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A FUNCTIONAL CALCULUS AND RESTRICTION THEOREM ON H-TYPE GROUPS

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Let L be the sublaplacian and T the partial laplacian with respect to central variables on H-type groups. We investigate a class of invariant differential operators by the joint functional calculus of L and T. We establish Stein–Tomas type restriction theorems for these operators. In particular, the asymptotic behaviors of restriction estimates are given.

1. Introduction

The restriction theorem for the Fourier transform plays an important role in harmonic analysis as well as in the theory of partial differential equations. The original version is credited to E. M. Stein and P. A. Tomas, and states that the transform of an L^p-function on $\mathbb{R}^n$ has a well-defined restriction to the unit sphere S^{n-1} which is square integrable on S^{n-1}. The result is listed as follows:

Theorem 1.1 [Stein 1993; Tomas 1975]. *Let $1 \le p \le \dfrac{2n+2}{n+3}$. Then the estimate*

$$\|\hat{f}\|_{L^2(S^{n-1})} \le C\|f\|_{L^p(\mathbb{R}^n)} \tag{1-1}$$

holds for all functions $f \in L^p(\mathbb{R}^n)$.

A simple duality argument shows that the estimate (1-1) is equivalent to the following estimate:

$$\|f * \widehat{d\sigma_r}\|_{p'} \le C_r\|f\|_p \tag{1-2}$$

for all Schwartz functions f on $\mathbb{R}^n$, where $1/p + 1/p' = 1$ and $d\sigma_r$ is the surface measure on the sphere with radius r.

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Moreover, according to the Knapp example [Stein 1993], the estimates (1-1) and (1-2) fail if $(2n+2)/(n+3) < p \leq 2$.

Many authors have worked on the topic and various new restriction theorems have been proved. The study of restriction theorems has recently obtained more and more attention. A survey of recent progress on restriction theorems can be found in [Tao 2004]. To generalize the restriction theorem on the Heisenberg group, D. Müller [1990] established the boundedness of the restriction operator with respect to the mixed L^p-norm and also gave a counterexample to show that the estimate between Lebesgue spaces for the restriction operator was necessarily trivial, due to the fact that the center of the Heisenberg group was of dimension one. Some extensions have been treated by S. Thangavelu [1991a; 1991b]. Restriction theorems have been also studied in the case of the Heisenberg motion group by P. K. Ratnakumar, R. Rawat and S. Thangavelu [Ratnakumar et al. 1997], where groups with center with dimension higher than one were first considered.

On an H-type group, let T be the laplacian on the center and L the sublaplacian. It is well known that L is positive and essentially self-adjoint. Let $L = \int_0^\infty \lambda \, dE(\lambda)$ be the spectral decomposition of L. Then the restriction operator can be formally written $\mathcal{P}_\lambda f = \delta_\lambda(L) f = \lim_{\epsilon \to \infty} \chi_{(\lambda-\epsilon,\lambda+\epsilon)}(L) f$ which is well defined for a Schwartz function f, where $\chi_{(\lambda-\epsilon,\lambda+\epsilon)}$ is the characteristic function of the interval $(\lambda-\epsilon, \lambda+\epsilon)$. Liu and Wang [2011] investigated the restriction theorem for the sublaplacian L on H-type groups with center whose dimension was greater than one. They gave the following result:

Theorem 1.2. *Let G be an H-type group with the underlying manifold $\mathbb{R}^{2n+m}$, where $m > 1$ is the dimension of the center. Suppose $1 \leq p \leq (2m+2)/(m+3)$. Then the following estimate*

$$\|\mathcal{P}_\lambda f\|_{p'} \leq C \lambda^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \|f\|_p, \lambda > 0$$

holds for all Schwartz functions f on G.

V. Casarino and P. Ciatti [2013a; 2013b] extended the results of Müller, Liu and Wang to Métivier groups. They proved the restriction theorem for the sublaplacian and the full laplacian on Métivier groups. In fact, they also investigated the joint functional calculus of L and T. The invariant differential operators related to the joint functional calculus of L and T on H-type groups do not have the homogeneous properties in general. Thus the asymptotic behaviors of restriction estimates for these operators are also interesting. Casarino and Ciatti [2013a; 2013b] did not discuss the asymptotic behavior of the full laplacian. In this article we will show restriction estimates for these operators on H-type groups. In particular, the asymptotic behaviors of restriction estimates are given.

The outline of the paper is as follows. In the second section, we provide the necessary background for the H-type group. In the third section, by introducing the joint functional calculus of L and T, the restriction operator can be computed explicitly. In the fourth section, we prove the restriction theorem on H-type groups. In the fifth section, we describe the restriction theorems for other operators with the form of the joint functional calculus of L and T. Finally, in the last section, we show that the range of p in the restriction theorem is sharp.

