \Omega$. As in (1.2), there exists a unique locally finite positive Borel measure μ on $\mathbb{R}^n$ with support in $\Delta(w, 2r)$ such that*

$$\int |\nabla u|^{p-2} \langle \nabla u, \nabla \theta \rangle \,\mathrm{d}x = -\int \theta \,\mathrm{d}\mu$$

whenever $\theta \in C_0^\infty(B(w, 2r))$. Moreover, there exists $c = c(p, n, M) \in [1, \infty)$ such that if $\tilde{r} = r/c$ then

$$c^{-1} r^{p-n} \mu(\Delta(w, \tilde{r})) \leq (u(a_{\tilde{r}}(w)))^{p-1} \leq c r^{p-n} \mu\big(\Delta\big(w, \tfrac{1}{2}\tilde{r}\big)\big),$$

where $a_{\tilde{r}}(w)$ is as in Definition 2.1.

3. Construction of Wolff snowflakes

In this section, following [Lewis et al. 2013] when $n \geq 3$ and [Bennewitz and Lewis 2005] when $n = 2$, we describe the construction of Wolff snowflakes in $\mathbb{R}^n$ which was originally introduced in [Wolff 1995]. To this end, let

$$\Omega_0 = \{(x', x_n) : x' \in \mathbb{R}^{n-1},\ x_n > 0\} \subset \mathbb{R}^n.$$

Set

$$Q(r) = \left\{x' \in \mathbb{R}^{n-1} : -\tfrac{1}{2}r \leq |x_i| \leq \tfrac{1}{2}r \ \text{ for } \ 1 \leq i \leq n-1\right\}.$$

Then $Q(r)$ is an $(n-1)$-dimensional cube with side length r and centered at 0. Let $\phi : \mathbb{R}^{n-1} \to \mathbb{R}$ be a piecewise linear function with support contained in $\left\{x' : |x'| < \frac{1}{2}\right\}$ satisfying

$$\|\nabla\phi\|_\infty \leq \theta_0. \tag{3.1}$$

For fixed large N, define $\psi(x') = N^{-1}\phi(Nx')$. Let $b > 0$ be a small constant and let Q be an $(n-1)$-dimensional cube with center a_Q and length $l(Q)$ contained in some hyperplane. Let $\operatorname{cch}(E)$ denote the closed convex hull. Let e be a unit normal to Q and define

$$P_Q = \operatorname{cch}(Q \cup \{a_Q + bl(Q)e\}) \quad \text{and} \quad \tilde{P}_Q = \operatorname{int}\operatorname{cch}(Q \cup \{a_Q - bl(Q)e\}).$$

We set $e = -e_n$ for $Q(1)$. We also define

$$\Lambda := \{x \in P_{Q(1)} \cup \tilde{P}_{Q(1)},\ x_n \geq \psi(x)\} \quad \text{and} \quad \partial := \{x \in \mathbb{R}^n,\ x' \in Q(1),\ x_n = \psi(x')\}.$$

We assume that $N = N(b, M)$ is so large that

$$d(\partial \setminus \partial\Omega_0, \partial[P_{Q(1)} \cup \tilde{P}_{Q(1)}]) \geq \tfrac{b}{100}.$$

From the construction, it can be easily seen that $\partial \subset Q(1) \times \left[-\frac{1}{2}, \frac{1}{2}\right]$ consists of a finite number of $(n-1)$-dimensional faces. We fix a Whitney decomposition of each face; we divide each face of ∂ into an $(n-1)$-dimensional cube Q, with side lengths 8^{-k}, $k = 1, 2, \ldots$, and $8^{-k} \approx$ to their distance from the edges of the face they lie on. We next choose a distinguished $(n-2)$-dimensional "side" for each $(n-1)$-dimensional cube.

Suppose Ω is a domain and $Q \subset \partial\Omega$ is an $(n-1)$-dimensional cube with distinguished side γ. Let e be a unit normal to $\partial\Omega$ on Q and assume that $P_Q \cap \Omega = \varnothing$ and $\tilde{P} \subset \Omega$. We form a new domain $\tilde{\Omega}$ as follows. Let $\mathcal{T}$ be the conformal affine map, i.e., composition of a translation, dilation, and rotation, with $\mathcal{T}(Q(1)) = Q$ which fixes dilation, $\mathcal{T}(0) = a_Q$ which fixes translation, $\mathcal{T}\left(\left\{x \in \partial Q(1) : x_1 = \frac{1}{2}\right\}\right)$ and $\mathcal{T}(-e_n)$ in the direction of e which fixes rotation. Let $\Lambda_Q = \mathcal{T}(\Lambda)$ and $\partial_Q = \mathcal{T}(\partial)$.

Then we define $\tilde{\Omega}$ through the relations

$$\tilde{\Omega} \cap (P_Q \cup \tilde{P}_Q) \quad \text{and} \quad \tilde{\Omega} \setminus (P_Q \cup \tilde{P}_Q) = \Omega \setminus (P_Q \cup \tilde{P}_Q).$$

Note that ∂_Q inherits from ∂ a natural subdivision into Whitney cubes with distinguished sides. This process is called "adding a blip to Ω along Q".

To use the process of "adding a blip" to construct a Wolff snowflake Ω_∞, starting from Ω_0, we first add a blip to Ω_0 along $Q(1)$ obtaining a new domain Ω_1. We then inherit a subdivision of $\partial\Omega_1 \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)})$ into Whitney cubes with distinguished sides, together with a finite set of edges E_1 (the edges of the faces of the graph are not in the Whitney cubes). Let G_1 be the set of all Whitney cubes in the subdivision. Then Ω_2 is obtained from Ω_1 by adding a blip along each $Q \in G_1$. From this process, we inherit a family of cubes $G_2 \subset \partial\Omega_2$ (each with a distinguished side) and a set of edges $E_2 \subset \partial\Omega_2$ of σ-finite $\mathcal{H}^{n-2}$ measure. Continuing by induction we get $(\Omega_m)_{m=n-1}^\infty$, $(G_m)_{m=n-1}^\infty$, and $(E_m)_{m=n-1}^\infty$, where

$$\partial\Omega_m \cap (P_{Q(1)} \cap \tilde{P}_{Q(1)}) = E_m \cup \bigcup_{Q \in G_m} Q \quad \text{for } m \geq n-1.$$

If $N = N(b, M)$ is large enough, then $\Omega_m \to \Omega_\infty$ in the Hausdorff distance sense. We call Ω_∞ a *Wolff snowflake*. We state a result which says that Wolff snowflakes are locally flat in the sense of Reifenberg.

Lemma 3.2 [Lewis et al. 2013, Lemma 7.1]. *If θ_0, N^{-1} are small enough, depending only n, then the Wolff snowflake domain Ω_∞ is $(c\theta_0, \infty)$-Reifenberg flat, where $c = c(n)$.*

4. Proof of Theorem 1.5

In this section we give a proof of Theorem 1.5 using some results from [Lewis et al. 2013; Azzam et al. 2016]. To this end, let Ω_∞ be a Wolff snowflake with constants θ_0, N as described in Section 3. For fixed p, $1 < p \neq 2 < \infty$, let u_∞ be the unique positive p-harmonic function in Ω_∞ with continuous boundary value zero on $\partial\Omega_\infty$ and $|x_n - u_\infty(x)| \to 0$ uniformly as $|x| \to \infty$. Let μ_∞ be the p-harmonic measure associated with u_∞ as in (1.2). A proof of existence and uniqueness of u_∞ can be found in [Lewis et al. 2013, Lemma 6.1]. Let Ω'_∞ be the restriction of Ω_∞ to $Q(1) \times [-1, 1]$ and let μ'_∞ be the restriction of μ_∞ to $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$.

The following lemma can be easily deduced by combining Lemma 7.4 and Proposition 7.6 from [Lewis et al. 2013] when $n \geq 3$ and combining Lemma 3.23 and Theorem 1 from [Bennewitz and Lewis 2005] when $n = 2$. Moreover, when $n \geq 3$ and $2 - \eta < p < 2$ it follows from Theorem 4 in [Lewis et al. 2013]. We first state a lemma.

Lemma 4.1. *When $n \geq 3$ let p be fixed, $2 < p < \infty$, and when $n \geq 2$ let p be fixed with $2 - \eta < p < 2$ for some $\eta > 0$. Let Ω'_∞ and μ'_∞ be described as above. Then* *for some $d > 0$ we have*

$$\lim_{r\to 0} \frac{\log \mu'_\infty(\Delta(x,r))}{\log r} \leq d < n-1 \quad \textit{for all } x \in \partial\Omega'_\infty \setminus \Lambda,$$

where $\Lambda \subset \partial\Omega'_\infty$ with $\mu'_\infty(\Lambda) = 0$. Moreover, $\mathcal{H}-\dim \mu'_\infty \leq d < n-1$.

Proof. We first show that there exist Wolff snowflakes for which $\mathcal{H}-\dim \mu < n-1$ in all cases stated in Lemma 4.1. To this end, as we discussed in Section 1, when $n \geq 3$ and $2 < p < \infty$ there exist Wolff snowflakes such that $\mathcal{H}-\dim \mu < n-1$ (see Theorems 2 and 3 in [Lewis et al. 2013]). When $n = 2$, it follows from [Bennewitz and Lewis 2005, Theorem 1] that there is a Wolff snowflake for which $\mathcal{H}-\dim \mu < 1$ whenever p is fixed with $2 < p < \infty$. Next, there exist Wolff snowflakes for which $\mathcal{H}-\dim \omega < n-1$, which is a well-known result of Wolff [1995] when $n \geq 3$. On the other hand, it is observed in [Lewis et al. 2013, Proposition 6.4] that there exists a Wolff snowflake such that the sign of $(n-1)-(\mathcal{H}-\dim \omega)$ equals the sign of $(n-1)-(\mathcal{H}-\dim \mu)$ for $p \in (2-\eta, 2)$. Therefore, combining these two results, we first conclude that there exists a Wolff snowflake for which $\mathcal{H}-\dim \omega < n-1$ when $2-\eta < p < 2$ for some $\eta > 0$. Using these observations and Lemma 7.4 in [Lewis et al. 2013] we finish the proof of lemma. $\square$

We are now ready to prove Theorem 1.5. Under the p-harmonic setting, we closely follow the arguments given in [Azzam et al. 2016] after Theorem 4.3. We first observe from Lemma 4.1, more specifically from the fact $\mathcal{H}-\dim \mu'_\infty \leq d < n-1$, and the definition of Hausdorff dimension of p-harmonic measure, that there is a Borel set $E \subset \partial\Omega'_\infty$ such that $\mu'_\infty(\mathbb{R}^n \setminus E) = 0$ and $\mathcal{H}^d(E) = 0$. From this observation and once again from Lemma 4.1 we also have

$$\lim_{r\to 0} \frac{\log \mu'_\infty(B(x,r))}{\log r} \leq d < n-1 \quad \text{for all } x \in E. \tag{4.2}$$

Note that Ω'_∞ is the restriction of Ω_∞ to $Q(1) \times [-1, 1]$; therefore,

$$\partial\Omega_\infty \setminus \{(x', x_n) \in \mathbb{R}^n : x_n = 0\} \subset \partial\Omega'_\infty.$$

For ease of notation we let

$$\mathfrak{R}^{n-1} := \{(x', x_n) \in \mathbb{R}^n : x' \in \mathbb{R}^{n-1} \text{ and } x_n = 0\}.$$

From (4.2) it follows that for α, $0 < \alpha < n-1-d$, one can find small enough ρ such that $\mu'_\infty(E_1) > 0$, where

$$E_1 = \left\{ x \in (E \cap \partial\Omega_\infty) \setminus \mathfrak{R}^{n-1} : \frac{\log \mu'_\infty(B(x,r))}{\log r} < n-1-\alpha \text{ for all } r \in (0,\rho] \right\}.$$

We next fix a point $\zeta_0 \in E_1$. By the regularity of p-harmonic measure we can find $\rho_0 \in (0, \rho]$ and a compact set $K \subset E_1 \cap B(\zeta_0, \rho_0)$ such that for all $x \in K$ and

$r \in (0, \rho_0)$ the following property holds:

$$\mu'_\infty(K) > 0 \quad \text{and} \quad \mu'_\infty(B(x, r)) > r^{n-1-\alpha}.$$

The construction yields that $K \subset \partial\Omega'_\infty \cap \partial\Omega_\infty$ and $\mathrm{cl}(\Omega'_\infty) \subset \mathrm{cl}(\Omega_\infty)$. Then using the fact that the support of μ'_∞ is contained in $(Q(1) \times [-1, 1]) \cap \partial\Omega_\infty$ we have

$$\mu_\infty(K) > 0 \quad \text{and} \quad \mu_\infty(B(x, r) \cap \partial\Omega_\infty) > r^{n-1-\alpha} \tag{4.3}$$

for all $x \in K$ and $r \in (0, \rho_0)$.

For a given number t, $4 \le t$, and given open set $O \subset \mathbb{R}^{n-1}$ we use $\mathcal{W}_t(O)$ to denote the set of maximal dyadic cubes $Q \subset O$ satisfying $tQ \cap K^{\mathrm{c}} = \varnothing$. Let $0 < \epsilon < \frac{1}{100}$ and let $\mathcal{I}$ be the family of cubes $Q \in \mathcal{W}_{\epsilon^{-2}}(K^{\mathrm{c}})$ such that

$$Q \cap (Q(1) \times [-1, 1]) \cap \partial\Omega_\infty \neq \varnothing.$$

Note that

$$l(Q) \approx \epsilon^2 \operatorname{dist}(Q, K) \quad \text{for all } Q \in \mathcal{I} \quad \text{and} \quad \partial\Omega'_\infty \setminus K \subset \bigcup_{Q \in \mathcal{I}} Q.$$

For each $Q \in \mathcal{I}$, fix some point $z_Q \in Q \cap \partial\Omega'_\infty$. We then define a new domain Ω^+_ϵ by

$$\Omega^+_\epsilon := \Omega'_\infty \cup \biggl(\bigcup_{Q \in \mathcal{I}} B_Q\biggr), \quad \text{where } B_Q = B(z_Q, \epsilon \operatorname{dist}(z_Q, K)).$$

It is observed in [Azzam et al. 2016, Lemma 2.2] that if θ, ϵ in the construction of the Wolff snowflake in Section 3 are small enough then Ω^+_ϵ is $(c\epsilon^{1/2}, r_0)$-Reifenberg flat and $K \subset \partial\Omega^+_\epsilon$, provided that the original domain Ω_∞ is (δ, r_0)-Reifenberg flat. Note that from Lemma 3.2 we have that Wolff snowflake domain Ω_∞ is $(c\theta_0, r_0)$-Reifenberg flat, where $r_0 = \infty$. Therefore if we choose θ and ϵ small enough and use Lemma 2.2 from [Azzam et al. 2016] then Ω^+_ϵ is a $(c\epsilon^{1/2}, \infty)$-Reifenberg flat domain satisfying

$$K \subset \partial\Omega'_\infty \cap \partial\Omega^+_\epsilon \quad \text{and} \quad \mathrm{cl}(\Omega_\infty) \subset \mathrm{cl}(\Omega^+_\epsilon). \tag{4.4}$$

Let u^+_ϵ be a positive p-harmonic function in Ω^+_ϵ with continuous boundary value zero on $\partial\Omega^+_\epsilon$. Let μ^+_ϵ be the p-harmonic measure associated with u^+_ϵ as in (1.2). From the construction of Ω^+_ϵ we have $u^+_\epsilon \ge u'_\infty$ on $\partial\Omega'_\infty$. Then it follows from the maximum principle for positive p-harmonic functions and (4.4) that $u^+_\epsilon \ge u'_\infty$ in Ω'_∞. This observation, Lemmas 2.6–2.8 and (4.3) yield

$$\mu^+_\epsilon(K) > 0 \quad \text{and} \quad \mu^+_\epsilon(B(x, r)) > r^{n-1-\alpha} \quad \text{for all } x \in K,\ r \in (0, \rho_0). \tag{4.5}$$

As μ^+_ϵ is a Radon measure which follows from Lemma 2.8 and satisfies (4.5) and Ω^+_ϵ is $(\hat{\delta}, \hat{r}_0)$-Reifenberg flat domain, it follows from [Azzam et al. 2016, Lemma 3.1] that $\mathcal{H}^{n-1}|_{\partial\Omega^+_\epsilon}$ is locally finite. Let $\Omega := \Omega^+_\epsilon$ be the $(\hat{\delta}, \hat{r}_0)$-Reifenberg flat domain with locally finite surface measure and let $\mu := \mu_\epsilon^+$ be the p-harmonic measure as above. From (4.5) we conclude that Theorem 1.5 is true. □

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ON THE GEOMETRY OF GRADIENT EINSTEIN-TYPE MANIFOLDS

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In this paper we introduce the notion of Einstein-type structure on a Riemannian manifold (M, g), unifying various particular cases recently studied in the literature, such as gradient Ricci solitons, Yamabe solitons and quasi-Einstein manifolds. We show that these general structures can be locally classified when the Bach tensor is null.

1. Introduction and main results

In the last years there has been an increasing interest in the study of Riemannian manifolds endowed with metrics satisfying some structural equations, possibly involving curvature and some globally defined vector fields. These objects naturally arise in several different frameworks; the most important and well studied examples are *Ricci solitons*, see, e.g., [Hamilton 1988; Perelman 2002; Ni and Wallach 2008; Naber 2010; Cao and Chen 2012; Brendle 2013] and references therein. Other examples are, for instance, *Ricci almost solitons* [Pigola et al. 2011], *Yamabe solitons* [Daskalopoulos and Sesum 2013; Cao et al. 2012], *Yamabe quasisolitons* [Huang and Li 2014; Wang 2013], *conformal gradient solitons* [Tashiro 1965; Catino et al. 2012], *quasi-Einstein manifolds* [Kim and Kim 2003; Case et al. 2011; Catino et al. 2013; He et al. 2012], and ρ*-Einstein solitons* [Catino and Mazzieri 2016; Catino et al. 2015].

In this paper we study Riemannian manifolds satisfying a general structural condition that includes all the aforementioned examples as particular cases, in order to hopefully provide a useful compendium that also gives a summary and unification of classification problems thoroughly studied over the past years.

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Towards this aim we consider a smooth, connected Riemannian manifold (M, g) of dimension $m \geq 3$, and we denote by Ric and S the corresponding *Ricci tensor* and *scalar curvature*, respectively (see the next section for the details). We denote by $\mathrm{Hess}(f)$ the Hessian of a function $f \in C^\infty(M)$ and by $\mathscr{L}_X g$ the Lie derivative of the metric g in the direction of the vector field X. We introduce the following:

Definition 1.1. We say that (M, g) is an *Einstein-type manifold* (or, equivalently, that (M, g) supports an *Einstein-type structure*) if there exist $X \in \mathfrak{X}(M)$ and $\lambda \in C^\infty(M)$ such that

$$\alpha\,\mathrm{Ric} + \frac{\beta}{2}\mathscr{L}_X g + \mu X^\flat \otimes X^\flat = (\rho S + \lambda) g, \tag{1-1}$$

for some constants $\alpha, \beta, \mu, \rho \in \mathbb{R}$, with $(\alpha, \beta, \mu) \neq (0, 0, 0)$. If $X = \nabla f$ for some $f \in C^\infty(M)$, we say that (M, g) is a *gradient Einstein-type manifold*. Accordingly (1-1) becomes

$$\alpha\,\mathrm{Ric} + \beta\,\mathrm{Hess}(f) + \mu\, df \otimes df = (\rho S + \lambda) g, \tag{1-2}$$

for some $\alpha, \beta, \mu, \rho \in \mathbb{R}$.

Here $\mathfrak{X}(M)$ denotes the set of smooth vector fields on M and $X^\flat$ the 1-form metrically dual to X.

We note that, from the definition, the term ρS could clearly be absorbed into the function λ. However, we keep them separate in order to explicitly include and highlight the case of ρ-Einstein solitons.

In the present paper we focus our analysis on the gradient case.

Leaving aside the case $\beta = 0$ that will be addressed separately, see Proposition 5.7, we say that the gradient Einstein-type manifold (M, g) is *nondegenerate* if $\beta \neq 0$ and $\beta^2 \neq (m-2)\alpha\mu$; otherwise, that is if $\beta \neq 0$ and $\beta^2 = (m-2)\alpha\mu$, we have a *degenerate* gradient Einstein-type manifold. Note that, in this last case, necessarily α and μ are not null. The above terminology is justified by the next observation:

$$\begin{gathered} (M, g) \text{ is conformally Einstein} \\ \Updownarrow \\ (M, g) \text{ is a degenerate, gradient, Einstein-type manifold,} \\ \text{for some } \alpha, \beta, \mu \neq 0. \end{gathered} \tag{1-3}$$

For the proof and for the notion of conformally Einstein manifold see Section 2.

In case f is constant we say that the Einstein-type structure is *trivial*. Note that, since $m \geq 3$, in this case (M, g) is Einstein. However, the converse is generally false; indeed, if (M, g) is Einstein, then for some constant $\Lambda \in \mathbb{R}$ we have $\mathrm{Ric} = \Lambda g$ and inserting into (1-2) we obtain

$$\beta\,\mathrm{Hess}(f) + \mu\, df \otimes df = (\rho S + \lambda - \Lambda\alpha) g.$$

Thus, if $\rho \neq 0$, (M, g) is a Yamabe quasisoliton and f is not necessarily constant (see [Huang and Li 2014; Wang 2013]).

We will also deal with the case $\alpha = 0$ separately, see Theorem 1.4. We explicitly remark that, from the definition, α and β cannot both be equal to zero.

As we have already noted, the class of manifolds satisfying Definition 1.1 gives rise to the previously quoted examples by specifying, in general not in a unique way, the values of the parameters and possibly the function λ. In particular we have:

(1) Einstein manifolds: $(\alpha, \beta, \mu, \rho) = (1, 0, 0, 1/m)$, $\lambda = 0$ (or, equivalently for $m \geq 3$, $\rho = 0$ and $\lambda = S/m$).

(2) Ricci solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0)$, $\lambda \in \mathbb{R}$.

(3) Ricci almost solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, 0)$, $\lambda \in C^\infty(M)$.

(4) Yamabe solitons: $(\alpha, \beta, \mu, \rho) = (0, 1, 0, 1)$, $\lambda \in \mathbb{R}$.

(5) Yamabe quasisolitons: $(\alpha, \beta, \mu, \rho) = (0, 1, -1/k, 1)$, $k \in \mathbb{R} \setminus \{0\}$, $\lambda \in \mathbb{R}$.

(6) conformal gradient solitons: $(\alpha, \beta, \mu, \rho) = (0, 1, 0, 0)$, $\lambda \in C^\infty(M)$.

(7) quasi-Einstein manifolds: $(\alpha, \beta, \mu, \rho) = (1, 1, -1/k, 0)$, $\lambda \in \mathbb{R}$, $k \neq 0$.

(8) ρ-Einstein solitons: $(\alpha, \beta, \mu, \rho) = (1, 1, 0, \rho)$, $\rho \neq 0$, $\lambda \in \mathbb{R}$.

Of course one may wonder about the existence of Einstein-type structures. We know from the literature positive answers to the various examples that we mentioned earlier. For the general case we can consider three different necessary conditions. The first two are the general integrability conditions (4-5) and (4-6) contained in Theorem 4.4 below. The third comes from the simple observation that, in the case $\mu \neq 0$, tracing (1-2) and defining $u = e^{f\mu/\beta}$, the existence of a gradient Einstein-type structure on (M, g) yields the existence of a positive solution of

$$Lu = \Delta u - \frac{\mu}{\beta^2}[m\lambda + (m\rho - \alpha)S]u = 0,$$

so that, by a well-known spectral result (see, for instance, [Fischer-Colbrie and Schoen 1980; Moss and Piepenbrink 1978]), the operator L is stable, or, in other words, the spectral radius of L, $\lambda_1^L(M)$, is nonnegative. Here we will not further pursue this direction.

As it appears in Definition 1.1, the fact that (M, g) is an Einstein-type manifold can be interpreted as a prescribed condition on the Ricci tensor of g (see, for instance, the nice survey [Bourguignon 1981]), that is, on the "trace part" of the Riemann tensor. Thus, it is reasonable to expect classification and rigidity results for these structures only assuming further conditions on the traceless part of the Riemann tensor, i.e., on the Weyl tensor. Indeed, most of the aforementioned papers pursue this direction, for instance, assuming that (M, g) is locally conformally flat or has harmonic Weyl tensor. In the spirit of the recent work of H.-D. Cao and Q. Chen [2013], we study the class of gradient Einstein-type manifolds with vanishing Bach tensor along the integral curves of f. We note that this condition is weaker than local conformal flatness (see Section 2).

It turns out that, as in the case of gradient Ricci solitons (see [Cao and Chen 2012; 2013; Cao et al. 2014]), the leading actor is a three tensor, D, that plays a fundamental role in relating the Einstein-type structure to the geometry of the underlying manifold. D naturally appears when writing the first two integrability conditions for the structure defining the differential system (1-2). Quite unexpectedly, the constant ρ and the function λ have no influence on this relation.

Our main purpose is to give local characterizations of complete, noncompact, nondegenerate gradient Einstein-type manifolds. Denoting with B the Bach tensor of (M, g) (see Section 2), our first result is

Theorem 1.2. *Let (M, g) be a complete, noncompact, nondegenerate, gradient, Einstein-type manifold of dimension $m \geq 3$. If $B(\nabla f, \cdot) = 0$ and f is a proper function, then, in a neighborhood of every regular level set of f, the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

In dimension four we improve this result, obtaining

Corollary 1.3. *Let (M^4, g) be a complete, noncompact, nondegenerate, gradient, Einstein-type manifold of dimension four. If $B(\nabla f, \cdot) = 0$ and f is a proper function, then, in a neighborhood of every regular level set of f, the manifold (M, g) is locally a warped product with three-dimensional fibers of constant curvature. In particular, (M^4, g) is locally conformally flat.*

As we will show in Section 7, the properness assumption is satisfied by some important subclasses of Einstein-type manifolds, under quite natural geometric assumptions. As a consequence, in the case of gradient Ricci solitons, we recover a local version of the results in [Cao and Chen 2013; Cao et al. 2014], while, in the cases of ρ-Einstein solitons and Ricci almost solitons, we prove two new classification theorems (see Theorem 7.1 and 7.2).

In the special case $\alpha = 0$ (which includes Yamabe solitons, Yamabe quasisolitons and conformal gradient solitons) we give a version of Theorem 1.2 in the following local result that provides a very precise description of the metric in this situation. Note that Theorem 1.4 and Corollary 1.5 also apply to the compact case.

Theorem 1.4. *Let (M, g) be a complete, gradient, Einstein-type manifold of dimension $m \geq 3$ with $\alpha = 0$. Then, in a neighborhood of every regular level set of f, the manifold (M, g) is locally a warped product with $(m-1)$-dimensional fibers. More precisely, every regular level set Σ of f admits a maximal open neighborhood $U \subset M^m$ on which f only depends on the signed distance r to the hypersurface Σ. In addition, the potential function f can be chosen in such a way that the metric g takes the form on U*

$$g = dr \otimes dr + \left(\frac{f'(r)}{f'(0)} e^{\mu f(r)}\right)^2 g^{\Sigma},$$

where g^{Σ} is the metric induced by g on Σ. As a consequence, f has at most two critical points on M^m and we have the following cases:

(1) *If f has no critical points, then (M, g) is globally conformally equivalent to a direct product $I \times N^{m-1}$ of some interval $I = (t_*, t^*) \subseteq \mathbb{R}$ with a $(m-1)$-dimensional complete Riemannian manifold (N^{m-1}, g^N). More precisely, the metric takes the form*

$$g = u^2(t)\big(dt^2 + g^N\big),$$

where $u : (t_*, t^*) \to \mathbb{R}$ is some positive smooth function.

(2) *If f has only one critical point $O \in M^m$, then (M, g) is globally conformally equivalent to the interior of a Euclidean ball of radius $t^* \in (0, +\infty]$. More precisely, on $M^m \setminus \{O\}$, the metric takes the form*

$$g = v^2(t)\big(dt^2 + t^2 g^{\mathbb{S}^{m-1}}\big),$$

where $v : (0, t^*) \to \mathbb{R}$ is some positive smooth function and $\mathbb{S}^{m-1}$ denotes the standard unit sphere of dimension $m-1$. In particular (M, g) is complete, noncompact, and rotationally symmetric.

(3) *If the function f has two critical points $N, S \in M^m$, then (M, g) is globally conformally equivalent to $\mathbb{S}^m$. More precisely, on $M^m \setminus \{N, S\}$, the metric takes the form*

$$g = w^2(t)\big(dt^2 + \sin^2(t) g^{\mathbb{S}^{m-1}}\big),$$

where $w : (0, \pi) \to \mathbb{R}$ is some smooth positive function. In particular (M, g) is compact and rotationally symmetric.

In this case we can obtain a stronger global result just assuming nonnegativity of the Ricci curvature, namely we have the following:

Corollary 1.5. *Any nontrivial, complete, gradient, Einstein-type manifold of dimension $m \geq 3$ with $\alpha = 0$ and nonnegative Ricci curvature is either rotationally symmetric or it is isometric to a Riemannian product $\mathbb{R} \times N^{m-1}$, where N^{m-1} is an $(m-1)$-dimensional Riemannian manifold with nonnegative Ricci curvature.*

The proof of Theorem 1.4 follows immediately from [Catino et al. 2012] by substituting $u = e^{\mu f}$ in the equation. This result covers the cases of Yamabe solitons [Cao et al. 2012] and conformal gradient solitons [Catino et al. 2012]. Concerning Yamabe quasisolitons, Corollary 1.5 improves the results in [Huang and Li 2014]. In particular, this shows that most of the assumptions in [Huang and Li 2014, Theorem 1.1] are not necessary.

The paper is organized as follows. In Section 2 we recall some useful definitions and properties of various geometric tensors and fix our conventions and notation. In Section 3 we collect some useful commutation relations for covariant derivatives of functions and tensors. In Section 4 we prove the two aforementioned integrability conditions that follow directly from the Einstein-type structures. In Section 5 we compute the squared norm of the tensor D in terms of D itself, the Bach tensor B and the potential function f. In Section 6 we relate the tensor D to the geometry of the regular level sets of the potential function f. Finally, in Section 7 we prove Theorem 1.2 and Corollary 1.3, and we give some geometric applications in the special cases of gradient Ricci solitons, ρ-Einstein solitons, and Ricci almost solitons.

2. Definitions and notation

In this section we recall some useful definitions and properties of various geometric tensors and fix our conventions and notation.

To perform computations, we freely use the method of the moving frame referring to a local orthonormal coframe of the m-dimensional Riemannian manifold (M, g). We fix the index range $1 \le i, j, \dots \le m$ and recall that the Einstein summation convention will be in force throughout.

We denote by R the *Riemann curvature tensor* (of type $(1, 3)$) associated to the metric g, and by Ric and S the corresponding *Ricci tensor* and *scalar curvature*, respectively. The components of the $(0, 4)$-versions of the Riemann tensor and of the *Weyl tensor* W are related by the formula

$$R_{ijkt} = W_{ijkt} + \frac{1}{m-2}(R_{ik}\delta_{jt} - R_{it}\delta_{jk} + R_{jt}\delta_{ik} - R_{jk}\delta_{it}) - \frac{S}{(m-1)(m-2)}(\delta_{ik}\delta_{jt} - \delta_{it}\delta_{jk}) \tag{2-1}$$

and they satisfy the symmetry relations

$$R_{ijkt} = -R_{jikt} = -R_{ijtk} = R_{ktij},$$
$$W_{ijkt} = -W_{jikt} = -W_{ijtk} = W_{ktij}.$$

A computation shows that the Weyl tensor is also totally trace-free. The *Schouten tensor* A is defined by

$$\mathrm{A} = \mathrm{Ric} - \frac{S}{2(m-1)}g.$$

Tracing we have $\mathrm{tr}(\mathrm{A}) = A_{tt} = ((m-2))/(2(m-1))S$.

Remark 2.1. Some authors adopt a different convention and define the Schouten tensor as $A/(m-2)$.

According to this convention the (components of the) Ricci tensor and the scalar curvature are respectively given by $R_{ij} = R_{itjt} = R_{titj}$ and $S = R_{tt}$. We note that, in terms of the Schouten tensor and of the Weyl tensor, the Riemann curvature tensor can be expressed in the form

$$R = \mathrm{W} + \frac{1}{m-2}\mathrm{A} \owedge g,$$

where $\owedge$ is the Kulkarni–Nomizu product; in components,

$$R_{ijkt} = W_{ijkt} + \frac{1}{m-2}(A_{ik}\delta_{jt} - A_{it}\delta_{jk} + A_{jt}\delta_{ik} - A_{jk}\delta_{it}).$$

Next we introduce the *Cotton tensor* C as the obstruction to the commutativity of the covariant derivative of the Schouten tensor, that is

$$C_{ijk} = A_{ij,k} - A_{ik,j} = R_{ij,k} - R_{ik,j} - \frac{1}{2(m-1)}(S_k\delta_{ij} - S_j\delta_{ik}). \tag{2-2}$$

We also recall that the Cotton tensor, for $m \geq 4$, can be defined as one of the possible divergences of the Weyl tensor; precisely

$$C_{ijk} = \left(\frac{m-2}{m-3}\right)W_{tikj,t} = -\left(\frac{m-2}{m-3}\right)W_{tijk,t}. \tag{2-3}$$

A computation shows that the two definitions (for $m \geq 4$) coincide.

Remark 2.2. It is worthwhile to recall that the Cotton tensor is skew-symmetric in the second and third indices (i.e., $C_{ijk} = -C_{ikj}$) and totally trace-free (i.e., $C_{iik} = C_{iki} = C_{kii} = 0$).

We are now ready to define the *Bach tensor* B, originally introduced by Bach [1921] in the study of conformal relativity. Its components are

$$B_{ij} = \frac{1}{m-2}(C_{jik,k} + R_{kt}W_{ikjt}), \tag{2-4}$$

that, in case $m \geq 4$, by (2-3) can be alternatively written as

$$B_{ij} = \frac{1}{m-3}W_{ikjt,tk} + \frac{1}{m-2}R_{kt}W_{ikjt}.$$

Note that if (M, g) is either locally conformally flat (i.e., $C = 0$ if $m = 3$ or $W = 0$ if $m \geq 4$) or Einstein, then $B = 0$. A computation shows that the Bach tensor is symmetric (i.e., $B_{ij} = B_{ji}$) and evidently trace-free (i.e., $B_{ii} = 0$). As a consequence we observe that we can write

$$B_{ij} = \frac{1}{m-2}(C_{ijk,k} + R_{kl}W_{ikjl}).$$

We recall that

Definition 2.3. The manifold (M, g) is *conformally Einstein* if its metric g can be pointwise conformally deformed to an Einstein metric $\tilde{g}$.

We observe that, if $\tilde{g} = e^{2a\varphi} g$, for some $\varphi \in C^\infty(M)$ and some constant $a \in \mathbb{R}$, then its Ricci tensor $\widetilde{\mathrm{Ric}}$ is related to that of g by the well-known formula (see [Besse 2008])

$$\widetilde{\mathrm{Ric}} = \mathrm{Ric} - (m-2)a\,\mathrm{Hess}(\varphi) + (m-2)a^2 d\varphi \otimes \varphi - [(m-2)a^2|\nabla\varphi|^2 + a\Delta\varphi]g. \tag{2-5}$$

Here the various operators (for their precise definitions see Section 3) are defined with respect to the metric g.

Now we can easily prove statement (1-3); indeed, suppose that $\beta \neq 0$ and $\beta^2 = (m-2)\alpha\mu$, that is, the Einstein-type structure is degenerate. Tracing (1-2) we obtain

$$\frac{1}{\alpha}(\rho S + \lambda) = \frac{1}{m}\Big(S + \frac{\beta}{\alpha}\Delta f + \frac{\mu}{\alpha}|\nabla f|^2\Big). \tag{2-6}$$

Choose $\varphi = f$ and $a = -\beta/((m-2)\alpha)$ in (2-5) to obtain

$$\widetilde{\mathrm{Ric}} = \frac{1}{\alpha}\bigg[\frac{\beta^2}{(m-2)\alpha} - \mu\bigg] df \otimes df + \frac{1}{\alpha}(\rho S + \lambda)g + \frac{\beta}{(m-2)\alpha}\Big(\Delta f - \frac{\beta}{\alpha}|\nabla f|^2\Big)g. \tag{2-7}$$

Inserting (2-6) into (2-7) and using the fact that the Einstein-type structure is degenerate yields

$$\widetilde{\mathrm{Ric}} = \frac{1}{\alpha}\bigg[\frac{\beta^2}{(m-2)\alpha} - \mu\bigg] df \otimes df + \frac{1}{m}\bigg[S + 2\frac{\beta}{\alpha}\frac{m-1}{m-2}\Delta f - \frac{\mu}{\alpha}(m-1)|\nabla f|^2\bigg]g.$$

Hence, since $\beta^2 = (m-2)\alpha\mu$,

$$\widetilde{\mathrm{Ric}} = \frac{1}{m}\bigg[S + 2\frac{\beta}{\alpha}\frac{m-1}{m-2}\Delta f - \frac{\mu}{\alpha}(m-1)|\nabla f|^2\bigg]g,$$

that is, $\tilde{g} = e^{-(2\beta)/((m-2)\alpha) f} g$ is an Einstein metric (this was also obtained in [Besse 2008, Theorem 1.159]).

Conversely suppose that $\tilde{g} = e^{2af} g$, $a \neq 0$, is an Einstein metric, so that $\widetilde{\mathrm{Ric}} = \Lambda\tilde{g}$, for some $\Lambda \in \mathbb{R}$. From (2-5)

$$\begin{aligned}\mathrm{Ric} - (m-2)a\,\mathrm{Hess}(f) + (m-2)a^2 df \otimes df \\ = \big[\Lambda e^{2af} + (m-2)a^2|\nabla f|^2 + a\Delta f\big]g.\end{aligned} \tag{2-8}$$

Tracing we get

$$\frac{S}{m-1} = \big[(m-2)a^2|\nabla f|^2 + a\Delta f\big] + a\Delta f + \frac{m}{m-1}\Lambda e^{2af}.$$

Thus, inserting into (2-8),

$$\mathrm{Ric} - (m-2)a\,\mathrm{Hess}(f) + (m-2)a^2 df \otimes df = \Big(\frac{S}{m-1} - a\Delta f - \frac{\Lambda}{m-1}e^{2af}\Big)g.$$

We choose $\alpha = 1$, $\beta = -(m-2)a$, $\mu = (m-2)a^2$, $\rho = 1/(m-1)$, and $\lambda(x) = -a\Delta f - \Lambda/(m-1)e^{2af}$. We note that $\beta \neq 0$ and

$$\beta^2 = (m-2)^2 a^2 = (m-2)\alpha\mu,$$

so this choice of α, β, μ, ρ, and λ yields a degenerate Einstein-type structure.

To conclude we note that the equivalence of degenerate gradient Ricci solitons and conformally Einstein metrics is well-known in conformal geometry (see [Catino 2012; Jauregui and Wylie 2015]).

3. Some basics on moving frames and commutation rules

In this section we collect some useful commutation relations for covariant derivatives of functions and tensors that will be used in the rest of the paper. All of these formulas are well-known to experts.

Let (M, g) be a Riemannian manifold of dimension $m \geq 3$. For the sake of completeness (see [Alías et al. 2016] for details) we recall that having fixed a (local) orthonormal coframe $\{\theta^i\}$, with dual frame $\{e_i\}$, the corresponding *Levi-Civita connection forms* $\{\theta^i_j\}$ are the 1-forms uniquely defined by the requirements

$$d\theta^i = -\theta^i_j \wedge \theta^j \text{ (first structure equations)}, \quad \text{and} \quad \theta^i_j + \theta^j_i = 0.$$

The *curvature forms* $\{\Theta^i_j\}$ associated to the connection are the 2-forms defined via the *second structure equations*

$$d\theta^i_j = -\theta^i_k \wedge \theta^k_j + \Theta^i_j.$$

They are skew-symmetric (i.e., $\Theta^i_j + \Theta^j_i = 0$) and they can be written as

$$\Theta^i_j = \tfrac{1}{2} R^i_{jkt} \theta^k \wedge \theta^t = \sum_{k<t} R^i_{jkt} \theta^k \wedge \theta^t,$$

where R^i_{jkt} are precisely the coefficients of the ((1, 3)-version of the) Riemann curvature tensor.

The *covariant derivative of a vector field* $X \in \mathfrak{X}(M)$ is defined by

$$\nabla X = (dX^i + X^j \theta^i_j) \otimes e_i = X^i_k \theta^k \otimes e_i,$$

while the *covariant derivative of a* 1*-form* ω is defined by

$$\nabla\omega = (d\omega_i - w_j \theta^j_i) \otimes \theta^i = \omega_{ik} \theta^k \otimes \theta^i.$$

The *divergence* of the vector field $X \in \mathfrak{X}(M)$ is the trace of the endomorphism $(\nabla X)^\sharp : TM \to TM$, that is,

$$\operatorname{div} X = \operatorname{tr}(\nabla X)^\sharp = g(\nabla_{e_i} X, e_i) = X^i_i.$$

For a smooth function f we can write

$$df = f_i\theta^i,$$

for some smooth coefficients $f_i \in C^\infty(M)$. The *Hessian* of f, Hess(f), is the $(0, 2)$-tensor defined as

$$\mathrm{Hess}(f) = \nabla df = f_{ij}\theta^j \otimes \theta^i,$$

with

$$f_{ij}\theta^j = df_i - f_t\theta_i^t.$$

Note that (see Lemma 3.1 below)

$$f_{ij} = f_{ji}.$$

The *Laplacian* of f, Δf, is the trace of the Hessian, in other words

$$\Delta f = \mathrm{tr}(\mathrm{Hess}(f)) = f_{ii}.$$

The moving frame formalism reveals extremely useful in determining the commutation rules of geometric tensors. Some of them will be essential in our computations.

Lemma 3.1. *Let $f \in C^3(M)$. The following equalities hold.*

(3-1) $\quad f_{ij} = f_{ji}.$

(3-2) $\quad f_{ijk} = f_{jik}.$

(3-3) $\quad f_{ijk} = f_{ikj} + f_t R_{tijk}.$

(3-4)
$$f_{ijk} = f_{ikj} + f_t W_{tijk} + \frac{1}{m-2}(f_t R_{tj}\delta_{ik} - f_t R_{tk}\delta_{ij} + f_j R_{ik} - f_k R_{ij}) - \frac{S}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij}).$$

(3-5)
$$f_{ijk} = f_{ikj} + f_t W_{tijk} + \frac{1}{m-2}(f_t A_{tj}\delta_{ik} - f_t A_{tk}\delta_{ij} + f_j A_{ik} - f_k A_{ij}).$$

In particular, tracing (3-3), we deduce

(3-6)
$$f_{itt} = f_{tti} + f_t R_{ti}.$$

For the Riemann curvature tensor we recall the classical Bianchi identities that in our formalism become

$$R_{ijkt} + R_{itjk} + R_{iktj} = 0 \quad \text{(first Bianchi identity)},$$
$$R_{ijkt,l} + R_{ijlk,t} + R_{ijtl,k} = 0 \quad \text{(second Bianchi identity)}.$$

For the derivatives of the curvature we have the well known formulas

Lemma 3.2.

$$R_{ijkt,lr} - R_{ijkt,rl} = R_{sjkt}R_{silr} + R_{iskt}R_{sjlr} + R_{ijst}R_{sklr} + R_{ijks}R_{stlr},$$
$$R_{ij,k} - R_{ik,j} = -R_{tijk,t} = R_{tikj,t},$$
$$R_{ij,kt} - R_{ij,tk} = R_{likt}R_{lj} + R_{ljkt}R_{li}.$$

The first Bianchi identities imply that

$$C_{ijk} + C_{jki} + C_{kij} = 0. \tag{3-7}$$

From the definition of the Cotton tensor we also deduce that

$$C_{ijk,t} = A_{ij,kt} - A_{ik,jt} = R_{ij,kt} - R_{ik,jt} - \frac{1}{2(m-1)}(S_{kt}\delta_{ij} - S_{jt}\delta_{ik}).$$

On the other hand, by Lemma 3.2 and Schur's identity $S_i = \frac{1}{2}R_{ik,k}$,

$$R_{ik,jk} = R_{ik,kj} + R_{tijk}R_{tk} + R_{tkjk}R_{ti} = \frac{1}{2}S_{ij} - R_{tk}R_{itjk} + R_{it}R_{tj}.$$

This enables us to obtain the following expression for the divergence of the Cotton tensor:

$$C_{ijk,k} = R_{ij,kk} - \frac{m-2}{2(m-1)}S_{ij} + R_{tk}R_{itjk} - R_{it}R_{tj} - \frac{1}{2(m-1)}\Delta S\delta_{ij}.$$

The previous relation also shows that

$$C_{ijk,k} = C_{jik,k}, \tag{3-8}$$

thus confirming the symmetry of the Bach tensor, see (2-4).

Taking the covariant derivative of (3-7) and using (3-8) we also deduce

$$C_{kij,k} = 0.$$

4. The tensor *D* and the integrability conditions

The main result of this section concerns two natural integrability conditions that follow directly from the Einstein-type structure; as in the case of Ricci solitons and Yamabe (quasi-)solitons, there is a natural tensor that turns out to play a fundamental role in relating the Einstein-type structure to the geometry of the underlying manifold. Quite surprisingly, as it is shown in Theorem 4.4, the presence of the constant ρ and of the function λ seems to be completely irrelevant.

Let (M, g) be gradient Einstein-type manifold of dimension $m \geq 3$. Equation (1-2) in components reads as

$$\alpha R_{ij} + \beta f_{ij} + \mu f_i f_j = (\rho S + \lambda)\delta_{ij}. \tag{4-1}$$

Tracing the previous relation we immediately deduce that

$$(\alpha - m\rho)S + \beta\Delta f + \mu|\nabla f|^2 = m\lambda. \tag{4-2}$$

Definition 4.1. We define the tensor D by its components

$$\begin{aligned} D_{ijk} = \frac{1}{m-2}(f_k R_{ij} - f_j R_{ik}) + \frac{1}{(m-1)(m-2)} f_t (R_{tk}\delta_{ij} - R_{tj}\delta_{ik}) \\ - \frac{S}{(m-1)(m-2)}(f_k\delta_{ij} - f_j\delta_{ik}). \end{aligned} \tag{4-3}$$

Note that D is skew-symmetric in the second and third indices (i.e., $D_{ijk} = -D_{ikj}$) and totally trace-free (i.e., $D_{iik} = D_{iki} = D_{kii} = 0$).

Remark 4.2. We explicitly note that our conventions for the Cotton tensor and for the tensor D differ from those in [Cao and Chen 2013].

Lemma 4.3. *Let (M, g) be a gradient Einstein-type manifold of dimension $m \geq 3$. The tensor D can be written as*

$$\begin{aligned} D_{ijk} = \frac{\beta}{\alpha}\bigg[\frac{1}{m-2}(f_j f_{ik} - f_k f_{ij}) + \frac{1}{(m-1)(m-2)} f_t (f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) \\ - \frac{\Delta f}{(m-1)(m-2)}(f_j\delta_{ik} - f_k\delta_{ij})\bigg]. \end{aligned} \tag{4-4}$$

The proof is just a simple computation, using the definitions of the tensors involved, (4-1) and (4-2).

The following theorem should be compared with [Cao and Chen 2013, Lemma 3.1 and Equation (4.1)], [Cao et al. 2014, Lemma 2.4 and Equation (2.12)] and [Huang and Li 2014, Proposition 2.2]. This result highlights the geometric relevance of D in this general situation and shows that, even in this more general framework, similar structural equations hold.

Theorem 4.4. *Let (M, g) be a gradient Einstein-type manifold with $\beta \neq 0$ of dimension $m \geq 3$. Then the following integrability conditions hold*

$$\alpha C_{ijk} + \beta f_t W_{tijk} = \bigg[\beta - \frac{(m-2)\alpha\mu}{\beta}\bigg] D_{ijk}, \tag{4-5}$$

$$\alpha B_{ij} = \frac{1}{m-2}\bigg\{\bigg[\beta - \frac{(m-2)\alpha\mu}{\beta}\bigg] D_{ijk,k} + \beta\Big(\frac{m-3}{m-2}\Big) f_t C_{jit} - \mu f_t f_k W_{itjk}\bigg\}. \tag{4-6}$$

Proof. We begin with the covariant derivative of (4-1) to get

$$\alpha R_{ij,k} + \beta f_{ij,k} + \mu(f_{ik} f_j + f_i f_{jk}) = (\rho S_k + \lambda_k)\delta_{ij}.$$

Skew-symmetrizing with respect to j and k and using (3-3) we obtain

$$\begin{aligned} \alpha(R_{ij,k} - R_{ik,j}) + \beta f_t R_{tijk} + \mu(f_{ik} f_j - f_{ij} f_k) \\ = \rho(S_k\delta_{ij} - S_j\delta_{ik}) + (\lambda_k\delta_{ij} - \lambda_j\delta_{ik}). \end{aligned} \tag{4-7}$$

To get rid of the two terms on the right-hand side of (4-7) we proceed as follows. First we trace the equation with respect to i and j and we use Schur's identity $S_k = 2R_{tk,t}$ to deduce

$$[\alpha - 2\rho(m-1)]S_k = 2\beta f_t R_{tk} + 2(m-1)\lambda_k - 2\mu(f_t f_{tk} - \Delta f f_k). \tag{4-8}$$

Second, from equations (4-1) and (4-2) we respectively have

$$f_{tk} = \frac{1}{\beta}\big[(\rho S + \lambda)\delta_{tk} - \alpha R_{tk} - \mu f_t f_k\big]bu \tag{4-9}$$

and

$$\Delta f = \frac{1}{\beta}\big[(m\rho - \alpha)S + m\lambda - \mu|\nabla f|^2\big].$$

Inserting the two previous relations into (4-8) and simplifying we deduce the following important equation

$$\begin{aligned}[\alpha - 2\rho(m-1)]S_k = 2\Big(\beta + \frac{\alpha\mu}{\beta}\Big) f_t R_{tk} + 2(m-1)\lambda_k \\ - \frac{2\mu}{\beta}[\alpha - \rho(m-1)]S f_k + \frac{2\mu}{\beta}(m-1)\lambda f_k.\end{aligned} \tag{4-10}$$

From (2-1) and (4-4) we deduce that

$$f_t R_{tijk} = f_t W_{tijk} - D_{ijk} - \frac{1}{m-1}(f_t R_{tk}\delta_{ij} - f_t R_{tj}\delta_{ik}). \tag{4-11}$$

Inserting now (4-11), (2-2), and (4-10) into (4-7) and simplifying we get (4-5).

Taking the divergence of (4-5) we obtain

$$\alpha C_{ijk,k} - \beta f_{tk} W_{itjk} - \beta\Big(\frac{m-3}{m-2}\Big) f_t C_{jit} = \Big[\beta - \frac{(m-2)\alpha\mu}{\beta}\Big] D_{ijk,k}.$$

Using the definition of the Bach tensor (2-4), (4-9), and the symmetries of W we immediately deduce (4-6). ☐

Remark 4.5. Equation (4-10) is the analogue of the fundamental relation $S_k = 2f_t R_{tk}$, valid for every gradient Ricci soliton.

Remark 4.6. In the case $\beta = 0$ (and thus $\alpha \neq 0$), by direct calculations, using (2-2), (4-3), and (4-1), one can show that $D = 0$ and equations (4-5) and (4-6) take the form

$$\alpha C_{ijk} = -\mu(f_j f_{ik} - f_k f_{ij}) - \frac{\mu}{m-1} f_t(f_{tj}\delta_{ik} - f_{tk}\delta_{ij}) + \frac{\mu\Delta f}{m-1}(f_j\delta_{ik} - f_k\delta_{ij}),$$

$$\alpha B_{ij} = \frac{1}{m-2}\big\{\alpha C_{ijk,k} - \mu f_t f_k W_{itjk}\big\}.$$

5. Vanishing of the tensor D

In this section we compute the squared norm of the tensor D in terms of D itself, the Bach tensor B, and the potential function f. Moreover, under the assumption of Theorem 1.2, we prove the vanishing of D. We begin with

Lemma 5.1. *Let (M, g) be a nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$. If $\alpha \neq 0$,*

$$\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 = -\beta(m-2) f_i f_j B_{ij} + \frac{\beta}{\alpha}\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right](f_i f_j D_{ijk})_k, \tag{5-1}$$

while if $\alpha = 0$

$$\left(\frac{m-2}{2}\right)|D|^2 = -(m-2) f_i f_j B_{ij} + (f_i f_j C_{ijk})_k. \tag{5-2}$$

Proof. We observe that, since D is totally trace-free and $D_{ijk} = -D_{ikj}$,

$$|D|^2 = D_{ijk} D_{ijk} = \frac{1}{m-2} D_{ijk}(f_k R_{ij} - f_j R_{ik}) = \frac{1}{m-2}(f_k R_{ij} D_{ijk} + f_j R_{ik} D_{ikj}),$$

so that

$$|D|^2 = \frac{2}{m-2} f_k R_{ij} D_{ijk}.$$

The nondegeneracy condition $\beta - (m-2)\alpha\mu/\beta \neq 0$ implies that, using (4-5) and the definition of the Bach tensor, we can write

$$\begin{aligned}\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 &= f_k R_{ij}(\alpha C_{ijk} + \beta f_t W_{tijk}) \\ &= \alpha f_k R_{ij} C_{ijk} - \beta f_i f_j R_{tk} W_{itjk} \\ &= \alpha f_k R_{ij} C_{ijk} - \beta(m-2) f_i f_j B_{ij} + \beta f_i f_j C_{ijk,k}.\end{aligned}$$

By the symmetries of the Cotton tensor we also have

$$\begin{aligned} f_i f_j C_{ijk,k} &= f_i (f_j C_{ijk})_k - f_i f_{jk} C_{ijk} \\ &= (f_i f_j C_{ijk})_k - f_{ik} f_j C_{ijk} \\ &= (f_i f_j C_{ijk})_k + f_{ij} f_k C_{ijk}.\end{aligned}$$

Therefore we obtain

$$\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 = \alpha f_k R_{ij} C_{ijk} - \beta(m-2) f_i f_j B_{ij} + \beta(f_i f_j C_{ijk})_k + \beta f_{ij} f_k C_{ijk}. \tag{5-3}$$

If $\alpha = 0$, using (4-1) in (5-3) we immediately get

$$\left(\frac{m-2}{2}\right)|D|^2 = -(m-2) f_i f_j B_{ij} + (f_i f_j C_{ijk})_k,$$

that is (5-2).

If $\alpha \neq 0$, using equations (4-1) and (4-5) in (5-3) and simplifying we deduce

$$\begin{aligned}\left(\frac{m-2}{2}\right)\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right]|D|^2 \\ = -\beta(m-2) f_i f_j B_{ij} + \frac{\beta}{\alpha}\left[\beta - \frac{(m-2)\alpha\mu}{\beta}\right](f_i f_j D_{ijk})_k,\end{aligned} \tag{5-4}$$

that is, (5-1). □

Remark 5.2. In the case $\alpha \neq 0$, (5-1) can be obtained in a direct way. One takes the second integrability condition (4-6), multiplies both members by $f_i f_j$ and simplifies, using the symmetries of the tensors involved and (4-5).

Theorem 5.3. *Let (M, g) be a complete nondegenerate gradient Einstein-type manifold of dimension $m \geq 3$. If $B(\nabla f, \cdot) = 0$ and f is proper, then $D = 0$.*

Proof. We define the vector field $Y = Y(\alpha)$ of components

$$Y_k = \begin{cases} (\beta/\alpha) f_i f_j D_{ijk} & \text{if } \alpha \neq 0, \\ f_i f_j C_{ijk} & \text{if } \alpha = 0. \end{cases} \tag{5-5}$$

By the symmetries of D and C we immediately have

$$g(Y, \nabla f) = 0. \tag{5-6}$$

If $B(\nabla f, \cdot) = 0$ and $\alpha \neq 0$, from (5-1) we obtain

$$\left(\frac{m-2}{2}\right)|D|^2 = \frac{\beta}{\alpha}(f_i f_j D_{ijk})_k,$$

while if $\alpha = 0$ from (5-2) we deduce

$$\left(\frac{m-2}{2}\right)|D|^2 = (f_i f_j C_{ijk})_k.$$

In both cases

$$\left(\frac{m-2}{2}\right)|D|^2 = \operatorname{div} Y. \tag{5-7}$$

Now let c be a regular value of f and Ω_c and Σ_c be, respectively, the corresponding sublevel set and level hypersurface, i.e., $\Omega_c = \{x \in M : f(x) \leq c\}$ and $\Sigma_c = \{x \in M : f(x) = c\}$. Integrating (5-7) on Ω_c and using the divergence theorem we get

$$\int_{\Omega_c} \left(\frac{m-2}{2}\right)|D|^2 = \int_{\Omega_c} \operatorname{div} Y = \int_{\Sigma_c} g(Y, \nu),$$

where ν is the unit normal to Σ_c. Since ν is in the direction of ∇f, using (5-6) and letting $c \to +\infty$ we immediately deduce

$$\int_M \left(\frac{m-2}{2}\right)|D|^2 = 0,$$

which implies $D = 0$ on M. □

Remark 5.4. The validity of Theorem 5.3 is based on that of the divergence theorem in this situation. Thus, instead of using properness of f, we can use [Gol′dshtein and Troyanov 1999, Theorem A] to obtain the above conclusion, that is $D \equiv 0$, under the following assumptions; for some $p > 1$, M is p-parabolic and the vector field $Y \in L^q(M)$, where q is the conjugate exponent of p. We note that a sufficient condition for p-parabolicity is

$$\frac{1}{\mathrm{vol}(\partial B_r)^{\frac{1}{p-1}}} \notin L^1(+\infty)$$

(see, e.g., [Troyanov 1999]), and, according to (5-5), $Y \in L^q(M)$ if for some pair of conjugate exponents P, P' we have

$$|\nabla f| \in L^{2Pq}(M) \quad \text{and} \quad |D| \in L^{P'q}(M) \text{ if } \alpha \neq 0$$

or

$$|\nabla f| \in L^{2Pq}(M) \quad \text{and} \quad |C| \in L^{P'q}(M) \text{ if } \alpha = 0.$$

Remark 5.5. A simple computation using the definition of the tensor D gives

$$f_i D_{ijk} = \frac{1}{m-1}(f_t f_k R_{tj} - f_t f_j R_{tk}),$$

and then

$$f_i f_j D_{ijk} = \frac{1}{m-1}(\mathrm{Ric}(\nabla f, \nabla f) f_k - |\nabla f|^2 f_t R_{tk}).$$

This shows that, in the case $\alpha \neq 0$, the vector field Y defined in (5-5) can be expressed in the remarkable form

$$Y = \frac{\beta}{\alpha(m-1)}\big[\mathrm{Ric}(\nabla f, \nabla f)\nabla f - |\nabla f|^2\big(\mathrm{Ric}(\nabla f, \cdot)^\sharp\big)\big],$$

where $\sharp$ denotes the usual musical isomorphism.

Moreover, in the special case of a gradient Ricci soliton (M, g, f, λ), using the fundamental relation $S_k = 2 f_t R_{tk}$, the vector field Y can also be written in the equivalent form

$$Y = \frac{1}{2(m-1)}\big[g(\nabla S, \nabla f)\nabla f - |\nabla f|^2 \nabla S\big].$$

We also observe that

$$g(Y,\nabla f)=0,\, g(Y,\nabla S)=\frac{1}{2(m-1)}\big[g(\nabla S,\nabla f)^2-|\nabla S|^2|\nabla f|^2\big]\le 0,$$

and that

$$|Y|^2=\frac{1}{4(m-1)^2}|\nabla f|^2\big[|\nabla S|^2|\nabla f|^2-g(\nabla S,\nabla f)^2\big]=-\frac{1}{2(m-1)}|\nabla f|^2 g(Y,\nabla S).$$

Remark 5.6. In case $\beta=0$ and $\mu\neq 0$, using Remark 4.6 and arguing as in Lemma 5.1, one can obtain the following identity

$$\frac{\alpha}{2\mu}|C|^2=(m-2)f_if_jB_{ij}-(f_if_jC_{ijk})_k.$$

Then, following the proof of Theorem 5.3, we obtain

Proposition 5.7. *Let (M,g) be a complete nondegenerate gradient Einstein-type manifold of dimension $m\ge 3$ and with $\beta=0$. If $B(\nabla f,\cdot)=0$ and f is proper, then $C=0$.*

6. D and the geometry of the level sets of f

In this section we relate the tensor D to the geometry of the regular level sets of the potential function f. Our first result highlights, in the case $\alpha\neq 0$, the link between the squared norm of the tensor D and the second fundamental form of the level sets of f. This should be compared with [Cao and Chen 2013, Proposition 3.1] and [Cao and Chen 2012, Lemma 4.1]. For the case $\alpha=0$ we refer to [Huang and Li 2014, Proposition 2.3].

From now on, we extend our index convention assuming $1\le i,j,k,\ldots\le m$ and $1\le a,b,c,\ldots\le m-1$.

Proposition 6.1. *Let (M,g) be a complete, m-dimensional, gradient, Einstein-type manifold with $\alpha,\beta\neq 0$ and $m\ge 3$. Let c be a regular value of f and let $\Sigma_c=\{x\in M\,|\,f(x)=c\}$ be the corresponding level hypersurface. For $p\in\Sigma_c$ choose an orthonormal frame such that $\{e_1,\ldots,e_{m-1}\}$ are tangent to Σ_c and $e_m=\nabla f/|\nabla f|$ (i.e., $\{e_1,\ldots,e_{m-1},e_m\}$ is a local first order frame along f). Then, in p, the squared norm of the tensor D can be written as*

$$|D|^2=\Big(\frac{\beta}{\alpha}\Big)^2\frac{2|\nabla f|^4}{(m-2)^2}|h_{ab}-h\delta_{ab}|^2+\frac{2|\nabla f|^2}{(m-1)(m-2)}R_{am}R_{am}, \tag{6-1}$$

where h_{ab} are the coefficients of the second fundamental tensor and h is the mean curvature of Σ_c.

Remark 6.2. Note that $|h_{ab}-h\delta_{ab}|^2$ is the squared norm of the traceless second fundamental tensor Φ of components $\Phi_{ab}=h_{ab}-h\delta_{ab}$.

Proof. First of all, we observe that in the chosen frame we have

$$df = f_a\theta^a + f_m\theta^m = |\nabla f|\theta^m,$$

since $f_a = 0$, $a = 1, \dots, m-1$.

The second fundamental tensor II of the immersion $\Sigma_c \hookrightarrow M$ is

$$II = h_{ab}\theta^b \otimes \theta^a \otimes \nu,$$

where the coefficients $h_{ab} = h_{ba}$ are defined as

$$\nabla e_m = \nabla\nu = \theta^a_m \otimes e_a = -\theta^m_a \otimes e_a = -h_{ab}\theta^b \otimes e_a$$

(see also [Alías et al. 2016]), so that

$$h_{ab} = g(II(e_a, e_b), \nu) = -g(\nabla_{e_a}\nu, e_b) = -(\nabla\nu)^\flat(e_a, e_b).$$

In the present setting we have

$$\nabla\nu = \frac{1}{|\nabla f|}\nabla(\nabla f) + \nabla\Big(\frac{1}{|\nabla f|}\Big) \otimes \nabla f$$

and

$$(\nabla\nu)^\flat = \frac{1}{|\nabla f|}\,\mathrm{Hess}(f) + d\Big(\frac{1}{|\nabla f|}\Big) \otimes df.$$

Thus, using (4-1), we deduce

$$h_{ab} = -\frac{1}{|\nabla f|} f_{ab} = \frac{1}{\beta|\nabla f|}[\alpha R_{ab} - (\rho S + \lambda)\delta_{ab}]. \tag{6-2}$$

The mean curvature h is defined as $h = h_{aa}/(m-1)$. Tracing (6-2) we get

$$h = \frac{1}{\beta|\nabla f|}\Big[\Big(\frac{\alpha}{m-1} - \rho\Big)S - \frac{\alpha}{m-1}R_{mm} - \lambda\Big]. \tag{6-3}$$

Now we compute the squared norm of the traceless second fundamental tensor Φ.

$$\begin{aligned}
|h_{ab} - h\delta_{ab}|^2 &= |h_{ab}|^2 - 2hh_{aa} + (m-1)h^2 = |h_{ab}|^2 - (m-1)h^2 \\
&= \frac{1}{\beta^2|\nabla f|^2}\Big\{\big[\alpha R_{ab} - (\rho S + \lambda)\delta_{ab}\big]^2 \\
&\qquad - (m-1)\Big[\Big(\frac{\alpha}{m-1} - \rho\Big)S - \frac{\alpha}{m-1}R_{mm} - \lambda\Big]^2\Big\} \\
&= \frac{\alpha^2}{\beta^2|\nabla f|^2}\Big\{|\mathrm{Ric}|^2 - 2R_{am}R_{am} - (R_{mm})^2 \\
&\qquad - \frac{1}{m-1}\big[S^2 - 2SR_{mm} + (R_{mm})^2\big]\Big\} \\
&= \frac{\alpha^2}{\beta^2|\nabla f|^2}\Big[|\mathrm{Ric}|^2 - 2R_{am}R_{am} - \frac{m}{m-1}(R_{mm})^2 \\
&\qquad - \frac{1}{m-1}S^2 + \frac{2}{m-1}SR_{mm}\Big].
\end{aligned} \tag{6-4}$$

On the other hand, from the definition of D we have

$$
\begin{aligned}
(6\text{-}5)\quad |D|^2 = {} & \frac{(f_k R_{ij} - f_j R_{ik})^2}{(m-2)^2} \\
& + \frac{(f_t R_{tk}\delta_{ij} - f_t R_{tj}\delta_{ik})^2}{(m-1)^2(m-2)^2} + \frac{S^2}{(m-1)^2(m-2)^2}(f_k\delta_{ij} - f_j\delta_{ik})^2 \\
& + \frac{2}{(m-1)(m-2)^2}(f_k R_{ij} - f_j R_{ik})(f_t R_{tk}\delta_{ij} - f_t R_{tj}) \\
& - \frac{2S}{(m-1)(m-2)^2}(f_k R_{ij} - f_j R_{ik})(f_k\delta_{ij} - f_j\delta_{ik}) \\
& - \frac{2S}{(m-1)^2(m-2)^2}(f_t R_{tk}\delta_{ij} - f_t R_{tj}\delta_{ik})(f_k\delta_{ij} - f_j\delta_{ik}) \\
= {} & \frac{2|\nabla f|^2}{(m-2)^2}\big(|\mathrm{Ric}|^2 - R_{am}R_{am} - R_{mm}R_{mm}\big) \\
& + \frac{2|\nabla f|^2}{(m-1)(m-2)^2}(R_{am}R_{am} + R_{mm}R_{mm}) \\
& + \frac{2S^2}{(m-1)(m-2)^2}|\nabla f|^2 \\
& + \frac{4|\nabla f|^2}{(m-1)(m-2)^2}(SR_{mm} - (R_{mm})^2 - R_{am}R_{am}) \\
& - \frac{4S|\nabla f|^2}{(m-1)(m-2)^2}(S - R_{mm}) - \frac{4S|\nabla f|^2}{(m-1)(m-2)^2}R_{mm}.
\end{aligned}
$$

Simplifying, rearranging, and comparing (6-4) and (6-5) we arrive at

$$\frac{(m-2)^2}{2|\nabla f|^2}|D|^2 = \left(\frac{\beta}{\alpha}\right)^2 |\nabla f|^2 |h_{ab} - h\delta_{ab}|^2 + \left(\frac{m-2}{m-1}\right) R_{am}R_{am},$$

which easily implies (6-1). □

Proposition 6.1 is one of the key ingredients in the proof of the following theorem, which generalizes [Cao and Chen 2013, Proposition 3.2] (compare also with [Huang and Li 2014, Proposition 2.4]). Our proof is similar to those in [Cao and Chen 2013; Huang and Li 2014], but the presence of μ and the nonconstancy of λ require extra care, in particular in showing that S is constant on Σ_c.

Theorem 6.3. *Let (M, g) be a complete, m-dimensional $(m \geq 3)$, gradient, Einstein-type manifold with $\alpha, \beta \neq 0$ and tensor $D \equiv 0$. Let c be a regular value of f and let $\Sigma_c = \{x \in M \,|\, f(x) = c\}$ be the corresponding level hypersurface. Choose any local orthonormal frame such that $\{e_1, \ldots, e_{m-1}\}$ are tangent to Σ_c and $e_m = \nabla f/|\nabla f|$ (i.e., $\{e_1, \ldots, e_{m-1}, e_m\}$ is a first order frame along f). Then*

(1) *$|\nabla f|^2$ is constant on Σ_c;*

(2) *$R_{am} = R_{ma} = 0$ for every $a = 1, \ldots, m-1$ and e_m is an eigenvector of* Ric;

(3) *Σ_c is totally umbilical;*

(4) *the mean curvature h is constant on Σ_c*;

(5) *the scalar curvature S and λ are constant on Σ_c*;

(6) *Σ_c is Einstein with respect to the induced metric*;

(7) *on Σ_c the (components of the) Ricci tensor of M can be written as $R_{ab} = (S-\Lambda_1)/(m-1)\delta_{ab}$, where $\Lambda_1 \in \mathbb{R}$ is an eigenvalue of multiplicity* 1 *or m (and in this latter case $S = m\Lambda_1$); in either case e_m is an eigenvector associated to Λ_1.*

Proof. If $D = 0$, from Proposition 6.1 we immediately deduce that

$$h_{ab} - h\delta_{ab} = 0, \tag{6-6}$$

that is, property (3), and

$$R_{am} = 0, \qquad (a = 1, \dots, m-1).$$

From (6-6) a simple computation using (6-2) and (6-3) shows that

$$R_{ab} = \frac{S - R_{mm}}{m-1}\delta_{ab}, \tag{6-7}$$

which also implies

$$\mathrm{Ric}(\nu, \nu) = \frac{R_{ij} f_i f_j}{|\nabla f|^2} = R_{mm} = R_{mm}|\nu|^2.$$

This complete the proof of (2). To prove (1) we take the covariant derivative of $\beta|\nabla f|^2$ and use (4-1).

$$\begin{aligned} \beta(|\nabla f|^2)_k &= 2\beta f_i f_{ik} \\ &= 2\big[(\rho S + \lambda - \mu|\nabla f|^2) f_k - \alpha f_t R_{tk}\big] \\ &= 2\big[(\rho S + \lambda - \mu|\nabla f|^2) f_k - \alpha f_c R_{ck} - \alpha|\nabla f| R_{mk}\big]. \end{aligned}$$

Evaluating the previous relation at $k = a$ and using property (2) we immediately get

$$\big(|\nabla f|^2\big)_a = 0,$$

that is (1). To prove (4) we start from the Codazzi equations, that in our setting read

$$-R_{mabc} = h_{ab,c} - h_{ac,b}.$$

Tracing with respect to a and c we get

$$-R_{maba} = -R_{mkbk} + R_{mmbm} = h_{ab,a} - h_{aa,b},$$

that is, using (2),

$$0 = -R_{mb} = h_{ab,a} - h_{aa,b}. \tag{6-8}$$

On the other hand, from (3) we have

$$h_{ab,a} = h_b \quad \text{and} \quad h_{aa,b} = (m-1)h_b,$$

so that (6-8) immediately implies

$$0 = (m-2)h_b, \quad \text{for} \quad b = 1, \dots, m-1,$$

that is (4). To show the validity of (5) we first observe that, evaluating (4-10) at $k = a$ and using (2), we deduce

$$[\alpha - 2\rho(m-1)]S_a - 2(m-1)\lambda_a = 0,$$

which implies

$$[\alpha - 2\rho(m-1)]S - 2(m-1)\lambda \quad \text{is constant on } \Sigma_c. \tag{6-9}$$

From (6-3), the constancy of h and of $|\nabla f|$ on Σ_c also give that

$$[\alpha - \rho(m-1)]S - \alpha R_{mm} - (m-1)\lambda \quad \text{is constant on } \Sigma_c. \tag{6-10}$$

Combining (6-9) and (6-10) we arrive at

$$S - 2R_{mm} \quad \text{is constant on } \Sigma_c. \tag{6-11}$$

Now we evaluate (4-10) at $k = m$, we use (2) and rearrange to deduce

$$\begin{aligned}
&[\alpha - 2\rho(m-1)]S_m \\
&\quad = 2\Big(\beta + \frac{\alpha\mu}{\beta}\Big)|\nabla f| R_{mm} + 2(m-1)\lambda_m - \frac{2\mu|\nabla f|}{\beta}\{[\alpha - \rho(m-1)]S - (m-1)\lambda\} \\
&\quad = 2\beta|\nabla f| R_{mm} + 2(m-1)\lambda_m - \frac{2\mu|\nabla f|}{\beta}\{[\alpha - \rho(m-1)]S - \alpha R_{mm} - (m-1)\lambda\}.
\end{aligned}$$

Since by (1) and (6-10) the quantity $(2\mu|\nabla f|)/\beta\{[\alpha - \rho(m-1)]S - \alpha R_{mm} - (m-1)\lambda\}$ is constant on Σ_c we infer that

$$[\alpha - 2\rho(m-1)]S_m - 2\beta|\nabla f| R_{mm} - 2(m-1)\lambda_m \quad \text{is constant on } \Sigma_c. \tag{6-12}$$

Now we take the covariant derivative of (6-12) and evaluate at $k = a$ to obtain

$$[\alpha - 2\rho(m-1)]S_{ma} - 2\beta|\nabla f| R_{mm,a} - 2(m-1)\lambda_{ma} = 0 \quad \text{on } \Sigma_c. \tag{6-13}$$

But $S_{ma} = S_{am}$ and $\lambda_{ma} = \lambda_{am}$, thus (6-13) can be written as

$$\{[\alpha - 2\rho(m-1)]S - 2(m-1)\lambda\}_{am} = 2\beta|\nabla f| R_{mm,a} \quad \text{on } \Sigma_c,$$

which implies, by (6-9), that

$$R_{mm} \quad \text{is constant on } \Sigma_c.$$

The previous relation, (6-11), and (6-9) show that S and λ are constant on Σ_c, that is (5). To prove (6) we start from the Gauss equations

$$^{\Sigma_c}R_{abcd} = R_{abcd} + h_{ac}h_{bd} - h_{ad}h_{bc},$$

which, by property (3), can be rewritten as

$$^{\Sigma_c}R_{abcd} = R_{abcd} + h^2(\delta_{ac}\delta_{bd} - \delta_{ad}\delta_{bc}). \tag{6-14}$$

Tracing (6-14) with respect to b and d gives

$$^{\Sigma_c}R_{ac} = R_{ac} - R_{amcm} + (m-2)h^2\delta_{ac}. \tag{6-15}$$

Tracing again we deduce that

$$^{\Sigma_c}S = S - 2R_{mm} + (m-1)(m-2)h^2 \quad \text{is constant on } \Sigma_c. \tag{6-16}$$

Now a simple computation using decomposition (2-1) of the Riemann tensor, (6-7) and the fact that $W_{amcm} = 0$ (see Proposition 6.4) shows that

$$R_{amcm} = \frac{1}{m-1}R_{mm}\delta_{ac}. \tag{6-17}$$

Next, inserting (6-7) and (6-17) into (6-15), we get

$$^{\Sigma_c}R_{ac} = \left[\frac{S-2R_{mm}}{m-1} + (m-2)h^2\right]\delta_{ac},$$

which shows the validity of (6). Now (7) is an easy consequence of the other properties. □

The next two results are analogues of, respectively, Lemmas 4.2 and 4.3 of [Cao and Chen 2013].

Proposition 6.4. *Let* (M, g) *be a complete*, *noncompact*, m*-dimensional* $(m \geq 3)$, *nondegenerate*, *Einstein-type manifold with* $\alpha \neq 0$. *If* $D = 0$ *then* $C = 0$, *unless* f *is locally constant.*

Proof. First of all, by analyticity, it is sufficient to prove the result where $\{\nabla f \neq 0\}$. We choose a local first order frame along f (so that $f_a = 0$, $a = 1, \dots, m-1$ and $f_m = |\nabla f|$). The vanishing of D implies, by the first integrability condition (4-5), that

$$\alpha C_{ijk} + \beta f_t W_{tijk} = 0,$$

which implies, since $\alpha \neq 0$,

$$C_{ijk} = -\frac{\beta}{\alpha} f_t W_{tijk} \tag{6-18}$$

and consequently

$$f_i C_{ijk} = f_m C_{mjk} = |\nabla f| C_{mjk} = 0, \quad (j, k = 1, \dots, m).$$

Thus

$$C_{mjk} = 0$$

at all points where $|\nabla f| \neq 0$. Using (3) and (4) of Theorem 6.3 we have

$$h_{ab,c} = 0,$$

and from the Codazzi equations we get

$$-R_{mabc} = h_{ab,c} - h_{ac,b} = 0.$$

Since $R_{am} = 0$ by (2) of Theorem 6.3, from the decomposition (2-1) we easily deduce

$$W_{ambc} = 0,$$

which implies by (6-18) that

$$C_{abc} = 0.$$

By the symmetries of C, to conclude it only remains to show that $C_{abm} = 0 = C_{amb}$. First we observe that $R_{am} = 0$ implies, by the definition of covariant derivative,

$$\begin{aligned} 0 &= dR_{am} \\ &= R_{km}\theta^k_a + R_{ak}\theta^k_m + R_{am,k}\theta^k \\ &= R_{bm}\theta^b_a + R_{mm}\theta^m_a + R_{ab}\theta^b_m + R_{am}\theta^m_m + R_{am,k}\theta^k \\ &= R_{mm}\theta^m_a + R_{ab}\theta^b_m + R_{am,k}\theta^k, \end{aligned}$$

so that, using (6-7),

$$\begin{aligned} R_{am,k}\theta^k = R_{am,b}\theta^b + R_{am,m}\theta^m &= R_{ab}\theta^m_b - R_{mm}\theta^m_a \\ &= \Big(\frac{S - R_{mm}}{m-1}\delta_{ab}\Big)\theta^m_b - R_{mm}\theta^m_a \\ &= \Big(\frac{S - mR_{mm}}{m-1}\Big)\theta^m_a. \end{aligned} \tag{6-19}$$

Now we want to show that $R_{am,m} = 0$. To see that we first evaluate (4-1) for $i = a$ and $j = m$, obtaining $f_{am} = 0$; then we take the covariant derivative of the same equation

$$\alpha R_{ij,k} + \beta f_{ijk} + \mu(f_{ik}f_j + f_i f_{jk}) = (\rho S_k + \lambda_k)\delta_{ij},$$

which for $i = k = m$ and $j = a$ gives (using $f_{am} = 0$)

$$\alpha R_{am,m} = -\beta f_{mam};$$

but

$$f_{mam} = f_{mma} + f_i R_{imam} = f_{mma},$$

while (4-2) and Theorem 6.3 tell us that the (globally defined) quantity Δf is constant on Σ_c, so that

$$(\Delta f)_a = 0.$$

On the other hand, from (4-1) and (6-7) we deduce

$$\beta f_{ab} = -\frac{1}{m-1}\{[\alpha - \rho(m-1)]S - \alpha R_{mm} - (m-1)\lambda\}\delta_{ab} \tag{6-20}$$

which implies, by tracing, that

$$\beta(\Delta f - f_{mm}) \quad \text{is constant on } \Sigma_c.$$

In particular

$$f_{mam} = f_{mma} = (\Delta f)_a = 0,$$

and thus

$$R_{am,m} = 0.$$

Getting back to (6-19) we now have

$$R_{am,b}\theta^b = \Big(\frac{S - mR_{mm}}{m-1}\Big)\theta_a^m,$$

and thus

$$\begin{aligned} R_{am,b} &= \Big(\frac{S - mR_{mm}}{m-1}\Big)\theta_a^m(e_b) \\ &= \frac{1}{|\nabla f|}\Big(\frac{mR_{mm} - S}{m-1}\Big) f_{ab}. \end{aligned} \tag{6-21}$$

Schur's identity implies

$$S_m = 2R_{im,i} = 2R_{am,a} + 2R_{mm,m}. \tag{6-22}$$

From the definition of C we have, using (6-7) and (6-21),

$$\begin{aligned} C_{abm} &= R_{ab,m} - R_{am,b} - \frac{1}{2(m-1)}S_m\delta_{ab} \\ &= \frac{S_m - R_{mm,m}}{m-1}\delta_{ab} + \frac{1}{|\nabla f|}\Big(\frac{s - mR_{mm}}{m-1}\Big) f_{ab} - \frac{1}{2(m-1)}S_m\delta_{ab} \\ &= \frac{1}{2(m-1)}S_m\delta_{ab} - \frac{1}{m-1}R_{mm,m}\delta_{ab} + \frac{1}{|\nabla f|}\Big(\frac{S - mR_{mm}}{m-1}\Big) f_{ab}. \end{aligned} \tag{6-23}$$

Using (6-22), (6-21), and (6-20) in (6-23) we arrive at

$$\begin{aligned} C_{abm} &= \frac{1}{m-1}R_{cm,c}\delta_{ab} + \frac{1}{|\nabla f|}\Big(\frac{S - mR_{mm}}{m-1}\Big) f_{ab} \\ &= -\frac{1}{m-1}\frac{1}{|\nabla f|}(S - mR_{mm,m}) f_{ab} + \frac{1}{|\nabla f|}\Big(\frac{S - mR_{mm}}{m-1}\Big) f_{ab} \\ &= 0, \end{aligned}$$

concluding the proof. □

In dimension four, we can prove the following:

Corollary 6.5. *Let (M^4, g) be a complete, noncompact, nondegenerate, Einstein-type manifold of dimension four with $\alpha \neq 0$. If $D = 0$ then $W = 0$, unless f is locally constant.*

Proof. From Proposition 6.4, we know that $C_{ijk} = 0$. Hence, from (4-5), we deduce $f_t W_{tijk} = 0$ for any $i, j, k = 1, \dots, 4$. For any $p \in M^4$ such that $\nabla f(p) \neq 0$, we choose an orthonormal frame $\{e_1, \dots, e_4\}$ such that $e_4 = \nabla f/|\nabla f|$, thus we have

$$W_{4ijk}(p) = 0, \quad \text{for } i, j, k = 1, \dots 4.$$

It remains to show that $W_{abcd}(p) = 0$ for any $a, b, c, d = 1, 2, 3$. This follows from the symmetries and the traceless property of the Weyl tensor (see, for instance, [Cao and Chen 2013, Lemma 4.3]). ☐

7. Proof of the main theorems and some geometric applications

In this last section we first prove Theorem 1.2 and Corollary 1.3. Then, we give some geometric applications in the special cases of gradient Ricci solitons, ρ-Einstein solitons, and Ricci almost solitons. We begin with

Proof of Theorem 1.2 and Corollary 1.3. From Theorem 5.3 we know that the tensor D has to vanish on M. Let Σ be a regular level set of the function $f : M^m \to \mathbb{R}$, i.e., $|\nabla f| \neq 0$ on Σ, which exists by Sard's Theorem and the fact that f is nontrivial. By Theorem 6.3 (1) we have that $|\nabla f|$ must be constant on Σ. Thus, in a neighborhood U of Σ which does not contain any critical point of f, the potential function f only depends on the signed distance r to the hypersurface Σ. Hence, by a suitable change of variable, we can express the metric g_{ij} as

$$ds^2 = dr^2 + g_{ab}(r, \theta) d\theta^a \otimes d\theta^b \qquad (r_* < r < r^*)$$

for some maximal $r_* \in [-\infty, 0)$ and $r^* \in (0, \infty]$, where $(\theta^2, \dots, \theta^m)$ is any local coordinates system on the level surface Σ. Moreover, by Theorem 6.3 (3)-(4), we have

$$\frac{\partial}{\partial r} g_{ab} = -2h_{ab} = \phi(r) g_{ab},$$

where $\phi(r) = -2h(r)$. Thus, it follows easily that

$$g_{ab}(r, \theta) = e^{\Phi(r)} g_{ab}(0, \theta), \quad \text{where} \quad \Phi(r) = \int_0^r \phi(r)\, dr.$$

This proves that on U the metric g takes the form of a warped product metric

$$ds^2 = dr^2 + w(r)^2 g^E, \quad r \in (r_*, r^*),$$

where w is some positive smooth function on U, and $g^E = g^\Sigma$ is the metric defined on the level surface Σ, which is Einstein, by Theorem 6.3 (6). This concludes the proof of Theorem 1.2.

The proof of Corollary 1.3 follows from the previous considerations combined with Corollary 6.5. ☐

Next we show that the properness assumption on the potential function f in Theorem 1.2 is automatically satisfied by some classes of Einstein-type manifolds.

First of all, let (M, g) be a complete, noncompact, *gradient Ricci soliton* with potential function f. Then, it is well known that f is always proper, provided that the soliton is either shrinking [Cao and Zhou 2010, Theorem 1.1], or steady with positive Ricci curvature and scalar curvature attaining its maximum at some point [Cao and Chen 2012, Proposition 2.3], or expanding with nonnegative Ricci curvature [Cao et al. 2014, Lemma 5.5]. Hence, in these cases, Theorem 1.2 provides a local version of the classification results obtained in [Cao and Chen 2013; Cao et al. 2014].

Secondly, if (M, g) is a complete, noncompact, *gradient shrinking ρ-Einstein soliton* with $\rho > 0$ and bounded scalar curvature, then it follows by [Catino et al. 2015, Lemma 3.2] that the potential function f is proper. Hence, Theorem 1.2 implies the following

Theorem 7.1. *Let (M, g) be a complete, noncompact, gradient shrinking ρ-Einstein soliton of dimension $m \geq 3$ with bounded scalar curvature and $\rho > 0$. If $B(\nabla f, \cdot) = 0$, then around any regular point of f the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

Finally, we want to show the following result concerning *gradient Ricci almost solitons* which are "strongly" shrinking.

Theorem 7.2. *Let (M, g) be a complete, noncompact, gradient Ricci almost soliton of dimension $m \geq 3$ with bounded Ricci curvature and with $\lambda \geq \underline{\lambda} > 0$, for some $\underline{\lambda}$. If $B(\nabla f, \cdot) = 0$, then around any regular point of f the manifold (M, g) is locally a warped product with $(m-1)$-dimensional Einstein fibers.*

Proof. By Theorem 1.2 it is sufficient to show that under these assumptions the potential function is proper. To do this we will apply a second variation argument as in [Cao and Zhou 2010, Theorem 1.1]. Let $r(x) = \operatorname{dist}(x, o)$, for some fixed origin $o \in M$. We will show that, for $r(x) \gg 1$,

$$f(x) \geq \tfrac{1}{2}\underline{\lambda}\big(r(x) - c\big)^2,$$

for some positive constant $c > 0$ depending only on m and on the geometry of g on the unit ball $B_o(1)$. Let $\gamma(s)$, $0 \leq s \leq s_0$ for some $s_0 > 0$, be any minimizing unit speed geodesic starting from $o = \gamma(0)$ and let $\dot\gamma(s)$ be the unit tangent vector of γ. Then by the second variation of the arc length, we have

$$\int_0^{s_0} \phi^2(s) \operatorname{Ric}(\dot\gamma, \dot\gamma)\, ds \leq (m-1) \int_0^{s_0} |\dot\phi(s)|^2\, ds,$$

for every nonnegative function $\phi : [0, s_0] \to \mathbb{R}$. We choose $\phi(s) = s$ on $[0, 1]$, $\phi(s) = 1$ on $[1, s_0 - 1]$, and $\phi(s) = s_0 - s$ on $[s_0 - 1, s_0]$. Then, since the solitons has bounded Ricci curvature, one has

$$\int_0^{s_0} \mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \le 2(m-1) + \max_{B_1(o)} |\,\mathrm{Ric}\,| + \max_{B_1(\gamma(s_0))} |\,\mathrm{Ric}\,| \le C,$$

for some positive constant C independent of s_0. On the other hand, from the soliton equation, we have

$$\nabla_{\dot\gamma}\nabla_{\dot\gamma} f = \lambda - \mathrm{Ric}(\dot\gamma, \dot\gamma).$$

Integrating along γ, we get

$$\dot f\big(\gamma(s_0)\big) - \dot f\big(\gamma(0)\big) = \int_0^{s_0} \lambda\, ds - \int_0^{s_0} \mathrm{Ric}(\dot\gamma, \dot\gamma)\, ds \ge \underline{\lambda} s_0 - C.$$

Integrating again, we obtain the desired estimate

$$f(\gamma(s_0)) \ge \tfrac{1}{2}\underline{\lambda}(s_0 - c)^2,$$

for some constant c. This concludes the proof of the theorem. □

Remark 7.3. From the above proof, if $\underline{\lambda} = \underline{\lambda}(r)$ is such that $1/\underline{\lambda}(r) = o(1/r^2)$ as $r \to +\infty$ we have $f(r) \to +\infty$ as $r \to +\infty$. This suffices to prove Theorem 7.2.

