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Boundedness for a maximal modified Bochner–Riesz operator between weighted L^2 spaces is proved. As a consequence, we have sufficient conditions for a.e. convergence of the modified Bochner–Riesz means at the critical exponent $p_\lambda = 2n/(n - 2\lambda - 1)$.

1. Introduction

This paper contains the results proved in the author's doctoral dissertation [Annoni 2010] and referenced by S. Lee and A. Seeger [2015], but yet unpublished in a mathematical journal. For $\lambda, R > 0$, let B_R^λ denote the Bochner–Riesz operators and m_λ the Fourier multipliers introduced in [Bochner 1936]:

$$B_R^\lambda(f)(x) = \int_{\mathbb{R}^n} \hat{f}(\xi) m_\lambda\left(\frac{|\xi|}{R}\right) e^{2\pi i \xi x}\, d\xi, \quad m_\lambda(t) = (1 - t^2)_+^\lambda.$$

For $p < 2$, results related to almost everywhere convergence and maximal operators have been proved by Tao [1998; 2002], Ashurov [1983], and Ahmedov, Ashurov, and Mahmud [Ashurov et al. 2010]. For $p \geq 2$, partial results on almost everywhere convergence of $B_R^\lambda(f)$ to f as $R \to \infty$ have been achieved in [Carbery 1983; Christ 1985]. Carbery, Rubio de Francia, and Vega [Carbery et al. 1988] obtained a.e. convergence in the range $2 \leq p < p_\lambda$ and $\lambda > 0$.

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In this paper, the situation at the critical exponent $p_\lambda = 2n/(n-2\lambda-1)$ is studied by considering the modified Bochner–Riesz multipliers $m_{\lambda,\gamma}$

$$m_{\lambda,\gamma}(t) = \frac{(1-t^2)_+^\lambda}{(1-\log(1-t^2))^\gamma},$$

which were introduced by Seeger [1987]. Seeger [1996] showed that $m_{\lambda,\gamma}$ is an $L^{p_\lambda}(\mathbb{R}^2)$ multiplier for $\gamma > 1/p'_\lambda$ (where $1/p_\lambda + 1/p'_\lambda = 1$). His results easily extend to dimensions $n \geq 3$ when $\lambda \geq (n-1)/(2(n+1))$ and had already been proven to be sharp in [Seeger 1987] when $n = 2$.

In order to investigate for which values of γ the means $B_R^{\lambda,\gamma}$ defined via $m_{\lambda,\gamma}$ converge a.e. for functions in L^{p_λ}, we study the maximal operator $B_*^{\lambda,\gamma}$. The following theorem is my main result.

Theorem 1.1. *Let $1 < 1+2\lambda < n$ and $0 \leq \mu < 2\gamma - 2$. Then there is a constant $C = C(n, \lambda, \gamma, \mu)$ such that*

$$\int_{\mathbb{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^2\, dx \tag{1}$$

for all $f \in L^2(\mathbb{R}^n, dx)$ and

$$\int_{\mathbb{R}^n} |B_*^{\lambda,\gamma}(f)(x)|^2 w_{\lambda,\mu}(x)\, dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 w_{\lambda,\mu}(x)\, dx \tag{2}$$

for all $f \in L^2(\mathbb{R}^n, w_{\lambda,\mu}(x)\, dx)$, where $w_{\lambda,\mu} = \omega_{\lambda,\mu}(|x|)$ and

$$\omega_{\lambda,\mu}(t) = \begin{cases} \dfrac{1}{t^{2\lambda+1}} & \text{if } 0 < t \leq 1, \\ \dfrac{1}{t^{2\lambda+1}(\log(et))^\mu} & \text{if } t > 1. \end{cases} \tag{3}$$

For $(2\lambda+1)/n < \mu$, we also have $L^{p_\lambda} \subseteq L^2 + L^2(w_{\lambda,\mu})$. Hence:

Corollary 1.2. *If $1 < 1+2\lambda < n$, $f \in L^{p_\lambda}(\mathbb{R}^n)$, and $\gamma > 1/p'_\lambda + 1/2$, we have*

$$\lim_{R\to\infty} B_R^{\lambda,\gamma}(f)(x) = f(x) \tag{4}$$

for almost every $x \in \mathbb{R}^n$. If $f \in L^p(\mathbb{R}^n)$ for $2 \leq p < p_\lambda$, then the condition $\gamma \geq 0$ suffices for (4) *to hold.*

When I first proved this result, it was natural to wonder whether the condition $\gamma > 1/p'_\lambda + 1/2$ was sharp. Lee, Rogers, and Seeger [Lee et al. 2014] have since proved among other things that, if

$$\frac{2(n+1)}{n-1} < p < \infty, \quad n \geq 2,$$

and $m \in B^2_{\alpha,q}$, then the maximal operator

$$M_m(f) := \sup_{t>0}\left|\left(\widehat{f}\, m(t|\cdot|)\right)^{\vee}\right|$$

is bounded from $L^{p,q'}$ to L^p. This can be applied to $m = m_{\lambda,\gamma}$ to conclude that the condition $\gamma > 1/p'_\lambda + 1/2$ in Corollary 1.2 can be replaced by $\gamma > 1/p'_\lambda$, if we further assume $(n-1)/(2(n+1)) < \lambda$.

Lee and Seeger [2015] have gone much further, proving that a.e. convergence of

$$S_t(f) := \left(\widehat{f}\, m_{\lambda,\gamma} \circ \rho(t(\cdot))\right)^{\vee}$$

to f (where ρ is an arbitrary homogeneous "distance" function, that is a homogeneous function that satisfies $\rho(\xi) > 0$ if $\xi \in \mathbb{R}^n \setminus \{0\}$ and $\rho(0) = 0$) holds for every $f \in L^{p_\lambda,q}$ when $q \geq 1$ if and only if $\gamma > 1/q'$, for all $0 < \lambda < (n-1)/2$. For $q = p_\lambda$ and $\rho(\xi) = |\xi|$, this implies Corollary 1.2. In particular, they proved that the condition $\gamma > 1/q'$ is sharp.

The sufficiency of the condition $\gamma > 1/q'$ in [Lee and Seeger 2015] is presented as a consequence of a boundedness estimate between appropriate homogeneous Herz spaces — see [Baernstein and Sawyer 1985; Gilbert 1972] — of a maximal operator defined via an arbitrary quasiradial multiplier $h \circ \rho$, provided that h lies in an appropriate Besov space. A particular case of the same theorem also implies a characterization of boundedness for certain convolution operators on L^2 spaces that are weighted with power weights. In order to prove the sufficiency of the condition on γ, both of our papers use the approach of [Carbery et al. 1988], to some extent. However, much of my work is necessary to deal with the weight $w_{\lambda,\mu}$, that isn't homogenous. The first choice of Lee and Seeger was to keep working with a homogeneous weight, but to use the observation that, for $p > 2$, the space $L^{p,2}$ is embedded in $L^2(|x|^{-n(1-2/p)}\,dx)$. By sharpening the analysis in [loc. cit.], this idea would only have yielded their result for $q = 2$. They solved the problem for all q by using Herz spaces, embedding theorems, and innovations that were needed to work with a more general "distance" function ρ and multiplier h.

The necessity of the condition $\gamma > 1/q'$ starts with the reminder that the operators S_t $(t > 0)$ are naturally defined on the Schwartz class $\mathcal{S}$ and extended on bigger spaces by using density. So, they proved that each operator S_t is continuous from $\mathcal{S}$ — equipped with the $L^{p_\lambda,q}$ norm and topology — to $\mathcal{S}'$ only if $\gamma > 1/q'$.

This paper. The proof of Theorem 1.1 follows closely the idea developed in [Carbery et al. 1988], but accounts for the necessity to work with nonhomogeneous weights.