2. Preliminaries

Definition 2.1 (H-type group). Let $\mathfrak{g}$ be a two step nilpotent Lie algebra endowed with an inner product $\langle\cdot,\cdot\rangle$. Its center is denoted by $\mathfrak{z}$. The algebra $\mathfrak{g}$ is said to be of H-type if $[\mathfrak{z}^\perp,\mathfrak{z}^\perp]=\mathfrak{z}$ and for every $t\in\mathfrak{z}$, the map $J_t:\mathfrak{z}^\perp\to\mathfrak{z}^\perp$ defined by

$$\langle J_t u, w\rangle := \langle t,[u,w]\rangle \text{ for all } u,w\in\mathfrak{z}^\perp$$

is an orthogonal map whenever $|t|=1$.

An H-type group is a connected and simply connected Lie group G whose Lie algebra is of H-type.

For a given $0\neq a\in\mathfrak{z}^*$, the dual of $\mathfrak{z}$, we can define a skew-symmetric mapping $B(a)$ on $\mathfrak{z}^\perp$ by

$$\langle B(a)u,w\rangle = a([u,w]) \text{ for all } u,w\in\mathfrak{z}^\perp.$$

We denote by z_a the element of $\mathfrak{z}$ determined by

$$\langle B(a)u,w\rangle = a([u,w]) = \langle J_{z_a}u,w\rangle.$$

Since $B(a)$ is skew-symmetric and nondegenerate, the dimension of $\mathfrak{z}^\perp$ is even, i.e., $\dim\mathfrak{z}^\perp = 2n$.

For a given $0\neq a\in\mathfrak{z}^*$, we can choose an orthonormal basis

$$\{E_1(a),E_2(a),\dots,E_n(a),\bar{E}_1(a),\bar{E}_2(a),\dots,\bar{E}_n(a)\}$$

of $\mathfrak{z}^\perp$ such that

$$B(a)E_i(a) = |z_a|J_{\frac{z_a}{|z_a|}}E_i(a) = |a|\bar{E}_i(a)$$

and

$$B(a)\bar{E}_i(a) = -|a|E_i(a).$$

We set $m=\dim\mathfrak{z}$. Throughout this paper we assume that $m>1$. We can choose an orthonormal basis $\{\epsilon_1,\epsilon_2,\dots,\epsilon_m\}$ of $\mathfrak{z}$ such that $a(\epsilon_1)=|a|$ and $a(\epsilon_j)=0$, $j=2,3,\dots,m$. Then we can denote the elements of $\mathfrak{g}$ by

$$(z,t)=(x,y,t)=\sum_{i=1}^{n}(x_iE_i+y_i\bar{E}_i)+\sum_{j=1}^{m}t_j\epsilon_j.$$

We identify G with its Lie algebra $\mathfrak{g}$ via the exponential map. The group law on H-type group G has the form

$$(z,t)(z',t') = (z+z', t+t'+\tfrac{1}{2}[z,z']), \tag{2-1}$$

where $[z,z']_j = \langle z, U^j z'\rangle$ for a suitable skew-symmetric matrix U^j, $j = 1, 2, \ldots, m$.

Theorem 2.2. *G is an H-type group with underlying manifold $\mathbb{R}^{2n+m}$, with the group law (2-1) and the matrix U^j, $j = 1, 2, \ldots, m$ satisfies the following conditions*:

(i) *U^j is a $2n \times 2n$ skew-symmetric and orthogonal matrix, $j = 1, 2, \ldots, m$.*

(ii) *$U^i U^j + U^j U^i = 0$, where $i, j = 1, 2, \ldots, m$ with $i \neq j$.*

Proof. See [Bonfiglioli and Uguzzoni 2004]. □

Remark 2.3. In particular, $\langle z, U^1 z'\rangle = \sum_{j=1}^{n}(x'_j y_j - y'_j x_j)$.

Remark 2.4. All the above expressions depend on a given $0 \neq a \in \mathfrak{z}^*$, but we will suppress a from them for simplification.

Remark 2.5. It is well know that H-type algebras are closely related to Clifford modules [Reimann 2001]. H-type algebras can be classified by the standard theory of Clifford algebras. Especially, on the H-type group G, there is a relation between the dimension of the center and its orthogonal complement space. That is $m+1 \leq 2n$ (see [Kaplan and Ricci 1983]).