To conclude, we note that Ricci almost solitons which are warped products were constructed in [Pigola et al. 2011, Remark 2.6].

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ON THE FOURIER–JACOBI MODEL FOR SOME ENDOSCOPIC ARTHUR PACKET OF $U(3) \times U(3)$: THE NONGENERIC CASE

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For a generic L-parameter of $U(n) \times U(n)$, it is conjectured that there is a unique representation in their associated relevant Vogan L-packet which produces the unique Fourier–Jacobi model. We investigated this conjecture for some nongeneric L-parameters of $U(3) \times U(3)$ and discovered that it is true for some nongeneric L-parameters and false for some nongeneric L-parameters. In the case when it holds, we specified such representation under the local Langlands correspondence for unitary groups.

1. Introduction

The local Gan–Gross–Prasad conjecture deals with certain restriction problems between p-adic groups. In this paper, we shall investigate it for some nongeneric case not treated before.

Let E/F be a quadratic extension of number fields and $G = U(3)$ be the quasisplit unitary group of rank 3 relative to E/F. Then $H = U(2) \times U(1)$ is the unique elliptic endoscopic group for G. Rogawski [1990] has defined a certain enlarged class of L-packets, or A-packets, of G using endoscopic transfer of one-dimensional characters of H to G. In more detail, let $\varrho = \otimes_v \varrho_v$ be a one-dimensional automorphic character of H. The A-packet $\Pi(\varrho) \simeq \otimes \Pi(\varrho_v)$ is the transfer of ϱ with respect to functoriality for an embedding of L-groups $\xi : {}^L H \to {}^L G$. Then for all places v of F, the packet $\Pi(\varrho_v)$ contains a certain nontempered representation $\pi^n(\varrho_v)$ and it contains an additional supercuspidal representation $\pi^s(\varrho_v)$ precisely when v remains prime in E. Gelbart and Rogawski [1991] showed that the representations in this A-packet arise in the Weil representation of G. Our goal is to study the branching rule of the representations in this A-packet.

For the branching problem, there is a fascinating conjecture, the so-called Gan–Gross–Prasad (GGP) conjecture, which was first proposed by Gross and Prasad [1992] for orthogonal groups and later they, together with Gan, extended it

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to all classical groups in [Gan et al. 2012]. Since it concerns our main theorem, we shall give a brief review on the GGP conjecture, especially for unitary groups.

Let E/F be a quadratic extension of nonarchimedean local fields of characteristic zero. Let V_{n+1} be a hermitian space of dimension $n+1$ over E and W_n a skew-hermitian space of dimension n over E. Let $V_n \subset V_{n+1}$ be a nondegenerate subspace of codimension 1, so that if we set

$$G_n = U(V_n) \times U(V_{n+1}) \quad \text{or} \quad U(W_n) \times U(W_n)$$

and

$$H_n = U(V_n) \quad \text{or} \quad U(W_n),$$

then we have a diagonal embedding

$$\Delta : H_n \hookrightarrow G_n.$$

Let π be an irreducible smooth representation of G_n. In the hermitian case, one is interested in computing

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \mathbb{C}).$$

We shall call this the *Bessel* case (B) of the GGP conjecture. For the GGP conjecture in the skew-hermitian case, we need to introduce a certain Weil representation ω_{ψ,χ,W_n} of H_n, where ψ is a nontrivial additive character of F and χ is a character of $E^\times$ whose restriction to $F^\times$ is the quadratic character $\omega_{E/F}$ associated to E/F by local class field theory. (For the exact definition of ω_{ψ,χ,W_n}, please see page 77.) In this case, one is interested in computing

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \omega_{\psi,\chi,W_n}).$$

We shall call this the *Fourier–Jacobi* case (FJ) of the GGP conjecture. To treat both cases using one notation, we shall let ν equal $\mathbb{C}$ or ω_{ψ,χ,W_n} in the respective cases.

By the results of [Aizenbud et al. 2010; Sun 2012], it is known that

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n}(\pi, \nu) \leq 1,$$

and so the next step is to specify irreducible smooth representations π such that

$$\mathrm{Hom}_{\Delta H_n}(\pi, \nu) = 1.$$

(A nonzero element of $\mathrm{Hom}_{\Delta H_n}(\pi, \nu)$ is called a *Bessel* (*Fourier–Jacobi*) model of π in the hermitian (skew-hermitian) case.)

Gan, Gross and Prasad [2012] brought this problem into a more general setting using the notion of relevant pure inner forms of G_n and Vogan L-packets. A pure inner form of G_n is a group of the form

$$G_n' = U(V_n') \times U(V_{n+1}') \quad \text{or} \quad U(W_n') \times U(W_n''),$$

where $V_n' \subset V_{n+1}'$ are n and $n+1$ dimensional hermitian spaces over E, and W_n', W_n'' are n-dimensional skew-hermitian spaces over E.

Furthermore, if

$$V_{n+1}'/V_n' \cong V_{n+1}/V_n \quad \text{or} \quad W_n' = W_n'',$$

we say that G_n' is a relevant pure inner form.
(Indeed, there are four pure inner forms of G_n and among them, only two are relevant.)

If G_n' is relevant, we set

$$H_n' = U(V_n') \quad \text{or} \quad U(W_n'),$$

so that we have a diagonal embedding

$$\Delta : H_n' \hookrightarrow G_n'.$$

Now suppose that ϕ is an L-parameter for the group G_n. Then the (relevant) Vogan L-packet Π_ϕ associated to ϕ consists of certain irreducible smooth representations of G_n and its (relevant) pure inner forms G_n' whose L-parameter is ϕ. We denote the relevant Vogan L-packet of ϕ by Π_ϕ^R.

With these notions, we can loosely state the result of Beuzart-Plessis [2014; 2012; 2015] for the *Bessel* case and Gan and Ichino [2016] for the *Fourier–Jacobi* case as follows:

Theorem 1.1. *For a tempered L-parameter ϕ of G_n, the following hold*:

(i) $\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n'}(\pi', \nu) = 1$.

(ii) *Using the local Langlands correspondence for unitary groups, we can pinpoint the unique $\pi' \in \Pi_\phi^R$ such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_n'}(\pi', \nu) = 1.$$

To emphasize its dependence on the number n, we denote the *Fourier–Jacobi* and *Bessel* cases of Theorem 1.1 as $(\mathrm{FJ})_n$ and $(\mathrm{B})_n$ respectively, and later we shall elaborate more on this notation. The GGP conjecture predicts that this theorem also holds for a generic L-parameter ϕ of G_n.

Our main theorem is to investigate $(\mathrm{FJ})_3$ for some L-parameter of G_3 involving a nongeneric L-parameter of $U(W_3)$. More precisely, we have:

Main Theorem. *For an irreducible smooth representation π_2 of $U(W_3)$, let $\pi = \pi^n(\varrho) \otimes \pi_2$ as a representation of G_3. Then*

(i) $\mathrm{Hom}_{\Delta H_3}(\pi, \omega_{\psi,\chi,W_3}) = 0$ *if π_2 is not a theta lift from $U(V_2)$.*

(ii) *Assume that* π_2 *is the theta lift from* $U(V_1')$ *and let* $\phi = \phi^n \otimes \phi_2$ *be the* L*-parameter of* π. *Then*

$$\sum_{\pi' \in \Pi_\phi^R} \dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_3'}(\pi', \omega_{\psi,\chi,W_3}) = 1.$$

(iii) *Using the local Langlands correspondence for unitary groups, we can explicitly describe the representation* $\pi' \in \Pi_\phi^R$ *appearing in (ii) such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta H_3'}(\pi', \omega_{\psi,\chi,W_3}) = 1.$$

Remark 1.2. As we shall see in Theorem 3.2, the L-parameter of $\pi^n(\varrho_v)$ is not only nontempered but also nongeneric. Thus if we choose the L-parameters of ϕ_2 in ϕ apart from those obtained by the theta lift from $U(V_2)$ to $U(W_3)$, then the first part of the Main Theorem tells us that the GGP conjecture may not be true for nongeneric L-parameters of G_n.

The proof of the Main Theorem is based on the following see-saw diagram:

$$\begin{tikzcd}
U(W_3) \times U(W_3) \arrow[d, dash] \arrow[drr, dash] & & U(V_2) \arrow[d, dash] \arrow[dll, dash] \\
U(W_3) & & U(V_1) \times U(V_1)
\end{tikzcd}$$

Since all elements in the A-packet $\Pi(\varrho)$ can be obtained by theta lift from $U(V_1)$, we can write $\pi^n(\varrho) = \Theta_{\psi,\chi,W_3,V_1}(\sigma)$ where σ is an irreducible smooth character of $U(V_1)$, and ψ,χ are some characters, which are needed to fix a relevant Weil representation. Then by the see-saw identity, we have

$$\mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \simeq \mathrm{Hom}_{U(V_1)}(\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee), \sigma).$$

From this, we see that for having $\mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \neq 0$, it should be preceded by $\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0$. This accounts for (i) in the Main Theorem because

$$\begin{aligned}\mathrm{Hom}_{U(W_3)}&(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \omega^\vee_{\psi,\chi,W_3}, \pi_2^\vee) \\ &\simeq \mathrm{Hom}_{U(W_3)}(\Theta_{\psi,\chi,W_3,V_1}(\sigma) \otimes \pi_2, \omega_{\psi,\chi,W_3}).\end{aligned}$$

If $\Theta_{\psi,\chi,V_2,W_3}(\pi_2^\vee) \neq 0$, then by the local theta correspondence, $\pi_2^\vee$ should be $\Theta_{\psi,\chi,W_3,V_2}(\pi_0)$, where π_0 is an irreducible representation of $U(V_2)$. By applying $(\mathrm{B})_1$, we can pinpoint π_0 and σ in the framework of local Langlands correspondence such that $\mathrm{Hom}_{U(V_1)}(\pi_0, \sigma) \neq 0$. Next we shall use the precise local theta correspondences for $(U(V_1), U(W_3))$ and $(U(V_1), U(W_1))$ in order to transfer the recipe for $(\mathrm{B})_1$ to $(\mathrm{FJ})_3$.

The rest of the paper is organized as follows: In Section 2, we shall give a brief sketch of the local Langlands correspondence for unitary groups. In Section 3, we collect some results on the local theta correspondence for unitary groups which we will use in the proof of our main results. In Section 4, we shall prove our Main Theorem.

Notations. We fix some notations we shall use throughout this paper:

- E/F is a quadratic extension of nonarchimedean local fields of characteristic zero.
- c is the nontrivial element of $\mathrm{Gal}(E/F)$.
- Fr_E is a Frobenius element of $\mathrm{Gal}(\bar{E}/E)$.
- Denote by $\mathrm{Tr}_{E/F}$ and $\mathrm{N}_{E/F}$ the trace and norm maps from E to F.
- δ is an element of E such that $\mathrm{Tr}_{E/F}(\delta) = 0$.
- Let ψ be an additive character of F and define

$$\psi^E(x) := \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big) \quad \text{and} \quad \psi_2^E(x) := \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

- Let χ be a character of $E^\times$ whose restriction to $F^\times$ is $\omega_{E/F}$, which is the quadratic character associated to E/F by local class field theory.
- For a linear algebraic group G, its F-points will be denoted by $G(F)$ or simply by G.

2. Local Langlands correspondence for unitary groups

By the recent work of Mok [2015] and Kaletha, Mínguez, Shin and White [2014], the local Langlands correspondence is now known for unitary groups conditional on the stabilization of the twisted trace formula and weighted fundamental lemma. The twisted trace formula has now been stabilized in [Waldspurger 2014a; Waldspurger 2014b] and [Moeglin and Waldspurger 2014a; Moeglin and Waldspurger 2014b] and the proof of the weighted fundamental lemma is an ongoing project of Chaudouard and Laumon. Since our main results are expressed using the local Langlands correspondence, we shall assume the local Langlands correspondence for unitary groups. In this section, we list some of its properties which are used in this paper. Indeed, much of this section is excerpts from Section 2 in [Gan and Ichino 2016].

Hermitian and skew-hermitian spaces. For $\varepsilon = \pm 1$, let V be a finite n-dimensional vector space over E equipped with a nondegenerate ε-hermitian c-sesquilinear form $\langle\cdot,\cdot\rangle_V : V \times V \to E$. That means for $v, w \in V$ and $a, b \in E$,

$$\langle av, bw\rangle_V = ab^c\langle v, w\rangle_V, \qquad \langle w, v\rangle_V = \varepsilon \cdot \langle v, w\rangle_V^c.$$

We define $\operatorname{disc} V = (-1)^{(n-1)n/2} \cdot \det V$, so that

$$\operatorname{disc} V \in \begin{cases} F^\times/\mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = +1, \\ \delta^n \cdot F^\times/\mathrm{N}_{E/F}(E^\times) & \text{if } \varepsilon = -1, \end{cases}$$

and we can define $\epsilon(V) = \pm 1$ by

$$\epsilon(V) = \begin{cases} \omega_{E/F}(\operatorname{disc} V) & \text{if } \varepsilon = +1, \\ \omega_{E/F}(\delta^{-n} \cdot \operatorname{disc} V) & \text{if } \varepsilon = -1. \end{cases} \tag{2-1}$$

By a theorem of Landherr, for a given positive integer n, there are exactly two isomorphism classes of ε-hermitian spaces of dimension n and they are distinguished from each other by $\epsilon(V)$. Let $U(V)$ be the unitary group of V defined by

$$U(V) = \{g \in \mathrm{GL}(V) \mid \langle gv, gw \rangle_V = \langle v, w \rangle_V \text{ for } v, w \in V\}.$$

Then $U(V)$ turns out to be the connected reductive algebraic group defined over F.

L-parameters and component groups. Let I_F be the inertia subgroup of $\mathrm{Gal}(\bar{F}/F)$. Let $W_F = I_F \rtimes \langle \mathrm{Fr}_F \rangle$ be the Weil group of F and $\mathrm{WD}_F = W_F \times \mathrm{SL}_2(\mathbb{C})$ be the Weil–Deligne group of F. For a homomorphism $\phi : \mathrm{WD}_F \to \mathrm{GL}_n(\mathbb{C})$, we say that it is a representation of WD_F if

(i) ϕ is continuous and $\phi(\mathrm{Fr}_F)$ is semisimple,

(ii) the restriction of ϕ to $\mathrm{SL}_2(\mathbb{C})$ is induced by a morphism of algebraic groups $\mathrm{SL}_2 \to \mathrm{GL}_n$.

If, moreover, the image of W_F is bounded then we say that ϕ is tempered. Define $\phi^\vee$ by $\phi^\vee(w) = {}^t\phi(w)^{-1}$ and call this the contragredient representation of ϕ. If E/F is a quadratic extension of local fields and ϕ is a representation of WD_E, fix $s \in W_F \setminus W_E$ and define a representation ϕ^c of WD_E by $\phi^c(w) = \phi(sws^{-1})$. The equivalence class of ϕ^c is independent of the choice of s. We say that ϕ is conjugate self-dual if there is an isomorphism $b : \phi \mapsto (\phi^\vee)^c$. Note that there is a natural isomorphism $(((\phi^\vee)^c)^\vee)^c \simeq \phi$ so that $(b^\vee)^c$ can be considered as an isomorphism from ϕ onto $(\phi^\vee)^c$. For $\varepsilon = \pm 1$, we say that ϕ is conjugate self-dual with sign ε if there exists such a b satisfying the extra condition $(b^\vee)^c = \varepsilon \cdot b$. Define $\mathrm{As}(\phi) : \mathrm{WD}_F \to \mathrm{GL}_{n^2}(\mathbb{C})$ by tensor induction of ϕ as follows:

$$\mathrm{As}(\phi)(w) = \begin{cases} \phi(w) \otimes \phi(s^{-1}ws) & \text{if } w \in \mathrm{WD}_E, \\ \iota \circ (\phi(s^{-1}w) \otimes \phi(ws)) & \text{if } w \in \mathrm{WD}_F \setminus \mathrm{WD}_E, \end{cases}$$

where ι is the linear isomorphism of $\mathbb{C}^n \otimes \mathbb{C}^n$ given by $\iota(x \otimes y) = y \otimes x$.

Then the equivalence class of $\mathrm{As}(\phi)$ is also independent of the choice of s. We set $\mathrm{As}^+(\phi) = \mathrm{As}(\phi)$ and $\mathrm{As}^-(\phi) = \mathrm{As}(\chi \otimes \phi)$.

Let V be an n-dimensional ε-hermitian space over E. An L-parameter for the unitary group $U(V)$ is a $\mathrm{GL}_n(\mathbb{C})$-conjugacy class of admissible homomorphisms

$$\phi : \mathrm{WD}_F \to {}^L U(V) = \mathrm{GL}_n(\mathbb{C}) \rtimes \mathrm{Gal}(E/F),$$

such that the composite of ϕ with the projection onto $\mathrm{Gal}(E/F)$ is the natural projection of WD_F to $\mathrm{Gal}(E/F)$.

The following proposition from [Gan et al. 2012, Section 8] enables us to remove the cumbersome $\mathrm{Gal}(E/F)$-factor in the definition of L-parameters of $U(V)$.

Proposition 2.1. *Restriction to the Weil–Deligne group* WD_E *gives a bijection between the set of L-parameters for $U(V)$ and the set of equivalence classes of conjugate self-dual representations*

$$\phi : \mathrm{WD}_E \to \mathrm{GL}_n(\mathbb{C})$$

of sign $(-1)^{n-1}$.

With this proposition, by an L-parameter for $U(V)$, we mean an n-dimensional conjugate self-dual representation ϕ of WD_E of sign $(-1)^{n-1}$.

Given an L-parameter ϕ of $U(V)$, we say that ϕ is generic if its adjoint L-function $L(s, \mathrm{Ad}\circ\phi) = L(s, \mathrm{As}^{(-1)^{n-1}}(\phi))$ is holomorphic at $s = 1$. Write ϕ as a direct sum

$$\phi = \bigoplus_i m_i \phi_i$$

of pairwise inequivalent irreducible representations ϕ_i of WD_E with multiplicities m_i. We say that ϕ is square-integrable if it has no multiplicity (i.e., $m_i = 1$ for all i) and ϕ_i is conjugate self-dual with sign $(-1)^{n-1}$ for all i. Furthermore, we can associate its component group S_ϕ to ϕ. As explained in [Gan et al. 2012, Section 8], S_ϕ is a finite 2-abelian group which can be described as

$$S_\phi = \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j$$

with a canonical basis $\{a_j\}$, where the product ranges over all j such that ϕ_j is conjugate self-dual with sign $(-1)^{n-1}$. If we denote the image of $-1 \in \mathrm{GL}_n(\mathbb{C})$ in S_ϕ by z_ϕ, we have

$$z_\phi = (m_j a_j) \in \prod_j (\mathbb{Z}/2\mathbb{Z}) a_j.$$

Local Langlands correspondence for unitary groups. The aim of the local Langlands correspondence for unitary groups is to classify the irreducible smooth representations of unitary groups. To state it, we first introduce some notations.

- Let V^+ and V^- be the n-dimensional ε-hermitian spaces with $\epsilon(V^+) = +1$, $\epsilon(V^-) = -1$ respectively.

- Let $\mathrm{Irr}(U(V^{\pm}))$ be the set of irreducible smooth representations of $U(V^{\pm})$.

Then a form of the local Langlands correspondence enhanced by Vogan [1993], says that for an L-parameter ϕ of $U(V^{\pm})$, there is the so-called Vogan L-packet Π_ϕ, a finite set consisting of irreducible smooth representations of $U(V^{\pm})$, such that

$$\mathrm{Irr}(U(V^{+})) \sqcup \mathrm{Irr}(U(V^{-})) = \bigsqcup_{\phi} \Pi_\phi,$$

where ϕ on the right-hand side runs over all equivalence classes of L-parameters for $U(V^{\pm})$. Then under the local Langlands correspondence, we may also decompose Π_ϕ as

$$\Pi_\phi = \Pi_\phi^{+} \sqcup \Pi_\phi^{-},$$

where for $\epsilon = \pm 1$, the L-packet Π_ϕ^{ϵ} consists of the representations of $U(V^{\epsilon})$ in Π_ϕ.

Furthermore, as explained in [Gan et al. 2012, Section 12], there is a bijection

$$J^{\psi}(\phi) : \Pi_\phi \to \mathrm{Irr}(S_\phi),$$

which is canonical when n is odd and depends on the choice of an additive character of $\psi : F^{\times} \to \mathbb{C}$ when n is even. More precisely, such bijection is determined by the $\mathrm{N}_{E/F}(E^{\times})$-orbit of nontrivial additive characters

$$\begin{cases} \psi^{E} : E/F \to \mathbb{C}^{\times} & \text{if } \varepsilon = +1, \\ \psi : F \to \mathbb{C}^{\times} & \text{if } \varepsilon = -1. \end{cases}$$

According to this choice, when n is even, we write

$$J^{\psi} = \begin{cases} J_{\psi^{E}} & \text{if } \varepsilon = +1, \\ J_{\psi} & \text{if } \varepsilon = -1, \end{cases}$$

and even when n is odd, we retain the same notation $J^{\psi}(\phi)$ for the canonical bijection. Hereafter, if a nontrivial additive character $\psi : F \to \mathbb{C}^{\times}$ is fixed, we define $\psi^{E} : E/F \to \mathbb{C}^{\times}$ by

$$\psi^{E}(x) := \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big),$$

and using these two characters, we fix once and for all a bijection

$$J^{\psi}(\phi) : \Pi_\phi \to \mathrm{Irr}(S_\phi),$$

as above.

With these fixed bijections, we can label all irreducible smooth representations of $U(V^{\pm})$ as $\pi(\phi, \eta)$ for some unique L-parameter ϕ of $U(V^{\pm})$ and $\eta \in \mathrm{Irr}(S_\phi)$.

Properties of the local Langlands correspondence. We briefly list some properties of the local Langlands correspondence for unitary groups, which we will use in this paper:

- $\pi(\phi, \eta)$ is a representation of $U(V^{\epsilon})$ if and only if $\eta(z_{\phi}) = \epsilon$.
- $\pi(\phi, \eta)$ is tempered if and only if ϕ is tempered.
- $\pi(\phi, \eta)$ is square-integrable if and only if ϕ is square-integrable.
- The component groups S_{ϕ} and $S_{\phi^{\vee}}$ are canonically identified. Under this canonical identification, if $\pi = \pi(\phi, \eta)$, then its contragredient representation $\pi^{\vee}$ is $\pi(\phi^{\vee}, \eta \cdot \nu)$ where
$$\nu(a_j) = \begin{cases} \omega_{E/F}(-1)^{\dim \phi_j} & \text{if } \dim_{\mathbb{C}} \phi \text{ is even,} \\ 1 & \text{if } \dim_{\mathbb{C}} \phi \text{ is odd.} \end{cases}$$
(This property follows from a result of Kaletha [2013, Theorem 4.9].)

3. Local theta correspondence

In this section, we state the local theta correspondence of unitary groups for two low rank cases. From now on, for $\epsilon = \pm 1$, we shall denote by V_n^{ϵ} the n-dimensional hermitian space with $\epsilon(V_n^{\epsilon}) = \epsilon$ and by W_n^{ϵ} the n-dimensional skew-hermitian space with $\epsilon(W_n^{\epsilon}) = \epsilon$, so that $W_n^{\epsilon} = \delta \cdot V_n^{\epsilon}$.

The Weil representation for unitary groups. In this subsection, we introduce the Weil representation.

Let E/F be a quadratic extension of local fields and let $(V_m, \langle\, , \rangle_{V_m})$ be an m-dimensional hermitian space and $(W_n, \langle\, , \rangle_{W_n})$ an n-dimensional skew-hermitian space over E. Define the symplectic space
$$\mathbb{W}_{V_m, W_n} := \operatorname{Res}_{E/F} V_m \otimes_E W_n$$
with the symplectic form
$$\langle v \otimes w, v' \otimes w' \rangle_{\mathbb{W}_{V_m, W_n}} := \tfrac{1}{2} \operatorname{Tr}_{E/F}(\langle v, v' \rangle_{V_m} \langle w, w' \rangle_{W_n}).$$
We also consider the associated symplectic group $\mathrm{Sp}(\mathbb{W}_{V_m, W_n})$ which preserves $\langle \cdot, \cdot \rangle_{\mathbb{W}_{V_m, W_n}}$ and the metaplectic group $\widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m, W_n})$ which sits in a short exact sequence:
$$1 \to \mathbb{C}^{\times} \to \widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m, W_n}) \to \mathrm{Sp}(\mathbb{W}_{V_m, W_m}) \to 1.$$
Let $\mathbb{X}_{V_m, W_n}$ be a Lagrangian subspace of $\mathbb{W}_{V_m, W_n}$ and fix an additive character $\psi : F \to \mathbb{C}^{\times}$. Then we have a Schrödinger model of the Weil representation ω_{ψ} of $\widetilde{\mathrm{Sp}}(\mathbb{W})$ on $\mathcal{S}(\mathbb{X}_{V_m, W_n})$, where $\mathcal{S}$ is the Schwartz–Bruhat function space.

If we set
$$\chi_{V_m} := \chi^m \quad \text{and} \quad \chi_{W_n} := \chi^n,$$

where χ is a character of $E^\times$ whose restriction to $F^\times$ is $\omega_{E/F}$, which is the quadratic character associated to E/F by local class field theory, then (χ_{V_m}, χ_{W_n}) gives a splitting homomorphism

$$\iota_{\chi_{V_m},\chi_{W_n}} : U(V_m) \times U(W_n) \to \widetilde{\mathrm{Sp}}(\mathbb{W}_{V_m,W_n}),$$

and so by composing this with ω_ψ, we have a Weil representation $\omega_\psi \circ \iota_{\chi_{V_m},\chi_{W_n}}$ of $U(V_m) \times U(W_n)$ on $\mathbb{S}(\mathbb{X}_{V_m,W_n})$.

When the choice of ψ and (χ_{V_m}, χ_{W_n}) is fixed as above, we simply write

$$\omega_{\psi,W_n,V_m} := \omega_\psi \circ \iota_{\chi_{V_m},\chi_{W_n}}.$$

Throughout the rest of the paper, we shall denote the Weil representation of $U(V_m) \times U(W_n)$ by ω_{ψ,W_n,V_m} with the choice of characters (χ_{V_m}, χ_{W_n}) understood as above.

Remark 3.1. When $m = 1$, the image of $U(V_1)$ in $\widetilde{\mathrm{Sp}}(\mathbb{W}_{V_1,W_n})$ coincides with the image of the center of $U(W_n)$, and so we regard the Weil representation of $U(V_1) \times U(W_n)$ as a representation of $U(W_n)$. In this case, we denote the Weil representation of $U(W_n)$ as ω_{ψ,W_n}. Furthermore, we can also use $\chi_{V_1} = \chi^{-1}$ for the choice of splitting homomorphism $\iota_{\chi_{V_1},\chi_{W_n}}$ instead of $\chi_{V_1} = \chi$. In this case, the Weil representation of $U(W_n)$ is $\omega^\vee_{\psi,W_n}$.

Local theta correspondence. Given a Weil representation ω_{ψ,W_n,V_m} of $U(V_m) \times U(W_n)$ and an irreducible smooth representation π of $U(W_n)$, the maximal π-isotypic quotient of ω_{ψ,V_m,W_n} is of the form

$$\Theta_{\psi,V_m,W_n}(\pi) \boxtimes \pi$$

for some smooth representation $\Theta_{\psi,V_m,W_n}(\pi)$ of $U(V_m)$ of finite length. By Howe duality[1], the maximal semisimple quotient $\theta_{\psi,V_m,W_n}(\pi)$ of $\Theta_{\psi,V_m,W_n}(\pi)$ is either zero or irreducible.

In this paper, we consider two kinds of theta correspondences for $(U(1) \times U(3))$ and $(U(2) \times U(3))$:

***Case* 1.** First we shall consider the theta correspondence for $U(V_1^\epsilon) \times U(W_3^{\epsilon'})$. The following is a compound of Theorem 3.4 and Theorem 3.5 in [Haan 2016].

Theorem 3.2. *Let ϕ be an L-parameter of $U(V_1^\pm)$. Then*:

(i) *For any $\epsilon, \epsilon' = \pm 1$ and any $\pi \in \Pi_\phi^{\epsilon'}$, we have $\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi)$ is nonzero and irreducible.*

[1]It was first proved by Waldspurger [1990] for all residual characteristics except $p = 2$. Recently, Gan and Takeda [2014; 2016] have made it available for all residual characteristics.

(ii) $\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi) =$

$$\begin{cases} \textit{a nontempered representation} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon', \\ \textit{a supercuspidal representation} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon', \end{cases}$$

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

(iii) *The L-parameter $\theta(\phi)$ of $\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi)$ has the following form*:

(3-1) $\theta(\phi) =$

$$\begin{cases} \theta^n(\phi) = \chi|\cdot|_E^{\frac{1}{2}} \oplus \phi\cdot\chi^{-2} \oplus \chi|\cdot|_E^{-\frac{1}{2}} & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon', \\ \theta^s(\phi) = \phi\cdot\chi^{-2} \oplus \chi\boxtimes \boldsymbol{S}_2 & \textit{if } \epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon', \end{cases}$$

where $\boldsymbol{S}_2$ is the standard 2-dimensional representation of $\mathrm{SL}_2(\mathbb{C})$.

(iv) *For ϵ, ϵ' such that $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi_{\theta^n(\phi)}.$$

(v) *For ϵ, ϵ' such that $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon'$, the theta correspondence $\pi \mapsto \theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi)$ gives an injection*

$$\Pi_\phi \hookrightarrow \Pi_{\theta^s(\phi)}.$$

Write

- $S_\phi = (\mathbb{Z}/2\mathbb{Z})a_1$,
- $S_{\theta^n(\phi)} = (\mathbb{Z}/2\mathbb{Z})a_1$ *if* $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = \epsilon\cdot\epsilon'$,
- $S_{\theta^s(\phi)} = (\mathbb{Z}/2\mathbb{Z})a_1 \times (\mathbb{Z}/2\mathbb{Z})a_2$ *if* $\epsilon\left(\frac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\right) = -\epsilon\cdot\epsilon'$,

where

$$\psi_2^E(x) = \psi(\mathrm{Tr}_{E/F}(\delta x)).$$

(*Note that $\theta^s(\phi)$ is a square-integrable L-parameter of $U(W_3^{\epsilon})$ and the summand $(\mathbb{Z}/2\mathbb{Z})a_2$ of $S_{\theta^s(\phi)}$ arises from the summand $\chi\boxtimes\mathbf{S}_2$ in $\theta^s(\phi)$.*)

Since we are only dealing with odd dimensional spaces, there are three canonical bijections:

- $J^\psi(\phi) : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi)$,
- $J^\psi(\theta^n(\phi)) : \Pi_{\theta^n(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta^n(\phi)})$,
- $J^\psi(\theta^s(\phi)) : \Pi_{\theta^s(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta^s(\phi)})$.

Using these maps, the following bijections and inclusions

$$\begin{gathered}\mathrm{Irr}(S_\phi) \longleftrightarrow \mathrm{Irr}(S_{\theta^n(\phi)})\\ \eta \mapsto \theta^n(\eta),\\ \mathrm{Irr}(S_\phi) \hookrightarrow \mathrm{Irr}(S_{\theta^s(\phi)})\\ \eta \mapsto \theta^s(\eta),\end{gathered}$$

induced by the theta correspondence can be explicated as follows:

$$\theta^n(\eta)(a_1) = \eta(a_1)\cdot\epsilon\big(\tfrac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\big), \tag{3-2}$$

$$\theta^s(\eta)(a_1) = \eta(a_1)\cdot\epsilon\big(\tfrac{1}{2}, \phi\otimes\chi^{-3}, \psi_2^E\big), \qquad \theta_2(\eta)(a_2) = -1. \tag{3-3}$$

Remark 3.3. Note that $\theta^n(\phi)$ is a nongeneric L-parameter. Gan and Ichino [2016, Proposition B.1 in Appendix] proved that an L-parameter is generic if and only if its associated L-packet Π_ϕ contains a generic representation (i.e., one possessing a Whittaker model). Together with Corollary 4.2.3 in [Gelbart and Rogawski 1990], which asserts that all elements in $\Pi_{\theta^n(\phi)}$ have no Whittaker models, we see that $\theta^n(\phi)$ is a nongeneric L-parameter.

***Case* 2.** Now we shall consider the theta correspondence for $U(V_2^{\epsilon'}) \times U(W_3^{\epsilon})$. The following summarizes some results of Gan and Ichino [2014; 2016], which are specialized to this case.

Theorem 3.4. *Let ϕ be an L-parameter for $U(V_2^{\pm})$. Then*:

(i) *Suppose that ϕ does not contain χ^3.*

  (a) *For any $\pi \in \Pi_\phi^{\epsilon'}$ and any $\epsilon \in \{\pm 1\}$, we have $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is nonzero and $\theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi\otimes\chi^{-1}) \oplus \chi^2.$$

  (b) *For each $\epsilon = \pm 1$, the theta correspondence $\pi \mapsto \theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi^\epsilon_{\theta(\phi)}.$$

(ii) *Suppose that ϕ contains χ^3.*

  (a) *For any $\pi \in \Pi_\phi^{\epsilon'}$, exactly one of $\Theta_{\psi, W_3^+, V_2^{\epsilon'}}(\pi)$ or $\Theta_{\psi, W_3^-, V_2^{\epsilon'}}(\pi)$ is nonzero.*

  (b) *If $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is nonzero, then $\theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ has L-parameter*

$$\theta(\phi) = (\phi\otimes\chi^{-1}) \oplus \chi^2.$$

  (c) *The theta correspondence $\pi \mapsto \theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ gives a bijection*

$$\Pi_\phi \longleftrightarrow \Pi_{\theta(\phi)}.$$

(iii) *We have fixed a bijection*

$$J_{\psi^E}(\phi) : \Pi_\phi \longleftrightarrow \mathrm{Irr}(S_\phi),$$

where

$$\psi^E(x) = \psi\big(\tfrac{1}{2}\,\mathrm{Tr}_{E/F}(\delta x)\big)$$

and there is the bijection

$$J^\psi(\theta(\phi)) : \Pi_{\theta(\phi)} \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}).$$

- *If ϕ does not contain χ^3, we have*

  $$S_{\theta(\phi)} = S_\phi \times (\mathbb{Z}/2\mathbb{Z})b_1,$$

  where the extra copy of $\mathbb{Z}/2\mathbb{Z}$ of $S_{\theta(\phi)}$ arises from the summand χ^2 in $\theta(\phi)$. Then for each ϵ, using the above bijections J and J_{ψ^E}, one has a canonical bijection

  $$\mathrm{Irr}(S_\phi) \longleftrightarrow \mathrm{Irr}^\epsilon(S_{\theta(\phi)}),$$
  $$\eta \longleftrightarrow \theta(\eta),$$

  induced by the theta correspondence, where $\mathrm{Irr}^\epsilon(S_{\theta(\phi)})$ is the set of irreducible characters η' of $S_{\theta(\phi)}$ such that $\eta'(z_{\theta(\phi)}) = \epsilon$ and the bijection is determined by

  $$\theta(\eta)|_{S_\phi} = \eta.$$

- *If ϕ contains χ^3, then $\phi \otimes \chi^{-1}$ contains χ^2, and so*

  $$S_{\theta(\phi)} = S_\phi.$$

  Thus, one has a canonical bijection

  $$\mathrm{Irr}(S_\phi) \longleftrightarrow \mathrm{Irr}(S_{\theta(\phi)}),$$
  $$\eta \longleftrightarrow \theta(\eta),$$

  induced by the theta correspondence and it is given by

  $$\theta(\eta) = \eta.$$

(iv) *If π is tempered and $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is nonzero, then $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi)$ is irreducible and tempered.*

4. Main Theorem

In this section, we prove our Main Theorem. To prove it, we first state the precise result of Beuzart-Plessis which we shall use in the proof of Theorem 4.1[2].

Theorem (B)$_n$. *Let $\phi = \phi^{(n+1)} \times \phi^{(n)}$ be a tempered L-parameter of $U(V^{\pm}_{n+1}) \times U(V^{\pm}_n)$ and write $S_{\phi^{(n+1)}} = \prod_i (\mathbb{Z}/2\mathbb{Z})a_i$ and $S_{\phi^{(n)}} = \prod_j (\mathbb{Z}/2\mathbb{Z})b_j$. Let*

$$\Delta : U(V^{\pm}_n) \hookrightarrow U(V^{\pm}_{n+1}) \times U(V^{\pm}_n)$$

be the diagonal map. Then for $\pi(\eta) \in \Pi^{R,\,\pm}_{\phi} = \Pi^{\pm}_{\phi^{(n+1)}} \times \Pi^{\pm}_{\phi^{(n)}}$, where $\eta \in \mathrm{Irr}(S_\phi) = \mathrm{Irr}(S_{\phi^{(n+1)}}) \times \mathrm{Irr}(S_{\phi^{(n)}})$,

$$\mathrm{Hom}_{\Delta(U(V^{\pm}_n))}(\pi(\eta), \mathbb{C}) = 1 \Leftrightarrow \eta = \eta^{\ddagger},$$

where

$$\begin{cases} \eta^{\ddagger}(a_i) = \epsilon\big(\tfrac{1}{2}, \phi^{(n+1)}_i \otimes \phi^{(n)}, \psi^E_{-2}\big), \\ \eta^{\ddagger}(b_j) = \epsilon\big(\tfrac{1}{2}, \phi^{(n+1)} \otimes \phi^{(n)}_j, \psi^E_{-2}\big), \end{cases}$$

where $\psi^E_{-2}(x) = \psi(-\mathrm{Tr}_{E/F}(\delta x))$.

Theorem 4.1. *Let $\phi^{(1)}, \phi^{(2)}$ be tempered L-parameters of $U(V^{\pm}_1)$ and $U(V^{\pm}_2)$, respectively and suppose that $\phi^{(2)}$ does not contain χ^{-3}. Let*

$$\theta^n(\phi^{(1)}) = \chi|\cdot|_E^{\frac{1}{2}} \oplus \phi^{(1)} \cdot \chi^{-2} \oplus \chi|\cdot|_E^{-\frac{1}{2}},$$
$$\theta^s(\phi^{(1)}) = \phi^{(1)} \cdot \chi^{-2} \oplus \chi \boxtimes \mathbf{S}_2$$

be the two L-parameters of $U(W^{\pm}_3)$ appearing in (3-1) and let

$$\theta(\phi^{(2)}) = \phi^{(2)} \otimes \chi \oplus \chi^{-2}$$

be the L-parameters of $U(W^{\pm}_3)$ appearing in Theorem 3.4 (ii), in which χ is replaced by χ^{-1}.

Write

- $S_{\phi^{(1)}} = S_{\theta^n(\phi^{(1)})} = (\mathbb{Z}/2\mathbb{Z})a_1$,
- $S_{\theta^s(\phi^{(1)})} = S_{\phi^{(1)}} \times (\mathbb{Z}/2\mathbb{Z})a_2$,
- $S_{\phi^{(2)}} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})b_1 & \text{if } \phi^{(2)} \text{ is irreducible,} \\ (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})b_2 & \text{if } \phi^{(2)} = \phi^{(2)}_1 \oplus \phi^{(2)}_2 \text{ is reducible,} \end{cases}$
- $S_{\theta(\phi^{(2)})} = S_{\phi^{(2)}} \times (\mathbb{Z}/2\mathbb{Z})c_1$,

where c_1 comes from the component χ^{-2} of $\theta(\phi^{(2)})$. We use the fixed character ψ to fix the local Langlands correspondence for $\Pi_{\phi^{(2)}} \leftrightarrow \mathrm{Irr}(S_{\phi^{(2)}})$.

---

[2]Recently, Gan and Ichino [2016] extended Beuzart-Plessis's work to the generic case relating it to the (FJ) case.

For $x = n, s$, let

$$\theta^x(\phi^{(1)}, \phi^{(2)}) = \theta^x(\phi^{(1)}) \times \theta(\phi^{(2)})$$

be an L-parameter of $G_3^{\pm} = U(W_3^{\pm}) \times U(W_3^{\pm})$ and $\pi^x(\eta) \in \Pi^{R,\,\epsilon}_{\theta^x(\phi^{(1)},\phi^{(2)})}$ be a representation of a relevant pure inner form of G_3. Then,

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^x(\eta), \omega_{\psi, W_3^{\epsilon}}) \neq 0 \Longleftrightarrow \eta = \eta_x^{\dagger},$$

where $\eta_x^{\dagger} \in \mathrm{Irr}(S_{\theta^x(\phi^{(1)},\phi^{(2)})}) = \mathrm{Irr}(S_{\theta^x(\phi^{(1)})}) \times \mathrm{Irr}(S_{\theta(\phi^{(2)})})$ is specified as follows:

(i) When $\phi^{(2)}$ is irreducible,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(c_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big), \end{cases}$$

$$\begin{cases} \eta_s^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(a_2) = -1, \\ \eta_s^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(c_1) = -\epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

(ii) When $\phi^{(2)} = \phi_1^{(2)} \oplus \phi_2^{(2)}$ is reducible,

$$\begin{cases} \eta_n^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_1^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(b_2) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_2^{(2)}, \psi_2^E\big), \\ \eta_n^{\dagger}(c_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big), \end{cases}$$

$$\begin{cases} \eta_s^{\dagger}(a_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(a_2) = -1, \\ \eta_s^{\dagger}(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_1^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(b_2) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_2^{(2)}, \psi_2^E\big), \\ \eta_s^{\dagger}(c_1) = -\epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big). \end{cases}$$

Furthermore,

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\Delta U(W_3^{\epsilon})}(\pi^x(\eta_x^{\dagger}), \omega_{\psi, W_3^{\epsilon}}) = 1.$$

Remark 4.2. When $x = n$ or s, we have $\eta_x^{\dagger}(z_{\theta^x(\phi^{(1)})}) = \eta_x^{\dagger}(z_{\theta(\phi^{(2)})})$ so that $\eta_x^{\dagger}$ always corresponds to a representation $\pi^x(\eta_x^{\dagger})$ of a relevant pure inner form of G_3.

Proof. For each $x = n, s$, we first prove the existence of some $\epsilon_x \in \{\pm 1\}$ and $\pi^x(\eta) \in \Pi^{R,\,\epsilon_x}_{\theta^x(\phi^{(1)},\phi^{(2)})}$ such that

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon_x})}(\pi^x(\eta), \omega_{\psi, W_3^{\epsilon_x}}) \neq 0.$$

For $a \in F^\times$, let L_a be the 1-dimensional hermitian space with form $a \cdot \mathrm{N}_{E/F}$. Then

$$V_2^+/V_1^+ \simeq V_2^-/V_1^- \simeq L_{-1}.$$

We consider the following see-saw diagram (ϵ, ϵ' will be determined soon):

$$\begin{tikzcd} U(W_3^\epsilon) \times U(W_3^\epsilon) \arrow[d, no head] \arrow[drr, no head] & & U(V_2^{\epsilon'}) \arrow[d, no head] \arrow[dll, no head] \\ U(W_3^\epsilon) & & U(V_1^{\epsilon'}) \times U(L_{-1}) \end{tikzcd} . \tag{4-1}$$

In this diagram, we shall use three theta correspondences:

(i) $U(V_2^{\epsilon'}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ^2, χ^3),

(ii) $U(V_1^{\epsilon'}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ, χ^3),

(iii) $U(L_{-1}) \times U(W_3^\epsilon)$ relative to the pair of characters (χ^{-1}, χ^3).

By (B)$_1$, there is a unique $\epsilon' \in \{\pm 1\}$ and a unique pair of component characters

$$(\eta_2, \eta_1) \in \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(2)})^\vee}) \times \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(1)})^\vee}),$$

such that

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2) \otimes \pi(\eta_1), \mathbb{C}) \neq 0.$$

Moreover, $\epsilon' = \eta_1(a_1) = \epsilon\big(\frac{1}{2}, (\phi^{(1)})^\vee \otimes (\phi^{(2)})^\vee, \psi_{-2}^E\big) = \epsilon\big(\frac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big)$.

By Theorem 3.4 (i), (iv) and [Atobe and Gan 2016, Theorem 4.1],

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is nonzero for any $\epsilon \in \{\pm 1\}$. Since

$$\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2)) \boxtimes \Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is the maximal $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))$-isotypic quotient of ω_{ψ, V_2, W_3} and we have that $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2)) \boxtimes \pi(\eta_2)$ is a $\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))$-isotypic quotient of ω_{ψ, V_2, W_3}, the representation $\pi(\eta_2)$ should be a quotient of $\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$. By Proposition 5.4 in [Atobe and Gan 2016],

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big)$$

is irreducible and thus we have $\Theta_{\psi, V_2^{\epsilon'}, W_3^\epsilon}\big(\Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big) = \pi(\eta_2)$.

Since

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2), \pi^\vee(\eta_1)) \neq 0,$$

by the see-saw identity and Remark 3.1, we have

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}\big(\Theta_{\psi, W_3^\epsilon, V_1^{\epsilon'}}(\pi^\vee(\eta_1)) \otimes \omega_{\psi, W_3^\epsilon}^\vee, \Theta_{\psi, W_3^\epsilon, V_2^{\epsilon'}}(\pi(\eta_2))\big) \neq 0,$$

and since $\Theta_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2))$ and $\omega^\vee_{\psi,W_3^\epsilon}$ are admissible, we have

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}\big(\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^\vee(\eta_1))\otimes\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2)),\omega_{\psi,W_3^\epsilon}\big)\neq 0.$$

By Theorem 3.2 (i) and Theorem 3.4 (i), the L-parameter of $\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^\vee(\eta_1))\otimes\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi(\eta_2))$ is

$$\begin{cases}\theta^n(\phi^{(1)},\phi^{(2)}) & \text{if } \epsilon=\epsilon\big(\tfrac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big)\cdot\epsilon\big(\tfrac12,\phi^{(1)}\otimes\phi^{(2)},\psi_2^E\big),\\ \theta^s(\phi^{(1)},\phi^{(2)}) & \text{if } \epsilon=-\epsilon\big(\tfrac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big)\cdot\epsilon\big(\tfrac12,\phi^{(1)}\otimes\phi^{(2)},\psi_2^E\big),\end{cases}$$

and by Theorem 3.2 (v) and Theorem 3.4 (iii), we see that their associated component characters are $\eta_n^\dagger$ and $\eta_s^\dagger$ in each case.

Next we shall prove that these are the unique representations which yield *Fourier–Jacobi* models in each of the L-packets, $\Pi_{\theta^n(\phi^{(1)},\phi^{(2)})}$ and $\Pi_{\theta^s(\phi^{(1)},\phi^{(2)})}$.

Since $\theta^s(\phi^{(1)},\phi^{(2)})$ is a tempered L-parameter, the uniqueness easily follows from $(\mathrm{FJ})_3$ in this case. Therefore, we shall only consider the nontempered L-parameter $\theta^n(\phi^{(1)},\phi^{(2)})$. Let $\pi_2\otimes\pi_1\in\Pi^{R,\,\epsilon}_{\theta^n(\phi^{(1)},\phi^{(2)})}=\Pi^\epsilon_{\theta^n(\phi^{(1)})}\times\Pi^\epsilon_{\theta(\phi^{(2)})}$ be a representation satisfying

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\pi_1,\omega_{\psi,W_3^\epsilon})\neq 0,$$

and in turn

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\omega^\vee_{\psi,W_3^\epsilon},\pi_1^\vee)\neq 0.$$

(The existence of such $\pi_2\otimes\pi_1$ was insured by the previous step.) Then by Theorem 3.2 (iv), we can write $\pi_2=\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^{(1)})$ for some $\pi^{(1)}\in\Pi^{\epsilon'}_{\theta^{(1)}}$ where

$$\epsilon'=\epsilon\cdot\epsilon\big(\tfrac12,\phi^{(1)}\otimes\chi^{-3},\psi_2^E\big).$$

Then by applying the see-saw duality in the see-saw diagram in (4-1), one has

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi_2\otimes\omega^\vee_{\psi,W_3^\epsilon},\pi_1^\vee)\simeq\mathrm{Hom}_{U(V_1^{\epsilon'})}(\pi^{(2)},\pi^{(1)})\neq 0,$$

where

$$\pi^{(2)}=\Theta_{\psi,V_2^{\epsilon'},W_3^\epsilon}(\pi_1^\vee).$$

Note that $\pi^{(2)}\neq 0$ and so it has the tempered L-parameter $(\phi^{(2)})^\vee$. Then by $(\mathrm{B})_1$, $(\pi^{(2)},\pi^{(1)})$ is the unique pair in the L-packet $\Pi_{(\phi^{(2)})^\vee}\times\Pi_{\phi^{(1)}}$ such that

$$\mathrm{Hom}_{U(V_1^{\epsilon'})}(\pi^{(2)},\pi^{(1)})\neq 0,$$

and so (π_2,π_1) should be $(\Theta_{\psi,W_3^\epsilon,V_1^{\epsilon'}}(\pi^{(1)}),\Theta^\vee_{\psi,W_3^\epsilon,V_2^{\epsilon'}}(\pi^{(2)}))$. This settles the uniqueness issue. □

Remark 4.3. When the L-parameter $\phi^{(2)}$ of $U(V_2^{\pm})$ contains χ^{-3}, we can write $\phi^{(2)} = \phi_0 \oplus \chi^{-3}$ for an L-parameter ϕ_0 of $U(V_1^{\pm})$. Then, we set

$$\theta(\phi^{(2)}) = \begin{cases} 3 \cdot \chi^{-2} & \text{if } \phi_0 = \chi^{-3}, \\ \phi_0 \cdot \chi \oplus 2 \cdot \chi^{-2} & \text{if } \phi_0 \neq \chi^{-3}, \end{cases}$$

and

$$S_{\theta(\phi^{(2)})} = \begin{cases} (\mathbb{Z}/2\mathbb{Z})b_1 & \text{if } \phi_0 = \chi^{-3}, \\ (\mathbb{Z}/2\mathbb{Z})b_1 \times (\mathbb{Z}/2\mathbb{Z})c_1 & \text{if } \phi_0 \neq \chi^{-3}. \end{cases}$$

If one develops a similar argument in this case, one could also have a recipe as in Theorem 4.4 for the nontempered case. However, we need some assumption on the irreducibility of the theta lifts because we cannot apply Proposition 5.4 in [Atobe and Gan 2016].

Theorem 4.4. *Let the notations be as in* Theorem 4.1 and assume this time that $\phi^{(2)}$ contains χ^{-3}. Assume that for $\pi \in \Pi^{\epsilon}_{\theta((\phi^{(2)})^\vee)}$, if $\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}(\pi)$ is nonzero, it is irreducible.

Then,

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi^n(\eta), \omega_{\psi, W_3^\epsilon}) \neq 0 \Longleftrightarrow \eta = \eta_n^\dagger,$$

where $\eta_n^\dagger \in \mathrm{Irr}(S_{\theta^n(\phi^{(1)},\phi^{(2)})}) = \mathrm{Irr}(S_{\theta^n(\phi^{(1)})}) \times \mathrm{Irr}(S_{\theta(\phi^{(2)})})$ is specified as follows:

- When $\phi_0 = \chi^{-3}$,

$$\begin{cases} \eta_n^\dagger(a_1) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\right), \\ \eta_n^\dagger(b_1) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\right). \end{cases}$$

- When $\phi_0 \neq \chi^{-3}$,

$$\begin{cases} \eta_n^\dagger(a_1) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\right), \\ \eta_n^\dagger(b_1) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\right), \\ \eta_n^\dagger(c_1) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\right). \end{cases}$$

Proof. We first prove the existence of some $\epsilon \in \{\pm 1\}$ and $\pi^n(\eta) \in \Pi^{R,\epsilon}_{\theta^n(\phi^{(1)},\phi^{(2)})}$ such that

$$\mathrm{Hom}_{\Delta U(W_3^\epsilon)}(\pi^n(\eta), \omega_{\psi, W_3^\epsilon}) \neq 0.$$

By (B)$_1$, there is a unique $\epsilon' \in \{\pm 1\}$ and a unique pair of component characters

$$(\eta_2, \eta_1) \in \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(2)})^\vee}) \times \mathrm{Irr}^{\epsilon'}(S_{(\phi^{(1)})^\vee}),$$

such that

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2) \otimes \pi(\eta_1), \mathbb{C}) \neq 0.$$

Moreover, $\eta_2(b_1) = \epsilon((\phi^{(1)})^\vee \otimes (\phi_0)^\vee, \psi_{-2}^E) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\right)$ and

$$\epsilon' = \eta_1(a_1) = \epsilon\left(\tfrac{1}{2}, (\phi^{(1)})^\vee \otimes (\phi^{(2)})^\vee, \psi_{-2}^E\right) = \epsilon\left(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\right).$$

By Theorem 3.4 (ii) and (iv),

$$\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$$

is a nonzero irreducible representation of $U(W_3^{\epsilon})$ for some $\epsilon \in \{\pm 1\}$ and by Theorem 4.1 in [Atobe and Gan 2016], $\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}\big(\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big)$ is nonzero. So by our assumption, it is irreducible and

$$\Theta_{\psi, V_2^{\epsilon'}, W_3^{\epsilon}}\big(\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big) = \pi(\eta_2).$$

Since

$$\mathrm{Hom}_{\Delta U(V_1^{\epsilon'})}(\pi(\eta_2), \pi^{\vee}(\eta_1)) \neq 0,$$

by the see-saw identity and Remark 3.1, we have

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}\big(\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \omega^{\vee}_{\psi, W_3^{\epsilon}}, \Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))\big) \neq 0,$$

and since $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$ and $\omega^{\vee}_{\psi, W_3^{\epsilon}}$ are admissible, we have

$$\mathrm{Hom}_{\Delta U(W_3^{\epsilon})}\big(\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2)), \omega_{\psi, W_3^{\epsilon}}\big) \neq 0.$$

Let $\Theta_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2)) = \pi(\eta_3)$ for some $\eta_3 \in \prod_{\theta((\phi^{(2)})^{\vee})}$. If $\phi_0 = \chi^{-3}$, then $\eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(3 \cdot b_1) = \eta_3(b_1)$ and if $\phi_0 \neq \chi^{-3}$, then $\eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(b_1)$. Thus in both cases, we have

$$\epsilon = \eta_3\big(z_{\theta((\phi^{(2)})^{\vee})}\big) = \eta_3(b_1) = \eta_2(b_1) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big).$$

Since

$$\epsilon' = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi^{(2)}, \psi_2^E\big) = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \phi_0, \psi_2^E\big) \cdot \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big),$$

we have $\epsilon \cdot \epsilon' = \epsilon\big(\tfrac{1}{2}, \phi^{(1)} \otimes \chi^{-3}, \psi_2^E\big)$ and so by Theorem 3.2 (iii) and Theorem 3.4 (i), the L-parameter of $\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$ is $\theta^n(\phi^{(1)}, \phi^{(2)})$.

Furthermore, by applying Theorem 3.2 (v) and Theorem 3.4 (iii), we see that the associated component character of

$$\Theta_{\psi, W_3^{\epsilon}, V_1^{\epsilon'}}(\pi^{\vee}(\eta_1)) \otimes \Theta^{\vee}_{\psi, W_3^{\epsilon}, V_2^{\epsilon'}}(\pi(\eta_2))$$

is exactly $\eta_n^{\dagger}$ in each case and it proves the existence part. The proof of the uniqueness part is essentially the same as the one in Theorem 4.1. □

Remark 4.5. It is remarkable that for the supercuspidal L-parameter $\theta^s(\phi^{(1)}, \phi^{(2)})$ with $\theta(\phi^{(2)})$ as above, the recipe, which is suggested in $(\mathrm{FJ})_3$, does not occur with the theta lift from $U(V_1^{\pm})$ and $U(V_2^{\pm})$. This is quite similar to Proposition 4.6 in [Haan 2016], which concerns the nongeneric aspect of *Bessel* cases of the GGP conjecture.

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A KIRCHBERG-TYPE TENSOR THEOREM FOR OPERATOR SYSTEMS

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We construct operator systems $\mathfrak{C}_I$ that are universal in the sense that all operator systems can be realized as their quotients. They satisfy the operator system lifting property. Without relying on the theorem by Kirchberg, we prove the Kirchberg-type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

Combining this with a result of Kavruk, we give a new operator system theoretic proof of Kirchberg's theorem and show that Kirchberg's conjecture is equivalent to its operator system analogue

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I.$$

It is natural to ask whether the universal operator systems $\mathfrak{C}_I$ are projective objects in the category of operator systems. We show that an operator system from which all unital completely positive maps into operator system quotients can be lifted is necessarily one-dimensional. Moreover, a finite-dimensional operator system satisfying a perturbed lifting property can be represented as the direct sum of matrix algebras. We give an operator system theoretic approach to the Effros–Haagerup lifting theorem.

1. Introduction

Every Banach space can be realized as a quotient of $\ell_1(I)$ for a suitable choice of index set I. Moreover, every linear map $\varphi : \ell_1(I) \to E/F$ lifts to $\tilde{\varphi} : \ell_1(I) \to E$ with $\|\tilde{\varphi}\| < (1+\varepsilon)\|\varphi\|$. On noncommutative sides, $\bigoplus_1 T_{n_i}$ (respectively $C^*(\mathbb{F})$) plays such a role in the category of operator spaces (respectively C^*-algebras). The purpose of this paper is to find operator systems that play such a role in the category of operator systems.

We construct operator systems $\mathfrak{C}_I$ that are universal in the sense that all operator systems can be realized as their quotients. The method of construction is motivated

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by [Blecher 1992, Proposition 3.1] and the coproduct of operator systems [Fritz 2014; Kerr and Li 2009]. The index set I is chosen to be sufficiently large that we can index the set $\mathcal{S}^+_{\|\cdot\|\le 1}$ of positive contractive elements in an operator system $\mathcal{S}$. The operator system $\mathfrak{C}_I$ is realized as the infinite coproduct of $\{M_k \oplus M_k\}_{k\in\mathbb{N}}$ admitting copies of $M_k \oplus M_k$ up to the cardinality of I.

We prove that the operator systems $\mathfrak{C}_I$ satisfy the operator system lifting property: for any unital C^*-algebra $\mathcal{A}$ with its closed ideal $\mathcal{I}$ and the quotient map $\pi : \mathcal{A} \to \mathcal{A}/\mathcal{I}$, every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{A}$. It is helpful to picture the situation using the commutative diagram

$$\begin{tikzcd} & \mathcal{A} \arrow[d, "\pi"] \\ \mathfrak{C}_I \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I} \end{tikzcd}$$

For a free group $\mathbb{F}$ and a Hilbert space H, Kirchberg [1994, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H).$$

The proof was later simplified in [Pisier 1996] and [Farenick and Paulsen 2012] using operator space theory and operator system theory, respectively. Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^*-algebra category: every C^*-algebra is a C^*-quotient of $C^*(\mathbb{F})$ and a C^*-subalgebra of $B(H)$ for suitable choices of $\mathbb{F}$ and H.

For suitable choices of I and H, every operator system is a subsystem of $B(H)$ by the Choi–Effros theorem [Choi and Effros 1977] and is a quotient of $\mathfrak{C}_I$ which is proved in Section 3. We will prove a Kirchberg-type tensor theorem

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$$

in Section 4. The proof is independent of Kirchberg's theorem. Combining this with Kavruk's idea [2012] we give a new operator system theoretic proof of Kirchberg's theorem.

We also prove that the operator system analogue

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

of Kirchberg's conjecture

$$C^*(\mathbb{F}) \otimes_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F})$$

is equivalent to Kirchberg's conjecture itself.

In the final section, we consider several lifting problems of completely positive maps. It is natural to ask whether the universal operator system $\mathfrak{C}_I$ is a projective object in the category of operator systems. In other words, for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, does every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$? The answer is negative in an extreme manner. An operator system satisfying such a lifting property is necessarily one-dimensional. This is essentially due to Archimedeanization of quotients [Paulsen and Tomforde 2009]. Even though some perturbation is allowed, there is also rigidity: for a finite-dimensional operator system E and a faithful state ω, the following are equivalent:

(i) If $\varepsilon > 0$ and $\varphi : E \to \mathcal{S}/\mathcal{J}$ is a completely positive map for an operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, then there exists a self-adjoint lifting $\tilde{\varphi} : E \to \mathcal{S}$ of φ such that $\tilde{\varphi} + \varepsilon\omega 1_{\mathcal{S}}$ is completely positive.

(ii) E is unitally completely order isomorphic to the direct sum of matrix algebras.

In order to prove it, we give a characterization of nuclearity via the projectivity and the minimal tensor product: an operator system $\mathcal{S}$ is nuclear if and only if

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 \to \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

is a quotient map for any quotient map $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$.

Finally, we present an operator system theoretic approach to the Effros–Haagerup lifting theorem [Effros and Haagerup 1985].

2. Preliminaries

Let $\mathcal{S}$ and $\mathcal{T}$ be operator systems. Following [Kavruk et al. 2011], henceforth abbreviated [KPTT1], an *operator system structure* on $\mathcal{S} \otimes \mathcal{T}$ is defined as a family of cones $M_n(\mathcal{S} \otimes_\tau \mathcal{T})^+$ satisfying

(T1) $(\mathcal{S}\otimes\mathcal{T}, \{M_n(\mathcal{S}\otimes_\tau\mathcal{T})^+\}_{n=1}^\infty, 1_{\mathcal{S}}\otimes 1_{\mathcal{T}})$ is an operator system denoted by $\mathcal{S}\otimes_\tau\mathcal{T}$,

(T2) $M_m(\mathcal{S})^+ \otimes M_n(\mathcal{T})^+ \subset M_{mn}(\mathcal{S} \otimes_\tau \mathcal{T})^+$ for all $m, n \in \mathbb{N}$, and

(T3) if $\varphi : \mathcal{S} \to M_m$ and $\psi : \mathcal{T} \to M_n$ are unital completely positive maps, then $\varphi \otimes \psi : \mathcal{S} \otimes_\tau \mathcal{T} \to M_{mn}$ is a unital completely positive map.

By an *operator system tensor product,* we mean a mapping $\tau : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$, such that for every pair of operator systems $\mathcal{S}$ and $\mathcal{T}$, we have that $\tau(\mathcal{S}, \mathcal{T})$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$, and denote it by $\mathcal{S} \otimes_\tau \mathcal{T}$. We call an operator system tensor product τ *functorial,* if the following property is satisfied:

(T4) For any operator systems $\mathcal{S}_1, \mathcal{S}_2, \mathcal{T}_1, \mathcal{T}_2$ and unital completely positive maps $\varphi : \mathcal{S}_1 \to \mathcal{T}_1$, $\psi : \mathcal{S}_2 \to \mathcal{T}_2$, the map $\varphi \otimes \psi : \mathcal{S}_1 \otimes \mathcal{S}_2 \to \mathcal{T}_1 \otimes \mathcal{T}_2$ is unital completely positive.

Given a linear mapping $\varphi : V \to W$ between vector spaces, its n-th amplification $\varphi_n : M_n(V) \to M_n(W)$ is defined as $\varphi_n([x_{i,j}]) = [\varphi(x_{i,j})]$. For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+ = \left\{ \begin{array}{l} X \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\varphi \otimes \psi)_n(X) \in M_{nkl}^+ \text{ for all unital} \\ \text{completely positive maps } \varphi : \mathcal{S} \to M_k,\, \psi : \mathcal{T} \to M_l \end{array} \right\}.$$

Then the family $\{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})^+\}_{n=1}^\infty$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. Moreover, if we let $\iota_{\mathcal{S}} : \mathcal{S} \to \mathcal{B}(\mathcal{H})$ and $\iota_{\mathcal{T}} : \mathcal{T} \to \mathcal{B}(\mathcal{K})$ be any unital complete order embeddings, then this is the operator system structure on $\mathcal{S} \otimes \mathcal{T}$ arising from the embedding $\iota_{\mathcal{S}} \otimes \iota_{\mathcal{T}} : \mathcal{S} \otimes \mathcal{T} \to \mathcal{B}(\mathcal{H} \otimes \mathcal{K})$ [KPTT1, Theorem 4.4]. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\min} \mathcal{T})\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the *minimal* tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\min} \mathcal{T}$.

The mapping $\min : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\min} \mathcal{T}$ is an injective, associative, symmetric and functorial operator system tensor product. The positive cone of the minimal tensor product is the largest among all possible positive cones of operator system tensor products at each matrix level [KPTT1, Theorem 4.6]. The operator system minimal tensor product $\mathcal{A} \otimes_{\min} \mathcal{B}$ of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a dense subsystem of C^*-minimal tensor product $\mathcal{A} \otimes_{C^*\min} \mathcal{B}$ [KPTT1, Corollary 4.10].

For operator systems $\mathcal{S}$ and $\mathcal{T}$, we put

$$D_n^{\max}(\mathcal{S}, \mathcal{T}) = \left\{ \alpha(P \otimes Q)\alpha^* : P \in M_k(\mathcal{S})^+,\ Q \in M_l(\mathcal{T})^+,\ \alpha \in M_{n,kl},\, k, l \in \mathbb{N} \right\}.$$

This is a matrix ordering on $\mathcal{S} \otimes \mathcal{T}$ with order unit $1_{\mathcal{S}} \otimes 1_{\mathcal{T}}$. Let $\{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty$ be the Archimedeanization of the matrix ordering $\{D_n^{\max}(\mathcal{S}, \mathcal{T})\}_{n=1}^\infty$. Then it can be written as

$$M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+ = \{X \in M_n(\mathcal{S} \otimes \mathcal{T}) : \forall \varepsilon > 0,\ X + \varepsilon I_n \otimes 1_{\mathcal{S}} \otimes 1_{\mathcal{T}} \in D_n^{\max}(\mathcal{S}, \mathcal{T})\}.$$

We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\max} \mathcal{T})^+\}_{n=1}^\infty, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the *maximal* operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\max} \mathcal{T}$.

The mapping $\max : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\max} \mathcal{T}$ is an associative, symmetric and functorial operator system tensor product. The positive cone of the maximal tensor product is the smallest among all possible positive cones of operator system tensor products at each matrix level [KPTT1, Theorem 5.5]. The operator system maximal tensor product $\mathcal{A} \otimes_{\max} \mathcal{B}$ of unital C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ is a dense subsystem of C^*-maximal tensor product $\mathcal{A} \otimes_{C^*\max} \mathcal{B}$ [KPTT1, Theorem 5.12].

For completely positive maps $\varphi : \mathcal{S} \to B(H)$ and $\psi : \mathcal{T} \to B(H)$, let $\varphi \cdot \psi : \mathcal{S} \otimes \mathcal{T} \to B(H)$ be the map given on simple tensors by $(\varphi \cdot \psi)(x \otimes y) = \varphi(x)\psi(y)$.

We put

$$M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})^+ = \left\{ \begin{array}{l} X \in M_n(\mathcal{S} \otimes \mathcal{T}) : (\varphi \cdot \psi)_n(X) \geq 0 \text{ for all completely positive} \\ \text{maps } \varphi : \mathcal{S} \to B(H),\ \psi : \mathcal{T} \to B(H) \text{ with commuting ranges} \end{array} \right\}.$$

Then the family $\{M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})^+\}_{n=1}^{\infty}$ is an operator system structure on $\mathcal{S} \otimes \mathcal{T}$. We call the operator system $(\mathcal{S} \otimes \mathcal{T}, \{M_n(\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T})\}_{n=1}^{\infty}, 1_{\mathcal{S}} \otimes 1_{\mathcal{T}})$ the *commuting* tensor product of $\mathcal{S}$ and $\mathcal{T}$ and denote it by $\mathcal{S} \otimes_{\mathrm{c}} \mathcal{T}$. If $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{S}$ is an operator system, then we have

$$\mathcal{A} \otimes_{\mathrm{c}} \mathcal{S} = \mathcal{A} \otimes_{\max} \mathcal{S}$$

[KPTT1, Theorem 6.7]. Hence, the maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems.

For an inclusion $\mathcal{S} \subset B(H)$, we let $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ be the operator system with underlying space $\mathcal{S} \otimes \mathcal{T}$ whose matrix ordering is induced by the inclusion $\mathcal{S} \otimes \mathcal{T} \subset B(H) \otimes_{\max} \mathcal{T}$. We call the operator system $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ the *enveloping left* operator system tensor product of $\mathcal{S}$ and $\mathcal{T}$. The mapping $\mathrm{el} : \mathcal{O} \times \mathcal{O} \to \mathcal{O}$ sending $(\mathcal{S}, \mathcal{T})$ to $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ is a left injective functorial operator system tensor product. Here, $\mathcal{S} \otimes_{\mathrm{el}} \mathcal{T}$ is independent of the choice of any injective operator system containing $\mathcal{S}$ instead of $B(H)$.

Given an operator system $\mathcal{S}$, we call $\mathcal{J} \subset \mathcal{S}$ the kernel, provided that it is the kernel of a unital completely positive map from $\mathcal{S}$ to another operator system. If we define a family of positive cones $M_n(\mathcal{S}/\mathcal{J})^+$ on $M_n(\mathcal{S}/\mathcal{J})$ as

$$M_n(\mathcal{S}/\mathcal{J})^+ := \big\{ [x_{i,j} + \mathcal{J}]_{i,j} : \forall \varepsilon > 0, \exists\, k_{i,j} \in \mathcal{J}, \varepsilon I_n \otimes 1_{\mathcal{S}} + [x_{i,j} + k_{i,j}]_{i,j} \in M_n(\mathcal{S})^+ \big\},$$

then $(\mathcal{S}/\mathcal{J}, \{M_n(\mathcal{S}/\mathcal{J})^+\}_{n=1}^{\infty}, 1_{\mathcal{S}/\mathcal{J}})$ satisfies all the conditions of an operator system, from Proposition 3.4 in [Kavruk et al. 2013], henceforth abbreviated [KPTT2]. We call this the quotient operator system. With this definition, the first isomorphism theorem can be proved: if $\varphi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive map with $\mathcal{J} \subset \ker \varphi$, then the map $\tilde{\varphi} : \mathcal{S}/\mathcal{J} \to \mathcal{T}$ given by $\tilde{\varphi}(x + \mathcal{J}) = \varphi(x)$ is a unital completely positive map from Proposition 3.6 in the same paper. In particular, when

$$M_n(\mathcal{S}/\mathcal{J})^+ = \big\{ [x_{i,j} + \mathcal{J}]_{i,j} : \exists\, k_{i,j} \in \mathcal{J},\ [x_{i,j} + k_{i,j}]_{i,j} \in M_n(\mathcal{S})^+ \big\}$$

for all $n \in \mathbb{N}$, we call the kernel $\mathcal{J}$ completely order proximinal.

Since the kernel $\mathcal{J}$ in an operator system $\mathcal{S}$ is a closed subspace, the operator space structure of $\mathcal{S}/\mathcal{J}$ can be interpreted in two ways: first, as the operator space quotient and second, as the operator space structure induced by the operator system quotient. The two matrix norms can be different. For a specific example, see [KPTT2, Example 4.4].

For a unital completely positive surjection $\varphi : \mathcal{S} \to \mathcal{T}$, we call $\varphi : \mathcal{S} \to \mathcal{T}$ *a complete order quotient map* [Han 2011, Definition 3.1] if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take an element P in $M_n(\mathcal{S})$ so that it satisfies

$$P + \varepsilon I_n \otimes 1_{\mathcal{S}} \in M_n(\mathcal{S})^+ \quad \text{and} \quad \varphi_n(P) = Q,$$

or equivalently, if for any Q in $M_n(\mathcal{T})^+$ and $\varepsilon > 0$, we can take a positive element P in $M_n(\mathcal{S})$ satisfying

$$\varphi_n(P) = Q + \varepsilon I_n \otimes 1_{\mathcal{S}}.$$

This definition is compatible with [Farenick et al. 2013, Proposition 3.2]: every strictly positive element lifts to a strictly positive element. An element $x \in \mathcal{S}$ is called *strictly positive* if there exists $\delta > 0$ such that $x \geq \delta 1_{\mathcal{S}}$. The map $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map if and only if the induced map $\widetilde{\varphi} : \mathcal{S}/\ker\varphi \to \mathcal{T}$ is a unital complete order isomorphism. In other operator system references, this is termed a complete quotient map. To avoid confusion with complete quotient maps in operator space theory, we use the terminology of *a complete order quotient map* throughout this paper. In this paper, we say that a linear map $\Phi : V \to W$ for operator spaces V and W is a *complete quotient map* if Φ_n maps the open unit ball of $M_n(V)$ onto the open unit ball of $M_n(W)$. When $\varphi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map (respectively a complete order embedding), we will use the special type arrow as $\varphi : \mathcal{S} \twoheadrightarrow \mathcal{T}$ (respectively $\varphi : \mathcal{S} \hookrightarrow \mathcal{T}$) throughout the paper.

The normed space dual $\mathcal{S}^*$ of an operator system $\mathcal{S}$ is matrix ordered by the cones

$$M_n(\mathcal{S}^*)^+ = \{\text{completely positive maps from } \mathcal{S} \text{ to } M_n\},$$

where we identify $[\varphi_{i,j}] \in M_n(\mathcal{S}^*)$ with the mapping $x \in \mathcal{S} \mapsto [\varphi_{i,j}(x)] \in M_n$. Unfortunately, duals of operator systems fail to be operator systems in general due to the lack of matrix order unit. When $\mathcal{S}$ is finite-dimensional, there exists a state ω_0 on $\mathcal{S}$ such that $(\mathcal{S}^*, \{M_n(\mathcal{S}^*)^+\}_{n\in\mathbb{N}}, \omega_0)$ is an operator system [Choi and Effros 1977, Corollary 4.5]. In fact, we can show that every faithful state on $\mathcal{S}$ plays such a role by the compactness of $\mathcal{S}^+_{\|\cdot\|=1}$.

Let f (respectively g) be a state on $M_n(\mathcal{S})$ (respectively $M_n(\mathcal{T})$). We identify a subsystem $M_n \otimes \mathbb{C}1_{\mathcal{S}}$ of $M_n(\mathcal{S})$ (respectively $M_n \otimes \mathbb{C}1_{\mathcal{T}}$ of $M_n(\mathcal{T})$) with M_n. We call (f, g) a compatible pair whenever $f|_{M_n} = g|_{M_n}$. An operator system structure is defined on the amalgamated direct sum $\mathcal{S} \oplus \mathcal{T}/\langle(1_{\mathcal{S}}, -1_{\mathcal{T}})\rangle$ identifying each order unit. For $s \in M_n(\mathcal{S})$ and $t \in M_n(\mathcal{T})$, we define

(1) $(s+t)^* = s^* + t^*$,

(2) $s + t \geq 0$ if and only if $f(s) + g(t) \geq 0$ for all compatible pairs (f, g).

This operator system is denoted by $\mathcal{S} \oplus_1 \mathcal{T}$ and called the coproduct of operator systems $\mathcal{S}$ and $\mathcal{T}$. The canonical inclusion from $\mathcal{S}$ (respectively $\mathcal{T}$) into $\mathcal{S} \oplus_1 \mathcal{T}$

is a complete order embedding. The coproducts of operator systems satisfy the universal property: for unital completely positive maps $\varphi : \mathcal{S} \to \mathcal{R}$ and $\psi : \mathcal{T} \to \mathcal{R}$, there is a unique unital completely positive map $\Phi : \mathcal{S} \oplus_1 \mathcal{T} \to \mathcal{R}$ that extends both φ and ψ, i.e., such that the diagram

$$\begin{tikzcd} \mathcal{S} \arrow[d, hook] \arrow[dr, "\varphi"] & \\ \mathcal{S} \oplus_1 \mathcal{T} \arrow[r, dashed, "\Phi"] & \mathcal{R} \\ \mathcal{T} \arrow[u, hook] \arrow[ur, "\psi"'] & \end{tikzcd}$$

commutes [Fritz 2014, Proposition 3.3]. The coproduct $\mathcal{S} \oplus_1 \mathcal{T}$ can be realized as a quotient operator system. The map

$$s + t \in \mathcal{S} \oplus_1 \mathcal{T} \mapsto 2(s, t) + \langle 1_{\mathcal{S}}, -1_{\mathcal{T}} \rangle \in \mathcal{S} \oplus \mathcal{T} / \langle 1_{\mathcal{S}}, -1_{\mathcal{T}} \rangle$$

is a unital complete order isomorphism [Kavruk 2014].

We refer to [KPTT1; KPTT2; Kavruk 2014; Fritz 2014] for general information on tensor products, quotients, duals and coproducts of operator systems.

3. Universal operator systems $\mathfrak{C}_I$

The coproduct of two operator systems can be generalized to any family of operator systems in a way parallel to [Fritz 2014]. Suppose that $\{\mathcal{S}_\iota\}_{\iota \in I}$ is a family of operator systems. We consider their algebraic direct sum $\bigoplus_{\iota \in I} \mathcal{S}_\iota$ consisting of finitely supported elements and its subspace

$$N = \operatorname{span}\Big\{ n_{\iota_1} - n_{\iota_2} \in \bigoplus_{\iota \in I} \mathcal{S}_\iota : \iota_1, \iota_2 \in I \Big\},$$

where

$$n_{\iota_0}(\iota) = \begin{cases} 1_{\mathcal{S}_{\iota_0}} & \text{if } \iota = \iota_0, \\ 0 & \text{otherwise.} \end{cases}$$

The algebraic quotient

$$\Big(\bigoplus_{\iota \in I} \mathcal{S}_\iota \Big) / N$$

can be regarded as an amalgamated direct sum of $\{\mathcal{S}_\iota\}_{\iota \in I}$ identifying all order units $1_{\mathcal{S}_\iota}$ over $\iota \in I$. We denote general elements in $M_n\big((\bigoplus_{\iota \in I} \mathcal{S}_\iota)/N \big)$ in brief by

$$\sum_{\iota \in F} x_\iota, \quad \text{where } x_\iota \in M_n(\mathcal{S}_\iota),\ F \text{ is a finite subset of } I.$$

Let ω_ι be a state on $M_n(\mathcal{S}_\iota)$ for each $\iota \in F$. We identify each subsystem $M_n \otimes \mathbb{C}1_{\mathcal{S}_\iota}$ of $M_n(\mathcal{S}_\iota)$ with M_n. Whenever $\omega_{\iota_1}|_{M_n} = \omega_{\iota_2}|_{M_n}$ for each $\iota_1, \iota_2 \in F$, we call the family $\{\omega_\iota\}_{\iota\in F}$ compatible. On $M_n\big(\big(\bigoplus_{\iota\in I}\mathcal{S}_\iota\big)/N\big)$, we define the involution by

$$\Big(\sum_{\iota\in F}x_\iota\Big)^* = \sum_{\iota\in F}x_\iota^*$$

and the positive cone as

$$\sum_{\iota\in F}x_\iota \in M_n\big(\big(\bigoplus_{\iota\in I}\mathcal{S}_\iota\big)/N\big)^+ \quad\Longleftrightarrow\quad \sum_{\iota\in F}\omega_\iota(x_\iota)\geq 0$$

for any compatible family $\{\omega_\iota\}_{\iota\in F}$ of states. The triple

$$\big(\big(\bigoplus_{\iota\in I}\mathcal{S}_\iota\big)/N,\ \big\{M_n\big(\big(\bigoplus_{\iota\in I}\mathcal{S}_\iota\big)/N\big)^+\big\}_{n\in\mathbb{N}},\ n_\iota+N\big)$$

is denoted by $\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}$ and called the coproduct of operator systems $\{\mathcal{S}_\iota\}_{\iota\in I}$. The following is an immediate generalization of the results on the coproduct of two operator systems studied in [Fritz 2014] to any family of operator systems.

Proposition 3.1. *Suppose that $\{\mathcal{S}_\iota\}_{\iota\in I}$ (respectively $\{\mathcal{A}_\iota\}_{\iota\in I}$) is a family of operator systems (respectively unital C^*-algebras). Then:*

(i) *$\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}$ is an operator system.*

(ii) *For any subset $J\subset I$, the inclusion*

$$\bigoplus\nolimits_1\{\mathcal{S}_\iota:\iota\in J\}\subset\bigoplus\nolimits_1\{\mathcal{S}_\iota:\iota\in I\}$$

is completely order isomorphic.

(iii) *For unital completely positive maps $\varphi_\iota:\mathcal{S}_\iota\to\mathcal{R}$, there exists a unique unital completely positive map $\Phi:\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}\to\mathcal{R}$ which extends all φ_ι, i.e., such that the diagram*

$$\begin{tikzcd} \mathcal{S}_\iota \arrow[d, hook] \arrow[dr, "\varphi_\iota"] & \\ \bigoplus_1\{\mathcal{S}_\iota:\iota\in I\} \arrow[r, dashed, "\Phi"] & \mathcal{R} \end{tikzcd}$$

commutes.

(iv) *We have $\sum_{\iota\in F}x_\iota\in M_n\big(\bigoplus_1\{\mathcal{S}_\iota:\iota\in I\}\big)^+$ if and only if there exist $\alpha_\iota\in M_n$ for $\iota\in F$ such that*

$$\sum_{\iota\in F}\alpha_\iota=0 \quad \textit{and} \quad x_\iota+\alpha_\iota\otimes 1_{\mathcal{S}_\iota}\in M_n(\mathcal{S}_\iota)^+.$$

(v) *The coproduct $\bigoplus_1\{\mathcal{A}_\iota:\iota\in I\}$ is an operator subsystem of the unital C^*-algebra free product $*_{\iota\in I}\mathcal{A}_\iota$.*

The following is an immediate generalization of [Kavruk 2012, Proposition 4.7] to any finite family of operator systems.

Proposition 3.2. *Suppose that $\mathcal{S}_1, \ldots, \mathcal{S}_n$ are operator systems and*

$$N = \operatorname{span}\{n_i - n_j \in \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n : 1 \leq i, j \leq n\} \quad (n_i(j) = \delta_{i,j} 1_{\mathcal{S}_i}).$$

Then, N is a kernel in $\mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n$ and the map

$$\sum_{i=1}^{n} x_i \in \mathcal{S}_1 \oplus_1 \cdots \oplus_1 \mathcal{S}_n \mapsto n(x_i) + N \in \mathcal{S}_1 \oplus \cdots \oplus \mathcal{S}_n / N$$

is a unital complete order isomorphism.

Suppose that I is an index set and $\{I_k\}_{k \in \mathbb{N}}$ is a sequence of index sets having the same cardinality as I. Let $M_k(C([0, 1]))_{\iota_k}$ denote the copy of $M_k(C([0, 1]))$ for each index $\iota_k \in I_k$. We denote the copy of $1 \in C([0, 1])$ (respectively $t \in C([0, 1])$) in $M_k(C([0, 1])_{\iota_k}$ by 1_{ι_k} (respectively t_{ι_k}). For each $\iota_k \in I_k$, we let $\mathfrak{C}_{\iota_k}$ be an operator subsystem of $M_k(C([0, 1])_{\iota_k}$ generated by

$$\{e_{ij} \otimes 1_{\iota_k} : 1 \leq i, j \leq k\} \quad \text{and} \quad \{e_{ij} \otimes t_{\iota_k} : 1 \leq i, j \leq k\}.$$

We define the operator system $\mathfrak{C}_I$ as the coproduct

$$\bigoplus\nolimits_1 \{\mathfrak{C}_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

The operator system $\mathfrak{C}_I$ depends only on the cardinality of the index set I.

Proposition 3.3. *The operator system $\mathfrak{C}_I$ is unitally completely order isomorphic to the coproduct*

$$\bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

Proof. It is sufficient to show that each $\mathfrak{C}_{\iota_k}$ is unitally completely order isomorphic to the direct sum $M_k \oplus M_k$. For $\alpha, \beta \in M_{nk}$, we have

$\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k}$ is positive in $M_n(\mathfrak{C}_{\iota_k})$
$\Leftrightarrow \alpha \otimes 1 + \beta \otimes t$ is positive in $M_n(M_k(C([0, 1])))$
$\Leftrightarrow \forall t \in [0, 1],\ f(t) = \alpha + t\beta \in M_{nk}^{+}$ (since $M_n(M_k(C([0, 1]))) \simeq C([0, 1], M_{nk})$)
$\Leftrightarrow \alpha, \alpha + \beta \in M_{nk}^{+}$ (because f is affine).

Hence, the mapping

$$\alpha \otimes 1_{\iota_k} + \beta \otimes t_{\iota_k} \in \mathfrak{C}_{\iota_k} \mapsto (\alpha, \alpha + \beta) \in M_k \oplus M_k, \quad \alpha, \beta \in M_k$$

is a unital complete order isomorphism. $\square$

A C^*-cover $(\mathcal{A}, \iota)$ of an operator system $\mathcal{S}$ is a unital C^*-algebra $\mathcal{A}$ with a unital complete order embedding $\iota : \mathcal{S} \hookrightarrow \mathcal{A}$ such that $\iota(\mathcal{S})$ generates $\mathcal{A}$ as a C^*-algebra. The enveloping C^*-algebra $C_e^*(\mathcal{S})$ is a C^*-cover of $\mathcal{S}$ satisfying the universal minimal property: for any C^*-cover $\iota : \mathcal{S} \hookrightarrow \mathcal{A}$, there is a unique unital $*$-homomorphism

$$\pi : \mathcal{A} \to C_e^*(\mathcal{S})$$

such that $\pi(\iota(x)) = x$ for all $x \in \mathcal{S}$ [Hamana 1979].

Let $\mathcal{S}$ be an operator subsystem of $\mathcal{T}$. We say that $\mathcal{S}$ is relatively weakly injective in $\mathcal{T}$ if

$$\mathcal{S} \otimes_{\mathrm{c}} \mathcal{R} \hookrightarrow \mathcal{T} \otimes_{\mathrm{c}} \mathcal{R}$$

for any operator system $\mathcal{R}$. The following are equivalent [Bhattacharya 2014, Theorem 4.1]:

(i) $\mathcal{S}$ is relatively weakly injective in $\mathcal{T}$.

(ii) $\mathcal{S} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty) \hookrightarrow \mathcal{T} \otimes_{\mathrm{c}} C^*(\mathbb{F}_\infty)$.

(iii) For any unital completely positive map $\varphi : \mathcal{S} \to B(H)$, there exists a unital completely positive map $\Phi : \mathcal{T} \to \varphi(\mathcal{S})''$ such that $\Phi|_{\mathcal{S}} = \varphi$.