In Section 2, Theorem 1.1 is reduced to Lemma 2.1, which is in turn reduced to Lemma 5.2 in Section 5. Lemma 5.2 is proved in Section 6.

In Section 3, an upper bound is given for the Fourier transform of $w^{(1)}_{\lambda,\mu}$, which is $w_{\lambda,\mu}$ smoothened in a neighborhood of the spherical surface $\|x\| = 1$. An analytic continuation argument is needed to prove that the upper bound holds for all $0 < \lambda < (n-1)/2$. This upper bound will be used to prove Lemma 5.2.

In Section 4, a new weight $\widetilde{w}_{N,\lambda,\mu}$ is exhibited that is comparable to $1/w_{\lambda,\mu}$ and that has an algebraic form needed in the computations of Section 5.

In Section 5, Lemma 2.1 is reduced to Lemma 5.2. Lemma 5.2 contains weighted Fourier inequalities for the special weight used in this paper. It is crucial that the "constants" appearing in both such inequalities have a certain functional form with respect to the parameter t. So, general results such as those in [Benedetto and Heinig 2003] were not sufficient.

Section 6 contains the proofs of Lemma 5.2 and Corollary 1.2.

We shall refer to [Carbery et al. 1988] for every piece of the proof that doesn't differ significantly. Yet, the reader can find more details of the proof contained in that reference in [Grafakos 2014, Subsection 10.5.2].

2. Reduction of Theorem 1.1 to Lemma 2.1

We will only need to show (2), as the proof of (1) is contained in [Grafakos 2014] for the case $\gamma = 0$ (which implies it for all $\gamma \geq 0$). Let φ, ψ be smooth functions, supported in $\left[-\frac{1}{2}, \frac{1}{2}\right]$ and $\left[\frac{1}{8}, \frac{5}{8}\right]$ respectively, with values in [0, 1], that satisfy

$$\varphi(t) + \sum_{k=0}^{\infty} \psi\left(\frac{1-t}{2^{-k}}\right) = 1$$

for all $t \in [0, 1)$. Let $e_1 = (1, 0, \dots, 0) \in \mathbb{R}^n$. We define $m_{\lambda,\gamma,00}(t) = m_{\lambda,\gamma}(te_1)\,\varphi(t)$ and

$$m_{\lambda,\gamma,k}(t) = 2^{k\lambda} m_{\lambda,\gamma}(te_1)\,\psi\left(\frac{1-t}{2^{-k}}\right), \quad k = 0, 1, 2, \dots \tag{5}$$

We define $\widetilde{m}_{\lambda,\gamma,k}$, $(S_{\lambda,\gamma,k})_t$, $(S_{\lambda,\gamma,k})_*$, and $G_{\lambda,\gamma,k}$ from $m_{\lambda,\gamma,k}$, analogous to how $\widetilde{m}^\delta$, S^δ_t, S^δ_*, and G^δ were defined from m^δ in [Carbery et al. 1988]. Similarly, we also define $(\widetilde{S}_{\lambda,\gamma,k})_t$, $(\widetilde{S}_{\lambda,\gamma,k})_*$, and $\widetilde{G}_{\lambda,\gamma,k}$ by using $\widetilde{m}_{\lambda,\gamma,k}$ instead of $m_{\lambda,\gamma,k}$. For $m_{\lambda,\gamma,k}$ we have the estimate

$$\sup_{0 \leq t \leq 1} \left| \frac{d^\ell}{dt^\ell} m_{\lambda,\gamma,k}(t) \right| \leq C_{\lambda,\gamma,\ell}\, \frac{2^{k\ell}}{k^\gamma} \tag{6}$$

for all $\ell \in \mathbb{Z}^+ \cup \{0\}$. As in [loc. cit.], these inequalities follow:

$$\|B^{\lambda,\gamma}_*\| \leq \|(S_{\lambda,\gamma,00})_*\| + \sum_{k=0}^{\infty} 2^{-k\lambda} \|(S_{\lambda,\gamma,k})_*\|, \tag{7}$$

$$\left\|(S_{\lambda,\gamma,k})_*(f)\right\|^2_{L^2(w_{\lambda,\mu})} \leq 2^{k+1} \left\|G_{\lambda,\gamma,k}(f)\right\|_{L^2(w_{\lambda,\mu})} \left\|\widetilde{G}_{\lambda,\gamma,k}(f)\right\|_{L^2(w_{\lambda,\mu})}. \tag{8}$$

By reasoning as in [loc. cit.], one then shows without difficulty that the right-hand side in (8) can be controlled by the left-hand side of the inequality in the result we are about to state:

Lemma 2.1. *For $k > 4$ we have*

$$\int_{\mathbb{R}^n} \int_1^2 |(S_{\lambda,\gamma,k})_{a\,t}(f)(x)|^2 \, \frac{dt}{t} \, w_{\lambda,\mu}(x)\, dx \le C_{n,\lambda,\mu,\gamma,k} \int_{\mathbb{R}^n} |f(x)|^2 w_{\lambda,\mu}(x)\, dx$$

for all $a > 0$ and for all functions f in $L^2(w_{\lambda,\mu})$, with

$$C_{n,\lambda,\mu,\gamma,k} = C_{n,\lambda,\mu,\gamma} \frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}.$$

We need not to worry about $k \le 4$ because it is easily verified that $w_{\lambda,\mu}$ is an A_2 weight under the conditions of Theorem 1.1 and therefore

$$\|(S_{\lambda,\gamma,k})_*(f)\|^2_{L^2(w_{\lambda,\mu})} < \infty$$

for every k. Inequality (8) and Lemma 2.1 then imply:

$$\text{(9)} \qquad \big\|(S_{\lambda,\gamma,k})_*\big\|_{L^2(w_{\lambda,\mu}) \to L^2(w_{\lambda,\mu})} \le C'(n,\lambda,\gamma)\Big(\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\Big)^{1/2}.$$

So, the right-hand side of (7) is finite if $\mu < 2\gamma - 2$. Theorem 1.1 is now proved modulo Lemma 2.1.

3. An upper bound for $|\widehat{w}_{\lambda,\mu}|$

The main result of this section will be used in Section 6. Let $\theta \in C^\infty(\mathbb{R})$ satisfy $0 \le \theta \le 1$, $\operatorname{supp}(\theta) \subset \big[\frac{9}{10}, \frac{11}{10}\big]$, $\theta \equiv 1$ on $\big[\frac{19}{20}, \frac{21}{20}\big]$. Now define

$$\text{(10)} \qquad \omega^{(1)}_{\lambda,\mu}(t) = \omega_{\lambda,\mu}(t)\big(1 - \theta(t)\big) + \theta(t).$$

and $w^{(1)}_{\lambda,\mu}(x) = \omega^{(1)}_{\lambda,\mu}(|x|)$ for all $x \in \mathbb{R}^n \setminus \{0\}$. Then $w^{(1)}_{\lambda,\mu}$ is smooth on $\mathbb{R}^n \setminus \{0\}$ and $w^{(1)}_{\lambda,\mu} \approx_{\lambda,\mu} w_{\lambda,\mu}$; that is, $w^{(1)}_{\lambda,\mu}(x)$ and $w_{\lambda,\mu}(x)$ are comparable with comparability constant depending on λ and μ only. The goal of this section is to prove this result:

Theorem 3.1. *Let $w_{\lambda,\mu}$ and $w^{(1)}_{\lambda,\mu}$ be defined as above. Then for every λ satisfying $\frac{n-1}{4} < \lambda < \frac{n-1}{2}$ and every $\mu \ge 0$ there exists a constant $C_{n,\lambda,\mu}$ such that*

$$\text{(11)} \qquad |\widehat{w}_{\lambda,\mu}(\xi)| \le \Omega_{\lambda,\mu}(\xi) := \begin{cases} C_{n,\lambda,\mu} \dfrac{1}{|\xi|^{n-2\lambda-1}\big(\log \frac{e}{|\xi|}\big)^{\mu}} & \text{if } |\xi| \le 1, \\ C_{n,\lambda,\mu} \dfrac{1}{|\xi|^{n-2\lambda-1}} & \text{if } |\xi| \ge 1, \end{cases}$$

and, for all λ satisfying $0 < \lambda < \frac{n-1}{2}$ and μ as above, there exists a constant $C'_{n,\lambda,\mu}$ such that

$$\text{(12)} \qquad |\widehat{w}^{(1)}_{\lambda,\mu}(\xi)| \le C'_{n,\lambda,\mu}\, \Omega_{\lambda,\mu}(\xi)$$

for all $\xi \in \mathbb{R}^n \setminus \{0\}$.