The left invariant vector fields which agree respectively with $\partial/\partial x_j$, $\partial/\partial y_j$ at the origin are given by

$$X_j = \frac{\partial}{\partial x_j} + \frac{1}{2}\sum_{k=1}^{m}\Big(\sum_{l=1}^{2n} z_l U^k_{l,j}\Big)\frac{\partial}{\partial t_k},$$

$$Y_j = \frac{\partial}{\partial y_j} + \frac{1}{2}\sum_{k=1}^{m}\Big(\sum_{l=1}^{2n} z_l U^k_{l,j+n}\Big)\frac{\partial}{\partial t_k},$$

where $z_l = x_l$, $z_{l+n} = y_l$, $l = 1, 2, \ldots, n$.

The vector fields $T_k = \partial/\partial t_k$, $k = 1, 2, \ldots, m$ correspond to the center of G. In terms of these vector fields, we introduce the sublaplacian L and full laplacian Δ respectively

$$L = -\sum_{j=1}^{n}(X_j^2 + Y_j^2) = -\Delta_z + \tfrac{1}{4}|z|^2 T - \sum_{k=1}^{m}\langle z, U^k \nabla_z\rangle T_k, \tag{2-2}$$

$$\Delta = L + T, \tag{2-3}$$

where

$$\Delta_z = \sum_{j=1}^{2n} \frac{\partial^2}{\partial z_j^2}, \quad T = -\sum_{k=1}^{m} \frac{\partial^2}{\partial t_k^2}, \quad \nabla_z = \Big(\frac{\partial}{\partial z_1}, \frac{\partial}{\partial z_2}, \dots, \frac{\partial}{\partial z_{2n}}\Big)^t.$$

3. The restriction operator

First we recall some results about the scaled special Hermite expansion. We refer the reader to [Thangavelu 1993, 2004] for details. Letting $\lambda > 0$, the twisted laplacian (or the scaled special Hermite expansion) L_λ is defined by

$$L_\lambda = -\Delta_z + \frac{\lambda^2|z|^2}{4} - i\lambda \sum_{j=1}^{n} \Big(x_j \frac{\partial}{\partial y_j} - y_j \frac{\partial}{\partial x_j}\Big),$$

where we identify $z = x + iy \in \mathbb{C}^n$ with $z = (x, y) \in \mathbb{R}^{2n}$.

For $f, g \in L^1(\mathbb{C}^n)$, we define the λ-twisted convolution by

$$f \times_\lambda g = \int_{\mathbb{C}^n} f(z-w)g(w)e^{\frac{1}{2}i\lambda \operatorname{Im} z\cdot\overline{w}}\,dw.$$

Set Laguerre function $\varphi_k^\lambda(z) = L_k^{n-1}\big(\frac{1}{2}\lambda|z|^2\big)e^{-\frac{1}{4}\lambda|z|^2}$, $k = 0, 1, 2, \dots$, where L_k^{n-1} is the Laguerre polynomial of type $(n-1)$ and degree k.

For any Schwartz function f on $\mathbb{C}^n$, we have the scaled special Hermite expansion

$$f(z) = \Big(\frac{\lambda}{2\pi}\Big)^n \sum_{k=0}^{\infty} f \times_\lambda \varphi_k^\lambda(z), \tag{3-1}$$

which is an orthogonal form. We also have

$$\|f\|^2 = \Big(\frac{\lambda}{2\pi}\Big)^n \sum_{k=0}^{\infty} \|f \times_\lambda \varphi_k^\lambda\|^2. \tag{3-2}$$

Moreover, $f \times_\lambda \varphi_k^\lambda$ is an eigenfunction of L_λ with the eigenvalue $(2k+n)\lambda$ and

$$\|f \times_\lambda \varphi_k^\lambda\|_2 \le (2k+n)^{n\left(\frac{1}{p}-\frac{1}{2}\right)-\frac{1}{2}} \lambda^{n\left(\frac{1}{p}-\frac{3}{2}\right)} \|f\|_p \quad \text{for } 1 \le p < \frac{6n+2}{3n+4} \tag{3-3}$$

(see [Thangavelu 1991b]).

Now we turn to the expression for the restriction operator. We may identify $\mathfrak{z}^*$ with $\mathfrak{z}$. Therefore, we will write $\langle a, t\rangle$ instead of $a(t)$ for $a \in \mathfrak{z}^*$ and $t \in \mathfrak{z}$.

