

*Pacific
Journal of
Mathematics*

**UNIQUENESS OF CONFORMAL RICCI FLOW
USING ENERGY METHODS**

THOMAS BELL

Volume 286 No. 2

February 2017

UNIQUENESS OF CONFORMAL RICCI FLOW USING ENERGY METHODS

THOMAS BELL

We analyze an energy functional associated to conformal Ricci flow along closed manifolds with constant negative scalar curvature. Given initial conditions we use this functional to demonstrate the uniqueness of both the metric and the pressure function along conformal Ricci flow.

1. Introduction

Uniqueness of Ricci flow on closed manifolds was originally proved by Hamilton [1982]. Chen and Zhu [2006] subsequently proved uniqueness on complete noncompact manifolds with bounded curvature. The method employed in [Chen and Zhu 2006] utilizes DeTurck Ricci flow. Recently Kotschwar [2014] used energy techniques to give another proof of the uniqueness on complete manifolds. Kotschwar's proof does not rely on DeTurck Ricci flow. A natural question is whether similar techniques can be applied to demonstrate uniqueness of other geometric flows. One such flow is conformal Ricci flow, introduced by Fischer [2004]. Ricci flow preserves many important properties of metrics, but it generally does not preserve the property of constant scalar curvature. Conformal Ricci flow is a modification of Ricci flow which is intended for this purpose, and for this reason it is restricted to the class of metrics of constant scalar curvature. Conformal Ricci flow is, like Ricci flow, a weakly parabolic flow of the metric on manifolds, except that conformal Ricci flow is coupled with an elliptic equation.

Let (M^n, g_0) be a smooth n -dimensional Riemannian manifold with a metric g_0 of constant scalar curvature s_0 . Conformal Ricci flow on M is defined as follows:

$$(1) \quad \begin{cases} \frac{\partial g}{\partial t} = -2 \operatorname{Ric}_{g(t)} + 2 \frac{s_0}{n} g(t) - 2p(t)g(t), & \text{on } M \times [0, T], \\ s(g(t)) = s_0 \end{cases}$$

Here $g(t)$, $t \in [0, T]$, is a family of metrics on M with $g(0) = g_0$, $s(g(t))$ is the scalar curvature of $g(t)$, and $p(t)$, $t \in [0, T]$, is a family of functions on M .

MSC2010: primary 53C25, 53C44; secondary 35K65.

Keywords: conformal Ricci flow, Ricci flow.

In [Fischer 2004; Lu et al. 2014] we see that (1) is equivalent to the following system:

$$(2) \quad \begin{cases} \frac{\partial g}{\partial t} = -2 \operatorname{Ric}_{g(t)} + 2 \frac{s_0}{n} g(t) - 2p(t)g(t) \\ ((n-1)\Delta_{g(t)} + s_0)p(t) = -\left\langle \operatorname{Ric}_{g(t)} - \frac{s_0}{n} g(t), \operatorname{Ric}_{g(t)} - \frac{s_0}{n} g(t) \right\rangle. \end{cases}$$

Throughout we use V to denote the following symmetric 2-tensor:

$$(3) \quad V(t) = \operatorname{Ric}_{g(t)} - \frac{s_0}{n} g(t) + p(t)g(t).$$

In this paper we use Kotschwar’s energy techniques to give a proof of the uniqueness of conformal Ricci flow for closed manifolds with metrics of constant negative scalar curvature. It is worth noting similarities to the study of certain elliptic-hyperbolic systems done by Andersson and Moncrief [2011]. The existence of solutions to conformal Ricci flow has been shown by Fischer [2004] and by Lu, Qing, and Zheng [2014], the latter paper using DeTurck conformal Ricci flow. More precisely we prove the following uniqueness theorem of conformal Ricci flow:

Theorem 1. *Let (M^n, g_0) be a closed manifold with constant negative scalar curvature s_0 . Suppose $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ are two solutions of (1) on $M \times [0, T]$ with $\tilde{g}(0) = g(0)$. Then $(\tilde{g}(t), \tilde{p}(t)) = (g(t), p(t))$ for $0 \leq t \leq T$.*

2. The differences between $g(t)$ and $\tilde{g}(t)$

Let $g(t)$ and $\tilde{g}(t)$ be as in Theorem 1. We treat g as our background metric and \tilde{g} as our alternative metric. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of g and \tilde{g} respectively. Similarly, let R and \tilde{R} represent the full Riemannian curvature tensors of g and \tilde{g} respectively.

Let $h = g - \tilde{g}$, and $A = \nabla - \tilde{\nabla}$. Explicitly, $A^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ where Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of ∇ and $\tilde{\nabla}$ respectively. Also let $S = R - \tilde{R}$ and $q = p - \tilde{p}$.

In this section we find bounds on $h, A, S, q, \nabla q$, and $\nabla \nabla q$ (see Propositions 3 and 5). Throughout this chapter we use the convention $X * Y$ to denote any finite sum of tensors of the form $X \cdot Y$. We use $C(X)$ to denote a finite sum of tensors of the form X .

2.1. Preliminary calculations. First we calculate some useful expressions for quantities which arise in the proofs of Propositions 3 and 5. We calculate

$$g^{ij} - \tilde{g}^{ij} = g^{ik}(\tilde{g}^{j\ell} \tilde{g}_{k\ell}) - \tilde{g}^{j\ell}(g^{ik} g_{k\ell}) = -g^{ik} \tilde{g}^{j\ell} h_{k\ell},$$

i.e.,

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h.$$

If X is any tensor which is not a function we have

$$(\nabla - \tilde{\nabla})X = A * X.$$

We check this when X is a $(1, 1)$ -tensor. Calculating in local coordinates we see

$$\begin{aligned} (\nabla_i - \tilde{\nabla}_i)X_j^k &= \partial_i X_j^k - \Gamma_{ij}^\ell X_\ell^k + \Gamma_{i\ell}^k X_j^\ell - \partial_i X_j^k + \tilde{\Gamma}_{ij}^\ell X_\ell^k - \tilde{\Gamma}_{i\ell}^k X_j^\ell \\ &= A_{i\ell}^k X_j^\ell - A_{ij}^\ell X_\ell^k = A * X. \end{aligned}$$

