Pacific Journal of Mathematics

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Volume 286 No. 2 February 2017

PACIFIC JOURNAL OF MATHEMATICS
Vol. 286, No. 2, 2017
dx.doi.org/10.2140/pjm.2017.286.277

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We analyze an energy functional associated to conformal Ricci flow along closed manifolds with constant negative scalar curvature. Given initial conditions we use this functional to demonstrate the uniqueness of both the metric and the pressure function along conformal Ricci flow.

1. Introduction

Uniqueness of Ricci flow on closed manifolds was originally proved by Hamilton [1982]. Chen and Zhu [2006] subsequently proved uniqueness on complete noncompact manifolds with bounded curvature. The method employed in [Chen and Zhu 2006] utilizes DeTurck Ricci flow. Recently Kotschwar [2014] used energy techniques to give another proof of the uniqueness on complete manifolds. Kotschwar's proof does not rely on DeTurck Ricci flow. A natural question is whether similar techniques can be applied to demonstrate uniqueness of other geometric flows. One such flow is conformal Ricci flow, introduced by Fischer [2004]. Ricci flow preserves many important properties of metrics, but it generally does not preserve the property of constant scalar curvature. Conformal Ricci flow is a modification of Ricci flow which is intended for this purpose, and for this reason it is restricted to the class of metrics of constant scalar curvature. Conformal Ricci flow is, like Ricci flow, a weakly parabolic flow of the metric on manifolds, except that conformal Ricci flow is coupled with an elliptic equation.

Let (M^n, g_0) be a smooth n-dimensional Riemannian manifold with a metric g_0 of constant scalar curvature s_0. Conformal Ricci flow on M is defined as follows:

$$(1)\qquad \begin{cases} \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}_{g(t)} + 2\dfrac{s_0}{n} g(t) - 2p(t)g(t), \\ s(g(t)) = s_0 \end{cases} \qquad \text{on } M \times [0, T].$$

Here $g(t)$, $t \in [0, T]$, is a family of metrics on M with $g(0) = g_0$, $s(g(t))$ is the scalar curvature of $g(t)$, and $p(t)$, $t \in [0, T]$, is a family of functions on M.

MSC2010: primary 53C25, 53C44; secondary 35K65.
Keywords: conformal Ricci flow, Ricci flow.

In [Fischer 2004; Lu et al. 2014] we see that (1) is equivalent to the following system:

$$(2)\qquad \begin{cases} \dfrac{\partial g}{\partial t} = -2\,\mathrm{Ric}_{g(t)} + 2\dfrac{s_0}{n} g(t) - 2p(t)g(t) \\ ((n-1)\Delta_{g(t)} + s_0)p(t) = -\Big\langle \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n} g(t), \mathrm{Ric}_{g(t)} - \dfrac{s_0}{n} g(t) \Big\rangle. \end{cases}$$

Throughout we use V to denote the following symmetric 2-tensor:

$$(3)\qquad V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n} g(t) + p(t)g(t).$$

In this paper we use Kotschwar's energy techniques to give a proof of the uniqueness of conformal Ricci flow for closed manifolds with metrics of constant negative scalar curvature. It is worth noting similarities to the study of certain elliptic-hyperbolic systems done by Andersson and Moncrief [2011]. The existence of solutions to conformal Ricci flow has been shown by Fischer [2004] and by Lu, Qing, and Zheng [2014], the latter paper using DeTurck conformal Ricci flow. More precisely we prove the following uniqueness theorem of conformal Ricci flow:

Theorem 1. *Let (M^n, g_0) be a closed manifold with constant negative scalar curvature s_0. Suppose $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ are two solutions of (1) on $M \times [0, T]$ with $\tilde{g}(0) = g(0)$. Then $(\tilde{g}(t), \tilde{p}(t)) = (g(t), p(t))$ for $0 \le t \le T$.*

2. The differences between $g(t)$ and $\tilde{g}(t)$

Let $g(t)$ and $\tilde{g}(t)$ be as in Theorem 1. We treat g as our background metric and $\tilde{g}$ as our alternative metric. Let ∇ and $\tilde{\nabla}$ be the Riemannian connections of g and $\tilde{g}$ respectively. Similarly, let R and $\tilde{R}$ represent the full Riemannian curvature tensors of g and $\tilde{g}$ respectively.

Let $h = g - \tilde{g}$, and $A = \nabla - \tilde{\nabla}$. Explicitly, $A^i_{jk} = \Gamma^i_{jk} - \tilde{\Gamma}^i_{jk}$ where Γ^i_{jk} and $\tilde{\Gamma}^i_{jk}$ are the Christoffel symbols of ∇ and $\tilde{\nabla}$ respectively. Also let $S = R - \tilde{R}$ and $q = p - \tilde{p}$.

In this section we find bounds on h, A, S, q, ∇q, and $\nabla\nabla q$ (see Propositions 3 and 5). Throughout this chapter we use the convention $X * Y$ to denote any finite sum of tensors of the form $X \cdot Y$. We use $C(X)$ to denote a finite sum of tensors of the form X.