Theorem 3.4. *Suppose that I is an index set and $\{I_k\}_{k\in\mathbb{N}}$ is a sequence of index sets having the same cardinality as I. Then,*

(i) *the unital C^*-algebra free product*

$$*_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k}$$

is a C^*-cover of $\mathfrak{C}_I$;

(ii) *the unital C^*-algebra free product*

$$*_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k}$$

is a C^*-envelope of $\mathfrak{C}_I$;

(iii) *for a unital C^*-algebra $\mathcal{A}$, every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}$ has completely positive extensions*

$$\Phi : *_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k} \to \mathcal{A} \quad \textit{and} \quad \Psi : *_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k} \to \mathcal{A};$$

(iv) *$\mathfrak{C}_I$ is relatively weakly injective in both*

$$*_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0,1]))_{\iota_k} \quad \textit{and} \quad *_{k\in\mathbb{N},\iota_k\in I_k} (M_k \oplus M_k)_{\iota_k}.$$

Proof. (i) By Proposition 3.1(iv), (v), we have

$$\begin{aligned}
&\sum_{\iota_k \in F} x_{\iota_k} \in M_n(\mathfrak{C}_I)^+ \\
&\qquad \Longleftrightarrow \exists \alpha_{\iota_k} \in M_n,\ \sum_{\iota_k \in F} \alpha_{\iota_k} = 0 \text{ and } \ x_{\iota_k} + \alpha_{\iota_k} \otimes 1_{\mathfrak{C}_{\iota_k}} \in M_n(\mathfrak{C}_{\iota_k})^+ \\
&\qquad \Longleftrightarrow \exists \alpha_{\iota_k} \in M_n,\ \sum_{\iota_k \in F} \alpha_{\iota_k} = 0 \text{ and } \ x_{\iota_k} + \alpha_{\iota_k} \otimes 1_{\mathfrak{C}_{\iota_k}} \in M_n(M_k(C([0,1]))_{\iota_k})^+ \\
&\qquad \Longleftrightarrow \sum_{\iota_k \in F} x_{\iota_k} \in M_n\Big(\bigoplus\nolimits_1 \{M_k(C([0,1]))_{\iota_k} : k \in \mathbb{N},\ \iota_k \in I_k\}\Big)^+ \\
&\qquad \Longleftrightarrow \sum_{\iota_k \in F} x_{\iota_k} \in M_n(*_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0,1]))_{\iota_k})^+.
\end{aligned}$$

Hence, $\mathfrak{C}_I$ is an operator subsystem of $*_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0,1]))_{\iota_k}$. By the Weierstrass approximation theorem, each $\mathfrak{C}_{\iota_k}$ generates $M_k(C([0,1]))_{\iota_k}$ as a C^*-algebra. Hence, $*_{k \in \mathbb{N}, \iota_k \in I_k} M_k(C([0,1]))_{\iota_k}$ is a C^*-cover of $\mathfrak{C}_I$.

(ii) The proof is motivated by [Farenick and Paulsen 2012, Theorem 2.6]. Suppose that

$$\mathfrak{C}_I \subset B(H) \quad \text{and} \quad *_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k} \subset B(K).$$

Let $\mathcal{A}$ be a C^*-algebra generated by $\mathfrak{C}_I$ in $B(H)$. By the Arveson extension theorem, the canonical inclusion from $\mathfrak{C}_I = \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}$ into $*_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k}$ extends to a unital completely positive map $\rho : \mathcal{A} \to B(K)$. Then letting $\rho = V^* \pi(\,\cdot\,) V$ be a minimal Stinespring decomposition of ρ for a $*$-representation $\pi : \mathcal{A} \to B(\widehat{K})$ and an isometry $V : K \to \widehat{K}$, we have the commutative diagram

$$\begin{tikzcd}
\mathcal{A} \arrow[rr, "\pi"] \arrow[drr, "\rho"] & & B(\widehat{K}) \arrow[d, "V^* \cdot V"] \\
\mathfrak{C}_I \arrow[u, hook] \quad \subset \quad *_{k \in \mathbb{N}, \iota_k \in I_k} (M_k \oplus M_k)_{\iota_k} & \subset & B(K)
\end{tikzcd}$$

For a unitary matrix U in $(M_k \oplus M_k)_{\iota_k}$ (U need not be unitary in $\mathcal{A}$), we can write $\pi(U)$ in the operator matrix form

$$\pi(U) = \begin{pmatrix} U & B \\ C & D \end{pmatrix}.$$

Since U is unitary in $B(K)$ and

$$1 = \|U\| \le \left\| \begin{pmatrix} U & B \\ C & D \end{pmatrix} \right\| = \|\pi(U)\| \le 1,$$

we have $B = 0 = C$ by the C^*-axiom. It follows that ρ is multiplicative on

$$\mathcal{U} := \{U \in (U(k) \oplus U(k))_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\}.$$

By the spectral theorem, every matrix can be written as a linear combination of unitary matrices. It follows that the set $\mathcal{U}$ generates $\mathcal{A}$ as a C^*-algebra. We can regard ρ as a surjective $*$-homomorphism from $\mathcal{A}$ onto $*_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k}$. Hence, $*_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k}$ is the universal quotient of all C^*-algebras generated by $\mathfrak{C}_I$.

(iii) Since each $\mathfrak{C}_{\iota_k}$ is unitally completely order isomorphic to $M_k \oplus M_k$ which is injective, there exists a unital completely positive projection

$$P_{\iota_k} : M_k(C([0, 1]))_{\iota_k} \to \mathfrak{C}_{\iota_k}.$$

By [Boca 1991, Theorem 3.1], the unital free products

$$*_{k\in\mathbb{N},\iota_k\in I_k}(\varphi|_{\mathfrak{C}_{\iota_k}} \circ P_{\iota_k}) : *_{k\in\mathbb{N},\iota_k\in I_k} M_k(C([0, 1]))_{\iota_k} \to \mathcal{A}$$

and

$$*_{k\in\mathbb{N},\iota_k\in I_k}\varphi|_{\mathfrak{C}_{\iota_k}} : *_{k\in\mathbb{N},\iota_k\in I_k}(M_k \oplus M_k)_{\iota_k} \to \mathcal{A}$$

are completely positive extensions of φ.

(iv) Let $\varphi : \mathfrak{C}_I \to B(H)$ be a unital completely positive map. The double commutant $\varphi(\mathcal{S})''$ of its range is a C^*-algebra. The relative weak injectivity follows from (iii) and [Bhattacharya 2014, Theorem 4.1]. □

Theorem 3.5. *Suppose that $\mathcal{S}$ is an operator system and $\mathcal{S}^+_{\|\cdot\|\le 1}$ is indexed by a set I. Then, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_I$. Furthermore, the kernel is completely order proximinal and every positive element $x \in M_k(\mathcal{S})$ can be lifted to a positive element $\tilde{x} \in M_k(\mathfrak{C}_I)$ with $\|\tilde{x}\| \le k^2\|x\|$.*

Proof. Let $\{I_k\}_{k\in\mathbb{N}}$ be a sequence of index sets with the same cardinality as I. Then each element in $M_k(\mathcal{S})^+_{\|\cdot\|\le 1}$ can be indexed by I_k. Suppose that $\mathcal{S} \subset B(H)$. Then for each index $\iota_k \in I_k$, we define a unital completely positive map $\Phi_{\iota_k} : M_k(C([0, 1]))_{\iota_k} \to B(H)$ as

$$\Phi_{\iota_k}(\alpha \otimes f) = \frac{1}{k}\begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \alpha \otimes f(x_{\iota_k}) \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} = \frac{1}{k}\sum_{i,j} \alpha_{i,j} f(x_{\iota_k})_{i,j},$$

where $x_{\iota_k} \in M_k(\mathcal{S})^+_{\|\cdot\|\le 1}$ and each e_i is a column vector. Let $\varphi_{\iota_k} : \mathfrak{C}_{\iota_k} \to \mathcal{S}$ be its restriction on $\mathfrak{C}_{\iota_k}$. By Proposition 3.1(iii), there exists a unital completely positive map $\Phi : \mathfrak{C}_I \to \mathcal{S}$ which extends all φ_{ι_k} over $\iota_k \in I_k$, $k \in \mathbb{N}$. Since $\mathcal{S}^+_{\|\cdot\|\le 1}$ is contained in the range of Φ, Φ is surjective.

Choose an element $x_{\iota_k} \in M_k(\mathcal{S})^+_{\|\cdot\|=1}$. From

$$\Phi_k(k[E_{ij} \otimes t_{\iota_k}]_{i,j}) = [k\Phi(E_{ij} \otimes t_{\iota_k})]_{i,j} = [x_{\iota_k}(i, j)]_{i,j} = x_{\iota_k}$$

and

$$k[E_{i,j} \otimes t_{\iota_k}]_{i,j} = k[E_{ij}]_{i,j} \otimes t_{\iota_k} = k\begin{pmatrix} e_1 \\ \vdots \\ e_n \end{pmatrix} \begin{pmatrix} e_1^t & \cdots & e_n^t \end{pmatrix} \otimes t_{\iota_k} \in M_{k^2}(C([0,1]))^+,$$

we see that $\Phi : \mathfrak{C}_I \to \mathcal{S}$ is a complete order quotient map whose kernel is completely order proximinal. Moreover, we have

$$\begin{aligned} \|k[E_{i,j} \otimes t_{\iota_k}]_{i,j}\|_{M_k(C([0,1]))} &= \|k[E_{i,j}]_{i,j}\| \\ &= k\left\| \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} \begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \right\| \\ &= k\left\| \begin{pmatrix} e_1^t & \cdots & e_k^t \end{pmatrix} \begin{pmatrix} e_1 \\ \vdots \\ e_k \end{pmatrix} \right\| = k^2. \end{aligned}$$ □

We define the operator system $\mathfrak{C}_1$ as the coproduct

$$\bigoplus\nolimits_1 \{M_k \oplus M_k : k \in \mathbb{N}\}.$$

Note that $\mathfrak{C}_1 = \mathfrak{C}_I$ when $|I| = 1$.

Theorem 3.6. *Suppose that an operator system $\mathcal{S}$ is a countable union of its finite-dimensional subsystems. Then, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$.*

Proof. First, we show that every finite-dimensional operator system is an operator system quotient of $\mathfrak{C}_{\mathbb{N}}$. Let E be a finite-dimensional operator system. We index a countable dense subset D_k of $M_k(E)^+_{\|\cdot\|\le 1}$ by $\mathbb{N}$. Define a unital completely positive map $\Phi : \mathfrak{C}_{\mathbb{N}} \to E$ as in Theorem 3.5. Since the range of Φ is a dense subspace of a finite-dimensional space E, Φ is surjective.

Choose $\varepsilon > 0$ and an element x in $M_k(E)^+_{\|\cdot\|\le 1}$. Since E is finite-dimensional, the inverse of $\widetilde{\Phi} : \mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi \to E$ is completely bounded. Let

$$\|\widetilde{\Phi}^{-1} : E \to \mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi\|_{cb} \le M.$$

Take $y \in D_k$ so that $\|x - y\| \le \varepsilon/(2M)$. Since $\|\widetilde{\Phi}_k^{-1}(x-y)\| \le \varepsilon/2$, we have

$$\widetilde{\Phi}_k^{-1}(x-y) + \frac{\varepsilon}{2} I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi} \in M_k(\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi)^+.$$

There exists a positive element z in $M_k(\mathfrak{C}_{\mathbb{N}})$ satisfying

$$z + \operatorname{Ker}\Phi_k = \widetilde{\Phi}_k^{-1}(x-y) + \varepsilon I_k \otimes 1_{\mathfrak{C}_{\mathbb{N}}/\operatorname{Ker}\Phi},$$

which implies

$$\Phi_k(z) = \widetilde{\Phi}_k(z + \operatorname{Ker} \Phi) = x - y + \varepsilon I_k \otimes 1_E.$$

As in the proof of Theorem 3.5, we can take a positive element $\tilde{y}$ in $M_k(\mathfrak{C}_{\mathbb{N}})$ such that $\Phi_k(\tilde{y}) = y$. It follows that

$$\Phi_k(z + \tilde{y}) = (x - y + \varepsilon I_k \otimes 1_E) + y = x + \varepsilon I_k \otimes 1_E.$$

Hence, E is an operator system quotient of $\mathfrak{C}_{\mathbb{N}}$.

Next, we show that $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$. We enumerate the coproduct summands of $\mathfrak{C}_{\mathbb{N}}$ as

$$(M_1 \oplus M_1)_1,\ (M_1 \oplus M_1)_2,\ (M_2 \oplus M_2)_1,\ (M_1 \oplus M_1)_3,\ (M_2 \oplus M_2)_2,\ (M_3 \oplus M_3)_1, \ldots$$

and denote them by $M_{a_k} \oplus M_{a_k}$. Since $k \geq a_k$, the identity map on M_{a_k} is factorized as $Q_k \circ J_k$ for unital completely positive maps

$$J_k : A \in M_{a_k} \mapsto A \oplus \omega(A) I_{k-a_k} \in M_k \quad (\omega : \text{a state on } M_{a_k})$$

and

$$Q_k : A \in M_k \mapsto [A_{i,j}]_{1 \leq i,j \leq a_k} \in M_{a_k}.$$

By the universal property of the coproduct, there exists a unital completely positive map $J : \mathfrak{C}_{\mathbb{N}} \to \mathfrak{C}_1$ (respectively $Q : \mathfrak{C}_1 \to \mathfrak{C}_{\mathbb{N}}$) which extends all $J_k \oplus J_k : M_{a_k} \oplus M_{a_k} \to M_k \oplus M_k$ (respectively $Q_k \oplus Q_k : M_k \oplus M_k \to M_{a_k} \oplus M_{a_k}$). Then, the identity map on $\mathfrak{C}_{\mathbb{N}}$ is factorized as $Q \circ J$. Hence, $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$.

Suppose that $\mathcal{S} = \bigcup_{k=1}^{\infty} E_k$ for finite dimensional subsystems E_k of $\mathcal{S}$. We can find complete order quotient maps $\Psi_k : \mathfrak{C}_1 \to E_k$. By the universal property of the coproduct, there exists a unital completely positive map $\Psi : \mathfrak{C}_{\mathbb{N}} \to \mathcal{S}$ which extends all Ψ_k. It is easy to check that Ψ is a complete order quotient map. Since $\mathfrak{C}_{\mathbb{N}}$ is an operator system quotient of $\mathfrak{C}_1$, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$. □

Theorem 3.7. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I}$ is a closed ideal in it. Every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{A}$, i.e., such that the diagram*

$$\begin{tikzcd} & \mathcal{A} \arrow[d, two heads] \\ \mathfrak{C}_I \arrow[ur, dashed, "\tilde{\varphi}"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I} \end{tikzcd}$$

commutes.

Proof. Let z_{ι_k} be the direct sum of two Choi matrices associated to the restrictions of $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$ on each two blocks M_k, that is,

$$z_{\iota_k} = [\varphi|_{(M_k \oplus M_k)_{\iota_k}}(E_{i,j} \oplus 0_k)]_{i,j} \oplus [\varphi|_{(M_k \oplus M_k)_{\iota_k}}(0_k \oplus E_{i,j})]_{i,j}.$$

Then z_{ι_k} belongs to the positive cone of $M_k(\mathcal{A}/I) \oplus M_k(\mathcal{A}/I)$. Then let $\tilde{z}_{\iota_k} \in M_k(\mathcal{A}) \oplus M_k(\mathcal{A})$ be a positive lifting z_{ι_k}. Its corresponding mapping

$$\tilde{\varphi}_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \mathcal{A}$$

is a completely positive lifting of $\varphi|_{(M_k \oplus M_k)_{\iota_k}}$. We let

$$\tilde{\varphi}_{\iota_k}(I_{2k}) = 1 + h, \quad h = h^+ - h^- \quad (h \in \mathcal{I}, \quad h^+, h^- \in \mathcal{I}^+)$$

and take a state ω on $(M_k \oplus M_k)_{\iota_k}$. Considering

$$\alpha \in (M_k \oplus M_k)_{\iota_k} \mapsto (1 + h^+)^{-\frac{1}{2}} (\tilde{\varphi}_{\iota_k}(\alpha) + \omega(\alpha) h^-)(1 + h^+)^{-\frac{1}{2}} \in \mathcal{A}$$

as in [KPTT2, Remark 8.3], we may assume that the lifting $\tilde{\varphi}_{\iota_k}$ is unital. By the universal property of the coproduct, there exists a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$ that extends all $\tilde{\varphi}_{\iota_k}$. □

The universal C^*-algebra $C^*_u(\mathcal{S})$ is the C^*-cover of $\mathcal{S}$ satisfying the universal property: if $\varphi : \mathcal{S} \to \mathcal{A}$ is a unital completely positive map for a unital C^*-algebra $\mathcal{A}$, then there exists a $*$-homomorphism $\pi : C^*_u(\mathcal{S}) \to \mathcal{A}$ such that $\pi \circ \iota = \varphi$ [Kirchberg and Wassermann 1998]. For a unital completely positive map $\varphi : \mathcal{S} \to \mathcal{T}$ and a complete order embedding $\iota : \mathcal{T} \to C^*_u(\mathcal{T})$, we denote the unique $*$-homomorphic extension of $\iota \circ \varphi : \mathcal{S} \to C^*_u(\mathcal{T})$ by $C^*_u(\varphi)$. We can regard $C^*_u(\,\cdot\,)$ as a functor from the category of operator systems to the category of C^*-algebras.

Corollary 3.8. *Let $\mathcal{S}$ be an operator system and $Q : \mathfrak{C}_I \to \mathcal{S}$ be a complete order quotient map. The following are equivalent*:

(i) *$\mathcal{S}$ has the operator system lifting property*;

(ii) *$C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$ has a unital $*$-homomorphic right inverse*;

(iii) *$C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$ has a unital completely positive right inverse.*

Proof. (i) ⇒ (ii). The inclusion $\iota : \mathcal{S} \subset C^*_u(\mathcal{S})$ lifts to a unital completely positive map $\tilde{\iota} : \mathcal{S} \to C^*_u(\mathfrak{C}_I)$. Its $*$-homomorphic extension $\rho : C^*_u(\mathcal{S}) \to C^*_u(\mathfrak{C}_I)$ is the right inverse of $C^*_u(Q) : C^*_u(\mathfrak{C}_I) \to C^*_u(\mathcal{S})$.

(ii) ⇒ (iii). Trivial.

(iii) ⇒ (i). Suppose that $\varphi : \mathcal{S} \to \mathcal{A}/\mathcal{I}$ is a unital completely positive map for a unital C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$. By Theorem 3.7, $\varphi \circ Q : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\psi : \mathfrak{C}_I \to \mathcal{A}$. Let $\rho : C^*_u(\mathfrak{C}_I) \to \mathcal{A}$ (respectively $\sigma : C^*_u(\mathcal{S}) \to \mathcal{A}/\mathcal{I}$) be a unique $*$-homomorphic extension of ψ (respectively φ). Suppose that r is a unital completely positive right inverse of $C^*_u(Q)$. We thus have the diagram

$$\begin{tikzcd}
C^*_u(\mathfrak{C}_I) \arrow[rr, "\rho"] \arrow[dr, shift left, "C^*_u(Q)"] & & \mathcal{A} \arrow[dd, two heads, "\pi"] \\
& C^*_u(\mathcal{S}) \arrow[ul, shift left, "r"] \arrow[dr, "\sigma"] & \\
\mathfrak{C}_I \arrow[uu, hook] \arrow[r, two heads, "Q"] & \mathcal{S} \arrow[u, hook, "\iota"] \arrow[r, "\varphi"] & \mathcal{A}/\mathcal{I}
\end{tikzcd}$$

Let us show that

$$\tilde{\varphi} := \rho \circ r \circ \iota : \mathcal{S} \to \mathcal{A}$$

is a lifting of φ. Since $\mathfrak{C}_I$ generates $C^*_u(\mathfrak{C}_I)$ as a C^*-algebra, $\pi \circ \psi = \varphi \circ Q$ implies that

$$\pi \circ \rho = \sigma \circ C^*_u(Q).$$

For $x \in \mathcal{S}$, we have

$$\pi \circ \tilde{\varphi}(x) = \pi \circ \rho \circ r(x) = \sigma \circ C^*_u(Q) \circ r(x) = \varphi(x). \qquad \square$$

4. A Kirchberg-type tensor theorem for operator systems

For a free group $\mathbb{F}$ and a Hilbert space H, Kirchberg [1994, Corollary 1.2] proved that

$$C^*(\mathbb{F}) \otimes_{\min} B(H) = C^*(\mathbb{F}) \otimes_{\max} B(H).$$

Kirchberg's theorem is striking if we recall that $C^*(\mathbb{F})$ and $B(H)$ are universal objects in the C^*-algebra category: every C^*-algebra is a C^*-quotient of $C^*(\mathbb{F})$ and a C^*-subalgebra of $B(H)$ for suitable choices of $\mathbb{F}$ and H. Every operator system is a quotient of $\mathfrak{C}_I$ and a subsystem of $B(H)$ for suitable choices of I and H. Hence we may say that

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H),$$

the proof of which will follow, is the Kirchberg-type theorem in the category of operator systems.

If $\mathcal{S}$ has the operator system local lifting property, $\mathcal{S} \otimes_{\min} B(H) = \mathcal{S} \otimes_{\max} B(H)$ [KPTT2, Theorem 8.6]. From this, Theorem 3.7 immediately yields that $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$. The proof of [KPTT2, Theorem 8.6] depends on Kirchberg's theorem. We give a direct proof of $\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H)$ that is independent of Kirchberg's theorem. By combining this with [Kavruk 2012], we present a new operator system theoretic proof of Kirchberg's theorem in Corollary 4.4.

Theorem 4.1. *For an index set I and a Hilbert space H, we have*

$$\mathfrak{C}_I \otimes_{\min} B(H) = \mathfrak{C}_I \otimes_{\max} B(H).$$

Proof. Let z be a positive element in $\mathfrak{C}_I \otimes_{\min} B(H)$. We write $z = \sum_{\iota_k \in F} z_{\iota_k}$ for a finite subset F of $\bigcup_{k=1}^{\infty} I_k$ and $z_{\iota_k} \in \mathfrak{C}_{\iota_k} \otimes B(H)$. By Proposition 3.1(ii) and the injectivity of the minimal tensor product, we can regard z as a positive element in

$$\bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H).$$

We apply Proposition 3.2 to $\bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\}$ to obtain the complete order isomorphism

$$\Phi : \sum_{\iota_k \in F} x_{\iota_k} \in \bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \mapsto |F|(x_{\iota_k})_{\iota_k \in F} + N \in \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N,$$

where $|F|$ denotes the number of elements of the set F and

$$N = \operatorname{span}\Big\{ n_{\iota_l} - n_{\iota'_m} \in \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \iota_l, \iota'_m \in F \Big\} \quad (n_{\iota_k}(\iota'_j) = \delta_{\iota_k, \iota'_j} I_k \oplus I_k).$$

Let

$$Q : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \twoheadrightarrow \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N$$

be the canonical quotient map. By [Farenick and Paulsen 2012, Proposition 1.15], its dual map

$$Q^* : \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N\Big)^* \hookrightarrow \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}\Big)^*$$

is a complete order embedding. The range of Q^* is the annihilator

$$N^{\perp} = \Big\{\varphi \in \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}\Big)^* : N \subset \operatorname{Ker} \varphi\Big\}.$$

The linear map $\gamma_k : M_k \to M_k^*$ defined as

$$\gamma_k(\alpha)(\beta) = \sum_{i,j=1}^{k} \alpha_{i,j} \beta_{i,j} = \operatorname{tr}(\alpha \beta^t)$$

is a complete order isomorphism [Paulsen et al. 2011, Theorem 6.2]. Define a complete order isomorphism

$$\Gamma : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \to \bigoplus_{\iota_k \in F} (M_k^* \oplus M_k^*)_{\iota_k} \simeq \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}\Big)^*$$

by

$$\langle \Gamma((\alpha_{\iota_k})), (\beta_{\iota_k}) \rangle = \Big\langle \Big((\gamma_k \oplus \gamma_k)\Big(\frac{\alpha_{\iota_k}}{2k}\Big)\Big), (\beta_{\iota_k}) \Big\rangle = \sum_{\iota_k \in F} \frac{1}{2k} \operatorname{tr}(\alpha_{\iota_k} \beta_{\iota_k}^t).$$

Then, Γ^{-1} maps the annihilator $N^{\perp}$ onto the operator subsystem

$$K = \Big\{(\alpha_{\iota_k}) \in \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} : \frac{\operatorname{tr}(\alpha_{\iota_l})}{l} = \frac{\operatorname{tr}(\alpha_{\iota'_m})}{m} \text{ for all } \iota_l, \iota'_m \in F\Big\}$$

of $\bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k}$. We have obtained complete order isomorphisms

$$\Big(\bigoplus\nolimits_1 \big\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\big\}\Big)^* \simeq \Big(\bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k}/N\Big)^* \simeq N^{\perp} \simeq K.$$

Considering the duals of the above isomorphisms, we obtain a complete order isomorphism

$$\Lambda : \bigoplus\nolimits_1 \big\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\big\} \to K^*,$$

which maps each $\sum_{\iota_k \in F} \beta_{\iota_k} \in \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\}$ to a functional

$$(\alpha_{\iota_k}) \in K \mapsto \sum_{\iota_k \in F} \frac{|F|}{2k} \operatorname{tr}(\beta_{\iota_k} \alpha_{\iota_k}^t) \in \mathbb{C}.$$

In particular, Λ maps the order unit to the state ω on K defined as

$$\omega((\alpha_{\iota_k})) = \sum_{\iota_k \in F} \frac{1}{2k} \operatorname{tr}(\alpha_{\iota_k}).$$

It enables us to make the identification

$$\bigoplus\nolimits_1 \big\{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\big\} \otimes_{\min} B(H) \simeq K^* \otimes_{\min} B(H),$$

where K^* is an operator system with an order unit ω. The linear map $\varphi : K \to B(H)$ corresponding to z in a canonical way is completely positive [KPTT2, Lemma 8.5]. By the Arveson extension theorem, $\varphi : K \to B(H)$ extends to a completely positive map $\tilde{\varphi} : \bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k} \to B(H)$. We have the commutative diagram

$$\begin{array}{ccc} \bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k} & \xrightarrow{\Phi^{-1}\circ Q} & \bigoplus_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \\ \Gamma \big\downarrow & & \big\downarrow \Lambda \\ \big(\bigoplus_{\iota_k \in F}(M_k \oplus M_k)_{\iota_k}\big)^* & \xrightarrow{\quad R \quad} & K^* \end{array}$$

where R denotes the restriction. It follows that

$$(\Phi^{-1} \circ Q) \otimes \mathrm{id} : \bigoplus_{\iota_k \in F} (M_k \oplus M_k)_{\iota_k} \otimes_{\min} B(H) \to \bigoplus\nolimits_1 \{(M_k \oplus M_k)_{\iota_k} : \iota_k \in F\} \otimes_{\min} B(H)$$

is a complete order quotient map. Maximal tensor products of complete order quotient maps are still complete order quotient maps [Han 2011, Theorem 3.4].

Hence, we obtain

$$\begin{tikzcd}
\bigoplus_{\iota_k\in F}(M_k\oplus M_k)_{\iota_k}\otimes_{\min} B(H) \arrow[r, equal] \arrow[d, two heads, "(\Phi^{-1}\circ Q)\otimes\mathrm{id}"'] & \bigoplus_{\iota_k\in F}(M_k\oplus M_k)_{\iota_k}\otimes_{\max} B(H) \arrow[d, two heads, "(\Phi^{-1}\circ Q)\otimes\mathrm{id}"] \\
\bigoplus_1\{(M_k\oplus M_k)_{\iota_k}:\iota_k\in F\}\otimes_{\min} B(H) & \bigoplus_1\{(M_k\oplus M_k)_{\iota_k}:\iota_k\in F\}\otimes_{\max} B(H)
\end{tikzcd}$$

The element z is also positive in $\mathfrak{C}_I\otimes_{\max} B(H)$. The same arguments apply to all matricial levels. □

The maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems. For this reason, the weak expectation property of C^*-algebras bifurcates into the weak expectation property and the double commutant expectation property of operator systems. We say that an operator system $\mathcal{S}$ has the *double commutant expectation property* provided that for every completely order isomorphic inclusion $\mathcal{S}\subset B(H)$, there exists a completely positive map $\varphi: B(H)\to\mathcal{S}''$ that fixes $\mathcal{S}$. For an operator system $\mathcal{S}$, the following are equivalent [KPTT2, Theorem 7.6; Kavruk 2012, Theorem 5.9]:

(i) $\mathcal{S}$ has the double commutant expectation property.

(ii) $\mathcal{S}$ is (el, c)-nuclear.

(iii) $\mathcal{S}\otimes_{\min} C^*(\mathbb{F}_\infty)=\mathcal{S}\otimes_{\max} C^*(\mathbb{F}_\infty)$.

(iv) $\mathcal{S}\otimes_{\min}(\ell^2_\infty\oplus_1\ell^3_\infty)=\mathcal{S}\otimes_{\mathrm{c}}(\ell^2_\infty\oplus_1\ell^3_\infty)$.

Theorem 4.2. *An operator system $\mathcal{S}$ has the double commutant expectation property if and only if it satisfies*

$$\mathcal{S}\otimes_{\min}\mathfrak{C}_I=\mathcal{S}\otimes_{\mathrm{c}}\mathfrak{C}_I.$$

Proof. $\Rightarrow$) Every operator system with the double commutant expectation property is (el, c)-nuclear. Since the minimal tensor product is injective [KPTT1, Theorem 4.6], we have

$$\begin{tikzcd}
B(H)\otimes_{\min}\mathfrak{C}_I \arrow[r, equal] & B(H)\otimes_{\max}\mathfrak{C}_I \\
\mathcal{S}\otimes_{\min}\mathfrak{C}_I \arrow[u, hook] & \mathcal{S}\otimes_{\mathrm{el=c}}\mathfrak{C}_I. \arrow[u, hook]
\end{tikzcd}$$

$\Leftarrow$) Fix two indices $\iota_2'\in I_2$ and $\iota_3'\in I_3$. Define a unital completely positive map $\Phi:\ell^2_\infty\oplus_1\ell^3_\infty\to\mathfrak{C}_I$ by

$$\begin{aligned}\Phi\big((a_1,a_2)+(b_1,b_2,b_3)\big)&=\mathrm{diag}(a_1,a_2,a_1,a_2)+\mathrm{diag}(b_1,b_2,b_3,b_1,b_2,b_3)\\&\in(M_2\oplus M_2)_{\iota_2'}\oplus_1(M_3\oplus M_3)_{\iota_3'}\subset\mathfrak{C}_I.\end{aligned}$$

For each index $\iota_k \neq \iota'_2, \iota'_3$, we take a state ω_{ι_k} on $(M_k \oplus M_k)_{\iota_k}$ and define a unital completely positive map

$$\psi_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \ell^2_\infty \oplus_1 \ell^3_\infty$$

as $\psi_{\iota_k}(\alpha) = \omega_{\iota_k}(\alpha) 1_{\ell^2_\infty \oplus_1 \ell^3_\infty}$. For ι'_2 and ι'_3 we also define unital completely positive maps

$$\psi_{\iota'_2} : (M_2 \oplus M_2)_{\iota'_2} \to \ell^2_\infty \oplus_1 \ell^3_\infty \quad \text{and} \quad \psi_{\iota'_3} : (M_3 \oplus M_3)_{\iota'_3} \to \ell^2_\infty \oplus_1 \ell^3_\infty$$

by $\psi_{\iota'_2}(\alpha \oplus \beta) = (\alpha_{11}, \alpha_{22})$ and $\psi_{\iota'_3}(\alpha \oplus \beta) = (\alpha_{11}, \alpha_{22}, \alpha_{33})$. By the universal property of the coproduct, there exists a unital completely positive map $\Psi : \mathfrak{C}_I \to \ell^2_\infty \oplus_1 \ell^3_\infty$ that extends all ψ_{ι_k}. The identity map on $\ell^2_\infty \oplus_1 \ell^3_\infty$ is factorized through unital completely positive maps as

$$\mathrm{id}_{\ell^2_\infty \oplus_1 \ell^3_\infty} = \Psi \circ \Phi.$$

By the hypothesis, we have completely positive maps

$$\mathcal{S} \otimes_{\min} (\ell^2_\infty \oplus_1 \ell^3_\infty) \xrightarrow{\mathrm{id}_{\mathcal{S}} \otimes \Phi} \mathcal{S} \otimes_{\min} \mathfrak{C}_I = \mathcal{S} \otimes_c \mathfrak{C}_I \xrightarrow{\mathrm{id}_{\mathcal{S}} \otimes \Psi} \mathcal{S} \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^3_\infty).$$

Since the positive cone of the commuting tensor product is the subcone of that of the minimal tensor product at each matrix level, we have

$$\mathcal{S} \otimes_{\min} (\ell^2_\infty \oplus_1 \ell^3_\infty) = \mathcal{S} \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^3_\infty).$$

By [Kavruk 2012, Theorem 5.9], $\mathcal{S}$ has the double commutant expectation property. □

Since the maximal tensor product and the commuting tensor product are two different means of extending the C^*-maximal tensor product from the category of C^*-algebras to operator systems, we can regard

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I \quad \text{and} \quad \mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$$

as operator system analogues of Kirchberg's conjecture

$$C^*(\mathbb{F}) \otimes_{\min} C^*(\mathbb{F}) = C^*(\mathbb{F}) \otimes_{\max} C^*(\mathbb{F}).$$

The former is not true and the latter is equivalent to Kirchberg's conjecture itself.

Corollary 4.3. (i) $\mathfrak{C}_I \otimes_c \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$. *In particular*, $\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I \neq \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$.

(ii) *The Kirchberg's conjecture has an affirmative answer if and only if*

$$\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I.$$

Proof. (i) Similarly to the proof of Theorem 4.2, we can show that the identity map on $\ell^2_\infty \oplus_1 \ell^2_\infty$ is factorized as

$$\mathrm{id}_{\ell^2_\infty \oplus_1 \ell^2_\infty} = \Psi \circ \Phi$$

for unital completely positive maps $\Phi : \ell^2_\infty \oplus_1 \ell^2_\infty \to \mathfrak{C}_I$ and $\Psi : \mathfrak{C}_I \to \ell^2_\infty \oplus_1 \ell^2_\infty$. Assume to the contrary that $\mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I = \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I$. Then, we have completely positive maps

$$\begin{aligned}(\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_c (\ell^2_\infty \oplus_1 \ell^2_\infty) &\xrightarrow{\Phi\otimes\Phi} \mathfrak{C}_I \otimes_{\mathrm{c}} \mathfrak{C}_I \\ &= \mathfrak{C}_I \otimes_{\max} \mathfrak{C}_I \xrightarrow{\Psi\otimes\Psi} (\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\max} (\ell^2_\infty \oplus_1 \ell^2_\infty).\end{aligned}$$

Since the positive cone of the maximal tensor product at each matrix level is the subcone of that of the commuting tensor product, we have

$$(\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\mathrm{c}} (\ell^2_\infty \oplus_1 \ell^2_\infty) = (\ell^2_\infty \oplus_1 \ell^2_\infty) \otimes_{\max} (\ell^2_\infty \oplus_1 \ell^2_\infty).$$

This contradicts

$$\mathrm{NC}(2) \otimes_{\mathrm{c}} \mathrm{NC}(2) \neq \mathrm{NC}(2) \otimes_{\max} \mathrm{NC}(2),$$

which was shown in [Farenick et al. 2014, Corollary 7.12]. Here, $\mathrm{NC}(n)$ is defined as the operator subsystem $\mathrm{span}\{1, h_1, \ldots, h_n\}$ of the universal C^*-algebra generated by self-adjoint contractions $h_1, \ldots, h_n$ as in Definition 6.1 of the same paper. It is unitally completely order isomorphic to the coproduct (involving n terms)

$$\ell^2_\infty \oplus_1 \cdots \oplus_1 \ell^2_\infty.$$

(ii) By [Kavruk 2012, Theorem 5.14], Kirchberg's conjecture has an affirmative answer if and only if $\ell^2_\infty \oplus_1 \ell^3_\infty$ has the double commutant expectation property. By Theorem 4.2 this is equivalent to $(\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\min} \mathfrak{C}_I = (\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_c \mathfrak{C}_I$. By [Kavruk 2012, Theorem 5.9] this is equivalent to $\mathfrak{C}_I$ having the double commutant expectation property, and another application of Theorem 4.2 gives the equivalence with $\mathfrak{C}_I \otimes_{\min} \mathfrak{C}_I = \mathfrak{C}_I \otimes_c \mathfrak{C}_I$. □

We say that an operator subsystem $\mathcal{S}$ of a unital C^*-algebra $\mathcal{A}$ *contains enough unitaries* if the unitaries in $\mathcal{S}$ generate $\mathcal{A}$ as a C^*-algebra. If $\mathcal{S} \subset \mathcal{A}$ contains enough unitaries and $\mathcal{S} \otimes_{\min} \mathcal{B} \hookrightarrow \mathcal{A} \otimes_{\max} \mathcal{B}$ completely order isomorphically for a unital C^*-algebra $\mathcal{B}$, then we have $\mathcal{A} \otimes_{\min} \mathcal{B} = \mathcal{A} \otimes_{\max} \mathcal{B}$ [KPTT2, Proposition 9.5].

Corollary 4.4 (Kirchberg). *Let $\mathbb{F}_\infty$ be a free group on a countably infinite number of generators and H be a Hilbert space. We have*

$$C^*(\mathbb{F}_\infty) \otimes_{\min} B(H) = C^*(\mathbb{F}_\infty) \otimes_{\max} B(H).$$

Proof. Since the identity map on $\ell^2_\infty \oplus_1 \ell^3_\infty$ is factorized through $\mathfrak{C}_I$ by unital completely positive maps, Theorem 4.1 immediately implies that

$$(\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\min} B(H) = (\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\max} B(H).$$

Alternatively, applying the proof of Theorem 4.1 to commutative algebras instead of matrix algebras, we obtain

$$\begin{tikzcd}
\ell^5_\infty \otimes_{\min} B(H) \arrow[r, equal] \arrow[d, two heads] & \ell^5_\infty \otimes_{\max} B(H) \arrow[d, two heads] \\
(\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\min} B(H) & (\ell^2_\infty \oplus_1 \ell^3_\infty) \otimes_{\max} B(H)
\end{tikzcd}$$

For the remaining proof, we follow [Kavruk 2012]. Since

$$\ell^2_\infty \oplus_1 \ell^3_\infty \subset C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$$

contains enough unitaries [Kavruk 2012, Theorem 4.8], we have

$$C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\min} B(H) = C^*(\mathbb{Z}_2 * \mathbb{Z}_3) \otimes_{\max} B(H)$$

by [KPTT2, Proposition 9.5]. The free group $\mathbb{F}_\infty$ embeds into the free product $\mathbb{Z}_2 * \mathbb{Z}_3$ [de la Harpe 2000]. By [Pisier 2003, Proposition 8.8], $C^*(\mathbb{F}_\infty)$ is a C^*-subalgebra of $C^*(\mathbb{Z}_2 * \mathbb{Z}_3)$ complemented by a unital completely positive map. $\square$

A wide class of operator systems shares the properties of $\mathfrak{C}_I$. Let $\mathcal{M} = \{\mathcal{M}_k\}_{k\in\mathbb{N}}$ be a sequence of direct sums of matrix algebras such that

$$\limsup_{k\to\infty} s(k) = \infty$$

when $\mathcal{M}_k = M_{d_1} \oplus \cdots \oplus M_{d_n}$ and $s(k) = \max\{d_1, \ldots, d_n\}$. Let $\mathcal{M}_{\iota_k}$ denote the copy of $\mathcal{M}_k$ for each index $\iota_k \in I_k$. We define the operator system $\mathfrak{C}_I(\mathcal{M})$ (respectively $\mathfrak{C}_1(\mathcal{M})$) as the coproduct

$$\bigoplus\nolimits_1 \{\mathcal{M}_{\iota_k} : k \in \mathbb{N}, \iota_k \in I_k\} \quad \text{(respectively } \bigoplus\nolimits_1 \{\mathcal{M}_k : k \in \mathbb{N}\}).$$

In particular, we have $\mathfrak{C}_I = \mathfrak{C}_I(\mathcal{M})$ and $\mathfrak{C}_1 = \mathfrak{C}_1(\mathcal{M})$ when $\mathcal{M}_k = M_k \oplus M_k$.

Theorem 4.5. *Suppose that $\mathcal{S}$ is an operator system.*

(i) *If $\mathcal{S}^+_{\|\cdot\|\le 1}$ is indexed by a set I, then $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.*

(ii) *If $\mathcal{S}$ is a countable union of its finite dimensional subsystems, then $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1(\mathcal{M})$.*

(iii) *$\mathfrak{C}_I(\mathcal{M})$ satisfies the operator system lifting property.*

(iv) *$\mathfrak{C}_I(\mathcal{M}) \otimes_{\min} B(H) = \mathfrak{C}_I(\mathcal{M}) \otimes_{\max} B(H)$.*

(v) *$\mathcal{S}$ has the double commutant expectation property if and only if* $\mathcal{S} \otimes_{\min} \mathfrak{C}_I(\mathcal{M}) = \mathcal{S} \otimes_{\mathrm{c}} \mathfrak{C}_I(\mathcal{M})$.

(vi) *Kirchberg's conjecture has an affirmative answer if and only if* $\mathfrak{C}_I(\mathcal{M}) \otimes_{\min} \mathfrak{C}_I(\mathcal{M}) = \mathfrak{C}_I(\mathcal{M}) \otimes_c \mathfrak{C}_I(\mathcal{M})$.

Proof. (i) We take a subsequence $\{\mathcal{M}_{n_k}\}_{k\in\mathbb{N}}$ so that $s(n_k) \geq 2k$ and put $\mathcal{N}_k = \mathcal{M}_{n_k}$. By Proposition 3.1(ii), $\mathfrak{C}_I(\mathcal{N})$ is an operator subsystem of $\mathfrak{C}_I(\mathcal{M})$. Take a state ω_l on $\mathcal{M}_l$ for each $l \neq n_k$. By the universal property of the coproduct, there exists a unital completely positive map $P : \mathfrak{C}_I(\mathcal{M}) \to \mathfrak{C}_I(\mathcal{N})$ such that

$$P(x) = \begin{cases} x & \text{if } x \in \mathcal{M}_{n_k} \\ \omega_l(x)1 & \text{if } x \in \mathcal{M}_l, l \neq n_k. \end{cases}$$

Since P is a unital completely positive projection, $\mathfrak{C}_I(\mathcal{N})$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.

We may assume that $s(k) \geq 2k$ for each $k \in \mathbb{N}$. We write $\mathcal{M}_k = M_{d_1} \oplus \cdots \oplus M_{d_m}$ and $d_l \geq 2k$. The identity map on $M_k \oplus M_k$ is factorized as $\mathrm{id}_{M_k \oplus M_k} = Q_k \circ J_k$ for the unital completely positive maps

$$J_k : A \in M_k \oplus M_k \mapsto (\omega(A) I_{d_1+\cdots+d_{l-1}}) \oplus A \oplus (\omega(A) I_{(d_l-2k)+d_{l+1}+\cdots+d_m}) \in \mathcal{M}_k$$

(where ω is a state on $M_k \oplus M_k$) and

$$Q_k : A_1 \oplus \cdots \oplus A_m \in \mathcal{M}_k \mapsto [(A_l)_{i,j}]_{1\leq i,j\leq k} \oplus [(A_l)_{i+k,j+k}]_{1\leq i,j\leq k} \in M_k \oplus M_k.$$

Let $J : \mathfrak{C}_I \to \mathfrak{C}_I(\mathcal{M})$ (respectively $Q : \mathfrak{C}_I(\mathcal{M}) \to \mathfrak{C}_I$) be the unital completely positive extension of $J_{\iota_k} : (M_k \oplus M_k)_{\iota_k} \to \mathcal{M}_{\iota_k}$ (respectively $Q_{\iota_k} : \mathcal{M}_{\iota_k} \to (M_k \oplus M_k)_{\iota_k}$) over $k \in \mathbb{N}$, $\iota_k \in I_k$. Then, the identity map on $\mathfrak{C}_I$ is factorized as $\mathrm{id}_{\mathfrak{C}_I} = Q \circ J$. Hence, $\mathfrak{C}_I$ is an operator system quotient of $\mathfrak{C}_I(\mathcal{M})$.

(ii) By Theorem 3.6, $\mathcal{S}$ is an operator system quotient of $\mathfrak{C}_1$. The remaining proof is similar to (i).

(iii), (iv) The proofs of Theorems 3.7 and 4.1 work generally for coproducts of direct sums of matrix algebras.

(v), (vi) The proofs of Theorem 4.2 and Corollary 4.3 work generally for coproducts of direct sums of matrix algebras which the identity map on $\ell_2^\infty \oplus_1 \ell_3^\infty$ factorizes through. $\square$

5. Liftings of completely positive maps

It is natural to ask whether the universal operator system $\mathfrak{C}_I$ is a projective object in the category of operator systems. In other words, for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, does every unital completely positive map $\varphi : \mathfrak{C}_I \to \mathcal{S}/\mathcal{J}$ lift to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{S}$? The answer is negative in an extreme manner.

Proposition 5.1. *An operator system $\mathcal{S}$ is one-dimensional if and only if for any operator system $\mathcal{T}$ and its kernel $\mathcal{J}$, every unital completely positive map $\varphi : \mathcal{S} \to \mathcal{T}/\mathcal{J}$ lifts to a completely positive map $\tilde{\varphi} : \mathcal{S} \to \mathcal{T}$.*

Proof. Let V^+ be the cone in $\mathbb{R}^3$ generated by

$$\{(x, y, 1) : (x-1)^2 + y^2 \le 1,\, y \ge 0\}$$

and the origin. The triple $V := (\mathbb{C}^3, V^+, (1, 1, 2))$ is an Archimedean ordered $*$-vector space. The positive cones of the operator system $\mathrm{OMAX}(V)$ introduced in [Paulsen et al. 2011] are given as

$$M_n(\mathrm{OMAX}(V))^+ = \{X \in M_n(V) : \forall \varepsilon > 0,\, X + \varepsilon I_n \otimes 1_V \in M_n^+ \otimes V^+\},$$

where

$$M_n^+ \otimes V^+ = \left\{ \sum_{i=1}^m \alpha_i \otimes v_i \in M_n \otimes V : m \in \mathbb{N},\, \alpha_i \in M_n^+,\, v_i \in V^+ \right\}.$$

Let

$$P : (x, y, z) \in \mathrm{OMAX}(V) \mapsto (x, y) \in \ell^2_\infty$$

be the projection. We take $\varepsilon > 0$ and an element

$$\alpha_1 \otimes (1, 0) + \alpha_2 \otimes (0, 1) \in M_n(\ell^2_\infty)^+ = M_n^+ \oplus M_n^+$$

for nonzero α_2. Since

$$\begin{aligned} &\alpha_1 \otimes (1, 0) + \alpha_2 \otimes (0, 1) + \varepsilon I_n \otimes (1, 1) \\ &\qquad = \left(\alpha_1 + \frac{\varepsilon}{2}\left(I_n - \frac{\alpha_2}{\|\alpha_2\|}\right)\right) \otimes (1, 0) + \alpha_2 \otimes \left(\frac{\varepsilon}{2\|\alpha_2\|}, 1\right) + \varepsilon I_n \otimes \left(\tfrac{1}{2}, 1\right) \end{aligned}$$

lifts to a positive element in $M_n(\mathrm{OMAX}(V))$, the projection $P : \mathrm{OMAX}(V) \to \ell^2_\infty$ is a complete order quotient map.

Suppose that $\dim \mathcal{S} \ge 2$. Let v be a positive element in $\mathcal{S}$ distinct from the scalar multiple of the identity. Considering $v - \lambda I$ for sufficiently large $\lambda > 0$, we may assume that the spectrum of v contains zero. Let ω_1 and ω_2 be states on $\mathcal{S}$ that extend, respectively, the Dirac measures

$$\begin{aligned} \delta_{\{0\}} &: \lambda 1 + \mu v \in \mathrm{span}\{1, v\} \mapsto \lambda \in \mathbb{C}, \\ \delta_{\{\|v\|\}} &: \lambda 1 + \mu v \in \mathrm{span}\{1, v\} \mapsto \lambda + \mu \|v\| \in \mathbb{C}. \end{aligned}$$

The unital completely positive map $\varphi : \mathcal{S} \to \ell^2_\infty$ defined by $\varphi = (\omega_1, \omega_2)$ cannot be lifted to a completely positive map, because the fiber of $\varphi(v) = (0, \|v\|)$ does not intersect V^+. $\square$

The absence of completely positive liftings in the above proof is essentially due to Archimedeanization of quotients [Paulsen and Tomforde 2009]. In Corollary 5.5, we will see that there is also rigidity, even though some perturbation is allowed.

A linear map between normed spaces is called a quotient map if it maps the open unit ball onto the open unit ball. Between Banach spaces, it suffices to show that the image of the open unit ball is dense in the open unit ball in Lemma A.2.1 of [Effros and Ruan 2000]. Suppose that $T : E \to F$ is a bounded linear surjection for normed spaces E and F. Let E_0 be a dense subspace of E, and $Q_0 : E_0 \to Q(E_0)$ be the surjective restriction of Q on E_0. Then, Q_0 is a quotient map if and only if $\overline{\ker Q_0} = \ker Q$ and Q is a quotient map [Defant and Floret 1993, 7.4]. This is called the quotient lemma.

Thanks to the quotient lemma, we can describe the 1-exactness of operator systems by incomplete tensor products. For an operator system $\mathcal{S}$ and a unital C^*-algebra $\mathcal{A}$ with its closed ideal $\mathcal{I}$, we denote the completion of $\mathcal{S} \otimes_{\min} \mathcal{A}$ by $\mathcal{S} \hat{\otimes}_{\min} \mathcal{A}$ and the closure of $\mathcal{S} \otimes \mathcal{I}$ in it by $\mathcal{S} \bar{\otimes} \mathcal{I}$. When

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$$

is a complete order quotient map with its kernel $\mathcal{S} \bar{\otimes} \mathcal{I}$ for any C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$, $\mathcal{S}$ is called *1-exact*.

Proposition 5.2. *Suppose that $\mathcal{S}$ is an operator system and $\mathcal{A}$ is a unital C^*-algebra with its closed ideal $\mathcal{I}$. Then the map*

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$$

is a complete order quotient map with its kernel $\mathcal{S} \bar{\otimes} \mathcal{I}$ if and only if the map

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$$

is a complete order quotient map. Hence, an operator system $\mathcal{S}$ is 1*-exact if and only if* $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any unital C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$.*

Proof. The operator space quotient and the operator system quotient of $\mathcal{S} \hat{\otimes}_{\min} \mathcal{A}$ by $\mathcal{S} \bar{\otimes} \mathcal{I}$ are completely isometric [KPTT2, Theorem 5.1]. Since $\mathcal{S} \otimes \mathcal{I}$ is the kernel of $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$, we can also consider both the operator space quotient and the operator system quotient of $\mathcal{S} \otimes_{\min} \mathcal{A}$ by $\mathcal{S} \otimes \mathcal{I}$. It is easy to check that the operator space quotient (respectively operator system quotient) $(\mathcal{S} \otimes_{\min} \mathcal{A})/(\mathcal{S} \otimes \mathcal{I})$ is an operator subspace (respectively operator subsystem) of the operator space quotient (respectively operator system quotient) $(\mathcal{S} \hat{\otimes}_{\min} \mathcal{A})/(\mathcal{S} \bar{\otimes} \mathcal{I})$. If $z \in \mathcal{S} \otimes \mathcal{A}$ and $z + \mathcal{S} \bar{\otimes} \mathcal{I}$ is positive in the operator system quotient $(\mathcal{S} \hat{\otimes}_{\min} \mathcal{A})/(\mathcal{S} \bar{\otimes} \mathcal{I})$, then there exists $x \in \mathcal{S} \bar{\otimes} \mathcal{I}$ such that

$$z + \frac{\varepsilon}{2} 1_{\mathcal{S}} \otimes 1_{\mathcal{A}} + x \in (\mathcal{S} \hat{\otimes}_{\min} \mathcal{A})^+.$$

Take $x_0 \in \mathcal{S} \otimes I$ with $\|x - x_0\| < \varepsilon/2$. Considering $(x_0 + x_0^*)/2$, we may assume that x_0 is self-adjoint. We have

$$z + \varepsilon 1_{\mathcal{S}} \otimes 1_{\mathcal{A}} + x_0 \in (\mathcal{S} \otimes_{\min} \mathcal{A})^+,$$

which implies that $z + \mathcal{S} \otimes \mathcal{I}$ is positive in $(\mathcal{S} \otimes_{\min} \mathcal{A})/(\mathcal{S} \otimes \mathcal{I})$. Hence, [KPTT2, Theorem 5.1] immediately implies that the operator space quotient and the operator system quotient of $\mathcal{S} \otimes_{\min} \mathcal{A}$ by $\mathcal{S} \otimes \mathcal{I}$ are completely isometric.

A unital linear map between operator systems is completely order isomorphic if and only if it is completely isometric, by [Effros and Ruan 2000, Corollary 5.1.2]. If a linear map between Banach spaces maps the open unit ball into the open unit ball densely, then it is a quotient map, from Lemma A.2.1 in the same paper. Combining them with the quotient lemma, we have equivalences:

$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map with its kernel $\mathcal{S} \bar{\otimes} \mathcal{I}$

$\iff$ the operator system quotient $(\mathcal{S} \hat{\otimes}_{\min} \mathcal{A})/(\mathcal{S} \bar{\otimes} \mathcal{I})$ is completely order isomorphic to $\mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$

$\iff$ the operator space quotient $(\mathcal{S} \hat{\otimes}_{\min} \mathcal{A})/(\mathcal{S} \bar{\otimes} \mathcal{I})$ is completely isometric to $\mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$ [KPTT2, Theorem 5.1; Effros and Ruan 2000, Corollary 5.1.2]

$\iff$ the map $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I}$ is a complete quotient map with its kernel $\mathcal{S} \bar{\otimes} \mathcal{I}$

$\iff$ the map $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete quotient map (quotient lemma, [Effros and Ruan 2000, Lemma A.2.1])

$\iff$ the operator space quotient $(\mathcal{S} \otimes_{\min} \mathcal{A})/(\mathcal{S} \otimes \mathcal{I})$ is completely isometric to $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$

$\iff$ the operator system quotient $(\mathcal{S} \otimes_{\min} \mathcal{A})/(\mathcal{S} \otimes \mathcal{I})$ is completely order isomorphic to $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ [KPTT2, Theorem 5.1; Effros and Ruan 2000, Corollary 5.1.2]

$\iff$ $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ is a complete order quotient map. □

As pointed out in [KPTT2, Section 5], the framework of short exact sequences

$$0 \to \mathcal{S} \bar{\otimes} \mathcal{I} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A} \to \mathcal{S} \hat{\otimes}_{\min} \mathcal{A}/\mathcal{I} \to 0$$

with complete tensor products is inappropriate if we replace ideals in C^*-algebras and C^*-quotients by kernels in operator systems and operator system quotients. Even a one-dimensional operator system does not satisfy such exactness. Instead of short exact sequences with complete tensor products, we make a replacement in

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$$

with incomplete tensor products.

Theorem 5.3. *Let $\mathcal{S}$ be an operator system. Then, the following are equivalent:*

(i) *$\mathcal{S}$ is nuclear.*

(ii) *If $\Phi : \mathcal{T}_1 \to \mathcal{T}_2$ is a complete order quotient map for operator systems $\mathcal{T}_1$ and $\mathcal{T}_2$, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 \to \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

is a complete order quotient map.

(iii) *If $\Phi : \mathfrak{C}_I \to \mathcal{T}$ is a complete order quotient map for an operator system $\mathcal{T}$, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathfrak{C}_I \to \mathcal{S} \otimes_{\min} \mathcal{T}$$

is a complete order quotient map.

(iv) *If $\Phi : \mathcal{T} \to E$ is a complete order quotient map for an operator system $\mathcal{T}$ and a finite-dimensional operator system E, then*

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T} \to \mathcal{S} \otimes_{\min} E$$

is a complete order quotient map.

Proof. (i) $\Rightarrow$ (ii). Maximal tensor products of complete order quotient maps are still complete order quotient maps [Han 2011, Theorem 3.4]. Combining this with the hypothesis, we have a complete order quotient map

$$\mathrm{id}_{\mathcal{S}} \otimes \Phi : \mathcal{S} \otimes_{\min} \mathcal{T}_1 = \mathcal{S} \otimes_{\max} \mathcal{T}_1 \twoheadrightarrow \mathcal{S} \otimes_{\max} \mathcal{T}_2 = \mathcal{S} \otimes_{\min} \mathcal{T}_2$$

(ii) $\Rightarrow$ (i). The proof is motivated by [Effros and Ruan 2000, Theorem 14.6.1]. Taking $\mathcal{T}_1$ as a unital C^*-algebra and $\mathcal{T}_2$ as its C^*-quotient, we see that $\mathcal{S}$ is a 1-exact operator system by Proposition 5.2. We take a finite-dimensional operator subsystem E of $\mathcal{S}$ and $\varepsilon > 0$. Then, E is a 1-exact operator system [KPTT2, Corollary 5.8], or equivalently, a 1-exact operator space [KPTT2, Proposition 5.5]. Let $E \subset B(\ell_2)$ and $P_n : \ell_2 \to \ell_2^n$ be the projection given by $P_n((\lambda_i)_{i=1}^{\infty}) = (\lambda_1, \dots, \lambda_n)$. For sufficiently large n, the truncation mapping

$$\varphi : x \in E \to P_n x P_n \in M_n$$

is injective with $\|\varphi^{-1}\|_{cb} < 1 + \varepsilon'$ for $\varepsilon' = \varepsilon/(1 + 2 \dim E)$ by [Pisier 1995], [Effros and Ruan 2000, Theorem 14.4.1]. Note that φ is unital completely positive and φ^{-1} is unital self-adjoint. By [Brown and Ozawa 2008, Corollary B.11], there exists a unital completely positive map $\psi : \varphi(E) \to \mathcal{S}$ with $\|\varphi^{-1} - \psi\|_{cb} \le 2\varepsilon' \dim E$. Though [Brown and Ozawa 2008, Corollary B.11] assumes that the range space is a C^*-algebra, its proof still works more generally when the range space is an operator system.

By choosing a faithful state ω on M_n, we can regard the dual space M_n^* as an operator system. Since ω is faithful on any operator subsystem, $(\varphi(E)^*, \omega|_{\varphi(E)})$

is also an operator system. The element z in $\varphi(E)^* \otimes_{\min} \mathcal{S}$ corresponding to $\psi : \varphi(E) \to \mathcal{S}$ canonically is positive [KPTT2, Lemma 8.5]. Since the duals of the complete order embeddings between finite-dimensional operator systems are complete order quotient maps [Farenick and Paulsen 2012, Proposition 1.15], the restriction $R : M_n^* \to \varphi(E)^*$ is a complete order quotient map. By the hypothesis,

$$R \otimes \mathrm{id}_{\mathcal{S}} : M_n^* \otimes_{\min} \mathcal{S} \to \varphi(E)^* \otimes_{\min} \mathcal{S}$$

is also a complete order quotient map. There exists a positive lifting $\tilde{z} \in M_n^* \otimes_{\min} \mathcal{S}$ of $z + \varepsilon' \omega|_{\varphi(E)} \otimes 1_{\mathcal{S}}$. The completely positive map $\tilde{\psi} : M_n \to \mathcal{S}$ corresponding to $\tilde{z}$ satisfies

$$\|\psi - \tilde{\psi}|_{\varphi(E)}\|_{cb} \le \varepsilon'.$$

By the Arveson extension theorem, $\varphi : E \to M_n$ extends to a unital completely positive map $\tilde{\varphi} : \mathcal{S} \to M_n$. We thus obtain a diagram

$$\begin{tikzcd}
\mathcal{S} \arrow[ddr, "\tilde{\varphi}"'] & E \arrow[l, "\iota"'] \arrow[r, "\iota"] \arrow[d, "\varphi"'] & \mathcal{S} \\
 & \varphi(E) \arrow[ur, "\psi"] \arrow[d, "\iota"] & \\
 & M_n \arrow[uur, "\tilde{\psi}"'] &
\end{tikzcd}$$

where the ι denote inclusions.

It follows that

$$\begin{aligned}
\|\tilde{\psi} \circ \tilde{\varphi}(x) - x\| &\le \|\tilde{\psi} \circ \varphi(x) - \psi \circ \varphi(x)\| + \|\psi \circ \varphi(x) - \varphi^{-1} \circ \varphi(x)\| \\
&\le \varepsilon' \|x\| + 2\varepsilon' \dim E \|x\| \\
&= \varepsilon \|x\|.
\end{aligned}$$

for all $x \in E$. Considering the directed set

$$\{(E, \varepsilon) : E \text{ is a finite-dimensional operator subsystem of } \mathcal{S}, \varepsilon > 0\}$$

with the standard partial order, we can take nets of unital completely positive maps $\varphi_\lambda : \mathcal{S} \to M_{n_\lambda}$ and completely positive maps $\psi'_\lambda : M_{n_\lambda} \to \mathcal{S}$ such that $\psi'_\lambda \circ \varphi_\lambda$ converges to the map $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology.

Since each φ_λ is unital, $\psi'_\lambda(I_{n_\lambda})$ converges to $1_{\mathcal{S}}$. Let us choose a state ω_λ on M_{n_λ} and set

$$\psi_\lambda(A) = \frac{1}{\|\psi'_\lambda\|} \psi'_\lambda(A) + \omega_\lambda(A) \Big(1_{\mathcal{S}} - \frac{1}{\|\psi'_\lambda\|} \psi'_\lambda(I_{n_\lambda}) \Big).$$

Then $\psi_\lambda : M_{n_\lambda} \to \mathcal{S}$ is a unital completely positive map such that $\psi_\lambda \circ \varphi_\lambda$ converges to the map $\mathrm{id}_{\mathcal{S}}$ in the point-norm topology. By Corollary 3.2 of [Han and Paulsen 2011], $\mathcal{S}$ is nuclear.

(ii) $\Rightarrow$ (iii), (ii) $\Rightarrow$ (iv). Trivial.

(iii) $\Rightarrow$ (ii). Choose a positive element z in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$ and $\varepsilon > 0$. By Theorem 3.5, we can take a complete order quotient map $\Psi : \mathfrak{C}_I \to \mathcal{T}_1$. By the assumption, there exists a positive element $\tilde{z}$ in $\mathcal{S} \otimes_{\min} \mathfrak{C}_I$ satisfying $(\mathrm{id}_{\mathcal{S}} \otimes \Phi \circ \Psi)(\tilde{z}) = z + \varepsilon 1$. Thus, $\mathrm{id}_{\mathcal{S}} \otimes \Psi(\tilde{z})$ is a positive lifting of $z + \varepsilon 1$.

(iv) $\Rightarrow$ (ii). Choose a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{T}_2$. Take E as a finite-dimensional operator subsystem of $\mathcal{T}_2$ generated by $\{y_i : 1 \le i \le n\}$ and $\mathcal{T}$ as $\Phi^{-1}(E)$. □

Remark 5.4. The equivalence of (i) and (ii) was already discovered by Kavruk independently. The proof depends on Kavruk's result that is not yet published.

Corollary 5.5. *Suppose that E is a finite-dimensional operator system and ω is a faithful state on E. The following are equivalent*:

(i) *If $\varepsilon > 0$ and $\varphi : E \to \mathcal{S}/\mathcal{J}$ is a completely positive map for an operator system $\mathcal{S}$ and its kernel $\mathcal{J}$, then there exists a self-adjoint lifting $\tilde{\varphi} : E \to \mathcal{S}$ of φ such that $\tilde{\varphi} + \varepsilon\omega 1_{\mathcal{S}}$ is completely positive.*

(ii) *E is unitally completely order isomorphic to the direct sum of matrix algebras.*

Proof. (i) $\Rightarrow$ (ii). Condition (i) can be rephrased to state that

$$\mathrm{id}_{E^*} \otimes \pi : E^* \otimes_{\min} \mathcal{S} \to E^* \otimes_{\min} \mathcal{S}/\mathcal{J}$$

is a complete order quotient map for any operator system $\mathcal{S}$ and its kernel $\mathcal{J}$. Hence, E^* is a finite-dimensional nuclear operator system. Every finite-dimensional nuclear operator system is unitally completely order isomorphic to the direct sum of matrix algebras [Han and Paulsen 2011, Corollary 3.7]. Suppose that E^* is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$ for some $n, k_i \in \mathbb{N}$. Taking their duals, we see that E is completely order isomorphic to $\bigoplus_{i=1}^n M_{k_i}$. Suppose that the isomorphism maps the order unit of E to a matrix A in $\bigoplus_{i=1}^n M_{k_i}$. Then, A is positive definite. Let

$$A = U^* \operatorname{diag}(\lambda_1, \ldots, \lambda_m) U, \quad \lambda_i > 0,\ m = \sum_{i=1}^n k_i$$

be a diagonalization of A. The mapping

$$\alpha \in \bigoplus_{i=1}^n M_{k_i} \mapsto U^* \operatorname{diag}\big(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}\big)\, \alpha \operatorname{diag}\big(\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_m}\big) U \in \bigoplus_{i=1}^n M_{k_i}$$

is a complete order isomorphism that maps the identity matrix to A.

(ii) $\Rightarrow$ (i) We may assume that $E = \bigoplus_{i=1}^n M_{k_i}$. Let A be a density matrix of ω and $\lambda > 0$ be its smallest eigenvalue. Suppose that $z \in \bigoplus_{i=1}^n M_{k_i}(\mathcal{S}/\mathcal{J})$ is the direct sum of Choi matrices corresponding to the restrictions of φ on each blocks M_{k_i}. There exists a lifting $\tilde{z} \in \bigoplus_{i=1}^n M_{k_i}(\mathcal{S})$ of z such that $\tilde{z} + \varepsilon\lambda I_m \otimes 1_{\mathcal{S}}$ ($m = \sum_{i=1}^n k_i$) is positive. Let $\tilde{\varphi} : E \to \mathcal{S}$ be a self-adjoint map corresponding to $\tilde{z}$. Then we have

$$\tilde{\varphi} + \varepsilon\omega 1_{\mathcal{S}} = \tilde{\varphi} + \varepsilon \mathrm{tr}(\,\cdot\, A)1_{\mathcal{S}} \geq_{cp} \tilde{\varphi} + \varepsilon\lambda \mathrm{tr}(\,\cdot\,)1_{\mathcal{S}} \geq_{cp} 0. \qquad \square$$

In the last statement of Proposition 5.2, an operator systems $\mathcal{S}$ is fixed, and a C^*-algebra $\mathcal{A}$ and its closed ideal $\mathcal{I}$ are considered to be variables in

$$\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}.$$

In the following, we switch their roles. As a result, we give an operator system theoretic proof of the Effros–Haagerup lifting theorem [Effros and Haagerup 1985, Theorem 3.2].

Theorem 5.6. *Suppose that $\mathcal{A}$ is a unital C^*-algebra and $\mathcal{I}$ is its closed ideal. The following are equivalent*:

(i) $\mathrm{id}_{\mathcal{S}} \otimes \pi : \mathcal{S} \otimes_{\min} \mathcal{A} \to \mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any operator system $\mathcal{S}$.*

(ii) $\mathrm{id}_{\mathcal{B}} \otimes \pi : \mathcal{B} \otimes_{\min} \mathcal{A} \to \mathcal{B} \otimes_{\min} \mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any unital C^*-algebra $\mathcal{B}$.*

(iii) $\mathrm{id}_{B(H)} \otimes \pi : B(H) \otimes_{\min} \mathcal{A} \to B(H) \otimes_{\min} \mathcal{A}/\mathcal{I}$ *is a complete order quotient map for a separable Hilbert space H.*

(iv) $\mathrm{id}_E \otimes \pi : E \otimes_{\min} \mathcal{A} \to E \otimes_{\min} \mathcal{A}/\mathcal{I}$ *is a complete order quotient map for any finite-dimensional operator system E.*

(v) *The sequence*

$$0 \to \mathcal{B} \otimes_{C^*\min} \mathcal{I} \to \mathcal{B} \otimes_{C^*\min} \mathcal{A} \to \mathcal{B} \otimes_{C^*\min} \mathcal{A}/\mathcal{I} \to 0$$

is exact for any C^*-algebra $\mathcal{B}$.

(vi) *For any finite-dimensional operator system E, every completely positive map $\varphi : E \to \mathcal{A}/\mathcal{I}$ lifts to a completely positive map $\tilde{\varphi} : E \to \mathcal{A}$.*

(vii) *For any finite-dimensional operator system E, every unital completely positive map $\varphi : E \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : E \to \mathcal{A}$.*

(viii) *For any index set I, every unital completely positive finite rank map $\varphi : \mathfrak{C}_I \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_I \to \mathcal{A}$ with* $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\tilde{\varphi}$.

(ix) *Every unital completely positive finite rank map $\varphi : \mathfrak{C}_1 \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\varphi} : \mathfrak{C}_1 \to \mathcal{A}$ with* $\mathrm{Ker}\,\varphi = \mathrm{Ker}\,\tilde{\varphi}$.

Proof. (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) and (viii) $\Rightarrow$ (ix) are trivial. (vi) $\Rightarrow$ (vii) follows from [KPTT2, Remark 8.3]. (ii) $\Leftrightarrow$ (v) follows from Proposition 5.2. For (iii) $\Rightarrow$ (iv) and (iv) $\Rightarrow$ (i), it is sufficient to consider the first matrix level.

(iii) $\Rightarrow$ (iv). Let $E \subset B(H)$ for a separable Hilbert space H. Take a strictly positive element z in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ which is an operator subsystem of $B(H) \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the assumption, there exists a positive lifting $\tilde{z}$ in $B(H) \otimes_{\min} \mathcal{A}$. Let $\{x_i : 1 \leq i \leq k\}$ be a self-adjoint basis of E and $\{\hat{x}_i : 1 \leq i \leq k\}$ be its dual basis. Each functional $\hat{x}_i$ on E extends to a continuous self-adjoint functional on $B(H)$ which we still denote by $\hat{x}_i$. The map $P := \sum_{i=1}^k \hat{x}_i \otimes x_i : B(H) \to B(H)$ is a self-adjoint projection onto E. Since

$$(\mathrm{id}_{B(H)} - P) \otimes \pi(\tilde{z}) = z - (P \otimes \mathrm{id}_{\mathcal{A}/\mathcal{I}})(z) = 0,$$

we have

$$(\mathrm{id}_{B(H)} - P) \otimes \mathrm{id}_{\mathcal{A}}(\tilde{z}) \in B(H) \otimes \mathcal{I}.$$

We write

$$(\mathrm{id}_{B(H)} - P) \otimes \mathrm{id}_{\mathcal{A}}(\tilde{z}) = \sum_{i=1}^n b_i \otimes h_i, \quad b_i \in B(H)_{sa}, h_i \in \mathcal{I}_{sa}.$$

Each h_i is decomposed into $h_i = h_i^+ - h_i^-$ for $h_i^+, h_i^- \in \mathcal{I}^+$. From

$$\begin{aligned} 0 \leq \tilde{z} &= (P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n b_i \otimes h_i^+ - \sum_{i=1}^n b_i \otimes h_i^- \\ &\leq (P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^- \end{aligned}$$

and

$$(\mathrm{id}_{B(H)} \otimes \pi)\left((P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| \otimes h_i^-\right) = z,$$

we see that

$$(P \otimes \mathrm{id}_{\mathcal{A}})(\tilde{z}) + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^+ + \sum_{i=1}^n \|b_i\| 1 \otimes h_i^- \in E \otimes_{\min} \mathcal{A}$$

is a positive lifting of z.

(iv) $\Rightarrow$ (i). Take a positive element $z = \sum_{i=1}^n x_i \otimes y_i$ in $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$. Let E be a finite-dimensional operator system generated by $\{x_i : 1 \leq i \leq n\}$. Since $E \otimes_{\min} \mathcal{A}/\mathcal{I}$ is an operator subsystem of $\mathcal{S} \otimes_{\min} \mathcal{A}/\mathcal{I}$, we have z also positive in $E \otimes_{\min} \mathcal{A}/\mathcal{I}$. By the hypothesis, there exists a positive element $\tilde{z}$ in $E \otimes_{\min} \mathcal{A}$ such that $(\mathrm{id}_E \otimes \pi)(\tilde{z}) = z$. This element is also positive in $\mathcal{S} \otimes_{\min} \mathcal{A}$.

(iv) $\Leftrightarrow$ (vi). Suppose that E is a finite dimensional operator system and $\varphi : E \to \mathcal{A}/\mathcal{I}$ is a completely positive map. The element z in $E^* \otimes_{\min} \mathcal{A}/\mathcal{I}$ corresponding to φ

is positive. Since E is finite-dimensional, we have $E^* \otimes_{\min} \mathcal{A} = E^* \hat{\otimes}_{\min} \mathcal{A}$. The kernel $E^* \otimes \mathcal{I}$ of $\mathrm{id}_{E^*} \otimes \pi$ is completely order proximinal in $E^* \otimes_{\min} \mathcal{A}$ [KPTT2, Corollary 5.1.5]. By the hypothesis, z lifts to a positive element $\tilde{z}$ in $E^* \otimes_{\min} \mathcal{A}$. The map $\tilde{\varphi} : E \to \mathcal{A}$ corresponding to $\tilde{z}$ is completely positive. The converse is merely the reverse of the argument.

(vii) $\Rightarrow$ (vi). The inclusion $\iota : \varphi(E) + \mathbb{C}1_{\mathcal{A}/\mathcal{I}} \subset \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\tilde{\iota} : \varphi(E) + \mathbb{C}1_{\mathcal{A}/\mathcal{I}} \to \mathcal{A}$. The map $\tilde{\iota} \circ \varphi$ is the completely positive lifting of φ.

(vii) $\Rightarrow$ (viii). Let $Q : \mathfrak{C}_I \to \mathfrak{C}_I / \operatorname{Ker} \varphi$ be a quotient map. We have a factorization $\varphi = \psi \circ Q$ for $\psi : \mathfrak{C}_I / \operatorname{Ker} \varphi \to \mathcal{A}/\mathcal{I}$. By the hypothesis, ψ lifts to a unital completely positive map $\tilde{\psi} : \mathfrak{C}_I / \operatorname{Ker} \varphi \to \mathcal{A}$. Then $\tilde{\psi} \circ Q$ is a unital completely positive lifting of φ and their kernels coincide.

(ix) $\Rightarrow$ (vii). By Theorem 3.6, there exists a complete order quotient map $\Phi : \mathfrak{C}_1 \to E$. The map $\varphi \circ \Phi : \mathfrak{C}_1 \to \mathcal{A}/\mathcal{I}$ lifts to a unital completely positive map $\psi : \mathfrak{C}_1 \to \mathcal{A}$ such that their kernels coincide. Since $\operatorname{Ker} \Phi \subset \operatorname{Ker} \psi$, we get that ψ induces a map $\tilde{\varphi} : E \to \mathcal{A}/\mathcal{I}$ which is a unital completely positive lifting of φ. □

The following theorem can be regarded as an operator system version of the quotient lemma.

Theorem 5.7. *Suppose that $\Phi : \mathcal{S} \to \mathcal{T}$ is a unital completely positive surjection for operator systems $\mathcal{S}$ and $\mathcal{T}$. Let S_0 be an operator subsystem that is dense in $\mathcal{S}$, $\mathcal{T}_0 := \Phi(\mathcal{S}_0)$, and $\Phi_0 = \Phi|_{\mathcal{S}_0} : \mathcal{S}_0 \to \mathcal{T}_0$ be the surjective restriction. Then, the following are equivalent*:

- (i) *$\Phi : \mathcal{S} \to \mathcal{T}$ is a complete order quotient map and for any $\varepsilon > 0$, $k \in \mathbb{N}$ and a self-adjoint element $x \in \operatorname{Ker} \Phi_k$, there exists a self-adjoint element $x_0 \in \operatorname{Ker}(\Phi_0)_k$ such that $x_0 + \varepsilon 1 \geq x$.*
- (ii) *$\Phi_0 : \mathcal{S}_0 \to \mathcal{T}_0$ is a complete order quotient map.*

Proof. The following arguments apply to all matricial levels.

(i) $\Rightarrow$ (ii). Choose $\varepsilon > 0$ and $\Phi_0(y_0) \in \mathcal{T}_0^+$ for a self-adjoint $y_0 \in \mathcal{S}_0$. By the hypothesis, there exist self-adjoint $x \in \operatorname{Ker} \Phi$ and $x_0 \in \operatorname{Ker} \Phi_0$ such that

$$y_0 + \tfrac{\varepsilon}{2} 1 + x \in \mathcal{S}^+ \quad \text{and} \quad x \leq x_0 + \tfrac{\varepsilon}{2} 1.$$

It follows that

$$y_0 + \varepsilon 1 + x_0 \geq y_0 + \tfrac{\varepsilon}{2} 1 + x \geq 0.$$

(ii) $\Rightarrow$ (i). Take $\varepsilon > 0$ and a self-adjoint element x in $\operatorname{Ker} \Phi$. Since $\mathcal{S}_0$ is dense in $\mathcal{S}$, there exists a self-adjoint element y_0 in $\mathcal{S}_0$ such that

$$x - \tfrac{\varepsilon}{3} 1 \leq y_0 \leq x + \tfrac{\varepsilon}{3} 1,$$

which implies that

$$\Phi_0(-y_0+\tfrac{\varepsilon}{3}1)=\Phi(-y_0+x+\tfrac{\varepsilon}{3}1)\in\mathcal{T}^+\cap\mathcal{T}_0=\mathcal{T}_0^+.$$

There exists an element x_0 in $\operatorname{Ker}\Phi_0$ such that $-y_0+\frac{2}{3}\varepsilon 1+x_0\geq 0$. From

$$x-\tfrac{\varepsilon}{3}1\leq y_0\leq\tfrac{2}{3}\varepsilon 1+x_0,$$

it follows that $x\leq\varepsilon 1+x_0$.

Now let $\Phi(y)\in\mathcal{T}^+$ for a self-adjoint $y\in\mathcal{S}$. There exists an element y_0 in $\mathcal{S}_0$ such that

$$y-\tfrac{\varepsilon}{3}1\leq y_0\leq y+\tfrac{\varepsilon}{3}1,$$

which implies that

$$\Phi_0(y_0+\tfrac{\varepsilon}{3}1)\geq\Phi(y)\geq 0.$$

There exists an element $x_0\in\operatorname{Ker}\Phi_0$ such that $y_0+x_0+\frac{2}{3}\varepsilon 1\geq 0$. It follows that

$$y+x_0+\varepsilon 1\geq y_0+x_0+\tfrac{2}{3}\varepsilon 1\geq 0. \qquad\square$$

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REMARKS ON QUANTUM UNIPOTENT SUBGROUPS AND THE DUAL CANONICAL BASIS

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We prove the tensor product decomposition of the half of the quantized universal enveloping algebra associated with a Weyl group element which was conjectured by Berenstein and Greenstein (preprint, 2014, arXiv 1411.1391; see Conjecture 5.5) using the theory of the dual canonical basis. In fact, based on the compatibility between the decomposition and the dual canonical basis, a weak multiplicity-free property between the factors is established.

1. Introduction

Let $\mathfrak{g}$ be a symmetrizable Kac–Moody Lie algebra and w be a Weyl group element. In [Kimura 2012], we studied the compatibility of the dual canonical basis and the quantum coordinate ring of the unipotent subgroup associated with a finite subset $\Delta_+ \cap w\Delta_-$, where Δ_+ (resp. Δ_-) is the set of positive (resp. negative) roots of $\mathfrak{g}$. The purpose of this paper is to study the compatibility of the dual canonical basis and the "quantum coordinate ring" of the pro-unipotent subgroup associated with a cofinite subset $\Delta_+ \cap w\Delta_+$.

Let $\boldsymbol{U}_q(\mathfrak{g})$ be the quantized enveloping algebra and

$$\boldsymbol{U}_q(\mathfrak{g}) \simeq \boldsymbol{U}_q^+(\mathfrak{g}) \otimes \boldsymbol{U}_q^0(\mathfrak{g}) \otimes \boldsymbol{U}_q^-(\mathfrak{g})$$

be its triangular decomposition. Let $\boldsymbol{U}_q^{\geq 0}(\mathfrak{g})$ be the subalgebra generated by $\boldsymbol{U}_q^+(\mathfrak{g})$ and $\boldsymbol{U}_q^0(\mathfrak{g})$. Let $T_w = T_{i_1} T_{i_2} \cdots T_{i_\ell} : \boldsymbol{U}_q \to \boldsymbol{U}_q$ be Lusztig's symmetry associated with a Weyl group element w, where $\boldsymbol{i} = (i_1, \dots, i_\ell)$ is a reduced word of w. It is known that $T_w \in \operatorname{Aut}(\boldsymbol{U}_q(\mathfrak{g}))$ does not depend on the choice of reduced word.

Berenstein and Greenstein [2014, Conjecture 5.5] conjectured the following tensor product decomposition of the half $\boldsymbol{U}_q^-$ in general. We show the multiplicity-free property of the multiplications of the dual canonical basis elements between the

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finite part and the cofinite part. We also prove the decomposition in the dual integral form $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ of the Lusztig integral form $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ with respect to Kashiwara's nondegenerate bilinear form.

Theorem 1.1. (1) *For a Weyl group element $w \in W$, multiplication in $\boldsymbol{U}_q^-$ defines an isomorphism of vector spaces over $\mathbb{Q}(q)$:*

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \otimes (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \xrightarrow{\sim} \boldsymbol{U}_q^-.$$

(2) *For a Weyl group element $w \in W$, we set*

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0})_{\mathcal{A}}^{\mathrm{up}} := \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w \boldsymbol{U}_q^{\geq 0}$$

and

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)_{\mathcal{A}}^{\mathrm{up}} := \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_w \boldsymbol{U}_q^-.$$

Then multiplication in $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ defines an isomorphism of free $\mathcal{A}$-modules:

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0})_{\mathcal{A}}^{\mathrm{up}} \otimes_{\mathcal{A}} (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)_{\mathcal{A}}^{\mathrm{up}} \xrightarrow{\sim} \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}.$$

Remark 1.2. (1) Theorem 1.1(1) can be shown directly in finite-type cases using the Poincaré–Birkhoff–Witt bases of $\boldsymbol{U}_q^-$ (see [Berenstein and Greenstein 2014, Proposition 5.3]). Hence it is a new result only in infinite-type cases.

(2) For the proof of Theorem 1.1(1), we use the dual canonical bases and the multiplication formula for them; in particular we will prove Theorem 1.1(2). After finishing this work, the author was informed of a proof which does not involve the theory of the dual canonical basis by Toshiyuki Tanisaki [2015, Proposition 2.10], who also proved the tensor product decomposition in Lusztig form, De Concini–Kac form and De Concini–Procesi form.

We note that the De Concini–Kac form (resp. De Concini–Procesi form) is related to the dual integral form of Lusztig's integral form with respect to the Kashiwara (resp. Lusztig) nondegenerate bilinear form on $\boldsymbol{U}_q^-$. Since the multiplicative structure of the dual canonical basis does not depend on the choice of nondegenerate bilinear form (and hence the definition of the dual canonical basis), our argument yields results for the tensor product decompositions of the De Concini–Kac form and the De Concini–Procesi form.

Remark 1.3. We note that the fact that $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$ has a Poincaré–Birkhoff–Witt basis was shown by Beck, Chari and Pressley [Beck et al. 1999, Proposition 2.3] in general. (Throughout that paper, it is assumed that the generalized Cartan matrix is of symmetric affine type, but it should be noted that the assumption is not used in the proof of [Beck et al. 1999, Proposition 2.3]. For more details, see Theorem 2.18). The injectivity in Theorem 1.1 can be easily proved by the linear independence of the Poincaré–Birkhoff–Witt monomials (see [Lusztig 1993, Theorem 40.2.1(a)]) and the triangular decomposition of the quantized enveloping algebra (see [Lusztig 1993, Section 3.2]). Hence the nontrivial assertion is the surjectivity in Theorem 1.1.

Theorem 1.4. (1) *For a Weyl group element $w \in W$ and for a reduced word $\boldsymbol{i} = (i_1, \dots, i_\ell)$ of w, we have*

$$\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

(2) *We have that $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ is compatible with the dual canonical basis; that is, $\boldsymbol{B}^{\mathrm{up}} \cap \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ is a $\mathbb{Q}(q)$-basis of $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$. In fact, there exists a subset $\mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \subset \mathscr{B}(\infty)$ such that*

$$\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^- = \bigoplus_{b \in \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

Using the theory of crystal bases, we can obtain the characterization of the subset $\mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-)$. For $w \in W$, we have the decomposition theorem of the crystal basis $\mathscr{B}(\infty)$ of $\boldsymbol{U}_q^-$ associated with a Weyl group element (and a reduced word) and the corresponding multiplication formula. We consider the map Ω_w associated with a Weyl group element which was introduced by Saito [1994] (and Baumann, Kamnitzer and Tingley [Baumann et al. 2014]):

$$\Omega_w := (\tau_{\leq w}, \tau_{>w}) : \mathscr{B}(\infty) \to \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \times \mathscr{B}(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-),$$

where $\tau_{\leq w}(b)$ and $\tau_{>w}(b)$ are defined by crystal bases as follows:

$$\begin{aligned}
L(b, \boldsymbol{i}) &:= \big(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\hat{\sigma}_{i_1}^* b), \dots, \varepsilon_{i_\ell}(\hat{\sigma}_{i_{\ell-1}}^* \cdots \hat{\sigma}_{i_1}^* b)\big) \in \mathbb{Z}_{\geq 0}^\ell, \\
b(\boldsymbol{c}, \boldsymbol{i}) &:= f_{i_1}^{(c_1)} T_{i_1}(f_{i_2}^{(c_2)}) \cdots T_{i_1} \cdots T_{i_{\ell-1}}(f_{i_\ell}^{(c_\ell)}) \bmod q \mathscr{L}(\infty) \in \mathscr{B}(\infty), \\
\tau_{\leq \boldsymbol{i}}(b) &:= b(L(b, \boldsymbol{i}), \boldsymbol{i}) \in \mathscr{B}(\infty), \\
\tau_{> \boldsymbol{i}}(b) &:= \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}_{i_\ell}^* \cdots \hat{\sigma}_{i_1}^* b \in \mathscr{B}(\infty).
\end{aligned}$$

The following is the multiplicity-free result of the multiplication of the dual canonical basis elements in the finite part and the cofinite part.

Theorem 1.5. *Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \dots, i_\ell)$ be a reduced word of w. For a crystal basis element $b \in \mathscr{B}(\infty)$, we have*

$$G^{\mathrm{up}}(\tau_{\leq \boldsymbol{i}}(b)) G^{\mathrm{up}}(\tau_{> \boldsymbol{i}}(b)) \in G^{\mathrm{up}}(b) + \sum_{L(b', \boldsymbol{i}) < L(b, \boldsymbol{i})} q \mathbb{Z}[q] G^{\mathrm{up}}(b'),$$

where $L(b', \boldsymbol{i}) < L(b, \boldsymbol{i})$ in the left lexicographic order on $\mathbb{Z}_{\geq 0}^\ell$ associated with a reduced word $\boldsymbol{i}$.

Using induction on the lexicographic order on each root space, we obtain the surjectivity in Theorem 1.1(2). In particular, Theorem 1.1(1) can be shown.

Since the subalgebras $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$ and $\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-$ are compatible with the dual canonical basis and since the dual bar-involution σ which characterizes the dual canonical basis is a (twisted) anti-involution, we obtain the tensor product factorization in the opposite order.

Corollary 1.6. *For a Weyl group element $w \in W$, multiplication in $\boldsymbol{U}_q^-$ defines an isomorphism of vector spaces*:

$$(\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^-) \otimes (\boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}) \xrightarrow{\sim} \boldsymbol{U}_q^-.$$

2. Review of quantum unipotent subgroups and the dual canonical basis

2A. ***Quantum universal enveloping algebra.*** In this subsection, we give a brief review of the definition of quantum universal enveloping algebra. The reader is referred to [Kashiwara 1991; 1993a; 1993b] for more details.

2A1. Let I be a finite index set.