If f is a function however, then we have the following:

$$(\nabla^i - \tilde{\nabla}^i)f = (g^{ij} - \tilde{g}^{ij})\partial_j f = -g^{ik}\tilde{g}^{j\ell}h_{k\ell}\partial_j f = -g^{ik}h_{k\ell}\tilde{\nabla}^\ell f,$$

or in other words

$$(\nabla - \tilde{\nabla})f = h * \tilde{\nabla}f.$$

We now calculate

$$\nabla\tilde{g}^{-1} = (\nabla - \tilde{\nabla})\tilde{g}^{-1} = \tilde{g}^{-1} * A.$$

The following calculation is also important.

$$\nabla_i h_{jk} = \nabla_i g_{jk} - \nabla_i \tilde{g}_{jk} = -(\nabla_i - \tilde{\nabla}_i)\tilde{g}_{jk}.$$

Thus we have

$$\nabla h = \tilde{g} * A.$$

Now we are able to calculate the following for a function f .

$$\begin{aligned} \nabla(\nabla - \tilde{\nabla})f &= \nabla(h * \tilde{\nabla}f) \\ &= \nabla h * \tilde{\nabla}f + h * (\nabla - \tilde{\nabla})\tilde{\nabla}f + h * \tilde{\nabla}\tilde{\nabla}f \\ &= \tilde{g} * A * \tilde{\nabla}f + h * A * \tilde{\nabla}f + h * \tilde{\nabla}\tilde{\nabla}f. \end{aligned}$$

Now let

$$\begin{aligned} (4) \quad U_{ijkl}^a &= g^{ab}\nabla_b \tilde{R}_{ijkl} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}_{ijkl} \\ &= g^{ab}(\nabla_b - \tilde{\nabla}_b)\tilde{R}_{ijkl} + (g^{ab} - \tilde{g}^{ab})\tilde{\nabla}_b \tilde{R}_{ijkl} \\ &= A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R}, \end{aligned}$$

and we may calculate

$$\begin{aligned} \nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) &= \nabla_a(g^{ab}\nabla_b \tilde{R} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) + g^{ab}\nabla_a \nabla_b (R - \tilde{R}) \\ &= \operatorname{div} U + \Delta S. \end{aligned}$$

We summarize the above calculations in the following lemma:

Lemma 2. *Using the notation defined at the beginning of this section,*

$$(5) \quad g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h,$$

$$(6) \quad (\nabla - \tilde{\nabla})X = A * X,$$

$$(7) \quad (\nabla - \tilde{\nabla})f = h * \tilde{\nabla}f,$$

$$(8) \quad \nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A,$$

$$(9) \quad \nabla h = \tilde{g} * A,$$

$$(10) \quad \nabla(\nabla - \tilde{\nabla})f = \tilde{g} * A * \tilde{\nabla}f + h * A * \tilde{\nabla}f + h * \tilde{\nabla}\tilde{\nabla}f,$$

$$(11) \quad U = A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R},$$

$$(12) \quad \nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) = \operatorname{div} U + \Delta S,$$

where U is defined in (4).

2.2. Bounds on time derivatives of h , A and S . In this subsection we derive bounds on the time derivatives of h , A and S . In particular we prove the following proposition. Here, as well as throughout this chapter, we let C denote a constant dependent only upon n while N denotes a constant with further dependencies.

Proposition 3. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of (1) on $M \times [0, T]$. Using the notation defined at the beginning of this section, there exist constants N_h , N_A and N_S such that*

$$(13) \quad \left| \frac{\partial}{\partial t} h \right| \leq N_h |h| + C(|S| + |q|),$$

$$(14) \quad \left| \frac{\partial}{\partial t} A \right| \leq N_A (|h| + |A|) + C(|\nabla S| + |\nabla q|),$$

$$(15) \quad \left| \frac{\partial}{\partial t} S - \Delta S - \operatorname{div} U \right| \leq N_S (|h| + |A| + |S| + |q|) + C|\nabla \nabla q|,$$

where U is defined in (4).

Proof. We start with the time derivative of h . By (1) we have

$$\begin{aligned} \frac{\partial}{\partial t} h_{ij} &= -2(R_{ij} - \tilde{R}_{ij}) + 2\frac{s_0}{n}(g_{ij} - \tilde{g}_{ij}) - 2(p g_{ij} - \tilde{p} \tilde{g}_{ij}) \\ &= -2S_{kij}^k + 2\frac{s_0}{n}h_{ij} - 2[(p - \tilde{p})g_{ij} + \tilde{p}(g_{ij} - \tilde{g}_{ij})] \\ &= -2S_{kij}^k + 2\frac{s_0}{n}h_{ij} - 2q g_{ij} - 2\tilde{p} h_{ij}. \end{aligned}$$

Hence

$$\frac{\partial}{\partial t} h = C(S) + C(s_0 h) + C(q) + \tilde{p} * h$$

and

$$(16) \quad \left| \frac{\partial}{\partial t} h \right| \leq C((|s_0| + |\tilde{p}|)|h| + |S| + |q|).$$

This proves (13).