2.1. ***Preliminary calculations.*** First we calculate some useful expressions for quantities which arise in the proofs of Propositions 3 and 5. We calculate

$$g^{ij} - \tilde{g}^{ij} = g^{ik}(\tilde{g}^{j\ell}\tilde{g}_{k\ell}) - \tilde{g}^{j\ell}(g^{ik}g_{k\ell}) = -g^{ik}\tilde{g}^{j\ell}h_{k\ell},$$

i.e.,

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h.$$

If X is any tensor which is not a function we have

$$(\nabla - \tilde{\nabla})X = A * X.$$

We check this when X is a (1, 1)-tensor. Calculating in local coordinates we see

$$\begin{aligned}(\nabla_i - \tilde{\nabla}_i)X^k_j &= \partial_i X^k_j - \Gamma^\ell_{ij} X^k_\ell + \Gamma^k_{i\ell} X^\ell_j - \partial_i X^k_j + \tilde{\Gamma}^\ell_{ij} X^k_\ell - \tilde{\Gamma}^k_{i\ell} X^\ell_j \\ &= A^k_{i\ell} X^\ell_j - A^\ell_{ij} X^k_\ell = A * X.\end{aligned}$$

If f is a function however, then we have the following:

$$(\nabla^i - \tilde{\nabla}^i) f = (g^{ij} - \tilde{g}^{ij})\partial_j f = -g^{ik}\tilde{g}^{j\ell} h_{k\ell} \partial_j f = -g^{ik} h_{k\ell} \tilde{\nabla}^\ell f,$$

or in other words

$$(\nabla - \tilde{\nabla}) f = h * \tilde{\nabla} f.$$

We now calculate

$$\nabla \tilde{g}^{-1} = (\nabla - \tilde{\nabla})\tilde{g}^{-1} = \tilde{g}^{-1} * A.$$

The following calculation is also important.

$$\nabla_i h_{jk} = \nabla_i g_{jk} - \nabla_i \tilde{g}_{jk} = -(\nabla_i - \tilde{\nabla}_i)\tilde{g}_{jk}.$$

Thus we have

$$\nabla h = \tilde{g} * A.$$

Now we are able to calculate the following for a function f.

$$\begin{aligned}\nabla(\nabla - \tilde{\nabla}) f &= \nabla(h * \tilde{\nabla} f) \\ &= \nabla h * \tilde{\nabla} f + h * (\nabla - \tilde{\nabla})\tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f \\ &= \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f.\end{aligned}$$

Now let

$$\begin{aligned}U^a_{ijk\ell} &= g^{ab}\nabla_b \tilde{R}_{ijk\ell} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= g^{ab}(\nabla_b - \tilde{\nabla}_b)\tilde{R}_{ijk\ell} + (g^{ab} - \tilde{g}^{ab})\tilde{\nabla}_b \tilde{R}_{ijk\ell} \\ &= A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R},\end{aligned} \tag{4}$$

and we may calculate

$$\begin{aligned}\nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) &= \nabla_a(g^{ab}\nabla_b \tilde{R} - \tilde{g}^{ab}\tilde{\nabla}_b \tilde{R}) + g^{ab}\nabla_a\nabla_b(R - \tilde{R}) \\ &= \operatorname{div} U + \Delta S.\end{aligned}$$

We summarize the above calculations in the following lemma:

Lemma 2. *Using the notation defined at the beginning of this section,*

$$g^{-1} - \tilde{g}^{-1} = \tilde{g}^{-1} * h, \tag{5}$$

$$(\nabla - \tilde{\nabla})X = A * X, \tag{6}$$

$$(\nabla - \tilde{\nabla})f = h * \tilde{\nabla} f, \tag{7}$$

$$\nabla \tilde{g}^{-1} = \tilde{g}^{-1} * A, \tag{8}$$

$$\nabla h = \tilde{g} * A, \tag{9}$$

$$\nabla(\nabla - \tilde{\nabla})f = \tilde{g} * A * \tilde{\nabla} f + h * A * \tilde{\nabla} f + h * \tilde{\nabla}\tilde{\nabla} f, \tag{10}$$

$$U = A * \tilde{R} + \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{R}, \tag{11}$$

$$\nabla_a(g^{ab}\nabla_b R - \tilde{g}^{ab}\tilde{\nabla}_b\tilde{R}) = \operatorname{div} U + \Delta S, \tag{12}$$

where U is defined in (4).

2.2. *Bounds on time derivatives of h, A and S.* In this subsection we derive bounds on the time derivatives of h, A and S. In particular we prove the following proposition. Here, as well as throughout this chapter, we let C denote a constant dependent only upon n while N denotes a constant with further dependencies.

Proposition 3. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of (1) on $M \times [0, T]$. Using the notation defined at the beginning of this section, there exist constants N_h, N_A and N_S such that*

$$\left|\frac{\partial}{\partial t} h\right| \leq N_h|h| + C(|S| + |q|), \tag{13}$$

$$\left|\frac{\partial}{\partial t} A\right| \leq N_A(|h| + |A|) + C(|\nabla S| + |\nabla q|), \tag{14}$$

$$\left|\frac{\partial}{\partial t} S - \Delta S - \operatorname{div} U\right| \leq N_S(|h| + |A| + |S| + |q|) + C|\nabla\nabla q|, \tag{15}$$

where U is defined in (4).

Proof. We start with the time derivative of h. By (1) we have

$$\begin{aligned}
\frac{\partial}{\partial t} h_{ij} &= -2(R_{ij} - \tilde{R}_{ij}) + 2\frac{s_0}{n}(g_{ij} - \tilde{g}_{ij}) - 2(p\, g_{ij} - \tilde{p}\, \tilde{g}_{ij}) \\
&= -2S^k_{kij} + 2\frac{s_0}{n} h_{ij} - 2[(p - \tilde{p})g_{ij} + \tilde{p}(g_{ij} - \tilde{g}_{ij})] \\
&= -2S^k_{kij} + 2\frac{s_0}{n} h_{ij} - 2q\, g_{ij} - 2\tilde{p}\, h_{ij}.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t} h = C(S) + C(s_0 h) + C(q) + \tilde{p} * h$$

and

$$\left|\frac{\partial}{\partial t}h\right| \le C((|s_0|+|\tilde{p}|)|h|+|S|+|q|). \tag{16}$$

This proves (13).