Definition 2.1. A *root datum* is a quintuple $(A, P, \Pi, P^\vee, \Pi^\vee)$ which consists of

(1) a square matrix $(a_{ij})_{i,j \in I}$, called the symmetrizable generalized Cartan matrix, that is, an I-indexed $\mathbb{Z}$-valued matrix which satisfies
   (a) $a_{ii} = 2$ for $i \in I$,
   (b) $a_{ij} \in \mathbb{Z}_{\leq 0}$ for $i \neq j$,
   (c) there exists a diagonal matrix $\mathrm{diag}(d_i)_{i \in I}$ such that $(d_i a_{ij})_{i,j \in I}$ is symmetric and d_i are positive integers;

(2) P : a free abelian group (the weight lattice);

(3) $\Pi = \{\alpha_i \mid i \in I\} \subset P$: the set of simple roots such that $\Pi \subset P \otimes_{\mathbb{Z}} \mathbb{Q}$ is linearly independent;

(4) $P^\vee = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$: the dual lattice (the coweight lattice) of P with perfect pairing $\langle \cdot , \cdot \rangle : P^\vee \otimes_{\mathbb{Z}} P \to \mathbb{Z}$;

(5) $\Pi^\vee = \{h_i \mid i \in I\} \subset P^\vee$: the set of simple coroots, satisfying
   (a) $a_{ij} = \langle h_i, \alpha_j \rangle$ for all $i, j \in I$,
   (b) there exists $\{\Lambda_i\}_{i \in I} \subset P$, called the set of fundamental weights, satisfying $\langle h_i, \Lambda_j \rangle = \delta_{ij}$ for $i, j \in I$.

We say $\Lambda \in P$ is *dominant* if $\langle h_i, \Lambda \rangle \geq 0$ for any $i \in I$ and denote by P_+ the set of dominant integral weights. Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset P$ be the root lattice. Let $Q_\pm = \pm \sum_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set $|\xi| = \sum_{i \in I} \xi_i$.

2A2. Let $(A, P, \Pi, P^\vee, \Pi^\vee)$ be a root datum. We set $\mathfrak{h} := P^\vee \otimes_{\mathbb{Z}} \mathbb{C}$. A triple $(\mathfrak{h}, \Pi, \Pi^\vee)$ is called a Cartan datum or a realization of a generalized Cartan matrix A.

It is known that there exists a symmetric bilinear form $(\cdot , \cdot)$ on $\mathfrak{h}^*$ satisfying

(1) $(\alpha_i, \alpha_i) = d_i a_{ij}$,

(2) $\langle h_i, \lambda \rangle = 2(\alpha_i, \lambda)/(\alpha_i, \alpha_i)$ for $i \in I$ and $\lambda \in \mathfrak{h}^*$.

Definition 2.2. Let $\mathfrak{g}$ be the *symmetrizable Kac–Moody Lie algebra* associated with a realization $(\mathfrak{h}, \Pi, \Pi^\vee)$ of a symmetrizable generalized Cartan matrix $A = (a_{ij})_{i,j\in I}$, that is, a Lie algebra which is generated by $\{e_i\}_{i\in I} \cup \{f_i\}_{i\in I} \cup \mathfrak{h}$ with the following relations:

(1) $[h_1, h_2] = 0$ for $h_1, h_2 \in \mathfrak{h}$,

(2) $[h, e_i] = \langle h, \alpha_i \rangle e_i$ and $[h, f_i] = -\langle h, \alpha_i \rangle f_i$ for $h \in \mathfrak{h}$ and $i \in I$,

(3) $[e_i, f_j] = \delta_{ij} \alpha_i^\vee$ for $i, j \in I$,

(4) $\mathrm{ad}(e_i)^{1-a_{ij}}(e_j) = 0$ and $\mathrm{ad}(f_i)^{1-a_{ij}}(f_j) = 0$ for $i \neq j$, where $\mathrm{ad}(x)(y) = [x, y]$.

Let $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) be the Lie subalgebra which is generated by $\{e_i\}_{i\in I}$ (resp. $\{f_i\}_{i\in I}$). We have the triangular decomposition and the root space decomposition

$$\mathfrak{g} = \mathfrak{n}_- \oplus \mathfrak{h} \oplus \mathfrak{n}_+ = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathfrak{h}^* \setminus \{0\}} \mathfrak{g}_\alpha,$$

where $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \mid [h, x] = \langle h, \alpha \rangle x \ \ \forall h \in \mathfrak{h}\}$. The set $\Delta := \{\alpha \in \mathfrak{h}^* \setminus \{0\} \mid \mathfrak{g}_\alpha \neq 0\}$ is called the root system of $\mathfrak{g}$.

2A3. We fix a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$. We introduce an indeterminate q. For $i \in I$, we set $q_i = q^{d_i}$. For $\xi = \sum \xi_i \alpha_i \in Q$, we set $q_\xi := \prod_{i \in I} q_i^{\xi_i}$.

For $n \in \mathbb{Z}$ and $i \in I$, we set

$$[n]_i := \frac{q_i^n - q_i^{-n}}{q_i - q_i^{-1}}$$

and $[n]_i! = [n]_i [n-1]_i \cdots [1]_i$ for $n > 0$ and $[0]! = 1$.

Definition 2.3. The *quantized enveloping algebra* $\boldsymbol{U}_q(\mathfrak{g})$ associated with a root datum $(A, P, \Pi, P^\vee, \Pi^\vee)$ is the $\mathbb{Q}(q)$-algebra which is generated by $\{e_i\}_{i\in I}$, $\{f_i\}_{i\in I}$ and $\{q^h \mid h \in P^\vee\}$ with the following relations:

(1) $q^0 = 1$ and $q^{h+h'} = q^h q^{h'}$ for $h, h' \in P^\vee$,

(2) $q^h e_i q^{-h} = q^{\langle h, \alpha_i \rangle} e_i$ and $q^h f_i q^{-h} = q^{-\langle h, \alpha_i \rangle} f_i$ for $i \in I$ and $h \in P^\vee$,

(3) $e_i f_j - f_j e_i = \delta_{ij}(k_i - k_i^{-1})/(q_i - q_i^{-1})$, where $k_i = q^{d_i h_i}$,

(4) $$\sum_{k=0}^{1-a_{ij}} (-1)^k e_i^{(1-a_{ij}-k)} e_j e_i^{(k)} = \sum_{k=0}^{1-a_{ij}} (-1)^k f_i^{(1-a_{ij}-k)} f_j f_i^{(k)} = 0$$ (q-Serre relations),

where $e_i^{(k)} = e_i^k/[k]_i!$ and $f_i^{(k)} = f_i^k/[k]_i!$ for $i \in I$ and $k \in \mathbb{Z}_{>0}$.

2A4. Let $\boldsymbol{U}_q^0$ be the subalgebra of $\boldsymbol{U}_q(\mathfrak{g})$ which is generated by $\{q^h \mid h \in P^\vee\}$; it is isomorphic to the group algebra

$$\mathbb{Q}(q)[P^\vee] := \bigoplus_{h \in P^\vee} \mathbb{Q}(q) q^h$$

over $\mathbb{Q}(q)$. For $\xi = \sum_{i \in I} \xi_i \alpha_i \in Q$, we set

$$k_\xi := \prod_{i \in I} k_i^{\xi_i} = \prod_{i \in I} q^{d_i \xi_i h_i}.$$

Let $\boldsymbol{U}_q^+$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{e_i\}_{i \in I}$, let $\boldsymbol{U}_q^-$ be the $\mathbb{Q}(q)$-subalgebra generated by $\{f_i\}_{i \in I}$, let $\boldsymbol{U}_q^{\geq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\boldsymbol{U}_q^0$ and $\boldsymbol{U}_q^+$, and let $\boldsymbol{U}_q^{\leq 0}$ be the $\mathbb{Q}(q)$-subalgebra generated by $\boldsymbol{U}_q^0$ and $\boldsymbol{U}_q^-$.

Theorem 2.4 [Lusztig 1993, Corollary 3.2.5]. *The multiplication of $\boldsymbol{U}_q$ induces the triangular decomposition of $\boldsymbol{U}_q(\mathfrak{g})$ as vector spaces over $\mathbb{Q}(q)$:*

$$\boldsymbol{U}_q(\mathfrak{g}) \cong \boldsymbol{U}_q^+ \otimes \boldsymbol{U}_q^0 \otimes \boldsymbol{U}_q^- \cong \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^0 \otimes \boldsymbol{U}_q^+. \tag{2-1}$$

2A5. For $\xi \in \pm Q$, we define $\boldsymbol{U}_q^\pm(\mathfrak{g})_\xi$ by

$$\boldsymbol{U}_q^\pm(\mathfrak{g})_\xi := \{x \in \boldsymbol{U}_q^\pm(\mathfrak{g}) \mid q^h x q^{-h} = q^{\langle h, \xi \rangle} x \text{ for } h \in P^\vee\}. \tag{2-2}$$

Then we have a root space decomposition

$$\boldsymbol{U}_q^\pm(\mathfrak{g}) = \bigoplus_{\xi \in Q_\pm} \boldsymbol{U}_q^\pm(\mathfrak{g})_\xi.$$

An element $x \in \boldsymbol{U}_q^\pm(\mathfrak{g})$ is called homogeneous if $x \in \boldsymbol{U}_q^\pm(\mathfrak{g})_\xi$ for some $\xi \in Q_\pm$.

2A6. We define a $\mathbb{Q}(q)$-algebra anti-involution $* : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$ by

$$*(e_i) = e_i, \quad *(f_i) = f_i, \quad *(q^h) = q^{-h}. \tag{2-3}$$

We call this the *star involution*.

We define a $\mathbb{Q}$-algebra automorphism $\overline{} : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$ by

$$\overline{e_i} = e_i, \quad \overline{f_i} = f_i, \quad \overline{q} = q^{-1}, \quad \overline{q^h} = q^{-h}. \tag{2-4}$$

We call this the *bar involution*.

These two involutions preserve $\boldsymbol{U}_q^+(\mathfrak{g})$ and $\boldsymbol{U}_q^-(\mathfrak{g})$, and we have $\overline{} \circ * = * \circ \overline{}$.

2A7. In this article, we choose the following comultiplication $\Delta = \Delta_-$ on $\boldsymbol{U}_q(\mathfrak{g})$:

$$\Delta(q^h) = q^h \otimes q^h, \quad \Delta(e_i) = e_i \otimes k_i^{-1} + 1 \otimes e_i, \quad \Delta(f_i) = f_i \otimes 1 + k_i \otimes f_i. \tag{2-5}$$

2A8. We define a $\mathbb{Q}(q)$-algebra structure on $\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-$ by

$$(x_1 \otimes y_1)(x_2 \otimes y_2) = q^{-(\mathrm{wt}(x_2),\mathrm{wt}(y_1))} x_1 x_2 \otimes y_1 y_2, \tag{2-6}$$

where x_i, y_i $(i = 1, 2)$ are homogeneous elements. Let $r = r_- : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-$ be the $\mathbb{Q}(q)$-algebra homomorphism defined by

$$r(f_i) = f_i \otimes 1 + 1 \otimes f_i \quad (i \in I).$$

We call this the twisted comultiplication. Then it is known that there exists a unique $\mathbb{Q}(q)$-valued nondegenerate symmetric bilinear form $(\cdot,\cdot) : \boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^- \to \mathbb{Q}(q)$ with the following properties:

$$(1,1)=1,\ (f_i, f_j)=\delta_{ij},\ (r(x), y_1 \otimes y_2)=(x, y_1 y_2),\ (x_1 \otimes x_2, r(y))=(x_1 x_2, y)$$

for homogeneous $x, y_1, y_2 \in \boldsymbol{U}_q^-$, where the form

$$(\cdot \otimes \cdot, \cdot \otimes \cdot) : (\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-) \otimes (\boldsymbol{U}_q^- \otimes \boldsymbol{U}_q^-) \to \mathbb{Q}(q)$$

is defined by $(x_1 \otimes x_2, y_1 \otimes y_2) = (x_1, y_1)(x_2 \otimes y_2)$ for $x_1, x_2, y_1, y_2 \in \boldsymbol{U}_q^-$.

2A9. For $i \in I$, we define the unique $\mathbb{Q}(q)$-linear map ${}_ir : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ (resp. $r_i : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$) by

$$({}_ir(x), y) = (x, f_i y),$$
$$(r_i(x), y) = (x, y f_i).$$

Lemma 2.5 [Lusztig 1993, Section 1.2.13]. *For $x, y \in \boldsymbol{U}_q^-$, we have q-boson relations*:

$${}_ir(xy) = {}_ir(x)y + q^{(\mathrm{wt}\, x, \alpha_i)} x\, {}_ir(y),$$
$$r_i(xy) = q^{(\mathrm{wt}\, y, \alpha_i)} r_i(x) y + x\, r_i(y).$$

Lemma 2.6 [Lusztig 1993, Proposition 3.1.6]. *We have*

$$[e_i, x] = \frac{r_i(x)k_i - k_i^{-1}\, {}_ir(x)}{q_i - q_i^{-1}} \quad \textit{for } x \in \boldsymbol{U}_q^-. \tag{2-7}$$

Using the q-boson relation, we obtain the following result.

Lemma 2.7 [Lusztig 1993, Lemma 38.1.2, Proposition 38.1.6]. *For each $i \in I$, any element $x \in \boldsymbol{U}_q^-$ can be written uniquely as*

$$x = \sum_{c \geq 0} f_i^{(c)} x_c \quad \textit{with } x_c \in \mathrm{Ker}({}_ir).$$

2B. ***Canonical basis and dual canonical basis.*** We give a brief review of the theory of the canonical basis and the dual canonical basis following Kashiwara. Note that Kashiwara called them the lower global basis and the upper global basis.

2B1. We define $\mathbb{Q}$-subalgebras $\mathcal{A}_0$, $\mathcal{A}_\infty$ and $\mathcal{A}$ of $\mathbb{Q}(q)$ by

$$\begin{aligned}\mathcal{A}_0 &:= \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = 0\},\\ \mathcal{A}_\infty &:= \{ f \in \mathbb{Q}(q) \mid f \text{ is regular at } q = \infty\},\\ \mathcal{A} &:= \mathbb{Q}[q^{\pm 1}].\end{aligned}$$

2B2. We introduce the crystal basis of $\boldsymbol{U}_q^-$. For more details, see [Kashiwara 1991, Section 3]. We define the Kashiwara operators $\tilde{e}_i$ and $\tilde{f}_i$ on $\boldsymbol{U}_q^-$ by

$$\begin{aligned}\tilde{e}_i x &= \sum_{c \geq 1} f_i^{(c-1)} x_c,\\ \tilde{f}_i x &= \sum_{c \geq 0} f_i^{(c+1)} x_c,\end{aligned}$$

and we set

$$\mathscr{L}(\infty) := \sum_{\substack{\ell \geq 0\\ i_1,\dots,i_\ell \in I}} \mathcal{A}_0 \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \subset \boldsymbol{U}_q^-,$$

$$\mathscr{B}(\infty) := \{ \tilde{f}_{i_1} \cdots \tilde{f}_{i_\ell} 1 \bmod q\mathscr{L}(\infty) \mid l \geq 0, i_1, \dots, i_\ell \in I\} \subset \mathscr{L}(\infty)/q\mathscr{L}(\infty).$$

Then $\mathscr{L}(\infty)$ is an $\mathcal{A}_0$-lattice with $\mathbb{Q}(q) \otimes_{\mathcal{A}_0} \mathscr{L}(\infty) \simeq \boldsymbol{U}_q^-$ that is stable under $\tilde{e}_i$ and $\tilde{f}_i$, and $\mathscr{B}(\infty)$ is a $\mathbb{Q}$-basis of $\mathscr{L}(\infty)/q\mathscr{L}(\infty)$. We also have induced maps $\tilde{f}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$ and $\tilde{e}_i : \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}$ with the property that $\tilde{f}_i \tilde{e}_i b = b$ for $b \in \mathscr{B}(\infty)$ with $\tilde{e}_i b \neq 0$. We call $(\mathscr{B}(\infty), \mathscr{L}(\infty))$ the (lower) crystal basis of $\boldsymbol{U}_q^-$ and call $\mathscr{L}(\infty)$ the (lower) crystal lattice. We denote $1 \bmod q\mathscr{L}(\infty)$ by u_∞.

2B3. It is also known that the star involution $* : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ induces an $\mathcal{A}_0$-linear isomorphism $* : \mathscr{L}(\infty) \to \mathscr{L}(\infty)$ and a bijection $* : \mathscr{B}(\infty) \to \mathscr{B}(\infty)$; see [Kashiwara 1991, Proposition 5.2.4; 1993b, Theorem 2.1.1]. We set

$$\begin{aligned}\tilde{f}_i^* &:= * \circ \tilde{f}_i \circ * : \mathscr{B}(\infty) \to \mathscr{B}(\infty),\\ \tilde{e}_i^* &:= * \circ \tilde{e}_i \circ * : \mathscr{B}(\infty) \to \mathscr{B}(\infty) \sqcup \{0\}.\end{aligned}$$

2B4. Let $\overline{\mathscr{L}(\infty)} = \{\overline{x} \mid x \in \mathscr{L}(\infty)\}$. Then the natural map

$$\mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is an isomorphism of $\mathbb{Q}$-vector spaces. Let G^{low} be the inverse of this isomorphism. The image

$$\boldsymbol{B}^{\mathrm{low}} = \{ G^{\mathrm{low}}(b) \mid b \in \mathscr{B}(\infty)\} \subset \mathscr{L}(\infty) \cap \overline{\mathscr{L}(\infty)} \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$$

is an $\mathcal{A}$-basis of $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}$ and is called the canonical basis or the lower global basis of $\boldsymbol{U}_q^-$.

2B5. The important property of the canonical basis is the following compatibility with the left and right ideals which are generated by Chevalley generators $\{f_i\}_{i\in I}$.

Theorem 2.8 [Lusztig 1993, Theorems 14.3.2 and 14.4.3; Kashiwara 1991, Theorem 7]. *For $i \in I$ and $n \geq 1$, $f_i^n \boldsymbol{U}_q^-$ and $\boldsymbol{U}_q^- f_i^n$ are compatible with the canonical basis; that is, $f_i^n \boldsymbol{U}_q^- \cap \boldsymbol{B}^{\mathrm{low}}$ (resp. $\boldsymbol{U}_q^- f_i^n \cap \boldsymbol{B}^{\mathrm{low}}$) is a basis of $f_i^n \boldsymbol{U}_q^-$ (resp. $\boldsymbol{U}_q^- f_i^n$). In fact, we have*

$$f_i^n \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i(b)\geq m}} \mathcal{A}G^{\mathrm{low}}(b),$$

$$\boldsymbol{U}_q^- f_i^n \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}} = \bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i^*(b)\geq m}} \mathcal{A}G^{\mathrm{low}}(b).$$

2B6. Let $\sigma : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^-$ be the $\mathbb{Q}$-linear map defined by

$$(\sigma(x), y) = \overline{(x, \overline{y})}$$

for arbitrary $x, y \in \boldsymbol{U}_q^-$. Let $\sigma(\mathscr{L}(\infty)) := \{\sigma(x) \mid x \in \mathscr{L}(\infty)\}$ and set the dual integral form:

$$\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} := \{x \in \boldsymbol{U}_q^- \mid (x, \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}) \subset \mathcal{A}\}.$$

$\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ has an $\mathcal{A}$-subalgebra of $\boldsymbol{U}_q^-$. The natural map

$$\mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \to \mathscr{L}(\infty)/q\mathscr{L}(\infty)$$

is also an isomorphism of $\mathbb{Q}$-vector spaces, so let G^{up} be the inverse of the above isomorphism. Then

$$\boldsymbol{B}^{\mathrm{up}} = \{G^{\mathrm{up}}(b) \mid b \in \mathscr{B}(\infty)\} \subset \mathscr{L}(\infty) \cap \sigma(\mathscr{L}(\infty)) \cap \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$$

is an $\mathcal{A}$-basis of $\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}$ and is called the dual canonical basis or the upper global basis of $\boldsymbol{U}_q^-$.

Proposition 2.9 [Kimura 2012, Proposition 4.26(1)]. *For $i \in I$ and $c \geq 1$, let $f_i^{\{c\}} = f_i^{(c)}/(f_i^{(c)}, f_i^{(c)})$. Then we have*

$$f_i^{\{c\}} = q_i^{c(c-1)/2} f_i^c.$$

2B7. For the dual canonical basis, we have the following expansion of left and right multiplication with respect to the Chevalley generators and their (shifted) powers.

Theorem 2.10 [Kashiwara 2012, Proposition 2.2; Oya 2015, Proposition 4.14 (ii)]. *For* $b \in \mathscr{B}(\infty)$, $i \in I$ *and* $c \geq 1$, *we have*

$$f_i^{\{c\}} G^{\mathrm{up}}(b) = q_i^{-c\varepsilon_i(b)} G^{\mathrm{up}}(\tilde{f}_i^c b) + \sum_{\varepsilon_i(b')<\varepsilon_i(b)+c} F_{i;b,b'}^{\{c\}}(q) G^{\mathrm{up}}(b'), \tag{2-8a}$$

$$G^{\mathrm{up}}(b) f_i^{\{c\}} = q_i^{-c\varepsilon_i^*(b)} G^{\mathrm{up}}(\tilde{f}_i^{*c} b) + \sum_{\varepsilon_i^*(b')<\varepsilon_i^*(b)+c} F_{i;b,b'}^{*\{c\}}(q) G^{\mathrm{up}}(b'), \tag{2-8b}$$

where

$$F_{i;b,b'}^{\{c\}}(q) := (f_i^{\{c\}} G^{\mathrm{up}}(b), G^{\mathrm{low}}(b')) = q_i^{c(c-1)/2} (G^{\mathrm{up}}(b), ({}_i r)^c G^{\mathrm{low}}(b')) \in q_i^{-c\varepsilon_i(b)} q\mathbb{Z}[q],$$

$$F_{i;b,b'}^{*\{c\}}(q) := (G^{\mathrm{up}}(b) f_i^{\{c\}}, G^{\mathrm{low}}(b')) = q_i^{c(c-1)/2} (G^{\mathrm{up}}(b), (r_i)^c G^{\mathrm{low}}(b')) \in q_i^{-c\varepsilon_i^*(b)} q\mathbb{Z}[q].$$

2C. ***Braid group action and the (dual) canonical basis.*** In this subsection, we recall the compatibility between Lusztig's braid symmetry and the (dual) canonical basis (for more details, see [Kimura 2012, Sections 4.4 and 4.6]).

2C1. *Braid group action on quantized enveloping algebra.* Let W be the Weyl group and $\{s_i\}_{i\in I}$ be the set of simple reflections, and let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function.

Following Lusztig [1993, Section 37.1.3], we define the $\mathbb{Q}(q)$-algebra automorphisms

$$T'_{i,\epsilon} : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$$

and

$$T''_{i,\epsilon} : \boldsymbol{U}_q(\mathfrak{g}) \to \boldsymbol{U}_q(\mathfrak{g})$$

for $i \in I$ and $\epsilon \in \{\pm 1\}$ by the following formulae:

$$T'_{i,\epsilon}(q^h) = q^{s_i(h)}; \tag{2-9a}$$

$$T'_{i,\epsilon}(e_j) = \begin{cases} -k_i^{\epsilon} e_i & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{\epsilon r} e_i^{(r)} e_j e_i^{(s)} & \text{for } j \neq i; \end{cases} \tag{2-9b}$$

$$T'_{i,\epsilon}(f_j) = \begin{cases} -e_i k_i^{-\epsilon} & \text{for } j = i, \\ \displaystyle\sum_{r+s=-\langle h_i,\alpha_j\rangle} (-1)^r q_i^{-\epsilon r} f_i^{(s)} f_j f_i^{(r)} & \text{for } j \neq i; \end{cases} \tag{2-9c}$$

and

(2-10a) $$T''_{i,-\epsilon}(q^h)=q^{s_i(h)};$$

(2-10b) $$T''_{i,-\epsilon}(e_j)=\begin{cases} -f_ik_i^{-\epsilon} & \text{for } j=i,\\ \sum_{r+s=-\langle h_i,\alpha_j\rangle}(-1)^r q_i^{\epsilon r}e_i^{(s)}e_je_i^{(r)} & \text{for } j\neq i;\end{cases}$$

(2-10c) $$T''_{i,-\epsilon}(f_j)=\begin{cases} -k_i^{\epsilon}e_i & \text{for } j=i,\\ \sum_{r+s=-\langle h_i,\alpha_j\rangle}(-1)^r q_i^{-\epsilon r}f_i^{(r)}f_jf_i^{(s)} & \text{for } j\neq i.\end{cases}$$

It is known that $\{T'_{i,\epsilon}\}_{i\in I}$ and $\{T''_{i,\epsilon}\}_{i\in I}$ satisfy the braid relation.

Lemma 2.11 [Lusztig 1993, Proposition 37.1.2(d), Section 37.2.4].

(1) *We have* $T'_{i,\epsilon}\circ T''_{i,-\epsilon}=T''_{i,-\epsilon}\circ T'_{i,\epsilon}=\mathrm{id}$.

(2) *We have* $*\circ T'_{i,\epsilon}\circ *=T''_{i,-\epsilon}$ *for* $i\in I$ *and* $\epsilon\in\{\pm 1\}$.

In the following, we write $T_i=T''_{i,1}$ and $T_i^{-1}=T'_{i,-1}$ as in [Saito 1994, Proposition 1.3.1].

2C2.

Proposition 2.12 [Lusztig 1993, Proposition 38.1.6, Lemma 38.1.5].

(1) *For* $i\in I$, *we have*

$$\boldsymbol{U}_q^-\cap T_i\boldsymbol{U}_q^-=\{x\in\boldsymbol{U}_q^-\mid {}_ir(x)=0\},$$
$$\boldsymbol{U}_q^-\cap T_i^{-1}\boldsymbol{U}_q^-=\{x\in\boldsymbol{U}_q^-\mid r_i(x)=0\}.$$

(2) *For* $i\in I$, *we have the following orthogonal decomposition with respect to* $(\cdot,\cdot)_-$:

$$\boldsymbol{U}_q^-=(\boldsymbol{U}_q^-\cap T_i\boldsymbol{U}_q^-)\oplus f_i\boldsymbol{U}_q^-=(\boldsymbol{U}_q^-\cap T_i^{-1}\boldsymbol{U}_q^-)\oplus \boldsymbol{U}_q^-f_i.$$

Corollary 2.13. *For* $i\in I$, *the subalgebra* $\boldsymbol{U}_q^-\cap T_i\boldsymbol{U}_q^-$ *(resp.* $\boldsymbol{U}_q^-\cap T_i\boldsymbol{U}_q^-$*) is compatible with the dual canonical basis*; *that is*, *we have*

$$\boldsymbol{U}_q^-\cap T_i\boldsymbol{U}_q^-\cap\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}=\bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i(b)=0}}\mathcal{A}G^{\mathrm{up}}(b),$$
$$\boldsymbol{U}_q^-\cap T_i^{-1}\boldsymbol{U}_q^-\cap\boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}}=\bigoplus_{\substack{b\in\mathscr{B}(\infty)\\ \varepsilon_i^*(b)=0}}\mathcal{A}G^{\mathrm{up}}(b).$$

2C3.

Proposition 2.14 [Saito 1994, Proposition 3.4.7, Corollary 3.4.8].

(1) *Let* $x \in \boldsymbol{U}_q^- \in \mathscr{L}(\infty) \cap T_i^{-1}\boldsymbol{U}_q^-$ *with* $b := x \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty)$. *We have*

$$T_i(x) \in \mathscr{L}(\infty) \cap T_i\boldsymbol{U}_q^-,$$
$$T_i(x) \equiv \tilde{f}_i^{*\varphi_i(b)}\tilde{e}_i^{\varepsilon_i(b)}b \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *Let*

$$\sigma_i : \{b \in \mathscr{B}(\infty) \mid \varepsilon_i^*(b) = 0\} \to \{b \in \mathscr{B}(\infty) \mid \varepsilon_i(b) = 0\}$$

be the map defined by $\sigma_i(b) = \tilde{f}_i^{*\varphi_i(b)}\tilde{e}_i^{\varepsilon_i(b)}b$. *Then* σ_i *is bijective and its inverse is given by*

$$\sigma_i^*(b) = (* \circ \sigma_i \circ *)(b) = \tilde{f}_i^{\varphi_i^*(b)}\tilde{e}_i^{*\varepsilon_i^*(b)}b.$$

The bijections σ_i and σ_i^* are called Saito crystal reflections. In [Saito 1994, Corollary 3.4.8], σ_i and σ_i^* are denoted by Λ_i and Λ_i^{-1}. Following Baumann, Kamnitzer and Tingley [Baumann et al. 2014, Section 5.5], for convenience, we extend σ_i and σ_i^* to $\mathscr{B}(\infty)$ by setting

$$\hat{\sigma}_i(b) := \sigma_i(\tilde{e}_i^{*\max}(b)),$$
$$\hat{\sigma}_i^*(b) := \sigma_i^*(\tilde{e}_i^{\max}(b)),$$

so we can consider $\hat{\sigma}_i$ and $\hat{\sigma}_i^*$ as maps from $\mathscr{B}(\infty)$ to itself.

2C4. Let ${}^i\pi : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \cap T_i\boldsymbol{U}_q^-$ (resp. $\pi^i : \boldsymbol{U}_q^- \to \boldsymbol{U}_q^- \cap T_i^{-1}\boldsymbol{U}_q^-$) be the orthogonal projection whose kernel is $f_i\boldsymbol{U}_q^-$ (resp. $\boldsymbol{U}_q^- f_i$) in Proposition 2.12(2). We have the following relations among the braid group action and the (dual) canonical basis.

Theorem 2.15 [Lusztig 1996, Theorem 1.2; Kimura 2012, Theorem 4.23].

(1) *For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i^*(b) = 0$, *we have*

$$T_i(\pi^i G^{\mathrm{low}}(b)) = {}^i\pi(G^{\mathrm{low}}(\sigma_i(b))),$$
$$(1-q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle} T_i G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\sigma_i b).$$

(2) *For* $b \in \mathscr{B}(\infty)$ *with* $\varepsilon_i(b) = 0$, *we have*

$$T_i^{-1}({}^i\pi G^{\mathrm{low}}(b)) = \pi^i(G^{\mathrm{low}}(\sigma_i^*(b))),$$
$$(1-q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle} T_i^{-1} G^{\mathrm{up}}(b) = G^{\mathrm{up}}(\sigma_i^* b).$$

We note that the constant term $(1-q_i^2)^{\langle h_i, \mathrm{wt}\, b\rangle}$ depends on the choice of nondegenerate bilinear form on $\boldsymbol{U}_q^-(\mathfrak{g})$.

2D. *Poincaré–Birkhoff–Witt bases.* Let $W = \langle s_i \mid i \in I \rangle$ be the Weyl group of $\mathfrak{g}$, where $\{s_i \mid i \in I\}$ is the set of simple reflections associated with $i \in I$, and let $\ell : W \to \mathbb{Z}_{\geq 0}$ be the length function. For a Weyl group element w, let

$$I(w) := \{(i_1, i_2, \dots, i_{\ell(w)}) \in I^{\ell(w)} \mid s_{i_1} \cdots s_{i_{\ell(w)}} = w\}$$

be the set of reduced words of w.

2D1. Let $\Delta = \Delta_+ \sqcup \Delta_-$ be the root system of the Kac–Moody Lie algebra $\mathfrak{g}$ and the decomposition into positive and negative roots.

For a Weyl group element $w \in W$, we set

$$\Delta_+(\leq w) := \Delta_+ \cap w\Delta_- = \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_-\},$$
$$\Delta_+(> w) := \Delta_+ \cap w\Delta_+ = \{\beta \in \Delta_+ \mid w^{-1}\beta \in \Delta_+\}.$$

It is well known that $\Delta_+(\leq w)$ and $\Delta_+(>w)$ are bracket closed; that is, for $\alpha, \beta \in \Delta_+(\leq w)$ (resp. $\alpha, \beta \in \Delta_+(>w)$) with $\alpha+\beta \in \Delta_+$, we have $\alpha+\beta \in \Delta_+(\leq w)$ (resp. $\in \Delta_+(>w)$).

For a reduced word $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell) \in I(w)$, we define positive roots $\beta_{\boldsymbol{i},k}$ $(1 \leq k \leq \ell)$ by the formula

$$\beta_{\boldsymbol{i},k} = s_{i_1} \cdots s_{i_{k-1}}(\alpha_{i_k}) \quad (1 \leq k \leq \ell).$$

It is well known that $\Delta_+(\leq w) = \{\beta_{\boldsymbol{i},k}\}_{1 \leq k \leq \ell}$ and we put a total order on $\Delta_+(\leq w)$. We note that the convex total order on $\Delta_+(\leq w)$ is associated with a reduced word $\boldsymbol{i} \in I(w)$.

2D2. For a Weyl group element $w \in W$ and a reduced word $\boldsymbol{i} = (i_1, i_2, \dots, i_\ell) \in I(w)$, we define the root vector $f_\epsilon(\beta_{\boldsymbol{i},k})$ associated with $\beta_{\boldsymbol{i},k} \in \Delta_+(\leq w)$ and a sign $\epsilon \in \{\pm 1\}$ by

$$f_\epsilon(\beta_{\boldsymbol{i},k}) := T^\epsilon_{i_1} T^\epsilon_{i_2} \cdots T^\epsilon_{i_{k-1}}(f_{i_k}),$$

and its divided power by

$$f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} := T^\epsilon_{i_1} T^\epsilon_{i_2} \cdots T^\epsilon_{i_{k-1}}(f^{(c)}_{i_k}) \quad \text{for } c \in \mathbb{Z}_{\geq 0}.$$

Theorem 2.16 [Lusztig 1993, Propositions 40.2.1 and 41.1.3].

(1) *For $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, $\epsilon \in \{\pm 1\}$ and $\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}$, we set*

$$f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := \begin{cases} f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} f_\epsilon(\beta_{\boldsymbol{i},2})^{(c_2)} \cdots f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} & \text{if } \epsilon = +1, \\ f_\epsilon(\beta_{\boldsymbol{i},\ell})^{(c_\ell)} f_\epsilon(\beta_{\boldsymbol{i},\ell-1})^{(c_{\ell-1})} \cdots f_\epsilon(\beta_{\boldsymbol{i},1})^{(c_1)} & \text{if } \epsilon = -1. \end{cases}$$

Then $\{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}}$ is linearly independent.

(2) *The subspace of* $\boldsymbol{U}_q^-(\mathfrak{g})$ *spanned by* $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *does not depend on the choice of reduced word* $\boldsymbol{i}\in I(w)$. *We denote this subspace by* $\boldsymbol{U}_q^-(\leq w,\epsilon)$. *The basis* $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *of* $\boldsymbol{U}_q^-(\leq w,\epsilon)$ *is called the Poincaré–Birkhoff–Witt basis or the lower Poincaré–Birkhoff–Witt basis.*

Definition 2.17. For a Weyl group element $w\in W$, a reduced word $\boldsymbol{i}=(i_1,\dots,i_\ell)\in I(w)$ and $\boldsymbol{c}=(c_1,\dots,c_\ell)\in\mathbb{Z}_{\geq 0}^{\ell}$, we set

$$\xi(\boldsymbol{c},\boldsymbol{i}) := -\sum_{1\leq k\leq \ell} c_k\beta_{\boldsymbol{i},k}\in Q_-.$$

We also have the following characterization of $\boldsymbol{U}_q^-(\leq w,\epsilon)$.

Theorem 2.18 [Beck et al. 1999, Proposition 2.3]. *For* $w\in W$, $\epsilon\in\{\pm 1\}$ *and* $\boldsymbol{i}=(i_1,\dots,i_\ell)\in I(w)$, *let*

$$\begin{aligned}&\boldsymbol{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^{\geq 0}\\ &\quad=\begin{cases}\boldsymbol{U}_q^-\cap T_{i_1}\cdots T_{i_\ell}\boldsymbol{U}_q^{\geq 0}=\{x\in\boldsymbol{U}_q^-\mid T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}(x)\in\boldsymbol{U}_q^{\geq 0}\} & \text{if } \epsilon=+1,\\ \boldsymbol{U}_q^-\cap T_{i_1}^{-1}\cdots T_{i_\ell}^{-1}\boldsymbol{U}_q^{\geq 0}=\{x\in\boldsymbol{U}_q^-\mid T_{i_\ell}\cdots T_{i_1}(x)\in\boldsymbol{U}_q^{\geq 0}\} & \text{if } \epsilon=-1.\end{cases}\end{aligned}$$

Then the Poincaré–Birkhoff–Witt basis $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ *forms a* $\mathbb{Q}(q)$*-basis of* $\boldsymbol{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^{\geq 0}$; *that is,* $\boldsymbol{U}_q^-(\leq w,\epsilon)=\boldsymbol{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^{\geq 0}$.

For the convenience of readers checking the notation, we give a proof of the above theorem.

Proof. Since the $\epsilon=-1$ case can be proved from the $\epsilon=+1$ case by applying the $*$-involution, it suffices for us to prove the claim for the $\epsilon=+1$ case. For $1\leq k\leq\ell$, we have

$$\begin{aligned}T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}T_{i_1}\cdots T_{i_{k-1}}(f_{i_k}^{(c_k)}) &= T_{i_\ell}^{-1}\cdots T_{i_k}^{-1}(f_{i_k}^{(c_k)})\\ &= (-1)^{c_k}T_{i_\ell}^{-1}\cdots T_{i_{k+1}}^{-1}(e_{i_k}^{(c_k)}k_{i_k}^{c_k}).\end{aligned}$$

Since $(i_k,\dots,i_\ell)$ is a reduced word, we have $T_{i_\ell}^{-1}\cdots T_{i_{k+1}}^{-1}(e_{i_k}^{(c_k)})\in\boldsymbol{U}_q^+$. Hence

$$T_{i_\ell}^{-1}\cdots T_{i_1}^{-1}T_{i_1}\cdots T_{i_{k-1}}(f_{i_k}^{(c_k)})\in\boldsymbol{U}_q^{\geq 0}.$$

So the inclusion $\boldsymbol{U}_q^-(\leq w,\epsilon)\subset\boldsymbol{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^{\geq 0}$ is shown, and it suffices to prove the opposite inclusion, that is, that the Poincaré–Birkhoff–Witt basis $\{f_\epsilon(\boldsymbol{c},\boldsymbol{i})\}_{\boldsymbol{c}\in\mathbb{Z}_{\geq 0}^{\ell}}$ spans $\boldsymbol{U}_q^-\cap T_{w^\epsilon}^{\epsilon}\boldsymbol{U}_q^{\geq 0}$.

Let $x\in\boldsymbol{U}_q^-\cap T_w\boldsymbol{U}_q^{\geq 0}$ be a homogeneous element. We write it as the sum

$$x=\sum_{c_1} f_{i_1}^{(c_1)}x_{c_1}$$

with $x_{c_1} \in \boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^-$. Then we have $T_{i_1}^{-1}(x_{c_1}) \in \boldsymbol{U}_q^-$. So we write it as the sum

$$T_{i_1}^{-1}(x_{c_1}) = \sum_{c_2} f_{i_2}^{(c_2)} x_{c_1,c_2}.$$

Repeating this process, we obtain elements $x_{c_1,\ldots,c_k} \in \boldsymbol{U}_q^- \cap T_{i_k} \boldsymbol{U}_q^-$ for $1 \leq k \leq \ell$ and

$$T_{i_k}^{-1}(x_{c_1,\ldots,c_k}) = \sum_{c_{k+1}} f_{i_{k+1}}^{(c_{k+1})} x_{c_1,\ldots,c_k,c_{k+1}}$$

for $1 \leq k < \ell$. Then we obtain

$$\begin{aligned}
&T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(x) \\
&\qquad = \sum_{c_1} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(x_{c_1}) \\
&\qquad = \sum_{c_1,c_2} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(f_{i_2}^{(c_2)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(x_{c_1,c_2}) \\
&\qquad = \sum_{c_1,c_2,\ldots,c_\ell} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(f_{i_2}^{(c_2)}) \cdots T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) x_{c_1,\ldots,c_\ell}.
\end{aligned}$$

By the assumption $x \in \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$, the left-hand side is in $\boldsymbol{U}_q^{\geq 0}$. By the triangular decomposition and $T_{i_\ell}^{-1} \cdots T_{i_k}^{-1}(f_{i_k}^{(c_k)}) \in \boldsymbol{U}_q^- \cap T_w \boldsymbol{U}_q^{\geq 0}$, we have that $x_{c_1,\ldots,c_\ell} \in \boldsymbol{U}_q^- \cap \boldsymbol{U}_q^{\geq 0} = \mathbb{Q}(q)$. Hence we obtain

$$x = \sum_{c_1,\ldots,c_\ell} x_{c_1,\ldots,c_\ell} f_{+1}(\boldsymbol{c}, \boldsymbol{i}) \in \boldsymbol{U}_q^-(\leq w, +1),$$

so $\boldsymbol{U}_q^- \cap T_{w^\epsilon}^{\epsilon} \boldsymbol{U}_q^{\geq 0} \subset \boldsymbol{U}_q^-(\leq w, \epsilon)$. □

Remark 2.19. The stronger assertion for Lusztig's integral form is proved in [Beck et al. 1999, Proposition 2.3].

2D3. *Poincaré–Birkhoff–Witt basis and crystal basis.*

Theorem 2.20. *For $w \in W$, $\boldsymbol{i} \in (i_1, \ldots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$:*

(1) *We have $f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \in \mathscr{L}(\infty)$ and*

$$b_\epsilon(\boldsymbol{c}, \boldsymbol{i}) := f_\epsilon(\boldsymbol{c}, \boldsymbol{i}) \bmod q\mathscr{L}(\infty) \in \mathscr{B}(\infty).$$

(2) *The map $\mathbb{Z}_{\geq 0}^{\ell} \to \mathscr{B}(\infty)$ which is defined by $\boldsymbol{c} \mapsto b_\epsilon(\boldsymbol{c}, \boldsymbol{i})$ is injective. We denote the image by $\mathscr{B}(w, \epsilon)$, and this does not depend on the choice of reduced word $\boldsymbol{i} \in I(w)$.*

2D4.

Proposition 2.21 [Kimura 2012, Proposition 4.26(2)]. *For $c \geq 1$ and $1 \leq k \leq \ell$, let*

$$f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)} / \big(f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}, f_\epsilon(\beta_{\boldsymbol{i},k})^{(c)}\big).$$

Then we have $f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^{\{c\}} = q_{i_k}^{c(c-1)/2} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},k})^c \in \boldsymbol{B}^{\mathrm{up}}$.

Definition 2.22 (dual Poincaré–Birkhoff–Witt basis). For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$, we set

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) := \frac{f_\epsilon(\boldsymbol{c}, \boldsymbol{i})}{(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))},$$

and $\{f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i})\}_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}}$ is called the dual Poincaré–Birkhoff–Witt basis or upper Poincaré–Birkhoff–Witt basis.

By the definition of the dual Poincaré–Birkhoff–Witt basis and the computation of $(f_\epsilon(\boldsymbol{c}, \boldsymbol{i}), f_\epsilon(\boldsymbol{c}, \boldsymbol{i}))$, we have

$$\begin{aligned}
&f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) \\
&= \begin{cases} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},2})^{\{c_2\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{\{c_\ell\}} & \text{if } \epsilon = +1, \\ f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell})^{\{c_\ell\}} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},\ell-1})^{\{c_{\ell-1}\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} & \text{if } \epsilon = -1 \end{cases} \\
&= \begin{cases} (1-q_{i_1}^2)^{\langle h_{i_1}, \xi(\boldsymbol{c}_{\geq 2}, \boldsymbol{i}_{\geq 2})\rangle} f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} T_{i_1}^{\epsilon}\big(f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\geq 2},2})^{\{c_2\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\geq 2},\ell})^{\{c_\ell\}}\big) \\ \hfill \text{if } \epsilon = +1, \\ (1-q_{i_1}^2)^{\langle h_{i_1}, \xi(\boldsymbol{c}_{\geq 2}, \boldsymbol{i}_{\geq 2})\rangle} T_{i_1}^{\epsilon}\big(f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\geq 2},\ell})^{\{c_\ell\}} \cdots f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i}_{\geq 2},2})^{\{c_2\}}\big) f_\epsilon^{\mathrm{up}}(\beta_{\boldsymbol{i},1})^{\{c_1\}} \\ \hfill \text{if } \epsilon = -1, \end{cases}
\end{aligned}$$

where $\boldsymbol{c}_{\geq 2} = (c_2, \dots, c_\ell) \in \mathbb{Z}_{\geq 0}^{\ell-1}$, $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$ and $\boldsymbol{i}_{\geq 2} = (i_2, \dots, i_\ell) \in I(w_{\geq 2})$.

Using the Levendorskii–Soibelman formula (see [Kimura 2012, Theorem 4.27]) and the definition of the dual canonical basis, we have the following result.

Theorem 2.23 [Kimura 2012, Theorems 4.25 and 4.29]. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. The Poincaré–Birkhoff–Witt basis satisfies the following properties*:

(1) *The subalgebra $\boldsymbol{U}_q^-(\leq w, \epsilon)$ is compatible with the dual canonical basis*; *that is, there exists a subset $\mathscr{B}(\leq w, \epsilon) := \mathscr{B}(\boldsymbol{U}_q^-(\leq w, \epsilon)) \subset \mathscr{B}(\infty)$ such that*

$$\boldsymbol{U}_q^-(\leq w, \epsilon) = \bigoplus_{b \in \mathscr{B}(\leq w, \epsilon)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

(2) *The transition matrix between the dual Poincaré–Birkhoff–Witt basis and the dual canonical basis is triangular with* 1*'s on the diagonal with respect to the* (*left*) *lexicographic order $\leq$ on $\mathbb{Z}_{\geq 0}^{\ell}$. More precisely, we have*

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) = G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) + \sum_{\boldsymbol{c}' < \boldsymbol{c}} d_{\boldsymbol{c},\boldsymbol{c}'}^{\boldsymbol{i}}(q) G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}', \boldsymbol{i}))$$

with

$$d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) := \big(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}(b_{\epsilon}(\boldsymbol{c}',\boldsymbol{i}))\big) \in q\mathbb{Z}[q].$$

Remark 2.24. In the symmetric case, we note that it can be shown that

$$d^{\boldsymbol{i}}_{\boldsymbol{c},\boldsymbol{c}'}(q) = \big(f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i}), G^{\mathrm{low}}(b_{\epsilon}(\boldsymbol{c}',\boldsymbol{i}))\big) \in q\mathbb{Z}_{\geq 0}[q],$$

by the positivity of the (twisted) comultiplication with respect to the canonical basis and Proposition 2.21.

In particular, we obtain a proof of the positivity of the transition matrix from the canonical basis into the lower Poincaré–Birkhoff–Witt basis in simply laced type for an arbitrary reduced word of the longest element w_0 using the orthogonality of the (lower) Poincaré–Birkhoff–Witt basis.

For "adapted" reduced words, it was proved by Lusztig [1990, Corollary 10.7]. For an arbitrary reduced word, it was proved by Kato [2014, Theorm 4.17] using the categorification of the Poincaré–Birkhoff–Witt basis via the Khovanov–Lauda–Rouquier algebra. It was also proved by Oya [2015, Theorem 5.2].

3. Proof of the surjectivity

3A. ***Multiplication formula for $U_q^-(\leq w, \epsilon)$.*** For a Weyl group element w, a reduced word $\boldsymbol{i} \in I(w)$ and $0 \leq p < \ell$, we consider a subalgebra which is generated by

$$\{f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})\}_{p+1 \leq k \leq \ell}.$$

It can be shown that this subalgebra is also compatible with the dual canonical basis. This can be proved using the transition matrix between the dual Poincaré–Birkhoff–Witt basis and the dual canonical basis.

In this subsection, we give statements for the $\epsilon = +1$ case. We can obtain the corresponding claims for the $\epsilon = -1$ case by applying the $*$-involution. So we denote $f^{\mathrm{up}}_{\epsilon}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$, $b_{\epsilon}(\boldsymbol{c},\boldsymbol{i})$ by $f^{\mathrm{up}}(\beta_{\boldsymbol{i},k})$, $f^{\mathrm{up}}(\boldsymbol{c},\boldsymbol{i})$, $b(\boldsymbol{c},\boldsymbol{i})$, omitting ϵ.

Proposition 3.1. *Let $w \in W$ and $\boldsymbol{i} \in I(w)$. For $\boldsymbol{c} \in \mathbb{Z}^{\ell}_{\geq 0}$ and $0 \leq p < \ell$, we set*

$$\tau_{\leq p}(\boldsymbol{c}) := (c_1, \dots, c_p, 0, \dots, 0) \in \mathbb{Z}^{\ell}_{\geq 0},$$
$$\tau_{> p}(\boldsymbol{c}) := (0, \dots, 0, c_{p+1}, \dots, c_{\ell}) \in \mathbb{Z}^{\ell}_{\geq 0}.$$

Then we have

$$G^{\mathrm{up}}\big(b(\tau_{\leq p}(\boldsymbol{c})), \boldsymbol{i}\big) G^{\mathrm{up}}\big(b(\tau_{> p}(\boldsymbol{c}), \boldsymbol{i})\big) \in G^{\mathrm{up}}\big(b(\boldsymbol{c},\boldsymbol{i})\big) + \sum_{\boldsymbol{d} < \boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}\big(b(\boldsymbol{d},\boldsymbol{i})\big).$$

Proof. By the transition from the dual canonical basis to the dual Poincaré–Birkhoff–Witt basis, we have

$$G^{\mathrm{up}}\big(b(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i})\big) \in f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i}) + \sum_{\boldsymbol{d}_{\leq p} < \tau_{\leq p}(\boldsymbol{c})} q\mathbb{Z}[q] f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i}),$$

$$G^{\mathrm{up}}\big(b(\tau_{> p}(\boldsymbol{c}), \boldsymbol{i})\big) \in f^{\mathrm{up}}(\tau_{> p}(\boldsymbol{c}), \boldsymbol{i}) + \sum_{\boldsymbol{d}_{> p} < \tau_{> p}(\boldsymbol{c})} q\mathbb{Z}[q] f^{\mathrm{up}}(\boldsymbol{d}_{> p}, \boldsymbol{i}),$$

and note that we have $\boldsymbol{d}_{\leq p} = \tau_{\leq p}(\boldsymbol{d}_{\leq p})$ and $\boldsymbol{d}_{>p} = \tau_{>p}(\boldsymbol{d}_{>p})$ by the Levendorskii–Soibelman formula in the right-hand sides.

Hence in the product of the right-hand sides, we have four kinds of terms:

$$\begin{aligned} f^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) &= f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i}) f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i}),\\ f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}) + \boldsymbol{d}_{>p}, \boldsymbol{i}) &= f^{\mathrm{up}}(\tau_{\leq p}(\boldsymbol{c}), \boldsymbol{i}) f^{\mathrm{up}}(\boldsymbol{d}_{>p}, \boldsymbol{i}),\\ f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}) + \boldsymbol{d}_{\leq p}, \boldsymbol{i}) &= f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i}) f^{\mathrm{up}}(\tau_{>p}(\boldsymbol{c}), \boldsymbol{i}),\\ f^{\mathrm{up}}(\boldsymbol{d}_{\leq p} + \boldsymbol{d}_{>p}, \boldsymbol{i}) &= f^{\mathrm{up}}(\boldsymbol{d}_{\leq p}, \boldsymbol{i}) f^{\mathrm{up}}(\boldsymbol{d}_{>p}, \boldsymbol{i}). \end{aligned}$$

We note that $\tau_{\leq p}(\boldsymbol{c}) + \boldsymbol{d}_{>p} < \boldsymbol{c}$, $\tau_{>p}(\boldsymbol{c}) + \boldsymbol{d}_{\leq p} < \boldsymbol{c}$ and $\boldsymbol{d}_{\leq p} + \boldsymbol{d}_{>p} < \boldsymbol{c}$ by the construction. Hence, using the transition from the dual Poincaré–Birkhoff–Witt basis to the dual canonical basis, we obtain the claim. □

3B. ***Compatibility of*** $\boldsymbol{U_q^-(>w, \epsilon)}$**.** For a Weyl group element, we consider the cofinite subset $\Delta_+ \cap w\Delta_+$ and corresponding quantum coordinate ring $\boldsymbol{U}_q^-(>w, \epsilon)$.

Definition 3.2. For $w \in W$ and $\epsilon \in \{\pm 1\}$, we set

$$\boldsymbol{U}_q^-(>w, \epsilon) = \boldsymbol{U}_q^- \cap T_{w^\epsilon}^{\epsilon} \boldsymbol{U}_q^-.$$

The following is the main result in this subsection.

Theorem 3.3. *For $w \in W$ and $\epsilon \in \{\pm 1\}$, $\boldsymbol{U}_q^-(>w, \epsilon)$ is compatible with the dual canonical basis*; *namely*, *$\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon) := \boldsymbol{B}^{\mathrm{up}} \cap \boldsymbol{U}_q^-(>w, \epsilon)$ is a $\mathbb{Q}(q)$-basis of $\boldsymbol{U}_q^-(>w, \epsilon)$.*

The proof of this theorem occupies the rest of this subsection, and we give the characterization of the subset $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$.

3B1. We provide an alternative description of $\boldsymbol{U}_q^-(>w, \epsilon)$ which is more convenient for proving the compatibility of the dual canonical basis.

Proposition 3.4. *For $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$, we have*

$$\boldsymbol{U}_q^-(>w, \epsilon) = \boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} \boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} T_{i_2}^{\epsilon} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1}^{\epsilon} \cdots T_{i_\ell}^{\epsilon} \boldsymbol{U}_q^-.$$

In fact, the right-hand side does not depend on the choice of reduced word $\boldsymbol{i} \in I(w)$. The above proposition can be shown by the following lemmas.

Lemma 3.5. *For a Weyl group element $w \in W$, a reduced word $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and a homogeneous element $x \in \boldsymbol{U}_q^-$, there exists $x_{\boldsymbol{c}} \in \boldsymbol{U}_q^- \cap T_{i_\ell}\boldsymbol{U}_q^-$ for $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell$ with*

(3-1)
$$\begin{aligned} T_w^{-1}(x) &= \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1} T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) T_{i_\ell}^{-1}(x_{\boldsymbol{c}}) \\ &\in \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(c_{\ell-1})}) e_{i_\ell}^{(c_\ell)} \boldsymbol{U}_q^{\leq 0}. \end{aligned}$$

Proof. Following the proof in [Beck et al. 1999, Proposition 2.3], we proceed by induction on the length $\ell(w)$. By Lemma 2.7, we have the decomposition

$$x = \sum f_{i_1}^{(c_1)} x_{c_1} \quad \text{with } x_{c_1} \in \boldsymbol{U}_q^- \cap T_{i_1}\boldsymbol{U}_q^-.$$

So we have $T_{i_1}^{-1}(x_{c_1}) \in \boldsymbol{U}_q^- \cap T_{i_1}^{-1}\boldsymbol{U}_q^{-1}$. Applying $T_{i_1}^{-1}$ to x, we obtain

$$T_{i_1}^{-1}(x) = \sum_{c_1 \geq 0} T_{i_1}^{-1}(f_{i_1}^{(c_1)}) T_{i_1}^{-1}(x_{c_1}),$$

so we obtain the claim for $\ell(w) = 1$.

By induction on the length ℓ, we assume that

$$T_{i_{\ell-1}}^{-1} \cdots T_{i_1}^{-1}(x) = \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_{\ell-1}}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}})$$

with $T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) \in \boldsymbol{U}_q^- \cap T_{i_{\ell-1}}^{-1}\boldsymbol{U}_q^-$. Since $T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) \in \boldsymbol{U}_q^-$, we have the decomposition

$$T_{i_{\ell-1}}^{-1}(x_{c_1,\dots,c_{\ell-1}}) = \sum_{c_\ell \geq 0} f_{i_\ell}^{(c_\ell)} x_{c_1,\dots,c_\ell} \quad \text{with } x_{c_1,\dots,c_\ell} \in \boldsymbol{U}_q^- \cap T_{i_\ell}\boldsymbol{U}_q^-.$$

So we obtain the following claim:

$$\begin{aligned} &T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(x) \\ &\quad = \sum_{\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^\ell} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(f_{i_1}^{(c_1)}) \cdots T_{i_\ell}^{-1} T_{i_{\ell-1}}^{-1}(f_{i_{\ell-1}}^{(c_{\ell-1})}) T_{i_\ell}^{-1}(f_{i_\ell}^{(c_\ell)}) T_{i_\ell}^{-1}(x_{c_1,\dots,c_\ell}). \end{aligned}$$

The second claim is clear from the definition of $\{T_i\}$ and the defining relations of $\boldsymbol{U}_q$. □

Lemma 3.6. *If $\ell(s_i w) > \ell(w)$, we have $\boldsymbol{U}_q^- \cap T_{s_i w}\boldsymbol{U}_q^- \subset \boldsymbol{U}_q^- \cap T_i\boldsymbol{U}_q^-$.*

Proof. Let $(i_1, \dots, i_\ell)$ be a reduced word of w such that $(i, i_1, \dots, i_\ell)$ is a reduced word of $s_i w$. For a homogeneous element $x \in \boldsymbol{U}_q^-$, we decompose $x = \sum_{c \geq 0} f_i^{(c)} x_c$

with $x_c \in \boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$. So we have

$$T_i^{-1} x = \sum_{c \geq 0} T_i^{-1}(f_i^{(c)}) T_i^{-1}(x_c) \in \sum_{c \geq 0} e_i^{(c)} \boldsymbol{U}_q^{\leq 0}$$

with $T_i^{-1}(x_c) \in \boldsymbol{U}_q^- \cap T_i^{-1} \boldsymbol{U}_q^-$. Apply T_w^{-1} to both sides, we have

$$\begin{aligned}
&T_w^{-1} T_i^{-1} x \\
&= \sum_{c \geq 0} T_w^{-1}(T_i^{-1}(f_i^{(c)})) T_w^{-1} T_i^{-1}(x_c) \\
&\in \sum_{(c,d_1,\dots,d_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1}} T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(e_i^{(c)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(d_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(d_{\ell-1})}) e_{i_\ell}^{(d_\ell)} \boldsymbol{U}_q^{\leq 0}.
\end{aligned}$$

Suppose $x \in \boldsymbol{U}_q^- \cap T_{s_i w} \boldsymbol{U}_q^-$ is a homogeneous element; that is, $T_w^{-1} T_i^{-1} x \in \boldsymbol{U}_q^- \cap T_{s_i w}^{-1} \boldsymbol{U}_q^-$. Since

$$\left\{T_{i_\ell}^{-1} \cdots T_{i_1}^{-1}(e_i^{(c)}) T_{i_\ell}^{-1} \cdots T_{i_2}^{-1}(e_{i_1}^{(d_1)}) \cdots T_{i_\ell}^{-1}(e_{i_{\ell-1}}^{(d_{\ell-1})}) e_{i_\ell}^{(d_\ell)} \mid (c, d_1, \dots, d_\ell) \in \mathbb{Z}_{\geq 0}^{\ell+1}\right\}$$

is linearly independent by the assumption $\ell(s_i w) > \ell(w)$, we have $x_c = 0$ for $c > 0$. So we obtain $x = x_0 \in \boldsymbol{U}_q^- \cap T_i \boldsymbol{U}_q^-$. □

Proof of Proposition 3.4. We proceed by induction on the length $\ell(w)$ of a Weyl group element. When $\ell(w) = 1$, this is tautological, so we have the claim. By the induction hypothesis, we can assume that

$$\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

Then we have

$$\begin{aligned}
&\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- \\
&\qquad = \boldsymbol{U}_q^- \cap T_{i_1}(\boldsymbol{U}_q^- \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-) = \boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.
\end{aligned}$$

By Lemma 3.6, we obtain the claim

$$\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- = \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-. \qquad □$$

3B2. Let w be a Weyl group element and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ be a reduced word. Following Saito [1994, Lemma 4.1.3] and Baumann, Kamnitzer and Tingley [Baumann et al. 2014, Proposition 5.24], we define the Lusztig datum of $b \in \mathscr{B}(\infty)$ in direction $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$ (($\boldsymbol{i}, \epsilon$)-Lusztig datum for short).

Definition 3.7 (($\boldsymbol{i}, \epsilon$)-Lusztig datum). For $w \in W$, $\boldsymbol{i} \in I(w)$ and $\epsilon \in \{\pm 1\}$, define

$$L_\epsilon(b, \boldsymbol{i}) = \begin{cases} \left(\varepsilon_{i_1}(b), \varepsilon_{i_2}(\hat{\sigma}_{i_1}^* b), \dots, \varepsilon_{i_\ell}(\hat{\sigma}_{i_{\ell-1}}^* \cdots \hat{\sigma}_{i_1}^* b)\right) \in \mathbb{Z}_{\geq 0}^\ell & \text{if } \epsilon = +1, \\ \left(\varepsilon_{i_1}^*(b), \varepsilon_{i_2}^*(\hat{\sigma}_{i_1} b), \dots, \varepsilon_{i_\ell}^*(\hat{\sigma}_{i_{\ell-1}} \cdots \hat{\sigma}_{i_1} b)\right) \in \mathbb{Z}_{\geq 0}^\ell & \text{if } \epsilon = -1. \end{cases}$$

By construction in Theorem 2.20, we have

$$\boldsymbol{c} = L_\epsilon(b_\epsilon(\boldsymbol{c}, \boldsymbol{i}), \boldsymbol{i})$$

for $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}^{\ell}$; that is, the map $b_\epsilon(-, \boldsymbol{i}) : \mathbb{Z}_{\geq 0}^{\ell} \to \mathscr{B}(\infty)$ is a section of the $(\boldsymbol{i}, \epsilon)$-Lusztig datum $L_\epsilon(-, \boldsymbol{i}) : \mathscr{B}(\infty) \to \mathbb{Z}_{\geq 0}^{\ell}$.

3B3. The following gives a characterization of $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$ in terms of the $(\boldsymbol{i}, \epsilon)$-Lusztig datum.

Theorem 3.8. *For $w \in W$ and $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$, we set*

$$\mathscr{B}(>w, \epsilon) = \{b \in \mathscr{B}(\infty) \mid L_\epsilon(b, \boldsymbol{i}) = 0\}.$$

Then we have

$$\boldsymbol{U}_q^-(>w, \epsilon) = \bigoplus_{b \in \mathscr{B}(>w, \epsilon)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

Proof. By Proposition 3.4, it suffices for us to prove the compatibility for the intersection $\boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} \boldsymbol{U}_q^- \cap T_{i_1}^{\epsilon} T_{i_2}^{\epsilon} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1}^{\epsilon} \cdots T_{i_\ell}^{\epsilon} \boldsymbol{U}_q^-$.

Since $\epsilon = -1$ can be obtained by applying the $*$-involution, we prove only the $\epsilon = +1$ case. We prove the claim by induction on the length $\ell(w)$. For $\ell(w) = 1$, it is the claim in Corollary 2.13. We consider the intersection

$$\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-.$$

By the induction hypothesis, $\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-$ is compatible with the dual canonical basis, and $\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^-$ is also compatible with the dual canonical basis, so the intersection $\boldsymbol{U}_q^- \cap T_{i_1}^{-1} \boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^-$ is compatible with the dual canonical basis. Applying Theorem 2.15, we obtain the claim for $\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^-$. Since

$$\begin{aligned} &\boldsymbol{U}_q^- \cap T_{i_1} \boldsymbol{U}_q^- \cap T_{i_1} T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_1} \cdots T_{i_\ell} \boldsymbol{U}_q^- \\ &\qquad = \boldsymbol{U}_q^- \cap T_{i_1} \big(\boldsymbol{U}_q^- \cap T_{i_2} \boldsymbol{U}_q^- \cap \cdots \cap T_{i_2} \cdots T_{i_\ell} \boldsymbol{U}_q^- \big), \end{aligned}$$

we obtain the description of $\boldsymbol{B}^{\mathrm{up}}(>w, +1)$. $\square$

3C. *Multiplication formula between $\boldsymbol{B}^{\mathrm{up}}(\leq w, \epsilon)$ and $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$.*

3C1. We generalize the (special cases of the) formula in Theorem 2.10 using the dual canonical basis $\boldsymbol{B}^{\mathrm{up}}(>w, \epsilon)$.

Theorem 3.9. *For $b \in \mathscr{B}(>w, \epsilon)$ and $\boldsymbol{c} \in \mathbb{Z}_{\geq 0}$, we have*

$$f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) G^{\mathrm{up}}(b) \in G^{\mathrm{up}}(\nabla_{\boldsymbol{i}, \epsilon}^{\boldsymbol{c}}(b)) + \sum_{L_\epsilon(b', \boldsymbol{i}) < \boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b) f_\epsilon^{\mathrm{up}}(\boldsymbol{c}, \boldsymbol{i}) \in G^{\mathrm{up}}(\nabla_{\boldsymbol{i}, \epsilon}^{\boldsymbol{c}}(b)) + \sum_{L_\epsilon(b', \boldsymbol{i}) < \boldsymbol{c}} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = -1,$$

where

$$\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b) = \begin{cases} \tilde{f}^{c_1}_{i_1}\sigma_{i_1}\cdots\tilde{f}^{c_{\ell-1}}_{i_{\ell-1}}\sigma_{i_{\ell-1}}\tilde{f}^{c_\ell}_{i_\ell}\sigma_{i_\ell}\sigma^*_{i_\ell}\cdots\sigma^*_{i_1}(b) & \text{if } \epsilon=+1, \\ \tilde{f}^{*c_1}_{i_1}\sigma^*_{i_1}\cdots\tilde{f}^{*c_{\ell-1}}_{i_{\ell-1}}\sigma^*_{i_{\ell-1}}\tilde{f}^{*c_\ell}_{i_\ell}\sigma^*_{i_\ell}\sigma_{i_\ell}\cdots\sigma_{i_1}(b) & \text{if } \epsilon=-1. \end{cases}$$

Proof. We proceed by induction on the length $\ell(w)$ of a Weyl group element. Since $\epsilon=-1$ can be obtained by applying the $*$-involution, it suffices for us to prove the $\epsilon=+1$ case. Let $w_{\geq 2}=s_{i_2}\cdots s_{i_\ell}\in W$ and $\boldsymbol{i}_{\geq 2}=(i_2,\dots,i_\ell)\in I(w_{\geq 2})$. Let $b\in\mathscr{B}(\infty)$ with $L_{+1}(b,\boldsymbol{i})=0$; that is, we have

$$\big(\varepsilon_{i_1}(b),\varepsilon_{i_2}(\sigma^*_{i_1}b),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_1}b)\big)=(0,\dots,0).$$

So let $b_{\geq 2}:=\sigma^*_{i_1}b$; then we have

$$\begin{aligned} L_{+1}(b_{\geq 2},\boldsymbol{i}_{\geq 2}) &= \big(\varepsilon_{i_2}(b_{\geq 2}),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}b_{\geq 2})\big) \\ &= \big(\varepsilon_{i_2}(\sigma^*_{i_1}b),\dots,\varepsilon_{i_\ell}(\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}\sigma^*_{i_1}b)\big)=(0,\dots,0)\in\mathbb{Z}^{\ell-1}_{\geq 0} \end{aligned}$$

by definition of the Lusztig datum.

By the induction hypothesis for $w_{\geq 2}\in W$ and $\boldsymbol{i}_{\geq 2}\in I(w_{\geq 2})$, we have

$$f^{\mathrm{up}}_\epsilon(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})G^{\mathrm{up}}(b_{\geq 2})-G^{\mathrm{up}}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},\epsilon}(b_{\geq 2})\big)\in\sum_{L_\epsilon(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}} q\mathbb{Z}[q]G^{\mathrm{up}}(b'_{\geq 2})$$

with $\boldsymbol{c}_{\geq 2}=(c_2,\dots,c_\ell)\in\mathbb{Z}^{\ell-1}_{\geq 0}$. Since $\boldsymbol{U}^-_q\cap T^{-1}_{i_1}\boldsymbol{U}^-_q$ is spanned by the dual canonical basis $\{G^{\mathrm{up}}(b)\mid\varepsilon^*_{i_1}(b)=0\}$ and since $f^{\mathrm{up}}_\epsilon(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})\in\boldsymbol{U}^-_q\cap T^{-1}_{i_1}\boldsymbol{U}^-_q$ and $G^{\mathrm{up}}(b_{\geq 2})\in\boldsymbol{U}^-_q\cap T^{-1}_{i_1}\boldsymbol{U}^-_q$, we obtain $\varepsilon^*_{i_1}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},\epsilon}(b_{\geq 2})\big)=0$ and $\varepsilon^*_{i_1}(b'_{\geq 2})=0$.

We have

$$\begin{aligned} &f^{\mathrm{up}}_{+1}(\boldsymbol{c},\boldsymbol{i})G^{\mathrm{up}}(b) \\ &\quad=(1-q^2_{i_1})^{\langle h_{i_1},\xi(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})+\mathrm{wt}(b_{\geq 2})\rangle}f^{\{c_1\}}_{i_1}T_{i_1}\big(f^{\mathrm{up}}_\epsilon(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})G^{\mathrm{up}}(b_{\geq 2})\big) \\ &\quad\in(1-q^2_{i_1})^{\langle h_{i_1},\xi(\boldsymbol{c}_{\geq 2},\boldsymbol{i}_{\geq 2})+\mathrm{wt}(b_{\geq 2})\rangle}f^{\{c_1\}}_{i_1} \\ &\qquad\qquad\times T_{i_1}\Bigg(G^{\mathrm{up}}\big(\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big)+\sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}}q\mathbb{Z}[q]G^{\mathrm{up}}(b'_{\geq 2})\Bigg) \\ &\quad=f^{\{c_1\}}_{i_1}\Bigg(G^{\mathrm{up}}\big(\sigma_{i_1}\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big)+\sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}}q\mathbb{Z}[q]G^{\mathrm{up}}(\sigma_{i_1}b'_{\geq 2})\Bigg). \end{aligned}$$

We note that

$$\begin{aligned} \tilde{f}^{c_1}_{i_1}\sigma_{i_1}\nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2}) &= \tilde{f}^{c_1}_{i_1}\sigma_{i_1}\tilde{f}^{c_1}_{i_2}\sigma_{i_2}\cdots\tilde{f}^{c_{\ell-1}}_{i_{\ell-1}}\sigma_{i_{\ell-1}}\tilde{f}^{c_\ell}_{i_\ell}\sigma_{i_\ell}\sigma^*_{i_\ell}\cdots\sigma^*_{i_2}(b_{\geq 2}) \\ &= \nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b) \end{aligned}$$

and

$$f_{i_1}^{\{c_1\}} G^{\mathrm{up}}\big(\sigma_{i_1} \nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b)) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b''),$$

$$f_{i_1}^{\{c_1\}} G^{\mathrm{up}}(\sigma_{i_1} b'_{\geq 2}) \in G^{\mathrm{up}}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'').$$

By Theorem 2.10,

$$f_{i_1}^{\{c_1\}} \Big(G^{\mathrm{up}}\big(\sigma_{i_1} \nabla^{\boldsymbol{c}_{\geq 2}}_{\boldsymbol{i}_{\geq 2},+1}(b_{\geq 2})\big) + \sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<c_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}(\sigma_{i_1} b'_{\geq 2}) \Big)$$

can be written in the form

$$G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},+1}(b)) + \sum_{L_{+1}(b'_{\geq 2},\boldsymbol{i}_{\geq 2})<\boldsymbol{c}_{\geq 2}} q\mathbb{Z}[q] G^{\mathrm{up}}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}) + \sum_{\varepsilon_{i_1}(b'')<c_1} q\mathbb{Z}[q] G^{\mathrm{up}}(b'').$$

Since we have $(c'_2, \dots, c'_\ell) = L_{+1}(b'_{\geq 2}, \boldsymbol{i}_{\geq 2}) < \boldsymbol{c}_{\geq 2}$, we obtain

$$L_{+1}(\tilde{f}_{i_1}^{c_1} \sigma_{i_1} b'_{\geq 2}, \boldsymbol{i}) = (c_1, c'_2, \dots, c'_\ell) < \boldsymbol{c}$$

and we have

$$L_{+1}(b'', \boldsymbol{i}) = (\varepsilon_{i_1}(b''), \dots) < L_{+1}(b, \boldsymbol{i}) = (c_1, c_2, \dots, c_\ell)$$

because $\varepsilon_{i_1}(b'') < c_1$. We obtain the claim. □

Using the transition in Theorem 2.23(2) from the Poincaré–Birkhoff–Witt basis to the dual canonical basis, we obtain the following multiplicity-free result.

Theorem 3.10. *Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. For $\boldsymbol{c} \in \mathbb{Z}^\ell_{\geq 0}$ and $b \in \mathscr{B}(>w, \epsilon)$, we have*

$$G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) G^{\mathrm{up}}(b) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)) + \sum_{L_\epsilon(b',\boldsymbol{i})<c} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = +1,$$

$$G^{\mathrm{up}}(b) G^{\mathrm{up}}(b_\epsilon(\boldsymbol{c}, \boldsymbol{i})) \in G^{\mathrm{up}}(\nabla^{\boldsymbol{c}}_{\boldsymbol{i},\epsilon}(b)) + \sum_{L_\epsilon(b',\boldsymbol{i})<c} q\mathbb{Z}[q] G^{\mathrm{up}}(b') \quad \text{if } \epsilon = -1.$$

3C2.

Definition 3.11. Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. We define maps $\tau_{\leq w,\epsilon} : \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon)$ and $\tau_{>w,\epsilon} : \mathscr{B}(\infty) \to \mathscr{B}(>w, \epsilon)$ by

$$\tau_{\leq w,\epsilon}(b) = b_\epsilon(L_\epsilon(b, \boldsymbol{i}), \boldsymbol{i}),$$

$$\tau_{>w,\epsilon}(b) = \begin{cases} \sigma_{i_1} \cdots \sigma_{i_\ell} \hat{\sigma}^*_{i_\ell} \cdots \hat{\sigma}^*_{i_1}(b) & \text{if } \epsilon = +1, \\ \sigma^*_{i_1} \cdots \sigma^*_{i_\ell} \hat{\sigma}_{i_\ell} \cdots \hat{\sigma}_{i_1}(b) & \text{if } \epsilon = -1. \end{cases}$$

Proposition 3.12. *We have a bijection as sets*:

$$\Omega_w := (\tau_{\leq w,\epsilon}, \tau_{>w,\epsilon}) : \mathscr{B}(\infty) \to \mathscr{B}(\leq w, \epsilon) \times \mathscr{B}(>w, \epsilon).$$

We prove the multiplication property of the dual canonical basis elements between $\boldsymbol{U}_q^-(\leq w, \epsilon)$ and $\boldsymbol{U}_q^-(>w, \epsilon)$.

Theorem 3.13. *Let $w \in W$, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $\epsilon \in \{\pm 1\}$. For $b \in \mathscr{B}(\infty)$,*

$$G^{\mathrm{up}}(\tau_{\leq w,\epsilon}(b))G^{\mathrm{up}}(\tau_{>w,\epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b',\boldsymbol{i})<L_\epsilon(b,\boldsymbol{i})} q\mathbb{Z}[q]G^{\mathrm{up}}(b') \text{ if } \epsilon = +1,$$

$$G^{\mathrm{up}}(\tau_{>w,\epsilon}(b))G^{\mathrm{up}}(\tau_{\leq w,\epsilon}(b)) \in G^{\mathrm{up}}(b) + \sum_{L_\epsilon(b',\boldsymbol{i})<L_\epsilon(b,\boldsymbol{i})} q\mathbb{Z}[q]G^{\mathrm{up}}(b') \text{ if } \epsilon = -1.$$

Proof. Since $\epsilon = -1$ can be obtained by applying the $*$-involution, it suffices for us to prove the $\epsilon = +1$ case. We proceed by induction on the length $\ell(w)$.

First we have

$$\begin{aligned} &G^{\mathrm{up}}(b) - f_{i_1}^{\{\varepsilon_{i_1}(b)\}} G^{\mathrm{up}}\big(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b\big) \\ &= G^{\mathrm{up}}(b) - (1-q_{i_1}^2)^{\langle h_{i_1}, \mathrm{wt}(\hat{\sigma}_{i_1}^* b)\rangle} f_{i_1}^{\{\varepsilon_{i_1}(b)\}} T_{i_1} G^{\mathrm{up}}(\hat{\sigma}_{i_1}^* b) \in \sum_{\varepsilon_{i_1}(b')<\varepsilon_{i_1}(b)} q\mathbb{Z}[q]G^{\mathrm{up}}(b'). \end{aligned}$$

By Theorem 3.10, we only have to compute the product

$$(1-q_i^2)^{\langle h_i, \mathrm{wt}(\hat{\sigma}_i^* b)\rangle} f_i^{\{\varepsilon_i(b)\}} T_i G^{\mathrm{up}}(\hat{\sigma}_i^* b) \times G^{\mathrm{up}}(\tau_{>w,+1}(b)).$$

We note that

$$G^{\mathrm{up}}(\tau_{>w,+1}(b)) = (1-q_i^2)^{\langle h_{i_1}, \mathrm{wt}(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b))\rangle} T_{i_1} G^{\mathrm{up}}(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)),$$

where $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$.

By the induction hypothesis for $w_{\geq 2} = s_{i_2} \cdots s_{i_\ell}$ and $\boldsymbol{i}_{\geq 2} = (i_2, \dots, i_\ell) \in I(w_{\geq 2})$,

$$\begin{aligned} &G^{\mathrm{up}}(\hat{\sigma}_{i_1}^* b) - G^{\mathrm{up}}\big(\tau_{\leq w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) \\ &\qquad \in \sum_{L_{+1}(b'',\boldsymbol{i}_{\geq 2})<L_{+1}(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\geq 2})} q\mathbb{Z}[q]G^{\mathrm{up}}(b''). \end{aligned}$$

Applying $(1-q_{i_1}^2)^{\langle h_{i_1}, \mathrm{wt}(\hat{\sigma}_{i_1}^* b)\rangle} T_{i_1}$, we obtain

$$\begin{aligned} &G^{\mathrm{up}}(\sigma_{i_1} \hat{\sigma}_{i_1}^* b) - G^{\mathrm{up}}\big(\sigma_{i_1} \tau_{\leq w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1} \tau_{>w_{\geq 2}}(\hat{\sigma}_{i_1}^* b)\big) \\ &\qquad \in \sum_{L_{+1}(b'',\boldsymbol{i}_{\geq 2})<L_{+1}(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\geq 2})} q\mathbb{Z}[q]G^{\mathrm{up}}(\sigma_{i_1} b''). \end{aligned}$$

We note that $\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b = \sigma_{i_1}\hat{\sigma}_{i_1}^* b$. Multiplying the second term on the left by $f_{i_1}^{\{\varepsilon_{i_1}(b)\}}$, we have

$$\begin{aligned}&f_{i_1}^{\{\varepsilon_{i_1}(b)\}} G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big)\\ &\in G^{\mathrm{up}}\big(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) + \sum_{\varepsilon_{i_1}(b')<\varepsilon_{i_1}(b)} q\mathbb{Z}[q] G^{\mathrm{up}}(b').\end{aligned}$$

Then we obtain

$$\begin{aligned}&f_{i_1}^{\{\varepsilon_{i_1}(b)\}}\Big(G^{\mathrm{up}}\big(\tilde{e}_{i_1}^{\varepsilon_{i_1}(b)} b\big) - G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big) G^{\mathrm{up}}\big(\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b)\big)\Big)\\ &\in \sum_{\varepsilon_{i_1}(b')<\varepsilon_{i_1}(b)} q\mathbb{Z}[q] G^{\mathrm{up}}(b') + \sum_{L_{+1}(b'',\boldsymbol{i}_{\ge 2})<L_{+1}(\hat{\sigma}_{i_1}^* b,\boldsymbol{i}_{\ge 2})} q\mathbb{Z}[q] G^{\mathrm{up}}\big(\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1} b''\big).\end{aligned}$$

By the construction, $\tilde{f}_{i_1}^{\varepsilon_{i_1}(b)}\sigma_{i_1}\tau_{\le w_{\ge 2}}(\hat{\sigma}_{i_1}^* b) = \tau_{\le w}(b)$ and $\sigma_{i_1}\tau_{> w_{\ge 2}}(\hat{\sigma}_{i_1}^* b) = \tau_{> w}(b)$; hence we obtain the claim. □

3D. ***Application.*** We give a slight refinement of Lusztig's result [1996, Proposition 8.3] in the dual canonical basis. The following can be shown in a similar manner using the multiplicity-free property of the multiplications of a triple of the dual canonical basis elements, so we only state the claims.

Theorem 3.14. *Let w be a Weyl group element, $\boldsymbol{i} = (i_1, \dots, i_\ell) \in I(w)$ and $p \in [0, \ell]$ be an integer. We consider the intersection*

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^- = (\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^-) \cap (\boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-).$$

(1) *The subalgebra*

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-$$

is compatible with the dual canonical basis; *that is*, *there exists a subset*

$$\mathscr{B}(\boldsymbol{U}_q^- \cap T_{i_{p+1}} \cdots T_{i_\ell} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-) \subset \mathscr{B}(\infty)$$

such that

$$\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^- = \bigoplus_{b\in\mathscr{B}\left(\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_p}} \boldsymbol{U}_q^-\right)} \mathbb{Q}(q) G^{\mathrm{up}}(b).$$

(2) *Multiplication in $\boldsymbol{U}_q^-(\mathfrak{g})^{\mathrm{up}}_{\mathcal{A}}$ defines an isomorphism of free $\mathcal{A}$-modules*:

$$\begin{aligned}\big(\boldsymbol{U}_q^-(s_{i_{p+1}}\cdots s_{i_\ell}, +1)\big)^{\mathrm{up}}_{\mathcal{A}} \otimes_{\mathcal{A}} \big(\boldsymbol{U}_q^- \cap T_{s_{i_{p+1}}\cdots s_{i_\ell}} \boldsymbol{U}_q^- \cap T^{-1}_{s_{i_1}\cdots s_{i_\ell}} \boldsymbol{U}_q^-\big)^{\mathrm{up}}_{\mathcal{A}}&\\ \otimes_{\mathcal{A}} \big(\boldsymbol{U}_q^-(s_{i_p}\cdots s_{i_1}, -1)\big)^{\mathrm{up}}_{\mathcal{A}} \to \boldsymbol{U}_q^-,&\end{aligned}$$

where

$$(\boldsymbol{U}_q^- \cap T_{i_{p+1}} \cdots T_{i_\ell} \boldsymbol{U}_q^- \cap T_{i_p}^{-1} \cdots T_{i_1}^{-1} \boldsymbol{U}_q^-)_{\mathcal{A}}^{\mathrm{up}} = \boldsymbol{U}_q^-(\mathfrak{g})_{\mathcal{A}}^{\mathrm{up}} \cap T_{i_{p+1}} \cdots T_{i_\ell} \boldsymbol{U}_q^- \cap T_{i_p}^{-1} \cdots T_{i_1}^{-1} \boldsymbol{U}_q^-.$$

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SCALAR INVARIANTS OF SURFACES IN THE CONFORMAL 3-SPHERE VIA MINKOWSKI SPACETIME

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For a surface in the 3-sphere, by identifying the conformal round 3-sphere as the projectivized positive light cone in Minkowski 5-spacetime, we use the conformal Gauss map and the conformal transform to construct the associate homogeneous 4-surface in Minkowski 5-spacetime. We then derive the local fundamental theorem for a surface in the conformal round 3-sphere from that of the associate 4-surface in Minkowski 5-spacetime. More importantly, following an idea of Fefferman and Graham, we construct local scalar invariants for a surface in the conformal round 3-sphere. One distinct feature of our construction is to link the classic work of Blaschke to the work of Bryant and Fefferman and Graham.

1. Introduction

It is well-known that all local scalar invariants of a (pseudo-)Riemannian metric are Weyl invariants, based on Weyl's classical invariant theory for the orthogonal groups. A conformal structure on a manifold is described by an equivalent class of conformal Riemannian metrics. Two metrics g_1 and g_2 on a manifold M are conformal to each other if $g_1 = \lambda^2 g_2$ for some positive smooth function λ on M. There are several ways to set the theory of local conformal invariants, but it is no longer straightforward to account for local scalar conformal invariants because of the lack of Weyl Theorem for the group of conformal transformations. To tackle this problem, Fefferman and Graham, in a seminal paper [1985], described an ingenious construction for a Ricci-flat homogeneous Lorentzian ambient spacetime for a given conformal manifold, where the conformal manifold is represented by the homogeneous null hypersurface in the ambient spacetime. Their construction was motivated by the model case in which the conformal round sphere $\mathbb{S}^n$ is the projectivized positive light cone $\mathbb{N}_+^{n+1}$ in Minkowski spacetime $\mathbb{R}^{1,n+1}$. Thus they initiated the program of using local scalar (pseudo-)Riemannian invariants of the ambient metrics at the homogeneous null hypersurface to fully account for local scalar conformal invariants. Readers are referred to their recent expository paper

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[Fefferman and Graham 2012] to learn all the developments of this program, and also to [Bailey et al. 1994; Gover 2001]. This program has lead to many significant advances in the global theory of conformal geometry, particularly via conformally invariant PDEs.

In this paper we build a model case for the study of local scalar invariants of submanifolds in a conformal manifold, following Fefferman and Graham's approach. The model case for us is to study 2-surfaces $\hat{x}$ in the conformal round 3-sphere $(\mathbb{S}^3, [g_0])$. As in [Fefferman and Graham 1985], the conformal round 3-sphere is represented by the positive light cone $\mathbb{N}^4_+$ in Minkowski 5-spacetime $\mathbb{R}^{1,4}$. Given an immersed surface

$$\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$$

or equivalently

$$y = (1, \hat{x}) : \mathrm{M}^2 \to \mathbb{N}^4_+,$$

to incorporate all metrics in $[g_0]$ on the 3-sphere we consider the homogeneous extension

$$x^{\mathbb{N}} = \alpha(1, \hat{x}) : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}.$$

Then we will use the conformal Gauss map ξ of $\hat{x}$ to choose a canonical null vector y^* at each given point $y \in x^{\mathbb{N}} \subset \mathbb{N}^4_+$ to extend $x^{\mathbb{N}}$ further into a homogeneous timelike 4-surface

$$\tilde{x} = \alpha y + \alpha\rho y^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4}.$$

We will also consider the associate ruled 3-surface

$$x^+ = \frac{1}{\sqrt{2}}(e^t y + e^{-t} y^*) : \mathbb{R} \times \mathrm{M}^2 \to \mathbb{H}^4 \subset \mathbb{R}^{1,4}$$

where $\mathbb{H}^4$ is the hyperboloid in Minkowski 5-spacetime. The main idea, inspired by Fefferman and Graham's work, is to use the geometry of the associate 4-surface $\tilde{x}$ in Minkowski spacetime $\mathbb{R}^{1,4}$ (the associate ruled 3-surface x^+ in the hyperboloid $\mathbb{H}^4$ and the spacelike surface as the image of the conformal Gauss map ξ in the de Sitter spacetime $S^{3,1}$ in Minkowski spacetime $\mathbb{R}^{1,4}$) to study the geometry of the surface $\hat{x}$ in the conformal round 3-sphere $\mathbb{S}^3$.

Our approach facilitates proofs of the local fundamental theorems (see Theorem 3.3.1 and [Wang 1992; 1998]) and produces local scalar invariants of surfaces in the conformal round 3-sphere. It is more interesting to find scalar invariants and the PDE problems similar to the study of Willmore surfaces [Blaschke 1929; Bryant 1984; Li and Yau 1982; Marques and Neves 2014].