Recall the definition of V from (3):

$$(17) \quad V(t) = \text{Ric}_{g(t)} - \frac{s_0}{n} g(t) + p(t)g(t).$$

We may define \tilde{V} similarly using our alternate metric \tilde{g} . Since V and \tilde{V} are symmetric 2-tensors, then by [Chow et al. 2006, p. 108] we may calculate

$$(18) \quad \frac{\partial}{\partial t} A_{ij}^k = \tilde{g}^{kl} (\tilde{\nabla}_i \tilde{V}_{j\ell} + \tilde{\nabla}_j \tilde{V}_{i\ell} - \tilde{\nabla}_\ell \tilde{V}_{ij}) - g^{kl} (\nabla_i V_{j\ell} + \nabla_j V_{i\ell} - \nabla_\ell V_{ij}).$$

We proceed to calculate

$$(19) \quad \begin{aligned} & \tilde{g}^{kl} \tilde{\nabla}_i \tilde{V}_{j\ell} - g^{kl} \nabla_i V_{j\ell} \\ &= \tilde{g}^{kl} (\tilde{\nabla}_i \tilde{R}_{j\ell}) - g^{kl} (\nabla_i R_{j\ell}) + \tilde{g}^{kl} \tilde{\nabla}_i (\tilde{p} \tilde{g}_{j\ell}) - g^{kl} \nabla_i (p g_{j\ell}) \\ &= (\tilde{g}^{kl} - g^{kl}) \tilde{\nabla}_i \tilde{R}_{j\ell} + g^{kl} (\tilde{\nabla}_i - \nabla_i) \tilde{R}_{j\ell} - g^{kl} \nabla_i (S_{m_j \ell}^m) + \delta_j^k \tilde{\nabla}_i \tilde{p} - \delta_j^k \nabla_i p \\ &= \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla} \tilde{p} + C(\nabla q), \end{aligned}$$

where we have used (7) to get the last equality. Similarly we find

$$(20) \quad \tilde{g}^{kl} \tilde{\nabla}_j \tilde{V}_{i\ell} - g^{kl} \nabla_j V_{i\ell} = \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla} \tilde{p} + C(\nabla q).$$

Now we consider

$$(21) \quad \begin{aligned} & -\tilde{g}^{kl} \tilde{\nabla}_\ell \tilde{V}_{ij} + g^{kl} \nabla_\ell V_{ij} \\ &= \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) + \tilde{g}^{kl} \tilde{g}_{ij} \tilde{\nabla}_\ell \tilde{p} - g^{kl} g_{ij} \nabla_\ell p \\ &= \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) + (\tilde{g}^{kl} - g^{kl}) \tilde{g}_{ij} \tilde{\nabla}_\ell \tilde{p} \\ &\quad + g^{kl} (\tilde{g}_{ij} - g_{ij}) \tilde{\nabla}_\ell \tilde{p} + g^{kl} g_{ij} (\tilde{\nabla}_\ell - \nabla_\ell) \tilde{p} \\ &\quad + g^{kl} g_{ij} \nabla_\ell (\tilde{p} - p) \\ &= \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) \\ &\quad + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{p} + C(\nabla q). \end{aligned}$$

Hence by (18), (19), (20) and (21),

$$\frac{\partial}{\partial t} A = \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{R} + A * \tilde{R} + C(\nabla S) + h * \tilde{\nabla} \tilde{p} + C(\nabla q) + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{p}$$

and

$$(22) \quad \left| \frac{\partial}{\partial t} A \right| \leq C ((|\tilde{g}^{-1}| |\tilde{\nabla} \tilde{R}| + |\tilde{\nabla} \tilde{p}| + |\tilde{g}^{-1}| |\tilde{g}| |\tilde{\nabla} \tilde{p}|) |h| + |\tilde{R}| |A| + |\nabla S| + |\nabla q|).$$

This proves (14).

By [Chow et al. 2006, equation (2.67)] we have

$$(23) \quad \begin{aligned} \frac{\partial}{\partial t} R_{ijk}^\ell &= g^{\ell m} (\nabla_i \nabla_k V_{jm} - \nabla_i \nabla_m V_{jk} - \nabla_j \nabla_k V_{im} + \nabla_j \nabla_m V_{ik}) \\ &\quad - g^{\ell m} (R_{ijk}^r V_{rm} + R_{ijm}^q V_{kq}) \\ &= g^{\ell m} (-\nabla_i \nabla_k R_{jm} + \nabla_i \nabla_m R_{jk} + \nabla_j \nabla_k R_{im} - \nabla_j \nabla_m R_{ik}) \\ &\quad + g^{\ell m} (-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p) \\ &\quad + g^{\ell m} (R_{ijk}^r R_{rm} + R_{ijm}^r R_{kr}) - \frac{S_0}{n} g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) p \\ &\quad + g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) p. \end{aligned}$$

Following the calculations in [Chow et al. 2006, pp. 119–120] we have

$$(24) \quad \begin{aligned} \Delta R_{ijk}^\ell &= g^{ab} \nabla_a \nabla_b R_{ijk}^\ell = g^{ab} (-\nabla_a \nabla_i R_{jkb}^\ell - \nabla_a \nabla_j R_{bik}^\ell) \\ &= g^{ab} (-\nabla_i \nabla_a R_{jkb}^\ell + R_{aij}^m R_{mbk}^\ell + R_{aib}^m R_{jmk}^\ell + R_{aik}^m R_{jbm}^\ell - R_{aim}^\ell R_{jkb}^m \\ &\quad - \nabla_j \nabla_a R_{bik}^\ell + R_{ajb}^m R_{mik}^\ell + R_{aji}^m R_{bmk}^\ell + R_{ajk}^m R_{bim}^\ell - R_{ajm}^\ell R_{bik}^m) \\ &= g^{\ell m} (-\nabla_i \nabla_k R_{jm} + \nabla_i \nabla_m R_{jk} + \nabla_j \nabla_k R_{im} - \nabla_j \nabla_m R_{ik}) \\ &\quad + g^{mr} (-R_{ir} R_{jmk}^\ell - R_{jr} R_{mik}^\ell) \\ &\quad + g^{ab} (R_{aij}^m R_{mbk}^\ell + R_{aik}^m R_{jbm}^\ell - R_{aim}^\ell R_{jkb}^m + R_{aji}^m R_{bmk}^\ell \\ &\quad + R_{ajk}^m R_{bim}^\ell - R_{ajm}^\ell R_{bik}^m). \end{aligned}$$