Recall the definition of V from (3):

$$V(t) = \mathrm{Ric}_{g(t)} - \frac{s_0}{n}g(t) + p(t)g(t). \tag{17}$$

We may define $\tilde{V}$ similarly using our alternate metric $\tilde{g}$. Since V and $\tilde{V}$ are symmetric 2-tensors, then by [Chow et al. 2006, p. 108] we may calculate

$$\frac{\partial}{\partial t}A^k_{ij} = \tilde{g}^{k\ell}(\tilde{\nabla}_i\tilde{V}_{j\ell}+\tilde{\nabla}_j\tilde{V}_{i\ell}-\tilde{\nabla}_\ell\tilde{V}_{ij}) - g^{k\ell}(\nabla_iV_{j\ell}+\nabla_jV_{i\ell}-\nabla_\ell V_{ij}). \tag{18}$$

We proceed to calculate

$$\begin{aligned}
&\tilde{g}^{k\ell}\tilde{\nabla}_i\tilde{V}_{j\ell} - g^{k\ell}\nabla_iV_{j\ell} \\
&\quad = \tilde{g}^{k\ell}(\tilde{\nabla}_i\tilde{R}_{j\ell}) - g^{k\ell}(\nabla_iR_{j\ell}) + \tilde{g}^{k\ell}\tilde{\nabla}_i(\tilde{p}\tilde{g}_{j\ell}) - g^{k\ell}\nabla_i(pg_{j\ell}) \\
&\quad = (\tilde{g}^{k\ell}-g^{k\ell})\tilde{\nabla}_i\tilde{R}_{j\ell} + g^{k\ell}(\tilde{\nabla}_i-\nabla_i)\tilde{R}_{j\ell} - g^{k\ell}\nabla_i(S^m_{mj\ell}) + \delta^k_j\tilde{\nabla}_i\tilde{p} - \delta^k_j\nabla_ip \\
&\quad = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q),
\end{aligned} \tag{19}$$

where we have used (7) to get the last equality. Similarly we find

$$\tilde{g}^{k\ell}\tilde{\nabla}_j\tilde{V}_{i\ell} - g^{k\ell}\nabla_jV_{i\ell} = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q). \tag{20}$$

Now we consider

$$\begin{aligned}
&-\tilde{g}^{k\ell}\tilde{\nabla}_\ell\tilde{V}_{ij} + g^{k\ell}\nabla_\ell V_{ij} \\
&\quad = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + \tilde{g}^{k\ell}\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} - g^{k\ell}g_{ij}\nabla_\ell p \\
&\quad = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + (\tilde{g}^{k\ell}-g^{k\ell})\tilde{g}_{ij}\tilde{\nabla}_\ell\tilde{p} \\
&\qquad + g^{k\ell}(\tilde{g}_{ij}-g_{ij})\tilde{\nabla}_\ell\tilde{p} + g^{k\ell}g_{ij}(\tilde{\nabla}_\ell-\nabla_\ell)\tilde{p} \\
&\qquad + g^{k\ell}g_{ij}\nabla_\ell(\tilde{p}-p) \\
&\quad = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) \\
&\qquad + \tilde{g}^{-1}*h*\tilde{g}*\tilde{\nabla}\tilde{p} + h*\tilde{\nabla}\tilde{p} + C(\nabla q).
\end{aligned} \tag{21}$$

Hence by (18), (19), (20) and (21),

$$\frac{\partial}{\partial t}A = \tilde{g}^{-1}*h*\tilde{\nabla}\tilde{R} + A*\tilde{R} + C(\nabla S) + h*\tilde{\nabla}\tilde{p} + C(\nabla q) + \tilde{g}^{-1}*h*\tilde{g}*\tilde{\nabla}\tilde{p}$$

and

$$\left|\frac{\partial}{\partial t}A\right| \leq C\big((|\tilde{g}^{-1}||\tilde{\nabla}\tilde{R}|+|\tilde{\nabla}\tilde{p}|+|\tilde{g}^{-1}||\tilde{g}||\tilde{\nabla}\tilde{p}|)|h|+|\tilde{R}||A|+|\nabla S|+|\nabla q|\big). \tag{22}$$

This proves (14).

By [Chow et al. 2006, equation (2.67)] we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{\ell} &= g^{\ell m}(\nabla_i\nabla_k V_{jm}-\nabla_i\nabla_m V_{jk}-\nabla_j\nabla_k V_{im}+\nabla_j\nabla_m V_{ik})\\ &\qquad\qquad - g^{\ell m}(R_{ijk}^{r}V_{rm}+R_{ijm}^{q}V_{kq})\\ &= g^{\ell m}(-\nabla_i\nabla_k R_{jm}+\nabla_i\nabla_m R_{jk}+\nabla_j\nabla_k R_{im}-\nabla_j\nabla_m R_{ik})\\ &\quad + g^{\ell m}(-g_{jm}\nabla_i\nabla_k p+g_{jk}\nabla_i\nabla_m p+g_{im}\nabla_j\nabla_k p-g_{ik}\nabla_j\nabla_m p)\\ &\quad + g^{\ell m}(R_{ijk}^{r}R_{rm}+R_{ijm}^{r}R_{kr})-\frac{s_0}{n}g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p\\ &\quad + g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p.\end{aligned} \tag{23}$$