We remark that the key to our construction of associate surfaces is the conformal Gauss map ξ to a given surface $\hat{x}$ in the conformal round 3-sphere. Conformal

Gauss maps have been introduced in several contexts [Blaschke 1929; Bryant 1984; Rigoli 1987]. We are searching for a definition that fits into the context of ambient spaces of Fefferman and Graham (Lemmas 2.3.1 and 2.3.2). It is fascinating to see how Blaschke [1929] introduced the conformal Gauss map as the map representing the family of mean curvature 2-spheres of the surface $\hat{x}$ and the conformal transform $\hat{x}^*$ (Definition 2.4.1) as the other envelope surface of the conformal Gauss map. One technical assumption for the null vector y^* to be well defined at each point $y \in x^{\mathbb{N}}$ is to require that the conformal Gauss map of the surface $\hat{x}$ induces a spacelike surface in the de Sitter spacetime $\mathbb{S}^{1,3}$, which is equivalent to require the surface $\hat{x}$ is free of umbilical point in the conformal 3-sphere $\mathbb{S}^3$.

It is nice to know that in our construction the associate 4-surface $\tilde{x}$ in Minkowski spacetime $\mathbb{R}^{1,4}$ is a minimal 4-surface (of vanishing mean curvature) if and only if the 2-surface $\hat{x}$ is a Willmore surface with no umbilical point in $\mathbb{S}^3$ (Theorem 3.2.1). The same statement also holds for the associate ruled 3-surface x^+ in the hyperboloid $\mathbb{H}^4$ (Theorem 3.4.1) and the conformal Gauss map surface ξ in de Sitter sapcetime $\mathbb{S}^{1,3}$ (Theorem 2.5.2).

Upon realizing that a different representative $\lambda^2 g_0$ in the conformal class $[g_0]$ on $\mathbb{S}^3$ is equivalent to a different parametrization for the associate surface

$$\tilde{x} = \alpha y_\lambda + \alpha\rho y_\lambda^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4}, \tag{1.0.1}$$

where $y_\lambda = \hat{\lambda}(1, \hat{x})$ and $\hat{\lambda} = \lambda \circ \hat{x}$ for a conformal factor λ, the real issue is how we use the geometry of the surface $\hat{x}$ in the 3-sphere $(\mathbb{S}^3, \lambda^2 g_0)$ to calculate the geometry of the associate surface $\tilde{x}$. The solution is to use as the realization of $(\mathbb{S}^3, \lambda^2 g_0)$ the following 3-sphere $\mathbb{S}^3_\lambda$ in the positive light cone $\mathbb{N}^4_+$:

$$\lambda(1, x) : \mathbb{S}^3 \to \mathbb{N}^4_+. \tag{1.0.2}$$

For the convenience of readers we present the calculations of the geometry of $\mathbb{S}^3_\lambda$ as a spacelike 3-surface in Minkowski spacetime in Appendix B. But it starts with the observation that the conformal Gauss map

$$\xi = H_\lambda y_\lambda + \vec{\boldsymbol{n}}_\lambda : \mathrm{M}^2 \to \mathbb{R}^{1,4}, \tag{1.0.3}$$

where H_λ is the mean curvature of $\hat{x}$ in $(\mathbb{S}^3, \lambda^2 g_0)$ and $\vec{\boldsymbol{n}}_\lambda$ is the unit normal to y_λ in $\mathbb{S}^3_\lambda \subset \mathbb{N}^4_+$, is independent of the conformal metric $\lambda^2 g_0$ (cf. Lemma 2.3.4).

Using the calculations in Appendix B, we are able to show in the proof of Theorem 4.3.2 that the data $\{m, \omega^\lambda, \Omega_\lambda, \Omega^*_\lambda\}$ that determine the first and second fundamental forms of the associate surface $\tilde{x}$ in Minkowski spacetime $\mathbb{R}^{1,4}$ can all be expressed in terms of covariant derivatives of the curvature of the surface $\hat{x}$ in $(\mathbb{S}^3, \lambda^2 g_0)$ and the covariant derivatives of curvature of $(\mathbb{S}^3, \lambda^2 g_0)$ (including 0th order). In the exact same spirit as in [Fefferman and Graham 1985; 2012], our construction of associate surfaces $\tilde{x}$ provides a way to capture local scalar conformal invariants of a surface $\hat{x}$. Namely, one can obtain local scalar conformal invariants of the surface $\hat{x}$ in the conformal round 3-sphere by computing the local scalar (pseudo-)Riemannian invariants of the associate surface $\tilde{x}$ at the homogeneous surface $x^{\mathbb{N}}$ in the light cone in Minkowski 5-spacetime. The first nontrivial one is

$$\tilde{\Delta}\tilde{H}|_{\rho=0} = 2\alpha^{-3}(\Delta_\lambda H_\lambda + |\mathring{I\!I}_\lambda|^2 H_\lambda + (\mathring{I\!I}_\lambda)^{ij}(R^\lambda)_{i3j3} - (R^\lambda)_{3i,}{}^{i}) \tag{1.0.4}$$

in a general parametrization (1.0.1), where $(R^\lambda)_{i3j3}$ and $(R^\lambda)_{3i}$ are the Riemann curvature and Ricci curvature of the metric $\lambda^2 g_0$ on $\mathbb{S}^3$. Due to the homogeneity of $\tilde{x}$ we automatically have

$$\mathcal{H}_\lambda = \Delta_\lambda H_\lambda + |\mathring{I\!I}_\lambda|^2 H_\lambda + (\mathring{I\!I}_\lambda)^{ij}(R^\lambda)_{i3j3} - (R^\lambda)_{3i,}^{\ i} = \hat{\lambda}^{-3}(\Delta H + |\mathring{I\!I}|^2 H) \tag{1.0.5}$$

which is the curvature that vanishes if and only if the surface $\hat{x}$ is Willmore. Notice that extra curvature terms do not show up when we work with either the round metric g_0 or the Euclidean metric. Similar formulas have appeared in the literature [Hu and Li 2004; Gover and Waldron 2015] and are also used by R. Graham and N. Reichert (work in progress).

In (4.2.7) we obtain this conformal scalar invariant of higher order:

$$\begin{aligned}|\nabla\tilde{h}|^2|_{\rho=0} = \alpha^{-4}\big(&|\nabla\Omega_\lambda|^2 + 8|dH_\lambda|^2 + 2\,\mathrm{Ric}^\lambda(\vec{\boldsymbol{n}}_\lambda, \nabla H_\lambda) + 3H_\lambda^2|\Omega_\lambda|^2 \\ &+3K_\lambda^T|\Omega_\lambda|^2 + 6\Omega_\lambda\cdot\mathrm{Hess}(H_\lambda)\big),\end{aligned} \tag{1.0.6}$$

where K_λ^T is the sectional curvature of $(\mathbb{S}^3, \lambda^2 g_0)$ at the tangent plane to the surface $\hat{x}$. Another higher-order invariant is

$$\begin{aligned}\tilde{\Delta}\tilde{\Delta}\tilde{H}|_{\rho=0} = 8\alpha^{-5}\big(&\Delta_\lambda\mathcal{H}_\lambda + 9|\omega^\lambda|^2\mathcal{H}_\lambda - 3\mathrm{Div}(\omega^\lambda)\mathcal{H}_\lambda \\ &-6\omega^\lambda(\nabla\mathcal{H}_\lambda) - 6\mathcal{H}_\lambda|\mathring{I\!I}_\lambda|^{-2}\mathring{I\!I}_\lambda\cdot\Omega_\lambda^*\big),\end{aligned} \tag{1.0.7}$$

where $\omega^\lambda = \langle dy_\lambda, y_\lambda^*\rangle$ and $\Omega_\lambda^* = -\langle dy_\lambda^*, d\xi\rangle$ are parts of the data that determine the geometry of the associate surface $\tilde{x}$; they are given in (3.1.3) and (4.3.6) as invariants of the surface $\hat{x}$ in $(\mathbb{S}^3, \lambda^2 g_0)$.

To end the introduction we remark that, for the sake of the production of local scalar invariants, the assumption of having no umbilical point in our construction is not an issue.

2. The associate surfaces in $\mathbb{R}^{1,4}$

In this section we introduce the associate surfaces in Minkowski space $\mathbb{R}^{1,4}$ for a given surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$. We then show that such associate surface is canonical in doing conformal geometry for the surface $\hat{x}$. The construction relies on the conformal Gauss map and the conformal transform of $\hat{x}$. It is also very interesting to see how Blaschke and Bryant came to the conformal Gauss map and the conformal transform in very different perspectives [Blaschke 1929; Bryant 1984].

2.1. ***Surfaces in the 3-sphere.*** Suppose that

$$\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$$

is an immersed surface with isothermal coordinate (u^1, u^2). Let

$$\boldsymbol{n} : \mathrm{M}^2 \to \mathbb{R}^4$$

be the unit normal vector at each point on the surface. Then we obtain the first fundamental form

$$I = \langle d\hat{x}, d\hat{x} \rangle = E|du|^2 \tag{2.1.1}$$

and the second fundamental form

$$II = -\langle d\hat{x}, d\boldsymbol{n} \rangle = e(du^1)^2 + 2f du^1 du^2 + g(du^2)^2. \tag{2.1.2}$$

Hence the mean curvature of the surface in the 3-sphere is

$$H = \frac{1}{2E}(e+g) \tag{2.1.3}$$

and the Gaussian curvature of the surface is

$$K = \frac{eg - f^2}{E^2} + 1. \tag{2.1.4}$$

Notice that

$$\boldsymbol{n}_{u^1} = -\frac{e}{E}\hat{x}_{u^1} - \frac{f}{E}\hat{x}_{u^2}, \quad \boldsymbol{n}_{u^2} = -\frac{f}{E}\hat{x}_{u^1} - \frac{g}{E}\hat{x}_{u^2}. \tag{2.1.5}$$

If one takes another conformal metric $\lambda^2 g_0$ on the 3-sphere $\mathbb{S}^3$, where λ is a positive function on $\mathbb{S}^3$, then the first fundamental form for the surface $\hat{x}$ is

$$I_\lambda = \hat{\lambda}^2 I, \tag{2.1.6}$$

where $\hat{\lambda} = \lambda \circ \hat{x}$ and the second fundamental form is

$$II_\lambda = \hat{\lambda} II - \lambda_{\boldsymbol{n}} I, \tag{2.1.7}$$

where $\lambda_{\boldsymbol{n}} = \boldsymbol{n}(\lambda)$. Hence

$$H_\lambda = \hat{\lambda}^{-1}\Big(H - \frac{\lambda_{\boldsymbol{n}}}{\hat{\lambda}}\Big) \quad \text{and} \quad \mathring{II}_\lambda = \hat{\lambda}\mathring{II}, \tag{2.1.8}$$

where $\mathring{II}$ is the traceless part of the second fundamental form II. Here we see the easy scalar conformal invariant $|\mathring{II}|^2$, which can be considered to be the counter part of the square of the length of Weyl curvature on a conformal manifold.

2.2. ***Minkowski 5-spacetime.*** Let $\mathbb{R}^{1,4}$ be the Minkowski 5-spacetime, where we use the notation

$$\mathbb{R}^{1,4} = \{(t, x) : t \in \mathbb{R} \text{ and } x \in \mathbb{R}^4\}$$

with the Lorentz inner product

$$\langle (t, x), (s, y)\rangle = -st + x \cdot y.$$

Recall the positive light cone is given by

$$\mathbb{N}^4_+ = \{(t, x) \in \mathbb{R}^{1,4} : -t^2 + |x|^2 = 0 \text{ and } t > 0\};$$

the hyperboloid is given as

$$\mathbb{H}^4 = \{(t, x) \in \mathbb{R}^{1,4} : -t^2 + |x|^2 = -1 \text{ and } t > 0\};$$

and the de Sitter 4-spacetime is given as

$$\mathbb{S}^{1,3} = \{(t, x) \in \mathbb{R}^{1,4} : -t^2 + |x|^2 = 1\}.$$

Given a surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$, we may consider the 2-surface

$$y = (1, \hat{x}) : \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$$

and the homogeneous extension

$$x^{\mathbb{N}} = \alpha y : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$$

for $\alpha \in \mathbb{R}^+$. There does not seem to be a way of doing “geometry" of the homogeneous 3-surface $x^{\mathbb{N}}$ in the positive light cone $\mathbb{N}^4_+$.

To motivate our choice of the associate surface in $\mathbb{R}^{1,4}$ of $\hat{x}$ we first introduce the so-called homogeneous coordinate for $\mathbb{R}^{1,4}$ used in the ambient space construction of [Fefferman and Graham 1985; 2012], that is,

$$(t, x) = x^0(1, \hat{x}) + x^0 x^\infty \tfrac{1}{2}(1, -\hat{x}) \tag{2.2.1}$$

where

$$x^0 = \frac{1}{2}(r + t), \quad x^0 x^\infty = (-r + t)$$

and $r = |x|$ and $x = r\hat{x}$. In this coordinate the Minkowski metric is

$$\tilde{\mathcal{G}}_0 = -2x^\infty (dx^0)^2 - 2x^0\, dx^0\, dx^\infty + (x^0)^2 \big(1 - \tfrac{1}{2}x^\infty\big)^2 g_0(\hat{x}).$$

Hence, given a surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$, we are looking to construct an associate homogeneous timelike 4-surface

$$\tilde{x} = \alpha y + \alpha\rho y^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4} \tag{2.2.2}$$

if we can have canonically the null vector y^* at a given null position y on $x^{\mathbb{N}}$. It is clear that the associate surface $\tilde{x}$ is ruled by the positive quadrants of timelike 2-planes in Minkowski spacetime. One may consider the intersection of $\tilde{x}$ with the hyperboloid $\mathbb{H}^4$:

$$x^+ = \frac{1}{\sqrt{2}}(e^t y + e^{-t} y^*) : \mathbb{R} \times \mathrm{M}^2 \to \mathbb{H}^4, \tag{2.2.3}$$

which is called the associate ruled 3-surface since it is a 3-surface in hyperbolic 4-space ruled by geodesics lines. Recall that a geodesic line in the hyperboloid $\mathbb{H}^4$ is the intersection of the hyperboloid with a timelike 2-subspaces in Minkowski spacetime. In the following we will introduce the canonical choice of such y^*.

2.3. ***Conformal Gauss maps.*** Let us consider any unit spacelike normal vector to the homogeneous null 3-surface $x^{\mathbb{N}} = \alpha y$ in $\mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$. That is to ask a unit spacelike 5-vector ξ to satisfy

$$\langle \xi, x^{\mathbb{N}} \rangle = 0, \quad \langle \xi, x^{\mathbb{N}}_{u^1} \rangle = 0, \quad \langle \xi, x^{\mathbb{N}}_{u^2} \rangle = 0, \tag{2.3.1}$$

which implies that

$$\xi = ay + \vec{\boldsymbol{n}},$$

where $\vec{\boldsymbol{n}} = (0, \boldsymbol{n})$ is the unit normal to the surface $\hat{x}$ in the standard unit round 3-sphere in $\{1\} \times \mathbb{R}^4 \subset \mathbb{R}^{1,4}$. It turns out that there is a unique choice if we insist that the map

$$\xi : \mathrm{M}^2 \to \mathbb{S}^{1,3} \subset \mathbb{R}^{1,4}$$

is (weakly) conformal. Namely we have

Lemma 2.3.1. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface. For a unit normal vector ξ to the homogeneous null 3-surface $x^{\mathbb{N}} = \alpha y : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$,*

$$\langle \xi_{u^1}, \xi_{u^2} \rangle = 0$$

if and only if

$$\xi = Hy + \vec{\boldsymbol{n}}$$

and

$$\langle d\xi, d\xi \rangle = \tfrac{1}{2} E |\mathring{I\!I}|^2 |du|^2. \tag{2.3.2}$$

Proof. It is simply a straightforward calculation. We know

$$\xi_{u^i} = a_{u^i}(1, \hat{x}) + a(0, \hat{x}_{u^i}) + (0, \boldsymbol{n}_{u^i}).$$

Hence we have

$$\langle \xi_{u^1}, \xi_{u^2} \rangle = -2af + \frac{1}{E}(fe + fg) = 0,$$

which is equivalent to $a = H$. For the rest we calculate

$$\langle \xi_{u^1}, \xi_{u^1} \rangle = \langle \xi_{u^2}, \xi_{u^2} \rangle = \frac{1}{E^2}\Big(f^2 + \Big(\frac{e-g}{2}\Big)^2\Big)E. \qquad \square$$

Another way to identify a unique unit spacelike normal vector to the homogeneous null 3-surface $x^{\mathbb{N}} = \alpha y : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+$ is the following:

Lemma 2.3.2. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface. Then, for a unit spacelike normal vector ξ to $x^{\mathbb{N}} = \alpha y : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$,*

$$\xi = Hy + \vec{\boldsymbol{n}} \tag{2.3.3}$$

if and only if

$$\langle \Delta\xi, y \rangle = 0. \tag{2.3.4}$$

Proof. We simply calculate, for $\xi = a(1, \hat{x}) + (0, \boldsymbol{n})$,

$$\Delta_0\xi = \xi_{u^1u^1} + \xi_{u^2u^2} = (\Delta_0 a)(1, \hat{x}) + 2\nabla a(0, \nabla\hat{x}) + a(0, \Delta_0\hat{x}) + (0, \Delta_0\boldsymbol{n})$$

and

$$\langle \Delta_0\xi, (1, \hat{x}) \rangle = -2aE + 2HE.$$

Notice that $\Delta = E^{-1}\Delta_0$. $\square$

Before we give a formal definition of the conformal Gauss map we remark that (2.3.4) is the integrability condition for the unit vector field ξ to be the conformal Gauss map (up to a sign) for the surface $\hat{x}$. This turns out to be the easiest way to see that $\hat{x}$ is Willmore if and only if the conformal Gauss map ξ of $\hat{x}$ is also the conformal Gauss map (up to a sign) of the conformal transform $\hat{x}^*$ (see Definition 2.4.1).

Definition 2.3.3. Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is a surface. Then we will call

$$\xi = Hy + \vec{\boldsymbol{n}} : \mathrm{M}^2 \to \mathbb{S}^{1,3} \subset \mathbb{R}^{1,4} \tag{2.3.5}$$

the conformal Gauss map according to Blaschke (cf. [Bryant 1984; Rigoli 1987]).

For a positive function λ on the sphere $\mathbb{S}^3$ we consider the conformal metric $\lambda^2 g_0$ on the sphere $\mathbb{S}^3$, which can be realized as the 3-sphere $\mathbb{S}^3_\lambda$: $\lambda(1, x) : \mathbb{S}^3 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$ in Minkowski spacetime. It is essential here that the surface $\hat{x}$ in the 3-sphere $\mathbb{S}^3$ with the conformal metric $\lambda^2 g_0$ is realized as the 2-surface $\hat{\lambda}(1, \hat{x})$:

$\mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$ inside the 3-sphere $\mathbb{S}^3_\lambda$. It is helpful to see the calculations in Appendix B about the geometry of the 3-sphere $\mathbb{S}^3_\lambda$ in Minkowski spacetime $\mathbb{R}^{1,4}$.

Lemma 2.3.4. *If one works with a conformal metric $\lambda^2 g_0$ in general, then*

$$\xi = \xi_\lambda = H_\lambda y_\lambda + \vec{\boldsymbol{n}}_\lambda, \tag{2.3.6}$$

where $\vec{\boldsymbol{n}}_\lambda = \vec{\boldsymbol{n}} + (\log\lambda)_{\boldsymbol{n}} y$ is the unit normal to the surface

$$y_\lambda = \hat{\lambda}(1, \hat{x}) : M^2 \to \mathbb{S}^3_\lambda \subset \mathbb{N}^4_+.$$

Proof. It is easily seen that the normal direction to the surface y_λ inside $\mathbb{S}^3_\lambda$ is $\lambda_{\boldsymbol{n}}(1, \hat{x}) + \lambda(0, \boldsymbol{n})$ and that $\langle \lambda_{\boldsymbol{n}}(1, \hat{x}) + \lambda(0, \boldsymbol{n}), \lambda_{\boldsymbol{n}}(1, \hat{x}) + \lambda(0, \boldsymbol{n})\rangle = \lambda^2$. Therefore the unit normal for the surface y_λ in $\mathbb{S}^3_\lambda$ is $\vec{\boldsymbol{n}}_\lambda = \vec{\boldsymbol{n}} + (\log\lambda)_{\boldsymbol{n}} y$. Hence (2.1.8) yields

$$H_\lambda y_\lambda + \vec{\boldsymbol{n}}_\lambda = H y + \vec{\boldsymbol{n}}. \qquad \square$$

In light of (2.3.2), the conformal Gauss map gives rise a spacelike 2-surface

$$\xi : \mathrm{M}^2 \to \mathbb{S}^{1,3} \subset \mathbb{R}^{1,4}$$

when the original surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is free of umbilical points. We will have more detailed discussions for the reasons to call ξ the conformal Gauss map in Section 2.7.

It is interesting that Blaschke came across the conformal Gauss map from a different perspective. He considered the family of mean curvature 2-spheres to the surface $\hat{x}$ in $\mathbb{S}^3$. A round 2-sphere in 3-sphere can be thought of as the intersection of a timelike hyperplane and the 3-sphere at time $t = 1$ in Minkowski spacetime $\mathbb{R}^{1,4}$ and a timelike hyperplane in $\mathbb{R}^{1,4}$ is described by a unit normal vector lying in de Sitter 4-spacetime $\mathbb{S}^{1,3}$. Given a direction $(H, H\hat{x} + \boldsymbol{n}) \in \mathbb{S}^{1,3}$, the hyperplane perpendicular to that in $\mathbb{R}^{1,4}$ is given by the first equation in (2.3.1):

$$\langle (s, z), (H, H\hat{x} + \boldsymbol{n})\rangle = 0, \tag{2.3.7}$$

which is

$$-sH + Hz \cdot \left(\hat{x} + \frac{1}{H}\boldsymbol{n}\right) = 0.$$

At the level $s = 1$ in the 3-sphere $|z| = 1$, we arrive at

$$1 - \hat{z} \cdot \left(\hat{x} + \frac{1}{H}\boldsymbol{n}\right) = 0.$$

We may rewrite this as

$$\left|\hat{z} - \left(\hat{x} + \frac{1}{H}\boldsymbol{n}\right)\right|^2 = \frac{1}{H^2} \tag{2.3.8}$$

which clearly is a round 2-sphere of mean curvature H when intersects with the 3-sphere $\mathbb{S}^3 \subset \mathbb{R}^4$ at $t = 1$ in $\mathbb{R}^{1,4}$. Hence the equations (2.3.1) exactly ask the surface $y = (1, \hat{x}) : \mathrm{M}^2 \to \mathbb{S}^3 \subset \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$ to be an envelope surface of the family of mean curvature 2-spheres described by the conformal Gauss map ξ.

It is known that a mean curvature sphere of a surface goes to the mean curvature sphere of the image surface under conformal transformations.

2.4. ***Conformal transforms.*** Assume that the surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is free of umbilical points. Then the conformal Gauss map induces a spacelike 2-surface in the de Sitter 4-space $\mathbb{S}^{1,3}$

$$\xi : \mathrm{M}^2 \to \mathbb{S}^{1,3} \subset \mathbb{R}^{1,4}.$$

One notices that the equations (2.3.1) imply that $y = (1, \hat{x})$ is naturally a null normal vector the surface ξ in the de Sitter 4-spacetime $\mathbb{S}^{1,3}$. Because

$$\langle y, \xi_{u^i} \rangle = -\langle \xi, y_{u^i} \rangle = 0.$$

Hence it is natural to take the other null normal vector y^* such that

$$\begin{aligned} &\langle y^*, y\rangle = -1, \quad \langle y^*, y^*\rangle = 0, \quad \langle y^*, \xi \rangle = 0, \\ &\langle y^*, \xi_{u^1} \rangle = 0 \quad \text{and} \quad \langle y^*, \xi_{u^2} \rangle = 0. \end{aligned} \tag{2.4.1}$$

We may write

$$y^* = \hat{\mu}^*(1, \hat{x}^*).$$

Definition 2.4.1. Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is a surface with no umbilical point. And suppose that

$$y^* = \hat{\mu}^*(1, \hat{x}^*) : \mathrm{M}^2 \to \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$$

satisfies the equations (2.4.1) for $y = (1, \hat{x})$. Then the surface

$$\hat{x}^* : \mathrm{M}^2 \to \mathbb{S}^3$$

is said to be the conformal transform of the surface $\hat{x}$ according to [Bryant 1984] (cf. [Blaschke 1929]).

It is important that the conformal transform $\hat{x}^*$ of a surface $\hat{x}$ is independent of the conformal factor λ. Notice that the equations in (2.4.1) remain the same except the first one when replacing y by y_λ. It is again very interesting to recall how Blaschke discovered the surface $\hat{x}^*$. From the above discussions it is now easy to see that the surface $\hat{x}^*$ is nothing but the other envelope surface of the family of round 2-spheres described by the conformal Gauss map ξ, i.e., the family of the mean curvature spheres of the surface $\hat{x}$, since y^* satisfies the last three equations in (2.4.1).

2.5. *The geometry of the surface ξ in $\mathbb{S}^{1,3}$.* Recall that the first fundamental form for the surface ξ in the de Sitter spacetime $\mathbb{S}^{1,3} \subset \mathbb{R}^{1,4}$ is

$$I^{\xi} = \langle d\xi, d\xi \rangle = m|du|^2, \tag{2.5.1}$$

where

$$m = \tfrac{1}{2} E |\mathring{II}|^2. \tag{2.5.2}$$

The first fundamental form I^{ξ} is usually called the Möbius metric on the surface $\hat{x}$. If one works with a conformal metric $\lambda^2 g_0$ instead, then the Möbius metric remains the same:

$$m = m_{\lambda} = \tfrac{1}{2} E_{\lambda} |\mathring{II}_{\lambda}|^2. \tag{2.5.3}$$

The second fundamental form for the surface ξ in $\mathbb{S}^{1,3}$ is given by

$$II^{\xi} = -\langle d\xi, dy \rangle y - \langle d\xi, dy^* \rangle y^* = \Omega y + \Omega^* y^* = \Omega_{\lambda} \hat{\lambda}^{-2} y_{\lambda} + \Omega^*_{\lambda} \hat{\lambda}^2 y^*_{\lambda}$$

and

$$\begin{aligned} \Omega_{ij} &= -\langle \xi_{u^i}, y_{u^j} \rangle, & \Omega^*_{ij} &= -\langle \xi_{u^i}, y^*_{u^j} \rangle, \\ (\Omega_{\lambda})_{ij} &= -\langle \xi_{u^i}, (y_{\lambda})_{u^j} \rangle = \hat{\lambda} \Omega_{ij}, & (\Omega^*_{\lambda})_{ij} &= -\langle \xi_{u^i}, (y^*_{\lambda})_{u^j} \rangle = \hat{\lambda}^{-1} \Omega^*_{ij}. \end{aligned} \tag{2.5.4}$$

In fact it is easy to calculate that

$$\Omega = \begin{bmatrix} \dfrac{e-g}{2} & f \\ f & \dfrac{g-e}{2} \end{bmatrix} = \mathring{II}. \tag{2.5.5}$$

Let us first calculate the mean curvature in the y^* direction. We notice that

$$\langle \Delta_0 \xi, \ y^*_{\lambda} \rangle = ((\Omega^*_{\lambda})_{11} + (\Omega^*_{\lambda})_{22})$$

while

$$\langle \Delta_0 \xi, \ y_{\lambda} \rangle = ((\Omega_{\lambda})_{11} + (\Omega_{\lambda})_{22}) = 0.$$

Based on the calculations

$$\begin{aligned} \langle \Delta_0 \xi, \ \xi \rangle &= -2m, \\ \langle \Delta_0 \xi, \ \xi_{u^1} \rangle &= \tfrac{1}{2} m_{u^1} - \tfrac{1}{2} m_{u^1} = 0, \\ \langle \Delta_0 \xi, \ \xi_{u^2} \rangle &= -\tfrac{1}{2} m_{u^2} + \tfrac{1}{2} m_{u^2} = 0, \end{aligned}$$

we obtain

$$\begin{aligned} \Delta_0 \xi &= -((\Omega^*_{\lambda})_{11} + (\Omega^*_{\lambda})_{22}) y_{\lambda} - 2m\xi \\ &= (-((\Omega^*_{\lambda})_{11} + (\Omega^*_{\lambda})_{22}) - 2m H_{\lambda}) y_{\lambda} - 2m \vec{\boldsymbol{n}}_{\lambda}. \end{aligned} \tag{2.5.6}$$

On the other hand, we directly calculate

$$\begin{aligned}(2.5.7)\quad \Delta_0\xi &= \Delta_0(H_\lambda y_\lambda + \vec{\boldsymbol{n}}_\lambda)\\ &= (\Delta_0 H_\lambda)y_\lambda + H_\lambda \Delta_0 y_\lambda + 2(H_\lambda)_{u^1}(y_\lambda)_{u^1} + 2(H_\lambda)_{u^2}(y_\lambda)_{u^2} + \Delta_0\vec{\boldsymbol{n}}_\lambda.\end{aligned}$$

It seems that the best way to calculate geometrically is to use the Lorentz orthogonal frame

$$\{y_\lambda, y_\lambda^\dagger, (y_\lambda)_{u^1}, (y_\lambda)_{u^2}, \vec{\boldsymbol{n}}_\lambda\},$$

where

$$(2.5.8)\quad \langle y_\lambda^\dagger, y_\lambda\rangle = -1, \quad \langle y_\lambda^\dagger, y_\lambda^\dagger\rangle = \langle y_\lambda^\dagger, (y_\lambda)_{u^1}\rangle = \langle y_\lambda^\dagger, (y_\lambda)_{u^2}\rangle = \langle y_\lambda^\dagger, \vec{\boldsymbol{n}}_\lambda\rangle = 0.$$

We find that

$$(2.5.9)\qquad y_\lambda^\dagger = \frac{1}{\lambda}\big(\tfrac{1}{2}|\nabla \log\lambda|^2 y + y^\dagger - \nabla\log\lambda\big),$$

where $y^\dagger = \frac{1}{2}(1, -\hat{x})$ and ∇ is the gradient on the standard round 3-sphere. We will apply the inner product with the null vector $y_\lambda^\dagger$ to both (2.5.6) and (2.5.7). To calculate $H_\lambda\langle\Delta_0 y_\lambda, y_\lambda^\dagger\rangle + \langle\Delta_0\vec{\boldsymbol{n}}_\lambda, y_\lambda^\dagger\rangle$ we rewrite

$$H_\lambda\langle\Delta_0 y_\lambda, y_\lambda^\dagger\rangle = -H_\lambda(\langle(y_\lambda)_{u^1}, (y_\lambda^\dagger)_{u^1}\rangle + \langle(y_\lambda)_{u^2}, (y_\lambda^\dagger)_{u^2}\rangle)$$

and

$$\langle\Delta_0\vec{\boldsymbol{n}}_\lambda, y_\lambda^\dagger\rangle = -\langle(\vec{\boldsymbol{n}}_\lambda)_{u^1}, (y_\lambda^\dagger)_{u^1}\rangle - \langle(\vec{\boldsymbol{n}}_\lambda)_{u^2}, (y_\lambda^\dagger)_{u^2}\rangle - \langle\vec{\boldsymbol{n}}_\lambda, (y_\lambda^\dagger)_{u^i}\rangle_{u^i}.$$

Meanwhile one may calculate

$$(2.5.10)\qquad \begin{cases}(\vec{\boldsymbol{n}}_\lambda)_{u^1} = -\dfrac{e_\lambda}{E_\lambda}(y_\lambda)_{u^1} - \dfrac{f_\lambda}{E_\lambda}(y_\lambda)_{u^2} - \langle(\vec{\boldsymbol{n}}_\lambda)_{u^1}, y_\lambda^\dagger\rangle y_\lambda\\ (\vec{\boldsymbol{n}}_\lambda)_{u^2} = -\dfrac{f_\lambda}{E_\lambda}(y_\lambda)_{u^1} - \dfrac{g_\lambda}{E_\lambda}(y_\lambda)_{u^2} - \langle(\vec{\boldsymbol{n}}_\lambda)_{u^2}, y_\lambda^\dagger\rangle y_\lambda.\end{cases}$$

Hence we have

$$\begin{aligned}(2.5.11)\quad & H_\lambda\langle\Delta_0 y_\lambda, y_\lambda^\dagger\rangle + \langle\Delta_0\vec{\boldsymbol{n}}_\lambda, y_\lambda^\dagger\rangle\\ &= E_\lambda^{-1}(\mathring{I\!I}_\lambda)_{ij}\langle(y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^j}\rangle - \langle\vec{\boldsymbol{n}}_\lambda, (y_\lambda^\dagger)_{u^i}\rangle_{u^i}\\ &= -E_\lambda^{-1}(\mathring{I\!I}_\lambda)_{ij}R^\lambda_{i3j3} + E_\lambda(R^\lambda)_{3i,}{}^{i}\end{aligned}$$

due to (B.6), (B.7), and (B.8). Now we obtain the mean curvature of the surface ξ in the de Sitter spacetime $\mathbb{S}^{1,3}$.

Lemma 2.5.1. *Suppose that $\hat{x}: \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point and that $\xi : \mathrm{M}^2 \to \mathbb{S}^{1,3}$ is the conformal Gauss map. Then the surface ξ is* *spacelike and its mean curvature is a null vector*

$$H^{\xi} = 2\hat{\lambda}^2 \frac{\mathcal{H}_\lambda}{|\mathring{II}_\lambda|^2} y^*_\lambda \tag{2.5.12}$$

for any positive function λ on the 3-sphere $\mathbb{S}^3$, where

$$\mathcal{H}_\lambda = \Delta_\lambda H_\lambda + |\mathring{II}_\lambda|^2 H_\lambda + (\mathring{II}_\lambda)^{ij}(R^\lambda)_{i3j3} - (R^\lambda)_{3i,}{}^{i}, \tag{2.5.13}$$

and $(R^\lambda)_{ijkl}$, $(R^\lambda)_{ij}$ are the Riemann curvature and Ricci curvature for the conformal metric $\lambda^2 g_0$ on the 3-sphere $\mathbb{S}^3$.

Proof. We perform inner product to (2.5.6) and (2.5.7) by the null vector $y^\dagger_\lambda$ and obtain that

$$(\Omega^*_\lambda)_{11} + (\Omega^*_\lambda)_{22} = E_\lambda(-\Delta_\lambda H_\lambda - |\mathring{II}_\lambda|^2 H_\lambda - (\mathring{II}_\lambda)^{ij}(R^\lambda)_{i3j3} + (R^\lambda)_{3i,}{}^{i}) \tag{2.5.14}$$

in the light of (2.5.11). Then one can easily calculate the mean curvature for ξ in $\mathbb{S}^{1,3}$. □

We remark that (2.5.12) actually shows that

$$\mathcal{H}_\lambda = \hat{\lambda}^{-3}(-\Delta H - |\mathring{II}|^2 H) \tag{2.5.15}$$

for a surface $\hat{x}$ in the conformal 3-sphere.

Most of the next theorem was known to Blaschke [1929] and Bryant [1984].

Theorem 2.5.2. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point. Then $\hat{x}$ is a Willmore surface in $\mathbb{S}^3$ if and only if the conformal Gauss map induces a minimal spacelike surface in the de Sitter spacetime $\mathbb{S}^{1,3}$. Moreover its conformal transform $\hat{x}^*$ is a dual Willmore surface in $\mathbb{S}^3$.*

Proof. Because Lemma 2.3.2 implies that ξ is also the conformal Gauss map (up to the sign) for $\hat{x}^*$ when H^ξ vanishes. The two dual Willmore surfaces are the two envelope surfaces of the family of round 2-spheres described by the conformal Gauss map ξ. □

Remark 2.5.3. It is also known to [Blaschke 1929] and [Bryant 1984] that if $\hat{x}$ is a minimal surface in $\mathbb{S}^3$, then $\hat{x}^* = -\hat{x}$; and that $\hat{x}$ is a Willmore surface if and only if $\hat{x}^{**} = \hat{x}$. An interesting question then arises: is it possible to have $\hat{x}^{***} = \hat{x}$, and if so what would that equality imply for the surface?

2.6. *Finding y^*_λ.* Let us now solve y^*_λ for $y_\lambda = \hat{\lambda}(1, \hat{x}) = \hat{\lambda} y$, where $\hat{\lambda} = \lambda \circ \hat{x}$ and λ is a positive function on the sphere $\mathbb{S}^3$. At each point on the surface we set

$$y^*_\lambda = \kappa y_\lambda + \kappa_\dagger y^\dagger_\lambda + b\vec{n}_\lambda + \frac{\omega^\lambda_1}{E_\lambda}(y_\lambda)_{u^1} + \frac{\omega^\lambda_2}{E_\lambda}(y_\lambda)_{u^2}.$$

We get from (2.4.1)

$$\left\{\begin{aligned} \kappa_\dagger &= 1, \\ -2\kappa_+\kappa_- + b^2 + \frac{(\omega_1^\lambda)^2 + (\omega_2^\lambda)^2}{E_\lambda} &= 0, \\ b &= H_\lambda, \\ -(\Omega_\lambda)_{11}\omega_1^\lambda - (\Omega_\lambda)_{12}\omega_2^\lambda &= (H_\lambda)_{u^1} E_\lambda, \\ -(\Omega_\lambda)_{21}\omega_1^\lambda - (\Omega_\lambda)_{22}\omega_2^\lambda &= (H_\lambda)_{u^2} E_\lambda. \end{aligned}\right. \tag{2.6.1}$$

Lemma 2.6.1. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point. Then*

$$y_\lambda^* = \tfrac{1}{2}(|\omega^\lambda|^2 + H_\lambda^2) y_\lambda + y_\lambda^\dagger + H_\lambda \vec{\boldsymbol{n}}_\lambda - (\mathring{I\!I})_\lambda^{-1} d H_\lambda \tag{2.6.2}$$

for any positive function λ on the 3*-sphere*, *where*

$$|\omega^\lambda|^2 = \frac{(\omega_1^\lambda)^2 + (\omega_2^\lambda)^2}{E_\lambda} = \frac{1}{m}((H_\lambda)_{u^1}^2 + (H_\lambda)_{u^2}^2).$$

In particular,

$$y^* = \tfrac{1}{2}(|\omega|^2 + H^2) y + \tfrac{1}{2}(1, -\hat{x}) + H(0, \mathbf{n}) - (0, (\mathring{I\!I})^{-1} d H), \tag{2.6.3}$$

and

$$x^* = a\hat{x} + \frac{H}{1-a}\boldsymbol{n} - \frac{1}{1-a}(\mathring{I\!I})^{-1} d H, \tag{2.6.4}$$

where

$$a = \frac{|\omega|^2 + H^2 - 1}{|\omega|^2 + H^2 + 1}. \tag{2.6.5}$$

Proof. One simply solves (2.6.1) if $\det \Omega_\lambda \neq 0$, which is equivalent to the fact that the surface has no umbilical point. □

2.7. ***Canonicity of y^*.*** We next show that the choice of y^* is canonical in terms of doing conformal geometry for the surface $\hat{x}$ in $\mathbb{S}^3$. Two issues are involved; the first concerns the symmetry of the conformal 3-sphere. To be precise, for a conformal transformation

$$\phi : \mathbb{S}^3 \to \mathbb{S}^3$$

and the transformed surface

$$\phi(\hat{x}) : \mathrm{M}^2 \to \mathbb{S}^3,$$

is it true that

$$\tilde{\phi}(\tilde{x}) = \alpha\tilde{\phi}(y) + \alpha\rho\tilde{\phi}(y^*) : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4}$$

is the associate 4-surface of $\phi(\hat{x})$ in $\mathbb{R}^{1,4}$, where $\tilde{\phi}$ is the corresponding Lorentz transformation on $\mathbb{R}^{1,4}$ to ϕ? The other issue is whether or not the associate surface $\tilde{x}$ is independent of metrics in the conformal class of the round 3-sphere. The first easy and important fact is that the conformal Gauss map is independent of the metrics in the conformal class.

Lemma 2.7.1. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface. Then the conformal Gauss map ξ is independent of the metrics in the conformal class of the round 3-sphere $\mathbb{S}^3$. Meanwhile, the conformal Gauss map for the transformed surface $\phi(\hat{x})$ is exactly $\tilde{\phi}(\xi)$, where $\tilde{\phi}$ is the Lorentz transformation on the Minkowski spacetime $\mathbb{R}^{1,4}$ corresponding to a conformal transformation ϕ on $\mathbb{S}^3$.*

Proof. First of all, one needs to realize that, for any given metric in the conformal class of the round 3-sphere, it simply amounts to consider the surface

$$y_\lambda = \hat{\lambda}(1, \hat{x}) : \mathrm{M}^2 \to \mathbb{N}^4_+$$

for some positive function $\lambda : \mathbb{S}^3 \to \mathbb{R}^+$ and $\hat{\lambda} = \lambda \circ \hat{x}$. But this might only alter the parametrization of the homogeneous null 3-surface $x^{\mathbb{N}} = \alpha\hat{\lambda}(1, \hat{x}) : \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{N}^4_+$. Hence it will not alter the conformal Gauss map. Of course we have already seen this from Lemma 2.3.4.

Next we consider the transformed surface $\phi(\hat{x})$. Recall that, given a conformal transformation ϕ of 3-sphere, we have a unique Lorentz transformation $\tilde{\phi}$ in the time and orientation preserving component of the Lorentz group on the Minkowski spacetime such that, for $\lambda(1, \hat{x}) \in \mathbb{R}^{1,4}$,

$$\tilde{\phi}(\lambda(1, \hat{x})) = \lambda\mu(1, \phi(\hat{x})) \tag{2.7.1}$$

for some positive number μ. By the definition, which requires that $\tilde{\phi}$ be a linear map and that

$$\langle \tilde{\phi}((t, \hat{x})), \tilde{\phi}((s, \hat{y})) \rangle = \langle (t, \hat{x}), (s, \hat{y}) \rangle,$$

we now easily see that $\tilde{\phi}(\xi)$ is the conformal Gauss map for the transformed surface $\phi(\hat{x})$. Since $\tilde{\phi}(\xi)$ is the unit normal vector field to the homogeneous null 3-surface $\tilde{\phi}(x)$ in $\mathbb{N}^4_+$ that is conformal map from M^2 to $\mathbb{S}^{1,3}$. □

Consequently:

Proposition 2.7.2. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point. Then the associate surface*

$$\tilde{x} = \alpha y_\lambda + \alpha\rho y^*_\lambda : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4},$$

for any $y_\lambda = \hat{\lambda}(1, \hat{x})$ and $y^* = \hat{\lambda}^{-1}\lambda^*(1, \hat{x}^*)$ defined by the equations (2.4.1), is independent of the metrics in conformal class of the round 3-sphere $\mathbb{S}^3$.

Proof. It suffices to verify that

$$(\hat{\lambda}y)^* = \hat{\lambda}^{-1}y^*. \tag{2.7.2}$$

Since it implies that the change of metrics in the conformal class will at most cause a possible change of parametrization of the associate surface $\tilde{x}$. □

Lemma 2.7.3. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point. Let $y_\lambda = \hat{\lambda}(1, \hat{x}) \in \mathbb{N}^4_+$ and let ϕ be a conformal transformation of 3-sphere. Then*

$$\tilde{\phi}(y_\lambda)^* = \tilde{\phi}(y_\lambda^*). \tag{2.7.3}$$

Hence

$$\phi(\hat{x}^*) = (\phi(\hat{x}))^*. \tag{2.7.4}$$

Proof. From Lemma 2.7.1 we know that the conformal Gauss map for the transformed surface $\phi(\hat{x})$ is $\tilde{\phi}(\xi)$. Then it is easy to verify (2.4.1) for $\tilde{\phi}(y^*)$ to be $\tilde{\phi}(y)^*$. Then the Equation (2.7.4) follows from (2.7.1) and (2.7.3):

$$\hat{\gamma}^*(1, (\phi(\hat{x}))^*) = \tilde{\phi}(y)^* = \tilde{\phi}(y^*) = \hat{\mu}^*\hat{\lambda}^*(1, \phi(\hat{x}^*)). \qquad \square$$

Therefore:

Proposition 2.7.4. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point. Let ϕ be a conformal transformation of 3-sphere. Then the associate 4-surface in $\mathbb{R}^{1,4}$ of the transformed surface $\phi(\hat{x})$ is exactly the 4-surface $\tilde{\phi}(\tilde{x})$ transformed from the associate 4-surface $\tilde{x}$ of the original surface $\hat{x}$ under the corresponding Lorentz transformation $\tilde{\phi}$ of ϕ.*

3. The geometry of the associate surfaces

In this section we calculate the first and second fundamental forms for the associate homogeneous timelike 4-surfaces $\tilde{x}$ in $\mathbb{R}^{1,4}$ as well as for the associate ruled surface x^+ in the hyperboloid $\mathbb{H}^4$, for a given immersed 2-surface $\hat{x}$ in $\mathbb{S}^3$.

3.1. ***The first fundamental form for $\tilde{x}$ in $\mathbb{R}^{1,4}$.*** To calculate the first fundamental form for the surface in the parametrization

$$\tilde{x} = \alpha y_\lambda + \alpha\rho y_\lambda^* \tag{3.1.1}$$

associated with a conformal metric $\lambda^2 g_0$ on the 3-sphere $\mathbb{S}^3$, we first calculate

$$d\tilde{x} = (y_\lambda + \rho y_\lambda^*)d\alpha + \alpha y_\lambda^* d\rho + (\alpha(y_\lambda)_{u^1} + \alpha\rho(y_\lambda^*)_{u^1})du^1 + (\alpha(y_\lambda)_{u^2} + \alpha\rho(y_\lambda^*)_{u^2})du^2.$$

Hence the first fundamental form for the associate 4-surface $\tilde{x}$ in the coordinates (α, ρ, u^1, u^2) is

$$\begin{aligned} I^{\tilde{x}} = \langle d\tilde{x}, d\tilde{x}\rangle = -2\rho d\alpha d\alpha - 2\alpha d\alpha d\rho \\ + 2\alpha^2\langle (y_\lambda^*, (y_\lambda)_{u^1}\rangle d\rho du^1 + 2\alpha^2\langle y_\lambda^*, (y_\lambda)_{u^2}\rangle d\rho du^2 \\ + \langle \alpha(y_\lambda)_{u^1} + \alpha\rho(y_\lambda^*)_{u^1}, \alpha(y_\lambda)_{u^1} + \alpha\rho(y_\lambda^*)_{u^1}\rangle (du^1)^2 \\ + \langle \alpha(y_\lambda)_{u^2} + \alpha\rho(y_\lambda^*)_{u^2}, \alpha(y_\lambda)_{u^2} + \alpha\rho(y_\lambda^*)_{u^2}\rangle (du^2)^2 \\ + 2\langle \alpha(y_\lambda)_{u^1} + \alpha\rho(y_\lambda^*)_{u^1}, \alpha(y_\lambda)_{u^2} + \alpha\rho(y_\lambda^*)_{u^2}\rangle du^1 du^2. \end{aligned}$$

In fact one may calculate

$$\begin{aligned} (y_\lambda)_{u^1} &= -\omega_1^\lambda y_\lambda - \frac{(\Omega_\lambda)_{11}}{m}\xi_{u^1} - \frac{(\Omega_\lambda)_{12}}{m}\xi_{u^2}, \\ (y_\lambda)_{u^2} &= -\omega_2^\lambda y_\lambda - \frac{(\Omega_\lambda)_{21}}{m}\xi_{u^1} - \frac{(\Omega_\lambda)_{22}}{m}\xi_{u^2}, \\ (y_\lambda^*)_{u^1} &= \omega_1^\lambda y_\lambda^* - \frac{(\Omega_\lambda)_{11}^*}{m}\xi_{u^1} - \frac{(\Omega_\lambda)_{12}^*}{m}\xi_{u^2}, \\ (y_\lambda^*)_{u^2} &= \omega_2^\lambda y_\lambda^* - \frac{(\Omega_\lambda)_{21}^*}{m}\xi_{u^1} - \frac{(\Omega_\lambda)_{22}^*}{m}\xi_{u^2}, \end{aligned} \tag{3.1.2}$$

where

$$\omega^\lambda = \langle dy_\lambda, y_\lambda^*\rangle = -I_\lambda(\Omega_\lambda^{-1} dH_\lambda) \tag{3.1.3}$$

based on (2.6.1). Now let us write $I^{\tilde{x}}$ in matrix form:

$$I_{\tilde{x}} = \begin{bmatrix} -2\rho & -\alpha & 0 & 0 \\ -\alpha & 0 & \alpha^2\omega_1^\lambda & \alpha^2\omega_2^\lambda \\ 0 & \alpha^2\omega_1^\lambda & \multicolumn{2}{c}{\multirow{2}{*}{$\alpha^2 F$}} \\ 0 & \alpha^2\omega_2^\lambda & & \end{bmatrix} \tag{3.1.4}$$

where

$$\begin{cases} F_{11} = \dfrac{1}{m}(p^2+q^2) + 2\rho(\omega_1^\lambda)^2, \\ F_{12} = F_{21} = \dfrac{1}{m}q(p+r) + 2\rho\omega_1^\lambda\omega_2^\lambda, \\ F_{22} = \dfrac{1}{m}(q^2+r^2) + 2\rho(\omega_2^\lambda)^2, \end{cases} \tag{3.1.5}$$

and

$$\begin{bmatrix} p & q \\ q & r \end{bmatrix} = \Omega_\lambda + \rho\Omega_\lambda^*.$$

It can be calculated that

$$\det I^{\tilde{x}} = -\frac{\alpha^6}{m^2}(pr-q^2)^2 = -\frac{\alpha^6}{4m^2}(E_\lambda^2|\Omega_\lambda+\rho\Omega_\lambda^*|^2-\rho^2((\Omega_\lambda^*)_{11}+(\Omega_\lambda^*)_{22})^2)^2 \tag{3.1.6}$$

which can tell us where the associate surface $\tilde{x}$ is degenerate. It is maybe a little surprising that it is actually not difficult to calculate the inverse of $I_{\tilde{x}}$. We present the calculations in Appendix A since they are straightforward calculations.

3.2. ***The second fundamental form for $\tilde{x}$ in $\mathbb{R}^{1,4}$.*** It is clear from the definition that the conformal Gauss map ξ is the unit normal vector for the associate 4-surface $\tilde{x}$ in $\mathbb{R}^{1,4}$. Hence the second fundamental form for $\tilde{x}$ in $\mathbb{R}^{1,4}$ is

$$II^{\tilde{x}} = -\langle d\tilde{x}, d\xi\rangle = (\alpha(\Omega_\lambda)_{ij}+\alpha\rho(\Omega_\lambda^*)_{ij})du^i du^j, \tag{3.2.1}$$

or in matrix form

$$II_{\tilde{x}} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha\Omega_\lambda+\alpha\rho\Omega_\lambda^* \end{bmatrix}.$$

Therefore the mean curvature for the associate 4-surface in $\mathbb{R}^{1,4}$ is

$$H^{\tilde{x}} = \mathrm{Tr}(I_{\tilde{x}})^{-1}II_{\tilde{x}}.$$

To calculate the mean curvature $H^{\tilde{x}}$ one only needs to know the low-right 2×2 block in the inverse of the matrix $I_{\tilde{x}}$. According to the calculations in Appendix A, particularly (A.3), (A.9) and (A.10), we therefore have

$$H^{\tilde{x}} = \frac{m}{\alpha(pr-q^2)^2}((q^2+r^2)p-2q^2(p+r)+(p^2+q^2)r) = \frac{m(p+r)}{\alpha(pr-q^2)}, \tag{3.2.2}$$

where

$$pr-q^2 = \det\Omega_\lambda - \rho\,\mathrm{Tr}\,\Omega_\lambda\Omega_\lambda^* + \rho^2\det\Omega_\lambda^*$$

and

$$p+r = \rho((\Omega_\lambda^*)_{11}+(\Omega_\lambda^*)_{22}) = -\rho E_\lambda\mathcal{H}_\lambda \tag{3.2.3}$$

in the light of (2.5.14).

Theorem 3.2.1. *Suppose that $\hat{x}: \mathrm{M}^2\to\mathbb{S}^3$ is an immersed surface with no umbilical point. Then $\hat{x}$ is a Willmore surface in $\mathbb{S}^3$ if and only if the associate 4-surface $\tilde{x}$ in $\mathbb{R}^{1,4}$ is minimal.*

Proof. Based on the above equations (3.2.3) and (3.2.2) we obtain

$$H^{\tilde{x}} = \frac{\rho \det \Omega_\lambda \mathcal{H}_\lambda}{\alpha(\det \Omega_\lambda - \rho \operatorname{Tr} \Omega_\lambda \Omega_\lambda^* + \rho^2 \det \Omega_\lambda^*)}. \qquad \square$$

3.3. ***Local fundamental theorem for surfaces in conformal 3-sphere.*** In this subsection we state and prove a local fundamental theorem for surfaces in conformal 3-sphere. In the previous section we have introduced the associate surface $\tilde{x}$ in Minkowski spacetime $\mathbb{R}^{1,4}$ from a given surface $\hat{x}$ in $\mathbb{S}^3$. From the geometric structure of the associate surface $\tilde{x}$ one can tell that its intersection with the positive light cone $\mathbb{N}^4_+$ is a homogeneous null 3-surface whose projectivization will recover the original surface $\hat{x}$ in $\mathbb{S}^3$.

Given a surface $\hat{x}$ in $\mathbb{S}^3$ with a isothermal coordinates (u^1, u^2) on the parameter space M^2, we have the first fundamental form I in matrix form

$$I = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix}$$

and the second fundamental fundamental II form in matrix form

$$II = \begin{bmatrix} e & f \\ f & g \end{bmatrix}$$

The local fundamental theorem for surfaces in Riemannian geometry states that, up to isometries of the standard round sphere $\mathbb{S}^3$, locally the surface is uniquely determined by the first fundamental form I and the second fundamental form II in the standard round sphere $\mathbb{S}^3$. Conversely, given a positive definite symmetric 2-form I and a symmetric 2-form II in the parameter domain, which satisfy some integrability conditions (Gauss-Codazzi equations), up to isometries, there is locally a unique surface $\hat{x}$ in the standard round sphere $\mathbb{S}^3$ whose first and second fundamental forms are I and II. We are looking for the analogous local fundamental theorem for surfaces in conformal round 3-sphere $\mathbb{S}^3$. The core idea of the local fundamental theorem in Riemannian geometry is to solve the structure equations, which are the equations of motion of Frenet frames on the surface and are determined from I and II.

Our strategy here is to use the local fundamental theorem for the associate surface $\tilde{x}$ in the Minkowski spacetime $\mathbb{R}^{1,4}$ to establish the local fundamental theorem for a surface $\hat{x}$ in the conformal sphere $\mathbb{S}^3$. Since the association introduced in previous subsections requires that the surface $\hat{x}$ has no umbilical point, we will always assume here that surfaces $\hat{x}$ have no umbilical point.

To summarize the previous discussions, given a surface $\hat{x}$ in $\mathbb{S}^3$, we have $I = E|du|^2$ and $II = e(du^1)^2 + 2f du^1 du^2 + g(du^2)^2$. We also have the so-called Möbius metric $I^\xi = m|du|^2 = \frac{1}{2}E|\mathring{II}|^2|du|^2$ induced from the Conformal Gauss map ξ of the surface $\hat{x}$, where

$$\mathring{II} = \begin{bmatrix} \frac{e-g}{2} & f \\ f & \frac{g-e}{2} \end{bmatrix}$$

is the traceless part of the second fundamental form II. We then construct the associate surface

$$\tilde{x} = \alpha y_\lambda + \alpha\rho y_\lambda^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbf{M}^2 : \mathbb{R}^{1,4}.$$

The first fundamental form $I^{\tilde{x}}$ for $\tilde{x}$ in matrix form is, from (3.1.4),

$$\begin{bmatrix} -2\rho & -\alpha & 0 & 0 \\ -\alpha & 0 & \alpha^2\omega_1^\lambda & \alpha^2\omega_2^\lambda \\ 0 & \alpha^2\omega_1^\lambda & \frac{\alpha^2}{m}(p^2+q^2)+2\alpha^2\rho(\omega_1^\lambda)^2 & \frac{\alpha^2}{m}q(p+r)+2\alpha^2\rho\omega_1^\lambda\omega_2^\lambda \\ 0 & \alpha^2\omega_2^\lambda & \frac{\alpha^2}{m}q(p+r)+2\alpha^2\rho\omega_1^\lambda\omega_2^\lambda & \frac{\alpha^2}{m}(q^2+r^2)+2\alpha^2\rho(\omega_2^\lambda)^2 \end{bmatrix},$$

where

$$\omega^\lambda = \omega_1^\lambda du^1 + \omega_2^\lambda du^2 = -d\log\hat{\lambda} - I(\Omega^{-1}(dH)) = d\log\hat{\lambda} + \omega.$$

The second fundamental form $II^{\tilde{x}}$ for $\tilde{x}$ in $\mathbb{R}^{1,4}$ in matrix form is, from (3.2.1),

$$\begin{bmatrix} 0 & 0 \\ 0 & \alpha\Omega_\lambda + \alpha\rho\Omega_\lambda^* \end{bmatrix},$$

where $\Omega_\lambda = \hat{\lambda}\Omega$ and $\Omega_\lambda^* = \hat{\lambda}^{-1}\Omega^*$. Notice that $I^{\tilde{x}}$ and $II^{\tilde{x}}$ are exactly determined by the Möbius metric $I^\xi = m|du|^2$, the 1-form ω, the traceless symmetric 2-tensor Ω and the symmetric 2-tensor Ω^*, plus the conformal factor $\hat{\lambda}$.

Next we write the equations for the motion of the Frenet frames on the associate surface $\tilde{x}$ according to $I^{\tilde{x}}$ and $II^{\tilde{x}}$. We consider the Frenet frame

$$\left\{ y_\lambda, y_\lambda^*, \tfrac{1}{\sqrt{m}}\xi_{u^1}, \tfrac{1}{\sqrt{m}}\xi_{u^2}, \xi \right\}$$

on the associate surface $\tilde{x}$, which are the orthonormal frames on $\tilde{x}$ with respect to the Minkowski metric $\tilde{\mathcal{G}}_0$ on $\mathbb{R}^{1,4}$. We now write

(3.3.1)
$$\frac{\partial}{\partial u^1}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix} = \begin{bmatrix} -\omega_1^\lambda & 0 & -\frac{1}{\sqrt{m}}(\Omega_\lambda)_{11} & -\frac{1}{\sqrt{m}}(\Omega_\lambda)_{12} & 0 \\ 0 & \omega_1^\lambda & -\frac{1}{m}(\Omega_\lambda^*)_{11} & -\frac{1}{m}(\Omega_\lambda^*)_{12} & 0 \\ \frac{1}{\sqrt{m}}(\Omega_\lambda)_{11} & \frac{1}{\sqrt{m}}(\Omega_\lambda^*)_{11} & 0 & -\frac{1}{2m}m_{u^2} & -\sqrt{m} \\ \frac{1}{\sqrt{m}}(\Omega_\lambda)_{21} & \frac{1}{\sqrt{m}}(\Omega_\lambda^*)_{21} & \frac{1}{2m}m_{u^2} & 0 & 0 \\ 0 & 0 & \sqrt{m} & 0 & 0 \end{bmatrix}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix}$$

and

(3.3.2)

$$\frac{\partial}{\partial u^2}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix} = \begin{bmatrix} -\omega_2^\lambda & 0 & -\frac{1}{\sqrt{m}}(\Omega_\lambda)_{21} & -\frac{1}{\sqrt{m}}(\Omega_\lambda)_{22} & 0 \\ 0 & \omega_2^\lambda & -\frac{1}{m}(\Omega_\lambda^*)_{21} & -\frac{1}{m}(\Omega_\lambda^*)_{22} & 0 \\ \frac{1}{\sqrt{m}}(\Omega_\lambda)_{21} & \frac{1}{\sqrt{m}}(\Omega_\lambda^*)_{21} & 0 & -\frac{1}{2m}m_{u^1} & 0 \\ \frac{1}{\sqrt{m}}(\Omega_\lambda)_{22} & \frac{1}{\sqrt{m}}(\Omega_\lambda^*)_{22} & \frac{1}{2m}m_{u^1} & 0 & -\sqrt{m} \\ 0 & 0 & 0 & \sqrt{m} & 0 \end{bmatrix}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix}.$$

Remember we also have the two trivial equations

$$\frac{\partial}{\partial \alpha}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix} = 0 \quad \text{and} \quad \frac{\partial}{\partial \rho}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix} = 0.$$

To solve the ODE systems (3.3.1) and (3.3.2), the necessary integrable condition is

$$\frac{\partial}{\partial u^1}\frac{\partial}{\partial u^2}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix} = \frac{\partial}{\partial u^2}\frac{\partial}{\partial u^1}\begin{bmatrix} y_\lambda \\ y_\lambda^* \\ \frac{1}{\sqrt{m}}\xi_{u^1} \\ \frac{1}{\sqrt{m}}\xi_{u^2} \\ \xi \end{bmatrix}. \tag{3.3.3}$$

It turns out that (3.3.3) is equivalent to the following six equations on the variables: the positive function m, the 1-form ω^λ, the traceless symmetric matrix Ω_λ and the symmetric matrix Ω_λ^*,

$$\begin{cases} (\Omega_\lambda)_{11,2} - (\Omega_\lambda)_{12,1} = \omega_1^\lambda(\Omega_\lambda)_{12} - \omega_2^\lambda(\Omega_\lambda)_{11}, \\ (\Omega_\lambda)_{12,2} - (\Omega_\lambda)_{22,1} = \omega_1^\lambda(\Omega_\lambda)_{22} - \omega_2^\lambda(\Omega_\lambda)_{12}, \end{cases} \tag{3.3.4}$$

(3.3.5)

$$\begin{cases} (\Omega_\lambda^*)_{11,2} - (\Omega_\lambda^*)_{12,1} = -\omega_1^\lambda(\Omega_\lambda^*)_{12} + \omega_2^\lambda(\Omega_\lambda^*)_{11} + \frac{1}{2}\frac{(\Omega_\lambda^*)_{11} + (\Omega_\lambda^*)_{22}}{|\Omega_\lambda|^2}(|\Omega_\lambda|^2)_{u^2}, \\ (\Omega_\lambda^*)_{12,2} - (\Omega_\lambda^*)_{22,1} = -\omega_1^\lambda(\Omega_\lambda^*)_{22} + \omega_2^\lambda(\Omega_\lambda^*)_{12} + \frac{1}{2}\frac{(\Omega_\lambda^*)_{11} + (\Omega_\lambda^*)_{22}}{|\Omega_\lambda|^2}(|\Omega_\lambda|^2)_{u^2}, \end{cases}$$

$$\omega_{1,2}^\lambda - \omega_{2,1}^\lambda = \frac{1}{m}((\Omega_\lambda)_{11} - (\Omega_\lambda)_{22})(\Omega_\lambda^*)_{12} - ((\Omega_\lambda^*)_{11} - (\Omega_\lambda^*)_{22})(\Omega_\lambda)_{12}), \tag{3.3.6}$$

and

$$(\mathcal{K} - 1) = \frac{1}{m^2}\operatorname{Tr}\Omega_\lambda\Omega_\lambda^*, \tag{3.3.7}$$

where $\mathcal{K}$ is the Gaussian curvature of the Möbius metric $I^{\xi} = m|du|^2$. Of course, as one may verify, (3.3.4), (3.3.5), (3.3.6) and (3.3.7) are exactly the Gauss-Codazzi equations for the surface ξ in the de Sitter spacetime $\mathbb{S}^{1,3}$ induced by the conformal Gauss map ξ of the surface $\hat{x}$ in conformal 3-sphere $\mathbb{S}^3$.

Now we are ready to state and prove the local fundamental theorem for surfaces in conformal round 3-sphere $\mathbb{S}^3$.

Theorem 3.3.1. *Suppose that, on a domain in $D \subset \mathbb{R}^2$, we are given*

- *a traceless symmetric 2-form Ω,*
- *a positive function m or equivalently E such that $m = \dfrac{-\det \Omega}{E}$,*
- *a 1-form ω,*
- *a symmetric 2-form Ω^*,*

and that they satisfy the integrability conditions (3.3.4)–(3.3.7). Then, for a given point p_0 in D, there exists an open neighborhood D_0 of p_0 in D, a parametrized surface $\hat{x} : D_0 \to \mathbb{S}^3$ with no umbilical point, and a positive function $\hat{\lambda} : D_0 \to \mathbb{R}^+$ with $\hat{\lambda}(p_0) = 1$, such that

- *$\Omega = \hat{\lambda}\mathring{II}$, where $\mathring{II}$ is the traceless part of the second fundamental form of $\hat{x}$ in the standard round $\mathbb{S}^3$,*
- *$m|du|^2 = \langle d\xi, d\xi\rangle$ is the Möbius metric induced by the conformal Gauss map ξ of $\hat{x}$,*
- *$\omega = -I((\mathring{II})^{-1}(dH)) - d\log\hat{\lambda}$, where I is the first fundamental form and H is the mean curvature of $\hat{x}$ in the standard round $\mathbb{S}^3$,*
- *$\Omega^* = -\hat{\lambda}^{-1}\langle d\xi, dy^*\rangle$, where $y^* = \frac{1}{1-\hat{x}\cdot\hat{x}^*}(1, \hat{x}^*)$ and $\hat{x}^*$ is the conformal transform of $\hat{x}$.*

The surface $\hat{x}$ is unique up to a conformal transformation of $\mathbb{S}^3$.

Proof. We start by choosing starting values for $y, y^*, \xi_{u^1}, \xi_{u^2}, \xi$ at $p_0 = (u_0^1, u_0^2)$, First we take a null vector

$$y(u_0^1, u_0^2) = y_0 = (1, \hat{x}_0)$$

for some $\hat{x}_0 \in \mathbb{S}^3 \subset \mathbb{R}^4$. Then we choose $\xi(u_0^1, u_0^2) = \xi_0 \in \mathbb{R}^{1,4}$ such that

$$\langle y_0, \xi_0\rangle = 0 \text{ and } \langle \xi_0, \xi_0\rangle = 1. \tag{3.3.8}$$

Next we choose $\xi_{u^1}(u_0^1, u_0^2) = \xi_0^1 \in \mathbb{R}^{1,4}$ and $\xi_{u^2}(u_0^1, u_0^2) = \xi_0^2 \in \mathbb{R}^{1,4}$ such that

$$\begin{aligned} &\langle \xi_0^1, \xi_0^1\rangle = \langle \xi_0^2, \xi_0^2\rangle = m(u_0^1, u_0^2), \\ &\langle \xi_0^1, \xi_0^2\rangle = \langle \xi_0, \xi_0^1\rangle = \langle \xi_0, \xi_0^2\rangle = \langle y_0, \xi_0^1\rangle = \langle y_0, \xi_0^2\rangle = 0. \end{aligned} \tag{3.3.9}$$

Finally choose the unique null vector $y^*(u_0^1, u_0^2) = y_0^*$ such that

$$\begin{aligned} &\langle y_0^*, y_0\rangle = -1, \\ &\langle y_0^*, y_0^*\rangle = \langle y_0^*, \xi_0\rangle = \langle y_0^*, \xi_0^1\rangle = \langle y_0^*, \xi_0^2\rangle = 0. \end{aligned} \tag{3.3.10}$$

Notice that for any other choice of $\{y_1, y_1^*, \xi_1^1, \xi_1^2, \xi_1\}$ satisfying the same orthonormal properties in (3.3.8)–(3.3.10), there is a Lorentz transformation that takes one to the other. With the integrability conditions assumed we may solve the systems (3.3.1) and (3.3.2) at least in an open neighborhood D_0 of p_0 in D. Using the uniqueness of solutions to systems of linear ODE one sees that the solution $\{y, y^*, \frac{1}{\sqrt{m}}\xi_{u^1}, \frac{1}{\sqrt{m}}\xi_{u^2}, \xi\}$ remains orthonormal in the Minkowski metric in D_0.

Now one should realize that the $y = \hat{\lambda}(1, \hat{x})$ here is with some positive $\hat{\lambda}$ (not necessarily identically 1 in D_0). It is then clear from all previous calculations that the rest of the statements in the theorem can be easily verified. □

3.4. ***The geometry of the associate ruled surface x^+ in hyperbolic space $\mathbb{H}^4$.*** In this section we want to discuss the geometry of the associate ruled 3-surface x^+ in $\mathbb{H}^4$, which is associated with a given surface $\hat{x}$ in the conformal 3-sphere. It's relation to the associate surface $\tilde{x}$ is very much analogous to the one between the ambient spacetime and the Poincaré-Einstein manifold of a given conformal manifold in the work of Fefferman and Graham. It is evidently useful to understand the geometry of the associate ruled 3-surface x^+ in $\mathbb{H}^4$.

It is rather easy now to do calculations for x^+ after we have calculated the first fundamental form for the associate 4-surface $\tilde{x}$ in Minkowski spacetime $\mathbb{R}^{1,4}$ in Section 3.1. We first have

$$dx^+ = \tfrac{1}{\sqrt{2}}(e^t y_\lambda - e^{-t} y_\lambda^*)\, dt + (e^t (y_\lambda)_{u^1} + e^{-t}(y_\lambda^*)_{u^1})\, du^1 + (e^t (y_\lambda)_{u^2} + e^{-t}(y_\lambda^*)_{u^2})\, du^2$$

and, using (3.1.2),

$$\begin{aligned} I^{x^+} = (dt)^2 - 2\omega_i^\lambda\, dt\, du^i + \Bigg(&\frac{e^{2t}}{2m}((\Omega_\lambda)_{i1}(\Omega_\lambda)_{j1} + (\Omega_\lambda)_{i2}(\Omega_\lambda)_{j2}) \\ &+ (\omega_i\omega_j + \frac{1}{m}((\Omega_\lambda)_{i1}(\Omega_\lambda^*)_{j1} + (\Omega_\lambda)_{i2}(\Omega_\lambda^*)_{j2})) \\ &+ \frac{e^{-2t}}{2m}((\Omega_\lambda^*)_{i1}(\Omega_\lambda^*)_{j1} + (\Omega_\lambda^*)_{i2}(\Omega_\lambda^*)_{j2})\Bigg)\, du^i\, du^j. \end{aligned}$$

One can calculate the determinant

$$\det I^{x^+} = \frac{1}{8m^2}(E_\lambda^2 |e^t\Omega_\lambda + e^{-t}\Omega_\lambda^*|^2 - e^{-2t}((\Omega_\lambda^*)_{11} + (\Omega_\lambda^*)_{22})^2)^2, \tag{3.4.1}$$

which can tell us where the associate ruled surface x^+ is degenerate.

To obtain the second fundamental form of the surface x^+ it suffices to see that the conformal Gauss map ξ is still the unit normal vector to the surface x^+ in the hyperboloid $\mathbb{H}^4$. Hence

$$II^{x^+} = -\langle dx^+, d\xi \rangle = \frac{1}{\sqrt{2}}(e^t \Omega_\lambda + e^{-t}\Omega_\lambda^*). \tag{3.4.2}$$

By the similar calculations as that in the previous section we have the mean curvature of the associate ruled surface x^+ as follows:

$$H^{x^+} = e^{-3t} \frac{\sqrt{2}\det \Omega_\lambda \mathcal{H}_\lambda}{(\det \Omega_\lambda - e^{-2t}\operatorname{Tr}\Omega_\lambda\Omega_\lambda^* + e^{-4t}\det \Omega_\lambda^*)}. \tag{3.4.3}$$

Theorem 3.4.1. *Suppose that $\hat{x}$ is an immersed surface in the conformal sphere $\mathbb{S}^3$ with no umbilical point and that x^+ is the associate ruled surface in the hyperboloid $\mathbb{H}^4$. Then $\hat{x}$ is a Willmore surface in the conformal sphere if and only if the associate ruled 3-surface x^+ in the hyperboloid is a minimal surface.*

4. Scalar invariants of surfaces in conformal round 3-sphere

In this section we want to introduce scalar local invariants for surfaces in conformal round 3-sphere $\mathbb{S}^3$. We will first recall what are scalar invariants for hypersurfaces in (pseudo-)Riemannian geometry. Inspired by the work of Fefferman and Graham on scalar local invariants in conformal geometry we are going to use the associate surface $\tilde{x}$ in the Minkowski $\mathbb{R}^{1,4}$ of a given surface $\hat{x}$ in the 3-sphere $\mathbb{S}^3$, where one considers the standard conformal 3-sphere as the projectivized positive light cone of the Minkowski spacetime to construct scalar local invariant.

4.1. ***Scalar invariants of 4-surfaces in*** $\mathbb{R}^{1,4}$**.** For our purpose we will focus on the discussion of scalar (pseudo-)Riemannian invariants of 4-surfaces $\tilde{x}$ in the Minkowski spacetime $\mathbb{R}^{1,4}$. Suppose that

$$\phi = \phi(v^2, v^3, v^4, v^5) : A \subset \mathbb{R}^4 \to \mathbb{R}^{1,4}$$

is a local parametrization of a surface $\tilde{x}$, where A is a domain in $\mathbb{R}^4$. Hence it induces a local coordinate

$$\tilde{\phi} = \tilde{\phi}(v^1, v^2, v^3, v^4, v^5) : B \subset (-\epsilon, \epsilon) \times A \to \mathbb{R}^{1,4}$$

for $\mathbb{R}^{1,4}$ such that

$$\phi(v^2, v^3, v^4, v^5) = \tilde{\phi}(0, v^2, v^3, v^4, v^5).$$

We will use the Capital Latin letters to stand for indices from 1 to 5 and Latin letters to stand for the indices from 2 to 5. We will use $v = (v^1, v^2, \dots, v^5)$ and $\hat{v} = (v^2, \dots, v^5)$. Hence the Minkowski metric in this coordinate is given as

$$\tilde{\mathcal{G}}_0 = \langle d\tilde{\phi}, d\tilde{\phi} \rangle = (\tilde{\mathcal{G}}_0)_{IJ}\, dv^I\, dv^J$$

and the first fundamental form for $\tilde{x}$ in $\mathbb{R}^{1,4}$ is given as

$$I^{\tilde{x}} = \langle d\phi, d\phi \rangle = \tilde{g}_{ij}\, dv^i\, dv^j = (\tilde{\mathcal{G}}_0)_{ij}|_{v^1=0}\, dv^i\, dv^j.$$

To be more restrictive we will assume that the surface $\tilde{x}$ is timelike and let

$$\xi : B \to \mathbb{S}^{1,3}$$

be a unit normal vector field on $\tilde{x}$ in $\mathbb{R}^{1,4}$. Then the second fundamental form for $\tilde{x}$ is given as

$$II^{\tilde{x}} = -\langle d\phi, d\xi \rangle = \tilde{h}_{ij}\, dv^i\, dv^j,$$

and we have

$$\xi_{v^i} = -\tilde{h}_{ik}\tilde{g}^{kj}\phi_{v^j}.$$

Definition 4.1.1. Let $\boldsymbol{i} : \mathrm{M}^{n-1} \to \mathrm{N}^n$ be an immersed hypersurface and let g be a (pseudo)-Riemannian metric on the ambient manifold N^n. A scalar (pseudo-) Riemannian invariant $\mathrm{I}(\boldsymbol{i}, \mathrm{N}^n, g)$ for the hypersurface $\boldsymbol{i}$ in N^n at a point p_0 on the surface $\boldsymbol{i}$ is a polynomial in the variables that are the coordinate partial derivatives of g_{IJ} of any order and the reciprocal of the determinant of g_{IJ} at the point p_0 such that the value of $\mathrm{I}(\boldsymbol{i}, \mathrm{N}^n, g)$ at p_0 is independent of choices of local coordinates $\tilde{\phi}$ of N^n which are induced from a parametrization ϕ of the surface $\boldsymbol{i}$ nearby the given point p_0.

The well-known examples of scalar Riemannian invariants for $\tilde{x}$ in $\mathbb{R}^{1,4}$ are

- $\tilde{H} = \tilde{g}^{ij}\tilde{h}_{ij}$,
- $|\tilde{h}|^2 = \tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{ij}\tilde{h}_{kl}$ and $\tilde{H}^2 = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{h}_{ij}\tilde{h}_{kl}$,
- $\tilde{\Delta}\tilde{H} = \tilde{g}^{kl}\tilde{g}^{ij}\tilde{h}_{ij,kl}$, $\mathrm{Div}\,\mathrm{Div}\,\tilde{h} = \tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{ij,kl}$, $\tilde{H}|\tilde{h}|^2 = \tilde{g}^{ik}\tilde{g}^{jl}\tilde{g}^{mn}\tilde{h}_{ij}\tilde{h}_{kl}\tilde{h}_{mn}$, $\mathrm{Tr}_{\tilde{g}}\,\tilde{h}^3 = \tilde{g}^{in}\tilde{g}^{jk}\tilde{g}^{km}\tilde{h}_{ij}\tilde{h}_{kl}\tilde{h}_{mn}$, and $\tilde{H}^3 = \tilde{g}^{ij}\tilde{g}^{kl}\tilde{g}^{mn}\tilde{h}_{ij}\tilde{h}_{kl}\tilde{h}_{mn}$,
- $|\tilde{\nabla}\tilde{h}|^2 = \tilde{g}^{ip}\tilde{g}^{jq}\tilde{g}^{kr}\tilde{h}_{ij,k}\tilde{h}_{pq,r}$, $\tilde{g}^{ip}\tilde{g}^{jr}\tilde{g}^{kq}\tilde{h}_{ij,k}\tilde{h}_{pq,r}$, $\tilde{g}^{ip}\tilde{g}^{jr}\tilde{g}^{kq}h_{ij,k}h_{pq,r}$ $|\tilde{\nabla}\tilde{H}|^2 = \tilde{g}^{ij}\tilde{g}^{pq}\tilde{g}^{kr}\tilde{h}_{ij,k}\tilde{h}_{pq,r}$, $|\widetilde{\mathrm{Div}}\,\tilde{h}|^2 = \tilde{g}^{ip}\tilde{g}^{jk}\tilde{g}^{qr}\tilde{h}_{ij,k}\tilde{h}_{pq,r}$, $\widetilde{\mathrm{Div}}\,\tilde{h}\cdot d\tilde{H}$,
- $\tilde{\Delta}\tilde{\Delta}\tilde{H}$

Each scalar invariant has an order. To find the order of each scalar invariant one simply scales the metric by a constant κ and see what is the dimension of the scalar invariant. For example, we can easily find that

$$\tilde{H}[\kappa^2\tilde{\mathcal{G}}_0] = \kappa^{-1}\tilde{H}[\tilde{\mathcal{G}}_0], \qquad |\tilde{h}|^2[\kappa^2\tilde{\mathcal{G}}_0] = \kappa^{-2}|\tilde{h}|^2[\tilde{\mathcal{G}}_0],$$
$$\tilde{\Delta}\tilde{H}[\kappa^2\tilde{\mathcal{G}}_0] = \kappa^{-3}\tilde{\Delta}\tilde{H}[\tilde{\mathcal{G}}_0], \quad |\tilde{\nabla}\tilde{h}|^2[\kappa^2\tilde{\mathcal{G}}_0] = \kappa^{-4}|\tilde{\nabla}\tilde{h}|^2[\tilde{\mathcal{G}}_0],$$
$$\tilde{\Delta}\tilde{\Delta}\tilde{H}[\kappa^2\tilde{\mathcal{G}}_0] = \kappa^{-5}\tilde{\Delta}\tilde{\Delta}\tilde{H}[\tilde{\mathcal{G}}_0].$$

To understand what are scalar Riemannian invariants $\mathrm{I}(\tilde{x}, \mathbb{R}^{1,4}, \tilde{\mathcal{G}}_0)$ we want to use the so-called Fermi coordinates. A Fermi coordinate is one such that 1) on the surface ϕ is a normal coordinate at a given point $\tilde{x}_0$; 2) the coordinate curves $\tilde{\phi}(t, v^2, v^3, v^4, v^5)$ is a geodesic perpendicular to the surface at $\phi(v^2, v^3, v^4, v^5)$ with unit speed (a line segment perpendicular to the surface in $\mathbb{R}^{1,4}$). Hence, for a Fermi coordinate,

$$\tilde{\phi}(v^1, \dots, v^5) = \phi(v^2, \dots, v^5) + v^1 \xi. \tag{4.1.1}$$

The following facts are well known.

Lemma 4.1.2. *Suppose that $\tilde{x}$ is a timelike hypersurface in $\mathbb{R}^{1,4}$. Suppose that $\tilde{\phi}$ is a Fermi coordinate at a given point $\tilde{x}_0$. Then*

$$\tilde{\mathcal{G}}_0 = \begin{bmatrix} 1 & 0 \\ 0 & [\mathcal{G}_{ij}] \end{bmatrix}$$

and

$$\mathcal{G}_{ij}(v^1, \hat{v}) = \tilde{g}_{ij}(\hat{v}) - 2\tilde{h}_{ij}(\hat{v})v^1 + \tilde{h}_{ik}(\hat{v})\tilde{h}_{jl}(\hat{v})\tilde{g}^{kl}(\hat{v})(v^1)^2,$$

where

$$\tilde{g}_{ij}(\hat{v}) = \eta_{ij} - \tfrac{2}{3}\tilde{R}_{ikjl}v^k v^l + \cdots,$$
$$\tilde{h}_{ij}(\hat{v}) = \tilde{h}_{ij}(0) + \tilde{h}_{ij,k}(0)v^k + \cdots;$$

here $\tilde{R}_{ijkl} = \tilde{h}_{ik}\tilde{h}_{jl} - \tilde{h}_{ij}\tilde{h}_{kl}$ is the Riemann curvature tensor for $\tilde{x}$ and η is the standard matrix of signature $\{-1, 1, 1, 1\}$. *All the coefficients in the Taylor expansions for G_{ij} are polynomials of $\tilde{h}_{ij}$ and the covariant derivatives of $\tilde{h}_{ij}$ at $\tilde{x}_0$.*

Therefore, in the light of Weyl's theorem on the invariants of orthogonal groups, we may conclude:

Proposition 4.1.3. *All scalar invariants* $\mathrm{I}(\tilde{x}, \mathbb{R}^{1,4}, \tilde{\mathcal{G}}_0)$ *of a surface $\tilde{x}$ in $\mathbb{R}^{1,4}$ are linear combinations of terms that are complete contractions of tensor product of the second fundamental form $\tilde{h}$ and the covariant derivatives of $\tilde{h}$.*

Proof. From the above lemma it is easily that all scalar invariants of a surface $\tilde{x}$ in $\mathbb{R}^{1,4}$ are polynomials of the first fundamental form $\tilde{g}$, the second fundamental form $\tilde{h}$ and covariant derivatives of the second fundamental form $\tilde{h}$, if we evaluate them in a Fermi coordinate for the surface. Then, by the Weyl theorem on the invariants of orthogonal groups, we know they are linear combinations of full contractions of $\tilde{h}$ and covariant derivatives of $\tilde{h}$. □

4.2. ***Scalar invariants of the homogeneous associate surface $\tilde{x}$ in $\mathbb{R}^{1,4}$.*** Let us work with the parametrization

$$\tilde{x} = \alpha\hat{\lambda}(1, \hat{x}) + \alpha\rho\hat{\lambda}^{-1}\frac{1}{1-a}(1, \hat{x}^*) = \alpha y_\lambda + \alpha\rho y_\lambda^*$$

and use the calculations given in Section 3.1 and Section 3.2. Now let us compute some scalar invariants for our associate surface $\tilde{x}$ on the light cone where $\rho = 0$. Then the first fundamental form is

$$I_{\tilde{x}}|_{\rho=0} = \begin{bmatrix} 0 & -\alpha & 0 & 0 \\ -\alpha & 0 & \alpha^2\omega_1^\lambda & \alpha^2\omega_2^\lambda \\ 0 & \alpha^2\omega_1^\lambda & \alpha^2 E_\lambda & 0 \\ 0 & \alpha^2\omega_2^\lambda & 0 & \alpha^2 E_\lambda \end{bmatrix}$$

from (3.1.4), whose inverse is

$$I_{\tilde{x}}^{-1}|_{\rho=0} = \begin{bmatrix} |\omega^\lambda|^2 & -\frac{1}{\alpha} & \frac{\omega_1^\lambda}{\alpha E_\lambda} & \frac{\omega_2^\lambda}{\alpha E_\lambda} \\ -\frac{1}{\alpha} & 0 & 0 & 0 \\ \frac{\omega_1^\lambda}{\alpha E_\lambda} & 0 & \frac{1}{\alpha^2 E_\lambda} & 0 \\ \frac{\omega_2^\lambda}{\alpha E_\lambda} & 0 & 0 & \frac{1}{\alpha^2 E_\lambda} \end{bmatrix}.$$

And the second fundamental form at $\rho = 0$ is

$$II_{\tilde{x}}|_{\rho=0} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha\Omega_\lambda \end{bmatrix}.$$

So the simplest (pseudo-)Riemannian invariants is the mean curvature $\tilde{H}$, but it is clear that

$$\tilde{H}|_{\rho=0} = \frac{1}{\alpha E_\lambda}((\Omega_\lambda)_{11} + (\Omega_\lambda)_{22}) = 0.$$

The first nontrivial one is

$$|\tilde{h}|^2|_{\rho=0} = \tilde{g}^{ik}\tilde{g}^{jl}\tilde{h}_{ij}\tilde{h}_{kl}|_{\rho=0} = \alpha^{-2}|\Omega_\lambda|^2, \tag{4.2.1}$$

which produces the first nontrivial invariant $|\mathring{II}|^2$ for the surface $\hat{x}$ in the conformal 3-sphere(cf. see the definition for scalar invariant of surfaces in the conformal 3-sphere in the next subsection). In fact the following nontrivial invariants without taking any derivative are all easy to calculate

$$\mathrm{Tr}_{I^{\tilde{x}}}\, \tilde{h}^k|_{\rho=0} = \alpha^{-k}\, \mathrm{Tr}_{I^{\hat{x}}_\lambda}\, \Omega_\lambda^k$$

for any $k = 2, 3, \ldots$. Obviously those are the ones that can been easily seen with no difficulty at all.

Next we want to calculate $|\nabla\tilde{H}|^2$ and $\tilde{\Delta}\tilde{H}$ at $\rho = 0$. To do so, let us first recall from Section 3.2 the mean curvature

$$\tilde{H} = \frac{\rho \det \Omega_\lambda \mathcal{H}_\lambda}{\alpha(\det \Omega_\lambda - \rho\, \mathrm{Tr}\, \Omega_\lambda\Omega_\lambda^* + \rho^2 \det \Omega_\lambda^*)}.$$

Hence $\tilde{H}_\alpha = \tilde{H}_{u^1} = \tilde{H}_{u^2} = 0$ and $|\nabla\tilde{H}|^2 = 0$ at $\rho = 0$, that is, $|\nabla\tilde{H}|^2$ gives no invariant for the surface $\hat{x}$. Let us set the convention to have a, b, c stand for α, ρ;

i, j, k stand for u^1, u^2, and A, B, C stand for all four variables. We then calculate, at $\rho = 0$,

$$\begin{aligned}
\text{(4.2.2)}\quad \tilde{\Delta}\tilde{H} &= \frac{1}{\sqrt{|\tilde{g}|}}\partial_A(\sqrt{|\tilde{g}|}\tilde{g}^{AB}\partial_B\tilde{H}) \\
&= \frac{1}{\alpha^3 E}\big(\partial_\alpha(\sqrt{|\tilde{g}|}\tilde{g}^{\alpha\rho}\partial_\rho\tilde{H}) + \partial_\rho(\sqrt{|\tilde{g}|}\tilde{g}^{\rho B}\partial_B\tilde{H}) + \partial_i(\sqrt{|\tilde{g}|}\tilde{g}^{i\rho}\partial_\rho\tilde{H})\big) \\
&= \frac{1}{\alpha^3 E}\big(\partial_\alpha(\sqrt{|\tilde{g}|}\tilde{g}^{\alpha\rho}\partial_\rho\tilde{H}) + \partial_\rho(\sqrt{|\tilde{g}|}g^{\rho\alpha}\partial_\alpha\tilde{H}) + \sqrt{|\tilde{g}|}(\partial_\rho\tilde{g}^{\rho\rho})\partial_\rho\tilde{H}\big) \\
&= 2\alpha^{-3}\mathcal{H}_\lambda,
\end{aligned}$$

where one needs to use the fact that $\tilde{g}^{\rho\rho}|_{\rho=0} = 0$ and $\partial_\rho\tilde{g}^{\rho\rho}|_{\rho=0} = \frac{2}{\alpha^2}$ based on calculations (A.8) in Appendix A. This confirms that $\mathcal{H}_\lambda$ is indeed a conformal invariant of order 3 for a surface $\hat{x}$ in the 3-sphere in general conformal metric $\lambda^2 g_0$.