Proof. We begin with the proof of (11) As $w_{\lambda,\mu}$ is radial, its Fourier transform is given by

$$\widehat{w}_{\lambda,\mu}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \omega_{\lambda,\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr,$$

where J_k denotes the k-th Bessel function. It is well known — see [Watson 1944] — that $|J_k(r)| \le C_k\, r^k$ when $r \le 2\pi$ and $|J_k(r)| \le C_k\, r^{-\frac{1}{2}}$ when $r \ge 2\pi$. We control $|\widehat{w}_{\lambda,\mu}(\xi)|$ in two cases:

<u>Case 1</u>: $\frac{1}{|\xi|} \le 1$. Then

$$|\widehat{w}_{\lambda,\mu}(\xi)| \le C_n\left(\int_0^{\frac{1}{|\xi|}} r^{-2\lambda-1+\frac{n-2}{2}+\frac{n}{2}}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{\frac{1}{|\xi|}}^1 r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_1^\infty \frac{r^{-2\lambda-1-\frac{1}{2}+\frac{n}{2}}}{(\log(er))^\mu}\, dr\right).$$

<u>Case 2</u>: $\frac{1}{|\xi|} \ge 1$. Then

$$|\widehat{w}_{\lambda,\mu}(\xi)| \le C_n\left(\int_0^1 r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1}\, dr\right) + C_n\left(\int_1^{\frac{1}{|\xi|}} \frac{1}{(\log(er))^\mu}\, r^{\frac{n-2}{2}+\frac{n}{2}-2\lambda-1}\, dr\right) + \frac{C_n}{|\xi|^{\frac{n-1}{2}}}\left(\int_{\frac{1}{|\xi|}}^\infty \frac{1}{(\log(er))^\mu}\, r^{-\frac{1}{2}+\frac{n}{2}-2\lambda-1}\, dr\right).$$

If $\lambda > \frac{n-1}{4}$ and $\lambda < \frac{n-1}{2}$, all integrals converge and (11) easily follows by using calculus.

The same holds with $w_{\lambda,\mu}$ replaced by $w^{(1)}_{\lambda,\mu}$ and the proof is almost identical. Then, an analytic continuation argument and the smoothness of $w^{(1)}_{\lambda,\mu}$ can be used to prove that (12) holds in the bigger range $0 < \lambda < \frac{n-1}{2}$. The argument involves many details that we omit but that may be split in two pieces.

In the first one, given any $\lambda' \in \left(0, \frac{n-1}{4}\right]$, we use more asymptotic estimates of the Bessel functions — see [Watson 1944] — and iterated integration by parts to rewrite the right-hand side of

$$\widehat{w}^{(1)}_{\lambda,\mu}(\xi) = \frac{2\pi}{|\xi|^{\frac{n-2}{2}}} \int_0^\infty \omega^{(1)}_{\lambda,\mu}(r)\, J_{\frac{n-2}{2}}(2\pi|\xi|r)\, r^{\frac{n}{2}}\, dr \tag{13}$$

in a way that also is well defined when λ ranges in a complex neighborhood $\mathcal{O}_{\lambda'}$ of the real interval $\left(\lambda', \frac{n-1}{2}\right)$. We can call such extension $\tilde{u}_{\lambda,\mu}(\xi)$, and show that

$$|\tilde{u}_{\lambda,\mu}(\xi)| \le C'_{n,\lambda,\mu}\, \Omega_{\lambda,\mu}(\xi),$$

as in (12).

In the second one, for the same value of λ' and the same neighborhood $\mathcal{O}_{\lambda'}$, we use the dominated convergence theorem to prove that, for a given test function φ defined on $\mathbb{R}^n$, the right-hand side of

$$\int_{\mathbb{R}^n} \widehat{\varphi}(\xi)\, w^{(1)}_{\lambda,\mu}(\xi)\, d\xi = \int_{\mathbb{R}^n} \varphi(\xi)\, \tilde{u}_{\lambda,\mu}(\xi)\, d\xi, \tag{14}$$

rewritten after the first piece of the argument, is holomorphic, hence analytic, on $\mathcal{O}_{\lambda'}$. It can be proved easily that the left-hand side of (14) is also analytic on $\mathcal{O}_{\lambda'}$. Since (14) holds when $\lambda \in \left(\frac{n-1}{4}, \frac{n-1}{2}\right)$, we conclude from the analytic continuation theorem that (14) also holds when $\lambda \in \left(\lambda', \frac{n-1}{2}\right)$. Then $\widehat{w}^{(1)}_{\lambda,\mu} = \tilde{u}_{\lambda,\mu}$, since φ is arbitrary. The arbitrariness of λ' concludes the proof. □

4. A useful weight comparable to $1/w_{\lambda,\mu}$

In this section we show that $1/w_{\lambda,\mu}$ is comparable to another weight which can be written in a more useful way for our purposes, a fact that will be used in the next section. More precisely, let $u_{\lambda,\mu}$ and $\widetilde{w}_{N,\lambda,\mu}$ be defined by:

$$u_{\lambda,\mu}(y) = \begin{cases} |y|^{-n-2\lambda-1}\left(\log \frac{e}{|y|}\right)^{\mu} & \text{if } |y| < 1, \\ |y|^{-n-2\lambda-1} & \text{if } |y| \geq 1. \end{cases} \tag{15}$$

$$\widetilde{w}_{N,\lambda,\mu}(x) = \int_{\mathbb{R}^n} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy, \tag{16}$$

where N is a large enough integer independent of x.

The goal of this section is to prove that there exist constants $C_{1,n,\lambda,\mu,N}$ and $C_{2,n,\lambda,\mu,N}$ such that

$$\frac{C_{1,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \leq \widetilde{w}_{N,\lambda,\mu}(x) \leq \frac{C_{2,n,\lambda,\mu,N}}{w_{\lambda,\mu}(x)} \tag{17}$$

for all $x \in \mathbb{R}^n \setminus \{0\}$. Let us write $\widetilde{w}_{N,\lambda,\mu} = \widetilde{w}_{N,\lambda,\mu,1} + \widetilde{w}_{N,\lambda,\mu,2}$, where

$$\widetilde{w}_{N,\lambda,\mu,1}(x) = \int_{|y| \leq \frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy, \tag{18}$$

$$\widetilde{w}_{N,\lambda,\mu,2}(x) = \int_{|y| > \frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy. \tag{19}$$

Observe that in (18),

$$C_1 |\langle x, y\rangle| \leq |e^{i\langle x,y\rangle} - 1| \leq |x|\ |y| \tag{20}$$

for an absolute constant $0 < C_1 < 1$. Now, we estimate $\widetilde{w}_{N,\lambda,\mu,1}$.