Combining (23) and (24) we have

$$(25) \quad \begin{aligned} \frac{\partial}{\partial t} R_{ijk}^\ell &= \Delta R_{ijk}^\ell + g^{mr} (R_{ir} R_{jmk}^\ell + R_{jr} R_{mik}^\ell) \\ &\quad + g^{ab} (-R_{aij}^m R_{mbk}^\ell - R_{aik}^m R_{jbm}^\ell + R_{aim}^\ell R_{jkb}^m \\ &\quad - R_{aji}^m R_{bmk}^\ell - R_{ajk}^m R_{bim}^\ell + R_{ajm}^\ell R_{bik}^m) \\ &\quad + g^{\ell m} (-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p) \\ &\quad + g^{\ell m} (R_{ijk}^r R_{rm} + R_{ijm}^r R_{kr}) - \frac{S_0}{n} g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) \\ &\quad + g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) p. \end{aligned}$$

Hence the evolution of S is

$$\begin{aligned}
(26) \quad \frac{\partial}{\partial t} S_{ijk}^\ell &= \Delta R_{ijk}^\ell - \tilde{\Delta} \tilde{R}_{ijk}^\ell \\
&+ g^{mr} (R_{ir} R_{jmk}^\ell + R_{jr} R_{imk}^\ell) - \tilde{g}^{mr} (\tilde{R}_{ir} \tilde{R}_{jmk}^\ell + \tilde{R}_{jr} \tilde{R}_{mik}^\ell) \\
&+ g^{ab} (-R_{aij}^m R_{mbk}^\ell - R_{aik}^m R_{jbm}^\ell + R_{aim}^r R_{jbk}^\ell - R_{aji}^m R_{bmk}^\ell \\
&\quad - R_{ajk}^m R_{bim}^\ell + R_{ajm}^\ell R_{bik}^m) \\
&- \tilde{g}^{ab} (-\tilde{R}_{aij}^m \tilde{R}_{mbk}^\ell - \tilde{R}_{aik}^m \tilde{R}_{jbm}^\ell + \tilde{R}_{aim}^r \tilde{R}_{jbk}^\ell - \tilde{R}_{aji}^m \tilde{R}_{bmk}^\ell \\
&\quad - \tilde{R}_{ajk}^m \tilde{R}_{bim}^\ell + \tilde{R}_{ajm}^\ell \tilde{R}_{bik}^m) \\
&+ g^{\ell m} (-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p) \\
&- \tilde{g}^{\ell m} (-\tilde{g}_{jm} \tilde{\nabla}_i \tilde{\nabla}_k \tilde{p} + \tilde{g}_{jk} \tilde{\nabla}_i \tilde{\nabla}_m \tilde{p} + \tilde{g}_{im} \tilde{\nabla}_j \tilde{\nabla}_k \tilde{p} - \tilde{g}_{ik} \tilde{\nabla}_j \tilde{\nabla}_m \tilde{p}) \\
&+ g^{\ell m} (R_{ijk}^r R_{rm} + R_{ijm}^r R_{kr}) - \tilde{g}^{\ell m} (\tilde{R}_{ijk}^r \tilde{R}_{rm} + \tilde{R}_{ijm}^r \tilde{R}_{kr}) \\
&- \frac{S_0}{n} g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) + \frac{S_0}{n} \tilde{g}^{\ell m} (\tilde{R}_{ijk}^r \tilde{g}_{rm} + \tilde{R}_{ijm}^r \tilde{g}_{kr}) \\
&+ g^{\ell m} (R_{ijk}^r g_{rm} + R_{ijm}^r g_{kr}) p - \tilde{g}^{\ell m} (\tilde{R}_{ijk}^r \tilde{g}_{rm} + \tilde{R}_{ijm}^r \tilde{g}_{kr}) \tilde{p}.
\end{aligned}$$

Looking at the individual components, we see

$$\begin{aligned}
(27) \quad \Delta R - \tilde{\Delta} \tilde{R} &= g^{ab} \nabla_a \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{R} \\
&= \nabla_a (g^{ab} \nabla_b R) - \nabla_a (\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + (\nabla_a - \tilde{\nabla}_a) (\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) \\
&= \nabla_a (g^{ab} \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + \tilde{g}^{-1} * A * \tilde{\nabla} \tilde{R},
\end{aligned}$$

while

$$\begin{aligned}
(28) \quad g^{-1} R R - \tilde{g}^{-1} \tilde{R} \tilde{R} &= (g^{-1} - \tilde{g}^{-1}) (\tilde{R} \tilde{R}) + g^{-1} (R R - \tilde{R} \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + g^{-1} (R - \tilde{R}) \tilde{R} + g^{-1} (R R - R \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R,
\end{aligned}$$

and

$$\begin{aligned}
(29) \quad g^{-1} g \nabla \nabla p - \tilde{g}^{-1} \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} &= (g^{-1} - \tilde{g}^{-1}) \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} + g^{-1} (g - \tilde{g}) \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
&\quad + g^{-1} g (\nabla \nabla p - \tilde{\nabla} \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
&\quad + g^{-1} g (\nabla - \tilde{\nabla}) (\tilde{\nabla} \tilde{p}) + g^{-1} g (\nabla \nabla p - \nabla \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
&\quad + g^{-1} g \nabla (\nabla - \tilde{\nabla}) \tilde{p} + g^{-1} g \nabla \nabla (p - \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
&\quad + h * A * \tilde{\nabla} \tilde{p} + C (\nabla \nabla q),
\end{aligned}$$

where in the last equality we used (10). We also have

$$(30) \quad g^{-1}gR - \tilde{g}^{-1}\tilde{g}\tilde{R} = (g^{-1} - \tilde{g}^{-1})\tilde{g}\tilde{R} + g^{-1}(g - \tilde{g})\tilde{R} + g^{-1}g(R - \tilde{R}) \\ = \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} + h * \tilde{R} + C(S),$$

and lastly

$$(31) \quad g^{-1}gRp - \tilde{g}^{-1}\tilde{g}\tilde{R}\tilde{p} = (g^{-1} - \tilde{g}^{-1})\tilde{g}\tilde{R}\tilde{p} + g^{-1}(g - \tilde{g})\tilde{R}\tilde{p} \\ + g^{-1}g(R - \tilde{R})\tilde{p} + g^{-1}gR(p - \tilde{p}) \\ = \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} * \tilde{p} + h * \tilde{R} * \tilde{p} + S * \tilde{p} + R * q.$$