Following the calculations in [Chow et al. 2006, pp. 119–120] we have

$$\begin{aligned}\Delta R_{ijk}^{\ell} &= g^{ab}\nabla_a\nabla_b R_{ijk}^{\ell} = g^{ab}(-\nabla_a\nabla_i R_{jbk}^{\ell}-\nabla_a\nabla_j R_{bik}^{\ell})\\ &= g^{ab}\big(-\nabla_i\nabla_a R_{jbk}^{\ell}+R_{aij}^{m}R_{mbk}^{\ell}+R_{aib}^{m}R_{jmk}^{\ell}+R_{aik}^{m}R_{jbm}^{\ell}-R_{aim}^{\ell}R_{jbk}^{m}\\ &\qquad -\nabla_j\nabla_a R_{bik}^{\ell}+R_{ajb}^{m}R_{mik}^{\ell}+R_{aji}^{m}R_{bmk}^{\ell}+R_{ajk}^{m}R_{bim}^{\ell}-R_{ajm}^{\ell}R_{bik}^{m}\big)\\ &= g^{\ell m}(-\nabla_i\nabla_k R_{jm}+\nabla_i\nabla_m R_{jk}+\nabla_j\nabla_k R_{im}-\nabla_j\nabla_m R_{ik})\\ &\quad + g^{mr}(-R_{ir}R_{jmk}^{\ell}-R_{jr}R_{mik})\\ &\quad + g^{ab}\big(R_{aij}^{m}R_{mbk}^{\ell}+R_{aik}^{m}R_{jbm}^{\ell}-R_{aim}^{\ell}R_{jbk}^{m}+R_{aji}^{m}R_{bmk}^{\ell}\\ &\qquad\qquad + R_{ajk}^{m}R_{bim}^{\ell}-R_{ajm}^{\ell}R_{bik}^{m}\big).\end{aligned} \tag{24}$$

Combining (23) and (24) we have

$$\begin{aligned}\frac{\partial}{\partial t}R_{ijk}^{\ell} &= \Delta R_{ijk}^{\ell}+g^{mr}(R_{ir}R_{jmk}^{\ell}+R_{jr}R_{mik}^{\ell})\\ &\quad + g^{ab}\big(-R_{aij}^{m}R_{mbk}^{\ell}-R_{aik}^{m}R_{jbm}^{\ell}+R_{aim}^{\ell}R_{jbk}^{m}\\ &\qquad\qquad -R_{aji}^{m}R_{bmk}^{\ell}-R_{ajk}^{m}R_{bim}^{\ell}+R_{ajm}^{\ell}R_{bik}^{m}\big)\\ &\quad + g^{\ell m}(-g_{jm}\nabla_i\nabla_k p+g_{jk}\nabla_i\nabla_m p+g_{im}\nabla_j\nabla_k p-g_{ik}\nabla_j\nabla_m p)\\ &\quad + g^{\ell m}(R_{ijk}^{r}R_{rm}+R_{ijm}^{r}R_{kr})-\frac{s_0}{n}g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})\\ &\quad + g^{\ell m}(R_{ijk}^{r}g_{rm}+R_{ijm}^{r}g_{kr})p.\end{aligned} \tag{25}$$

Hence the evolution of S is

$$
\begin{aligned}
\frac{\partial}{\partial t} S_{ijk}^{\ell} = {} & \Delta R_{ijk}^{\ell} - \tilde{\Delta} \tilde{R}_{ijk}^{\ell} \\
& + g^{mr}(R_{ir} R_{jmk}^{\ell} + R_{jr} R_{jmk}^{\ell}) - \tilde{g}^{mr}(\tilde{R}_{ir} \tilde{R}_{jmk}^{\ell} + \tilde{R}_{jr} \tilde{R}_{mik}^{\ell}) \\
& + g^{ab}\big(-R_{aij}^{m} R_{mbk}^{\ell} - R_{aik}^{m} R_{jbm}^{\ell} + R_{aim}^{\ell} R_{jbk}^{m} - R_{aji}^{m} R_{bmk}^{\ell} \\
& \qquad - R_{ajk}^{m} R_{bim}^{\ell} + R_{ajm}^{\ell} R_{bik}^{m}\big) \\
& - \tilde{g}^{ab}\big(-\tilde{R}_{aij}^{m} \tilde{R}_{mbk}^{\ell} - \tilde{R}_{aik}^{m} \tilde{R}_{jbm}^{\ell} + \tilde{R}_{aim}^{\ell} \tilde{R}_{jbk}^{m} - \tilde{R}_{aji}^{m} \tilde{R}_{bmk}^{\ell} \\
& \qquad - \tilde{R}_{ajk}^{m} \tilde{R}_{bim}^{\ell} + \tilde{R}_{ajm}^{\ell} \tilde{R}_{bik}^{m}\big) \\
& + g^{\ell m}(-g_{jm} \nabla_i \nabla_k p + g_{jk} \nabla_i \nabla_m p + g_{im} \nabla_j \nabla_k p - g_{ik} \nabla_j \nabla_m p) \\
& - \tilde{g}^{\ell m}(-\tilde{g}_{jm} \tilde{\nabla}_i \tilde{\nabla}_k \tilde{p} + \tilde{g}_{jk} \tilde{\nabla}_i \tilde{\nabla}_m \tilde{p} + \tilde{g}_{im} \tilde{\nabla}_j \tilde{\nabla}_k \tilde{p} - \tilde{g}_{ik} \tilde{\nabla}_j \tilde{\nabla}_m \tilde{p}) \\
& + g^{\ell m}(R_{ijk}^{r} R_{rm} + R_{ijm}^{r} R_{kr}) - \tilde{g}^{\ell m}(\tilde{R}_{ijk}^{r} \tilde{R}_{rm} + \tilde{R}_{ijm}^{r} \tilde{R}_{kr}) \\
& - \frac{s_0}{n} g^{\ell m}(R_{ijk}^{r} g_{rm} + R_{ijm}^{r} g_{kr}) + \frac{s_0}{n} \tilde{g}^{\ell m}(\tilde{R}_{ijk}^{r} \tilde{g}_{rm} + \tilde{R}_{ijm}^{r} \tilde{g}_{kr}) \\
& + g^{\ell m}(R_{ijk}^{r} g_{rm} + R_{ijm}^{r} g_{kr}) p - \tilde{g}^{\ell m}(\tilde{R}_{ijk}^{r} \tilde{g}_{rm} + \tilde{R}_{ijm}^{r} \tilde{g}_{kr}) \tilde{p}.
\end{aligned} \tag{26}
$$