Lemma 3.1. *Let $0 \ne a \in \mathfrak{z}^*$. If $f(z,t) = e^{-i\langle a,t\rangle}\varphi(z)$, then*

$$Lf(z,t) = e^{-i\langle a,t\rangle} L_{|a|}\varphi(z).$$

Proof. Because $\langle a, t\rangle = |a|t_1$ and $\langle z, U^1\nabla_z\rangle = \sum_{j=1}^n (y_j \frac{\partial}{\partial x_j} - x_j \frac{\partial}{\partial y_j})$, Lemma 3.1 is easily deduced from the expression (2-2). □

Set $e_k^a(z,t) = e^{-i\langle a,t\rangle}\varphi_k^{|a|}(z)$. For $f \in \mathscr{S}(G)$, let

$$f^a(z) = \int_{\mathbb{R}^m} f(z,t)e^{i\langle a,t\rangle}\,dt$$

be the Fourier transform of f with respect to the central variable t. It is easy to obtain

$$f * e_k^a(z,t) = e^{-i\langle a,t\rangle} f^a \times_{|a|} \varphi_k^{|a|}(z). \tag{3-4}$$

Note that $f * e_k^a$ is an eigenfunction of T with the eigenvalue $|a|^2$. Furthermore, it follows from Lemma 3.1 that $f * e_k^a$ is an eigenfunction of L with the eigenvalue $(2k+n)|a|$. Thus $f * e_k^a$ is a joint eigenfunction of the operators L and T.

For a Schwartz function f on an H-type group, using the inversion formula for the Fourier transform together with (3-1) and (3-4), we have

$$\begin{aligned}
f(z,t) &= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m} f^a(z)e^{-i\langle a,t\rangle}\,da \\
&= \frac{1}{(2\pi)^m}\int_{\mathbb{R}^m}\biggl(\frac{|a|^n}{(2\pi)^n}\sum_{k=0}^{\infty} f^a \times_{|a|} \varphi_k^{|a|}(z)\biggr)e^{-i\langle a,t\rangle}\,da \\
&= \frac{1}{(2\pi)^{n+m}}\int_{\mathbb{R}^m}\sum_{k=0}^{\infty} f * e_k^a(z,t)|a|^n\,da \\
&= \int_0^\infty\biggl(\frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty}\lambda^{n+m-1}\int_{S^{m-1}} f * e_k^{\lambda\tilde{a}}(z,t)\,d\sigma(\tilde{a})\biggr)\,d\lambda.
\end{aligned}$$

The operators L and T extend to a pair of strongly commuting self-adjoint operators. Therefore, they admit a joint spectral decomposition. By the spectral theorem, we can define the joint functional calculus of L and T. The joint functional calculus of L and T was investigated in [Casarino and Ciatti 2013a]. As in that paper, we define the operator $\delta_\mu(h(L,T))$ for a suitable function $h : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ as

$$h(L,T)f(z,t) = \\ \int_0^\infty\biggl(\frac{1}{(2\pi)^{n+m}}\sum_{k=0}^{\infty} h\bigl((2k+n)\lambda,\lambda^2\bigr)\lambda^{n+m-1}\int_{S^{m-1}} f * e_k^{\lambda\tilde{a}}(z,t)\,d\sigma(\tilde{a})\biggr)\,d\lambda,$$

where we make the assumption on h that the expression on the right-hand side is a well-defined distribution for all Schwartz functions f. We also suppose $h((2k+n)\lambda,\lambda^2)$ is a strictly monotonic differentiable positive function of λ on $\mathbb{R}_+$, with the domain (A,B) where $0 \le A < B \le \infty$. Then for each $\mu \in (A,B)$, the equation

$$h((2k+n)\lambda,\lambda^2) = \mu$$

may be solved for each k. We denote the solution by $\lambda = \lambda_k(\mu)$ and λ'_k denotes the derivative of λ_k. Replacing λ with μ in the integral, we obtain

$$h(L,T)f(z,t) = \int_A^B \mu\left(\frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda'_k(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a})\right) d\mu,$$

which is the spectral decomposition of $h(L,T)$.

Thus, given a Schwartz function f, the spectral decomposition with respect to $h(L,T)$ is

$$f(z,t) = \int_A^B \left(\frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda'_k(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t) d\sigma(\tilde{a})\right) d\mu.$$

We can also use this equation to introduce the spectral resolution of $h(L,T)$, which is defined by

$$\mathcal{P}_\mu^h f(z,t) = \delta_\mu(h(L,T))f(z,t) = \lim_{\epsilon\to 0^+} \frac{1}{2\epsilon}\chi_{(\mu-\epsilon,\mu+\epsilon)}(h(L,T))f, \tag{3-5}$$

where f is a Schwartz function and $\chi_{(\mu-\epsilon,\mu+\epsilon)}$ is the characteristic function of the interval $(\mu-\epsilon, \mu+\epsilon)$. We easily find

$$\mathcal{P}_\mu^h f(z,t) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)|\lambda'_k(\mu)| \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a}).$$