The next invariant we want to calculate is $\tilde{\Delta}\tilde{\Delta}\tilde{H}$. To do so we observe, again from (A.8), that

$$\text{(4.2.3)}\qquad \begin{aligned}
&\partial_\rho|_{\rho=0}\tilde{g}^{\rho\alpha} = -\frac{2}{\alpha}|\omega^\lambda|^2, &\qquad &\partial_\rho|_{\rho=0}\tilde{g}^{\rho\rho} = \frac{2}{\alpha^2}, \\
&\partial_\rho|_{\rho=0}\tilde{g}^{\rho i} = -\frac{2}{\alpha^2}\frac{\omega_i^\lambda}{E_\lambda}, &\qquad &\partial_\rho\partial_\rho|_{\rho=0}\tilde{g}^{\rho\rho} = \frac{8}{\alpha^2}|\omega^\lambda|^2.
\end{aligned}$$

After a lengthy calculation we get

$$\begin{aligned}
\text{(4.2.4)}\quad \tilde{\Delta}\tilde{\Delta}\tilde{H}|_{\rho=0} = 8\alpha^{-5}\Big(&\Delta_\lambda\mathcal{H}_\lambda + 9|\omega^\lambda|^2\mathcal{H}_\lambda - 3\mathrm{Div}(\omega^\lambda)\mathcal{H}_\lambda \\
&- 6\omega^\lambda(\nabla\mathcal{H}_\lambda) - \frac{3\,\mathrm{Tr}(\Omega_\lambda\Omega_\lambda^*)}{2m^2}|\Omega_\lambda|^2\mathcal{H}_\lambda\Big).
\end{aligned}$$

This tells us that the quantity in parentheses is a conformal invariant of order 5 for the surface $\hat{x}$ in the 3-sphere.

We can also calculate the covariant derivatives of the second fundamental forms for the associate surface. We first list the relevant Christoffel symbols for the calculation

$$\text{(4.2.5)}\qquad \begin{aligned}
&\tilde{\Gamma}^k_{\alpha\alpha} = \tilde{\Gamma}^k_{\rho\rho} = \tilde{\Gamma}^k_{\alpha\rho} = 0, \\
&\tilde{\Gamma}^k_{\alpha j} = \alpha^{-1}\delta_{jk}, \\
&\tilde{\Gamma}^k_{\rho j} = \frac{1}{2E_\lambda}\Big((\omega_k^\lambda)_{u^j} - (\omega_j^\lambda)_{u^k} + \frac{1}{m}((\Omega_\lambda)_{jl}(\Omega_\lambda^*)_{kl} + (\Omega_\lambda)_{kl}(\Omega_\lambda^*)_{jl})\Big), \\
&\tilde{\Gamma}^k_{ij} = (\Gamma_\lambda)^k_{ij} - \omega_k^\lambda\delta_{ij}.
\end{aligned}$$

Then we calculate

$$
\begin{aligned}
&\tilde h_{ab,C} = 0,\\
&\tilde h_{ai,b} = 0,\\
&\tilde h_{\alpha j,k} = -(\Omega_\lambda)_{jk},\\
&\tilde h_{\rho j,k} = -\frac{\alpha}{2E_\lambda}\Big((\Omega_\lambda)_{ij}((\omega_i^\lambda)_{u^k} - (\omega_k^\lambda)_{u^i}) + \frac{1}{m}((\Omega_\lambda)_{kl}(\Omega_\lambda^*)_{il} + (\Omega_\lambda)_{il}(\Omega_\lambda^*)_{kl})\Big),\\
&\tilde h_{ij,\alpha} = -(\Omega_\lambda)_{ij},\\
&\tilde h_{ij,\rho} = \alpha(\Omega_\lambda^*)_{ij}\\
&\qquad -\frac{\alpha}{2E}\Big((\Omega_\lambda)_{lj}((\omega_l^\lambda)_{u^i} - (\omega_i^\lambda)_{u^l}) + \frac{1}{m}((\Omega_\lambda)_{kl}(\Omega_\lambda^*)_{ki} + (\Omega_\lambda)_{ki}(\Omega_\lambda^*)_{kl})\Big)\\
&\qquad -\frac{\alpha}{2E}\Big((\Omega_\lambda)_{il}((\omega_l^\lambda)_{u^j} - (\omega_j^\lambda)_{u^l}) + \frac{1}{m}((\Omega_\lambda)_{kl}(\Omega_\lambda^*)_{kj} + (\Omega_\lambda)_{kj}(\Omega_\lambda^*)_{kl})\Big),\\
&\tilde h_{ij,k} = \alpha(\Omega_\lambda)_{ij,k} + \alpha(\Omega_\lambda)_{lj}\omega_l^\lambda\delta_{ik} + \alpha(\Omega_\lambda)_{il}\omega_l^\lambda\delta_{jk}.
\end{aligned}
\tag{4.2.6}
$$

The easy ones are

$$\phi_\alpha = \tilde h_{\alpha j,k}\tilde g^{jk} = 0, \quad \phi_\rho = \tilde h_{\rho j,k}\tilde g^{jk} = \frac{1}{\alpha}\mathcal{H}_\lambda$$

in the light of (2.5.14). At the same time

$$
\begin{aligned}
\phi_i &= \tilde h_{iB,C}\tilde g^{BC} = \tilde h_{ij,C}\tilde g^{jC} + \tilde h_{ib,k}\tilde g^{bk} = \tilde h_{ij,k}\tilde g^{jk} + \tilde h_{ij,\alpha}\tilde g^{j\alpha} + \tilde h_{i\alpha,k}\tilde g^{\alpha k}\\
&= \frac{1}{\alpha E_\lambda}(\Omega_\lambda)_{ij,j} + \frac{3}{\alpha E_\lambda}(\Omega_\lambda)_{ij}\omega_j^\lambda - \frac{1}{\alpha E_\lambda}(\Omega_\lambda)_{ij}\omega_j^\lambda - \frac{1}{\alpha E_\lambda}(\Omega_\lambda)_{ij}\omega_j^\lambda\\
&= \frac{1}{\alpha E_\lambda}(\Omega_\lambda)_{ij,j} + \frac{1}{\alpha E_\lambda}(\Omega_\lambda)_{ij}\omega_j^\lambda = 0
\end{aligned}
$$

due to the integrability condition (3.3.4). Thus $|\widetilde{\mathrm{Div}}\,\tilde h|^2 (= 0)$ does not give any invariant on the surface $\hat x$, nor does $\widetilde{\mathrm{Div}}\,\tilde h \cdot d\tilde H (= 0)$, because $\tilde g^{\rho\rho}|_{\rho=0} = 0$.

We want to calculate $|\tilde\nabla\tilde h|^2$ since we have all the covariant derivatives $\tilde h_{AB,C}$ in (4.2.6). The calculation is direct yet very long. We omit details here.

$$
\begin{aligned}
|\tilde\nabla\tilde h|^2|_{\rho=0} = \alpha^{-4}\Big(&|\nabla\Omega|^2 + 8|dH|^2 - 6\Omega\cdot\Omega^*\\
&- \frac{2}{E_\lambda^3}(\Omega_\lambda)_{ij}\omega_k^\lambda(R^\lambda)_{3ijk} - \frac{6}{E_\lambda^3}(\Omega_\lambda)_{ij}(\Omega_\lambda)_{ki,j}\omega_k^\lambda\Big),
\end{aligned}
$$

where the Codazzi equation for the surface $\hat x$ in $(\mathbb{S}^3,\ \lambda^2 g_0)$

$$(\Omega_\lambda)_{ij,k} = (\Omega_\lambda)_{ik,j} + (R^\lambda)_{3ijk} + (H_\lambda)_{u^j}E_\lambda\delta_{ik} - (H_\lambda)_{u^k}E_\lambda\delta_{ij}$$

has been used. At this point we like to write each term as local scalar invariant of the surface $\hat{x}$ in $(\mathbb{S}^3,\ \lambda^2 g_0)$. We first calculate

$$\begin{aligned}
&(\Omega_\lambda)_{ij}\omega_k^\lambda(R^\lambda)_{3ijk}\\
&= (\Omega_\lambda)_{ij}\omega_1^\lambda(R^\lambda)_{3ij1} + (\Omega_\lambda)_{ij}\omega_2^\lambda(R^\lambda)_{3ij2}\\
&= E_\lambda((\Omega_\lambda)_{11}\omega_1^\lambda(R^\lambda)_{31} + (\Omega_\lambda)_{21}\omega_1^\lambda(R^\lambda)_{32} + (\Omega_\lambda)_{22}\omega_2^\lambda(R^\lambda)_{32} + (\Omega_\lambda)_{12}\omega_2^\lambda(R^\lambda)_{31})\\
&= E_\lambda(\Omega_\lambda)_{ij}\omega_j^\lambda(R^\lambda)_{3i} = -E_\lambda^2(H_\lambda)_{u^i}(R^\lambda)_{3i} = -\operatorname{Ric}^\lambda(\vec{\boldsymbol{n}}_\lambda, \lambda H_\lambda).
\end{aligned}$$

Then we deal with the last term:

$$\begin{aligned}
(\Omega_\lambda)_{ij}(\Omega_\lambda)_{ki,j}\omega_k^\lambda &= (\Omega_\lambda)_{ij}((\Omega_\lambda)_{ki}\omega_k^\lambda)_{,j} - (\Omega_\lambda)_{ij}(\Omega_\lambda)_{ki}\omega_{k,j}^\lambda\\
&= -E_\lambda(\Omega_\lambda)_{ij}(H_\lambda)_{i,j} - \tfrac{1}{2}E_\lambda^3|\Omega_\lambda|^2\operatorname{Div}(\omega^\lambda),
\end{aligned}$$

where

$$\begin{aligned}
\operatorname{Div}(\omega^\lambda) &= E_\lambda^{-1}\omega_{i,i}^\lambda = E_\lambda^{-1}(\omega_i^\lambda)_{u^i}\\
&= E_\lambda^{-1}(\langle \Delta_0 y_\lambda, y_\lambda^*\rangle + \langle (y_\lambda)_{u^i}, (y_\lambda^*)_{u^i}\rangle)\\
&= H_\lambda^2 - |\omega^\lambda|^2 + (R^\lambda)_{1212} + E^{-1}\langle (y_\lambda)_{u^i}, (y_\lambda^*)_{u^i}\rangle\\
&= H_\lambda^2 + 2\frac{\Omega_\lambda\cdot\Omega_\lambda^*}{|\Omega_\lambda|^2} + E_\lambda^{-1}(R^\lambda)_{1212},\\
\Delta_0 y_\lambda &= 2E_\lambda H_\lambda\vec{\boldsymbol{n}}_\lambda + 2E_\lambda y_\lambda^\dagger - (R^\lambda)_{1212}y_\lambda,
\end{aligned}$$

and

$$\sum_{i=1}^{2} E^{-1}\langle (y)_{u^i}, (y^*)_{u^i}\rangle = |\omega^\lambda|^2 + 2\frac{\Omega_\lambda\cdot\Omega_\lambda^*}{|\Omega_\lambda|^2}.$$

So we have obtained

$$\begin{aligned}
|\nabla\tilde{h}|^2|_{\rho=0} = \alpha^{-4}(|\nabla\Omega_\lambda|^2 &+ 8|dH_\lambda|^2 + 2\operatorname{Ric}^\lambda(\vec{\boldsymbol{n}}_\lambda, \nabla H_\lambda) + 3H_\lambda^2|\Omega|^2\\
&+ 3K_\lambda^T|\Omega_\lambda|^2 + 6\Omega_\lambda\cdot\operatorname{Hess}(H_\lambda)),
\end{aligned}\tag{4.2.7}$$

where

$$K_\lambda^T = E_\lambda^{-1}(R^\lambda)_{1212}$$

is the sectional curvature of $(\mathbb{S}^3, \lambda^2 g_0)$ of the tangent plane to the surface $\hat{x}$.

4.3. ***Scalar invariants for surfaces in the conformal round 3-sphere.*** Let us start with the definition of scalar invariants for surfaces in conformal sphere.

Definition 4.3.1. Let $\boldsymbol{i} : \mathrm{M}^{n-1} : \mathrm{N}^n$ be an immersed hypersurface and let $[g]$ be a class of conformal metrics on the ambient manifold $\mathrm{N}|^n$. $\mathrm{I}_c(\boldsymbol{i}, \mathrm{N}^n, g)$ is said to be a scalar conformal invariant of the hypersurface $\boldsymbol{i}$ in the conformal manifold $(\mathrm{N}^n, [g])$ if it is a scalar Riemannian invariant and

$$\mathrm{I}_c(\boldsymbol{i}, \mathrm{N}^n, \lambda^2 g) = \lambda^{-k}\mathrm{I}_c(\boldsymbol{i}, \mathrm{N}^n, g).\tag{4.3.1}$$

for any positive function λ on N^n, where k is the order of the invariant $\mathrm{I}_c(\boldsymbol{i}, \mathrm{N}^n, g)$.

Recall that, for an immersed surface $\hat{x}$ in $\mathbb{S}^3$, we have

$$\mathring{I\!I}(\hat{x}, \mathbb{S}^3, \lambda^2 g_0) = \lambda \mathring{I\!I}(\hat{x}, \mathbb{S}^3, g_0).$$

Hence it is easy to observe that

$$|\mathring{I\!I}|^2(\hat{x}, \mathbb{S}^3, \lambda^2 g_0) = \lambda^{-2} g_0^{ik} \lambda^{-2} g_0^{jl} \lambda \mathring{I\!I}_{ij} \lambda \mathring{I\!I}_{kl} = \lambda^{-2} \|\mathring{I\!I}\|^2(\hat{x}, \mathbb{S}^3, g_0)$$

and

$$\mathrm{Tr}_{\lambda^2 g_0}(\mathring{I\!I})^k(\hat{x}, \mathbb{S}^3, \lambda^2 g_0) = \lambda^{-k}\, \mathrm{Tr}_{g_0}(\mathring{I\!I})^k(\hat{x}, \mathbb{S}^3, g_0) \text{ for all } k = 2, 3, \ldots.$$

On the other hand, it does not seem easy to directly verify that $\mathcal{H}_\lambda$ is a conformal invariant for a surface in the conformal 3-sphere, though this is a well-known one. We have verified this in computing the mean curvature (cf. (2.5.12)) of the surface ξ in the de Sitter spacetime $\mathbb{S}^{1,3}$ as well as in the above calculation of $\tilde{\Delta}\tilde{H}$ (cf. (4.2.2)) of the homogeneous associate surface $\tilde{x}$. In general it takes tremendous, if not impossible, to verify whether an invariant $\mathrm{I}(\hat{x}, \mathbb{S}^3, \lambda^2 g_0)$ is conformally invariant, complicated by the six integrability conditions. The most important application of the construction of associate homogeneous surfaces is the following:

Theorem 4.3.2. *Suppose that $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ is an immersed surface with no umbilical point, and let*

$$\tilde{x} = \alpha y + \alpha \rho y^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4}$$

be the associate surface for $\hat{x}$, where $\hat{x}^*$ is the conformal transform of $\hat{x}$. Then any scalar (pseudo)-Riemannian invariant $\mathrm{I}(\tilde{x}, \mathbb{R}^{1,4}, \tilde{\mathcal{G}}_0)$ evaluated at $\rho = 0$, if it is nontrivial, is a scalar conformal invariant $\mathrm{I}_c(\hat{x}, \mathbb{S}^3, \lambda^2 g_0)$ multiplied with $|\mathring{I\!I}_\lambda|^{2n}$ for some integer n.

Proof. For any invariant $\mathrm{I}(\tilde{x}, \mathbb{R}^{1,4}, \tilde{\mathcal{G}}_0)$, we know that it is a full contraction of tensor product of the second fundamental form and the covariant derivatives. For a choice of representative $\lambda^2 g_0$ on $\mathbb{S}^3$, in the corresponding parametrization (3.1.1), we claim that

$$\mathrm{I}(\tilde{x}, \mathbb{R}^{1,4}, \tilde{\mathcal{G}}_0)|_{\rho=0} = \alpha^{-k} \mathrm{I}(\hat{x}, \mathbb{S}^3, \lambda^2 g_0) |\mathring{I\!I}_\lambda|^{2n} \tag{4.3.2}$$

for a positive integer k and a nonnegative integer n, due to the homogeneity of the associate surface. To see the right side of (4.3.2) is indeed a scalar Riemannian invariant multiplied with factor $|\mathring{I\!I}_\lambda|^{2n}$ for some integer n, we consider the tensors that determines the first and second fundamental forms of the associate surface in that parametrization. We recall from (2.5.5) that

$$\Omega_\lambda = \mathring{I\!I}_\lambda$$

is the traceless part of the second fundamental form for the surface $\hat{x}$ in the 3-sphere with the conformal metric $\lambda^2 g_0$. We also know from (3.1.3) that

$$\omega^\lambda = -I_\lambda((\mathring{I\!I}_\lambda)^{-1}(dH_\lambda)) = -\frac{2}{|\mathring{I\!I}_\lambda|^2}\mathring{I\!I}_\lambda(\nabla H_\lambda),$$

which causes us to include the possibly negative n in the right side of (4.3.2). We may also recall from (2.5.3) that

$$m = \tfrac{1}{2}E_\lambda|\mathring{I\!I}_\lambda|^2.$$

Next we want to show that Ω^*_λ is also a tensor product of covariant derivatives of the 1-form ω^λ, covariant derivatives of the second fundamental form $I\!I_\lambda$ and covariant derivatives of Riemann curvature tensor of the conformal metric $\lambda^2 g_0$ on the 3-sphere(including 0th order). Recall the definition

$$(\Omega^*_\lambda)_{ij} = \langle y^*_\lambda, \xi_{u^i u^j}\rangle.$$

We use the same idea in the calculation of the trace of Ω^* in Section 2.5. Hence we write

$$\xi_{u^iu^j} = -(\Omega^*_\lambda)_{ij}y_\lambda - (\Omega_\lambda)_{ij}y^*_\lambda + (\Gamma_m)^k_{ij}\xi_{u^k} - m\delta_{ij}\xi. \tag{4.3.3}$$

From (2.6.2) we know that

$$\langle y^*_\lambda, y^\dagger_\lambda\rangle = -\tfrac{1}{2}(|\omega^\lambda|^2 + H^2_\lambda).$$

Using $\xi = H_\lambda y_\lambda + \vec{\boldsymbol{n}}_\lambda$ from Lemma 2.3.4 and (B.6), we have

$$\langle \xi_{u^k}, y^\dagger_\lambda\rangle = -(H_\lambda)_{u^k} + (R^\lambda)_{3k}$$

and

$$\langle \xi, y^\dagger_\lambda\rangle = -H_\lambda.$$

Therefore we derive from (4.3.3) that

$$\begin{aligned}(4.3.4)\quad &\langle \xi_{u^iu^j}, y^\dagger_\lambda\rangle\\ &= (\Omega^*_\lambda)_{ij} + \tfrac{1}{2}(|\omega^\lambda|^2 + H^2_\lambda)(\Omega_\lambda)_{ij} + (\Gamma_m)^k_{ij}(-H_{u^k} + (R^\lambda)_{3k}) + Hm\delta_{ij},\end{aligned}$$

where

$$(\Gamma_m)^k_{ij} = \Gamma^k_{ij} + \tfrac{1}{2}|\Omega_\lambda|^{-2}(|\Omega_\lambda|^2_{u^i}\delta_{jk} + |\Omega_\lambda|^2_{u^j}\delta_{ik} - |\Omega_\lambda|^2_{u^k}\delta_{ij})$$

represents the Christoffel symbols for the Möbius metric $m|du|^2$. On the other hand we have

$$\xi_{u^iu^j} = (H_\lambda)_{u^iu^j}y_\lambda + (H_\lambda)_{u^i}(y_\lambda)_{u^j} + (H_\lambda)_{u^j}(y_\lambda)_{u^i} + H_\lambda(y_\lambda)_{u^iu^j} + (\vec{\boldsymbol{n}}_\lambda)_{u^iu^j}$$

which implies

$$
\begin{aligned}
(4.3.5)\quad & \langle \xi_{u^i u^j}, y_\lambda^\dagger \rangle \\
&= -(H_\lambda)_{u^i u^j} + H_\lambda \langle (y_\lambda)_{u^i u^j}, y_\lambda^\dagger \rangle + \langle (\vec{\boldsymbol{n}}_\lambda)_{u^i u^j}, y_\lambda^\dagger \rangle \\
&= -(H_\lambda)_{u^i u^j} - H_\lambda \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^j} \rangle - \langle (\vec{\boldsymbol{n}}_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^j} \rangle - \langle \vec{\boldsymbol{n}}_\lambda, (y_\lambda^\dagger)_{u^i} \rangle_{u^j} \\
&= -(H_\lambda)_{u^i u^j} + \frac{1}{E_\lambda} (\Omega_\lambda)_{ik} \langle (y_\lambda)_{u^k}, (y_\lambda^\dagger)_{u^j} \rangle - \langle \vec{\boldsymbol{n}}_\lambda, (y_\lambda^\dagger)_{u^i} \rangle_{u^j} \\
&= -(H_\lambda)_{u^i u^j} - \frac{1}{E_\lambda} (\Omega_\lambda)_{ik} (R^\lambda)_{i3k3} + ((R^\lambda)_{3i})_{u^j},
\end{aligned}
$$

by (B.7) and (B.6). Thus, comparing (4.3.4) and (4.3.5), we have

$$
\begin{aligned}
(4.3.6)\quad (\Omega_\lambda^*)_{ij} &= -(H_\lambda)_{u^i, u^j} - H_\lambda m \delta_{ij} - \frac{1}{E_\lambda} (\Omega_\lambda)_{ik} (R^\lambda)_{j3k3} + ((R^\lambda)_{3i})_{,u^j} \\
&\quad - \tfrac{1}{2} (|\omega^\lambda|^2 + H_\lambda^2)(\Omega_\lambda)_{ij} \\
&\quad + \tfrac{1}{2} |\Omega_\lambda|^{-2} (|\Omega_\lambda|^2_{u^i} \delta_{jk} + |\Omega_\lambda|^2_{u^j} \delta_{ik} - |\Omega_\lambda|^2_{u^k} \delta_{ij})((H_\lambda)_{u^k} - (R^\lambda)_{3k}).
\end{aligned}
$$

The last factor that goes into the left side of Equation (4.3.2) is the reciprocal of the determinant:

$$
\det \tilde{g}|_{\rho=0} = -\frac{\alpha^6}{m^2} (pr - q^2)^2|_{\rho=0} = \frac{\alpha^6}{m^2} (\det \Omega_\lambda)^2 = \alpha^6 E_\lambda^2 = \alpha^6 \det I_\lambda^{\hat{x}}.
$$

due to (3.1.6), where $I_\lambda^{\hat{x}} = (\hat{x})^*(\lambda^2 g_0) = E_\lambda |du|^2$.

To verify that the right side of (4.3.2) is actually a conformal invariant, for a positive functions λ on 3-sphere, we simply compare the right side of (4.3.2) evaluated at $\alpha = 1$ with that evaluated at $\alpha = \hat{\lambda}$ and $\lambda = 1$. We then observe that

$$
\mathrm{I}(\hat{x}, \mathbb{S}^3, \lambda^2 g_0) = \hat{\lambda}^{-k} \mathrm{I}(\hat{x}, \mathbb{S}^3, g_0).
$$

Therefore it is a conformal scalar invariant for the surface $\hat{x}$ in the 3-sphere. □

Appendix A: The inverse of $I^{\tilde{x}}$ in general parametrizations

We consider the general parametrization

$$
\tilde{x} = \alpha y_\lambda + \alpha \rho y_\lambda^* : \mathbb{R}^+ \times \mathbb{R}^+ \times \mathrm{M}^2 \to \mathbb{R}^{1,4}.
$$

Then the first fundamental form in matrix form is

$$
(A.1)\qquad I_{\tilde{x}} = \begin{bmatrix} -2\rho & -\alpha & 0 & 0 \\ -\alpha & 0 & \alpha^2 \omega_1^\lambda & \alpha^2 \omega_2^\lambda \\ 0 & \alpha^2 \omega_1^\lambda & \multicolumn{2}{c}{\multirow{2}{*}{$\alpha^2 F$}} \\ 0 & \alpha^2 \omega_2^\lambda & & \end{bmatrix}
$$

where

$$\text{(A.2)} \quad \begin{cases} F_{11} = \dfrac{1}{m}(p^2+q^2) + 2\rho(\omega_1^\lambda)^2 \\ F_{12} = F_{21} = \dfrac{1}{m} q(p+r) + 2\rho\omega_1^\lambda \omega_2^\lambda \\ F_{22} = \dfrac{1}{m}(q^2+r^2) + 2\rho(\omega_2^\lambda)^2 \end{cases} \text{and} \begin{cases} F_{11}^* = \dfrac{1}{m}(p^2+q^2) \\ F_{12}^* = F_{21} = \dfrac{1}{m} q(p+r) \\ F_{22}^* = \dfrac{1}{m}(q^2+r^2) \end{cases}$$

and

$$\begin{bmatrix} p & q \\ q & r \end{bmatrix} = \Omega_\lambda + \rho\Omega_\lambda^*.$$

It is easily seen that

$$\text{(A.3)} \qquad (F^*)^{-1} = \frac{m}{(pr-q^2)^2} \begin{bmatrix} r^2+q^2 & -q(p+r) \\ -q(p+r) & p^2+q^2 \end{bmatrix}$$

and

$$F|_{\rho=0} = F^*|_{\rho=0} = E \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Let

$$(I^{\tilde{x}})^{-1} = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix}.$$

Therefore, for example,

$$\text{(A.4)} \qquad \begin{aligned} -2\rho a_{11} - \alpha a_{12} \qquad\qquad &= 1, \\ -\alpha a_{11} + \alpha^2\omega_1 a_{13} + \alpha^2\omega_2 a_{14} &= 0, \\ \alpha^2\omega_1 a_{12} + \alpha^2 F_{11} a_{13} + \alpha^2 F_{21} a_{14} &= 0, \\ \alpha^2\omega_2 a_{12} + \alpha^2 F_{12} a_{13} + \alpha^2 F_{22} a_{14} &= 0. \end{aligned}$$

Subtracting the first of these equations multiplied by α from the second equation multiplied by 2, we get

$$\text{(A.5)} \qquad \alpha^2 a_{12} + 2\alpha^2\rho\omega_1 a_{13} + 2\alpha^2\rho\omega_2 a_{14} = -\alpha.$$

Subtracting (A.5) multiplied by ω_1 from the third equation in (A.4) as well as subtracting (A.5) multiplied by ω_2 from the fourth equation in (A.4), we get

$$\text{(A.6)} \qquad \alpha^2 F^* \begin{bmatrix} a_{13} \\ a_{14} \end{bmatrix} = \begin{bmatrix} \alpha\omega_1 \\ \alpha\omega_2 \end{bmatrix}$$

Plugging back the values of a_{13} and a_{14} into (A.5) we have

$$\text{(A.7)} \qquad \begin{cases} a_{12} = \alpha^{-1}\left(-1 - 2\rho[\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\right), \\ a_{11} = -\dfrac{\alpha a_{12} + 1}{2\rho} = [\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}. \end{cases}$$

Similarly one gets

$$\alpha^2 F^*\begin{bmatrix}a_{23}\\ a_{24}\end{bmatrix} = \begin{bmatrix}-2\rho\omega_1\\ -2\rho\omega_2\end{bmatrix}, \quad \alpha^2 F^*\begin{bmatrix}a_{33}\\ a_{34}\end{bmatrix} = \begin{bmatrix}1\\ 0\end{bmatrix}, \quad \alpha^2 F^*\begin{bmatrix}a_{43}\\ a_{44}\end{bmatrix} = \begin{bmatrix}0\\ 1\end{bmatrix}$$

which yield respectively

$$\text{(A.8)} \qquad \begin{cases} a_{21} = \alpha^{-1}\left(-1 - 2\rho[\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\right), \\ a_{22} = \dfrac{2\rho}{\alpha^2}\left(1 + 2\rho[\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\right), \end{cases}$$

$$\text{(A.9)} \qquad \begin{cases} a_{31} = \alpha^{-1}[\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}, \\ a_{32} = -\dfrac{2\rho}{\alpha^2}([\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}1\\ 0\end{bmatrix}, \end{cases}$$

$$\text{(A.10)} \qquad \begin{cases} a_{41} = \alpha^{-1}(F^*)^{-1}\begin{bmatrix}0\\ 1\end{bmatrix}, \\ a_{42} = \dfrac{2\rho}{\alpha^2}\left(1 + 2\rho[\omega_1, \omega_2](F^*)^{-1}\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}\right). \end{cases}$$

Appendix B: The geometry of the 3-sphere $\mathbb{S}^3_\lambda$ in $\mathbb{R}^{1,4}$

Let us calculate the Gauss Theorem for the 3-sphere $\mathbb{S}^3_\lambda$ in Minkowski spacetime $\mathbb{R}^{1,4}$. There is nothing new or difficult about the calculation, but this helps to understand better about the geometry of the 3-sphere $\mathbb{S}^3_\lambda \subset \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$. Crucial to our approach is that the induced metric on $\mathbb{S}^3_\lambda$ is exactly the conformal metric $\lambda^2 g_0$. We consider the Fermi parametrization induced from a parametrization of the surface $\hat{x} : \mathrm{M}^2 \to \mathbb{S}^3$ such that

$$\text{(B.1)} \qquad y_\lambda = \lambda(\hat{x}(u^1, u^2, u^3))(1, \hat{x}(u^1, u^2, u^3)) : \mathrm{M}^3 \to \mathbb{S}^3_\lambda \subset \mathbb{N}^4_+ \subset \mathbb{R}^{1,4}$$

with

$$\text{(B.2)} \qquad \hat{x}(u^1, u^2, 0) = \hat{x}(u^1, u^2) \text{ and } (y_\lambda)_{u^3}|_{u^3=0} = \vec{\boldsymbol{n}}_\lambda.$$

Notice that y_λ here is the extension of $\hat{\lambda}(1, \hat{x})$ before. We use the two null normal vectors $\{y_\lambda, y^\dagger_\lambda\}$ where

$$\text{(B.3)} \qquad \langle y^\dagger_\lambda, y_\lambda\rangle - 1, \langle y^\dagger_\lambda, (y_\lambda)_{u^1}\rangle = \langle y^\dagger_\lambda, (y_\lambda)_{u^2}\rangle = \langle y^\dagger_\lambda, (y_\lambda)_{u^3}\rangle = 0.$$

The first fundamental form is $I^{\mathbb{S}^3_\lambda} = \lambda^2 g_0 = \langle dy_\lambda, dy_\lambda\rangle$, and the second is $II^{\mathbb{S}^3_\lambda} = -\langle dy_\lambda, dy_\lambda^\dagger\rangle y_\lambda^\dagger - \langle dy_\lambda, dy_\lambda\rangle y_\lambda$. To find the curvature of the metric $g_\lambda = \lambda^2 g_0$ we calculate

$$\nabla^\lambda_{\partial_{u^j}} \nabla^\lambda_{\partial_{u^i}} \partial_{u^k} - \nabla^\lambda_{\partial_{u^i}} \nabla^\lambda_{\partial_{u^j}} \partial_{u^k} = R^\lambda(\partial_{u^i}, \partial_{u^j})\partial_{u^k} = (R^\lambda)_{ijk}{}^{l}\partial_{u^l}. \tag{B.4}$$

First

$$\nabla^\lambda_{\partial_{u^j}} \partial_{u^k} = (y_\lambda)_{u^k u^j} - \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle y_\lambda - \langle (y_\lambda)_{u^j}, (y_\lambda)_{u^k}\rangle y_\lambda^\dagger; \tag{B.5}$$

then

$$\begin{aligned}\partial_{u^i} \nabla^\lambda_{\partial_{u^j}} \partial_{u^k} &= (y_\lambda)_{u^k u^j u^i} - \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle_{u^i} y_\lambda - \langle (y_\lambda)_{u^j}, (y_\lambda)_{u^k}\rangle_{u^i} y_\lambda^\dagger \\ &\quad - \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle (y_\lambda)_{u^i} - \langle (y_\lambda)_{u^j}, (y_\lambda)_{u^k}\rangle (y_\lambda^\dagger)_{u^i}\end{aligned}$$

and

$$\begin{aligned}\nabla^\lambda_{\partial_{u^i}} \nabla^\lambda_{\partial_{u^j}} \partial_{u^k} &= (\partial_{u^i} \nabla^\lambda_{\partial_{u^j}} \partial_{u^k})^{T\mathbb{S}^3_\lambda} \\ &= (y_\lambda)^{T\mathbb{S}^3_\lambda}_{u^k u^j u^i} - \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle (y_\lambda)_{u^i} - \langle (y_\lambda)_{u^j}, (y_\lambda)_{u^k}\rangle (y_\lambda^\dagger)_{u^i}\end{aligned}$$

Hence

$$\begin{aligned}(R^\lambda)_{ijk}{}^{l}\partial_{u^l} &= \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle (y_\lambda)_{u^i} + \langle (y_\lambda)_{u^j}, (y_\lambda)_{u^k}\rangle (y_\lambda^\dagger)_{u^i} \\ &\quad - \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^k}\rangle (y_\lambda)_{u^j} - \langle (y_\lambda)_{u^i}, (y_\lambda)_{u^k}\rangle (y_\lambda^\dagger)_{u^j}.\end{aligned}$$

One notices that $\langle (y_\lambda^\dagger)_{u^i}, y_\lambda^\dagger\rangle = 0$ and $\langle (y_\lambda^\dagger)_{u^i}, y_\lambda\rangle = 0$ and concludes that

$$(y_\lambda^\dagger)_{u^i} = (g_\lambda)^{ml}\langle (y_\lambda^\dagger)_{u^i}, (y_\lambda)_{u^m}\rangle (y_\lambda)_{u^l}..$$

Therefore

$$\begin{aligned}(R^\lambda)_{ijk}{}^{l}\partial_{u^l} &= (\langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle \delta_i{}^l + (g_\lambda)_{jk}(g_\lambda)^{ml}\langle (y_\lambda^\dagger)_{u^i}, (y_\lambda)_{u^m}\rangle \\ &\quad - \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^k}\rangle \delta_j{}^l - (g_\lambda)_{ik}(g_\lambda)^{ml}\langle (y_\lambda^\dagger)_{u^j}, (y_\lambda)_{u^m}\rangle)\partial_{u^l}\end{aligned}$$

and

$$\begin{aligned}(R^\lambda)_{ijkl} &= (R^\lambda)_{ijk}{}^{n}(g_\lambda)_{nl} \\ &= \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^k}\rangle (g_\lambda)_{il} + \langle (y_\lambda^\dagger)_{u^i}, (y_\lambda)_{u^l}\rangle (g_\lambda)_{jk} \\ &\quad - \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^k}\rangle (g_\lambda)_{jl} - \langle (y_\lambda^\dagger)_{u^j}, (y_\lambda)_{u^l}\rangle (g_\lambda)_{ik}.\end{aligned}$$

On the surface $\hat{x}$, where $u^3 = 0$, we have

$$[(g_\lambda)_{ij}] = \begin{bmatrix} E_\lambda & 0 & 0 \\ 0 & E_\lambda & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Therefore we have, for $i, j \in \{1, 2\}$,

$$\begin{aligned} -\langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^j} \rangle - \langle (y_\lambda)_{u^3}, (y_\lambda^\dagger)_{u^3} \rangle E_\lambda \delta_{ij} &= (R^\lambda)_{i3j3}, \\ -\langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^3} \rangle E_\lambda \delta_{jl} + \langle (y_\lambda)_{u^l}, (y_\lambda^\dagger)_{u^3} \rangle E_\lambda &= (R^\lambda)_{3jjl}, \\ -\langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^i} \rangle E_\lambda - \langle (y_\lambda)_{u^j}, (y_\lambda^\dagger)_{u^j} \rangle E_\lambda &= (R^\lambda)_{ijij}. \end{aligned}$$

We obtain, for $i, j \in \{1, 2\}$,

$$\langle \vec{\boldsymbol{n}}_\lambda, (y_\lambda^\dagger)_{u^i} \rangle = \frac{1}{E_\lambda} (R^\lambda)_{ijj3} = -(R^\lambda)_{i3}, \tag{B.6}$$

and for $i \neq j$,

$$\begin{aligned} \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^j} \rangle &= -(R^\lambda)_{i3j3} \\ \langle (y_\lambda)_{u^i}, (y_\lambda^\dagger)_{u^i} \rangle &= -(R^\lambda)_{i3i3} + \tfrac{1}{2}((R^\lambda)_{33} - (R^\lambda)_{1212}) \\ \langle (y_\lambda)_{u^3}, (y_\lambda^\dagger)_{u^3} \rangle &= -\tfrac{1}{2}((R^\lambda)_{33} - (R^\lambda)_{1212}) \end{aligned} \tag{B.7}$$

Finally, for the induced Fermi coordinate from an isothermal coordinate, we can easily see that

$$\begin{aligned} (R^\lambda)_{3i,}{}^{i} = \frac{1}{E_\lambda} \left(\sum_{i=1}^{2} R^\lambda \right)_{3i,i} &= \frac{1}{E_\lambda} \sum_{i=1}^{2} (((R^\lambda)_{3i})_{u^i} - (R^\lambda)_{3k} (\Gamma_\lambda)_{ii}^k) \\ &= \frac{1}{E_\lambda} \sum_{i=1}^{2} (((R^\lambda)_{3i})_{u^i}. \end{aligned} \tag{B.8}$$

Indeed, we have $\sum_{i=1}^2 (\Gamma_\lambda)_{ii}^k = 0$ for $k = 1, 2$, where $(\Gamma_\lambda)_{ij}^k$ are the Christoffel symbols for the conformal metric $I_\lambda = E_\lambda |du|^2$ in isothermal coordinates.

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ACTION OF INTERTWINING OPERATORS ON PSEUDOSPHERICAL K-TYPES

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We give a concrete description of the two-fold cover of a simply connected, split real reductive group and its maximal compact subgroup as Chevalley groups. We study the representations of the maximal compact subgroups called pseudospherical representations, which appear with multiplicity one in the principal series representation. We introduce a family of canonically defined intertwining operators and compute their action on pseudospherical *K*-types, obtaining explicit formulas of the Harish-Chandra *c*-function.

1. Introduction

Assume that $\underline{G}$ is the split real form of a simply connected complex algebraic group. It turns out that $\underline{G}$ admits a unique nontrivial two-fold cover (or double cover) G, which is the nonlinear group we wish to study. Such coverings are well-studied. There are several general results about coverings of algebraic groups in [Steinberg 1968]. We are interested in *pseudospherical principal series representations*, that is, principal series representations that contain a pseudospherical K-type. These representations are defined for G and are related to a conjectural Shimura correspondence for split real groups; see [Adams et al. 2007]. Pseudospherical representation can refer to three definitions: Let $G = PK$ be an Iwasawa decomposition with $P = MAN$ a minimal parabolic subgroup. We have pseudospherical representations of M, pseudospherical representations of K and pseudospherical representations of G; see the definition at the beginning of Section 3.

The intertwining operators between two principal series representations, when considered as integral operators, reveal many properties of the principal series representations, such as reducibility points. The intertwining operators play an important role in the general Plancherel formula for semisimple Lie groups developed by Harish-Chandra. They are also related to the theory of Eisenstein series. A nice discussion of the formalism can be found in [Schiffmann 1971]. In this paper, we normalize the intertwining operators between two pseudospherical principal

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series, in a way that it is independent of the choice of representative in $N_K(A)$ of $w \in W = N_K(A)/Z_K(A)$, and obtain a canonical definition. We are interested in the action of intertwining operators on pseudospherical K-types. We compute explicitly the *Harish-Chandra c-function* associated to this action, which is our main result (Theorem 6.5). There is an analogous result in the p-adic case obtained by H. Y. Loke and G. Savin [2010].

The structure of this paper is arranged as follows: In Section 2, we recall some basic facts on Chevalley groups and their covering groups. We define the maximal compact subgroup K of the covering group G using Steinberg symbols. We calculate the structure of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the nontrivial two-fold cover of $\mathrm{SL}(2, \mathbb{R})$, making a comparison between Kubota cocycles and Steinberg symbols and writing down the exponential map from the Lie algebra to the cover. In Section 3, we define the pseudospherical representation following [Adams et al. 2007] and list some properties regarding the action of W on it. In Section 4, we define a family of canonical intertwining operators among pseudospherical principal series. In Section 5 we compute the intertwining operators of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, which are important for the general groups. Finally, we calculate the action of intertwining operators on pseudospherical K-types and obtain our main result in Section 6.

2. Chevalley groups and their covering groups

In Section 2A, we recall the well-known construction of the Chevalley groups. In Section 2B, we state a number of results for the covering group of a Chevalley group that we will need in later sections. In particular, we give the generators and relations of the double cover in terms of the Hilbert symbol. We define the minimal parabolic subgroup $P = NAM$ and the maximal compact subgroup K in terms of the Steinberg symbol; see Proposition 2.6. In Section 2C, we specialize our discussion in Section 2B to the case $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ and make a comparison between the definition based on Kubota symbols and the definition based on Steinberg symbols; see Proposition 2.8. We also compute explicitly an exponential map from the Lie algebra to the cover; see Proposition 2.11.

2A. ***Construction of a Chevalley group.*** In this section, we recall the construction of Chevalley groups following [Steinberg 1968]. Let $\mathfrak{g}$ be a semisimple Lie algebra over $\mathbb{C}$, and $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, and Φ the corresponding root system. We use $\alpha, \beta, \gamma, \ldots$ to denote the roots. Let B be the Killing form on $\mathfrak{g}$. Since it is nondegenerate, there exists $H'_\alpha \in \mathfrak{h}$ such that $B(H, H'_\alpha) = \alpha(H)$ for all $H \in \mathfrak{h}$. Define $(\alpha, \beta) = B(H'_\alpha, H'_\beta)$ for all $\alpha, \beta \in \Phi$. The Cartan integer $\langle \alpha, \beta \rangle$ is defined to be $2(\alpha, \beta)/(\beta, \beta)$. The root system Φ is invariant under all reflections w_α ($\alpha \in \Phi$), where w_α is the reflection across the hyperplane orthogonal to α. These reflections generate the Weyl group W.

For each α, define $H_\alpha = 2H'_\alpha/(\alpha,\alpha)$ and $H_i = H_{\alpha_i}$, where $\Delta = \{\alpha_1, \dots, \alpha_l\}$ is a set of simple roots. By [Steinberg 1968], one can choose $X_\alpha \in \mathfrak{g}_\alpha$ such that

- $[H_\beta, X_\alpha] = \langle \alpha, \beta \rangle X_\alpha$,
- $[X_\alpha, X_{-\alpha}] = H_\alpha$ is an integer linear combination of the H_i, and
- $[X_\alpha, X_\beta] = N_{\alpha\beta} X_{\alpha+\beta}$, where $N_{\alpha\beta}$ is an integer which is 0 if $\alpha+\beta$ is not a root.

The collection of H_i and X_α is called a *Chevalley basis* of the complex semisimple Lie algebra $\mathfrak{g}$. It is important that the integer span, $\mathfrak{g}_\mathbb{Z}$, of the basis elements is stable under the Lie bracket.

Let L_0 be the root lattice, i.e, the integer span of all roots in Φ, and let L_1 be the weight lattice, which is the set of all $\mu \in \mathfrak{h}^*$ such that $\mu(H_\alpha) \in \mathbb{Z}$ for all roots α. Assume $(\mathfrak{g}, V)$ is a complex finite-dimensional representation of $\mathfrak{g}$. One can show that its weight lattice L_V is contained between L_0 and L_1. To construct the Chevalley group based on the representation $(\mathfrak{g}, V)$, choose a full-rank lattice M in V which is invariant under the set

$$\{X_\alpha^n/n! : n \in \mathbb{Z}_{\geq 0},\ \alpha \in \Phi\},$$

where we are thinking of $X_\alpha^n/n!$ as a member of $\mathrm{End}(V)$. One can show (see [Steinberg 1968]) that such a lattice exists. For any field k, set V^k to be the vector space $M \otimes_\mathbb{Z} k$ on which $X_\alpha^n/n!$ acts in a natural way. Since the representation V has a finite number of weights, there is some n for each α such that $X_\alpha^n \in \mathrm{End}(V^k)$ is zero. Therefore, for $t \in k$ and $\alpha \in \Phi$,

$$x_\alpha(t) = \exp(tX_\alpha) = 1 + tX_\alpha + \frac{(tX_\alpha)^2}{2!} + \frac{(tX_\alpha)^3}{3!} + \cdots \in \mathrm{GL}(V^k)$$

is a finite sum and hence is well-defined.

Define the *Chevalley group* to be the subgroup $G(k)$ of $\mathrm{GL}(V^k)$ generated by $x_\alpha(t)$, with $t \in k,\ \alpha \in \Phi$. We say G is *simply connected* if $L_V = L_1$. Note that this definition is different from simply-connectedness in the topological sense. We assume all Chevalley groups are simply connected for the rest of this paper.

Define

$$w_\alpha(t) = x_\alpha(t)x_{-\alpha}(-t^{-1})x_\alpha(t) \quad \text{and} \quad h_\alpha(t) = w_\alpha(t)w_\alpha(-1) \quad \text{for } t \in k^\times.$$

Let T (the Cartan subgroup, or maximal torus) be the subgroup of G generated by $h_\alpha(t)$, with $t \in k^\times,\ \alpha \in \Phi$. By [Steinberg 1968, Lemma 28], $h_\alpha(t)$ is multiplicative as a function of t, and simply-connectedness implies that any element of T can be written uniquely as $h_1(t_1)h_2(t_2)\cdots h_l(t_l)$ for some $t_1, \dots, t_l \in k^\times$, where $h_i(t_i) = h_{\alpha_i}(t_i)$.

Now let us describe the generators and relations of a simply connected Chevalley group G over k:

(A) $x_\alpha(t)x_\alpha(u) = x_\alpha(t+u)$,

(B) $(x_\alpha(t), x_\beta(u)) = \prod_{\substack{i,j>0 \\ i\alpha+j\beta\in\Phi}} x_{i\alpha+j\beta}(c_{ij}t^i u^j)$,

(B′) $w_\alpha(t)x_\alpha(u)w_\alpha(-t) = x_{-\alpha}(-t^{-2}u)$,

(C) $h_\alpha(t)h_\alpha(u) = h_\alpha(tu)$.

Here the c_{ij} are integers depending on α, β and the chosen ordering, but not on t or u. By [Steinberg 1968, Theorem 8], if Φ is not of type A_1, then (A), (B), (C) form a complete set of relations for G constructed from Φ and k; if Φ is of type A_1, then (A), (B′), (C) form a complete set of relations. By [Steinberg 1968, Lemma 37], (B′) is also true when Φ is not of type A_1, and it implies that

$$w_\alpha(t) = w_{-\alpha}(-t^{-1}), \quad w_\alpha(1)h_\alpha(t)w_\alpha(-1) = h_\alpha(t^{-1}),$$

which we will use later.

2B. ***Covering groups.*** To study the covering group of a simply connected Chevalley group, we need some preparations. First, a central extension of a group G is a couple (π, G'), where G' is a group, and π is a homomorphism of G' onto G such that $\operatorname{Ker}\pi$ is a subset of the center of G'. A central extension (π, E) of a group G is *universal* if for any central extension (π', E') of G there exists a unique homomorphism $\phi : E \to E'$ such that $\pi' \circ \phi = \pi$. It is easy to see that if a universal central extension exists, it is unique up to isomorphism.

Theorem 2.1 [Steinberg 1968, Theorem 10]. *Let Φ be an irreducible root system and k a field such that $|k| > 4$ and if* $\operatorname{rank}\Phi = 1$, *then $|k| > 9$. Let G be the corresponding simply connected Chevalley group abstractly defined by the relations* (A), (B), (B′), (C), *let E be the group defined by the relations* (A), (B), (B′) (*we use* (B′) *only if* $\operatorname{rank}\Phi = 1$), *and let π be the natural homomorphism from E to G. Then (π, E) is a universal central extension of G.*

From now on, we use $x_\alpha(t)$, $w_\alpha(t)$, $h_\alpha(t)$ to denote the elements in the central extension of G, and $\underline{x}_\alpha(t)$, $\underline{w}_\alpha(t)$, $\underline{h}_\alpha(t)$ to denote the elements in G.

The next theorem gives a complete description of $C = \operatorname{Ker}\pi$:

Theorem 2.2 [Steinberg 1968, Theorem 12]. *Keep the assumptions in the previous theorem. $C = \operatorname{Ker}\pi$ is isomorphic to the abstract group A generated by the symbols $f(t, u)$ $(t, u \in k^*)$ subject to the relations*

(a) $f(t,u)f(tu,v) = f(t,uv)f(u,v), \quad f(1,u) = f(u,1) = 1$,

(b) $f(t,u)f(t,-u^{-1}) = f(t,-1)$,

(c) $f(t,u) = f(u^{-1}, t)$,

(d) $f(t,u) = f(t, -tu)$,

(e) $f(t,u) = f(t, (1-t)u)$.

In the case when Φ is not of type C_n $(n \geq 1)$ the relations above may be replaced by

(ab′) $f(t,u)f(t',u) = f(tt',u)$, $f(t,u)f(t,u') = f(t,uu')$,

(c′) $f(t,u) = f(u,t)^{-1}$,

(d′) $f(t,-t) = 1$,

(e′) $f(t,1-t) = 1$.

The isomorphism is given by

$$\phi : f(t,u) \mapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1},$$

where α is a fixed long root. One can write

$$h_\alpha(t)h_\alpha(u) = f(t,u)h_\alpha(tu)$$

if we identify $C = \operatorname{Ker}\pi$ with A via ϕ.

Remark. Because all long roots are conjugate by W, the isomorphism ϕ does not depend on the choice of a long root α.

Remark. These relations are satisfied by the norm residue symbol in class field theory.

For the application to real groups, we specialize our result to the case when $k = \mathbb{R}$, and consider the double cover. First, recall the real Hilbert quadratic symbol $(\ ,\ )_{\mathbb{R}}$. It is a map from $\mathbb{R}^* \times \mathbb{R}^*$ to $\mu_2 = \{\pm 1\}$. For $t, u \in \mathbb{R}^*$, $(t,u) = 1$ if and only if $x^2 - ty^2 - uz^2$ has a nontrivial solution $(x,y,z) \in \mathbb{R}^3$. It is easy to see that $(t,u) = 1$ unless both of t and u are negative. Assume G' is a *double cover* of G, more precisely, a central extension (p, G') of G such that $\operatorname{Ker} p$ is of order 2 and such that it *does not split*, i.e, there is no homomorphism $i : G \to G'$ such that $p \circ i = \mathrm{id}_G$. Since (π, E) is the universal central extension of G, there exists a homomorphism $q : E \to G'$ such that $p \circ q = \pi$. Any such q maps C onto $\operatorname{Ker} p$, that is, $\operatorname{Ker} p$ is a quotient of $C \cong A$. Passing to quotient, we use $\bar{f}(t,u) \in \mu_2$ to denote the image of $f(t,u) \in A$. Since the $\bar{f}(t,u)$ satisfy (a), (b), (c), (d), (e), G' is unique up to isomorphism. On the other hand, the Hilbert symbol (t,u) satisfies the relations that $\bar{f}(t,u)$ satisfies, hence $\bar{f}(t,u) = (t,u)$. Thus we have:

Corollary 2.3. *Assume G is a simply connected Chevalley group over $\mathbb{R}$. Then there exists a unique (up to isomorphism) double cover (p, G') of G. Moreover, an isomorphism $\phi : \mu_2 \to \operatorname{Ker} p$ is given by*

$$(t,u) \mapsto h_\alpha(t)h_\alpha(u)h_\alpha(tu)^{-1},$$

where α is a fixed long root and (t,u) is the real Hilbert quadratic symbol. One can write

$$h_\alpha(t)h_\alpha(u)=(t,u)h_\alpha(tu)$$

by identifying Ker p *and* μ_2 *via* ϕ. *Combining with* (A), (B), (B′), *we get a complete set of relations for* G'.

In the universal cover E, let T be the subgroup generated by $h_\alpha(t)$, $\alpha\in\Phi$, $t\in k^\times$. It is called the *metaplectic torus* of E. We also refer to the image of T in any cover of G as the metaplectic torus. The proposition below lists some relations in T.

Proposition 2.4. *Keep the assumptions in Theorems 2.1 and 2.2. Assume furthermore that Φ is not of type C_n. Then*

$$\begin{aligned}
h_\alpha(t)h_\alpha(u)&=f(t,u)h_\alpha(tu) && \text{if } \alpha \text{ is long};\\
h_\alpha(t)h_\alpha(u)&=f(t,u)^{n_\Phi}h_\alpha(tu) && \text{if } \alpha \text{ is short};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\alpha,\beta\rangle} && \text{if } \alpha,\beta \text{ are long};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\alpha,\beta\rangle} && \text{if } \alpha \text{ is long}, \beta \text{ is short};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{\langle\beta,\alpha\rangle} && \text{if } \alpha \text{ is short}, \beta \text{ is long};\\
(h_\alpha(t),h_\beta(u))&=f(t,u)^{n_\Phi\cdot\langle\alpha,\beta\rangle} && \text{if } \alpha,\beta \text{ are short}.
\end{aligned}$$

Here $n_\Phi=\max_{\alpha,\beta\in\Phi}(\alpha,\alpha)/(\beta,\beta)$ *and we identify* $f(t,u)$ *with its image in* C *via* ϕ.

Proof. By [Steinberg 1968, Lemma 37],

$$(h_\alpha(t),h_\beta(u))=h_\beta(t^{\langle\beta,\alpha\rangle}u)h_\beta(t^{\langle\beta,\alpha\rangle})^{-1}h_\beta(u)^{-1}$$

for any α,β. If β is long, the right-hand side is $f(u,t^{\langle\beta,\alpha\rangle})^{-1}$, which is equal to $f(t,u)^{\langle\beta,\alpha\rangle}$ since Φ is not of type C_n. Taking the inverse on both sides, we get $(h_\beta(u),h_\alpha(t))=f(u,t)^{\langle\beta,\alpha\rangle}$. Now assume β is short, α is long. Then

$$\begin{aligned}
h_\beta(u)h_\beta(t^{\langle\beta,\alpha\rangle})h_\beta(t^{\langle\beta,\alpha\rangle}u)^{-1}&=(h_\beta(u),h_\alpha(t))=f(u,t)^{\langle\alpha,\beta\rangle}\\
&=f(u,t^{\langle\alpha,\beta\rangle})=f(u,t^{\langle\beta,\alpha\rangle})^{\frac{(\alpha,\alpha)}{(\beta,\beta)}}=f(u,t^{\langle\beta,\alpha\rangle})^{n_\Phi}.
\end{aligned}$$

Because $\langle\beta,\alpha\rangle=\pm1$, $t^{\langle\beta,\alpha\rangle}$ runs through all the elements in $k^\times$. Finally, if both of α,β are short,

$$\begin{aligned}
(h_\alpha(t),h_\beta(u))&=(h_\beta(u)h_\beta(t^{\langle\beta,\alpha\rangle})h_\beta(t^{\langle\beta,\alpha\rangle}u)^{-1})^{-1}\\
&=f(u,t^{\langle\beta,\alpha\rangle})^{-n_\Phi}=f(t,u)^{n_\Phi\cdot\langle\beta,\alpha\rangle}.
\end{aligned}$$

□

Remark. Assume G is a real group and G' is its double cover. The relations above are still true if we replace $f(t,u)$ by (t,u). Because the Hilbert symbol (t,u) is bimultiplicative, by the proof of Proposition 2.4, one can remove the assumption that Φ is not of type C_n.

Proposition 2.5 [Steinberg 1968, Lemma 37]. *Let $c = c(\alpha, \beta) = \pm 1$ be independent of t and u. Then*

$$h_\alpha(t) x_\beta(u) h_\alpha(t)^{-1} = x_\beta(t^{\langle \beta, \alpha \rangle} u),$$
$$w_\alpha(1) h_\beta(t) w_\alpha(-1) = h_{w_\alpha \beta}(ct) h_{w_\alpha \beta}(c)^{-1}.$$

The next proposition gives a description of maximal compact subgroups in the setting of Chevalley groups:

Proposition 2.6. *Assume $k = \mathbb{C}$ or $\mathbb{R}$. Then there exists an automorphism σ of E such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for any $\alpha \in \Phi$, and an automorphism $\underline{\sigma}$ of G such that $\underline{\sigma}\, \underline{x}_\alpha(t) = \underline{x}_{-\alpha}(-t)$ for any $\alpha \in \Phi$. We have $\sigma h_\alpha(t) = h_\alpha(t^{-1})$, $\underline{\sigma}\, \underline{h}_\alpha(t) = \underline{h}_\alpha(t^{-1})$. Moreover, the group K of fixed points of σ is a subgroup of E containing $C = \operatorname{Ker} \pi$, the group $\underline{K}$ of fixed points of $\underline{\sigma}$ is a maximal compact subgroup of G, and $K = \pi^{-1}(\underline{K})$.*

Proof. This is basically [Steinberg 1968, Theorem 16], which proves the existence of $\underline{\sigma}$ and $\underline{K}$ for G. In particular, $x_\alpha(t) \mapsto x_{-\alpha}(-t)$, for all $\alpha \in \Phi$, preserves the relations (A) and (B). Hence $\underline{\sigma}$ can be lifted to an automorphism of E, which we denote by σ, such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for any $\alpha \in \Phi$. By the definition of $w_\alpha(t)$, $\sigma w_\alpha(t) = w_{-\alpha}(-t)$. So

$$\sigma h_\alpha(t) = \sigma w_\alpha(t) w_\alpha(-1) = \sigma w_\alpha(t) \sigma w_\alpha(-1) = w_{-\alpha}(-t) w_{-\alpha}(1).$$

Since $w_\alpha(t) = w_{-\alpha}(-t^{-1})$ for any $\alpha \in \Phi$, $t \in k^\times$, the last term is $w_\alpha(t^{-1}) w_\alpha(-1) = h_\alpha(t^{-1})$. Thus $\sigma h_\alpha(t) = h_\alpha(t^{-1})$ as in the linear case. Next, with the notation of Theorem 2.2, C is generated by $f(t, u)$, $t, u \in k^\times$, if we identify the groups A, C via ϕ. We have $h_\alpha(t) h_\alpha(u) = f(t, u) h_\alpha(tu)$. Let σ act on both sides. Then one has $h_\alpha(t^{-1}) h_\alpha(u^{-1}) = \sigma f(t, u) h_\alpha(t^{-1} u^{-1})$, which implies that $\sigma f(t, u) = f(t^{-1}, u^{-1})$. By relation (c) in Theorem 2.2, $f(t^{-1}, u^{-1}) = f(u, t^{-1}) = f(t, u)$ and hence σ fixes C. □

For the rest of this paper, we use $\underline{G}$ to denote a simply connected Chevalley group over $\mathbb{R}$, and G to denote the *double cover* of $\underline{G}$. For any subgroup H of G, let $\underline{H}$ be the image of H under the covering projection $p : G \to \underline{G}$. Define the real metaplectic torus T to be the subgroup of G generated by $h_\alpha(t)$, with $\alpha \in \Phi$ and $t \in \mathbb{R}^*$. Let $A \cong (\mathbb{R}^+)^l$ be the subgroup of T generated by $h_\alpha(t)$, with $\alpha \in \Phi$, $t > 0$. Here l is the rank of Φ. By the remark on page 196, $p|_A : A \to \underline{A}$ is an isomorphism, and hence for simplicity we just use A to denote this group. Let M be the subgroup of T generated by $h_\alpha(-1)$, with $\alpha \in \Phi$. It is easy to see that A is in the center of T, and T is the direct product of A and M. Note that M is a central extension of $\underline{M} \cong (\mathbb{Z}/2\mathbb{Z})^l$ by $\mu_2 = \{\pm 1\}$. Let Δ be a set of simple roots, and let Φ^+ be the corresponding set of positive roots. Let N be the group generated by $x_\alpha(t)$, with $\alpha \in \Phi^+$, $t \in \mathbb{R}$. Then $p|_N : N \to \underline{N}$ is an isomorphism, and hence for simplicity we just use N to denote this group. Define P to be subgroup of G generated by N and T, which we call a minimal parabolic subgroup (or Borel subgroup). We have the Langlands decomposition $P = NAM$. By Proposition 2.6, there exists an automorphism σ of G such that $\sigma x_\alpha(t) = x_{-\alpha}(-t)$ for all $\alpha \in \Phi$. Similarly for $\underline{G}$. The group K of fixed points of σ is a maximal compact subgroup of G which is the double cover of $\underline{K}$. It is easy to see that M (resp. $\underline{M}$) is a subgroup of K (resp. $\underline{K}$). One has $Z_{\underline{K}}(A) = \underline{M}$, which implies that $Z_K(A) = M$. Define the Weyl group W to be $N_K(A)/Z_K(A) = N_K(A)/M$. Then W is isomorphic to $N_{\underline{K}}(A)/\underline{M}$.

Lemma 2.7. *The $w_\alpha(1)$ lie in $N_K(A)$, for any $\alpha \in \Phi$, and their images in $N_K(A)/M$ generate W.*

Proof. Since $\sigma w_\alpha(1) = w_{-\alpha}(-1) = w_\alpha(1)$, we have $w_\alpha(1) \in K$. Also, by the second relation in Proposition 2.5, $w_\alpha(1)$ normalizes A. Each $w_\alpha(1)$ corresponds to the reflection s_α through the hyperplane determined by α, which gives an isomorphism between $W = N_K(A)/Z_K(A)$ and the Weyl group $\widehat{W}$ defined in the abstract root system setting. In particular, the $w_\alpha(1)$, for $\alpha \in \Phi$, generate W. □

2C. *The group* SL(2, $\mathbb{R}$) *and its double cover* $\widetilde{\mathrm{SL}}$(2, $\mathbb{R}$). In this section, we recall some basic facts about SL(2, $\mathbb{R}$) and its double cover $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, which are important for the study of representation theory of general covering groups.

$\underline{G} = \mathrm{SL}(2, \mathbb{R})$ may be described in Steinberg symbols: Let $X = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$, $Y = \left(\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right)$, $H = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ be the $\mathfrak{sl}_2$ triple. For $t \in \mathbb{R}$, define

$$\underline{x}(t) = \exp(tX) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, \quad \underline{w}(t) = \underline{x}(t)\underline{y}(-t^{-1})\underline{x}(t) = \begin{pmatrix} 0 & t \\ -t^{-1} & 0 \end{pmatrix},$$
$$\underline{y}(t) = \exp(tY) = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}, \quad \underline{h}(t) = \underline{w}(t)\underline{w}(-1) = \begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}.$$

Let N be the subgroup generated by $\underline{x}(t)$, $t \in \mathbb{R}$, and let A be the subgroup generated by $\underline{h}(t)$, $t > 0$. Then $\underline{K} = \mathrm{SO}(2)$ consists of $r_\phi = \left(\begin{smallmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{smallmatrix}\right)$, $\phi \in \mathbb{R}$, and $\underline{G} = NA\underline{K}$. Let $\underline{M} = \{\underline{h}(\pm 1)\} \in \underline{K}$. Then the subgroup $\underline{P}$ of upper-triangular matrices has the Langlands decomposition $\underline{P} = NA\underline{M}$.

By Corollary 2.3, there exists a unique nontrivial double cover $G = \widetilde{\mathrm{SL}}(2, \mathbb{R})$ of $\underline{G} = \mathrm{SL}(2, \mathbb{R})$, that is, a central extension of $\underline{G}$ by $\mu_2 = \{\pm 1\}$. We use p to denote the covering map. It is generated by the symbols $x(t)$, $y(t)$ satisfying the same relations as that of $\underline{G}$, except that $h(t)h(u) = (t, u)h(tu)$, where $(\,,\,)$ is the real Hilbert quadratic symbol. The map $\phi : N \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$, $\underline{x}(t) \mapsto x(t)$, $t \in \mathbb{R}$, is a group homomorphism; $\psi : A \to \widetilde{\mathrm{SL}}(2, \mathbb{R})$, $\underline{h}(t) \mapsto h(t)$, $t > 0$, is also a group homomorphism. Moreover, ϕ is the only homomorphism from N to $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ satisfying $p \circ \phi = \mathrm{Id}_N$. Assume ϕ' is another one. Consider $f : N \to \mu_2$, $n \mapsto \phi(n)\phi'(n)^{-1}$. Then we have $f(\underline{x}(t)) = f(\underline{x}(t/2)^2) = f(\underline{x}(t/2))^2 = 1$. So f is trivial, whence $\phi = \phi'$. A similar fact holds for ψ. We still denote the images of ϕ, ψ by N, A. Let K be the subgroup fixed by the automorphism σ of G, where σ sends $x(t)$ to $y(-t)$, $y(t)$ to $x(-t)$. Then K is a double cover of SO(2). We have the Iwasawa decomposition $G = NAK$. Let $M = \{\pm h(\pm 1)\} \subset K$. It is isomorphic to C_4, the cyclic group of order four, and it is an extension of $\underline{M}$ by μ_2. The group $P = NAM$ is an extension of $\underline{P}$ by μ_2.

We may also describe the group structure of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$ using Kubota cocycles. The only reason we introduce this is that Kubota cocycles make some calculations involving K more explicit. They will be used in Section 5. As a set, $\widetilde{\mathrm{SL}}(2, \mathbb{R}) = \mathrm{SL}(2, \mathbb{R}) \times \mu_2$. The group law is given by

$$(g, \varepsilon)(g', \varepsilon') = (gg', \varepsilon\varepsilon' c(g, g')).$$

Here c, called the *Kubota cocycle*, is given by the formula

$$c(g, g') = (x(g), x(g'))(-x(g)x(g'), x(gg')),$$

where

$$x\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{cases} c & \text{if } c \neq 0, \\ d & \text{if } c = 0, \end{cases}$$

and $(\, , \,)$ is the quadratic Hilbert symbol. The map $x(t) \mapsto (\underline{x}(t), 1)$, $y(t) \mapsto (\underline{y}(t), 1)$ gives an isomorphism between the two definitions. Direct calculation using the Kubota cocycle shows that $w(t) \mapsto (\underline{w}(t), 1)$ and $h(t) \mapsto (\underline{h}(t), \mathrm{sgn}(t))$. Thus:

Proposition 2.8. *We may write*

$$x(t) = (\underline{x}(t), 1), \quad y(t) = (\underline{y}(t), 1), \quad w(t) = (\underline{w}(t), 1), \quad h(t) = (\underline{h}(t), \mathrm{sgn}(t)).$$

The exponential map

$$\underline{\exp} : \mathfrak{sl}(2, \mathbb{R}) \to \mathrm{SL}(2, \mathbb{R})$$

is given by the exponents of matrices. In particular,

$$\underline{\exp}(tX) = \underline{x}(t), \quad \underline{\exp}(tH) = \underline{h}(e^t), \quad \underline{\exp}(-tZ) = r_t,$$

where $Z = X - Y$.

Proposition 2.9. *Let $\underline{e} : \mathbb{R} \to \mathrm{SO}(2)$ be the homomorphism sending ϕ to r_ϕ. Then there exists a unique homomorphism $e : \mathbb{R} \to K$ such that $p \circ e = \underline{e}$. It is given by $e(\phi) = (r_\phi, \epsilon(\phi/2))$, where $\epsilon : \mathbb{R}/2\pi\mathbb{Z} \to \pm 1$ is defined by $\epsilon(\theta) = \mathrm{sgn}(\sin\theta \sin 2\theta)$ when $\theta \neq 0, \pi/2, \pi, 3\pi/2$, $\epsilon(0) = 1$, $\epsilon(\pi/2) = -1$, $\epsilon(\pi) = -1$, $\epsilon(3\pi/2) = 1$.*

Proof. It is clear that e is of the form appearing in the proposition for some $\epsilon : \mathbb{R}/2\pi\mathbb{Z} \to \pm 1$. By working out $\exp(\theta)\exp(\theta) = \exp(2\theta)$, one sees that

$$\epsilon(\theta) = (x(r_\theta), x(r_\theta))(-1, x(r_{2\theta})) = (-1, x(r_\theta)x(r_{2\theta})) = \mathrm{sgn}(x(r_\theta)x(r_{2\theta})).$$

Direct calculations show that $x(r_\theta)x(r_{2\theta}) = \sin\theta \sin 2\theta$ when $\theta \neq 0, \pi/2, \pi, 3\pi/2$, $0 \le \theta < 2\pi$, and $\epsilon(0) = 1$, $\epsilon(\pi/2) = -1$, $\epsilon(\pi) = -1$, $\epsilon(3\pi/2) = 1$. □

Corollary 2.10. *For any integer n, the map $\sigma_{n/2} : K \to S^1$, $(r_\phi, \epsilon(\phi/2)) \mapsto e^{in\phi/2}$ is a character of K. In particular, $\sigma = \sigma_{n/2}|_M$ is a character of M satisfying $\sigma(I,1) = 1$, $\sigma(-I,-1) = i^n$, $\sigma(I,-1) = (-1)^n$, $\sigma(-I,1) = (-i)^n$.*

There exists a unique exponential map

$$\exp : \mathfrak{sl}(2,\mathbb{R}) \to \widetilde{\mathrm{SL}}(2,\mathbb{R})$$

such that $p \circ \exp = \underline{\exp}$. For any $X \in \mathfrak{sl}(2,\mathbb{R})$, let $\gamma(t)$ be the unique one-parameter subgroup of $\underline{G}$ whose tangent vector at the identity is equal to X. Since p is a covering map, $t \mapsto \gamma(t)$ can be partially lifted to a continuous map $\tilde{\gamma} : I \to G$ such that it pushes forward to $\gamma|_I$ for some neighborhood $I \subset \mathbb{R}$ around 0 and $\tilde{\gamma}(0) = 1 \in G$. Since γ is a continuous homomorphism, one can extend $\tilde{\gamma}$ to a homomorphism from $\mathbb{R}$ to G which lifts γ. We define $\exp(X)$ to be $\tilde{\gamma}(X)$.

Proposition 2.11. *We have*

$$\exp(tX) = (\underline{x}(t), 1), \quad \exp(tH) = (\underline{h}(e^t), 1), \quad \exp(-tZ) = (r_t, \epsilon(t/2)).$$

Proof. The first two are obvious and the third follows from Proposition 2.9. □

2D. *Connections between $\widetilde{\mathbf{SL}}(2,\mathbb{R})$ and general covering groups.* Let G be the unique nontrivial two-fold cover of a split real group $\underline{G}$. For each root α,

$$\Phi_\alpha : \widetilde{\mathrm{SL}}(2,\mathbb{R}) \to G$$

is defined to be the homomorphism sending $x(t)$ to $x_\alpha(t)$, $y(t)$ to $x_{-\alpha}(t)$, and $h(t)$ to $h_\alpha(t)$.

We now state a definition from [Adams et al. 2007], which will be used later:

Definition. A root α is said to be *metaplectic* if Φ_α does not factor through $\mathrm{SL}(2,\mathbb{R})$.

The next proposition follows from the first two equations in Proposition 2.4:

Proposition 2.12. *If G is not of type G_2, then α is metaplectic if and only if it is long. If G is of type G_2, then all roots are metaplectic.*

3. Pseudospherical Representations

For each $\alpha \in \Phi$, let $m_\alpha = h_\alpha(-1) \in G$ and $Z_\alpha = X_\alpha - X_{-\alpha} \in \mathfrak{g}$. Then we have $\exp(-\pi Z_\alpha) = m_\alpha$ by Propositions 2.8 and 2.11. The following definition is from Definition 4.9 and Lemma 4.11 of [Adams et al. 2007]:

Definition (pseudospherical representations). An irreducible representation σ of M is pseudospherical if the eigenvalues of $\sigma(m_\alpha)$ belong to $\{\pm i\}$ when α is a metaplectic root, and $\{1\}$ otherwise. An irreducible representation μ of K is pseudospherical if the eigenvalues of $d\mu(iZ_\alpha)$ belong to $\left\{\pm\frac{1}{2}\right\}$ when α is a metaplectic root, and $\{0\}$ otherwise. A representation of G is pseudospherical if it contains a pseudospherical K-type.

Remark. When $G = \widetilde{\mathrm{SL}}(n, \mathbb{R})$, the double cover of $\mathrm{SL}(n, \mathbb{R})$, the *Spinor representation* of $K = \mathrm{Spin}(n)$ is pseudospherical.

Remark. If G is simply laced or of type G_2, then every irreducible genuine representation of M is pseudospherical. In fact, all roots are metaplectic in this case, and so $m_\alpha^2 = h_\alpha(-1)h_\alpha(-1) = -1 \in \mu_2 \subset Z(G)$. So $\sigma(m_\alpha)^2 = \sigma(-1) = -I$ and hence its eigenvalues are $\pm i$ with multiplicities.

Also notice that eigenvalues in the pseudospherical conditions for M and K are compatible: $\log(1) = 2\pi i\mathbb{Z}$, $\log(\pm i) = \pm i\pi/2 + 2\pi i\mathbb{Z}$.

Example. In the case $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, the representation $\mu = \sigma_{1/2}$ is a pseudospherical representation of $K = \widetilde{\mathrm{SO}}(2)$ whose restriction $\sigma = \sigma_{1/2}|_M$ to M is a pseudospherical representation of M. In fact,

$$d\mu(Z_\alpha) = \lim_{t\to 0} \frac{\mu(\exp(tZ_\alpha)) - 1}{t}.$$

But $\mu(\exp(tZ_\alpha)) = \sigma_{1/2}(r_{-t}, \epsilon(-t/2)) = e^{-it/2}$, so the limit is

$$\lim_{t\to 0} \frac{e^{-it/2} - 1}{t} = -\frac{i}{2}.$$

Thus $d\mu(iZ_\alpha) = \frac{1}{2}$ and

$$\sigma(m_\alpha) = \sigma(h_\alpha(-1)) = \sigma(r_\pi, \epsilon(\pi/2)) = e^{i\pi/2} = i.$$

Below is a fundamental fact on pseudospherical representations:

Theorem 3.1 [Adams et al. 2007, Proposition 5.2]. *Let σ be a pseudospherical representation of M. There is a unique pseudospherical representation μ_σ of K such that $\mu_\sigma|_M = \sigma$ and this defines a bijection between pseudospherical representations of M and K.*

Now we want to define an action of W on irreducible representations (σ, V) of M that do not factor through $\underline{M}$ (or equivalently, $\sigma(-1) \neq 1$). We call such representations *genuine representations*. We use $\Pi_g(M)$ to denote the set of isomorphism classes of genuine representations of M. We will show that W fixes every isomorphism class of irreducible genuine pseudospherical representations of M. This is proved in [Adams et al. 2007, Lemma 4.11(3)]. We repeat the argument below for completeness.

Proposition 3.2. *Let $Z(M) \supset \mu_2$ be the center of M, and let $\Pi_g(Z(M))$ be the set of genuine characters of $Z(M)$, that is, those characters χ satisfying $\chi(-1) = -1$.*

For every $\chi \in \Pi_g(Z(M))$, there is a unique representation $\sigma(\chi)$ of M such that $\sigma(\chi)|_{Z(M)} = \chi \cdot I$. The map $\chi \mapsto \sigma(\chi)$ defines a bijection $\Pi_g(Z(M)) \to \Pi_g(M)$. The dimension of $\sigma(\chi)$ is $|M/Z(M)|^{1/2}$ and $\mathrm{Ind}_{Z(M)}^{M}(\chi) \cong |M/Z(M)|^{1/2}\sigma(\chi)$.

Proof. The key point of the proof in [Adams et al. 2007] is that if σ is a genuine representation of M, then the character $\operatorname{tr}\sigma$ is supported on $Z(M)$. Suppose m does not belong to $Z(M)$. Choose $h \in M$ such that $hmh^{-1} \neq m$. Since $\underline{M}$ is abelian, $p(hmh^{-1}) = p(h)p(m)p(h)^{-1} = p(m)$, so $hmh^{-1} = -m$. Taking the trace on both sides, we have $\operatorname{tr}\sigma(m) = \chi(-1)\operatorname{tr}\sigma(m)$. Since χ is genuine, $\chi(-1) = -1$, so $\operatorname{tr}\sigma(m) = 0$.

Therefore, every irreducible genuine representation of M is uniquely determined by its central character. Fix $\chi \in \Pi_g(Z(M))$. Let $I(\chi) = \mathrm{Ind}_{Z(M)}^{M}(\chi)$. This has central character χ, so it is a multiple of the irreducible representation $\sigma(\chi)$ of M with central character χ. Put $I(\chi) = n\sigma(\chi)$. By Frobenius reciprocity,

$$\mathrm{Hom}_M(I(\chi), I(\chi)) = \mathrm{Hom}_{Z(M)}(I(\chi)|_{Z(M)}, \chi),$$

which has dimension $|M/Z(M)|$. On the other hand, by Schur's lemma,

$$\mathrm{Hom}_M(I(\chi), I(\chi)) = \mathrm{Hom}_M(n\sigma(\chi), n\sigma(\chi)),$$

which has dimension n^2. Therefore $n = |M/Z(M)|^{1/2}$ and the dimension of $\sigma(\chi)$ is $|M/Z(M)|^{1/2}$. □

Since $N_K(A)$ acts on M by conjugation, it also acts on its center $Z(M)$, which factors down to $W = N_K(A)/M$. Thus we have an action of W on $\Pi_g(Z(M))$. By the proposition above, this gives rise to an action of W on $\Pi_g(M)$. More precisely, pick a representative $\hat{w} \in N_K(A)$ of $w \in W$. Then $\hat{w}\sigma(m) = \sigma(\hat{w}^{-1}m\hat{w})$ is a representation of M. Up to isomorphism, it is independent of the choice of a representative of w because different representatives $\hat{w}$ give the same central character, hence isomorphic representations. Therefore one can denote this representation by $w\sigma$, as an isomorphism class in $\Pi_g(M)$.

Proposition 3.3. *The action of the Weyl group W on the isomorphism classes of irreducible genuine representations of M fixes each isomorphism class of pseudospherical representations.*

Proof. Assume (σ, V) is a genuine representation of M. For all $w \in W$, choose a representative $\hat{w}$ of w in $N_K(A)$. By Theorem 3.1, there is a unique pseudospherical representation (μ_σ, V) of K such that $\mu_\sigma|_M = \sigma$. Let $\phi : V \to V,\ v \mapsto \mu_\sigma(\hat{w}^{-1})v$. Then

$$\phi(\mu_\sigma(k)v) = \mu_\sigma(\hat{w}^{-1})\mu_\sigma(k)v = \mu_\sigma(\hat{w}^{-1}k\hat{w})\mu_\sigma(\hat{w}^{-1})v = (\hat{w}\mu_\sigma)(k)\phi(v)$$

for any $k \in K$. Thus ϕ is a K-isomorphism, and restricting it to M, we get

$$\sigma \cong (\hat{w}\mu_\sigma)|_M = \hat{w}\sigma.$$

□

4. Principal series representations and intertwining operators

In this section, let G be the double cover of a split real group. We define the principal series representation of G and the intertwining operator. Most of the results are well-known in the linear group case; see [Schiffmann 1971]. The discussion for covering groups is almost identical to the linear case. The highlight is that the intertwining maps can be defined in a canonical way using the theory of pseudospherical representations.

Let χ be a character of A, and let δ be the modular character of A such that

$$\int_N f(a^{-1}na)\,dn = \delta(a)\int_N f(n)\,dn$$

for any $a \in A$ and any compact supported function f on N. Here we fix a Haar measure on N, which is topologically isomorphic to $\mathbb{R}^{|\Phi^+|}$. Since δ depends on N, we will write δ_N instead of δ when needed. The character δ is equal to the product of the roots in Φ^+, considered as multiplicative characters of A. Let (σ, V) be a pseudospherical representation of M.

Definition. Let $I(P, \sigma, \chi)$, the space of principal series, be the space of smooth functions $f : G \to V$ such that

$$f(namx) = \delta(a)^{1/2}\chi(a)\sigma(m) f(x)$$

for all $n \in N$, $a \in A$, $m \in M$, and $x \in G$. Then G acts on $I(P, \sigma, \chi)$ by right translation: $\rho(g) f(x) = f(xg)$. This defines a representation of G called the *principal series representation, or induced representation*, of G. For simplicity, we denote this representation by $I(\sigma, \chi)$ or $I(\chi)$ when there is no confusion.

Assume χ is a character of A. For all $w \in N_K(A)$, $w\chi(a) = \chi(w^{-1}aw)$ is another character of A. This action factors down to W. Note that $w_1(w_2\chi) = (w_1w_2)\chi$. In other words, we have an action of the Weyl group W on $\Pi(A) =$ the set of characters of A.

By Theorem 3.1, there is an irreducible representation (μ_σ, V) of K such that $\mu_\sigma|_M = \sigma$. For any $f \in I(P, \sigma, \chi)$ and any $w \in W$, pick a representative $\hat{w} \in N_K(A)$ of w, and define a function

$$M(w, \sigma, \chi) f(x) = \mu_\sigma(\hat{w}) \int_{N\cap \hat{w}N\hat{w}^{-1}\backslash N} f(\hat{w}^{-1}nx)\,dn.$$

Note that $n \to f(\hat{w}^{-1}nx)$ is left $(N \cap \hat{w}N\hat{w}^{-1})$-invariant, so the integral makes sense. Also it is well-defined, i.e, independent of the choice of a representative of w in $N_K(A)$ due to the normalizing factor μ_σ. For simplicity, we write w in place of $\hat{w}$ when there is no confusion. Let us remark that N_w, which is equal to

$N \cap wNw^{-1} \setminus N$, corresponds to those positive roots that are sent to negative by w^{-1}, and it has one-to-one correspondence with $B \setminus Bw^{-1}N$.

Let $S(w)$ be the set of χ such that the above integral is absolutely convergent for any $x \in G$, $f \in I(P, \sigma, \chi)$. We are going to show that $M(w, \sigma, \chi)$ maps $I(P, \sigma, \chi)$ into $I(P, \sigma, w\chi)$ for $\chi \in S(w)$. This map is called the *intertwining map*. For simplicity, we sometimes denote this map by $M(w, \chi)$ or $M(w)$.

Lemma 4.1. *Let w be an element in W, and let δ_w be a character of A such that*

$$\int_{N_w} f(a^{-1}na)\,dn = \delta_w(a) \int_{N_w} f(n)\,dn$$

for any $a \in A$ and any integrable function f on N. Then $(w\delta)^{1/2}\delta_w = \delta^{1/2}$.

Proof. For a simple reflection w, take $Q = P \cup Pw^{-1}P$ and let L, U be its Levi factor and unipotent radical. Note that $U = N \cap wNw^{-1}$. We have $\delta_N = \delta_U \delta_{N/U}$ and $\delta_{wNw^{-1}} = \delta_U \delta_{wNw^{-1}/U}$. But $\delta_{wNw^{-1}} = w\delta_N$ and $\delta_{wNw^{-1}/U} = \delta_{N/U}^{-1}$. The conclusion now follows from simple algebraic manipulations. □

Proposition 4.2. *If $\chi \in S(w)$, then $M(w, \sigma, \chi)$ maps $I(P, \sigma, \chi)$ into $I(P, \sigma, w\chi)$.*

Proof. $M(w, \sigma, \chi) f(nx) = M(w, \sigma, \chi) f(x)$ is obvious. Next we have

$$\begin{aligned} M(w, \sigma, \chi) f(ax) &= \mu_\sigma(w) \int_{N_w} f(w^{-1}nax)\,dn \\ &= \mu_\sigma(w) \int_{N_w} f((w^{-1}aw)w^{-1}(a^{-1}na)x)\,dn \\ &= \mu_\sigma(w) w\delta(a)^{1/2} w\chi(a) \int_{N_w} f(w^{-1}(a^{-1}na)x)\,dn \\ &= \mu_\sigma(w) w\delta(a)^{1/2} w\chi(a)\delta_w(a) \int_{N_w} f(w^{-1}nx)\,dn \\ &= \delta(a)^{1/2} w\chi(a) M(w, \sigma, \chi) f(x). \end{aligned}$$

Similarly,

$$\begin{aligned} M(w, \sigma, \chi) f(mx) &= \mu_\sigma(w) \int_{N_w} f(w^{-1}nmx)\,dn \\ &= \mu_\sigma(w) \int_{N_w} f((w^{-1}mw)w^{-1}(m^{-1}nm)x)\,dn \\ &= \mu_\sigma(w)\sigma(w^{-1}mw) \int_{N_w} f(w^{-1}(m^{-1}nm)x)\,dn \\ &= \sigma(m)\mu_\sigma(w) \int_{N_w} f(w^{-1}nx)\,dn \\ &= \sigma(m) M(w, \sigma, \chi) f(x). \end{aligned}$$

□

Assume that the Haar measures on the N_w are normalized so that, when $l(w_1w_2) = l(w_1) + l(w_2)$,

$$\int_{N_{w_1w_2}} f(n)\,dn = \int_{N_{w_1}\times N_{w_2}} f(w_1n_2w_1^{-1}n_1)\,dn_1\,dn_2$$

for any integrable function f on $N_{w_1w_2}$. Under this assumption, the following proposition holds:

Proposition 4.3. *Assume $w_1, w_2 \in W$ such that $l(w_1w_2) = l(w_1) + l(w_2)$. Then*

$$S(w_1w_2) = S(w_2) \cap w_2^{-1} S(w_1),$$

and for $\chi \in S(w)$ regular (only fixed by the trivial element in W),

$$M(w_1, \sigma, w_2\chi) \circ M(w_2, \sigma, \chi) = M(w_1w_2, \sigma, \chi).$$

Proof. Since χ is regular, the dimension of $\mathrm{Hom}_G(I(\chi), I(w\chi))$ is one for any $w \in W$, by Frobenius reciprocity. So it suffices to show that $(M(w_1)\circ M(w_2)f)(1) = M(w_1w_2)f(1)$:

$$\begin{aligned}(M(w_1)\circ M(w_2)f)(1) &= \mu_\sigma(w_1)\int_{N_{w_1}} (M(w_2)f)(w_1^{-1}n_1)\,dn_1\\ &= \mu_\sigma(w_1)\mu_\sigma(w_2)\int_{N_{w_1}} dn_1 \int_{N_{w_2}} f(w_2^{-1}n_2w_1^{-1}n_1)\,dn_2\\ &= \mu_\sigma(w_1w_2)\int_{N_{w_1}} dn_1 \int_{N_{w_2}} f(w_2^{-1}w_1^{-1}\cdot w_1n_2w_1^{-1}n_1)\,dn_2.\end{aligned}$$

By the assumption on the Haar measures, the last expression is equal to

$$\mu_\sigma(w_1w_2)\int_{N_{w_1w_2}} f(w_2^{-1}w_1^{-1}n)\,dn = M(w_1w_2)f(1). \qquad \square$$

5. Representations of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$

We carry out the detailed study of principal series and intertwining maps in the SL_2 case first, since it is the fundamental building block of the general case. The results in Section 5A are well-known, but we list them here for the purpose of making a comparison with the nonlinear case.