<u>Case 1</u>: $\frac{1}{|x|} \leq 1$. Given a positive constant $C > 0$, in view of (20),

$$\begin{aligned}
\widetilde{w}_{N,\lambda,\mu,1}(x) &= \int_{|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N \, |y|^{-n-2\lambda-1} \Big(\log\frac{e}{|y|}\Big)^{\mu} \, dy \\
&\geq C_{C,N} \int_{\Omega(x)} |x|^N |y|^N |y|^{-n-2\lambda-1} \Big(\log\frac{e}{|y|}\Big)^{\mu} \, dy \\
&= C_{n,C,N} \, |x|^N \frac{|S^{n-1}|}{e^{2\lambda+1-N}} \int_{e|x|}^{\infty} s^{2\lambda-N} (\log s)^{\mu} \, ds,
\end{aligned}$$

where $\Omega(x) = \big\{y : |y| \leq \frac{1}{|x|} \text{ and } C \leq \big|\big\langle \frac{x}{|x|}, \frac{y}{|y|}\big\rangle\big|\big\}$. In order for this integral to converge, we need $N > 2\lambda + 1$. Later we will also need N to be even. So, we set $N = N_\lambda := 2\lceil 2\lambda+1\rceil$. It easily follows that there exist constants $C_{n,\lambda,N}$ and $C_{\lambda,\mu,N}$ such that $\widetilde{w}_{N,\lambda,\mu,1}(x) \geq C_{n,\lambda,N}/w_{\lambda,\mu}(x)$ for all $x \in \mathbb{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$. An easier computation and (20) yield $\widetilde{w}_{N,\lambda,\mu,1}(x) \leq C'_{n,\lambda,N}/w_{\lambda,\mu}(x)$ in Case 1 for all $x \in \mathbb{R}^n$ satisfying $|x| \geq C_{\lambda,\mu,N}$. So, on $\{x \in \mathbb{R}^n : |x| \geq \max\{1, C_{\lambda,\mu,N}\}\}$ we have

$$\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,N} 1/w_{\lambda,\mu} \tag{21}$$

<u>Case 2</u>: $\frac{1}{|x|} > 1$. Let us use the decomposition $\widetilde{w}_{N,\lambda,\mu,1}(x) = I + II$, where

$$\begin{aligned}
I &= \int_{|y|\leq 1} |e^{i\langle x,y\rangle} - 1|^N \, |y|^{-n-2\lambda-1} \Big(\log\frac{e}{|y|}\Big)^{\mu} \, dy \approx_{n,\lambda,\mu,N} |x|^N, \\
II &= \int_{1<|y|\leq\frac{1}{|x|}} |e^{i\langle x,y\rangle} - 1|^N \, |y|^{-n-2\lambda-1} \, dy \approx_{n,\lambda,N} |x|^{2\lambda+1},
\end{aligned}$$

This proves that

$$\widetilde{w}_{N,\lambda,\mu,1}(x) \approx_{n,\lambda,\mu,N} |x|^N + |x|^{2\lambda+1} \approx_{n,\lambda,N} |x|^{2\lambda+1} = \frac{1}{w_{\lambda,\mu}(x)} \tag{22}$$

on $\{x \in \mathbb{R}^n : |x| \leq 1\}$. If $C_{\lambda,\mu,N} \leq 1$, then relations (21) and (22) immediately imply that $\widetilde{w}_{N,\lambda,\mu,1} \approx_{n,\lambda,\mu,N} 1/w_{\lambda,\mu}$ on $\mathbb{R}^n$. Otherwise, just observe that both functions $\widetilde{w}_{N,\lambda,\mu,1}$ and $1/w_{\lambda,\mu}$ are positive and continuous on the compact annulus $1 \leq |x| \leq C_{\lambda,\mu,N}$. We still have to show that $\widetilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} 1/w_{\lambda,\mu}$. Let us define

$$\widetilde{\widetilde{w}}_{\lambda,\mu,2}(x) = \int_{|y|>\frac{1}{|x|}} u_{\lambda,\mu}(y) \, dy. \tag{23}$$

Then

$$\widetilde{w}_{N,\lambda,\mu,2}(x) \leq 2^N \, \widetilde{\widetilde{w}}_{\lambda,\mu,2}(x). \tag{24}$$

We will prove that the inverse inequality also holds (with a constant different from 2^N), so that we have $\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{N,\lambda,\mu,n} \tilde{w}_{N,\lambda,\mu,2}$. Now, let us prove that

$$\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{N,\lambda,\mu,n} 1/w_{\lambda,\mu}.$$

Case 1: $\frac{1}{|x|} > 1$. Then:

$$\tilde{\tilde{w}}_{\lambda,\mu,2}(x) = \frac{|S^{n-1}|}{2\lambda+1}\,|x|^{2\lambda+1} \approx_{\lambda,n} |x|^{2\lambda+1} = \frac{1}{w_{\lambda,\mu}(x)}.$$

Case 2: $\frac{1}{|x|} \leq 1$. Then:

$$\begin{aligned}\tilde{\tilde{w}}_{\lambda,\mu,2}(x) &= C_{n,\lambda} + \frac{|S^{n-1}|}{e^{2\lambda+1}} \int_e^{e|x|} t^{2\lambda} (\log t)^{\mu}\, dt \\ &\approx_{\lambda,\mu,n} |x|^{2\lambda+1}\,(\log(e|x|))^{\mu} = \frac{1}{w_{\lambda,\mu}(x)}.\end{aligned}$$

This concludes the proof that $\tilde{\tilde{w}}_{\lambda,\mu,2} \approx_{\lambda,\mu,n} 1/w_{\lambda,\mu}$ on $\mathbb{R}^n \setminus \{0\}$. Now we need to prove that there exists a constant $C_{N,\lambda,\mu,n}$ such that the inequality

$$\tilde{\tilde{w}}_{\lambda,\mu,2} \leq C_{N,\lambda,\mu,n}\, \tilde{w}_{N,\lambda,\mu,2}$$

holds on $\mathbb{R}^n \setminus \{0\}$. Since both $\tilde{\tilde{w}}_{\lambda,\mu,2}$ and $\tilde{w}_{N,\lambda,\mu,2}$ are radial, it will be enough to prove that the functions $t \mapsto \tilde{w}_{N,\lambda,\mu,2}(te_1)$ and $t \mapsto \tilde{\tilde{w}}_{\lambda,\mu,2}(te_1)$ are comparable on $\mathbb{R}^+$, where $e_1 = (1,0,\ldots,0)$. Observe that $|e^{i\langle te_1,y\rangle} - 1| > \sqrt{2}$ on $G^t := \bigcup_{k\in\mathbb{Z}} G_k^t$, where

$$G_k^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k+1)\pi}{2t}, \frac{(4k+3)\pi}{2t}\right) \right\}$$

for all $t > 0$ and $k \in \mathbb{Z}$. Therefore $u_{\lambda,\mu}(y) \approx_N |e^{i\langle te_1,y\rangle} - 1|^N u_{\lambda,\mu}(y)$ on G^t. In particular, there exists a constant C_N such that

$$\int_{G^t} u_{\lambda,\mu}(y)\, dy \leq C_N \int_{G^t} |e^{i\langle te_1,y\rangle} - 1|^N u_{\lambda,\mu}(y)\, dy.$$

If $t > 0$ and $k \in \mathbb{Z} \setminus \{0\}$ we define

$$R_k^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{(4k-1)\pi}{2t}, \frac{(4k+1)\pi}{2t}\right) \right\}$$

and

$$R_0^t := \left\{ y \in \mathbb{R}^n : \langle e_1, y\rangle \in \left[\frac{-\pi}{2t}, \frac{\pi}{2t}\right) \text{ and } |y| > \frac{1}{t} \right\}.$$