Looking at the individual components, we see

$$
\begin{aligned}
\Delta R - \tilde{\Delta} \tilde{R} &= g^{ab} \nabla_a \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_a \tilde{\nabla}_b \tilde{R} \\
&= \nabla_a (g^{ab} \nabla_b R) - \nabla_a (\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + (\nabla_a - \tilde{\nabla}_a)(\tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) \\
&= \nabla_a (g^{ab} \nabla_b R - \tilde{g}^{ab} \tilde{\nabla}_b \tilde{R}) + \tilde{g}^{-1} * A * \tilde{\nabla} \tilde{R},
\end{aligned} \tag{27}
$$

while

$$
\begin{aligned}
g^{-1} RR - \tilde{g}^{-1} \tilde{R} \tilde{R} &= (g^{-1} - \tilde{g}^{-1})(\tilde{R} \tilde{R}) + g^{-1}(RR - \tilde{R} \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + g^{-1}(R - \tilde{R}) \tilde{R} + g^{-1}(RR - R \tilde{R}) \\
&= \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R,
\end{aligned} \tag{28}
$$

and

$$
\begin{aligned}
g^{-1} g \nabla \nabla p - \tilde{g}^{-1} \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} &= (g^{-1} - \tilde{g}^{-1}) \tilde{g} \tilde{\nabla} \tilde{\nabla} \tilde{p} + g^{-1}(g - \tilde{g}) \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
& \qquad + g^{-1} g (\nabla \nabla p - \tilde{\nabla} \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} \\
& \qquad + g^{-1} g (\nabla - \tilde{\nabla})(\tilde{\nabla} \tilde{p}) + g^{-1} g (\nabla \nabla p - \nabla \tilde{\nabla} \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
& \qquad + g^{-1} g \nabla (\nabla - \tilde{\nabla}) \tilde{p} + g^{-1} g \nabla \nabla (p - \tilde{p}) \\
&= \tilde{g}^{-1} * h * \tilde{g} * \tilde{\nabla} \tilde{\nabla} \tilde{p} + h * \tilde{\nabla} \tilde{\nabla} \tilde{p} + A * \tilde{\nabla} \tilde{p} \\
& \qquad + h * A * \tilde{\nabla} \tilde{p} + C(\nabla \nabla q),
\end{aligned} \tag{29}
$$

where in the last equality we used (10). We also have

$$
\begin{aligned}
g^{-1}gR - \tilde g^{-1}\tilde g\tilde R &= (g^{-1}-\tilde g^{-1})\tilde g\tilde R + g^{-1}(g-\tilde g)\tilde R + g^{-1}g(R-\tilde R)\\
&= \tilde g^{-1} * h * \tilde g * \tilde R + h * \tilde R + C(S),
\end{aligned} \tag{30}
$$

and lastly

$$
\begin{aligned}
g^{-1}gRp - \tilde g^{-1}\tilde g\tilde R\tilde p &= (g^{-1}-\tilde g^{-1})\tilde g\tilde R\tilde p + g^{-1}(g-\tilde g)\tilde R\tilde p\\
&\qquad + g^{-1}g(R-\tilde R)\tilde p + g^{-1}gR(p-\tilde p)\\
&= \tilde g^{-1} * h * \tilde g * \tilde R * \tilde p + h * \tilde R * \tilde p + S * \tilde p + R * q.
\end{aligned} \tag{31}
$$

Now by (26), (27), (28), (29), (30) and (31) we see

$$
\begin{aligned}
\frac{\partial}{\partial t} S = \nabla_a(g^{ab}\nabla_b R - \tilde g^{ab}\tilde\nabla_b\tilde R) &+ \tilde g^{-1} * A * \tilde\nabla\tilde R + \tilde g^{-1} * h * \tilde R * \tilde R\\
&+ S*\tilde R + S*R + \tilde g^{-1}*h*\tilde g*\tilde\nabla\tilde\nabla\tilde p + h*\tilde\nabla\tilde\nabla\tilde p + A*\tilde\nabla\tilde p\\
&+ h*A*\tilde\nabla\tilde p + C(\nabla\nabla q) + \tilde g^{-1}*h*\tilde g*\tilde R + h*\tilde R + C(S)\\
&+ \tilde g^{-1} * h * \tilde g * \tilde R * \tilde p + h * \tilde R * \tilde p + S * \tilde p + R * q.
\end{aligned}
$$

Hence by (12) we have

$$
\begin{aligned}
\Big|\frac{\partial}{\partial t} S - \Delta S - \operatorname{div} U\Big| \le C\Big(\big(&|\tilde g^{-1}||\tilde R|^2 + |\tilde g^{-1}||\tilde g||\tilde\nabla\tilde\nabla\tilde p| + |\tilde\nabla\tilde\nabla\tilde p|\\
&+ |\tilde g^{-1}||\tilde g||\tilde R| + |\tilde R| + |\tilde g^{-1}||\tilde g||\tilde R||\tilde p| + |\tilde R||\tilde p|\big)|h|\\
&+ (|\tilde g^{-1}||\tilde\nabla\tilde R| + |\tilde\nabla\tilde p| + |h||\tilde\nabla\tilde p|)|A|\\
&+ (|\tilde R| + |R| + 1 + |\tilde p|)|S| + |R||q| + |\nabla\nabla q|\Big).
\end{aligned} \tag{32}
$$