Now by (26), (27), (28), (29), (30) and (31) we see

$$\frac{\partial}{\partial t} S = \nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) + \tilde{g}^{-1} * A * \tilde{\nabla} \tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\ + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\ + h * A * \tilde{\nabla} \tilde{p} + C(\nabla \nabla q) + \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} + h * \tilde{R} + C(S) \\ + \tilde{g}^{-1} * h * \tilde{g} * \tilde{R} * \tilde{p} + h * \tilde{R} * \tilde{p} + S * \tilde{p} + R * q.$$

Hence by (12) we have

$$(32) \quad \left| \frac{\partial}{\partial t} S - \Delta S - \operatorname{div} U \right| \leq C((|\tilde{g}^{-1}| |\tilde{R}|^2 + |\tilde{g}^{-1}| |\tilde{g}| |\tilde{\nabla} \tilde{\nabla} \tilde{p}| + |\tilde{\nabla} \tilde{\nabla} \tilde{p}| \\ + |\tilde{g}^{-1}| |\tilde{g}| |\tilde{R}| + |\tilde{R}| + |\tilde{g}^{-1}| |\tilde{g}| |\tilde{R}| |\tilde{p}| + |\tilde{R}| |\tilde{p}|) |h| \\ + (|\tilde{g}^{-1}| |\tilde{\nabla} \tilde{R}| + |\tilde{\nabla} \tilde{p}| + |h| |\tilde{\nabla} \tilde{p}|) |A| \\ + (|\tilde{R}| + |R| + 1 + |\tilde{p}|) |S| + |R| |q| + |\nabla \nabla q|).$$

This proves (15). □

Remark 4. Upon closer observation we notice the following dependencies:

$$N_h = N_h(n, s_0, |\tilde{p}|). \\ N_A = N_A(n, s_0, |\tilde{g}|, |\tilde{g}^{-1}|, |\tilde{R}|, |\tilde{\nabla} \tilde{R}|, |\tilde{\nabla} \tilde{p}|). \\ N_S = N_S(n, s_0, |\tilde{g}|, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla} \tilde{R}|, |\tilde{p}|, |\tilde{\nabla} \tilde{p}|, |\tilde{\nabla} \tilde{\nabla} \tilde{p}|).$$

M is closed, so $M \times [0, T]$ is compact. Thus, given two metrics g and \tilde{g} , all of these quantities are bounded.

2.3. Bounds on q and its spatial derivatives. We turn our attention now to finding bounds on the differences between our pressure functions p and \tilde{p} . We have the following proposition:

Proposition 5. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of (1) on $M \times [0, T]$. Then there exist constants N_q and \hat{N}_q such that*

$$(33) \quad \int_M |q|^2 d\mu \leq N_q \int_M (|h|^2 + |A|^2 + |S|^2) d\mu,$$

$$(34) \quad \int_M |\nabla q|^2 d\mu \leq N_q \int_M (|h|^2 + |A|^2 + |S|^2) d\mu,$$

$$(35) \quad \int_M |\nabla \nabla q|^2 d\mu \leq \hat{N}_q \int_M (|h|^2 + |A|^2 + |S|^2) d\mu.$$

Proof. We let f represent any smooth function or tensor on M . This is general, but in this paper we represent f by the function q , the difference of the pressure functions. Since M is compact we have

$$\int_M ((n-1)\Delta + s_0)(f) \cdot f d\mu = s_0 \int_M |f|^2 d\mu - (n-1) \int_M \langle \nabla f, \nabla f \rangle d\mu.$$

Since $s_0 < 0$, taking the absolute value gives

$$(36) \quad \left| \int_M ((n-1)\Delta + s_0)(f) \cdot f d\mu \right| = |s_0| \int_M |f|^2 d\mu + (n-1) \int_M |\nabla f|^2 d\mu.$$

Now we deal specifically with p, \tilde{p} and q . By (2) we have the following equations for the pressure functions p and \tilde{p} :

$$(37) \quad ((n-1)\Delta + s_0)p = -\left\langle \text{Ric} - \frac{s_0}{n}g, \text{Ric} - \frac{s_0}{n}g \right\rangle.$$

$$(38) \quad ((n-1)\tilde{\Delta} + s_0)\tilde{p} = -\left\langle \tilde{\text{Ric}} - \frac{s_0}{n}\tilde{g}, \tilde{\text{Ric}} - \frac{s_0}{n}\tilde{g} \right\rangle.$$

Now we calculate

$$(39) \quad \begin{aligned} \Delta p - \tilde{\Delta} \tilde{p} &= g^{ab} \nabla_a \nabla_b p - \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{p} \\ &= (g^{-1} - \tilde{g}^{-1}) \tilde{\nabla} \tilde{\nabla} \tilde{p} + g^{-1} (\nabla - \tilde{\nabla}) \tilde{\nabla} \tilde{p} + g^{-1} \nabla (\nabla - \tilde{\nabla}) \tilde{p} + \Delta(p - \tilde{p}) \\ &= \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} + h * A * \tilde{\nabla} \tilde{p} + \Delta q. \end{aligned}$$