As

$$\int_{R_k^t} u_{\lambda,\mu}(y)\, dy \leq \int_{G_{k-1}^t} u_{\lambda,\mu}(y)\, dy$$

for all $k \in \mathbb{Z}^+$, and

$$\int_{R_k^t} u_{\lambda,\mu}(y)\, dy \leq \int_{G_k^t} u_{\lambda,\mu}(y)\, dy$$

for all $k \in \mathbb{Z}^-$, we also have

$$\int_{\bigcup_{k\in\mathbb{Z}\setminus\{0\}} R_k^t} u_{\lambda,\mu}(y)\,dy \le \int_{G^t} u_{\lambda,\mu}(y)\,dy \le C_N \int_{G^t} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy.$$

Since

$$\left\{y : |\langle e_1, y\rangle| > \frac{\pi}{2t}\right\} = G^t \cup \bigcup_{k\in\mathbb{Z}\setminus\{0\}} R_k^t,$$

we have

$$\int_{|\langle e_1,y\rangle|>\frac{\pi}{2t}} u_{\lambda,\mu}(y)\,dy \le 2C_N \int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy.$$

Since $u_{\lambda,\mu}$ is radial, we can replace e_1 by e_j in the inequality above for $j = 2, \ldots, n$. Let $|y|_\infty := \sup_{1\le j\le n} |\langle e_j, y\rangle|$. Then

$$\int_{\{y\in\mathbb{R}^n : |y|_\infty > \frac{\pi}{2t}\}} u_{\lambda,\mu}(y)\,dy \le 2nC_N \int_{|y|>\frac{1}{t}} |e^{i\langle te_1,y\rangle}-1|^N u_{\lambda,\mu}(y)\,dy. \tag{25}$$

Inequality (25) and the Lemma 4.1 easily imply — see (19) and (23) for details — that $\tilde{\tilde{w}}_{\lambda,\mu,2}(te_1) \le C_{n,\lambda,\mu,N} \cdot \tilde{w}_{N,\lambda,\mu,2}(te_1)$.

Lemma 4.1. *Let $u_{\lambda,\mu}$ be as in* (15). *Then, for all $n \in \mathbb{Z}^+$, $\lambda \in \mathbb{R}$, and $C > 1$ there exists a constant $D = D(n, \lambda, C) \in \mathbb{R}$ such that $u_{\lambda,\mu}\big(\frac{y}{C}\big) \le D\, u_{\lambda,\mu}(y)$ for all $y \in \mathbb{R}^n \setminus \{0\}$. We can choose $D = C^{n+2\lambda+1}(\log(eC))^\mu$.*

The proof of Lemma 4.1 is left to the reader. This completes the proof that $\tilde{w}_{N,\lambda,\mu,2} \approx_{n,\lambda,\mu,N} \frac{1}{w_{\lambda,\mu}}$ on $\mathbb{R}^n \setminus \{0\}$ and therefore the proof of (17), that is the claim of this section.

5. Reduction of Lemma 2.1 to Lemma 5.2

By duality, the inequality in Lemma 2.1 can be expressed as

$$\left\| \int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\cdot))(x)\,\frac{dt}{t} \right\|_{L^2(\frac{dx}{w_{\lambda,\mu}(x)})} \le C\,\|h(t,x)\|_{L^2(\frac{dt}{t}\,\frac{dx}{w_{\lambda,\mu}(x)})} \tag{26}$$

for all functions $h(t,x)$ in the appropriate space, where

$$C = C_{n,\lambda,\mu,\gamma,k} = \sqrt{C_{n,\lambda,\mu,\gamma}\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}}.$$

In view of the result of Section 4, for every $f \in L^2\big(\mathbb{R}^n, \frac{1}{w_{\lambda,\mu}}\big)$,

$$\|f\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \approx \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_{\lambda,\mu}(y) \left| \left(\sum_{j=0}^{N_\lambda/2} \hat{f}(g_{j,y}(\xi))\,b_j \right) \right|^2 dy\,d\xi, \tag{27}$$

where

$$g_{j,y}(\xi) = \left(\xi - \left(\frac{N_\lambda/2 - j}{2\pi}\right)y\right), \quad b_j = (-1)^j \binom{N_\lambda/2}{j},$$

Plancherel's identity was used, and the implicit comparability constants depend on λ, μ, n only. We can substitute the left-hand side of (26) by using (27) on the function

$$f(x) = \int_1^2 (S_{\lambda,\gamma,k})_{at}(h(t,\cdot))(x)\,\frac{dt}{t}.$$

For such a function we have

$$\tag{28} \begin{aligned} &\left|\left(\sum_{j=0}^{N_\lambda/2} \widehat{f}(g_{j,y}(\xi))\,b_j\right)\right|^2 \\ &\qquad = \left|\int_1^2 \left(\sum_{j=0}^{N_\lambda/2} \widehat{h}(t, g_{j,y}(\xi))\, m_{\lambda,\gamma,k}(at|g_{j,y}(\xi)|)\, b_j\right)\frac{dt}{t}\right|^2. \end{aligned}$$

Since $m_{\lambda,\gamma,k}$ is supported in $\left[1 - \frac{5}{8\cdot 2^k}, 1 - \frac{1}{8\cdot 2^k}\right]$, the Cauchy–Schwarz inequality in the t variable allows us to control the right-hand side of (28) by

$$\tag{29} \frac{C_\lambda}{2^k}\int_1^2 \left|\sum_{j=0}^{N_\lambda/2} \widehat{h}(t, g_{j,y}(\xi))\cdot m_{\lambda,\gamma,k}(a\,t|g_{j,y}(\xi)|)\, b_j\right|^2 \frac{dt}{t} =: H_{k,\lambda,\gamma}(y,\xi).$$

So, if we can show

$$\tag{30} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} u_{\lambda,\mu}(y)\, H_{k,\lambda,\gamma}(y,\xi)\, dy\, d\xi \le C\,\frac{2^{k(2\lambda-1)}}{k^{2\gamma-\mu}}\,\|h(t,x)\|^2_{L^2(\frac{dt}{t}\,\frac{dx}{\omega_{\lambda,\mu}(|x|)})}$$

for a constant $C := C_{n,\lambda,\mu,\gamma}$, then (26) is proved. But (30) follows from the following pointwise (with respect to t) estimate:

$$\tag{31} \|(S_{\lambda,\gamma,k})_t(h)(x)\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})} \le C_{n,\lambda,\mu,\gamma}\,\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\,\|h\|^2_{L^2(\frac{dx}{\omega_{\lambda,\mu}(|x|)})}$$

if (31) holds for all $t > 0$ rather than just $t \in [1,2]$ (which allowed us to drop the parameter a), and for all $h \in L^2(\mathbb{R}^n, dx/w_{\lambda,\mu}(x))$. In order to see that (31) implies (30), just use (27) with $f(x) = \big(\widehat{h}(\cdot)\,m_{\lambda,\gamma,k}(t|\cdot|)\big)^{\vee}(x) = (S_{\lambda,\gamma,k})_t(h)(x)$, to rewrite the left-hand side of (31).

By duality, (31) is equivalent to

$$\tag{32} \|(S_{\lambda,\gamma,k})_t(h)\|^2_{L^2(w_{\lambda,\mu})} \le C_{n,\lambda,\mu,\gamma}\,\frac{2^{2k\lambda}}{k^{2\gamma-\mu}}\,\|h\|^2_{L^2(w_{\lambda,\mu})}$$

for all $h \in L^2(w_{\lambda,\mu}(x)\,dx)$, $t > 0$. So, the latter also yields the inequality in Lemma 2.1 for every f in the appropriate space and every $a > 0$. We now need to prove (32).