This proves (15). □

Remark 4. Upon closer observation we notice the following dependencies:

$$N_h = N_h(n, s_0, |\tilde p|).$$
$$N_A = N_A(n, s_0, |\tilde g|, |\tilde g^{-1}|, |\tilde R|, |\tilde\nabla\tilde R|, |\tilde\nabla\tilde p|).$$
$$N_S = N_S(n, s_0, |\tilde g|, |\tilde g^{-1}|, |h|, |R|, |\tilde R|, |\tilde\nabla\tilde R|, |\tilde p|, |\tilde\nabla\tilde p|, |\tilde\nabla\tilde\nabla\tilde p|).$$

M is closed, so $M \times [0, T]$ is compact. Thus, given two metrics g and $\tilde g$, all of these quantities are bounded.

2.3. ***Bounds on q and its spatial derivatives.*** We turn our attention now to finding bounds on the differences between our pressure functions p and $\tilde p$. We have the following proposition:

Proposition 5. *Let $(g(t), p(t))$ and $(\tilde{g}(t), \tilde{p}(t))$ be two solutions of (1) on $M \times [0, T]$. Then there exist constants N_q and $\hat{N}_q$ such that*

$$\int_M |q|^2 \, d\mu \le N_q \int_M (|h|^2 + |A|^2 + |S|^2) \, d\mu, \tag{33}$$

$$\int_M |\nabla q|^2 \, d\mu \le N_q \int_M (|h|^2 + |A|^2 + |S|^2) \, d\mu, \tag{34}$$

$$\int_M |\nabla\nabla q|^2 \, d\mu \le \hat{N}_q \int_M (|h|^2 + |A|^2 + |S|^2) \, d\mu. \tag{35}$$

Proof. We let f represent any smooth function or tensor on M. This is general, but in this paper we represent f by the function q, the difference of the pressure functions. Since M is compact we have

$$\int_M ((n-1)\Delta + s_0)(f) \cdot f \, d\mu = s_0 \int_M |f|^2 \, d\mu - (n-1) \int_M \langle \nabla f, \nabla f \rangle \, d\mu.$$

Since $s_0 < 0$, taking the absolute value gives

$$\left| \int_M ((n-1)\Delta + s_0)(f) \cdot f \, d\mu \right| = |s_0| \int_M |f|^2 \, d\mu + (n-1) \int_M |\nabla f|^2 \, d\mu. \tag{36}$$

Now we deal specifically with p, $\tilde{p}$ and q. By (2) we have the following equations for the pressure functions p and $\tilde{p}$:

$$((n-1)\Delta + s_0)p = -\left\langle \mathrm{Ric} - \frac{s_0}{n} g, \mathrm{Ric} - \frac{s_0}{n} g \right\rangle. \tag{37}$$

$$((n-1)\tilde{\Delta} + s_0)\tilde{p} = -\left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g} \right\rangle. \tag{38}$$

Now we calculate

$$\begin{aligned} \Delta p - \tilde{\Delta}\tilde{p} &= g^{ab}\nabla_a\nabla_b p - \tilde{g}^{ab}\tilde{\nabla}_a\tilde{\nabla}_b\tilde{p} \\ &= (g^{-1} - \tilde{g}^{-1})\tilde{\nabla}\tilde{\nabla}\tilde{p} + g^{-1}(\nabla - \tilde{\nabla})\tilde{\nabla}\tilde{p} + g^{-1}\nabla(\nabla - \tilde{\nabla})\tilde{p} + \Delta(p - \tilde{p}) \\ &= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \Delta q. \end{aligned} \tag{39}$$

We also compute

$$\begin{aligned} &-\left\langle \mathrm{Ric} - \frac{s_0}{n} g, \mathrm{Ric} - \frac{s_0}{n} g \right\rangle + \left\langle \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g}, \tilde{\mathrm{Ric}} - \frac{s_0}{n} \tilde{g} \right\rangle \\ &\quad = -(g^{ik}g^{j\ell}R_{ij}R_{k\ell} - \tilde{g}^{ik}\tilde{g}^{j\ell}\tilde{R}_{ij}\tilde{R}_{k\ell}) + 2\frac{s_0}{n}(g^{ij}R_{ij} - \tilde{g}^{ij}\tilde{R}_{ij}) \\ &\quad = -(g^{-1} - \tilde{g}^{-1})\tilde{g}^{-1}\tilde{R}\tilde{R} - g^{-1}(g^{-1} - \tilde{g}^{-1})\tilde{R}\tilde{R} - g^{-1}g^{-1}(R - \tilde{R})\tilde{R} \\ &\qquad - g^{-1}g^{-1}R(R - \tilde{R}) + 2\frac{s_0}{n}(g^{-1} - \tilde{g}^{-1})\tilde{R} + 2\frac{s_0}{n}g^{-1}(R - \tilde{R}) \\ &\quad = \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\ &\qquad + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S). \end{aligned} \tag{40}$$

Combining (37), (38), (39) and (40), we see that q satisfies the following elliptic equation at each time $t \in [0, T]$:

$$
\begin{aligned}
(41)\quad Lq &= ((n-1)\Delta + s_0)(q) \\
&= \tilde{g}^{-1} * h * \tilde{\nabla}\tilde{\nabla}\tilde{p} + A * \tilde{\nabla}\tilde{p} + h * A * \tilde{\nabla}\tilde{p} + \tilde{g}^{-1} * \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} \\
&\quad + \tilde{g}^{-1} * h * \tilde{R} * \tilde{R} + S * \tilde{R} + S * R + \tilde{g}^{-1} * h * \tilde{R} + C(S).
\end{aligned}
$$