We also compute

$$(40) \quad \begin{aligned} &-\left\langle \text{Ric} - \frac{s_0}{n}g, \text{Ric} - \frac{s_0}{n}g \right\rangle + \left\langle \tilde{\text{Ric}} - \frac{s_0}{n}\tilde{g}, \tilde{\text{Ric}} - \frac{s_0}{n}\tilde{g} \right\rangle \\ &= -(g^{ik} g^{jl} R_{ij} R_{kl} - \tilde{g}^{ik} \tilde{g}^{jl} \tilde{R}_{ij} \tilde{R}_{kl}) + 2 \frac{s_0}{n} (g^{ij} R_{ij} - \tilde{g}^{ij} \tilde{R}_{ij}) \\ &= -(g^{-1} - \tilde{g}^{-1}) \tilde{g}^{-1} \tilde{R} \tilde{R} - g^{-1} (g^{-1} - \tilde{g}^{-1}) \tilde{R} \tilde{R} - g^{-1} g^{-1} (R - \tilde{R}) \tilde{R} \\ &\quad - g^{-1} g^{-1} R (R - \tilde{R}) + 2 \frac{s_0}{n} (g^{-1} - \tilde{g}^{-1}) \tilde{R} + 2 \frac{s_0}{n} g^{-1} (R - \tilde{R}) \\ &= \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\ &\quad + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S). \end{aligned}$$

Combining (37), (38), (39) and (40), we see that q satisfies the following elliptic equation at each time $t \in [0, T]$:

$$(41) \quad Lq = ((n - 1)\Delta + s_0)(q) \\ = \tilde{g}^{-1} * h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} + h * A * \tilde{\nabla} \tilde{p} + \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\ + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S).$$

Hence

$$(42) \quad |Lq| = |((n - 1)\Delta + s_0)(q)| \leq N(|h| + |A| + |S|).$$

To find estimates for q and ∇q , we combine (36) and (42):

$$|s_0| \int_M |q|^2 d\mu + (n-1) \int_M |\nabla q|^2 d\mu = \left| \int_M ((n-1)\Delta + s_0)(q) \cdot q d\mu \right| \\ \leq \int_M N(|h| + |A| + |S|) |q| d\mu \\ \leq \frac{|s_0|}{2} \int_M |q|^2 d\mu + N \int_M (|h|^2 + |A|^2 + |S|^2) d\mu.$$

Thus

$$\frac{|s_0|}{2} \int_M |q|^2 d\mu + (n - 1) \int_M |\nabla q|^2 d\mu \leq N \int_M (|h|^2 + |A|^2 + |S|^2) d\mu,$$

and we proved (33) and (34).

To find an appropriate bound for $|\nabla \nabla q|$ we use interior regularity theory for elliptic PDEs. From (41) we see that $Lq = f$ is an elliptic equation. We then have the following estimate from [Rauch 1991, p. 229]:

$$|q|_{H^2(W)} \leq K(|Lq|_{L^2(M)} + |q|_{H^1(M)}),$$

where W is any compactly supported open subset of M and K depends only upon the coefficients of the operator L , the subset W and the manifold M . Since M is a closed manifold we may in fact choose $W = M$. Thus we have

$$(43) \quad |q|_{H^2(M)} \leq K(|Lq|_{L^2(M)} + |q|_{H^1(M)}).$$

Upon squaring both sides we observe

$$(44) \quad \int_M |\nabla \nabla q|^2 d\mu \leq |q|_{H^2(M)}^2 \leq K^2 \left(\int_M |Lq|^2 d\mu + |q|_{H^1(M)}^2 \right).$$

Now (33) and (34) imply that

$$(45) \quad |q|_{H^1(M)}^2 \leq N \int_M (|h|^2 + |A|^2 + |S|^2) d\mu.$$

Combining (42), (44) and (45) we have

$$\int_M |\nabla \nabla q|^2 d\mu \leq N \int_M (|h|^2 + |A|^2 + |S|^2) d\mu,$$

and we have proved (35). \square

Remark 6. We observe the following dependencies:

$$\begin{aligned} N_q &= N_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla} \tilde{p}|, |\tilde{\nabla} \tilde{\nabla} \tilde{p}|), \\ \hat{N}_q &= \hat{N}_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla} \tilde{p}|, |\tilde{\nabla} \tilde{\nabla} \tilde{p}|, K), \end{aligned}$$

where K is from (43).

3. Energy estimates

We now define the energy functional

$$(46) \quad \mathcal{E}(t) = \int_M (|h|^2 + |A|^2 + |S|^2) d\mu,$$

as well as the following:

$$(47) \quad \mathcal{H}(t) = \int_M |h|^2 d\mu.$$

$$(48) \quad \mathcal{A}(t) = \int_M |A|^2 d\mu.$$

$$(49) \quad \mathcal{S}(t) = \int_M |S|^2 d\mu.$$

$$(50) \quad \mathcal{D}(t) = \int_M |\nabla S|^2 d\mu.$$

Note that $\mathcal{E}(t) = \mathcal{H}(t) + \mathcal{A}(t) + \mathcal{S}(t)$. We now estimate the evolution of the energy functional under conformal Ricci flow, $\mathcal{E}'(t)$, by first estimating the evolutions of \mathcal{H} , \mathcal{A} and \mathcal{S} .