We denote by $(K_{\lambda,\gamma,k})_t(x)$ the kernel of the operator $(S_{\lambda,\gamma,k})_t$, i.e., the inverse Fourier transform of the multiplier $m_{\lambda,\gamma,k}(t\,|\cdot|)$. $(K_{\lambda,\gamma,k})_t$ is radial on $\mathbb{R}^n$, and it is convenient to decompose it radially as

$$(K_{\lambda,\gamma,k})_t = (K_{\lambda,\gamma,k})_1^{(0)} + \sum_{j=1}^{\infty} (K_{\lambda,\gamma,k})_t^{(j)},$$

where

$$(K_{\lambda,\gamma,k})_1^{(0)}(x) = (K_{\lambda,\gamma,k})_t(x)\,\theta(2^{-(k+3)}x/t),$$
$$(K_{\lambda,\gamma,k})_t^{(j)}(x) = (K_{\lambda,\gamma,k})_t(x)\,\big(\theta(2^{-(j+k+3)}\,x/t) - \theta(2^{-(k+2+j)}\,x/t)\big),$$

for some radial smooth function θ supported in the ball $B(0,2)$ and equal to one on $B(0,1)$.

To prove estimate (32) we make use of the subsequent lemmas.

Lemma 5.1. *For all $M \geq 2n$ there is a constant $C_{\lambda,\gamma,k,M} = C_{\lambda,\gamma,k,M}(n,\theta)$ such that for all $j = 0,1,2,\ldots,$*

$$\sup_{\xi\in\mathbb{R}^n} |\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-jM}}{k^\gamma} \tag{33}$$

and also

$$|\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+l)M}}{k^\gamma} \tag{34}$$

whenever $\big|t\,|\xi| - 1\big| \geq 2^{l-k-3}$ and $l \geq 4$. Also,

$$|\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi)| \leq C_{\lambda,\gamma,M}\,\frac{2^{-(j+k+3)M}}{k^\gamma}(1+t\,|\xi|)^{-M} \tag{35}$$

whenever $|t\,\xi| \leq \frac{1}{8}$ or $|t\,\xi| \geq \frac{15}{8}$.

Proof. The proof for $t = 1$ follows the lines of the proof of Lemma 10.5.5 in [Grafakos 2014, p. 413]. Just observe that estimate (10.5.9) in p. 409 of that reference is now replaced by (6), which explains why the factor $1/k^\gamma$ appears. The general case (any $t > 0$) is straightforward in view of the fact that

$$\widehat{(K_{\lambda,\gamma,k})_t^{(j)}}(\xi) = \widehat{(K_{\lambda,\gamma,k})_1^{(j)}}(t\,\xi). \qquad \square$$

Lemma 5.2. *The inequalities*

$$\int_{||t\,\xi|-1|\,<\,\varepsilon} |\widehat{f}(\xi)|^2\,d\xi \leq C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t)\,\varepsilon \int_{\mathbb{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda,\mu}(x)} \tag{36}$$

and

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2\,\frac{d\xi}{(1+|t\,\xi|)^M} \leq C_{n,\lambda,\mu,M}\,\omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2\,\frac{dx}{w_{\lambda,\mu}(x)}, \tag{37}$$

hold for all Schwartz functions f, $t > 0$, $M \geq 2n$, all $0 < \varepsilon < 2$, λ, and μ as in Theorem 1.1.

The proof of Lemma 5.2 is postponed to Section 6.

By reasoning as in [Grafakos 2014, p. 414] and using the estimates in Lemmas 5.1 and 5.2 instead of those in Lemma 10.5.5 in [op. cit., p. 413] and Lemma 10.5.6 in [op. cit., p. 414], we can prove

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2 \, dx \le C \frac{2^{-2jM}}{2^k k^{2\gamma}} \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \tag{38}$$

for another constant $C = C_{n,\lambda,\mu,\gamma,M}$. By duality, this is equivalent to

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f)(x)|^2 \, w_{\lambda,\mu}(x) \, dx \le C \frac{2^{-2jM}}{2^k k^{2\gamma}} \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \, dx. \tag{39}$$

Given a Schwartz function f, we write

$$f_0 = f \chi_{Q_0^{(n,k,j,t)}},$$

where $Q_0^{(n,k,j,t)}$ is a cube centered at the origin of side length $C_n 2^{j+k+4} t$ (note that $\operatorname{supp}(K_{\lambda,\gamma,k})_t^{(j)} \subseteq B(0, 2^{j+k+4} t)$). Then for $x \in Q_0^{(n,k,j,t)}$ we have the inequality

$$|x| \le \sqrt{n} \, C_n 2^{j+k+4} t;$$

hence, (39) implies

$$\begin{aligned} \int_{\mathbb{R}^n} &|((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2 \, w_{\lambda,\mu}(x) \, dx \\ &\le C_{n,\lambda,\mu,\gamma,M} \frac{2^{-2jM}}{2^k k^{2\gamma}} \frac{\omega_{\lambda,\mu}(t)}{\omega_{\lambda,\mu}(\sqrt{n} \, C_n 2^{j+k+4} t)} \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2 \, w_{\lambda,\mu}(x) \, dx \end{aligned} \tag{40}$$

because the function $1/\omega_{\lambda,\mu}$ is increasing. A simple computation shows that

$$\sup_{t>0} \frac{\omega_{\lambda,\mu}(at)}{\omega_{\lambda,\mu}(t)} = \frac{1}{a^{2\lambda+1}} \quad \text{and} \quad \sup_{t>0} \frac{\omega_{\lambda,\mu}(at)}{\omega_{\lambda,\mu}(t)} = \frac{(\log(e/a))^{\mu}}{a^{2\lambda+1}} \tag{41}$$

if $a > 1$ and if $a \le 1$, respectively. Therefore, for all j and k such that $j + k \ge C'_n$ for a suitable purely dimensional constant C'_n,

$$\sup_{t>0} \frac{\omega_{\lambda,\mu}(t)}{\omega_{\lambda,\mu}(\sqrt{n} \, C_n 2^{j+k+4} t)} \le C''_{n,\lambda,\mu} 2^{(j+k)(2\lambda+1)} (j^{\mu} + k^{\mu}), \tag{42}$$

where we used the hypothesis on j and k and the fact that

$$(j+k)^{\mu} \le C_{\mu} (j^{\mu} + k^{\mu}).$$

It follows from (42) and (40) that $\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_0)(x)|^2 \, w_{\lambda,\mu}(x) \, dx$ is bounded by

$$C \, 2^{j(2\lambda+1-2M)} \frac{2^{2k\lambda}}{k^{2\gamma}} (j^{\mu} + k^{\mu}) \int_{Q_0^{(n,k,j,t)}} |f_0(x)|^2 \, w_{\lambda,\mu}(x) \, dx,$$

for $C = C_{n,\lambda,\mu,\gamma,M}$, provided that

$$j + k \geq C'_n. \tag{43}$$

Now write $\mathbb{R}^n \setminus Q_0^{(n,k,j,t)}$ as a mesh of cubes $Q_i^{(n,k,j,t)}$, indexed by $i \in \mathbb{Z} \setminus \{0\}$, of side lengths $C_n 2^{j+k+4} t$ (the same side length of $Q_0^{(n,k,j,t)}$) and centers c_{Q_i}. By using (33), reasoning as in [Grafakos 2014, p. 415] as well as simply noting that $2^{2k\lambda}(j^\mu + k^\mu) \geq 1$, we can find that the pieces

$$\int_{\mathbb{R}^n} |((K_{\lambda,\gamma,k})_t^{(j)} * f_i)(x')|^2 \, w_{\lambda,\mu}(x') \, dx'$$

are bounded by

$$C_{\lambda,\mu,\gamma,M} \, 2^{-2jM} \, \frac{2^{2k\lambda}}{k^{2\gamma}} (j^\mu + k^\mu) \int_{Q_i^{(n,k,j,t)}} |f_i(x)|^2 \, w_{\lambda,\mu}(x) \, dx$$

whenever f_i is supported in $Q_i^{(n,k,j,t)}$ and, in turn, that

$$\begin{aligned} &\|(K_{\lambda,\gamma,k})_t^{(j)} * f\|_{L^2(w_{\lambda,\mu})} \\ &\qquad \leq C''_{n,\lambda,\mu,\gamma,M} \, 2^{j(\lambda+\frac{1}{2}-M)} \, \frac{2^{k\lambda}}{k^\gamma} (j^{\frac{\mu}{2}} + k^{\frac{\mu}{2}}) \, \|f\|_{L^2(w_{\lambda,\mu})} \end{aligned} \tag{44}$$