Specifically, for the full laplacian Δ, $h(\xi,\eta) = \xi+\eta$, so we have $\mu = (2k+n)\lambda+\lambda^2$, which yields

$$\lambda_k(\mu) = \frac{1}{2}\sqrt{4\mu+(2k+n)^2} - \frac{2k+n}{2} \quad \text{and} \quad \lambda'_k(\mu) = \frac{1}{\sqrt{4\mu+(2k+n)^2}}. \tag{3-6}$$

Therefore,

$$\mathcal{P}_\mu^\Delta f(z,t) = \frac{1}{(2\pi)^{n+m}} \sum_{k=0}^{\infty} \lambda_k^{n+m-1}(\mu)\lambda'_k(\mu) \int_{S^{m-1}} f * e_k^{\lambda_k(\mu)\tilde{a}}(z,t)\, d\sigma(\tilde{a}).$$

4. The restriction theorem

Our main result is the following theorem.

Theorem 4.1. *Let G be an H-type group with the underlying manifold $\mathbb{R}^{2n+m}$, where $m > 1$ is the dimension of the center. Let $h(\xi,\eta) = \xi^\alpha + \eta^\beta$, $\alpha, \beta > 0$. Then* *for* $1 \le p \le (2m+2)/(m+3)$, *we have for all Schwartz functions* f:

$$\text{if } \alpha < 2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha > 2\beta \begin{cases} \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha = 2\beta \left\{ \|\mathcal{P}^h_\mu f\|_{p'} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \qquad 0 < \mu < \infty. \right.$$

First, we have the following abstract statement.

Proposition 4.2. *The function* $h((2k+n)\lambda, \lambda^2)$ *is a strictly monotonic differentiable positive function of* λ *on* $\mathbb{R}_+$, *with the domain* (A, B) *where* $0 \le A < B \le \infty$. *Then for* $1 \le p \le (2m+2)/(m+3)$, *the estimate*

$$\|\mathcal{P}^h_\mu f\|_{p'} \le C_\mu \|f\|_p$$

holds, *where*

$$C_\mu \le C \sum_{k=0}^{\infty} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \tag{4-1}$$

for all Schwartz functions f *and all positive* $\mu \in (A, B)$.

The proof of Proposition 4.2 coincides essentially with Theorem 4.1 in [Casarino and Ciatti 2013a] and we omit it. To obtain our Theorem 4.1, it suffices to show the convergence of the series in (4-1). Next we will exploit the following estimates, which can be easily proved by comparing the sums with the corresponding integrals:

Lemma 4.3. *Fix* $\nu \in \mathbb{R}$. *There exists* $C_\nu > 0$ *such that for* $A > 0$ *and* $n \in \mathbb{Z}_+$, *we have*

$$\sum_{\substack{m\in\mathbb{N} \\ 2m+n \ge A}} (2m+n)^\nu \le C_\nu A^{\nu+1}, \quad \nu < -1; \tag{4-2}$$

$$\sum_{\substack{m\in\mathbb{N} \\ 2m+n \le A}} (2m+n)^\nu \le C_\nu A^{\nu+1}, \quad \nu > -1. \tag{4-3}$$

Now Theorem 4.1 follows from the result in the following lemma.

Lemma 4.4. *Let* $h(\xi, \eta) = \xi^\alpha + \eta^\beta$, $\alpha, \beta > 0$. *The series in* (4-1) *has the estimate*

$$\text{if } \alpha < 2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & \mu > 1, \\ C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha > 2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & \mu > 1, \\ C_\mu \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha = 2\beta \begin{cases} C_\mu \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}, & 0 < \mu < \infty. \end{cases}$$

Proof. The function $h(\xi,\eta) = \xi^\alpha + \eta^\beta$, $\alpha, \beta > 0$, so $\mu = (2k+n)^\alpha \lambda_k^\alpha(\mu) + \lambda_k^{2\beta}(\mu)$, which yields

$$\lambda_k'(\mu) = \frac{1}{\alpha(2k+n)^\alpha \lambda_k^{\alpha-1}(\mu) + 2\beta\lambda_k^{2\beta-1}(\mu)}.$$

To study the convergence of this series, we need to distinguish three cases according to the relation of α and 2β: $\alpha < 2\beta$, $\alpha > 2\beta$ and $\alpha = 2\beta$. In order not to burden the exposition, we only prove the case $\alpha < 2\beta$, and the other cases are analogous.