5A. ***Linear case.*** Let $\underline{G} = \mathrm{SL}(2, \mathbb{R})$, and let $\underline{P}$ be the standard parabolic subgroup with Langlands decomposition $\underline{P} = NA\underline{M}$. Then $\underline{M}$ has only two characters. Let σ be any of them. For any complex number s, define a character χ of A by $\chi(a) = a^s$, where $a = \mathrm{diag}(a, a^{-1})$. The modular character $\delta(a)$ of A is a^2. So the space

$I(\underline{P}, \sigma, \chi)$ of principal series in this case is the collection of functions f such that

$$f(namx) = a^{s+1}\sigma(m) f(x).$$

For simplicity, we denote it by $I(\sigma, s)$. Let $\underline{K} = \mathrm{SO}(2)$. For any $n \in \mathbb{Z}$, $\tau_n(r_\phi) = e^{in\phi}$ is a character of $\underline{K}$. Define f_s^n such that

$$f_s^n(nak) = a^{s+1}\tau_n(k).$$

Then $f_s^n \in I(\tau_n|_{\underline{M}}, s)$. When n is even, $\tau_n|_{\underline{M}}$ is trivial, denoted by σ_0. When n is odd, $\tau_n|_{\underline{M}}$ is nontrivial, denoted by σ_1. We say f_s^n is of $\underline{K}$-type n.

The Weyl group W is of order two. Let w be its nontrivial element. Now we define the intertwining map $M(\sigma, s) : I(\sigma, s) \to I(\sigma, -s)$. For $f \in I(\sigma, s)$,

$$\begin{aligned} M(\sigma, s) f(x) &= \int_N f(wnx)\, dn &&\text{when } \sigma = \sigma_0, \\ M(\sigma, s) f(x) &= \tau_1(w)^{-1} \int_N f(wnx)\, dn &&\text{when } \sigma = \sigma_1. \end{aligned}$$

It does not depend on the choice of a representative element of w in $N_K(A)$. We have $M(\sigma, s) f_s^n = c_n(s) f_{-s}^n$ for some constant $c_n(s) = (M(\sigma, s) f_s^n)(1)$. The following proposition is well-known. We will give a proof of a more general proposition in the next subsection.

Proposition 5.1.
$$c_n(s) = \sqrt{\pi}\,\frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+n+1}{2}\right)\Gamma\left(\frac{s-n+1}{2}\right)}.$$

5B. ***Nonlinear case.*** Let $G = \widetilde{\mathrm{SL}}(2, \mathbb{R})$, and let $P = NAM$ be its standard parabolic subgroup which is the double cover of $\underline{P}$. Let K be the double cover of $\underline{K} = \mathrm{SO}(2)$. We are going to study the principal series of G and calculate the intertwining map using the Kubota cocycle. Let σ be a character of M, and define a character χ of A by

$$\chi(a) = a^s,$$

where $s \in \mathbb{C}$, $a = (\mathrm{diag}(a, a^{-1}), 1) \in A$. Since $\delta_N(a) = a^2$ and $\delta_{\bar{N}}(a) = a^{-2}$, $I(P, \sigma, \chi)$, which we denote by $I(\sigma, s)$ for simplicity, consists of functions f such that

$$f(namx) = a^{s+1}\sigma(m) f(x).$$

$I(\bar{P}, \sigma, \chi)$, which we denote by $\bar{I}(\sigma, s)$ for simplicity, consists of functions f such that

$$f(\bar{n}amx) = a^{s-1}\sigma(m) f(x).$$

For any $n \in \mathbb{Z}$, define $f_s^{n/2}$ such that

$$f_s^{n/2}(nak) = a^{s+1}\sigma_{n/2}(k),$$

where $\sigma_{n/2}$ is a character of K defined in Proposition 2.9. Then $f_s^{n/2} \in I(\sigma_{n/2}|_M, s)$. We say $f_s^{n/2}$ is of K-type $n/2$. Similarly, define $\bar{f}_s^{n/2}$ such that

$$\bar{f}_s^{n/2}(\bar{n}ak) = a^{s-1}\sigma_{n/2}(k).$$

Then $\bar{f}_s^{n/2} \in \bar{I}(\sigma_{n/2}|_M, s)$.

There are four different characters σ of M: $\sigma_0|_M, \sigma_{1/2}|_M, \sigma_1|_M, \sigma_{3/2}|_M$. Define the intertwining map $M(\sigma, s) : I(\sigma, s) \to I(\sigma, -s)$ by

$$M(\sigma, s) f(x) = \sigma_{i/2}(w)^{-1} \int_N f(wnx)\, dn, \quad \sigma = \sigma_{i/2}|_M,\ i = 0, 1, 2, 3.$$

This definition is canonical. Define $T : \bar{I}(\sigma, s) \to I(\sigma, -s)$ by

$$T f(x) = \sigma_{i/2}(w)^{-1} f(wx), \quad \sigma = \sigma_{i/2}|_M,\ i = 0, 1, 2, 3.$$

Also define intertwining maps $A(\sigma, s) : I(\sigma, s) \to \bar{I}(\sigma, s)$ such that

$$A(\sigma, s) f(x) = \int_{\bar{N}} f(\bar{n}x)\, d\bar{n}$$

and $\bar{A}(\sigma, s) : \bar{I}(\sigma, s) \to I(\sigma, s)$ such that

$$\bar{A}(\sigma, s) f(x) = \int_N f(nx)\, dn.$$

Then we have

$$M(\sigma, s) = T \circ A(\sigma, s).$$

$M(\sigma, s)$ sends $f_s^{n/2}$ to $c_{n/2}(s) f_{-s}^{n/2}$ for some constant $c_{n/2}(s)$. It is sometimes called the *Harish-Chandra c-function*. It is easy to see that $c_{n/2}(s) = A(\sigma, s) f_s^{n/2}(1)$. For simplicity, we sometimes use $I(s)$ in place of $I(\sigma, s)$ and $A(s)$ in place of $A(\sigma, s)$.

Define a pairing $(\,,\,) : I(s) \times I(-\bar{s}) \to \mathbb{C}$ by

$$(f, g) = \int_K f(k)\overline{g(k)}\, dk.$$

There is also a pairing $(\,,\,) : \bar{I}(s) \times \bar{I}(-\bar{s}) \to \mathbb{C}$ defined using the same formula. The following lemma follows from formal calculations:

Lemma 5.2. *For $f \in I(s)$ and $g \in \bar{I}(-\bar{s})$, we have $(A(s)f, g) = (f, \bar{A}(-\bar{s})g)$.*

Proposition 5.3. *For those $s \in i\mathbb{R}$ such that $I(s)$ is irreducible, $\bar{A}(s) \circ A(s)$ is a nonnegative constant.*

Proof. By Schur's lemma, $\bar{A}(s) \circ A(s)$ is a constant, say $\lambda(s)$. When $s \in i\mathbb{R}$, $s = -\bar{s}$, so by Lemma 5.2 $(A(s)f, g) = (f, \bar{A}(s)g)$. Taking $g = A(s)f$, we get $(A(s)f, A(s)f) = (f, \bar{A}(s) \circ A(s)f) = \bar{\lambda}(s)(f, f)$, hence $\lambda(s)$ is nonnegative. □

Below is a nice property of the c-function:

Proposition 5.4. $c_{n/2}(s) = c_{-n/2}(s)$.

Proof. Let $d = \mathrm{diag}(1, -1) \in \mathrm{GL}_2$. The conjugation action of d on SL_2 satisfies $d r_\phi d^{-1} = r_{-\phi}$. This action lifts to the covering group and it gives an inverse on K. Hence the conjugation of functions by d gives an intertwining map $I(\sigma, s) \to I(\sigma^{-1}, s)$ which we denote by $d(s)$. We have

$$M(\sigma^{-1}, s) \circ d(s) = d(-s) \circ M(\sigma, s).$$

In fact, for any $f \in I(\sigma, s)$,

$$\begin{aligned} M(\sigma^{-1}, s) \circ d(s) f(x) &= \sigma_{-n/2}(w)^{-1} \int_N (d(s) f)(wnx)\, dn \\ &= \sigma_{-n/2}(w)^{-1} \int_N f(dwnxd^{-1})\, dn \\ &= \sigma_{n/2}(w) \int_N f(w^{-1} dnxd^{-1})\, dn. \end{aligned}$$

Since $w^{-1} = mw$ for some $m \in M$, the last expression is

$$\sigma_{n/2}(w)\sigma(m) \int_N f(wdnxd^{-1})\, dn = \sigma_{n/2}(w^{-1}) \int_N f(wdnxd^{-1})\, dn.$$

On the other hand,

$$\begin{aligned} d(-s) \circ M(\sigma, s) f(x) &= \sigma_{n/2}(w)^{-1} \int_N f(wndxd^{-1})\, dn \\ &= \sigma_{n/2}(w)^{-1} \int_N f(wdnxd^{-1})\, dn, \end{aligned}$$

hence the two operators are equal.

Now take $f = f_s^{n/2}$. Then

$$\begin{aligned} M(\sigma^{-1}, s) \circ d(s) f_s^{n/2} &= c_{-n/2}(s) f_{-s}^{-n/2}, \\ d(-s) \circ M(\sigma, s) f_s^{n/2} &= c_{n/2}(s) f_{-s}^{-n/2}. \end{aligned}$$

Thus $c_{n/2}(s) = c_{-n/2}(s)$. □

Now we calculate $c_{n/2}(s)$:

Proposition 5.5.
$$c_{n/2}(s) = \sqrt{\pi} \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2} + \frac{n}{4}\right)\Gamma\left(\frac{s+1}{2} - \frac{n}{4}\right)}.$$

Proof. We have $c_{n/2}(s) = \int_{\bar{N}} f_s^{n/2}(\bar{n})\, d\bar{n}$. For $\bar{n} = \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right) \in \bar{N}$,

$$\begin{aligned}\bar{n} &= \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} y^{1/2} & 0 \\ 0 & y^{-1/2} \end{pmatrix} (r_\phi, 1) \\ &= \begin{pmatrix} y^{1/2}\cos\phi + xy^{-1/2}\sin\phi & -y^{1/2}\sin\phi + xy^{-1/2}\cos\phi \\ y^{-1/2}\sin\phi & y^{-1/2}\cos\phi \end{pmatrix}.\end{aligned}$$

Then

$$\bar{n}\cdot i = \frac{i}{ti+1} = \frac{t}{t^2+1} + \frac{1}{t^2+1} i = x + yi,$$

so $a(t) = y^{1/2} = 1/\sqrt{t^2+1},\ \tan\phi = t,\ \phi = \phi(t) = \arctan t$. Then

$$c_{n/2}(s) = \int_{\mathbb{R}} f_s^{n/2}\left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right) dt = \int_{\mathbb{R}} \frac{1}{(t^2+1)^{(s+1)/2}} \sigma_{n/2}(r_{\phi(t)}, 1)\, dt.$$

Since $\phi(t) \in (-\pi/2, \pi/2)$, we have $\epsilon(\phi(t)/2) = \operatorname{sgn}(\sin(\phi(t)/2)\sin(\phi(t))) = 1$, hence $\sigma_{n/2}(r_{\phi(t)}, 1) = e^{in\phi(t)/2}$. Finally, by substituting those expressions into the last integral, we get

$$c_{n/2}(s) = \int_{\mathbb{R}} \frac{1}{(t^2+1)^{(s+1)/2}} e^{in(\arctan t)/2} = \int_{\mathbb{R}} \frac{1}{(t^2+1)^{(s+1)/2}} \left(\frac{1-it}{\sqrt{t^2+1}}\right)^{-n/2} dt.$$

Now the proposition follows from the lemma below. □

Lemma 5.6. *For any $n \in \mathbb{Z}$,*

$$\int_{\mathbb{R}} \frac{1}{(t^2+1)^{(s+1)/2}} \left(\frac{1-it}{\sqrt{t^2+1}}\right)^{-n/2} dt = \sqrt{\pi}\, \frac{\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s+1}{2}\right)}{\Gamma\left(\frac{s+1}{2}+\frac{n}{4}\right)\Gamma\left(\frac{s+1}{2}-\frac{n}{4}\right)}.$$

Proof. The integral is absolutely convergent for $\operatorname{Re}(s) > 0$. The integrand is equal to

$$(1+it)^{(-2s-n-2)/4}(1-it)^{(-2s+n-2)/4},$$

which we denote by $f(t)$. By Lebesgue's dominated convergence theorem,

$$\lim_{y\to 0} \int_{\mathbb{R}} f(t) e^{-ity}\, dt = \int_{\mathbb{R}} f(t)\, dt.$$

Let $2u = \frac{1}{4}(2s+n+2),\ 2v = \frac{1}{4}(2s-n+2)$. By [Erdélyi et al. 1954],

$$\hat{f}(y) = \int_{\mathbb{R}} f(t) e^{-ity}\, dt = 2\pi 2^{-u-v}\Gamma(2v)^{-1} y^{u+v-1} W_{v-u, 1/2-v-u}(2y)$$

for $y > 0$. Here W is the Whittaker function

$$W_{\rho,\sigma}(z) = \frac{\Gamma(-2\sigma)}{\Gamma\left(\frac{1}{2}-\sigma-\rho\right)} M_{\rho,\sigma}(z) + \frac{\Gamma(2\sigma)}{\Gamma\left(\frac{1}{2}+\sigma-\rho\right)} M_{\rho,-\sigma}(z),$$

where

$$M_{\rho,\sigma}(z) = z^{1/2+\sigma} e^{-z/2} F\big(\tfrac{1}{2}+\sigma-\rho, 2\sigma+1, z\big),$$

$$F(a,b,z) = 1 + \sum_{k\geq 1} \frac{a(a+1)\cdots(a+k-1)}{b(b+1)\cdots(b+k-1)} \frac{z^k}{k!}.$$

So

$$W_{v-u,1/2-v-u}(2y) = \frac{\Gamma(-1+2u+2v)}{\Gamma(2u)} (2y)^{1-u-v} e^{-y} F(1-2v, 2-2u-2v, 2y) + \frac{\Gamma(1-2u-2v)}{\Gamma(1-2v)} (2y)^{u+v} e^{-y} F(2u, 2u+2v, 2y).$$

Thus

$$y^{u+v-1} W_{v-u,1/2-v-u}(2y) \to \frac{\Gamma(-1+2u+2v)}{\Gamma(2u)} 2^{1-u-v}$$

as $y \to 0$. It follows that

$$\int_{\mathbb{R}} f(t)\,dt = \lim_{y\to 0} \hat{f}(y) = 2\pi 2^{-u-v} \Gamma(2v)^{-1} \frac{\Gamma(-1+2u+2v)}{\Gamma(2u)} 2^{1-u-v} = \pi 2^{1-s} \frac{\Gamma(s)}{\Gamma\big(\frac{s+1}{2}+\frac{n}{4}\big)\Gamma\big(\frac{s+1}{2}-\frac{n}{4}\big)}.$$

By the double formula,

$$\Gamma(s) = \frac{1}{\sqrt{\pi}} 2^{s-1} \Gamma\Big(\frac{s}{2}\Big)\Gamma\Big(\frac{s+1}{2}\Big),$$

hence

$$\int_{\mathbb{R}} f(t)\,dt = \sqrt{\pi} \frac{\Gamma\big(\frac{s}{2}\big)\Gamma\big(\frac{s+1}{2}\big)}{\Gamma\big(\frac{s+1}{2}+\frac{n}{4}\big)\Gamma\big(\frac{s+1}{2}-\frac{n}{4}\big)}. \qquad \square$$

Now we consider a slightly more general situation, which will be used in the next section. Let (σ, V) be a finite-dimensional representation of M which is the restriction of a representation (μ, V) of K. Let $I(\sigma, s)$ be the space of functions $f : G \to V$ such that $f(namg) = a^{s+1}\sigma(m) f(g)$. For $f \in I(\sigma, s)$, define

$$M(s) f(x) = \mu(w)^{-1} \int_N f(wnx)\,dn.$$

By Proposition 4.2, $M(s)$ maps $I(\sigma, s)$ into $I(\sigma, -s)$. For $v \in V$, define

$$f_s^v(nak) = a^{s+1}\mu(k)v.$$

Then $v \mapsto f_s^v$ is an embedding of (μ, V) into $I(\sigma, s)$, as a K-subrepresentation.

Proposition 5.7. *Assume (μ, V) is a direct sum of $\sigma_{\pm n/2}$ for a fixed integer n. Then $(M(s) f_s^v)(1) = c_{n/2}(s)v$.*

Proof. If v belongs to one of those summands, then by the definition of $M(s)$ and Proposition 5.5, $(M(s)f_s^v)(1) = c_{n/2}(s)v$. Because $c_{n/2}(s) = c_{-n/2}(s)$, this is valid for any $v \in V$. □

6. Action of intertwining operators on pseudospherical K-types

This section contains the main result of this paper. Let G be the unique nontrivial two-fold cover of a split real group $\underline{G}$. Assume σ is a pseudospherical representation of M and μ_σ is the pseudospherical representation of K corresponding to σ. We note that the multiplicity of μ_σ in $I(P, \sigma, \chi)$ is one and then calculate the action of the intertwining operator on it, obtaining explicit formulas of the Harish-Chandra c-function.

The following lemma is fairly simple; see Definition 5.5 of [Adams et al. 2007].

Lemma 6.1. *As a K-representation, the multiplicity of μ_σ in $I(P, \sigma, \chi)$ is* 1.

Proof. It is easy to see that, as a K-representation, $I(P, \sigma, \chi)$ is isomorphic to $\mathrm{Ind}_M^K(\sigma)$. By Frobenius reciprocity, $\mathrm{Hom}_K(\mu_\sigma, \mathrm{Ind}_M^K(\sigma)) = \mathrm{Hom}_M(\sigma, \sigma)$, which is isomorphic to $\mathbb{C}$ by Schur's lemma. □

Let ϕ be the unique element in $\mathrm{Hom}_K(\mu_\sigma, I(P, \sigma, \chi))$ such that $(\phi v)(1) = v$ for all $v \in V$, and let ψ be the unique element in $\mathrm{Hom}_K(\mu_\sigma, I(P, \sigma, w\chi))$ such that $(\psi v)(1) = v$ for all $v \in V$. Then $M(w, \sigma, \chi)(\phi v) = c \cdot (\psi v)$ for some nonzero constant $c \in \mathbb{C}$ which does not depend on v.

Let $s = (s_1, \dots, s_l) \in \mathbb{C}^l$ and take $\chi = \chi_s$ to be the character of A such that

$$\chi_s(h_1(t_1) \cdots h_l(t_l)) = t_1^{s_1} \cdots t_l^{s_l}, \quad t_i > 0.$$

We write $I(P, \sigma, s)$ instead of $I(P, \sigma, \chi)$. Let $ws \in \mathbb{C}^l$ be such that $w\chi_s = \chi_{ws}$. We write $M(w, s)$ for the intertwining map instead of $M(w, \sigma, \chi_s)$.

Lemma 6.2. *Define a function $f_{P,\mu_\sigma,s}^v : G \to V$ such that*

$$f_{P,\mu_\sigma,s}^v(nak) = \chi_s(a)\delta_N(a)^{1/2}\mu_\sigma(k)v.$$

Then $f_{P,\mu_\sigma,s}^v$ is well-defined and lies in $I(P, \sigma, s)$.

Proof. For simplicity, we write f_s^v in place of $f_{P,\mu_\sigma,s}^v$ when there is no confusion. Since the Iwasawa decomposition is unique (this is not true in the p-adic case), f_s^v is well-defined. It is evident that $f_s^v(nx) = f_s^v(x)$. For any $a \in A$, we have $f_s^v(ax) = f_s^v(an(x)a(x)k(x))$. Since T normalizes N, it is equal to $\chi_s(a)\delta_N(a)^{1/2} f_s^v(x)$. Finally, since T normalizes N and A is contained in the center of T,

$$\begin{aligned} f_s^v(mx) &= f_s^v(mn(x)a(x)k(x)) = f_s^v(n'(x)ma(x)k(x)) \\ &= f_s^v(n'(x)a(x)mk(x)) = \sigma(m) f_s^v(x). \end{aligned}$$

Thus $f_s^v \in I(P, \sigma, s)$. □

Lemma 6.3. *Define $\phi : \mu_\sigma \to I(P, \sigma, s)$, $v \mapsto f_s^v$. Then ϕ is a K-intertwining map.*

Proof. We need to show $\phi(\sigma(k)v) = R(k)\phi(v)$. For $x \in G$, let $x = n(x)a(x)k(x)$ be the Iwasawa decomposition of x. Then

$$\phi(\sigma(k)v)(x) = f_s^{\sigma(k)v}(x) = \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x))\sigma(k)v.$$

On the other hand,

$$\begin{aligned} R(k)\phi(v)(x) = f_s^v(xk) &= \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x)k)v \\ &= \chi_s(a(x))\delta_N(a(x))^{1/2}\sigma(k(x))\sigma(k)v, \end{aligned}$$

which proves the identity. □

Proposition 6.4. *Assume σ is a genuine pseudospherical representation of M. Then $\mu_\sigma|_{K_\alpha} = m\sigma_{1/2} \oplus m'\sigma_{-1/2}$ for some integers m, m' when α is metaplectic, and $\mu_\sigma|_{K_\alpha} = m \cdot 1$ for some integer m when α is not metaplectic. Here $K_\alpha = \Phi_\alpha(\widetilde{\mathrm{SO}}(2))$.*

Proof. For each α, K_α is generated by $\exp(tZ_\alpha)$, $t \in \mathbb{R}$. By the definition at the beginning of Section 3, the eigenvalues of $\mu(\exp(tZ_\alpha))$ are $e^{\pm it/2}$ with multiplicities for α metaplectic, and 1 otherwise. On the other hand, for each $n \in \mathbb{Z}$, $\sigma_{n/2} : K_\alpha \to S^1$, $\exp(tZ_\alpha) \mapsto e^{-int/2}$ is a character of K_α, and $K_\alpha \cong S^1$ has no other characters. Thus $\mu_\sigma|_{K_\alpha}$ is a direct sum of $\sigma_{\pm 1/2}$ when α is metaplectic and 1 otherwise. □

Let $G_\alpha = \Phi_\alpha(\widetilde{\mathrm{SL}}(2, \mathbb{R})) \subset G$. Then $G_\alpha \cong \widetilde{\mathrm{SL}}(2, \mathbb{R})$ when α is metaplectic, and $G_\alpha \cong \mathrm{SL}(2, \mathbb{R})$ when α is not metaplectic. Let T_α be the image of the metaplectic torus of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$, and let N_α be the image of the unipotent radical of the standard parabolic subgroup of $\widetilde{\mathrm{SL}}(2, \mathbb{R})$. Consider $Q = P \cup Pw_\alpha P$, where $P = NT = NAM$ is a minimal parabolic subgroup of G. Then $U = N \cap w_\alpha N w_\alpha^{-1}$ is the unipotent radical of Q. We have $\delta_N(t) = \delta_U(t)\delta_{N/U}(t)$ for $t \in T$. In particular, taking $t \in T_\alpha$, we get $\delta_U(t) = 1$, hence $\delta_N(t) = \delta_{N/U}(t) = \delta_{N_\alpha}(t)$. Thus $\delta_N(t) = \delta_{N_\alpha}(t)$ for $t \in T_\alpha$.

Now we get to the main result of this paper; a similar result on double covers of p-adic groups can be found in [Loke and Savin 2010].

Theorem 6.5 (action of intertwining operators on pseudospherical K-types). *Let $M(w, s) : I(P, \sigma, s) \to I(P, \sigma, ws)$ be the intertwining map. Then $M(w, s) f_s^v = c(w, s) f_{ws}^v$ for some constant $c(w, s)$. Moreover, let $\Delta = \{\alpha_1, \ldots, \alpha_l\}$ be the set of simple roots, and let $w_i = w_{\alpha_i}$. Then in the case when Φ is simply laced or of type G_2,*

$$c(w_i, s) = c_{1/2}(s_i) \quad \textit{for all } i.$$

Otherwise,

$$\begin{aligned} c(w_i, s) &= c_0(s_i) && \textit{when } \alpha_i \textit{ is short}, \\ c(w_i, s) &= c_{1/2}(s_i) && \textit{when } \alpha_i \textit{ is long}. \end{aligned}$$

Here, for $\nu \in \mathbb{C}$,

$$c_0(\nu) := \sqrt{\pi}\,\frac{\Gamma\left(\frac{\nu}{2}\right)}{\Gamma\left(\frac{\nu+1}{2}\right)}, \quad c_{1/2}(\nu) := \sqrt{\pi}\,\frac{\Gamma\left(\frac{\nu}{2}\right)\Gamma\left(\frac{\nu+1}{2}\right)}{\Gamma\left(\frac{\nu}{2}+\frac{3}{4}\right)\Gamma\left(\frac{\nu}{2}+\frac{1}{4}\right)}.$$

Proof. The idea is reduction to the SL_2 case.

The multiplicities of (μ_σ, V) in $I(P, \sigma, s)$ and $I(P, \sigma, ws)$ are both 1, hence $M(w, s) f_s^v = c(w, s) f_{ws}^v$ for some constant $c(w, s)$. Evaluating at $g = 1$ on both sides, we get $M(w, s) f_s^v(1) = c(w, s)v$. For $w = w_i$, there is a map from $I(P, \sigma, s)$ to $I(\sigma, s_i)$ given by restricting functions on G to G_{α_i}, where $I(\sigma, s_i)$ is the space of functions $f : G_{\alpha_i} \to V$ such that $f(namx) = a^{s_i+1}\sigma(m) f(x)$ (here a stands for $h_{\alpha_i}(a)$). Since $N_{w_i} = N \cap w_i N w_i^{-1} \setminus N = N_{\alpha_i}$, $M(w_i, s)$ induces a map from $I(\sigma, s_i)$ to $I(\sigma, -s_i)$, and $f_s^v|_{G_{\alpha_i}}$ satisfies $f_s^v(nak) = a^{s_i+1}\mu_\sigma(k)v$ for $n \in N_{\alpha_i}$, $a \in A_{\alpha_i}$, $k \in K_{\alpha_i}$.

By Proposition 2.12, when Φ is simply laced or of type G_2, all roots are metaplectic By Proposition 6.4,

$$\mu_\sigma|_{K_{\alpha_i}} = m\sigma_{1/2} \oplus m'\sigma_{-1/2}$$

for some positive integers m, m'. Applying Proposition 5.7, we see that $c(w_i, s) = c_{1/2}(s_i)$.

Now assume Φ is of type B_n, C_n, or F_4. If α_i is long, then it is metaplectic, by the same argument as the paragraph above, and we have $c(w_i, s) = c_{1/2}(s_i)$; if α_i is short, then it is not metaplectic by Proposition 2.12. Hence by Proposition 6.4,

$$\mu_\sigma|_{K_{\alpha_i}} = m \cdot 1$$

for some positive integer m. Applying Proposition 5.7 again, $c(w_i, s) = c_0(s_i)$. □

Remark. We may write any $w \in W$ as a reduced product of simple reflections: $w = w_1 w_2 \cdots w_n$. Then by Proposition 4.3,

$$M(w, s) = M(w_1, w_2 \cdots w_n s) M(w_2, w_3 \cdots w_n s) \cdots M(w_{n-1}, w_n s) M(w_n, s),$$

which implies

$$c(w, s) = c(w_1, w_2 \cdots w_n s) c(w_2, w_3 \cdots w_n s) \cdots c(w_{n-1}, w_n s) c(w_n, s).$$

Define

$$\overline{M}(w, s) = \frac{M(w, s)}{c(w, s)}.$$

Then

$$\overline{M}(ww', s) = \overline{M}(w, w's) \circ \overline{M}(w', s)$$

for any $w, w' \in W$. These are called *normalized intertwining operators* and their composition law behaves like the Weyl group.

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LOCAL SYMMETRIC SQUARE L-FACTORS OF REPRESENTATIONS OF GENERAL LINEAR GROUPS

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This paper develops a theory of local symmetric square L-factors of representations of general linear groups. We will prove a certain characterization of a pole of symmetric square L-factors of square-integrable representations, the uniqueness of certain trilinear forms and the nonexistence of Whittaker models of higher exceptional representations.

Introduction

The purpose of this paper is to elaborate on the Rankin–Selberg construction of the twisted symmetric square L-functions of general linear groups, developed in [Bump and Ginzburg 1992; Takeda 2014]. We will mainly focus on the local aspects here.

Fix an integer $n \geq 2$. The setup involves an exceptional representation θ of an appropriate double cover $\overline{G}$ of a general linear group $G = G_n = \mathrm{GL}_n(F)$ over a nonarchimedean local field F of characteristic zero. This rather mysterious representation, which is the smallest genuine representation of this covering group in many senses, was first constructed in generality by Kazhdan and Patterson [1984].

We can associate to each representation φ of the Weil–Deligne group WD_F of F the local L-factor $L(s, \varphi)$ of Artin type. Let sym^2 and Λ^2 be the symmetric and exterior square representations of $\mathrm{GL}_n(\mathbb{C})$. Given an irreducible admissible representation π of G, we can define its local symmetric and exterior square L-factors as $L(s, \mathrm{sym}^2 \circ \phi(\pi))$ and $L(s, \Lambda^2 \circ \phi(\pi))$, where ϕ stands for the local Langlands correspondence between irreducible admissible representations of G and n-dimensional representations of WD_F.

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The factorization

$$L(s,\phi(\pi)\otimes\phi(\pi)) = L(s,\Lambda^2\circ\phi(\pi))L(s,\mathrm{sym}^2\circ\phi(\pi)) \tag{0-1}$$

is an easy consequence of the Langlands formalism. Assume that π is an irreducible square-integrable self-dual representation of G. Then $L(s,\phi(\pi)\otimes\phi(\pi))$ has a simple pole at $s=0$, and hence exactly one of the symmetric or exterior square L-factors of π has a pole at $s=0$.

It is known that $L(s,\Lambda^2\circ\phi(\pi))$ has a pole at $s=0$ only if n is even. Let ψ be a nontrivial additive character of F. When n is even, the L-factor $L(s,\Lambda^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if π admits a nonzero linear form λ on π which satisfies

$$\lambda\left(\pi\left(\begin{bmatrix} h & hX \\ 0 & h\end{bmatrix}\right)v\right) = \psi(\mathrm{tr}(X))\lambda(v)$$

for all $v\in\pi$, $h\in G_{n/2}$ and $X\in \mathrm{M}_{n/2}(F)$ (see [Kewat and Raghunathan 2012; Kewat 2011; Lapid and Mao 2017]). A linear form with this property is called a Shalika functional. As is well known, the space of Shalika functionals on any irreducible admissible representation is at most one-dimensional (see [Jacquet and Rallis 1996]).

We will prove analogous results for symmetric square L-factors. We call π distinguished if there is a nonzero G-invariant linear functional on $\pi\otimes\theta\otimes\theta^\vee$. The following theorem, which is a special case of Theorem 3.19, indicates that this notion of distinction is closely connected with the symmetric square L-factor.

Theorem A. *Let π be an irreducible admissible square-integrable representation of G with central character ω_π. Then $L(s,\mathrm{sym}^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if $\omega_\pi^2=1$ and $\pi\otimes\omega_\pi$ is distinguished.*

It should be noted that if n is even, π is distinguished and $\chi^2=1$, then $\omega_\pi^2=1$ and $\pi\otimes\chi$ is distinguished (see Lemma 1.12). Thus in the case of even n the L-factor $L(s,\mathrm{sym}^2\circ\phi(\pi))$ has a pole at $s=0$ if and only if π is distinguished. Notice that $L(s,\mathrm{sym}^2\circ\phi(\pi\otimes\chi)) = L(s,\mathrm{sym}^2\circ\phi(\pi))$.

As with many L-factors, the symmetric square L-factor may currently be defined not only by the local Langlands correspondence, but also via integral representations or through analysis of Fourier coefficients of Eisenstein series. Henniart [2010] has shown that the first and third definitions agree. We will define the symmetric square L-factor of irreducible admissible generic representations via the integral representation (see Definitions 3.10 and 3.12) and show that this approach gives the same L-factor at least for square-integrable representations (see Theorem 3.18).

Now the following corollary can trivially be deduced from Theorem A and the relevant result for $L(s,\Lambda^2\circ\phi(\pi))$, alluded to above.

Corollary A. *Let π be an irreducible square-integrable representation of G with central character ω_π.*

(1) *Assume that n is odd. Then π is distinguished if and only if ω_π is trivial and π is self-dual.*

(2) *Assume that n is even and ω_π is nontrivial. Then π is distinguished if and only if π is self-dual.*

(3) *Assume that n is even and ω_π is trivial. Then π is self-dual if and only if either a nonzero G-invariant linear functional on $\pi \otimes \theta \otimes \theta^\vee$ or a nonzero Shalika functional on π exists. Moreover, if one of the two functionals exists, then the other does not.*

The following theorem is included in Theorem 2.14.

Theorem B. *If π is an irreducible admissible unitary representation of G, then the space of G-invariant linear functionals on $\pi \otimes \theta \otimes \theta^\vee$ is at most one-dimensional.*

The unitarity assumption is expected to be unnecessary. Sun [2012] proved uniqueness of another trilinear form. Our proof of Theorem B is a refinement of the proof of the generic uniqueness in [Kable 2001, Theorem 6.1], combined with the same idea as in the proof of [Matringe 2014, Proposition 2.3]. Though the hypothesis is essential to this method, we can prove a somewhat stronger uniqueness, which is entirely analogous to the well-known theorem of Bernstein [1984] and its twisted analogue [Ok 1997] (cf. Remark 2.15(1) and [Anandavardhanan et al. 2004; Matringe 2014]).

Since $\overline{G}$ has a subgroup N, which is isomorphic to the group of upper unitriangular matrices of G, we can consider Jacquet modules, Whittaker models and derivatives of representations of $\overline{G}$.

Theorem C. *If $n \geq 3$, then the exceptional representations of $\overline{G}$ carry no Whittaker functionals.*

This result has been proved by Kazhdan and Patterson for nonarchimedean local fields of odd residual characteristic (see Theorem I.3.5 of [Kazhdan and Patterson 1984]). When $n = 3$, this is Lemma 6 of [Flicker et al. 1990]. We will give a different proof which covers the dyadic case. Eyal Kaplan indicated another proof, which uses Lemma 6 of [Flicker et al. 1990] together with induction. It is noteworthy that our proof covers the twisted case as well.

Theorem C completes the computation of derivatives of the exceptional representations. For all nonarchimedean local fields of characteristic zero, the first derivative has been computed by Kable [2001], and the second derivative has been considered by Bump and Ginzburg [1992]. Theorem C combined with the periodicity (see Theorem 5.1 of [Kable 2001]) implies that the third and higher derivatives of the exceptional representations are zero.

Our proof of Theorem A uses a local functional equation, which is a direct consequence of the generic uniqueness, and a stronger uniqueness result given in Theorem 2.14(2). The proofs of these uniqueness results rely upon the knowledge of derivatives of the exceptional representations. The local functional equation and uniqueness principle have not been previously discussed in the dyadic case because of a gap in this knowledge for the exceptional representations over dyadic fields. One of the contributions of this paper is to remove this restriction.

Takeda [2014] has recently constructed twisted exceptional representations and generalized the Rankin–Selberg integral to represent the twisted symmetric square L-functions. In the case of even n the results described so far except for Corollary A will be proved for twisted symmetric square L-factors and twisted exceptional representations (cf. Remark 3.20). When n is odd, we will discuss the symmetric square L-factors without twisting. In order to deal with the twisted case, we only have to analyze the representation of $\overline{G}$ induced from a twisted exceptional representation of $\overline{G}_{n-1}$. Though this analysis is not very difficult, if somewhat involved, we think that our formulation keeps our exposition a reasonable length and sufficient for future applications (cf. Theorem 3.19).

1. Exceptional representations

In this section we aim to review those properties of the exceptional representations that will be required below. Since the proper home for the exceptional representation is not really $\mathrm{GL}_r(F)$, but rather its covering group, we begin this section by recalling some relevant facts from the theory of the covering groups.

1A. ***Notation.*** The notation introduced here will be used constantly in later sections. Throughout, F will be a local field of characteristic 0. We write $|x|$ for the normalized absolute value of an element x of F. There is a quadratic Hilbert symbol $(\ ,\ )$ on $F^\times \times F^\times$ which takes values in $\mu_2 = \{\pm 1\}$. This symbol is symmetric and bimultiplicative, and its left kernel is the subgroup $F^{\times 2}$ of squares in $F^\times$. In the nonarchimedean case the symbols $\mathfrak{o}$ and q will denote, respectively, the ring of integers of F and the cardinality of the residue field of F.

By a character of a locally compact group H we mean any continuous homomorphism of H into $\mathbb{C}^\times$.

Definition 1.1. A character χ of $F^\times$ is said to be unitary (resp. quadratic, even, odd) if $\chi(a)$ is a complex number of modulus 1 for every $a \in F^\times$ (resp. $\chi^2 = 1$, $\chi(-1) = 1$, $\chi(-1) = -1$). When $a \in F^\times$, we define a quadratic character χ_a of $F^\times$ by $\chi_a(b) = (a, b)$ for $b \in F^\times$.

For each positive integer r, we denote by $G_r = \mathrm{GL}_r(F)$ the group of invertible matrices of size r, by T_r its subgroup of diagonal matrices, by B_r its subgroup of upper triangular matrices, by N_r its subgroup of upper triangular matrices with unit diagonal, by Z_r its subgroup of scalar matrices, by $\mathscr{P}_r$ its subgroup consisting of matrices whose last row is $(0, 0, \ldots, 0, 1) \in F^r$ and by $\mathscr{Y}_r$ the unipotent radical of $\mathscr{P}_r$. Put $\mathcal{P}_r = Z_r \cdot \mathscr{P}_r$. We denote the group of permutation matrices in G_r by W_r and identify it with the Weyl group of G_r. For a representation π of G_r we will denote its central character, if it exists, by ω_π unless otherwise mentioned.

We fix a maximal compact subgroup K_r of G_r. Let $K_r = \mathrm{GL}_r(\mathfrak{o})$ in the p-adic case. When $m < r$, we shall systematically regard G_m as a subgroup of G_r via the embedding into the upper left corner. We here allow the specific case $m = 0$ so that G_0 is the identity group. For a parabolic subgroup P of G_r we denote by δ_P the modulus function of P and extend it to the right K_r-invariant function on G_r.

By a standard parabolic subgroup of G_r we shall mean a parabolic subgroup of G_r which contains B_r. A composition of r is an ordered partition of r. To such a composition $\boldsymbol{r} = (r_1, \ldots, r_k)$ of r, we associate the standard parabolic subgroup $P_{\boldsymbol{r}} = M_{\boldsymbol{r}} U_{\boldsymbol{r}}$ of G_r, where $U_{\boldsymbol{r}}$ is the unipotent radical of $P_{\boldsymbol{r}}$ and the group $M_{\boldsymbol{r}} = G_{r_1} \times \cdots \times G_{r_k}$, regarded as embedded in the natural way as a block-diagonal subgroup of G_r, is a Levi subgroup of $P_{\boldsymbol{r}}$.

We define the subgroup $G_r^{\square}$ of G_r by

$$G_r^{\square} = \{g \in G_r \mid \det g \in F^{\times 2}\}.$$

Further we define the subgroup $M_{\boldsymbol{r}}^{\square}$ of $M_{\boldsymbol{r}}$ by

$$M_{\boldsymbol{r}}^{\square} = \left\{\mathrm{diag}[m_1, \ldots, m_k] \in M_{\boldsymbol{r}} \mid m_i \in G_{r_i}^{\square} \text{ for } i = 1, 2, \ldots, k\right\}.$$

Put

$$\mathscr{Z}_r = \{z^{e(r)} \mid z \in Z_r\},$$

where $e(r)$ is 1 or 2 according to whether r is odd or even. Set

$$\mathscr{T}_r = \left\{t \in T_r \mid t_{r-2i+1} t_{r-2i+2}^{-1} \in F^{\times 2} \text{ for } i = 1, 2, \ldots, \left[\tfrac{r}{2}\right]\right\},$$

writing a diagonal matrix $t \in T_r$ in the form $\mathrm{diag}[t_1, t_2, \ldots, t_r]$. We define the two compositions of r by

$$\boldsymbol{e}(r) = (2, 2, 2, \ldots, 2, 2), \quad \boldsymbol{o}(r) = (1, 2, 2, \ldots, 2, 1)$$

if r is even, and by

$$\boldsymbol{e}(r) = (1, 2, 2, \ldots, 2, 2), \quad \boldsymbol{o}(r) = (2, 2, 2, \ldots, 2, 1)$$

if r is odd. Lastly, we define the subgroup $\mathscr{M}_r$ of $M_{\boldsymbol{e}(r)}$ by $\mathscr{M}_r = Z_r \cdot M_{\boldsymbol{e}(r)}^{\square}$.

1B. ***The double covers of general linear groups.*** A central double covering $p_r : \bar{G}_r \to G_r$ corresponds in the usual way to a class in the cohomology group $H^2(G_r, \mu_2)$, where G_r acts trivially on the coefficients μ_2, and choosing a cocycle to represent this class is equivalent to choosing a section $\boldsymbol{s}_r : G_r \to \bar{G}_r$ of the map p_r. We shall choose $\boldsymbol{s}_r$ in such a way that the resulting cocycle σ_r agrees with the one constructed by Banks, Levy and Sepanski in [Banks et al. 1999, Section 3]. Let μ_2 inject into the center of $\bar{G}_r$. Then we can write typical elements of $\bar{G}_r$ uniquely in the form $\zeta \boldsymbol{s}_r(g)$ for $g \in G_r$ and $\zeta \in \mu_2$. The composition rule is given by

$$\zeta \boldsymbol{s}_r(g) \cdot \zeta' \boldsymbol{s}_r(g') = \zeta\zeta' \sigma_r(g, g') \boldsymbol{s}_r(gg') \quad (g, g' \in G_r,\ \zeta, \zeta' \in \mu_2).$$

The 2-cocycles $\{\sigma_r\}_{r=1}^{\infty}$ are well behaved with respect to restriction and satisfy a nice block formula on all standard Levi subgroups, i.e., if $r = r_1 + \cdots + r_k$ and $g_i, g_i' \in G_{r_i}$ for $i = 1, 2, \ldots, k$, then

$$\sigma_r\left[\begin{bmatrix} g_1 & & \\ & \ddots & \\ & & g_k \end{bmatrix}, \begin{bmatrix} g_1' & & \\ & \ddots & \\ & & g_k' \end{bmatrix}\right] = \prod_{i=1}^{k} \sigma_{r_i}(g_i, g_i') \prod_{j<l} (\det g_j, \det g_l').$$

The 2-cocycle σ_1 is trivial and σ_2 is the Kubota 2-cocycle on G_2.

For any subgroup H of G_r we write $\widetilde{H}$ for its preimage $p_r^{-1}(H)$. An irreducible admissible representation of $\widetilde{H}$ is said to be genuine if it does not descend to a representation of H. Since the restriction of σ_r to any copy of G_{r_i} embedded along the diagonal in G_r agrees with the 2-cocycle σ_{r_i}, we can naturally identify $\widetilde{G}_{r_i}$ with $\bar{G}_{r_i}$. The block-compatibility of σ_r guarantees that the map

$$(\zeta_1 \boldsymbol{s}_{r_1}(g_1), \ldots, \zeta_k \boldsymbol{s}_{r_k}(g_k)) \mapsto (\zeta_1 \cdots \zeta_k) \boldsymbol{s}_r(\mathrm{diag}[g_1, \ldots, g_k])$$

is a surjective group homomorphism $\widetilde{G}_{r_1}^{\square} \times \cdots \times \widetilde{G}_{r_k}^{\square} \to \widetilde{M}_{\boldsymbol{r}}^{\square}$, which gives the decomposition

$$\text{(1-1)} \quad \widetilde{M}_{\boldsymbol{r}}^{\square} \simeq \widetilde{G}_{r_1}^{\square} \times \widetilde{G}_{r_2}^{\square} \times \cdots \times \widetilde{G}_{r_k}^{\square} \big/ \big\{(\zeta_1, \zeta_2, \ldots, \zeta_k) \mid \zeta_i \in \mu_2,\ \zeta_1\zeta_2\cdots\zeta_k = 1\big\}.$$

Remark 1.2. (1) The center of $\bar{G}_r$ is $\widetilde{\mathscr{Z}}_r$.

(2) The center of $\widetilde{T}_r$ is $\widetilde{\mathscr{Z}}_r \widetilde{T}_r^{\square}$.

(3) The preimage $\widetilde{\mathscr{T}}_r$ is a maximal abelian subgroup of $\widetilde{T}_r$.

(4) It is known that

$$\sigma_r(ugu', g'u'') = \sigma_r(g, u'g') \quad (g, g' \in G_r,\ u, u', u'' \in N_r).$$

In particular, the restriction of $\boldsymbol{s}_r$ to N_r is a group homomorphism, by which we view subgroups of N_r as those of $\bar{G}_r$. If P is a standard parabolic subgroup of G_r with unipotent radical U, then

$$\tilde{p}\boldsymbol{s}_r(u)\tilde{p}^{-1} = \boldsymbol{s}_r(p_r(\tilde{p})up_r(\tilde{p})^{-1}) \quad (u \in U,\ \tilde{p} \in \widetilde{P}).$$

If F is nonarchimedean, then there are an open subgroup $\mathcal{K}_r$ of K_r and a map $\kappa_r : K_r \to \mu_2$ such that $k \mapsto \kappa_r(k)\boldsymbol{s}_r(k)$ is a group homomorphism from $\mathcal{K}_r$ to $\overline{G}_r$ by Proposition 0.1.2 of [Kazhdan and Patterson 1984]. The topology of $\overline{G}_r$ as a locally compact group is determined by this embedding. If the residual characteristic of F is odd, then we can take $\mathcal{K}_r = K_r$. The splitting $\mathcal{K}_r \to \overline{G}_r$ is not unique. We shall fix what Kazhdan and Patterson refer to as the canonical lift of K_r to $\overline{G}_r$ (see [Kazhdan and Patterson 1984, Proposition 0.I.3]).

1C. ***Lifts of the main involution.*** When φ is an automorphism of G_r, a lift of φ to $\overline{G}_r$ is an automorphism $\tilde{\varphi}$ of $\overline{G}_r$ such that $\tilde{\varphi}(\zeta) = \zeta$ and $p_r(\tilde{\varphi}(\tilde{g})) = \varphi(p_r(\tilde{g}))$ for all $\zeta \in \mu_2$ and $\tilde{g} \in \overline{G}_r$. The lift of any topological automorphism of G_r to $\overline{G}_r$ is a topological automorphism by Corollary 1 of [Kable 1999]. We consider a lift of the automorphism $g \mapsto {}^{\iota}g$ of G_r defined by ${}^{\iota}g = w_0^{(r)}\, {}^{t}g^{-1} w_0^{(r)}$, where ${}^{t}g$ is the transpose of the matrix g and $w_0^{(r)} \in W_r$ is the longest element.

Proposition 1.3 [Kable 1999]. *There exists a lift $\tilde{g} \mapsto {}^{\iota}\tilde{g}$ of the automorphism $g \mapsto {}^{\iota}g$ to $\overline{G}_r$ satisfying*

$$ {}^{\iota}\boldsymbol{s}_r(t) = \boldsymbol{s}_r({}^{\iota}t) \prod_{i>j} (t_i, t_j), \quad {}^{\iota}\tilde{z} = \tilde{z}^{-1}, \quad {}^{\iota}({}^{\iota}\tilde{g}) = \tilde{g}, \quad {}^{\iota}\boldsymbol{s}_r(u) = \boldsymbol{s}_r({}^{\iota}u) $$

for all $t = \operatorname{diag}[t_1, \dots, t_r] \in T_r$, $\tilde{z} \in \widetilde{\mathscr{Z}}_r$, $\tilde{g} \in \overline{G}_r$ and $u \in N_r$. All lifts are of the form $\tilde{g} \mapsto \varrho(\det p_r(\tilde{g}))\, {}^{\iota}\tilde{g}$, where ϱ is an arbitrary quadratic character of $F^{\times}$. Moreover, if the residual characteristic of F is odd and $f : K_r \to \overline{G}_r$ is a homomorphism, then $f({}^{\iota}k) = {}^{\iota}f(k)$ for all $k \in K_r$.

Proof. Kable has determined the lifts of the main involution and proved their basic properties. However, we need to keep track of his computations, using the cocycle defined in [Banks et al. 1999]. To that end, we recall how our cocycle σ_r is constructed. Put $\mathbb{G}_k = \mathrm{SL}_k(F)$ and define the embedding of G_r into $\mathbb{G}_{r+1}$ by $\jmath_r(g) = \operatorname{diag}[g, (\det g)^{-1}]$. There is a double cover $\bar{\mathbb{G}}_k$ of $\mathbb{G}_k$ by a theorem of Matsumoto [1969]. Banks, Levy and Sepanski [Banks et al. 1999] defined an explicit cocycle τ_k that represents the cohomology class of this cover and defined σ_r by

$$ \sigma_r(g, g') = \tau_{r+1}(\jmath_r(g), \jmath_r(g'))(\det g, \det g'). \tag{1-2} $$

The cocycle τ_{r+1} satisfies

$$ \tau_{r+1}(u, u') = (\det t, \det t') \prod_{1 \le i < j \le r} (t_i, t'_j) = \prod_{r \ge i \ge j \ge 1} (t_i, t'_j) = \prod_{r+1 \ge i \ge j \ge 1} (u_i, u'_j) $$

for $t = \operatorname{diag}[t_1, \dots, t_r]$ and $t' = \operatorname{diag}[t'_1, \dots, t'_r]$ by the block-compatibility of σ_r. Here we write

$$ u = \jmath_r(t) = \operatorname{diag}[u_1, \dots, u_{r+1}], $$

and similarly for $u' = \jmath_r(t')$. Kable chooses a cocycle on G_{r+1} which agrees with τ_{r+1} on the torus (see [Kable 1999, (3)]) and defines his cocycle on G_r by the relation [Kable 1999, (4)]. When $m = 0$ and $A = \mu_2$, it is the same as (1-2). Since he does not impose any other condition on his cocycle, we can apply all of his results to our cocycle.

Finally, we prove the last statement. We can define a quadratic character $\varrho_0 : K_r \to \mu_2$ by $\varrho_0(k) = {}^\iota f(k) f({}^\iota k)^{-1}$ for $k \in K_r$. Since $\mathrm{SL}_r(\mathfrak{o})$ is a perfect group, there is a quadratic character $\varrho_1 : \mathfrak{o}^\times \to \mu_2$ such that $\varrho_0(k) = \varrho_1(\det k)$ for all $k \in K_r$. Similarly, there is a quadratic character $\varrho_2 : \mathfrak{o}^\times \to \mu_2$ such that $f(k) = \varrho_2(\det k)\kappa_r(k)\boldsymbol{s}_r(k)$ for all $k \in K_r$. If $k \in K_r \cap T_r$, then $\kappa_r(k) = 1$ by (1.6) of [Takeda 2014], and

$$\varrho_1(\det k) = \varrho_0(k) = {}^\iota(\varrho_2(\det k)\boldsymbol{s}_r(k))(\varrho_2(\det {}^\iota k)\boldsymbol{s}_r({}^\iota k))^{-1} = {}^\iota\boldsymbol{s}_r(k)\boldsymbol{s}_r({}^\iota k)^{-1} = 1.$$

Therefore ϱ_1 must be trivial and hence ϱ_0 is trivial. □

Let π be a representation of $\widetilde{H}$. Taking a preimage $\tilde{g}$ of $g \in G_r$ in $\overline{G}_r$, we define the representation ${}^g\pi$ of ${}^g\widetilde{H} = \tilde{g}\widetilde{H}\tilde{g}^{-1}$ by ${}^g\pi(\tilde{h}) = \pi(\tilde{g}^{-1}\tilde{h}\tilde{g})$ for $\tilde{h} \in {}^g\widetilde{H}$, where conjugation is independent of the choice of $\tilde{g}$. We define a subgroup ${}^\iota\widetilde{H}$ of $\overline{G}_r$ by ${}^\iota\widetilde{H} = \{{}^\iota\tilde{h} \mid \tilde{h} \in \widetilde{H}\}$ and define a representation ${}^\iota\pi$ of ${}^\iota\widetilde{H}$ on the same space by ${}^\iota\pi(\tilde{h}) = \pi({}^\iota\tilde{h})$. If f is a function on $\widetilde{H}$, then we define a function ${}^\iota f$ on ${}^\iota\widetilde{H}$ by ${}^\iota f(\tilde{h}) = f({}^\iota\tilde{h})$ for $\tilde{h} \in {}^\iota\overline{H}$. If H is a subgroup of $M_{\boldsymbol{r}}$ containing $M_{\boldsymbol{r}}^\square$, where $\boldsymbol{r}$ is a composition of r, then H normalizes $U_{\boldsymbol{r}}$ in view of Remark 1.2(4) and we can construct, out of its pull-back to $\widetilde{H}U_{\boldsymbol{r}}$, the induced representation $\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi$. Here the induction is normalized in order that $\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi$ is unitarizable whenever π is unitarizable. Observe that ${}^\iota\delta_{P_{\boldsymbol{r}}} = \delta_{{}^\iota P_{\boldsymbol{r}}}$ and ${}^\iota P_{\boldsymbol{r}} = P_{\overleftarrow{\boldsymbol{r}}}$, where $\overleftarrow{\boldsymbol{r}} = (r_k, r_{k-1}, \dots, r_1)$. Note that $f \mapsto {}^\iota f$ gives a $\overline{G}_r$-equivariant isomorphism

$$(1\text{-}3) \qquad {}^\iota\big(\mathrm{Ind}_{\widetilde{H}U_{\boldsymbol{r}}}^{\overline{G}_r}\pi\big) \simeq \mathrm{Ind}_{{}^\iota\widetilde{H}\,{}^\iota U_{\boldsymbol{r}}}^{\overline{G}_r}{}^\iota\pi.$$

1D. ***The Weil representations of $\widetilde{G}_2^\square$.*** The Weil representation of $\overline{G}_2$ can be identified as the original example of the exceptional representation. Fix a nontrivial additive character ψ of F. Put $\mu_\psi(a) = \gamma(\psi_a)/\gamma(\psi)$ for $a \in F^\times$, where $\psi_a(x) = \psi(ax)$ and $\gamma(\psi)$ is the Weil constant associated to ψ. Recall that

$$\mu_\psi(ab) = \mu_\psi(a)\mu_\psi(b)(a,b), \quad \mu_\psi(ab^2) = \mu_\psi(a)$$

for $a, b \in F^\times$.

We will denote the space of Schwartz functions in k variables by $\mathscr{S}(F^k)$. For $x \in F^k$ we define the $\mathbb{C}$-linear functional e_x on $\mathscr{S}(F^k)$ by $e_x(\Phi) = \Phi(x)$. The Weil representation Ω^ψ associated to ψ is a genuine representation of the metaplectic double cover of $\mathrm{SL}_2(F)$ realized on the space $\mathscr{S}(F)$. The explicit action of the

Borel subgroup of $\mathrm{SL}_2(F)$ is given by

$$\Omega^\psi\left[\boldsymbol{s}_2\left[\begin{pmatrix} a & 0 \\ 0 & a^{-1}\end{pmatrix}\right]\right]\Phi(x)=\mu_\psi(a)|a|^{1/2}\Phi(xa), \tag{1-4}$$

$$\Omega^\psi\left[\boldsymbol{s}_2\left[\begin{pmatrix} 1 & b \\ 0 & 1\end{pmatrix}\right]\right]\Phi(x)=\psi(bx^2)\Phi(x) \tag{1-5}$$

for $\Phi\in\mathscr{S}(F)$, $a\in F^\times$ and $b,x\in F$. It is well known that Ω^ψ is reducible and written as the direct sum $\Omega^\psi=\Omega^\psi_1\oplus\Omega^\psi_{-1}$, where Ω^ψ_1 (resp. Ω^ψ_{-1}) is an irreducible representation realized in the space of even (resp. odd) Schwartz functions in one variable. For a character ϱ of $F^\times$ one can extend $\Omega^\psi_{\varrho(-1)}$ to an irreducible representation Ω^ψ_ϱ of $\widetilde{G}^\square_2$ by setting

$$\Omega^\psi_\varrho(\boldsymbol{s}_2(a\mathbf{1}_2))=\varrho(a)\mu_\psi(a) \tag{1-6}$$

for $a\in F^\times$. When ϱ is trivial, we will sometimes write $\Omega^\psi_+=\Omega^\psi_\varrho$. For $a\in F^\times$ we put $d(a)=\mathrm{diag}[a,1]\in G_2$.

Proposition 1.4. *Let ϱ be a character of $F^\times$.*

(1) *If $a\in F^\times$, then ${}^{d(a^{-1})}\Omega^\psi_\varrho\simeq\Omega^{\psi_a}_\varrho$.*

(2) *The representation $\mathrm{Ind}^{\bar{G}_2}_{\widetilde{G}^\square_2}\Omega^\psi_\varrho$ is irreducible and its equivalence class is independent of ψ.*

(3) *If $\varPhi\in\mathrm{Ind}^{\bar{G}_2}_{\widetilde{G}^\square_2}\Omega^\psi_\varrho$ and $e_1(\varPhi(\tilde{p}))=0$ for all $\tilde{p}\in\widetilde{\mathscr{P}}_2$, then $\varPhi=0$.*

Proof. We will prove only the last part, for the other results are recalled or derived in Section 2.2 of [Takeda 2014]. By (1-4), (1-6) and the assumption on $\varPhi$,

$$\begin{aligned}0&=e_1\big(\varPhi(\boldsymbol{s}_2(d(a^2))\tilde{p})\big)\\&=e_1\big(\Omega^\psi_\varrho(\boldsymbol{s}_2(d(a^2)))\varPhi(\tilde{p})\big)\\&=(a,-1)\varrho(a)\mu_\psi(a)^2|a|^{1/2}e_a(\varPhi(\tilde{p}))\end{aligned}$$

for all $a\in F^\times$ and $\tilde{p}\in\widetilde{\mathscr{P}}_2$. Therefore $e_a(\varPhi(\tilde{p}))=0$ for all $a\in F^\times$, and so in view of continuity, $e_a(\varPhi(\tilde{p}))=0$ for all $a\in F$. Bear in mind that $\varPhi$ is a $\mathscr{S}(F)$-valued function on $\bar{G}_2$. We conclude that $\varPhi(\tilde{p})=0$ for all $\tilde{p}\in\widetilde{\mathscr{P}}_2$. Since $G_2=G^\square_2\cdot\mathscr{P}_2$, we conclude that $\varPhi=0$. $\square$

1E. ***Exceptional representations.*** We can define a genuine character ξ^ψ_r of $\widetilde{\mathscr{T}}_r$ by

$$\xi^\psi_r(\boldsymbol{s}_r(t))=\prod_{i=0}^{[r/2]-1}\mu_\psi(t_{r-2i})^{-1}.$$

The exceptional representation θ_r^ψ is the unique irreducible subrepresentation of

$$\mathscr{I}_r^\psi=\mathrm{Ind}_{\widetilde{\mathscr{T}}_r N_r}^{\bar{G}_r}\xi_r^\psi\otimes\delta_{B_r}^{-1/4}$$

(see Theorem I.2.9 of [Kazhdan and Patterson 1984]). Next we recall Takeda's construction [2014] of the twisted exceptional representations.

Definition 1.5. Fix a positive integer r and a character χ of $F^\times$. In light of (1-1) and Remark 1.2(1) we can define the genuine representation $\Upsilon_{r,\chi}^\psi$ of $\widetilde{\mathscr{M}}_r$ to be the tensor product

$$\Upsilon_{r,\chi}^\psi=(\xi_r^\psi|_{\tilde{Z}_r})\boxtimes\Omega_+^{\psi^{-1}}\boxtimes\cdots\boxtimes\Omega_+^{\psi^{-1}}\quad\text{or}\quad\Upsilon_{r,\chi}^\psi=\Omega_\chi^{\psi^{-1}}\boxtimes\cdots\boxtimes\Omega_\chi^{\psi^{-1}}$$

according to whether r is odd or even. Put

$$I_{r,\chi}^\psi=\mathrm{Ind}_{\widetilde{\mathscr{M}}_r U_{e(r)}}^{\bar{G}_r}\Upsilon_{r,\chi}^\psi\otimes\delta_{P_{e(r)}}^{-1/4}.$$

By the Langlands theorem [Ban and Jantzen 2013] the representation $I_{r,\chi}^\psi$ has a unique irreducible subrepresentation, which we denote by $\theta_{r,\chi}^\psi$. Exceptional representations of $\bar{G}_r$ are twists of these representations $\theta_{r,\chi}^\psi$ by characters of $F^\times$.

Remark 1.6. Proposition 1.4(2) implies that the equivalence class of $\theta_{r,\chi}^\psi$ is independent of ψ whenever r is even. We will sometimes suppress the superscript ψ and write $\theta_{r,\chi}=\theta_{r,\chi}^\psi$ when r is even.

Remark 1.7. Whenever r is odd, the representation $\theta_{r,\chi}^\psi$ is defined independently of χ contrary to what one might guess from the notation. If χ is trivial, then by (1-4), (1-5), (1-6) and the invariant distribution theorem, the map $\Phi\mapsto e_0\circ\Phi$ gives a $\bar{G}_r$-intertwining embedding $I_{r,\chi}^\psi\hookrightarrow\mathscr{I}_r^\psi$ and hence $\theta_r^\psi\simeq\theta_{r,\chi}^\psi$. We may therefore omit the subscript χ from the notation either if r is odd or if χ is trivial. In view of Remark 1.6 we may write θ_r when r is even and χ is trivial. We trust this will cause no confusion.

A little more generally, we assume that χ is even. Then we can define a character ϱ of $F^{\times 2}$ by $\varrho(a^2)=\chi(a)$ for $a\in F^\times$. We extend ϱ to a character of $F^\times$ and denote it also by ϱ. If r is even, then since the map $\Phi\mapsto e_0\circ\Phi$ gives a $\bar{G}_r$-intertwining embedding $I_{r,\chi}^\psi\hookrightarrow\mathscr{I}_r^\psi\otimes\varrho$, we conclude $\theta_r\simeq\theta_{r,\chi}\otimes\varrho^{-1}$.

The notion of principal series representations of $\bar{G}_r$ is introduced in Section 1.1 of [Kazhdan and Patterson 1984]. The following result is an easy consequence of an analogue of the Stone–von Neumann theorem, which states that the genuine irreducible representations of the two-step nilpotent group $\widetilde{T}_r$ are parametrized by the genuine characters of its center $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^\square$ (cf. [Kazhdan and Patterson 1984; Bump and Ginzburg 1992, Proposition 1.1]).

Lemma 1.8. *Let $\widetilde{\mathcal{T}}_1$ and $\widetilde{\mathcal{T}}_2$ be maximal abelian subgroups of $\widetilde{T}_r$. Let ξ_i be a genuine character of $\widetilde{\mathcal{T}}_i$. If the restrictions of ξ_1 and ξ_2 to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ coincide, then* $\operatorname{Ind}_{\widetilde{\mathcal{T}}_1 N_r}^{\bar{G}_r}\xi_1 \simeq \operatorname{Ind}_{\widetilde{\mathcal{T}}_2 N_r}^{\bar{G}_r}\xi_2$.

1F. *Distinction by pairs of exceptional representations.*

Lemma 1.9. *Let χ and μ be characters of $F^\times$.*

(1) $(\theta_r^{\psi})^\vee \simeq \theta_r^{\psi^{-1}}$.

(2) *If r is odd and $a \in F^\times$, then* $\theta_r^{\psi_a} \simeq \theta_r^{\psi} \otimes \chi_a^{(r-1)/2}$.

(3) *If r is even, then* $\theta_{r,\chi}\otimes\mu \simeq \theta_{r,\chi\mu^2}$.

(4) ${}^{\iota}\theta_{r,\chi}^{\psi} \simeq \theta_{r,\chi^{-1}}^{\psi^{-1}}$.

Proof. We have $(\xi_r^{\psi})^{-1} = \xi_r^{\psi^{-1}}$ simply because $\mu_{\psi^{-1}} = \mu_{\psi}^{-1}$. Assertion (1) therefore follows from Theorem 5.1(5) of [Kable 2001].

Note that $\mu_{\psi_a} = \chi_a \cdot \mu_{\psi}$. The restrictions of $\xi_r^{\psi_a}$ and $\xi_r^{\psi}\cdot(\chi_a\circ\det)^{(r-1)/2}$ to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ agree when r is odd. Assertion (2) follows from Lemma 1.8.

Since $\Omega_\chi^{\psi}\otimes\mu \simeq \Omega_{\chi\mu^2}^{\psi}$ by definition, assertion (3) readily follows.

Finally, we will prove (4). First assume that χ is trivial. Since the restrictions of ${}^{\iota}\xi_r^{\psi}$ and $\xi_r^{\psi^{-1}}$ to $\widetilde{\mathscr{Z}}_r\widetilde{T}_r^{\square}$ coincide, we see that

$$ {}^{\iota}\mathscr{I}_r^{\psi} \simeq \operatorname{Ind}_{{}^{\iota}\widetilde{\mathscr{T}}_r N_r}^{\bar{G}_r} {}^{\iota}\xi_r^{\psi}\otimes\delta_{B_r}^{-1/4} \simeq \mathscr{I}_r^{\psi^{-1}} $$

by (1-3) and Lemma 1.8, from which assertion (4) follows.

Next assume that r is even. Since

$$ {}^{\iota}g = \begin{pmatrix}1&0\\0&-1\end{pmatrix} g \begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1} $$

for $g \in \mathrm{SL}_2(F)$, Proposition 1.3 shows that

$$ {}^{\iota}\tilde{g} = \begin{pmatrix}1&0\\0&-1\end{pmatrix} \tilde{g} \begin{pmatrix}1&0\\0&-1\end{pmatrix}^{-1} $$

for all elements $\tilde{g} \in \bar{G}_2$ such that $\det p_2(\tilde{g}) = 1$. Proposition 1.4(1) tells us that ${}^{\iota}\Omega^{\psi^{-1}} \simeq \Omega^{\psi}$ and so ${}^{\iota}\Omega_\chi^{\psi^{-1}} \simeq \Omega_{\chi^{-1}}^{\psi}$. If $g = \operatorname{diag}[g_1,\dots,g_{r/2}] \in M_{\boldsymbol{e}(r)}^{\square}$, then ${}^{\iota}g = w_r^{-1}\operatorname{diag}[{}^{\iota}g_1,\dots,{}^{\iota}g_{r/2}]w_r$, where the matrix w_r is defined in (2-1), and hence

$$ {}^{\iota}\Upsilon_{r,\chi}^{\psi} \simeq {}^{w_r}\Upsilon_{r,\chi^{-1}}^{\psi^{-1}} \simeq \Upsilon_{r,\chi^{-1}}^{\psi^{-1}} $$

(cf. Proposition 2.9 of [Takeda 2015]). □

We define the notion of distinction in our current setup. No subgroup of G_r appears, but the exceptional representations play the role of “restriction to the subgroup”.

Definition 1.10. We assume χ to be trivial whenever r is odd. Let π be an admissible representation of G_r. We say that π is χ-distinguished if there is a nonzero G_r-invariant linear form on $\pi \otimes \theta^{\psi}_{r,\chi} \otimes \theta^{\psi^{-1}}_r$. We say that π is distinguished if there is a nonzero G_r-invariant linear form on $\pi \otimes \theta^{\psi}_r \otimes \theta^{\psi^{-1}}_r$.

Remark 1.11. This notion of distinction is independent of the choice of ψ on account of Lemma 1.9(1)–(2) and Remark 1.6.

Lemma 1.12. *Let π be an irreducible admissible representation of G_r. Let χ be a character of $F^\times$.*

(1) *If r is odd and π is distinguished, then the central character ω_π of π is trivial and $\pi^\vee$ is distinguished.*

(2) *If r is even and π is χ-distinguished, then $\omega_\pi^2\chi^r$ is trivial, $\pi^\vee$ is χ^{-1}-distinguished and $\pi \otimes \mu$ is $\chi\mu^{-2}$-distinguished for all characters μ of $F^\times$.*

Remark 1.13. By Theorem 3.19, if π is square-integrable and χ-distinguished, then $\pi \simeq \pi^\vee \otimes \chi^{-1}$. It is expected that all irreducible admissible χ-distinguished representations π satisfy $\pi \simeq \pi^\vee \otimes \chi^{-1}$ (cf. [Flicker 1991, Proposition 12; Jacquet and Rallis 1996, Theorem 1.1, Proposition 6.1]).

Proof. For π to be χ-distinguished, the product of the three central characters must be trivial on $F^{\times e(r)}$ as $\widetilde{\mathscr{Z}}_r$ is the center of $\overline{G}_r$. This gives the stated conditions on ω_π (see Lemma 1.9(1) and (1-6)). We can easily deduce the remaining parts from the relevant properties of exceptional representations stated in Lemma 1.9(3)–(4). $\square$

1G. *The intertwining operator.* We will fix, once and for all, a positive integer $n \geq 2$ and write $G = G_n$ and $G' = G_{n-1}$. Put $\ell = \left[\frac{n}{2}\right]$. We embed G' into G via the map $h \mapsto \begin{pmatrix} h & \\ & 1\end{pmatrix}$. We omit the subscript n and adapt the same notation adding a prime $'$ for G'; that is,

$$\mathscr{P} = \mathscr{P}_n, \qquad \mathscr{T} = \mathscr{T}_n, \qquad \mathscr{Z} = \mathscr{Z}_n, \qquad N = N_n, \qquad B' = B_{n-1},$$
$$T' = T_{n-1}, \qquad \mathscr{Z}' = \mathscr{Z}_{n-1}, \qquad \xi^\psi = \xi^\psi_n, \qquad \theta^\psi_\chi = \theta^\psi_{n,\chi}, \qquad \theta^\psi = \theta^\psi_n,$$

and so on.

For each character ϱ of $F^\times$ we define a genuine character ζ^ψ_ϱ of $\widetilde{\mathscr{Z}}$ by

$$\zeta^\psi_\varrho(\boldsymbol{s}(z\mathbf{1}_n)) = \varrho(z)^{-1}\mu_\psi(z)^\ell$$

for $z \in F^{\times e(n)}$. Then we can extend θ^ψ_{n-1} to the representation $\theta^\psi_{n-1} \boxtimes \zeta^\psi_\varrho$ of the semidirect product $(\overline{G}' \times \widetilde{\mathscr{Z}}) \ltimes \mathscr{Y}$ by letting $\widetilde{\mathscr{Z}}$ act by ζ^ψ_ϱ and letting $\mathscr{Y}$ act trivially. For $s \in \mathbb{C}$ we consider the induced representation

$$I_\psi(s, \varrho) = \operatorname{Ind}^{\overline{G}}_{\widetilde{\mathscr{Z}}\widetilde{\mathscr{P}}}(\theta^\psi_{n-1} \boxtimes \zeta^\psi_\varrho) \otimes \delta^{s/4}_{\mathcal{P}}.$$

We define the intertwining operator

$$M(s,\varrho) : I_\psi(s,\varrho) \to J_\psi(-s,\varrho)$$

for $\Re s \gg 0$ by the integrals

$$M(s,\varrho) f^{(s)}(\tilde{g}) = \int_{\iota\mathscr{Y}} f^{(s)}(\boldsymbol{s}(\delta)^{-1} y\tilde{g})\, \mathrm{d}y$$

and by meromorphic continuation otherwise, where

$$J_\psi(s,\varrho) = \operatorname{Ind}_{\widetilde{\mathscr{Z}}^{\iota}\widetilde{\mathscr{P}}}^{\bar{G}}({}^{\delta}\theta_{n-1}^{\psi} \boxtimes \zeta_\varrho^{\psi}) \otimes \delta_{\iota\mathcal{P}}^{s/4}, \quad \delta = \begin{pmatrix} & 1 \\ \mathbf{1}_{n-1} & \end{pmatrix}. \tag{1-7}$$

The operator $M(s,\varrho)$ is holomorphic at $s=1$ due to the analysis in Sections 4.5 and 4.6 of [Takeda 2015] (see Lemma 3.2).

Lemma 1.14. *If $\widetilde{\mathcal{T}}'$ is a maximal abelian subgroup of $\widetilde{T}'$, then $\widetilde{\mathcal{T}}'\widetilde{\mathscr{Z}}$ is a maximal abelian subgroup of $\widetilde{T}$.*

Proof. Suppose that $\tilde{t} \in \widetilde{T}$ commutes with all elements in $\widetilde{\mathscr{Z}}\widetilde{\mathcal{T}}'$. We can write $\tilde{t} = \boldsymbol{s}(z\mathbf{1}_n)\cdot\tilde{t}'$ ($z \in F^\times$, $\tilde{t}' \in \widetilde{T}'$). If n is odd, then $\widetilde{\mathscr{Z}} = \widetilde{Z}$ and hence $\tilde{t}'$ commutes with all elements in $\widetilde{\mathcal{T}}'$, so that $\tilde{t}' \in \widetilde{\mathcal{T}}'$. If n is even, then since $\widetilde{\mathcal{T}}'$ contains $\widetilde{\mathscr{Z}}' = \widetilde{Z}'$, we have

$$(z,z')^{(n-2)/2} = \sigma\left[z\mathbf{1}_n, \begin{bmatrix} z'\mathbf{1}_{n-1} & \\ & 1\end{bmatrix}\right] = \sigma\left[\begin{bmatrix} z'\mathbf{1}_{n-1} & \\ & 1\end{bmatrix}, z\mathbf{1}_n\right] = (z,z')^{n/2}$$

for all $z' \in F^\times$, so that z must be a square, and hence $\tilde{t}' \in \widetilde{\mathcal{T}}'$. □

Lemma 1.15. *Let ϱ be a quadratic character of $F^\times$. The representation $I_\psi(1,\varrho)$ has a unique irreducible quotient, which is isomorphic to $\theta^{\psi^{-1}} \otimes \varrho$. Moreover, the quotient map*

$$I_\psi(1,\varrho) \to \theta^{\psi^{-1}} \otimes \varrho$$

is realized as the intertwining operator $M(1,\varrho)$.

Proof. Let W and W' denote the Weyl groups of G and G', respectively. Let $w_0 \in W$ and $w_0' \in W'$ be the longest elements. Since θ_{n-1}^{ψ} is a quotient of the principal series representation

$$\operatorname{Ind}_{w_0'\widetilde{\mathcal{T}}'N'}^{\bar{G}'} {}^{w_0'}\xi_{n-1}^{\psi} \otimes \delta_{B'}^{1/4}$$

by Theorem I.2.9 of [Kazhdan and Patterson 1984], the representation $I_\psi(1,\varrho)$ is a quotient of

$$\operatorname{Ind}_{w_0'\widetilde{\mathcal{T}}'\widetilde{\mathscr{Z}}N}^{\bar{G}}({}^{w_0'}\xi_{n-1}^{\psi} \boxtimes \zeta_\varrho^{\psi}) \otimes \delta_B^{1/4} \simeq \big(\operatorname{Ind}_{w_0\widetilde{\mathcal{T}}N}^{\bar{G}} {}^{w_0}\xi^{\psi^{-1}} \otimes \delta_B^{1/4}\big) \otimes \varrho,$$

where we use Lemma 1.8 and the assumption on ϱ, observing that the inducing characters agree on $\widetilde{\mathscr{Z}}\widetilde{T}^{\square}$. Therefore the first part follows. Similarly, $J_\psi(-1,\varrho)$ is a submodule of $\mathscr{I}^{\psi^{-1}} \otimes \varrho$, and hence $\theta^{\psi^{-1}} \otimes \varrho$ is a submodule of $J_\psi(-1, \varrho)$. We have an injective $\mathbb{C}$-linear map from $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$ to

$$\mathrm{Hom}_{\overline{G}}\big((\mathrm{Ind}^{\overline{G}}_{w_0 \widetilde{\mathscr{T}} N}{}^{w_0}\xi^{\psi^{-1}} \otimes \delta_B^{1/4}) \otimes \varrho, \mathscr{I}^{\psi^{-1}} \otimes \varrho\big).$$

Since the latter space is one-dimensional by Proposition I.2.2 of [Kazhdan and Patterson 1984],

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho)) \leq 1.$$

Since $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho)$ is a subspace of $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$ and since $\dim_{\mathbb{C}} \mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho) \geq 1$, these spaces are equal. Because $M(1, \varrho)$ gives a nonzero element in $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), J_\psi(-1, \varrho))$, it is proportional to the basis vector in $\mathrm{Hom}_{\overline{G}}(I_\psi(1, \varrho), \theta^{\psi^{-1}} \otimes \varrho)$. □

2. Derivatives of exceptional representations

Throughout this section we suppose that F is a nonarchimedean local field of characteristic 0.

2A. *Whittaker models of exceptional representations.* For an l-group $\mathcal{G}$, its closed subgroup H and a smooth representation ρ of H we define $\mathrm{ind}^{\mathcal{G}}_H \rho$ to be the space of all functions $f : \mathcal{G} \to \rho$ such that $f(hg) = \rho(h) f(g)$ for all $h \in H$ and $g \in \mathcal{G}$ and such that f is right invariant under some compact open subgroup of $\mathcal{G}$. Define c-$\mathrm{ind}^{\mathcal{G}}_H \rho$ to be the subspace of $\mathrm{ind}^{\mathcal{G}}_H \rho$ which consists of functions with compact support modulo H. The group $\mathcal{G}$ acts on both of these by right translation.

Definition 2.1. If U is a closed subgroup of $\mathcal{G}$, Ψ a character of U and π a smooth representation of $\mathcal{G}$, then we call the quotient space $\pi_{U,\Psi} = \pi/\pi(U, \Psi)$ the Jacquet module of π with respect to U and Ψ, where $\pi(U, \Psi)$ is the space spanned by the vectors of the form $\pi(u)v - \Psi(u)v$ for $v \in \pi$ and $u \in U$. When $\mathcal{G} = \overline{G}_r$ and $U = N_r$, a Ψ-Whittaker functional on π is a complex linear functional λ on π which satisfies $\lambda(\pi(u)v) = \Psi(u)\lambda(v)$ for all $v \in \pi$ and $u \in N_r$. The space of Ψ-Whittaker functionals on π can be identified with the space of complex linear functionals on $\pi_{N_r,\Psi}$.

We say that a character Ψ of N_r is generic if it is nontrivial on $U_{\boldsymbol{r}}$ for all compositions $\boldsymbol{r}$ of r. We define, as usual, a generic character ψ_r of N_r by

$$\psi_r(u) = \psi(u_{1,2} + u_{2,3} + \cdots + u_{r-1,r}), \quad u \in N_r.$$

Remark 2.2. (1) The space $\mathrm{Ind}^{\overline{G}_r}_{N_r} \Psi$ consists of all smooth functions W on $\overline{G}_r$ satisfying $W(u\tilde{g}) = \Psi(u)W(\tilde{g})$ for all $u \in N_r$ and $\tilde{g} \in \overline{G}_r$. The group $\overline{G}_r$ acts on this space by right translation, and a nontrivial intertwining map $\pi \to \mathrm{Ind}^{\overline{G}_r}_{N_r} \Psi$ is called the Ψ-Whittaker model of π. Note that π has a nonzero Ψ-Whittaker functional λ if and only if π has a Ψ-Whittaker model Λ. To obtain a model from a functional, set $\Lambda(\tilde{g}, v) = \lambda(\pi(\tilde{g})v)$, and to obtain a functional from a model, set $\lambda(v) = \Lambda(\tilde{e}, v)$, where $\tilde{e}$ denotes the identity element of $\overline{G}_r$.

(2) The group $\widetilde{T}_r$ acts transitively on the set of generic characters of N_r thanks to Remark 1.2(4). For $\tilde{t} \in \widetilde{T}_r$ the $\mathbb{C}$-linear map $v \mapsto \pi(\tilde{t})v$ is an isomorphism of $\pi_{N_r,\Psi}$ and $\pi_{N_r,{}^{\tilde{t}}\Psi}$.

(3) The vector space $\pi_{N_r,\Psi}$ can be identified with ${}^{\iota}\pi_{N_r,{}^{\iota}\Psi}$.

(4) For $a \in F^\times$ we define a character ψ_a of N_2 by $\psi_a\big[\big(\begin{smallmatrix}1 & x\\ 0 & 1\end{smallmatrix}\big)\big] = \psi(ax)$. Recall that

$$\dim_{\mathbb{C}}(\Omega^{\psi}_{\chi})_{N_2,\psi_a} = \begin{cases} 1 & \text{if } a \in F^{\times 2}, \\ 0 & \text{if } a \in F^\times \setminus F^{\times 2} \end{cases}$$

(Proposition 2.16 of [Takeda 2014]). When $a \in F^\times$, the complex linear maps on $(\Omega^{\psi}_{\chi})_{N_2,\psi_{a^2}}$ are scalar multiples of e_a in view of (1-5).

We define a matrix $w_r \in G_r$ by

$$w_r = \begin{pmatrix} & & & \mathbf{1}_2 \\ & & \mathbf{1}_2 & \\ & \iddots & & \\ \mathbf{1}_2 & & & \end{pmatrix} \quad \text{or} \quad w_r = \begin{pmatrix} & & & & \mathbf{1}_2 \\ & & & \mathbf{1}_2 & \\ & & \iddots & & \\ & \mathbf{1}_2 & & & \\ 1 & & & & \end{pmatrix} \tag{2-1}$$

according to whether r is even or odd. Put $k = \left[\frac{r}{2}\right]$ and

$$J^{\psi}_{r,\chi} = \operatorname{Ind}^{\overline{G}_r}_{{}^{\iota}\widetilde{\mathscr{M}}_r U_{\overleftarrow{e(r)}}} {}^{w_r}\Upsilon^{\psi}_{r,\chi} \otimes \delta^{1/4}_{P_{\overleftarrow{e(r)}}}, \quad \mathcal{J}^{\psi}_{r,\chi} = \operatorname{Ind}^{\overline{G}_r}_{\widetilde{\mathscr{M}}_r U_{e(r)}} \Upsilon^{\psi^{-1}}_{r,\chi} \otimes \delta^{1/4}_{P_{e(r)}}.$$

Lemma 2.3. *If Ψ is generic, then the space $(J^{\psi}_{r,\chi})_{N_r,\Psi}$ is one-dimensional.*

Remark 2.4. Kazhdan and Patterson [1984] studied Whittaker functionals on the principal series representations of $\overline{G}_r$. Its space of Whittaker functionals is not one-dimensional:

$$\dim_{\mathbb{C}}(\mathscr{I}^{\psi}_r)_{N_r,\Psi} = [F^\times : F^{\times 2}]^k$$

(see Lemma I.3.2 of that paper).

Proof. From Remark 2.2(2) we may assume that $\Psi = \psi_r$. We will apply Theorem 5.2 of [Bernstein and Zelevinsky 1977] to $J^{\psi}_{r,\chi}$ with

$$G = \overline{G}_r, \quad M = {}^{\iota}\widetilde{\mathscr{M}}_r, \quad U = U_{\overleftarrow{e(r)}}, \quad \theta = 1, \quad N = \{e\}, \quad V = N_r.$$

If we set

$$P = MU = {}^{\iota}\widetilde{\mathscr{M}}_r U_{\overleftarrow{e(r)}}, \quad Q = NV = N_r, \quad V' = M \cap {}^{w^{-1}}V, \quad \psi' = {}^{w^{-1}}\psi_r|_{V'}$$

for $w \in G$, then the space $(J^{\psi}_{r,\chi})_{N_r,\Psi}$ is glued from ${}^{w}(({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'})$, where wP runs through the Q-orbits on G/P such that Ψ is trivial on ${}^{w}U \cap V$. Fix a set Σ of representatives of $F^{\times 2}\backslash F^{\times}$. The Q-orbits satisfying this condition are of the form $\boldsymbol{s}_r(\iota_r(a)w_r^{-1})P$ for $a = (a_1, \dots, a_k) \in \Sigma^{\oplus k}$, where

$$\iota_r(a) = \mathrm{diag}[d(a_1), \dots, d(a_k)] \quad \text{or} \quad \iota_r(a) = \mathrm{diag}[d(a_1), \dots, d(a_k), 1]$$

according to whether r is even or odd. If $w = \iota_r(a)w_r^{-1}$, then

$$ {}^{w_r^{-1}}V' = M_{\boldsymbol{e}(r)} \cap N_r \simeq N_2^{\oplus k}, \quad {}^{w_r^{-1}}\psi' = \psi_{a_1} \oplus \psi_{a_2} \oplus \cdots \oplus \psi_{a_k}.$$

In light of Remark 2.2(4) the space $({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'}$ is zero unless $-a_i \in F^{\times 2}$ for all $i = 1, 2, \dots, k$, and when this is the case, $({}^{w_r}\Upsilon^{\psi}_{r,\chi})_{V',\psi'}$ is one-dimensional. □

Lemma 2.5. *Fix a preimage $\tilde{w}_r$ of w_r in $\overline{G}_r$. The integral*

$$\lambda^{\Psi}_{x}(\Phi) = \int_{\overleftarrow{U_{e(r)}}} e_x(\Phi(\tilde{w}_r^{-1}u))\overline{\Psi(u)}\,\mathrm{d}u$$

converges absolutely for all $\Phi \in \mathcal{J}^{\psi}_{r,\chi}$, $x \in F^k$ and characters Ψ of N_r.

Proof. We may assume χ to be unitary. Define a function f_0 on G_r by

$$\begin{aligned} f_0(g) &= \delta_{P_{e(r)}}(g)^{3/4} \prod_{i=1}^{k} \left| \frac{t_{r-2i+1}}{t_{r-2i+2}} \right|^{1/4} \\ &= \delta_{B_r}(g)^{1/2} \prod_{i=1}^{k} |t_{r-2i+1}|^{\alpha_i} |t_{r-2i+2}|^{\beta_i} \times \begin{cases} 1 & \text{if } r \text{ is even,} \\ |t_1|^{(r-1)/4} & \text{if } r \text{ is odd,} \end{cases} \end{aligned}$$

writing g in the form utk with $t = \mathrm{diag}[t_1, \dots, t_r] \in T_r$, $u \in N_r$ and $k \in K_r$, where $\alpha_i = i - \frac{1}{4}(r+3)$ and $\beta_i = i - \frac{1}{4}(r+1)$. In view of (1-4) we can find a positive constant c such that $|e_x(\Phi(\tilde{g}))| \le c f_0(p_r(\tilde{g}))$ for all $\tilde{g} \in \overline{G}_r$. Since

$$\tfrac{1}{4}(r-1) \ge \beta_k > \alpha_k > \beta_{k-1} > \alpha_{k-1} > \cdots > \beta_1 > \alpha_1,$$

the integral

$$\int_{U_{\overleftarrow{e(r)}}} f_0(w_r^{-1}u)\,\mathrm{d}u$$

is convergent by applying Proposition IV.2.1 of [Waldspurger 2003] with $P = B_r$ and $P' = w_r^{-1}B_r w_r$. □

Lemma 2.6. *If $\Phi \in \mathcal{J}^{\psi}_{r,\chi}$, $b \in (F^{\times})^{\oplus k}$, Ψ is a generic character of N_r and $\lambda^{\Psi}_{b}(\mathcal{J}^{\psi}_{r,\chi}(\tilde{p})\Phi) = 0$ for all $\tilde{p} \in \widetilde{\mathscr{P}}_r$, then $\Phi = 0$.*

Proof. The proof proceeds as in that of Proposition 3.2 of [Jacquet and Shalika 1983], where an analogous result was proved for standard modules of general linear groups. There is no harm in assuming that $\Psi = \psi_r$ in view of Remark 2.2(3).

The case $r=1$ is trivial. Proposition 1.4(3) proves the case $r=2$. We suppose that $r>2$, assuming the result up to $r-2$. Take a preimage $\tilde{w}_{r-2}$ of w_{r-2} in $\overline{G}_r$. Put $\tilde{w}=\tilde{w}_{r-2}\tilde{w}_r^{-1}$ and $b'=(b_1,\dots,b_{k-1})\in(F^\times)^{\oplus k-1}$. We define the $\mathbb{C}$-linear map $e^*_{b'}:\mathscr{S}(F^k)\to\mathscr{S}(F)$ by the relation

$$e_x(e^*_{b'}(\Phi))=\Phi(b_1,\dots,b_{k-1},x)$$

for $x\in F$. For each $\tilde{g}\in\overline{G}_r$ we define the map on $\mathcal{J}^{\psi}_{r,\chi}$ by

$$\Phi\mapsto W^*(\tilde{g},\Phi)=\int_{U_{\overleftarrow{e(r-2)}}} e^*_{b'}(\Phi(\tilde{w}_{r-2}^{-1}u\tilde{g}))\overline{\psi_r(u)}\,\mathrm{d}u\in\mathscr{S}(F).$$

Observe that

$$\lambda^{\Psi}_b(\mathcal{J}^{\psi}_{r,\chi}(\tilde{g})\Phi)=\int_{U_{(2,r-2)}} e_{b_k}(W^*(\tilde{w}u\tilde{g},\Phi))\overline{\psi_r(u)}\,\mathrm{d}u.$$

Hence the integrals are absolutely convergent in view of Lemma 2.5.