(in view of the argument in [Grafakos 2014]). Observe that condition (43) is satisfied if we assume $k \geq C'_n$, which we can as the convergence of (7) only depends on the estimates we have for k big enough. So, for $k \geq C'_n$, by using (44) and summing over $j = 0, 1, 2, \ldots$, we deduce (32) if we just choose $M > n/2$ (remember that $n > 2\lambda + 1$). In turn, (32) is equivalent to (31), which is equivalent to (26), which is equivalent to the inequality in Lemma 2.1. Therefore, this completes the proof of the lemma, modulo Lemma 5.2

6. Proof of Lemma 5.2

6.1. ***Proof of inequality* (36).** We reduce estimate (36) by duality to

$$\int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 \, w_{\lambda,\mu}(\xi) \, d\xi \leq C_{n,\lambda,\mu} \, \omega_{\lambda,\mu}(t) \, \varepsilon \int_{||tx|-1| \leq \varepsilon} |g(x)|^2 \, dx \tag{45}$$

for functions g supported in the annulus $\big||tx|-1\big| \leq \varepsilon$. In Section 3 we observed that

$$w_{\lambda,\mu} \approx_{\lambda,\mu} w_{\lambda,\mu}^{(1)}$$

and proved in Theorem 3.1 that the function $|\widehat{w}_{\lambda,\mu}^{(1)}|$ is bounded by a scalar multiple of $\Omega_{\lambda,\mu}$ (see (12)) in the whole range $\lambda \in (0, (n-1)/2)$. Therefore, we can start to prove (45) as follows:

$$
\begin{aligned}
(46)\quad \int_{\mathbb{R}^n} |\widehat{g}(\xi)|^2 w_{\lambda,\mu}(\xi)\, d\xi &\approx_{\lambda,\mu} \int_{\mathbb{R}^n} \big(\widehat{g}\,\overline{\widehat{g}}\big)^{\vee}(x)\, \widehat{w}^{(1)}_{\lambda,\mu}(x)\, dx \\
&\le C_{n,\lambda,\mu} \int_{\mathbb{R}^n} (|g| * |\tilde{\bar{g}}|)(x)\, \Omega_{\lambda,\mu}(x)\, dx \\
&= C_{n,\lambda,\mu} \iint_{\substack{||t y|-1| \le \varepsilon \\ ||t x|-1| \le \varepsilon}} |g(x)|\, |\tilde{\bar{g}}(y)|\, \Omega_{\lambda,\mu}(x-y)\, dx\, dy \\
&\le C_{n,\lambda,\mu} B(n,\lambda,\mu,\varepsilon,t) \|g\|^2_{L^2}
\end{aligned}
$$

where $\widetilde{g}(x) = g(-x)$ and

$$
(47)\qquad B(n,\lambda,\mu,\varepsilon,t) = \frac{1}{t^n} \sup_{\{x : ||x|-1| \le \varepsilon\}} \int_{||y|-1| \le \varepsilon} \Omega^t_{\lambda,\mu}(y-x)\, dy,
$$

where $\Omega^t_{\lambda,\mu}(x) := \Omega_{\lambda,\mu}(x/t)$. The last inequality of (46) is proved by interpolation between the $L^1(S) \to L^1(S)$ and $L^\infty(S) \to L^\infty(S)$ estimates for the linear operator

$$
L_{\lambda,\mu,t,\varepsilon}(g)(x) = \int_S g(y)\, \Omega_{\lambda,\mu}(y-x)\, dy,
$$

where

$$
S = \big\{ y \in \mathbb{R}^n : \big| |t y| - 1 \big| \le \varepsilon \big\},
$$

using the Cauchy–Schwarz inequality. It remains to establish that

$$
(48)\qquad B(n,\lambda,\mu,\varepsilon,t) \le C_{n,\lambda,\mu}\, \omega_{\lambda,\mu}(t)\, \varepsilon.
$$

Then we reason as in [Grafakos 2014, pp. 417, 418]: we apply a rotation and a change of variable to the integrals in (47) to push the dependence on x to the domain of integration, then control the supremum in (47) by integrating $\Omega_{\lambda,\mu}$ over the bigger set

$$
\big\{ y : \big| |y - e_1| - 1 \big| \le 2\varepsilon \big\},
$$

finally we split this latter integral over the sets S_0, S_ℓ, S_* defined in [op. cit.] to be

$$
\begin{aligned}
S_0 &= \big\{ y \in \mathbb{R}^n : \big| |y - e_1| - 1 \big| \le 2\varepsilon,\ |y| \le \varepsilon \big\}, \\
S_\ell &= \big\{ y \in \mathbb{R}^n : \big| |y - e_1| - 1 \big| \le 2\varepsilon,\ \ell\varepsilon \le |y| \le (\ell+1)\varepsilon \big\}, \\
S_* &= \big\{ y \in \mathbb{R}^n : \big| |y - e_1| - 1 \big| \le 2\varepsilon,\ |y| \ge 1 \big\}.
\end{aligned}
$$

In the end, matters reduce to proving the estimates

$$\int_{S_0} \Omega^t_{\lambda,\mu}(y)\,dy \le C'_{n,\lambda,\mu}\, t^n\, \omega_{\lambda,\mu}(t)\, \varepsilon^{2\lambda+1}, \tag{49}$$

$$\sum_{\ell=1}^{[\frac{1}{\varepsilon}]+1} \int_{S_\ell} \Omega^t_{\lambda,\mu}(y)\,dy \le C_{\lambda,\mu}\, t^n\, \varepsilon\, \omega_{\lambda,\mu}(t), \tag{50}$$

$$\int_{S_*} \Omega^t_{\lambda,\mu}(y)\,dy \le C_n\, \varepsilon\, \omega_{\lambda,\mu}(t)\, t^n. \tag{51}$$

In proving the inequalities above, we can assume without loss of generality that $t \ge 2$, because when $t < 2$ the proof of Lemma 5.2 is an immediate consequence of Lemma 10.5.6 in [op. cit., p. 414]. We can also assume that $t \ge C_{n,\lambda,\mu}$, due to the compactness of $[2, C_{n,\lambda,\mu}]$ and the continuity and positivity of the functions involved. For a suitable constant $C_{n,\lambda,\mu}$ and $t \ge \max\{2, C_{n,\lambda,\mu}\}$, (49) is proved by using calculus (note that the integrand in (49) is radial and the domain of integration is a sphere); (50) is proved by using the maximum of the integrand over each set S_ℓ, then by comparing the sum with an integral, finally by using calculus to estimate the integral; (51) is proved by using the maximum of the integrand over S_*. The condition that $t \ge 2 > \varepsilon$ was used in both (49) and (50) and (41) was used in (51).