If $\alpha < 2\beta$, then when $\mu \le 1$, it is easy to see that $\lambda_k(\mu) \sim \mu^{\frac{1}{\alpha}}/(2k+n)$ and $\lambda_k'(\mu) \sim \mu^{\frac{1}{\alpha}-1}/(2k+n)$, so that the series

$$\begin{aligned} C_\mu &\le C \sum_{k=0}^{\infty} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \\ &\le C \sum_{k=0}^{\infty} (2k+n)^{2n(\frac{1}{p}-\frac{1}{2})-1} \left(\frac{\mu^{\frac{1}{\alpha}}}{2k+n}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \frac{\mu^{\frac{1}{\alpha}-1}}{2k+n} \\ &\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \sum_{k=0}^{\infty} \frac{1}{(2k+n)^{2m\left(\frac{1}{p}-\frac{1}{2}\right)+1}} \\ &\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \end{aligned} \tag{4-4}$$

converges.

When $\mu > 1$, we split the sum into two parts, the sum over those k such that $(2k+n)^\alpha \lambda_k^\alpha(\mu) \ge \lambda^{2\beta}(\mu)$ and those such that $(2k+n)^\alpha \lambda_k^\alpha(\mu) < \lambda^{2\beta}(\mu)$. They are denoted by I and II respectively.

For the first part, $(2k+n)^\alpha \lambda_k^\alpha(\mu) \ge \lambda^{2\beta}(\mu)$ implies

$$\lambda_k(\mu) \sim \frac{\mu^{\frac{1}{\alpha}}}{2k+n}, \quad \lambda_k'(\mu) \sim \frac{\mu^{\frac{1}{\alpha}-1}}{2k+n}, \quad \text{and} \quad 2k+n \ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}.$$

Then we control the first part I by

$$
\begin{aligned}
\mathrm{I} &\le C \sum_{2k+n\ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \\
&\le C \sum_{2k+n\ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \left(\frac{\mu^{\frac{1}{\alpha}}}{2k+n}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \frac{\mu^{\frac{1}{\alpha}-1}}{2k+n} \\
&\le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \sum_{2k+n\ge \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} \frac{1}{(2k+n)^{2m\left(\frac{1}{p}-\frac{1}{2}\right)+1}}.
\end{aligned}
$$

By (4-2), we have

$$
\mathrm{I} \le C\mu^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \frac{1}{\mu^{\frac{2\beta-\alpha}{2\alpha\beta}\left(2m\left(\frac{1}{p}-\frac{1}{2}\right)\right)}} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}. \tag{4-5}
$$

For the second part, $(2k+n)^{\alpha}\lambda_k^{\alpha}(\mu) < \lambda^{2\beta}(\mu)$ implies

$$
\lambda_k(\mu) \sim \mu^{\frac{1}{2\beta}}, \quad \lambda_k'(\mu) \sim \mu^{\frac{1}{2\beta}-1}, \quad \text{and} \quad 2k+n < \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}.
$$

Then we control the second part II by

$$
\begin{aligned}
\mathrm{II} &\le C \sum_{2k+n< \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lambda_k^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}(\mu)|\lambda_k'(\mu)| \\
&\le C \sum_{2k+n< \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \left(\mu^{\frac{1}{2\beta}}\right)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \mu^{\frac{1}{2\beta}-1} \\
&\le C\mu^{\frac{1}{\beta}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1} \sum_{2k+n< \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1}.
\end{aligned}
$$

Because $1 \le p \le (2m+2)/(m+3)$, we obtain $2n\left(\frac{1}{p}-\frac{1}{2}\right)-1 \ge -1$. Hence, by (4-3) we get

$$
\sum_{2k+n< \mu^{\frac{2\beta-\alpha}{2\alpha\beta}}} (2k+n)^{2n\left(\frac{1}{p}-\frac{1}{2}\right)-1} \lesssim \mu^{\frac{2\beta-\alpha}{2\alpha\beta}\left(2n\left(\frac{1}{p}-\frac{1}{2}\right)\right)} = \mu^{\left(\frac{2}{\alpha}-\frac{1}{\beta}\right)n\left(\frac{1}{p}-\frac{1}{2}\right)}.
$$

Thus, for the second part we also have

$$
\mathrm{II} \le C\mu^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}. \tag{4-6}
$$

Finally, the estimate for the case $\alpha < 2\beta$ follows from (4-4), (4-5) and (4-6). This completes the proof of the first case.

Combining Proposition 4.2 and Lemma 4.4, Theorem 4.1 comes out easily. □

Especially, in the case $\Delta = L + T,\ h(\xi, \eta) = \xi + \eta$, we obtain the restriction theorem associated with the full laplacian on H-type groups.