Hence

$$
(42)\qquad |Lq| = \big|((n-1)\Delta + s_0)(q)\big| \le N(|h| + |A| + |S|).
$$

To find estimates for q and ∇q, we combine (36) and (42):

$$
\begin{aligned}
|s_0| \int_M |q|^2\, d\mu + (n-1) \int_M |\nabla q|^2\, d\mu &= \left| \int_M ((n-1)\Delta + s_0)(q) \cdot q\, d\mu \right| \\
&\le \int_M N(|h| + |A| + |S|)|q|\, d\mu \\
&\le \frac{|s_0|}{2} \int_M |q|^2\, d\mu + N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu.
\end{aligned}
$$

Thus

$$
\frac{|s_0|}{2} \int_M |q|^2\, d\mu + (n-1) \int_M \big|\nabla q|^2\, d\mu \le N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu,
$$

and we proved (33) and (34).

To find an appropriate bound for $|\nabla\nabla q|$ we use interior regularity theory for elliptic PDEs. From (41) we see that $Lq = f$ is an elliptic equation. We then have the following estimate from [Rauch 1991, p. 229]:

$$
|q|_{H^2(W)} \le K(|Lq|_{L^2(M)} + |q|_{H^1(M)}),
$$

where W is any compactly supported open subset of M and K depends only upon the coefficients of the operator L, the subset W and the manifold M. Since M is a closed manifold we may in fact choose $W = M$. Thus we have

$$
(43)\qquad |q|_{H^2(M)} \le K(|Lq|_{L^2(M)} + |q|_{H^1(M)}).
$$

Upon squaring both sides we observe

$$
(44)\qquad \int_M |\nabla\nabla q|^2\, d\mu \le |q|^2_{H^2(M)} \le K^2 \left(\int_M |Lq|^2\, d\mu + |q|^2_{H^1(M)} \right).
$$

Now (33) and (34) imply that

$$|q|^2_{H^1(M)} \le N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu. \tag{45}$$

Combining (42), (44) and (45) we have

$$\int_M |\nabla\nabla q|^2\, d\mu \le N \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu,$$

and we have proved (35). □

Remark 6. We observe the following dependencies:

$$N_q = N_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|).$$
$$\hat{N}_q = \hat{N}_q(n, s_0, |\tilde{g}^{-1}|, |h|, |R|, |\tilde{R}|, |\tilde{\nabla}\tilde{p}|, |\tilde{\nabla}\tilde{\nabla}\tilde{p}|, K),$$

where K is from (43).

3. Energy estimates

We now define the energy functional

$$\mathcal{E}(t) = \int_M (|h|^2 + |A|^2 + |S|^2)\, d\mu, \tag{46}$$

as well as the following:

$$\mathcal{H}(t) = \int_M |h|^2\, d\mu. \tag{47}$$

$$\mathcal{A}(t) = \int_M |A|^2\, d\mu. \tag{48}$$

$$\mathcal{S}(t) = \int_M |S|^2\, d\mu. \tag{49}$$

$$\mathcal{D}(t) = \int_M |\nabla S|^2\, d\mu. \tag{50}$$

Note that $\mathcal{E}(t) = \mathcal{H}(t) + \mathcal{A}(t) + \mathcal{S}(t)$. We now estimate the evolution of the energy functional under conformal Ricci flow, $\mathcal{E}'(t)$, by first estimating the evolutions of $\mathcal{H}$, $\mathcal{A}$ and $\mathcal{S}$.

3.1. *Evolution of $\mathcal{H}(t)$.*

Lu, Qing and Zheng [2014] give the evolution of the volume element under conformal Ricci flow

$$\frac{\partial}{\partial t}\, d\mu_{g(t)} = -np(t)\, d\mu_{g(t)}. \tag{51}$$

Hence by (13) and (47) we have

$$\begin{aligned}\mathcal{H}'(t) &\leq N\int_M |h|^2\,d\mu + \int_M 2\Big\langle \frac{\partial h}{\partial t}, h\Big\rangle\,d\mu\\ &\leq N\mathcal{H}(t) + \int_M 2|h|\Big|\frac{\partial h}{\partial t}\Big|\,d\mu\\ &\leq N\mathcal{H}(t) + N\int_M (|S|\,|h| + |h|^2 + |q|\,|h|)\,d\mu.\end{aligned}$$

Now we know that $N(|S|\,|h| + |q|\,|h|) \leq N(|h|^2 + |S|^2 + |q|^2)$. Hence

$$\begin{aligned}\mathcal{H}'(t) &\leq N\mathcal{H}(t) + N\int_M (|S|^2 + |q|^2)\,d\mu\\ &\leq N\mathcal{H}(t) + N\int_M (|S|^2 + |h|^2 + |A|^2)\,d\mu\\ &\leq N\mathcal{H}(t) + N\mathcal{S}(t) + N\mathcal{A}(t) = N\mathcal{E}(t).\end{aligned} \tag{52}$$

3.2. ***Evolution of $\mathcal{A}(t)$.*** By (14), (48) and (51) we have

$$\begin{aligned}\mathcal{A}'(t) &\leq N\mathcal{A}(t) + \int_M 2|A|\Big|\frac{\partial A}{\partial t}\Big|\,d\mu\\ &\leq N\mathcal{A}(t) + \int_M (N|h|\,|A| + N|A|^2 + C|\nabla S|\,|A| + C|\nabla q|\,|A|)\,d\mu.\end{aligned}$$