3.1. Evolution of $\mathcal{H}(t)$. Lu, Qing and Zheng [2014] give the evolution of the volume element under conformal Ricci flow

$$(51) \quad \frac{\partial}{\partial t} d\mu_{g(t)} = -np(t) d\mu_{g(t)}.$$

Hence by (13) and (47) we have

$$\begin{aligned} \mathcal{H}'(t) &\leq N \int_M |h|^2 d\mu + \int_M 2 \left\langle \frac{\partial h}{\partial t}, h \right\rangle d\mu \\ &\leq N\mathcal{H}(t) + \int_M 2|h| \left| \frac{\partial h}{\partial t} \right| d\mu \\ &\leq N\mathcal{H}(t) + N \int_M (|S||h| + |h|^2 + |q||h|) d\mu. \end{aligned}$$

Now we know that $N(|S||h| + |q||h|) \leq N(|h|^2 + |S|^2 + |q|^2)$. Hence

$$\begin{aligned} (52) \quad \mathcal{H}'(t) &\leq N\mathcal{H}(t) + N \int_M (|S|^2 + |q|^2) d\mu \\ &\leq N\mathcal{H}(t) + N \int_M (|S|^2 + |h|^2 + |A|^2) d\mu \\ &\leq N\mathcal{H}(t) + N\mathcal{S}(t) + N\mathcal{A}(t) = N\mathcal{E}(t). \end{aligned}$$

3.2. Evolution of $\mathcal{A}(t)$. By (14), (48) and (51) we have

$$\begin{aligned} \mathcal{A}'(t) &\leq N\mathcal{A}(t) + \int_M 2|A| \left| \frac{\partial A}{\partial t} \right| d\mu \\ &\leq N\mathcal{A}(t) + \int_M (N|h||A| + N|A|^2 + C|\nabla S||A| + C|\nabla q||A|) d\mu. \end{aligned}$$

Now

$$N|h||A| + C|\nabla S||A| + C|\nabla q||A| \leq N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2.$$

Hence we have that

$$\begin{aligned} (53) \quad \mathcal{A}'(t) &\leq N\mathcal{A}(t) + \int_M (N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2) d\mu \\ &\leq N\mathcal{A}(t) + N\mathcal{H}(t) + \mathcal{D}(t) + N \int_M (|h|^2 + |A|^2 + |S|^2) d\mu \\ &\leq N\mathcal{A}(t) + N\mathcal{H}(t) + N\mathcal{S}(t) + \mathcal{D}(t) = N\mathcal{E}(t) + \mathcal{D}(t). \end{aligned}$$

3.3. Evolution of $\mathcal{S}(t)$. By (15), (49) and (51) we have

$$\begin{aligned} \mathcal{S}'(t) &\leq N \int_M |S|^2 d\mu + \int_M 2 \left\langle \frac{\partial S}{\partial t}, S \right\rangle d\mu \\ &\leq N\mathcal{S}(t) + \int_M (2\langle \Delta S + \operatorname{div} V, S \rangle + N(|h| + |A| + |S| + |q|)|S| + C|\nabla \nabla q||S|) d\mu \\ &\leq N\mathcal{S}(t) + \int_M (2\langle \Delta S + \operatorname{div} V, S \rangle + N(|h|^2 + |A|^2 + |S|^2 + |q|^2 + |\nabla \nabla q|^2)) d\mu. \end{aligned}$$

Now by (33) and (35) we have

$$\begin{aligned} S'(t) &\leq N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) \\ &\quad + \int_M (2\langle \Delta S + \operatorname{div} V, S \rangle + N(|A|^2 + |S|^2 + |h|^2)) d\mu \\ &\leq N\mathcal{S}(t) + N\mathcal{H}(t) + N\mathcal{A}(t) + \int_M 2\langle \Delta S + \operatorname{div} V, S \rangle d\mu. \end{aligned}$$

Upon integrating by parts we get

$$\begin{aligned} S'(t) &\leq N\mathcal{E}(t) - 2 \int_M \langle \nabla S + V, \nabla S \rangle d\mu \\ &\leq N\mathcal{E}(t) - 2 \int_M |\nabla S|^2 d\mu + \int_M 2|V||\nabla S| d\mu. \end{aligned}$$

Now we know that

$$2|V||\nabla S| \leq |\nabla S|^2 + |V|^2 \leq |\nabla S|^2 + N(|h|^2 + |A|^2),$$

hence

$$(54) \quad S'(t) \leq N\mathcal{E}(t) + N \int_M (|h|^2 + |A|^2) d\mu - \int_M |\nabla S|^2 d\mu \leq N\mathcal{E}(t) - \mathcal{D}(t).$$

3.4. Proof of main theorem. We are now ready to prove Theorem 1.

Proof. By (52), (53) and (54) we know that

$$\mathcal{H}'(t) \leq N\mathcal{E}(t), \quad \mathcal{A}'(t) \leq N\mathcal{E}(t) + \mathcal{D}(t) \quad \text{and} \quad S'(t) \leq N\mathcal{E}(t) - \mathcal{D}(t),$$

so

$$\mathcal{E}'(t) \leq N\mathcal{E}(t).$$

Our initial condition $\tilde{g}(0) = g(0)$ tells us that at $t = 0$ we have $|h| = |A| = |S| = 0$. Therefore by the smoothness and integrability of our solutions we know

$$\lim_{t \rightarrow 0^+} \mathcal{E}(t) = 0,$$

so by Gronwall's inequality we know that $\mathcal{E} \equiv 0$ on $[0, T]$. Thus for $t \in [0, T]$ we have that $h \equiv 0$ and $g(t) \equiv \tilde{g}(t)$. Also, $\mathcal{E} \equiv 0$ implies $A \equiv 0$ and $S \equiv 0$, so (33) forces $q \equiv 0$. Thus $p(t) \equiv \tilde{p}(t)$. Therefore $(\tilde{g}(t), \tilde{p}(t)) = (g(t), p(t))$, $t \in [0, T]$. \square

4. Further research

The arguments in this paper are only valid for conformal Ricci flow on a compact manifold with constant positive scalar curvature. In particular, if $s_0 \geq 0$ we do not have the equality (36). It is worth discovering whether or not there is some other way to compute the bounds on q and its derivatives, namely equations (33), (34) and (35).

It is also interesting to consider complete noncompact manifolds of constant scalar curvature. Previous results in Ricci flow and parabolic PDE suggest that in this case we will not achieve uniqueness of conformal Ricci flow without some sort of bound on the curvature of the manifold.