Corollary 4.5. *For $1 \le p \le (2m+2)/(m+3)$, the estimates*

$$\|\mathcal{P}^{\Delta}_{\mu} f\|_{p'} \le C\mu^{(2n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \mu > 1$$

and

$$\|\mathcal{P}^{\Delta}_{\mu} f\|_{p'} \le C\mu^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ 0 < \mu \le 1$$

hold for all Schwartz functions f.

5. Examples

Similarly to what we have done so far in Theorem 4.1, we now discuss other operators with the form of the joint functional calculus of L and T. We obtain the following results. We omit the arguments which are really similar to that of Theorem 4.1.

Example 5.1. Let $h(\xi, \eta) = (\xi^{\alpha} + \eta^{\beta})^{-1},\ \alpha, \beta > 0$. For $1 \le p \le (2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha < 2\beta \begin{cases} \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha > 2\beta \begin{cases} \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu > 1, \\ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0 < \mu \le 1, \end{cases}$$

$$\text{if } \alpha = 2\beta \left\{ \|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \quad 0 < \mu < \infty. \right.$$

Example 5.2. Let $h(\xi, \eta) = (1+\xi)^{-1}$. For $1 \le p \le (2m+2)/(m+3)$, the estimates

$$\|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C\mu^{-2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \text{ when } \mu \to 0^{+},$$

and

$$\|\mathcal{P}^{h}_{\mu} f\|_{p'} \le C(1-\mu)^{2(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p,\ \text{ when } \mu \to 1^{-},$$

hold for all Schwartz functions f.

More generally, we have:

Example 5.3. Let $h(\xi,\eta)=(\xi^\alpha+\eta^\beta)^\gamma$, $\alpha,\beta,\gamma>0$. For $1\le p\le(2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha<2\beta \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha>2\beta, \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha=2\beta \left\{\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \quad 0<\mu<\infty.\right.$$

Example 5.4. Letting $h(\xi,\eta)=(\xi^\alpha+\eta^\beta)^{-\gamma}$, $\alpha,\beta,\gamma>0$, then for $1\le p\le (2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha<2\beta \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha>2\beta \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & \mu>1,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, & 0<\mu\le 1,\end{cases}$$

$$\text{if } \alpha=2\beta \left\{\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p, \quad 0<\mu<\infty.\right.$$

Example 5.5. Let $h(\xi,\eta)=(1+\xi^\alpha+\eta^\beta)^{-\gamma}$, $\alpha,\beta,\gamma>0$. Then for $1\le p\le (2m+2)/(m+3)$, we have for all Schwartz functions f:

$$\text{if } \alpha\le 2\beta \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p & \text{when } \mu\to 0^+,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C(1-\mu^{\frac{1}{\gamma}})^{\frac{2}{\alpha}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p & \text{when } \mu\to 1^-,\end{cases}$$

$$\text{if } \alpha>2\beta \begin{cases}\|\mathcal{P}_\mu^h f\|_{p'}\le C\mu^{-\frac{2}{\alpha\gamma}(n+m)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p & \text{when } \mu\to 0^+,\\ \|\mathcal{P}_\mu^h f\|_{p'}\le C(1-\mu^{\frac{1}{\gamma}})^{\frac{2}{\alpha}\left(n+\frac{\alpha}{2\beta}m\right)\left(\frac{1}{p}-\frac{1}{2}\right)-1}\|f\|_p & \text{when } \mu\to 1^-.\end{cases}$$

6. Sharpness of the range p

In this section we only give an example to show that the range of p in the restriction theorem associated with the full laplacian Δ is sharp. The example is constructed similarly to the counterexample of Müller [1990], which shows that the estimates between Lebesgue spaces for the operators $\mathcal{P}_\mu^\Delta$ are necessarily trivial.

Let $\varphi\in C_c^\infty(\mathbb{R}^m)$ be a radial function such that $\varphi(a)=\psi(|a|)$, where $\psi\in C_c^\infty(\mathbb{R})$, with $\psi=1$ on a neighborhood of the point n and $\psi=0$ near 0. Let h be a Schwartz function on $\mathbb{R}^m$ and define

$$f(z,t)=\int_{\mathbb{R}^m}\varphi(a)\hat{h}(a)e^{-\frac{|a|}{4}|z|^2}e^{-i\langle a,t\rangle}|a|^n\,da.$$

Denote

$$\begin{aligned}g(z,t)&=\int_{\mathbb{R}^m}\varphi(a)e^{-\frac{|a|}{4}|z|^2}e^{-i\langle a,t\rangle}|a|^n\,da\\&=\int_{\mathbb{R}^{m+2n}}\varphi(a)e^{-\frac{|\xi|^2}{|a|}}e^{-i(\langle a,t\rangle+\langle\xi,z\rangle)}\,d\xi\,da.\end{aligned}$$