Now

$$N|h|\,|A| + C|\nabla S|\,|A| + C|\nabla q|\,|A| \leq N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2.$$

Hence we have that

$$\begin{aligned}\mathcal{A}'(t) &\leq N\mathcal{A}(t) + \int_M (N|h|^2 + N|A|^2 + |\nabla S|^2 + |\nabla q|^2)\,d\mu\\ &\leq N\mathcal{A}(t) + N\mathcal{H}(t) + \mathcal{D}(t) + N\int_M (|h|^2 + |A|^2 + |S|^2)\,d\mu\\ &\leq N\mathcal{A}(t) + N\mathcal{H}(t) + N\mathcal{S}(t) + \mathcal{D}(t) = N\mathcal{E}(t) + \mathcal{D}(t).\end{aligned} \tag{53}$$

3.3. ***Evolution of $\mathcal{S}(t)$.*** By (15), (49) and (51) we have

$$\begin{aligned}\mathcal{S}'(t) &\leq N\int_M |S|^2\,d\mu + \int_M 2\Big\langle \frac{\partial S}{\partial t}, S\Big\rangle\,d\mu\\ &\leq N\mathcal{S}(t) + \int_M \big(2\langle \Delta S + \operatorname{div} V, S\rangle + N(|h| + |A| + |S| + |q|)|S| + C|\nabla\nabla q|\,|S|\big)\,d\mu\\ &\leq N\mathcal{S}(t) + \int_M \big(2\langle \Delta S + \operatorname{div} V, S\rangle + N(|h|^2 + |A|^2 + |S|^2 + |q|^2 + |\nabla\nabla q|^2)\big)\,d\mu.\end{aligned}$$

Now by (33) and (35) we have

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{S}(t)+N\mathcal{H}(t)+N\mathcal{A}(t)\\ &\qquad +\int_M \big(2\langle \Delta S+\operatorname{div} V,\, S\rangle + N(|A|^2+|S|^2+|h|^2)\big)\,d\mu\\ &\le N\mathcal{S}(t)+N\mathcal{H}(t)+N\mathcal{A}(t)+\int_M 2\langle \Delta S+\operatorname{div} V,\, S\rangle\,d\mu.\end{aligned}$$

Upon integrating by parts we get

$$\begin{aligned}\mathcal{S}'(t) &\le N\mathcal{E}(t)-2\int_M \langle \nabla S+V,\, \nabla S\rangle\,d\mu\\ &\le N\mathcal{E}(t)-2\int_M |\nabla S|^2\,d\mu+\int_M 2|V||\nabla S|\,d\mu.\end{aligned}$$

Now we know that

$$2|V||\nabla S| \le |\nabla S|^2+|V|^2 \le |\nabla S|^2+N(|h|^2+|A|^2),$$

hence

$$\mathcal{S}'(t)\le N\mathcal{E}(t)+N\int_M (|h|^2+|A|^2)\,d\mu-\int_M |\nabla S|^2\,d\mu \le N\mathcal{E}(t)-\mathcal{D}(t). \tag{54}$$

3.4. ***Proof of main theorem.*** We are now ready to prove Theorem 1.

Proof. By (52), (53) and (54) we know that

$$\mathcal{H}'(t)\le N\mathcal{E}(t),\quad \mathcal{A}'(t)\le N\mathcal{E}(t)+\mathcal{D}(t)\quad\text{and}\quad \mathcal{S}'(t)\le N\mathcal{E}(t)-\mathcal{D}(t),$$

so

$$\mathcal{E}'(t)\le N\mathcal{E}(t).$$

Our initial condition $\tilde{g}(0)=g(0)$ tells us that at $t=0$ we have $|h|=|A|=|S|=0$. Therefore by the smoothness and integrability of our solutions we know

$$\lim_{t\to 0^+}\mathcal{E}(t)=0,$$

so by Gronwall's inequality we know that $\mathcal{E}\equiv 0$ on $[0,T]$. Thus for $t\in[0,T]$ we have that $h\equiv 0$ and $g(t)\equiv\tilde{g}(t)$. Also, $\mathcal{E}\equiv 0$ implies $A\equiv 0$ and $S\equiv 0$, so (33) forces $q\equiv 0$. Thus $p(t)\equiv\tilde{p}(t)$. Therefore $(\tilde{g}(t),\tilde{p}(t))=(g(t),p(t))$, $t\in[0,T]$. □

4. Further research

The arguments in this paper are only valid for conformal Ricci flow on a compact manifold with constant positive scalar curvature. In particular, if $s_0\ge 0$ we do not have the equality (36). It is worth discovering whether or not there is some other way to compute the bounds on q and its derivatives, namely equations (33), (34) and (35).

It is also interesting to consider complete noncompact manifolds of constant scalar curvature. Previous results in Ricci flow and parabolic PDE suggest that in this case we will not achieve uniqueness of conformal Ricci flow without some sort of bound on the curvature of the manifold.

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Received March 10, 2015. Revised April 1, 2016.

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The Pacific Journal of Mathematics (ISSN 0030-8730) at the University of California, c/o Department of Mathematics, 798 Evans Hall #3840, Berkeley, CA 94720-3840, is published twelve times a year. Periodical rate postage paid at Berkeley, CA 94704, and additional mailing offices. POSTMASTER: send address changes to Pacific Journal of Mathematics, P.O. Box 4163, Berkeley, CA 94704-0163.

PJM peer review and production are managed by EditFLOW® from Mathematical Sciences Publishers.

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PACIFIC JOURNAL OF MATHEMATICS

Volume 286 No. 2 February 2017