By combining estimates (49), (50), and (51), we obtain (48). This concludes the proof of (45) and, therefore, of (36). □

6.2. ***Proof of inequality* (37).** Inequality (37) is already known for $t \le 1$; see equation (10.5.22) in [Grafakos 2014, p. 414]. Indeed, if $0 < t \le 1$ then $\omega_{\lambda,\mu}(t) = 1/t^{2\lambda+1}$, and (37) follows by dilation from the case $t = 1$, the one shown in [op. cit.]. For $t > 1$ define:

$$A^t_1 = \Big\{\xi \in \mathbb{R}^n : |\xi| \le \frac{1}{t}\Big\}, \qquad A^t_2 = \Big\{\xi \in \mathbb{R}^n : \frac{1}{t} < |\xi| \le \frac{2+\sqrt{t}}{t}\Big\},$$
$$A^t_3 = \Big\{\xi \in \mathbb{R}^n : \frac{2+\sqrt{t}}{t} < |\xi| \le \frac{2+t}{t}\Big\}, \quad A^t_4 = \Big\{\xi \in \mathbb{R}^n : \frac{2+t}{t} < |\xi|\Big\}.$$

We will prove (37) by proving that

$$I_j := \int_{A^t_j} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M}\,d\xi \le C_{n,\lambda,\mu,M}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \tag{52}$$

for each $j = 1, 2, 3, 4$. For $j = 1$, first observe that $1/(1+|t\,\xi|)^M \approx_M 1$ on A^t_1 and then argue as in the proof of (36), at the beginning of this section. By duality, we reduce (52) with $j = 1$ to

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 w_{\lambda,\mu}(\xi)\,d\xi \le C_{n,\lambda,\mu}\, \omega_{\lambda,\mu}(t) \int_{A^t_1} |f(x)|^2\,dx \tag{53}$$

for all functions f supported in the ball A_1^t. By proceeding as in (46), we can prove that

$$\int_{\mathbb{R}^n} |\widehat{f}(\xi)|^2 w_{\lambda,\mu}(\xi)\, d\xi \le B'(n,\lambda,\mu,t)\|f\|_{L^2}^2$$

for every f supported in A_1^t, where $B'(n,\lambda,\mu,t)$, now, is defined by

$$\begin{aligned} B'(n,\lambda,\mu,t) &= \sup_{\{x:|x|\le\frac{1}{t}\}} \int_{|y|\le\frac{1}{t}} \Omega_{\lambda,\mu}(y-x)\, dy \\ &= \frac{1}{t^n} \sup_{\{x:|x|\le 1\}} \int_{|y+x|\le 1} \Omega_{\lambda,\mu}\Big(\frac{y}{t}\Big)\, dy \end{aligned} \tag{54}$$

and all we still need to show is that

$$B'(n,\lambda,\mu,t) \le C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t). \tag{55}$$

Since $|x| \le 1$ and $|x+y| \le 1$ we have $|y| \le 2$. So, (55) is a consequence of

$$\frac{1}{t^n} \int_{|y|\le 2} \Omega_{\lambda,\mu}\Big(\frac{y}{t}\Big)\, dy \le C_{n,\lambda,\mu}\,\omega_{\lambda,\mu}(t), \tag{56}$$

which can be proved similarly to (49).

When $j = 2$, we use

$$I_2 \le \sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \frac{1}{(2+\ell)^M} \int_{\frac{1+\ell}{t}<|\xi|\le\frac{2+\ell}{t}} |\widehat{f}(\xi)|^2\, d\xi. \tag{57}$$

Next, we apply estimate (36) on each of the latter integrals. We are already assuming that $t > 1$. Since $\omega_{\lambda,\mu}(t) \approx_{\lambda,\mu,J} 1$ on any compact subinterval J of $(0,\infty)$, we can in fact assume $t \ge 3$. Now we control the right-hand side of (57) with

$$\begin{aligned} C_{n,\lambda,\mu} \sum_{\ell=0}^{\lceil\sqrt{t}\rceil} \frac{1}{(2+\ell)^M}\, \omega_{\lambda,\mu}\Big(\frac{2t}{3+2\ell}\Big) \frac{1}{3+2\ell} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \\ \le C_{n,\lambda,\mu,M}\,\omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}, \end{aligned} \tag{58}$$

provided $M > 2\lambda + 1$. This proves that (52) holds for $j = 2$.

If $j = 3$ then $(2+\sqrt{t})/t < |\xi|$, which implies that

$$\frac{1}{(1+|t\xi|)^M} \le \frac{1}{(3+\sqrt{t})^M}.$$

Then apply (36). Observe that, as long as $t > 1$, we have that the quantity $\tilde{t}$ that now plays the role of t in (36) is bounded above and below by absolute constants, so $\omega_{\lambda,\mu}(\tilde{t}) \approx_{\lambda,\mu} 1$. In addition, for t in the same range, we have $\tilde{\varepsilon} \le 1$ ($\tilde{\varepsilon}$ being the quantity that now plays the role of ε in (36)). These considerations imply that

$$I_3 \le \frac{1}{(3+\sqrt{t})^M} \int_{\frac{2+\sqrt{t}}{t} < |\xi| \le \frac{2+t}{t}} |\widehat{f}(\xi)|^2 \, d\xi \tag{59}$$
$$\le C'_{n,\lambda,\mu} \frac{1}{(3+\sqrt{t})^M} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}$$
$$\le C''_{n,\lambda,\mu,M}\, \omega_{\lambda,\mu}(t) \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)},$$

last inequality holding for a suitable constant $C''_{n,\lambda,\mu,M}$, provided that $M > 4\lambda + 2$.

It only remains to prove (52) with $j = 4$. We have

$$I_4 \le \sum_{\ell=\lfloor t \rfloor+1}^{\infty} \int_{\frac{1+\ell}{t} < |\xi| \le \frac{2+\ell}{t}} |\widehat{f}(\xi)|^2 \frac{1}{(1+|t\,\xi|)^M} \, d\xi. \tag{60}$$

Again, we apply (36) to the integral in the last term of (60), which is therefore controlled by

$$C_{n,\lambda,\mu} \sum_{\ell=\lfloor t \rfloor+1}^{\infty} \frac{1}{(2+\ell)^M} \frac{1}{\left(\frac{2t}{3+2\ell}\right)^{2\lambda+1}} \frac{1}{(3+2\ell)} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)}$$
$$\le C_{n,\lambda,\mu,M} \frac{1}{t^{2\lambda+1}} \int_{\mathbb{R}^n} |f(x)|^2 \frac{dx}{w_{\lambda,\mu}(x)} \frac{1}{(1+t)^{M-2\lambda-1}},$$

which yields the desired inequality, provided that $M > 2\lambda + 1$. By choosing any $M > 4\lambda + 2$ (as required after (59)), we conclude the proof of (37) and of the claimed statement.

Proof of Corollary 1.2. The proof in [Carbery et al. 1988] can be used with $m_{\lambda,\gamma}$ instead of m_1^λ to account for the case where $\gamma \ge 0$ and $2 \le p < p_\lambda$. When $p = p_\lambda$ and $\gamma > 1/p'_\lambda + 1/2$, values of μ satisfying $(2\lambda+1)/n < \mu < 2\gamma - 2$ exist. For such μ, since $1 < 1 + 2\lambda < n$, we can use Theorem 1.1. Since (4) trivially holds for all $f \in \mathcal{S}$, the boundedness of $B_*^{\lambda,\gamma}$ implies that it also holds for every $f \in L^2(\mathbb{R}^n, dx)$ and every $f \in L^2(\mathbb{R}^n, w_{\lambda,\mu})$. But then it must hold for every $f \in L^2 + L^2(w_{\lambda,\mu})$. Since $(2\lambda+1)/n < \mu$ we have $L^{p_\lambda} \subseteq L^2 + L^2(w_{\lambda,\mu})$, concluding the proof. □

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