Hence $\widehat{g(\xi,a)}=\varphi(a)e^{-\frac{|\xi|^2}{|a|}}$, which shows that $\hat{g}$ and consequently g are Schwartz functions. On the other hand, we have $f=h*_t g$, where $*_t$ denotes the involution about the central variable. By Lemma 3.1, we have $\Delta\big(e^{-i\langle a,t\rangle}e^{-\frac{1}{4}|a||z|^2}\big)=(n\lambda+\lambda^2)e^{-i\langle a,t\rangle}e^{-\frac{1}{4}|a||z|^2}$. Therefore, we write f by the integration with polar coordinates as

$$\begin{aligned}f(z,t)&=\int_0^\infty\bigg(\lambda^{n+m-1}\psi(\lambda)e^{-\frac{\lambda}{4}|z|^2}\int_{S^{m-1}}\hat{h}(\lambda w)e^{-i\lambda\langle w,t\rangle}\,d\sigma(w)\bigg)\,d\lambda\\&=\int_0^\infty\bigg(\lambda_\Delta(\mu)^{n+m-1}\lambda'_\Delta(\mu)\psi(\lambda_\Delta(\mu))e^{-\frac{\lambda_\Delta(\mu)}{4}|z|^2}\\&\qquad\qquad\int_{S^{m-1}}\hat{h}(\lambda_\Delta(\mu)w)e^{-i\lambda_\Delta(\mu)\langle w,t\rangle}\,d\sigma(w)\bigg)\,d\mu\\&=\int_0^\infty\mathcal{P}_\mu^\Delta f(z,t)\,d\mu,\end{aligned}$$

where

$$\begin{aligned}\mathcal{P}_\mu^\Delta f(z,t)&=\lambda_\Delta(\mu)^{n+m-1}\lambda'_\Delta(\mu)\psi(\lambda_\Delta(\mu))e^{-\frac{\lambda_\Delta(\mu)}{4}|z|^2}\\&\qquad\times\int_{S^{m-1}}\hat{h}(\lambda_\Delta(\mu)w)e^{-i\lambda_\Delta(\mu)\langle w,t\rangle}\,d\sigma(w),\\\lambda_\Delta(\mu)&=\frac{\sqrt{n^2+4\mu}-n}{2}.\end{aligned}$$

Therefore, letting $\mu=2n^2$, we have $\lambda_\Delta(2n^2)=n,\ \lambda'_\Delta(2n^2)=1/(3n)$ and

$$\begin{aligned}\mathcal{P}_{2n^2}^\Delta f(z,t)&=\tfrac{1}{3}n^{n+m-2}e^{-\frac{n|z|^2}{4}}\int_{S^{m-1}}\hat{h}(nw)e^{-in\langle w,t\rangle}\,d\sigma(w)\\&=\tfrac{1}{3}n^{n-1}e^{-\frac{n|z|^2}{4}}h*\widehat{d\sigma_n}(t).\end{aligned}$$

From the restriction theorem associated the full laplacian on H-type groups, we have the estimate $\|\mathcal{P}_{2n^2}^\Delta f\|_{L^{p'}(G)}\le C\|f\|_{L^p(G)}$.

Because of

$$\|\mathcal{P}^{\Delta}_{2n^2} f\|_{L^{p'}(G)} = C\|h * \widehat{d\sigma_n}\|_{L^{p'}(\mathbb{R}^m)} \tag{6-1}$$

and

$$\|f\|_{L^p(G)} \le \|h\|_{L^p(\mathbb{R}^m)}\|g\|_{L^1_t L^p_z} \lesssim \|h\|_{L^p(\mathbb{R}^m)}, \tag{6-2}$$

where the mixed Lebesgue norm is defined by

$$\|g\|_{L^1_t L^p_z} = \left(\int_{\mathbb{R}^{2n}} \left(\int_{\mathbb{R}^m} |f(z,t)|\, dt\right)^p dz\right)^{\frac{1}{p}},$$

we have $\|h * \widehat{d\sigma_n}\|_{L^{p'}(\mathbb{R}^m)} \le C\|h\|_{L^p(\mathbb{R}^m)}$.

From the sharpness of the Stein–Tomas theorem which is guaranteed by the Knapp counterexample, this would imply $p \le (2m+2)/(m+3)$. Hence the range of p can not be extended. With the same tricks we can prove the range of p for the restriction theorem associated with the functional calculus is also sharp.

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