Suppose that $\lambda^{\Psi}_b(\mathcal{J}^{\psi}_{r,\chi}(\tilde{p})\Phi)=0$ for all $\tilde{p}\in\widetilde{\mathscr{P}}_r$. If we replace $\tilde{p}$ by $\boldsymbol{s}_2(g)\tilde{p}$, then a simple computation yields

$$\int_{M_{2,r-2}(F)} e_{b_k}\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx))\,\mathrm{d}x=0$$

for all $g\in G^{\square}_2$, where

$$\varepsilon=\begin{pmatrix}0&0&\cdots&0&0\\1&0&\cdots&0&0\end{pmatrix}\in\mathrm{M}_{2,r-2}(F).$$

Replacing g by $\mathrm{diag}[b_k^{-2}a^2,1]g$, we obtain

$$\int_{M_{2,r-2}(F)} e_{a}\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx))\,\mathrm{d}x=0$$

for all $a\in F^\times$, and so by continuity, this holds for all $a\in F$.

For $x\in M_{2,r-2}(F)$ we define $\mathcal{F}_x\in\mathscr{S}(F)$ by

$$\mathcal{F}_x(y)=e_y\left(W^*\left[\tilde{w}\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde{p},\Phi\right]\right)\psi(\mathrm{tr}({}^t\varepsilon gx)),\quad y\in F.$$

Since the integral

$$\int_{M_{2,r-2}(F)}\left|e_y\left(\Omega^{\psi^{-1}}_{\chi}(\boldsymbol{s}_2(g))\mathcal{F}_x\right)\right|\mathrm{d}x$$

is convergent uniformly in y,

$$\begin{aligned}0&=\int_F\int_{M_{2,r-2}(F)} e_y\big(\Omega_\chi^{\psi^{-1}}(\boldsymbol{s}_2(g))\mathcal{F}_x\big)\overline{\Phi(y)}\,\mathrm{d}x\,\mathrm{d}y\\&=\int_{M_{2,r-2}(F)}\int_F e_y\big(\Omega_\chi^{\psi^{-1}}(\boldsymbol{s}_2(g))\mathcal{F}_x\big)\overline{\Phi(y)}\,\mathrm{d}y\,\mathrm{d}x\\&=\int_{M_{2,r-2}(F)}\int_F \mathcal{F}_x(y)\overline{e_y\big(\Omega_{\bar\chi^{-1}}^{\psi^{-1}}(\boldsymbol{s}_2(g))^{-1}\Phi\big)}\,\mathrm{d}y\,\mathrm{d}x\\&=\int_F\overline{e_y\big(\Omega_{\bar\chi^{-1}}^{\psi^{-1}}(\boldsymbol{s}_2(g))^{-1}\Phi\big)}\int_{M_{2,r-2}(F)}\mathcal{F}_x(y)\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

for all $\Phi\in\mathscr{S}(F)$, where $\bar\chi$ is defined by $\bar\chi(a)=\overline{\chi(a)}$ for $a\in F^\times$. We get

$$\int_{M_{2,r-2}(F)}\mathcal{F}_x(y)\,\mathrm{d}x=0$$

for all $g\in G_2^\square$, $\tilde p\in\widetilde{\mathscr{P}}_r$ and $y\in F$. Since this integral is absolutely convergent, we may apply the Fourier inversion to conclude that for all $\tilde p\in\widetilde{\mathscr{P}}_r$

$$\int_{M_{2,r-3}(F)} e_y\left(W^*\left[\tilde w\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & 0 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde p,\varPhi\right]\mathrm{d}x\right)=0.$$

We can prove that for any j with $1\le j<r-2$ the relation

$$\int_{M_{2,r-2-j}(F)} e_y\left(W^*\left[\tilde w\boldsymbol{s}_r\left[\begin{bmatrix}\mathbf{1}_2 & 0 & x\\ & \mathbf{1}_{r-2}\end{bmatrix}\right]\tilde p,\varPhi\right]\mathrm{d}x\right)=0$$

for all $\tilde p\in\widetilde{\mathscr{P}}_r$ implies the same relation with j replaced by $j+1$ by arguing exactly as on p. 118 of [Jacquet and Shalika 1983]. We ultimately get $W^*(\tilde w\tilde p,\varPhi)=0$ for all $\tilde p\in\widetilde{\mathscr{P}}_r$.

Substituting $\boldsymbol{s}_r(\mathrm{diag}[\mathbf{1}_2,p'])\tilde p$ for $\tilde p$, we see that $W^*(\boldsymbol{s}_r(p')\tilde w\tilde p,\varPhi)=0$ for all $p'\in\mathscr{P}_{r-2}$ and $\tilde p\in\widetilde{\mathscr{P}}_r$. The induction hypothesis applied to $\mathcal{J}_{r-2,\chi}^{\psi}$ gives $W^*(\boldsymbol{s}_r(g')\tilde w\tilde p,\varPhi)=0$ for all $g'\in G_{r-2}$ and $\tilde p\in\widetilde{\mathscr{P}}_r$. But then $W^*(u\tilde w\tilde p,\varPhi)=0$ for all $u\in U_{(r-2,2)}$ and $\tilde p\in\widetilde{P}_{(2,r-2)}$, and so by continuity, $W^*(\tilde g,\varPhi)=0$ for all $\tilde g\in\overline{G}_r$. We obtain $\varPhi=0$ by induction on r. □

Lemma 2.7. *When $r>2$, the representation $J_{r,\chi}^{\psi}$ is reducible.*

Proof. The periodicity of $\theta_{r,\chi}$ stated in [Kazhdan and Patterson 1984, Theorem I.2.9(e)] or [Takeda 2014, Proposition 2.36] shows that $(\theta_{r,\chi}^{\psi})_{U_{e(r)},1}\not\simeq(J_{r,\chi}^{\psi})_{U_{e(r)},1}$, which completes our proof. □

Proposition 2.8. *If $r>2$, then $(\theta_{r,\chi}^{\psi})_{N_r,\psi_r}=0$.*

Proof. Take a subrepresentation V_0 of $J_{r,\chi}^{\psi}$ such that $\theta_{r,\chi}^{\psi}=J_{r,\chi}^{\psi}/V_0$. There are $b\in(F^\times)^{\oplus k}$ and a generic character $\varPsi$ such that $\lambda_0(\varPhi)=\lambda_b^{\varPsi}({}^\iota\varPhi)$ gives a ψ_r-Whittaker functional on $J_{r,\chi}^{\psi}$. Suppose that $\theta_{r,\chi}^{\psi}$ admits a nonzero ψ_r-Whittaker functional λ. We can view λ as a linear form on $J^{\psi}_{r,\chi}$ which vanishes on V_0. Since λ is a scalar multiple of λ_0 by the uniqueness of the Whittaker model of $J^{\psi}_{r,\chi}$ (see Lemma 2.3), if $\Phi \in V_0$, then $\lambda_0(J^{\psi}_{r,\chi}(\tilde{g})\Phi) = 0$ for all $\tilde{g} \in \overline{G}_r$, and hence $\Phi = 0$ by Lemma 2.6. Thus $V_0 = 0$, which contradicts Lemma 2.7. $\square$

2B. ***The restriction to the group*** $\widetilde{\mathscr{P}}$. Define the character ν_r of $\overline{G}_r$ by $\nu_r(\tilde{g}) = |\det p_r(\tilde{g})|$ for $\tilde{g} \in \overline{G}_r$. We denote its restriction to $\widetilde{\mathscr{P}}_r$ by the same symbol. The five functors $\Phi^{\pm}$, $\Psi^{\pm}$ and $\hat{\Phi}^{+}$ play an important role in the theory of representations of $\widetilde{\mathscr{P}}_r$. These functors are the exact analogues of the functors described in [Zelevinsky 1980]. Although the theory is stated for G_r, the same principle works in the setting of the double covers $\overline{G}_r$ (see [Bump and Ginzburg 1992; Kable 2001]). Given a smooth representation π of $\overline{G}_r$ we write $\Psi^{+}\pi$ for the representation of $\widetilde{\mathscr{P}}_{r+1}$ on the same space such that $\mathscr{Y}_{r+1}$ acts trivially and $\overline{G}_r$ acts by $\pi \otimes \nu_r^{1/2}$. For a smooth representation τ of $\widetilde{\mathscr{P}}_r$ put

$$\Phi^{+}(\tau) = \text{c-ind}^{\widetilde{\mathscr{P}}_{r+1}}_{\widetilde{\mathscr{P}}_r \mathscr{Y}_{r+1}} \tau \otimes \nu_r^{1/2} \boxtimes (\psi_{r+1}|_{\mathscr{Y}_{r+1}}), \quad \Phi^{-}(\tau) = \tau_{\mathscr{Y}_r, \psi_r|_{\mathscr{Y}_r}},$$

$$\hat{\Phi}^{+}(\tau) = \text{ind}^{\widetilde{\mathscr{P}}_{r+1}}_{\widetilde{\mathscr{P}}_r \mathscr{Y}_{r+1}} \tau \otimes \nu_r^{1/2} \boxtimes (\psi_{r+1}|_{\mathscr{Y}_{r+1}}), \qquad \Psi^{-}(\tau) = \tau_{\mathscr{Y}_r, 1}.$$

The actions of the groups $\widetilde{\mathscr{P}}_{r-1}$ and $\overline{G}_{r-1}$ on $\Phi^{-}(\tau)$ and $\Psi^{-}(\tau)$ are normalized respectively in order that the following results hold (see Propositions 4.2 and 4.3 of [Kable 2001]):

Lemma 2.9. *If ρ, τ and κ are smooth representations of $\overline{G}_{r-1}$, $\widetilde{\mathscr{P}}_r$ and $\widetilde{\mathscr{P}}_{r-1}$, respectively, then*

$$\operatorname{Hom}_{\widetilde{\mathscr{P}}_r}(\tau, \Psi^{+}(\rho)) = \operatorname{Hom}_{\overline{G}_{r-1}}(\Psi^{-}(\tau), \rho), \quad \Psi^{+}(\rho)^{\vee} \simeq \nu_r^{-1} \otimes \Psi^{+}(\rho^{\vee}),$$
$$\operatorname{Hom}_{\widetilde{\mathscr{P}}_r}(\Phi^{+}(\kappa), \tau) = \operatorname{Hom}_{\widetilde{\mathscr{P}}_{r-1}}(\kappa, \Phi^{-}(\tau)), \quad \Phi^{+}(\kappa)^{\vee} \simeq \nu_r^{-1} \otimes \hat{\Phi}^{+}(\nu_{r-1} \otimes \kappa^{\vee}),$$
$$\operatorname{Hom}_{\widetilde{\mathscr{P}}_r}(\tau, \hat{\Phi}^{+}(\kappa)) = \operatorname{Hom}_{\widetilde{\mathscr{P}}_{r-1}}(\Phi^{-}(\tau), \kappa), \quad \Phi^{-}(\tau)^{\vee} \simeq \Phi^{-}(\tau^{\vee}).$$

Definition 2.10. Let π be an admissible representation of $\overline{G}_r$. For $i = 1, 2, \ldots, r$ the i-th derivative of a smooth representation π of $\overline{G}_r$ is a representation of $\overline{G}_{r-i}$ defined by $\pi^{(i)} = \Psi^{-}(\Phi^{-})^{i-1}(\pi|_{\widetilde{\mathscr{P}}_r})$. If $\pi^{(h)} \neq 0$ and $\pi^{(j)} = 0$ for all $j > h$, then we call the number h the depth of π and call $\pi^{(h)}$ the highest derivative of π. It is convenient to introduce the shifted derivatives $\pi^{[i]} = \pi^{(i)} \otimes \nu_{r-i}^{1/2}$.

If π is irreducible, then so is its highest derivative by Theorem 8.1 of [Zelevinsky 1980].

We identify the multiplicative group $F^{\times}$ with the center Z_r of the group G_r for $r > 0$. When π is an irreducible admissible representation of G_r, its central exponent is the real number $e(\pi)$ defined by $|\omega_\pi(z)| = |z|^{e(\pi)}$ for $z \in F^{\times}$. In the next subsection we will use the following consequence of the unitarizability criterion given in Section 7.3 of [Bernstein 1984].

Proposition 2.11 (Bernstein). *Let π be an irreducible unitary representation of G_r of depth h. Then $\pi^{[h]}$ is an irreducible unitary representation of G_{r-h} and all the central exponents of irreducible subquotients of $\pi^{[k]}$ are strictly positive for all $k = 1, 2, \dots, h-1$.*

Thanks to Proposition 2.8, we have the following generalization of Theorem 5.4 of [Kable 2001] to the dyadic and twisted cases. The exceptional representations are very small in the following sense:

Theorem 2.12. *If $3 \le k \le r$, then the k-th derivatives of the exceptional representations of $\overline{G}_r$ are zero.*

2C. *Uniqueness of invariant trilinear forms.*

Proposition 2.13 (Kable). (1) $(\theta_{r,\chi}^{\psi^{-1}})^{[2]} \simeq \theta_{r-2,\chi}^{\psi^{-1}}$.

(2) *If r is odd, then* $(\theta_r^{\psi})^{(1)} \otimes \nu_{r-1}^{1/4} \simeq \theta_{r-1}$.

(3) *If r is even, then*

$$\theta_r^{(1)} \otimes \nu_{r-1}^{1/4} \simeq \bigoplus_{a \in F^{\times 2} \backslash F^{\times}} (\theta_{r-1}^{\psi} \otimes \chi_a).$$

(4) *If r is even and χ is odd, then* $\theta_{r,\chi}^{(1)} = 0$.

Proof. After Bump and Ginzburg [1992] showed that the second derivative of an exceptional representation must again be exceptional, Kable identified it precisely [2001, Theorem 5.3]. Although they discussed only the case when χ is trivial, one can similarly prove the twisted case. The second and third assertions are Theorem 5.2 of [Kable 2001]. The last assertion is obvious as Ω_{χ}^{ψ} is supercuspidal if χ is odd. □

Here and throughout the rest of this paper we will retain the notation from Section 1G.

Theorem 2.14. *Let ϱ be a character of $F^{\times}$, π an irreducible admissible representation of G and ϑ an exceptional representation of $\overline{G}$.*

(1) *For all but finitely many values of q^{-s} we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \vartheta \otimes I_{\psi}(s, \varrho), \mathbb{C}) \le \dim_{\mathbb{C}} \pi^{(n)}.$$

(2) *Assume that χ is trivial if n is odd. If π and χ are unitary, then*

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \theta_{\chi}^{\psi} \otimes \theta^{\psi^{-1}}, \mathbb{C}) \le 1,$$

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \theta_{\chi}^{\psi} \otimes I_{\psi}(1, \varrho), \mathbb{C}) \le 1.$$

Remark 2.15. (1) One can view the second inequality of (2) as an analogue of Bernstein's theorem that $\dim_{\mathbb{C}} \mathrm{Hom}_{\mathscr{P}}(\pi \otimes \pi^\vee, \mathbb{C}) = 1$ for all irreducible admissible representations π of G, in view of

$$\begin{aligned}\mathrm{Hom}_G(\pi \otimes \pi^\vee \otimes \mathrm{Ind}_{\mathcal{P}}^G \delta_{\mathcal{P}}^{1/2}, \mathbb{C}) &\simeq \mathrm{Hom}_G(\pi \otimes \pi^\vee, \mathrm{Ind}_{\mathcal{P}}^G \delta_{\mathcal{P}}^{-1/2}) \\ &\simeq \mathrm{Hom}_{\mathcal{P}}(\pi \otimes \pi^\vee, \mathbb{C}) \\ &\simeq \mathrm{Hom}_{\mathscr{P}}(\pi \otimes \pi^\vee, \mathbb{C}).\end{aligned}$$

(2) Matringe [2014, Proposition 2.3] proved that if E is a quadratic extension of F and if π is an irreducible admissible unitary representation of $\mathrm{GL}_n(E)$, then the space of $\mathscr{P}$-invariant linear functionals on π is at most one-dimensional (cf. Theorem 1.1 of [Anandavardhanan et al. 2004]). This is an analogue of the second part in the context of Asai L-factors.

(3) When χ is trivial and F is not dyadic, Kable [2001, Theorem 6.1] proved the first part by modifying the proof of [Bump and Ginzburg 1992, Theorem 5.1], and moreover, if π is generic and unitary, then his result implies the second part. Actually, our proof combines his argument and the idea of [Matringe 2014]. Since the restriction to nondyadic F entered only through the lack of Theorem 2.12, his computation is now applicable to the dyadic case, and even to the twisted case.

Proof. Since $\widetilde{\mathscr{Z}}$ is the center of $\bar{G}$, the space $\mathrm{Hom}_G(\pi \otimes \vartheta \otimes I_\psi(4s, \varrho), \mathbb{C})$ is zero unless the product of the three central characters is trivial on $F^{\times e(n)}$. Assume that this is the case. Then the space is isomorphic to

$$\begin{aligned}\mathrm{Hom}_{\bar{G}}(\pi \otimes \vartheta, I_{\psi^{-1}}(-4s, \varrho^{-1})) &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}}(\pi|_{\mathscr{P}} \otimes \vartheta|_{\widetilde{\mathscr{P}}}, \Psi^+\theta_{n-1}^{\psi^{-1}} \otimes \nu^{-s}) \\ &\simeq \mathrm{Hom}_{\mathscr{P}}(\pi|_{\mathscr{P}} \otimes \vartheta|_{\widetilde{\mathscr{P}}} \otimes \Psi^+\theta_{n-1}^{\psi}, \nu^{1-s})\end{aligned}$$

by the Frobenius reciprocity and Lemma 2.9. Recall that

$$(\theta_{n-1}^{\psi^{-1}})^\vee \simeq \theta_{n-1}^{\psi}.$$

For $1 \le k \le n$ and exceptional representations θ of $\bar{G}_k$ and θ' of $\bar{G}_{k-1}$ we shall consider the space

$$\mathcal{H}_{k,\theta,\theta'}(\pi, s) = \mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}}) \otimes \theta|_{\widetilde{\mathscr{P}}_k} \otimes \Psi^+\theta', \nu_k^{1-s}\big).$$

Assume that $k \ge 2$. Since there is a short exact sequence

$$0 \to \Phi^+\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k}) \to \theta|_{\widetilde{\mathscr{P}}_k} \to \Psi^+\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k}) \to 0$$

as recorded in Section 3 of [Bernstein and Zelevinsky 1977], we have an exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\to \mathcal{H}_{k,\theta,\theta'}(\pi,s)\\ &\to \mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Phi^+\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big).\end{aligned}$$

Lemma 2.9 shows that

$$\begin{aligned}\text{(2-2)}\quad &\mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta',\Psi^+(\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee)\otimes\nu_k^{-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{G}_{k-1}}\big(\Psi^-\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta'\big),\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1}^{-s}\big)\\ &\simeq \mathrm{Hom}_{G_{k-1}}(\pi^{[n-k+1]}\otimes\theta'\otimes\theta^{(1)},\nu_{k-1}^{-s}).\end{aligned}$$

Lemma 2.9 again shows that

$$\begin{aligned}&\mathrm{Hom}_{\mathscr{P}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Phi^+\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\otimes\Psi^+\theta',\nu_k^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_k}\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta',\hat{\Phi}^+(\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1})\otimes\nu_k^{-s}\big)\\ &\simeq \mathrm{Hom}_{\widetilde{\mathscr{P}}_{k-1}}\big(\Phi^-\big((\Phi^-)^{n-k}(\pi|_{\mathscr{P}})\otimes\Psi^+\theta'\big),\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})^\vee\otimes\nu_{k-1}^{1-s}\big)\\ &\simeq \mathrm{Hom}_{\mathscr{P}_{k-1}}\big((\Phi^-)^{n-k+1}(\pi|_{\mathscr{P}})\otimes(\theta'|_{\widetilde{\mathscr{P}}_{k-1}}\otimes\nu_{k-1}^{1/2})\otimes\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k}),\nu_{k-1}^{1-s}\big).\end{aligned}$$

Now we use Theorem 2.12. It implies that $\Phi^-(\theta|_{\widetilde{\mathscr{P}}_k})\simeq\Psi^+\theta^{(2)}$ (see [Kable 2001, (6.8)]). The last space is isomorphic to $\mathcal{H}_{k-1,\theta',\theta^{[2]}}(\pi,s)$ and

$$\begin{aligned}&\dim_{\mathbb{C}}\mathcal{H}_{k,\theta,\theta'}(\pi,s)\\ &\quad\le\dim_{\mathbb{C}}\mathcal{H}_{k-1,\theta',\theta^{[2]}}(\pi,s)+\dim_{\mathbb{C}}\mathrm{Hom}_{G_{k-1}}\big(\pi^{[n-k+1]}\otimes\theta'\otimes\Psi^-(\theta|_{\widetilde{\mathscr{P}}_k}),\nu_{k-1}^{-s}\big).\end{aligned}$$

We can see by comparing the central characters that the latter dimension must be zero except for finitely many q^{-s}. From this point onwards the exceptional representations with respect to which the spaces $\mathcal{H}_{k,\theta,\theta'}(\pi,s)$ are formed will not play a significant role and we shall allow ourselves to omit them from the notation. Then

$$\begin{aligned}\dim_{\mathbb{C}}\mathrm{Hom}_G(\pi\otimes\vartheta\otimes I_\psi(4s,\varrho),\mathbb{C})&\le\dim_{\mathbb{C}}\mathcal{H}_n(\pi,s)\\ &\le\cdots\\ &\le\dim_{\mathbb{C}}\mathcal{H}_1(\pi,s)=\dim_{\mathbb{C}}\pi^{(n)}\end{aligned}$$

for all but finitely many q^{-s} by descending induction.

Next we prove (2). Since we obtain the injective map

$$\text{(2-3)}\qquad \mathrm{Hom}_G(\pi\otimes\varrho\otimes\theta_\chi^\psi\otimes\theta^{\psi^{-1}},\mathbb{C})\hookrightarrow\mathrm{Hom}_G(\pi\otimes\theta_\chi^\psi\otimes I_\psi(1,\varrho),\mathbb{C})$$

by composition with the quotient map in Lemma 1.15, we get the first inequality from the second. The proof of the second inequality is a variation on the proof of Proposition 2.2 of [Matringe 2014]. Let h denote the depth of π. Note that $(\Phi^-)^h(\pi|_{\mathscr{P}}) = 0$ and hence $\mathcal{H}_{n-h}(\pi, s) = 0$. If θ is a unitary exceptional representation, then $\theta^{(1)} \otimes \nu_{k-1}^{1/4}$ is zero or a unitary exceptional representation or a sum of such by Proposition 2.13. Thus the space (2-2) must vanish at $s = \frac{1}{4}$ for $k = n, n-1, \dots, n-h+2$ as the central characters do not match by Proposition 2.11. We conclude that

$$\dim_{\mathbb{C}} \mathrm{Hom}_G(\pi \otimes \vartheta \otimes I_\psi(1, \varrho), \mathbb{C}) \le \dim_{\mathbb{C}} \mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes (\theta')^{(1)}, \nu_{n-h}^{-1/4}\big)$$

for some unitary exceptional representations θ of $\bar{G}_{n-h}$ and θ' of $\bar{G}_{n-h+1}$.

Our task is to prove that the right-hand side is at most one. Without loss of generality we may suppose that $\theta' = \theta^{\psi}_{n-h+1}$ by replacing θ by $\theta \otimes \eta$ for some unitary character η of $F^\times$ in view of Proposition 2.13(4) and Remark 1.7. Then the space is zero by Proposition 2.13 unless the product of the central characters of $\pi^{[h]}$ and ϑ is quadratic. By comparing the central characters, one can find a nonzero element a_0 in F such that

$$\mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes (\theta')^{(1)}, \nu_{n-h}^{-1/4}\big) \simeq \mathrm{Hom}_{G_{n-h}}\big(\pi^{[h]} \otimes \theta \otimes \theta^{\psi}_{n-h}, \chi_{a_0}\big).$$

Notice that the central characters of $\theta^{\psi}_{n-h} \otimes \chi_a$ ($a \in F^{\times 2} \backslash F^\times$) are mutually different if $n-h$ is odd. Now our proof is complete by induction. □

3. Twisted symmetric square L-factors

One of the most significant uses of exceptional representations in number theory so far is as an ingredient in the Rankin–Selberg integral for the symmetric square L-function of an irreducible cuspidal automorphic representation of a general linear group found by Bump and Ginzburg [1992]. Let F for the moment be a local field of characteristic zero.

3A. *A normalization of the intertwining operator.*

Definition 3.1. A normalized intertwining operator is defined by

$$N(s, \varrho) = \frac{b(-s, \varrho^{-1})}{a(s, \varrho)} M(s, \varrho),$$

where

$$a(s, \varrho) = L\big(\tfrac{1}{2}n(s-1)+1, \varrho^2\big), \quad b(s, \varrho) = L\big(\tfrac{1}{2}n(s+1), \varrho^2\big).$$

Lemma 3.2. *The operator $M^*(s, \varrho) = a(s, \varrho)^{-1} M(s, \varrho)$ is entire.*

Proof. This is proved in Sections 4.5 and 4.6 of [Takeda 2015]. □

Lemma 3.3. *If we put* $\mathcal{M} = M_{(n-1,1)}$, *then*

$$^{\iota}\bigl(\mathrm{Ind}^{\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\widetilde{G}'}\,\theta^{\psi}_{n-1}\boxtimes\zeta^{\psi}_{\varrho}\bigr)\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\,{}^{\delta}\theta^{\psi^{-1}}_{n-1}\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}},$$

where the matrix δ *is defined in* (1-7).

Proof. Recall the longest element w_0' of the Weyl group of G'. The automorphism $g\mapsto w_0'\,{}^tg^{-1}w_0'$ of G stabilizes the subgroup G'. Its restriction to G' is the main involution ι' of G'. Since $\tilde g\mapsto\delta^{-1}\,{}^{\iota}\tilde g\delta$ is a lift of this automorphism, its restriction to $\overline{G}'$ differs from the lift of ι' only by twisting by a quadratic character η of $F^\times$ on account of Proposition 1.3. It follows from Lemma 1.9(4) that

$$\theta^{\psi}_{n-1}\simeq{}^{\iota'}\theta^{\psi^{-1}}_{n-1}\simeq{}^{\iota}({}^{\delta}\theta^{\psi^{-1}}_{n-1})\otimes\eta.$$

Thus ${}^{\iota}\theta^{\psi}_{n-1}\simeq{}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta$. Since ${}^{\iota}\zeta^{\psi}_{\varrho}=\zeta^{\psi^{-1}}_{\varrho^{-1}}$, we obtain

$$^{\iota}\bigl(\mathrm{Ind}^{\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\widetilde{G}'}\,\theta^{\psi}_{n-1}\boxtimes\zeta^{\psi}_{\varrho}\bigr)\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\bigl({}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta\bigr)\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

by (1-3). If n is odd, then ${}^{\delta}\theta^{\psi^{-1}}_{n-1}\otimes\eta\simeq{}^{\delta}\theta^{\psi^{-1}}_{n-1}$ by Lemma 1.9(3).

Suppose that n is even. Take a genuine character ξ' of ${}^{\iota}\widetilde{\mathscr{T}}'$ in such a way that

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota\widetilde{G}'}\,{}^{\delta}\theta^{\psi^{-1}}_{n-1}\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

is the unique irreducible subrepresentation of

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

(cf. Lemma 1.14). Since the restrictions of ξ' and $\xi'\cdot(\eta\circ\det)$ to $\widetilde{\mathscr{Z}}\widetilde{T}^{\square}=\widetilde{T}^{\square}$ coincide,

$$\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}\simeq\mathrm{Ind}^{\iota\widetilde{\mathcal{M}}}_{\widetilde{\mathscr{Z}}\,\iota(\widetilde{\mathscr{T}}'N')}\,\xi'\cdot(\eta\circ\det)\boxtimes\zeta^{\psi^{-1}}_{\varrho^{-1}}$$

by Lemma 1.8, which concludes our proof. □

Lemma 3.3 gives an important isomorphism,

$$J^{\psi}_{s,\varrho}:{}^{\iota}J_{\psi}(s,\varrho)\simeq I_{\psi^{-1}}(s,\varrho^{-1}).$$

The isomorphism depends on s in a fairly trivial way.

Definition 3.4. We call a right $\widetilde{K}$-finite function $(s,\tilde g)\mapsto f^{(s)}(\tilde g)$ on $\mathbb{C}\times\overline{G}$ a holomorphic section of $I_{\psi}(s,\varrho)$ if $f^{(s)}(\tilde g)$ is holomorphic in s for each $\tilde g\in\overline{G}$ and $f^{(s)}\in I_{\psi}(s,\varrho)$ for each $s\in\mathbb{C}$. A holomorphic section $f^{(s)}$ is a standard section if its restriction to $\mathbb{C}\times\widetilde{K}$ does not depend on s. We call a function $f^{(s)}$ on $\mathbb{C}\times\overline{G}$ a meromorphic section of $I_{\psi}(s,\varrho)$ if there is a nonzero entire function β such that $\beta(s)f^{(s)}$ is a holomorphic section.

We call $h^{(s)}$ a meromorphic section of $J_\psi(s,\varrho)$ if $j^\psi_{s,\varrho}({}^\iota h^{(s)})$ is. We define a $\mathbb{C}$-linear map

$$\hat{N}(s,\varrho)=\hat{N}_\psi(s,\varrho):I_\psi(s,\varrho)\to I_{\psi^{-1}}(-s,\varrho^{-1})$$

by $\hat{N}(s,\varrho)f^{(s)}=j^\psi_{-s,\varrho}({}^\iota N(s,\varrho)f^{(s)})$, where

$$[{}^\iota N(s,\varrho)f^{(s)}](\tilde{g})=[N(s,\varrho)f^{(s)}]({}^\iota\tilde{g}).$$

We can define a meromorphic function $\alpha_\psi(s,\varrho)$ by

$$\hat{N}_{\psi^{-1}}(-s,\varrho^{-1})\hat{N}_\psi(s,\varrho)=\alpha_\psi(s,\varrho)\cdot\mathrm{Id}.$$

Lemma 3.5. *The function $\alpha_\psi(s,\varrho)$ has neither pole nor zero.*

Proof. We can view $I_\psi(s,\varrho)$ as a subrepresentation of $\mathrm{Ind}^{\overline{G}}_{\widetilde{\mathscr{T}}N}\,\mu_s$, where μ_s is an extension to $\widetilde{\mathscr{T}}$ of the genuine character of $\widetilde{\mathscr{Z}}\widetilde{T}^{\square}$ defined by

$$\mu_s(\boldsymbol{s}(t))=\varrho(t_n)^{-1}|t_n|^{-(n-1)s/4}\prod_{i=0}^{l-1}\mu_\psi(t_{n-2i})\prod_{j=1}^{n-1}|t_j|^{(2j-n+s)/4}$$

for $t=\mathrm{diag}[t_1,\dots,t_n]\in\mathscr{Z}T^{\square}$. Theorem I.2.6 of [Kazhdan and Patterson 1984] shows that

$$\begin{aligned}\frac{a(s,\varrho)a(-s,\varrho^{-1})}{b(s,\varrho)b(-s,\varrho^{-1})}&\alpha_\psi(s,\varrho)\\ &\approx\prod_{j=1}^{n-1}\frac{L\big(j+\frac{1}{2}n(s-1),\varrho^2\big)L\big(j+\frac{1}{2}n(-1-s),\varrho^{-2}\big)}{L\big(j+\frac{1}{2}n(s-1)+1,\varrho^2\big)L\big(j+\frac{1}{2}n(-1-s)+1,\varrho^{-2}\big)}\\ &=\frac{L\big(1+\frac{1}{2}n(s-1),\varrho^2\big)L\big(1+\frac{1}{2}n(-1-s),\varrho^{-2}\big)}{L\big(\frac{1}{2}n(s+1),\varrho^2\big)L\big(\frac{1}{2}n(1-s),\varrho^{-2}\big)}\\ &=\frac{a(s,\varrho)a(-s,\varrho^{-1})}{b(s,\varrho)b(-s,\varrho^{-1})},\end{aligned}$$

where $\approx$ denotes equality up to multiplication by invertible functions. $\square$

3B. ***Semi-Whittaker functions.*** When $r>2$, the exceptional representations of $\overline{G}_r$ fail to possess Whittaker models with respect to generic characters of N_r, but they have models with respect to certain degenerate characters of N_r. We define the degenerate characters of N_r by

$$\psi_{\boldsymbol{e},r}(u)=\psi(u_{1,2}+u_{3,4}+\cdots+u_{r-1,r}),$$
$$\psi_{\boldsymbol{o},r}(u)=\psi(u_{2,3}+u_{4,5}+\cdots+u_{r-2,r-1})$$

when r is even. When r is odd, we define the degenerate characters by

$$\psi_{\boldsymbol{e},r}(u) = \psi(u_{2,3} + u_{4,5} + \cdots + u_{r-1,r}),$$
$$\psi_{\boldsymbol{o},r}(u) = \psi(u_{1,2} + u_{3,4} + \cdots + u_{r-2,r-1}).$$

It is important to note that $\psi_r = \psi_{\boldsymbol{e},r} \cdot \psi_{\boldsymbol{o},r}$ and $\psi_r^{-1} = {}^{\iota}\psi_{\boldsymbol{e},r} \cdot {}^{\iota}\psi_{\boldsymbol{o},r}$.

Recall that χ is assumed to be trivial whenever r is odd. We define the $\mathbb{C}$-linear functional ϵ_j on $\mathscr{S}(F^j)$ by

$$\epsilon_j(\Phi) = \Phi(1, 1, \ldots, 1)$$

for $\Phi \in \mathscr{S}(F^j)$. The functional $\Phi \mapsto \epsilon_k(\Phi(\tilde{e}))$ gives a $\overline{\psi_{\boldsymbol{e},r}}$-Whittaker functional on $I^{\psi}_{r,\chi}$ by (1-5), where $k = \left[\frac{r}{2}\right]$. The $\overline{\psi_{\boldsymbol{e},r}}$-Whittaker functional corresponds to a $\overline{G}_r$-intertwining map

$$Q = Q^{\psi}_{r,\chi} : I^{\psi}_{r,\chi} \to \operatorname{Ind}^{\overline{G}_r}_{N_r} \overline{\psi_{\boldsymbol{e},r}}$$

(see Remark 2.2(1)). One can see from the proof of Proposition 1.4(3) that Q is injective. Note that

$$Q(\boldsymbol{s}_r(zu)\tilde{g}, \Theta) = \overline{\psi_{\boldsymbol{e},r}(u)} \frac{\chi(z)^k}{\mu_\psi(z)^k} Q(\tilde{g}, \Theta)$$

with $z \in F^{\times e(r)}$, $u \in N_r$, $\tilde{g} \in \overline{G}_r$ and $\Theta \in I^{\psi}_{r,\chi}$. When $r = n$, we will suppress the subscript r.

For $f \in I_\psi(s, \varrho)$ we define a $\overline{\psi_{\boldsymbol{o}}}$-Whittaker function $R(f) = R^{\psi}_{s,\varrho}(f)$ by $R(\tilde{g}, f) = \epsilon_{\ell'}(f(\tilde{g}))$ for $\tilde{g} \in \overline{G}$, where $\ell' = \left[\frac{n-1}{2}\right]$. Note that

$$R(\boldsymbol{s}(zu)\tilde{g}, f) = \varrho(z)^{-1}\mu_\psi(z)^{\ell}\overline{\psi_{\boldsymbol{o}}(u)} R(\tilde{g}, f) \quad (z \in F^{\times e(n)},\ u \in N,\ \tilde{g} \in \overline{G}).$$

Lemma 3.6.

(1) *There is a $\overline{G}_r$-intertwining embedding $\hat{Q} = \hat{Q}^{\psi}_{r,\chi} : \theta^{\psi}_{r,\chi} \to \operatorname{Ind}^{\overline{G}_r}_{N_r} {}^{\iota}\psi_{\boldsymbol{e},r}$.*

(2) *There is a $\overline{G}$-intertwining embedding $\hat{R} = \hat{R}^{\psi}_{s,\varrho} : J_\psi(s, \varrho) \to \operatorname{Ind}^{\overline{G}}_{N} {}^{\iota}\psi_{\boldsymbol{o},r}$.*

Proof. Lemma 1.9(4) gives an isomorphism $\imath^{\psi}_{r,\chi} : {}^{\iota}\theta^{\psi}_{r,\chi} \simeq \theta^{\psi^{-1}}_{r,\chi^{-1}}$. We obtain a ${}^{\iota}\psi_{\boldsymbol{e},r}$-Whittaker model of $\theta^{\psi}_{r,\chi}$ and ${}^{\iota}\psi_{\boldsymbol{o},r}$-Whittaker model of $J_\psi(s, \varrho)$ by setting

$$\hat{Q}^{\psi}_{r,\chi}(\tilde{g}, \Theta) = Q^{\psi^{-1}}_{r,\chi^{-1}}({}^{\iota}\tilde{g}, \imath^{\psi}_{r,\chi}(\Theta)), \quad \hat{R}^{\psi}_{s,\varrho}(\tilde{g}, h) = R^{\psi^{-1}}_{s,\varrho^{-1}}({}^{\iota}\tilde{g}, \jmath^{\psi}_{s,\varrho}({}^{\iota}h))$$

for $\tilde{g} \in \overline{G}_r$, $\Theta \in \theta^{\psi}_{r,\chi}$ and $h \in J_\psi(s, \varrho)$. $\square$

3C. ***The local zeta integrals.*** Let π be an irreducible admissible generic representation of G and $\mathcal{W}^{\psi}(\pi)$ its ψ_n-Whittaker model. For $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_{\chi}^{\psi}$ and a meromorphic section $f^{(s)}$ of $I_{\psi}(s, \chi^{\ell}\omega_{\pi})$ the integral

$$Z(W, \Theta, f^{(s)}) = \int_{\mathscr{Z}N\backslash G} W(g) Q(g, \Theta) R(g, f^{(s)})\, \mathrm{d}g$$

makes sense at least formally. For a meromorphic section $h^{(s)}$ of $J_{\psi}(s, \chi^{\ell}\omega_{\pi})$ we define the integral $Z(W, \Theta, h^{(s)})$ by

$$Z(W, \Theta, h^{(s)}) = \int_{\mathscr{Z}N\backslash G} W(g) \hat{Q}(g, \Theta) \hat{R}(g, h^{(s)})\, \mathrm{d}g.$$

We will use the following estimate for Whittaker functions.

Proposition 3.7 [Jacquet and Shalika 1990, Proposition 3, p. 177]. *If π is an irreducible admissible unitary generic representation of G, then for each $1 \le j \le n-1$ there is a finite set C_j of characters of $F^{\times}$ with positive real parts, and for each $\chi \in C_j$, an integer n_{χ} with the following property: Let X_j be the set of functions of the form $\chi(a)(\log|a|)^k$ with $0 \le k \le n_{\chi}$ and X the functions on $(F^{\times})^{\oplus n-1}$ which are products of functions in the X_j. Then for each $W \in W^{\psi}(\pi)$ there are Schwartz functions $\phi_{\xi} \in \mathcal{S}(F^{n-1} \times K)$ such that for $g = tk$*

$$W(g) = \delta_{B'}(t)^{1/2} \sum_{\xi \in X} \phi_{\xi}\left(\frac{t_1}{t_2}, \ldots, \frac{t_{n-1}}{t_n}, k\right) \xi\left(\frac{t_1}{t_2}, \ldots, \frac{t_{n-1}}{t_n}\right).$$

In the following proposition by "local Euler factor" in the p-adic case we mean a function of the form $P(q^{-s})^{-1}$, where P is a polynomial satisfying $P(0) = 1$, and in the archimedean case we mean a product of functions of the form $\pi^{-s/2}\Gamma\big(\frac{1}{2}(s+b)\big)$ for constants $b \in \mathbb{C}$.

Proposition 3.8 (cf. [Bump and Ginzburg 1992; Takeda 2014]). *Let F be a* (*not necessarily nonarchimedean*) *local field of characteristic zero. Let π be an irreducible admissible generic representation of G. We assume χ to be trivial if n is odd.*

(1) *There is $\beta \in \mathbb{R}$ such that the integrals $Z(W, \Theta, f^{(s)})$ converge absolutely in the right half-plane $\Re s > \beta$ for all $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_{\chi}^{\psi}$ and holomorphic sections $f^{(s)}$ of $I_{\psi}(s, \chi^{\ell}\omega_{\pi})$.*

(2) *$Z(W, \Theta, f^{(s)})$ possesses a meromorphic continuation to $\mathbb{C}$. If F is nonarchimedean and $f^{(s)}$ is a standard section, then it represents a rational function of $q^{-s/4}$.*

(3) *There is a local Euler factor $L(s)$ such that $Z(W, \Theta, f^{(2s-1)})/L\big(\frac{s}{2}\big)$ is entire for all $W \in \mathcal{W}^{\psi}(\pi)$, $\Theta \in \theta_{\chi}^{\psi}$ and holomorphic sections $f^{(s)}$ of $I_{\psi}(s, \chi^{\ell}\omega_{\pi})$.*

(4) *For each point* $s_0\in\mathbb{C}$ *there are* $W\in\mathcal{W}^\psi(\pi)$, $\Theta\in\theta^\psi_\chi$ *and a holomorphic section* $f^{(s)}$ *of* $I_\psi(s,\chi^\ell\omega_\pi)$ *such that* $Z(W,\Theta,f^{(s)})$ *does not have a zero at* $s=s_0$.

(5) *If* π *and* χ *are unitary, then* $Z(W,\Theta,f^{(s)})$ *converges absolutely for the closed right half-plane* $\Re s\geq 1$.

(6) *Suppose that* F *is nonarchimedean,* χ *is unitary and* π *is square-integrable. Then* $Z(W,\Theta,f^{(s)})$ *converges absolutely for* $\Re s\geq -1$.

Proof. The paper [Bump and Ginzburg 1992] deals with some basic local theory, and Proposition 5.5 of [Takeda 2014] discusses the twisted case. Strictly speaking, our zeta integrals are slightly different from those treated in [Bump and Ginzburg 1992] and [Takeda 2014] when n is even. However, the arguments can easily be modified to deal with our integrals.

Assertions (2) and (4) are in Proposition 5.2 and Theorem 7.2 of [Bump and Ginzburg 1992], respectively. It is easy to see from the proof of [Bump and Ginzburg 1992, Proposition 5.2] that the integral $Z(W,\Theta,f^{(s)})$ is a finite sum of products of entire functions and Tate integrals. The exponents of the quasicharacters occurring in the Tate integrals are finite in number and are independent of the choice of W, Q and $f^{(s)}$, which verifies (1) and (3).

Finally, we assume χ to be unitary and prove (5) and (6). Since Z^2 and $T'^{\square}$ have finite indices in Z and T', it suffices to prove the convergence of the integral

$$\int_{T'^{\square}}\left|W(t't)Q(t't,\Theta)R(t't,f^{(s)})\right|\delta_B(t')^{-1}\,\mathrm{d}t'$$

for $\Re s\geq -1$ and all $t\in T$. We may assume that $t=1$, taking Proposition 1.4(1) into account. From (1-4) there are positive constants c and c' such that

$$|Q(\tilde t',\Theta)|\leq c\delta_B^{1/4}(\tilde t'),\quad |R(\tilde t',f^{(s)}))|\leq c'\delta_B^{1/4}(\tilde t')\delta_{\mathcal{P}}(\tilde t')^{(\Re s+1)/4}$$

for all $\tilde t'\in\widetilde{T}'^{\square}$. Therefore all that is required is to show that if π is unitary generic or square-integrable, then the integral

$$\int_{T'^{\square}}|W(t')|\delta_B(t')^{-1/2}\delta_{\mathcal{P}}(t')^{(\Re s+1)/4}\,\mathrm{d}t$$

is convergent for $\Re s\geq 1$ or $\Re s\geq -1$, respectively. Note that $\delta_B(t')=\delta_{B'}(t')\delta_{\mathcal{P}}(t')$ for $t'\in T'$. Since the integrals

$$\int_{F^\times}|a|^\delta\left|\log|a|\right|^k|\Phi(a)|\,\mathrm{d}a$$

are convergent for all $0<\delta\in\mathbb{R}$, $0\leq k\in\mathbb{Z}$ and $\Phi\in\mathscr{S}(F)$, Proposition 3.7 proves (5). The proof of (6) proceeds exactly as in that of Lemma 2 of [Kable 2004]. $\square$

Corollary 3.9. *Assume that F is nonarchimedean. Let π be an irreducible generic unitary representation of G and χ a unitary character of $F^\times$. Assume that χ is trivial if n is odd. Put $\varrho = \chi^\ell \omega_\pi$. If $\varrho^2 = 1$, then the following conditions are equivalent*:

(a) *$\pi \otimes \varrho$ is χ-distinguished*;

(b) $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho, \mathbb{C}) = \mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho), \mathbb{C})$;

(c) *the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ factors through the quotient*
$$\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho) \to \pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho.$$

Proof. Proposition 3.8(4)–(5) combined with Theorem 2.14(2) shows that the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ gives a basis vector in the one-dimensional vector space $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes I_\psi(1, \varrho), \mathbb{C})$. Since $\mathrm{Hom}_G(\pi \otimes \theta_\chi^\psi \otimes \theta^{\psi^{-1}} \otimes \varrho, \mathbb{C})$ is its subspace, the equivalence of the three conditions is evident. □

3D. *Good sections.*

Definition 3.10. Assume that ϱ is unitary. Let $s_0 \in \mathbb{C}$ and $f^{(s)}$ be a meromorphic section of $I_\psi(s, \varrho)$. When $\Re s_0 > -1$, we say that $f^{(s)}$ is good at $s = s_0$ if it is holomorphic at $s = s_0$. When $\Re s_0 < 0$, we say that $f^{(s)}$ is good at $s = s_0$ if $\hat{N}(s, \varrho) f^{(s)}$ is holomorphic at $s = s_0$. We call $f^{(s)}$ a good section if it is good at every point $s_0 \in \mathbb{C}$.

The following result can be proved in the same way as in the proof of Proposition 3.1 of [Yamana 2014] by utilizing Lemmas 3.2 and 3.5.

Proposition 3.11. (1) *Holomorphic sections are good sections.*

(2) *If $f^{(s)}$ is a good section of $I_\psi(s, \varrho)$, then $b(s, \varrho)^{-1} f^{(s)}$ is a holomorphic section.*

(3) *If $f^{(s)}$ is a meromorphic section which is good at $s = s_0$, then there is a good section $F^{(s)}$ such that $f^{(s)} - F^{(s)}$ has a zero of any prescribed order at $s = s_0$.*

(4) *Given a meromorphic section $f^{(s)}$ of $I_\psi(s, \varrho)$ the following conditions are equivalent*:

- *$f^{(s)}$ is a good section of $I_\psi(s, \varrho)$*;
- *$h^{(s)} = \hat{N}(-s, \varrho) f^{(-s)}$ is a good section of $I_{\psi^{-1}}(s, \varrho^{-1})$*;
- *there exist holomorphic sections $f_1^{(s)}$ of $I_\psi(s, \varrho)$ and $f_2^{(-s)}$ of $I_{\psi^{-1}}(-s, \varrho^{-1})$ such that*
$$f^{(s)} = f_1^{(s)} + \hat{N}_{\psi^{-1}}(-s, \varrho^{-1}) f_2^{(-s)}.$$

Definition 3.10 coincides in the strip $-1 < \Re s_0 < 0$ by Proposition 3.11(2).

3E. ***The twisted symmetric square L-factors.*** In Sections 3E–3G we will assume F to be nonarchimedean. Let π be an irreducible admissible generic representation of G. Suppose that χ is trivial if n is odd. Proposition 3.8(2) tells us that if $f^{(s)}$ is a standard section of $I_\psi(s, \chi^\ell \omega_\pi)$ multiplied by an element of $\mathbb{C}[q^{-s/4}, q^{s/4}]$ or a section obtained by applying the normalized intertwining operator to such a section of $I_{\psi^{-1}}(-s, \chi^{-\ell}\omega_\pi^{-1})$, then $Z(W, \Theta, f^{(2s-1)})$ is a rational function of $q^{-s/2}$. Let $\mathcal{I}(\pi, \chi)$ be the subspace of $\mathbb{C}(q^{-s/2})$ spanned by these local integrals. One can see from Propositions 3.8(2) and 3.11(2) that each such rational function can be written with a common denominator. That is, $\mathcal{I}(\pi, \chi)$ is a fractional $\mathbb{C}[q^{-s/2}, q^{s/2}]$-ideal. Proposition 3.8(4) shows that it contains 1. It is not difficult to see that $\mathcal{I}(\pi, \chi)$ is independent of the choice of ψ. With these properties of $\mathcal{I}(\pi, \chi)$ in hand, we can now define the twisted symmetric square L-factor.

Definition 3.12. The ideal $\mathcal{I}(\pi, \chi)$ has a unique generator of the form $Q_{\pi,\chi}(q^{-s/2})^{-1}$, where the polynomial $Q_{\pi,\chi}$ satisfies $Q_{\pi,\chi}(0) = 1$. We will define the twisted symmetric L-factor by $L(s, \pi, \mathrm{sym}^2 \otimes \chi) = Q_{\pi,\chi}(q^{-s/2})^{-1}$.

We expect that $Q_{\pi,\chi}(q^{-s/2})$ is a polynomial of q^{-s}. It may be worth noting the simple fact that $\delta_{\mathcal{P}}(t)^s$ is a power of q^{-2s} for $t \in \mathscr{T}$.

In other words, $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ is the minimal factor such that the ratios $Z(W, \Theta, f^{(2s-1)})/L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ are entire for all $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta_\chi^\psi$ and good sections $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$, simply because any holomorphic section can be expressed as a linear combination of standard sections with coefficients entire functions of s.

Remark 3.13. Recall that ν_k is the character of G_k defined by $\nu_k(g) = |\det g|$. Since $I_\psi(s, \varrho) \otimes \nu^y \simeq I_\psi(s + 4y, \varrho\nu_1^{-ny})$,

$$L(s, \pi \otimes \nu^y, \mathrm{sym}^2 \otimes \chi) = L(s + 2y, \pi, \mathrm{sym}^2 \otimes \chi)$$

for all $y \in \mathbb{C}$. If n is even and μ is a character of $F^\times$, then Lemma 1.9(3) implies

$$L(s, \pi \otimes \mu, \mathrm{sym}^2 \otimes \chi) = L(s, \pi, \mathrm{sym}^2 \otimes \chi\mu^2).$$

3F. ***Local functional equations.*** The need for normalizing $M(s, \chi^\ell \omega_\pi)$ and the need for including sections of the second type are clear from the following result:

Proposition 3.14. *Suppose that F is nonarchimedean. Let π be an irreducible admissible generic representation of G. We assume χ to be trivial if n is odd. Then there is a nowhere-vanishing entire function $\mathcal{E}(s, \pi, \chi, \psi)$ such that*

$$\frac{Z(W, \Theta, N(s, \chi^\ell \omega_\pi) f^{(s)})}{L\big(\tfrac{1}{2}(1-s), \pi^\vee, \mathrm{sym}^2 \otimes \chi^{-1}\big)} = \mathcal{E}\big(\tfrac{1}{2}(1+s), \pi, \chi, \psi\big) \frac{Z(W, \Theta, f^{(s)})}{L\big(\tfrac{1}{2}(1+s), \pi, \mathrm{sym}^2 \otimes \chi\big)}$$

for $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta_\chi^\psi$ and meromorphic sections $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$.

Proof. The generic uniqueness in Theorem 2.14(1) produces the functional equation above. It is well known that the contragredient representation $\pi^\vee$ of π is isomorphic to ${}^\iota\pi$, and we shall allow ourselves to confuse the two. The image of $\mathcal{W}^\psi(\pi)$ under the map $W \mapsto {}^\iota W$ is precisely the space $\mathcal{W}^{\psi^{-1}}(\pi^\vee)$. If $h^{(s)}$ is a meromorphic section of $J_\psi(s,\chi^\ell\omega_\pi)$, then

$$\begin{aligned} Z(W,\Theta,h^{(s)}) &= \int_{\mathscr{Z}N\backslash G} W({}^\iota g)\hat{Q}^\psi_\chi({}^\iota g,\Theta)\hat{R}^\psi_{s,\chi^\ell\omega_\pi}({}^\iota g,h^{(s)})\,\mathrm{d}g \\ &= \int_{\mathscr{Z}N\backslash G} {}^\iota W(g)Q^{\psi^{-1}}_{\chi^{-1}}(g,\iota^\psi_\chi(\Theta))R^{\psi^{-1}}_{s,\chi^{-\ell}\omega_\pi^{-1}}(g,J^\psi_{s,\chi^\ell\omega_\pi}({}^\iota h^{(s)}))\,\mathrm{d}g \\ &= Z({}^\iota W,\iota^\psi_\chi(\Theta),J^\psi_{s,\chi^\ell\omega_\pi}({}^\iota h^{(s)})) \end{aligned}$$

by the proof of Lemma 3.6. This combined with Proposition 3.11(4) shows that the ratios on both sides of the functional equation are holomorphic and nonzero everywhere on $\mathbb{C}$, and hence so is its factor of proportionality $\mathcal{E}\big(\tfrac{1}{2}(1+s),\pi,\chi,\psi\big)$. $\square$

3G. *Poles of the symmetric square L-factor and distinction.* We will continue to assume F to be nonarchimedean.

Lemma 3.15. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$. Assume that χ is trivial when n is odd.*

(1) *$L(s,\pi,\mathrm{sym}^2\otimes\chi)$ is holomorphic for $\Re s>0$.*

(2) *If $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has a pole at $s=0$, then $\chi^n\omega_\pi^2$ is trivial.*

(3) *$L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has at most a simple pole on $\Re s=0$.*

Proof. Recall that $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has the same poles as the family of local integrals $Z(W,\Theta,f^{(2s-1)})$ for good sections. Therefore the poles of $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ in $\Re s\geq 0$ are contained in the poles of good sections of $I_\psi(2s-1,\chi^\ell\omega_\pi)$ with multiplicity by Proposition 3.8(6). Our assertions now amount to the relevant analytic properties of $b(2s-1,\chi^\ell\omega_\pi)=L(ns,\chi^n\omega_\pi^2)$ in view of Proposition 3.11(2). $\square$

Lemma 3.16. (We keep the notation of Lemma 3.15.) *Assume that $\chi^n\omega_\pi^2=1$. Then there are $W\in\mathcal{W}^\psi(\pi)$, $\Theta\in\theta^\psi_\chi$ and a good section $f^{(s)}$ of $I_\psi(s,\chi^\ell\omega_\pi)$ such that*

$$M_\psi(1,\chi^\ell\omega_\pi)f^{(1)}=0,\qquad \lim_{s\to 1} Z(W,\Theta,N(s,\chi^\ell\omega_\pi)f^{(s)})\neq 0.$$

Proof. Proposition 3.8(4) enables us to choose $W\in\mathcal{W}^\psi(\pi)$, $\Theta\in\theta^\psi_\chi$ and a holomorphic section $h^{(s)}$ of $I_{\psi^{-1}}(s,\chi^\ell\omega_\pi)$ so that $Z({}^\iota W,\iota^\psi_\chi(\Theta),h^{(-1)})\neq 0$. Put $f^{(-s)}=\hat{N}_{\psi^{-1}}(s,\chi^\ell\omega_\pi)h^{(s)}$. Then $f^{(s)}$ is a good section in view of Proposition 3.11(1) and (4). Lemma 3.5 shows that

$$\begin{aligned}\lim_{s\to 1} M_\psi(s,\chi^\ell\omega_\pi)f^{(s)} &= \lim_{s\to -1} M_\psi(-s,\chi^\ell\omega_\pi)\hat{N}_{\psi^{-1}}(s,\chi^\ell\omega_\pi)h^{(s)}\\ &= \lim_{s\to -1}\alpha_{\psi^{-1}}(s,\chi^\ell\omega_\pi)\frac{a(-s,\chi^\ell\omega_\pi)}{b(s,\chi^\ell\omega_\pi)}{}^\iota\big((J^\psi_{s,\chi^\ell\omega_\pi})^{-1}(h^{(s)})\big)\\ &= 0\end{aligned}$$

and

$$\begin{aligned}&\lim_{s\to 1} Z(W,\Theta,N_\psi(s,\chi^\ell\omega_\pi)f^{(s)})\\ &\quad= \lim_{s\to -1} Z\big(W,\Theta,N_\psi(-s,\chi^\ell\omega_\pi)\hat{N}_{\psi^{-1}}(s,\chi^\ell\omega_\pi)h^{(s)}\big)\\ &\quad= \lim_{s\to -1}\alpha_{\psi^{-1}}(s,\chi^\ell\omega_\pi)Z\big(W,\Theta,{}^\iota\big((J^\psi_{s,\chi^\ell\omega_\pi})^{-1}(h^{(s)})\big)\big)\\ &\quad= \alpha_{\psi^{-1}}(-1,\chi^\ell\omega_\pi)Z({}^\iota W,\iota^\psi_\chi(\Theta),h^{(-1)})\\ &\quad\neq 0\end{aligned}$$

(see the proof of Proposition 3.14). □

Theorem 3.17. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$.*

(1) *Assume that n is even. Then $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ has a pole at $s=0$ if and only if π is χ-distinguished.*

(2) *Assume that n is odd. Then $L(s,\pi,\mathrm{sym}^2)$ has a pole at $s=0$ if and only if ω_π is quadratic and $\pi\otimes\omega_\pi$ is distinguished.*

Proof. First we shall prove the "only if" part, which, in view of Lemma 1.12, is equivalent to showing that π is χ-distinguished if $L(s,\pi^\vee,\mathrm{sym}^2\otimes\chi^{-1})$ has a pole at $s=0$. Then $\chi^n\omega_\pi^2$ is trivial by Lemma 3.15(2). In the case of odd n we may assume that ω_π is trivial at the cost of replacing π by $\pi\otimes\omega_\pi$ if necessary. If n is even, then $\theta^\psi_\chi\otimes\chi^\ell\omega_\pi\simeq\theta^\psi_\chi$ by Lemma 1.9(3). We get

$$Z(W,\Theta,M_\psi(1,\chi^\ell\omega_\pi)f^{(1)}) = cZ(W,\Theta,f^{(1)})$$

by evaluating the functional equation stated in Proposition 3.14 at $s=1$, where

$$c = 2\frac{a(1,\chi^\ell\omega_\pi)\mathcal{E}(1,\pi,\chi,\psi)\operatorname{Res}_{s=0}L(s,\pi^\vee,\mathrm{sym}^2\otimes\chi^{-1})}{L(1,\pi,\mathrm{sym}^2\otimes\chi)\operatorname{Res}_{s=-1}b(s,\chi^\ell\omega_\pi)}\neq 0.$$

Since the zeta integral is convergent by Proposition 3.8(6), the functional

$$W\otimes\Theta\otimes f\mapsto Z(W,\Theta,M_\psi(1,\chi^\ell\omega_\pi)f)$$

factors through the quotient

$$\pi\otimes\theta^\psi_\chi\otimes I_\psi(1,\chi^\ell\omega_\pi)\to\pi\otimes\theta^\psi_\chi\otimes\theta^{\psi^{-1}}$$

by Lemma 1.15, and hence so does $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$. Therefore π is χ-distinguished by Corollary 3.9.

Next suppose that $L(s, \pi, \mathrm{sym}^2 \otimes \chi)$ is holomorphic at $s = 0$ and that $\chi^n \omega_\pi^2$ is trivial. If we take $W \in \mathcal{W}^\psi(\pi)$, $\Theta \in \theta_\chi^\psi$ and a good section $f^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$ as in Lemma 3.16, then the functional equation in Proposition 3.14 shows that $Z(W, \Theta, f^{(1)}) \neq 0$. Thus the functional $W \otimes \Theta \otimes f \mapsto Z(W, \Theta, f)$ fails to factor through the quotient, and hence π cannot be χ-distinguished by Corollary 3.9. $\square$

3H. *Shahidi's symmetric square L-factor.* Let π be an irreducible admissible generic representation of G and χ a character of $F^\times$. We can define the twisted symmetric square L-factor by the Langlands–Shahidi method. We refer to [Shahidi 1990] for its precise definition. Henniart [2010] showed that this L-factor coincides with the Artin L-factor $L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)$, where ϕ denotes the local Langlands correspondence.

If F is a nonarchimedean local field of odd residual characteristic, π and χ are unramified and the order of ψ is 0, then there are a K-fixed Whittaker function $W^0 \in \mathcal{W}^\psi(\pi)$, a K-fixed semi-Whittaker function $\Theta^0 \in \theta_\chi^\psi$ and a K-fixed good section $f_0^{(s)}$ of $I_\psi(s, \chi^\ell \omega_\pi)$ such that

$$Z(W^0, \Theta^0, f_0^{(2s-1)}) = L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi), \tag{3-1}$$
$$Z\big(W^0, \Theta^0, N(2s-1, \chi^\ell \omega_\pi) f_0^{(2s-1)}\big) = L(1-s, \mathrm{sym}^2 \circ \phi(\pi^\vee) \otimes \chi^{-1})$$

by Theorem 4.1 and Proposition 5.6 of [Bump and Ginzburg 1992] (cf. [Takeda 2014]). Though our zeta integral is slightly different if n is even, one can easily see that the unramified computation of our integral is reduced to their computation.

Thus $L(s, \pi, \mathrm{sym}^2 \otimes \chi)^{-1}$ is divisible by $L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)^{-1}$ if π and χ are unramified. However, the coincidence of the two L-factors still remains open even in the unramified case. Nevertheless, we can prove that the two L-factors agree in the square-integrable case.

Theorem 3.18. *Suppose that F is nonarchimedean. Let π be an irreducible square-integrable representation of G and χ a character of $F^\times$. Suppose that χ is the trivial character if n is odd. Then*

$$L(s, \pi, \mathrm{sym}^2 \otimes \chi) = L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi).$$

Proof. We may assume that χ is unitary, taking Remark 3.13 into account. The proof is similar to those of [Kewat and Raghunathan 2012, Theorem 1.1] and [Kable 2004, Theorem 6]. Although the statement is purely local, its proof uses the global functional equations for both Shahidi's L-function and the Rankin–Selberg integrals.

Let p_0 be the residual characteristic of F and q the cardinality of the residue field of F. We can find a number field $\mathbb{F}$ which has a unique place v_0 lying over p_0

and such that the completion $\mathbb{F}_{v_0}$ of $\mathbb{F}$ at v_0 is isomorphic to F. By Lemma 6.5 of Chapter 1 of [Arthur and Clozel 1989] there is an irreducible cuspidal automorphic representation Π of $G(\mathbb{A})$ such that the local component Π_{v_0} of Π at v_0 is isomorphic to π, where $\mathbb{A}$ denotes the adèle ring of $\mathbb{F}$. Take a nontrivial additive character $\Psi : \mathbb{F}\backslash\mathbb{A} \to \mathbb{C}^\times$ and a Hecke character $\mathcal{X}$ of $\mathbb{A}^\times$ such that $\Psi_{v_0} = \psi$ and $\mathcal{X}_{v_0} = \chi$. We define the completed twisted symmetric square L-function by the infinite product

$$L(s, \Pi, \mathcal{X}, \mathrm{sym}^2) = \prod_v L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v).$$

The L-function $L(s, \Pi, \mathcal{X}, \mathrm{sym}^2)$ admits a meromorphic continuation to the entire complex plane and satisfies a functional equation

$$L(s, \Pi, \mathcal{X}, \mathrm{sym}^2) = \varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2) L(1-s, \Pi^\vee, \mathcal{X}^{-1}, \mathrm{sym}^2)$$

by Theorem 7.7 of [Shahidi 1990], where the function $\varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2)$ is entire and nonvanishing. The double cover $\overline{G}_\mathbb{A}$ of $G(\mathbb{A})$ and its global exceptional representation $\theta^\Psi_\mathcal{X}$ are constructed in [Kazhdan and Patterson 1984; Takeda 2014]. Note that $\overline{G}_\mathbb{A}$ is split over $G(F)$ and $\theta^\Psi_\mathcal{X}$ is an automorphic representation of $\overline{G}_\mathbb{A}$, which is isomorphic to the restricted tensor product $\bigotimes'_v \theta^{\Psi_v}_{\mathcal{X}_v}$. Let S_∞ be the set of archimedean places of $\mathbb{F}$ and S_r the set of finite places v for which Π_v or Ψ_v or $\theta^{\Psi_v}_{\mathcal{X}_v}$ is ramified. We set $S = S_\infty \cup S_r$.

We form the global induced representation and global intertwining operator. They have decompositions

$$I_\Psi(s, \mathcal{X}^\ell \omega_\Pi) \simeq \textstyle\bigotimes'_v I_{\Psi_v}(s, \mathcal{X}^\ell_v \omega_{\Pi_v}), \quad M(s, \mathcal{X}^\ell \omega_\Pi) = \bigotimes'_v M(s, \mathcal{X}^\ell_v \omega_{\Pi_v}).$$

The global functional equation of the completed Hecke L-function yields

$$M(s, \mathcal{X}^\ell \omega_\Pi) = \varepsilon\big(\tfrac{1}{2}n(s-1)+1, \mathcal{X}^n \omega^2_\Pi\big) \textstyle\bigotimes'_v N(s, \mathcal{X}^\ell_v \omega_{\Pi_v}). \tag{3-2}$$

For any holomorphic section $f^{(s)}$ of $I_\Psi(s, \mathcal{X}^\ell \omega_\Pi)$ we form the associated Eisenstein series $E(f^{(s)})$ on $G(F)\backslash\overline{G}_\mathbb{A}$ by

$$E(\tilde{g}, f^{(s)}) = \sum_{\gamma \in \mathcal{P}(F)\backslash G(F)} \sum_{\delta \in \mathscr{Z}\backslash Z(F)} f^{(s)}(\delta\gamma\tilde{g}),$$

where $\mathscr{Z} = \{z^{e(n)} \mid z \in Z(F)\}$. The series converges absolutely for $\Re s$ sufficiently large. By the theory of Eisenstein series, it can be continued to a meromorphic function on all of $\mathbb{C}$ satisfying the functional equation

$$E(f^{(s)}) = E(M(s, \mathcal{X}^\ell \omega_\Pi) f^{(s)}).$$

For $\varphi \in \Pi$, $\mathbf{\Theta} \in \theta^{\Psi}_{\mathcal{X}}$ and a meromorphic section $\boldsymbol{f}^{(s)}$ of $I_{\Psi}(s, \mathcal{X}^{\ell}\omega_{\Pi})$ we can consider the global zeta integral defined by

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = \int_{\mathscr{Z}_{\mathbb{A}}\boldsymbol{G}(F)\backslash \boldsymbol{G}(\mathbb{A})} \varphi(g)\mathbf{\Theta}(g)E(g, \boldsymbol{f}^{(s)})\,\mathrm{d}g,$$

where $\mathscr{Z}_{\mathbb{A}} = \{z^{\boldsymbol{e}(\boldsymbol{n})} \mid z \in Z(\mathbb{A})\}$. This integral converges absolutely for all s away from the poles of the Eisenstein series and defines a meromorphic function in s satisfying

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = Z(\varphi, \mathbf{\Theta}, M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}).$$

The $\psi_{\boldsymbol{n}}$-Whittaker coefficient of φ and the semi-Whittaker coefficients of $\mathbf{\Theta}$ and $\boldsymbol{f}^{(s)}$ are defined by

$$W^{\psi}(g, \varphi) = \int_{N(F)\backslash N(\mathbb{A})} \varphi(ug)\overline{\psi(u)}\,\mathrm{d}u,$$

$$Q^{\psi}(\tilde{g}, \mathbf{\Theta}) = \int_{N(F)\backslash N(\mathbb{A})} \mathbf{\Theta}(\boldsymbol{s}(u)\tilde{g})\psi_{\boldsymbol{e}}(u)\,\mathrm{d}u,$$

$$R^{\psi}(\tilde{g}, \boldsymbol{f}^{(s)}) = \int_{N(F)\backslash N(\mathbb{A})} \boldsymbol{f}^{(s)}(\boldsymbol{s}(u)\tilde{g})\psi_{\boldsymbol{o}}(u)\,\mathrm{d}u.$$

In the case of even n the Rankin–Selberg integral differs slightly from those considered by Bump and Ginzburg [1992] or by Takeda [2014], but it can be unfolded to an adelic integral of the product of $W^{\psi}(\varphi)$, $Q^{\psi}(\mathbf{\Theta})$ and $R^{\psi}(\boldsymbol{f}^{(s)})$ in the same manner as in [Bump and Ginzburg 1992]. If $W^{\psi}(\varphi) = \bigotimes_v W_v$, $\mathbf{\Theta} = \bigotimes_v \Theta_v$ and $\boldsymbol{f}^{(s)} = \bigotimes_v f_v^{(s)}$ are factorizable, then

$$Z(\varphi, \mathbf{\Theta}, \boldsymbol{f}^{(s)}) = \prod_v Z(W_v, \Theta_v, f_v^{(s)}),$$

$$Z(\varphi, \mathbf{\Theta}, M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}) = \prod_v Z(W_v, \Theta_v, M(s, \mathcal{X}_v^{\ell}\omega_{\Pi_v})f_v^{(s)}).$$

The first factorization was proved by the author and Eyal Kaplan [Kaplan and Yamana 2016]. We here prove the second one. Put $\boldsymbol{h}^{(-s)} = M(s, \mathcal{X}^{\ell}\omega_{\Pi})\boldsymbol{f}^{(s)}$. Unfolding the Eisenstein series, we have

$$\begin{aligned} Z(\varphi, \mathbf{\Theta}, \boldsymbol{h}^{(-s)}) &= \int_{\mathscr{Z}_{\mathbb{A}}{}^{\iota}\mathscr{P}(F)\backslash \boldsymbol{G}(\mathbb{A})} \varphi(g)\Theta(g)\boldsymbol{h}^{(-s)}(g)\,\mathrm{d}g \\ &= \int_{\mathscr{Z}_{\mathbb{A}}\mathscr{P}(F)\backslash \boldsymbol{G}(\mathbb{A})} \varphi({}^{\iota}g)\Theta({}^{\iota}g)\boldsymbol{h}^{(-s)}({}^{\iota}g)\,\mathrm{d}g. \end{aligned}$$

Substituting the Fourier expansion

$$\varphi({}^{\iota}g) = {}^{\iota}\varphi(g) = \sum_{\gamma \in N(F)\backslash \mathscr{P}(F)} W^{\psi^{-1}}(\gamma g, {}^{\iota}\varphi) = \sum_{\gamma \in N(F)\backslash {}^{\iota}\mathscr{P}(F)} W^{\psi}(\gamma\, {}^{\iota}g, \varphi),$$

we get

$$Z(\varphi, \boldsymbol{\Theta}, \boldsymbol{h}^{(-s)}) = \int_{\mathscr{Z}_{\mathbb{A}} N(F)\backslash G(\mathbb{A})} W^{\psi}({}^{\iota}g, \varphi)\Theta({}^{\iota}g)\boldsymbol{h}^{(-s)}({}^{\iota}g)\,\mathrm{d}g,$$

where our formal manipulations can be justified by the absolute convergence of this integral for $\Re s \ll 0$, which can be checked by a gauge estimate. For $i = 1, 2, \dots, n-1$ we put

$$\mathscr{U}^{(i)} = \left\{ \begin{pmatrix} \mathbf{1}_{n-i} & b \\ 0 & u \end{pmatrix} \,\middle|\, b \in \mathrm{M}_{n-i,i},\ u \in N_i \right\}.$$

Proposition 1.3 enables us to lift the main involution of $G(\mathbb{A})$ to $\bar{G}_{\mathbb{A}}$. For $i = 1, 2, \dots, n-1$ we define

$$Q_i(\tilde{g}) = \int_{\mathscr{U}^{(i)}(F)\backslash\mathscr{U}^{(i)}(\mathbb{A})} {}^{\iota}\Theta(\boldsymbol{s}(u)\tilde{g})\overline{\psi_{\boldsymbol{e}}(u)}\,\mathrm{d}u,$$

$$R_i(\tilde{g}, -s) = \int_{\mathscr{U}^{(i)}(F)\backslash\mathscr{U}^{(i)}(\mathbb{A})} {}^{\iota}\boldsymbol{h}^{(-s)}(\boldsymbol{s}(u)\tilde{g})\overline{\psi_{\boldsymbol{o}}(u)}\,\mathrm{d}u,$$

$$Z_i(\varphi, \Theta, \boldsymbol{h}^{-s}) = \int_{\mathscr{Z}_{\mathbb{A}} N(F)\mathscr{U}^{(i)}(\mathbb{A})\backslash G(\mathbb{A})} {}^{\iota}W^{\psi}(g, \varphi)Q_i(g)R_i(g, -s)\,\mathrm{d}g.$$

Let $\mathscr{N}_i$ be the subgroup of N consisting of matrices whose only nonzero off-diagonal elements are in the $(n-i)$-th column. When i is odd, Propositions 2.4 and 2.5 of [Bump and Ginzburg 1992] and Lemma 3.11 of [Takeda 2014] state that $Q_i(\boldsymbol{s}(u)\tilde{g})$ is independent of $u \in \mathscr{N}_i(\mathbb{A})$ and equal to $Q_{i+1}(\tilde{g})$, and hence

$$\begin{aligned} Z_i(\varphi, \Theta, \boldsymbol{h}^{(-s)}) &= \int_{\mathscr{Z}_{\mathbb{A}} N(F)(\mathbb{A})\mathscr{U}^{(i+1)}(\mathbb{A})\backslash G(\mathbb{A})} W^{\psi}(g, \varphi)Q_{i+1}(g) \\ &\qquad\qquad \times \int_{\mathscr{N}_i(F)\backslash\mathscr{N}_i(\mathbb{A})} R_i(\boldsymbol{s}(u)g, -s)\overline{\psi(u)}\,\mathrm{d}u\,\mathrm{d}g \\ &= Z_{i+1}(\varphi, \Theta, \boldsymbol{h}^{(-s)}). \end{aligned}$$

When i is even, Propositions 2.4 and 2.5 of [Bump and Ginzburg 1992] and Lemma 3.11 of [Takeda 2014] again imply that $Z_i(\varphi, \Theta, h^{(-s)}) = Z_{i+1}(\varphi, \Theta, h^{(-s)})$. Consequently,

$$\begin{aligned} Z(\varphi, \Theta, \boldsymbol{h}^{(-s)}) &= Z_1(\varphi, \Theta, \boldsymbol{h}^{(-s)}) = \cdots = Z_{n-1}(\varphi, \Theta, \boldsymbol{h}^{(-s)}) \\ &= \int_{\mathscr{Z}_{\mathbb{A}} N(\mathbb{A})\backslash G(\mathbb{A})} W^{\psi}(g, \varphi)Q^{\psi^{-1}}({}^{\iota}g, {}^{\iota}\Theta)R^{\psi^{-1}}({}^{\iota}g, {}^{\iota}\boldsymbol{h}^{(-s)})\,\mathrm{d}g. \end{aligned}$$

Since the semi-Whittaker function of Θ is the Whittaker function of the $M_{\boldsymbol{e}(n)}$-part of the constant term of Θ along $P_{\boldsymbol{e}(n)}$, one can verify that it is factorizable, and similarly for $\boldsymbol{h}^{(-s)}$, which gives rise to the factorization we want.

There are $W_i \in \mathcal{W}^{\psi}(\pi)$, $\Theta_i \in \theta_{\chi}^{\psi}$ and good sections $f_i^{(s)}$ such that

$$\sum_i Z(W_i, \Theta_i, f_i^{(2s-1)}) = L(s, \pi, \mathrm{sym}^2 \otimes \chi).$$

On substituting each of these triplets into the functional equation in Proposition 3.14 and summing the results, we find that

$$\sum_i Z\big(W_i, \Theta_i, N(2s-1, \chi^{\ell}\omega_{\pi}) f_i^{(2s-1)}\big) = \mathcal{E}(s, \pi, \chi, \psi) L(1-s, \pi^{\vee}, \mathrm{sym}^2 \otimes \chi^{-1}). \tag{3-3}$$

For $v \in S \setminus \{v_0\}$ we choose $W_v \in \mathcal{W}^{\Psi_v}(\Pi_v)$, $\Theta_v \in \theta_{\mathcal{X}_v}^{\Psi_v}$ and standard sections $f_v^{(s)}$ such that $Z(W_v, \Theta_v, f_v^{(s)})$ is not identically zero. Put

$$\boldsymbol{W}_i = W_i \otimes \Big(\bigotimes_{v \in S \setminus \{v_0\}} W_v\Big) \otimes \Big(\bigotimes_{v \notin S} W_v^0\Big),$$

$$\boldsymbol{\Theta}_i = \Theta_i \otimes \Big(\bigotimes_{v \in S \setminus \{v_0\}} \Theta_v\Big) \otimes \Big(\bigotimes_{v \notin S} \Theta_v^0\Big),$$

$$\boldsymbol{f}_i^{(s)} = f_i^{(s)} \otimes \Big(\bigotimes_{v \in S \setminus \{v_0\}} f_v^{(s)}\Big) \otimes \Big(\bigotimes_{v \notin S} f_{v,0}^{(s)}\Big).$$

Further set

$$A(s) = L(s, \Pi, \mathcal{X}, \mathrm{sym}^2)^{-1} \sum_i Z(\boldsymbol{W}_i, \boldsymbol{\Theta}_i, \boldsymbol{f}_i^{(2s-1)}) = a(s)\alpha(s)a(s, \pi, \chi),$$

where

$$a(s, \pi, \chi) = L(s, \pi, \mathrm{sym}^2 \otimes \chi)/L(s, \mathrm{sym}^2 \circ \phi(\pi) \otimes \chi)$$

and

$$a(s) = \prod_{v \in S_r \setminus \{v_0\}} \frac{Z(W_v, \Theta_v, f_v^{(2s-1)})}{L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v)},$$

$$\alpha(s) = \prod_{v \in S_\infty} \frac{Z(W_v, \Theta_v, f_v^{(2s-1)})}{L(s, \mathrm{sym}^2 \circ \phi(\Pi_v) \otimes \mathcal{X}_v)}.$$

Similarly, we put

$$B(s) = L(s, \Pi^{\vee}, \mathcal{X}^{-1}, \mathrm{sym}^2)^{-1} \sum_i Z\big(\boldsymbol{W}_i, \boldsymbol{\Theta}_i, M(1-2s, \mathcal{X}^{\ell}\omega_{\Pi}) \boldsymbol{f}_i^{(1-2s)}\big).$$

Note that

$$B(s) = \varepsilon(1-ns, \mathcal{X}^n \omega_{\Pi}^2) \mathcal{E}(1-s, \pi, \chi, \psi) b(s) \beta(s) a(s, \pi^{\vee}, \chi^{-1})$$

by (3-2) and (3-3), where $b(s)$ (resp. $\beta(s)$) is a product of the ratios

$$Z\big(W_v, \Theta_v, N(1-2s, \mathcal{X}_v^{\ell}\omega_{\Pi_v}) f_v^{(1-2s)}\big)/L\big(s, \mathrm{sym}^2 \circ \phi(\Pi_v^{\vee}) \otimes \mathcal{X}_v^{-1}\big)$$

over $v \in S_r \setminus \{v_0\}$ (resp. $v \in S_\infty$). Plugging the functional equation of Shahidi's L-function into the functional equation of the global zeta integral, we are led to

$$\varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2) A(s) = B(1-s),$$

that is,

$$\begin{aligned}(3\text{-}4)\quad & \varepsilon(s, \Pi, \mathcal{X}, \mathrm{sym}^2) a(s) \alpha(s) a(s, \pi, \chi) \\ &= \mathcal{E}(s, \pi, \chi, \psi) \varepsilon(n(s-1)+1, \mathcal{X}^n \omega_\Pi^2) b(1-s) \beta(1-s) a(1-s, \pi^\vee, \chi^{-1}).\end{aligned}$$

To prove Theorem 3.18, it is enough to prove that $a(s, \pi, \chi)$ is entire and nowhere vanishing. First suppose that $a(s, \pi, \chi)$ has a zero at $s = s_0$. This means that $a(s, \pi, \chi)$ has zeros at $s_0 + k(2\pi\sqrt{-1})/\log q$ for all $k \in \mathbb{Z}$. We claim that all but finitely many of these zeros must also be zeros of $A(s)$. This fails to happen only if all but finitely many zeros are canceled by the poles of $a(s)\alpha(s)$. The function $\alpha(s)$ can contribute only finitely many poles on any line with real part constant by Proposition 3.8(3), and this set of poles is independent of the choice of W_v, Θ_v and $f_v^{(s)}$ at the archimedean places. Hence $a(s)$ must have infinitely many poles of this form. Since the poles of $a(s)$ are of the form $s_j + m(4\pi\sqrt{-1})/\log q_v$ for $m \in \mathbb{Z}$ with $v \in S_r \setminus \{v_0\}$, there are a place v and $s_j \in \mathbb{C}$ and two integers $m_1 \neq m_2$ such that

$$s_0 + k_1 \frac{2\pi\sqrt{-1}}{\log q} = s_j + m_1 \frac{4\pi\sqrt{-1}}{\log q_v}, \quad s_0 + k_2 \frac{2\pi\sqrt{-1}}{\log q} = s_j + m_2 \frac{4\pi\sqrt{-1}}{\log q_v}$$

for some $k_1, k_2 \in \mathbb{Z}$ (in fact, there are infinitely many distinct integers with this property). Then $\log q_v/\log q$ is rational, which contradicts $(q_v, q) = 1$. Thus the points $s_0 + k(2\pi\sqrt{-1})/\log q$ are zeros of $A(s)$ for all but finitely many k.

Since $L(s, \mathrm{sym} \circ \phi(\pi) \otimes \chi)$ is holomorphic in the region $\Re s > 0$ by Proposition 7.2 of [Shahidi 1990], the function $a(s, \pi, \chi)$ is nonvanishing in the region $\Re s > 0$. Thus $\Re s_0 \leq 0$. From (3-4) we see that all but finitely many of the points $1 - s_0 + k(2\pi\sqrt{-1})/\log q$ are zeros of the function $B(s)$. Since $a(s, \pi^\vee, \chi^{-1})$ is nonzero for $\Re s > 0$, these zeros have to be the zeros of $b(s)\beta(s)$. Arguing as above, these cannot be zeros of $b(s)$ for infinitely many k. Since the poles of

$$\prod_{v \in S_\infty} L(s, \mathrm{sym} \circ \phi(\Pi_v^\vee) \otimes \mathcal{X}_v^{-1})$$

lie along horizontal lines, this product can contribute only finitely many poles on any vertical line. Thus these must be common zeros of functions

$$\prod_{v \in S_\infty} Z\big(W_v, \Theta_v, N(1-2s, \mathcal{X}_v^\ell \omega_{\Pi_v}) f_v^{(1-2s)}\big)$$

for all W_v, Θ_v and $f_v^{(s)}$. This contradicts Proposition 3.8(4) in view of the proof of Proposition 3.14.

Suppose that $a(s,\pi,\chi)$ has a pole at $s=s_0$. Since $L(s,\pi,\mathrm{sym}^2\otimes\chi)$ is holomorphic in the region $\Re s>0$ by Lemma 3.15(1), we obtain $\Re s_0\leq 0$. By Proposition 3.8(5) the product $a(s)\alpha(s)$ is holomorphic in $\Re s\geq 1$ and the function $b(1-s)\beta(1-s)$ is holomorphic in $\Re s\leq 0$. Therefore $A(s)$ is holomorphic in $\Re s\geq 1$ and $\Re s\leq 0$ by (3-4), so that the pole of $a(s,\pi,\chi)$ must be canceled by the zeros of $a(s)\alpha(s)$. Arguing as above, we can see that $s_0+k(4\pi\sqrt{-1})/\log q$ cannot be zeros of $a(s)$ for infinitely many integers k. Since the poles of

$$\prod_{v\in S_\infty} L(s,\mathrm{sym}^2\circ\phi(\Pi_v)\otimes\mathcal{X}_v)$$

lie along horizontal lines, this product can contribute only finitely many poles on any vertical line. Thus these must be common zeros of functions

$$\prod_{v\in S_\infty} Z(W_v,\Theta_v,f_v^{(2s-1)})$$

for all W_v, Θ_v and $f_v^{(s)}$, which contradicts Proposition 3.8(4). ☐

3I. *Proof of Theorem A and Corollary A*.

Theorem 3.19. *Let π be an irreducible square-integrable representation of G and χ a unitary character of $F^\times$.*

(1) *Assume that n is even. Then $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$ if and only if π is χ-distinguished.*

(2) *Assume that n is odd. Then $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$ if and only if $\omega_\pi^2=\chi^{-n}$ and $\pi\otimes(\omega_\pi^{-1}\chi^{-(n-1)/2})$ is distinguished.*

Proof. Theorems 3.17 and 3.18 prove the first part. The factorization (0-1) is extended to the twisted case as follows:

$$L(s,\phi(\pi)\otimes\phi(\pi)\otimes\chi)=L(s,\Lambda^2\circ\phi(\pi)\otimes\chi)L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi).$$

It is a consequence of Proposition 8.1 and Theorem 8.2 of [Jacquet et al. 1983] that $L(s,\phi(\pi)\otimes\phi(\pi)\otimes\chi)$ has a simple pole at $s=0$ exactly when $\pi\simeq\pi^\vee\otimes\chi^{-1}$.

Suppose that n is odd. If $L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)$ has a pole at $s=0$, then $\pi\simeq\pi^\vee\otimes\chi^{-1}$ and hence $\omega_\pi^2=\chi^{-n}$. Put $\mu=\omega_\pi\chi^{(n-1)/2}$ and $\pi'=\pi\otimes\mu^{-1}$. If $\omega_\pi^2=\chi^{-n}$, then $\mu^2=\chi^{-1}$, $\omega_{\pi'}=\omega_\pi\mu^{-n}=\omega_\pi\chi^{(n-1)/2}\mu^{-1}=1$ and

$$L(s,\mathrm{sym}^2\circ\phi(\pi)\otimes\chi)=L(s,\mathrm{sym}^2\circ\phi(\pi'))=L(s,\pi',\mathrm{sym}^2).$$

The equivalence now amounts to a combination of Theorems 3.17 and 3.18. ☐

When $\pi\simeq\pi^\vee$, one of the L-factors on the right-hand side of the factorization (0-1) must have a pole at $s=0$, and the other does not.

If n is odd or ω is nontrivial, then $L(s, \Lambda^2 \circ \phi(\pi))$ cannot have a pole at $s = 0$ by Theorems 4.3 and 6.1 of [Kewat and Raghunathan 2012], so that $\pi \simeq \pi^\vee$ if and only if $L(s, \mathrm{sym}^2 \circ \phi(\pi))$ has a pole at $s = 0$. Thus Lemma 1.12(1) and Theorem 3.17(2) prove Corollary A(1). Theorem 3.19(1) proves Corollary A(2).

Assume that n is even and ω is trivial. Then $L(s, \Lambda^2 \circ \phi(\pi))$ has a pole at $s = 0$ if and only if π admits a nontrivial Shalika model by Proposition 3.4 of [Lapid and Mao 2017]. This combined with Theorem 3.19(1) proves Corollary A(3).

Remark 3.20. In the proof of Corollary A we limit ourselves to the nontwisted case even when n is even, because of the lack of knowledge of suitable generalizations of the results for the twisted exterior square L-factors.

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