References

- [Andersson and Moncrief 2011] L. Andersson and V. Moncrief, “Einstein spaces as attractors for the Einstein flow”, *J. Differential Geom.* **89**:1 (2011), 1–47. MR Zbl
- [Chen and Zhu 2006] B.-L. Chen and X.-P. Zhu, “Uniqueness of the Ricci flow on complete noncompact manifolds”, *J. Differential Geom.* **74**:1 (2006), 119–154. MR Zbl
- [Chow et al. 2006] B. Chow, P. Lu, and L. Ni, *Hamilton’s Ricci flow*, Graduate Studies in Mathematics **77**, American Mathematical Society, Providence, RI, 2006. MR Zbl
- [Fischer 2004] A. E. Fischer, “An introduction to conformal Ricci flow”, *Classical Quantum Gravity* **21**:3 (2004), S171–S218. MR Zbl
- [Hamilton 1982] R. S. Hamilton, “Three-manifolds with positive Ricci curvature”, *J. Differential Geom.* **17**:2 (1982), 255–306. MR Zbl
- [Kotschwar 2014] B. Kotschwar, “An energy approach to the problem of uniqueness for the Ricci flow”, *Comm. Anal. Geom.* **22**:1 (2014), 149–176. MR Zbl
- [Lu et al. 2014] P. Lu, J. Qing, and Y. Zheng, “A note on conformal Ricci flow”, *Pacific J. Math.* **268**:2 (2014), 413–434. MR Zbl
- [Rauch 1991] J. Rauch, *Partial differential equations*, Graduate Texts in Mathematics **128**, Springer, 1991. MR Zbl

Received March 10, 2015. Revised April 1, 2016.

THOMAS BELL
DEPARTMENT OF MATHEMATICS
BRIGHAM YOUNG UNIVERSITY
275 TMCB
PROVO, UT 84602
UNITED STATES
thomas@mathematics.byu.edu

PACIFIC JOURNAL OF MATHEMATICS

Founded in 1951 by E. F. Beckenbach (1906–1982) and F. Wolf (1904–1989)

msp.org/pjm

EDITORS

Don Blasius (Managing Editor)
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
blasius@math.ucla.edu

Paul Balmer
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
balmer@math.ucla.edu

Robert Finn
Department of Mathematics
Stanford University
Stanford, CA 94305-2125
finn@math.stanford.edu

Sorin Popa
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
popa@math.ucla.edu

Vyjayanthi Chari
Department of Mathematics
University of California
Riverside, CA 92521-0135
chari@math.ucr.edu

Kefeng Liu
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
liu@math.ucla.edu

Igor Pak
Department of Mathematics
University of California
Los Angeles, CA 90095-1555
pak.pjm@gmail.com

Paul Yang
Department of Mathematics
Princeton University
Princeton NJ 08544-1000
yang@math.princeton.edu

Daryl Cooper
Department of Mathematics
University of California
Santa Barbara, CA 93106-3080
cooper@math.ucsb.edu

Jiang-Hua Lu
Department of Mathematics
The University of Hong Kong
Pokfulam Rd., Hong Kong
jhlu@maths.hku.hk

Jie Qing
Department of Mathematics
University of California
Santa Cruz, CA 95064
qing@cats.ucsc.edu

PRODUCTION

Silvio Levy, Scientific Editor, production@msp.org

SUPPORTING INSTITUTIONS

ACADEMIA SINICA, TAIPEI
CALIFORNIA INST. OF TECHNOLOGY
INST. DE MATEMÁTICA PURA E APLICADA
KEIO UNIVERSITY
MATH. SCIENCES RESEARCH INSTITUTE
NEW MEXICO STATE UNIV.
OREGON STATE UNIV.

STANFORD UNIVERSITY
UNIV. OF BRITISH COLUMBIA
UNIV. OF CALIFORNIA, BERKELEY
UNIV. OF CALIFORNIA, DAVIS
UNIV. OF CALIFORNIA, LOS ANGELES
UNIV. OF CALIFORNIA, RIVERSIDE
UNIV. OF CALIFORNIA, SAN DIEGO
UNIV. OF CALIF., SANTA BARBARA

UNIV. OF CALIF., SANTA CRUZ
UNIV. OF MONTANA
UNIV. OF OREGON
UNIV. OF SOUTHERN CALIFORNIA
UNIV. OF UTAH
UNIV. OF WASHINGTON
WASHINGTON STATE UNIVERSITY

These supporting institutions contribute to the cost of publication of this Journal, but they are not owners or publishers and have no responsibility for its contents or policies.


See inside back cover or msp.org/pjm for submission instructions.

The subscription price for 2017 is US \$450/year for the electronic version, and \$625/year for print and electronic. Subscriptions, requests for back issues and changes of subscriber address should be sent to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163, U.S.A. The Pacific Journal of Mathematics is indexed by Mathematical Reviews, Zentralblatt MATH, PASCAL CNRS Index, Referativnyi Zhurnal, Current Mathematical Publications and Web of Knowledge (Science Citation Index).

The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFlow® from Mathematical Sciences Publishers.

PUBLISHED BY

 **mathematical sciences publishers**
nonprofit scientific publishing

<http://msp.org/>

© 2017 Mathematical Sciences Publishers

PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 2 February 2017

Almost everywhere convergence for modified Bochner–Riesz means at the critical index for $p \geq 2$	257
MARCO ANNONI	
Uniqueness of conformal Ricci flow using energy methods	277
THOMAS BELL	
A functional calculus and restriction theorem on H-type groups	291
HEPING LIU and MANLI SONG	
Identities involving cyclic and symmetric sums of regularized multiple zeta values	307
TOMOYA MACHIDE	
Conformally Kähler Ricci solitons and base metrics for warped product Ricci solitons	361
GIDEON MASCHLER	
Calculating Greene’s function via root polytopes and subdivision algebras	385
KAROLA MÉSZÁROS	
Classifying resolving subcategories	401
WILLIAM SANDERS	
The symplectic plactic monoid, crystals, and MV cycles	439
JACINTA TORRES	
A note on torus actions and the Witten genus	499
MICHAEL WIEMELER	



0030-8730(201702)286:2;1-X