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IDENTITIES INVOLVING CYCLIC AND SYMMETRIC SUMS OF REGULARIZED MULTIPLE ZETA VALUES

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There are two types of regularized multiple zeta values: harmonic and shuffle types. The first purpose of the present paper is to give identities involving cyclic sums of regularized multiple zeta values of both types for depth less than 5. Michael Hoffman, in "Quasi-symmetric functions and mod *p* multiple harmonic sums" (*Kyushu Journal of Mathematics* 69 (2015), 345–366) proved an identity involving symmetric sums of regularized multiple zeta values of harmonic type for arbitrary depth. The second purpose is to prove Hoffman's identity for shuffle type. We also give a connection between the identities involving cyclic sums and symmetric sums, for depth less than 5.

1. Introduction and statement of results

Multiple zeta values (MZVs) are real numbers that are variations of special values of the Riemann zeta function $\zeta_1(s)=\sum_{m=1}^{\infty} 1/m^s$ with integer arguments. Regularized multiple zeta values (RMZVs) are generalizations of MZVs, which are defined in [Ihara et al. 2006] as constant terms of certain polynomials. There are two types of RMZVs: harmonic and shuffle types. It is known that these values satisfy a great many relations over $\mathbb{Q}$, including, for example, extended harmonic and shuffle relations, Drinfeld associator relations, and Kawashima's relations (e.g., see [Drinfeld 1990; Ihara et al. 2006; Kawashima 2009]). New classes of relations are being studied, but their exact structure is not yet fully understood.

The first purpose (Theorem 1.1) of the present paper is to give identities involving cyclic sums of RMZVs of both types for depth less than 5. Hoffman [1992, Theorem 2.2] proved an identity involving symmetric sums of MZVs for arbitrary depth, and then, he extended it to RMZVs of harmonic type [Hoffman 2015, Theorem 2.3]. The second purpose (Theorem 1.2) is to prove Hoffman's identity for shuffle type. We also show that Theorem 1.1 yields Theorem 1.2, for depth less than 5 (see Corollary 1.3).

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We will begin by introducing the notation and terminology that will be used to state our results. An MZV is a convergent series defined by

$$\zeta_n(\mathbf{l}_n) := \sum_{m_1>\cdots>m_n>0} \frac{1}{m_1^{l_1}\cdots m_n^{l_n}},$$

where $\mathbf{l}_n = (l_1, \ldots, l_n)$ is an (ordered) index set of positive integers with $l_1 \ge 2$. In other words, MZVs are images under the real-valued function ζ_n with the domain $\{(l_1, \ldots, l_n) \in \mathbb{N}^n \mid l_1 \ge 2\}$, where $\mathbb{N}$ denotes the set of positive integers. We call $w_n(\mathbf{l}_n) = l_1 + \cdots + l_n$ the weight, and $d_n(\mathbf{l}_n) = n$ the depth. Ihara, Kaneko, and Zagier [Ihara et al. 2006] extended MZVs to two types of RMZV (harmonic and shuffle) with two different renormalization procedures for divergent series $\zeta_n(\mathbf{l}_n)$ of $l_1 = 1$. The former and latter types are denoted by $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, and they inherit the harmonic and shuffle relation structures, respectively. The following are a few examples of these values: $\zeta_1^*(1) = \zeta_1^{ш}(1) = \zeta_2^{ш}(1, 1) = 0$ and $\zeta_2^*(1, 1) = -\zeta_1(2)/2 \neq 0$. In other words, RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$ are images under two different extension functions of ζ_n to the domain $\mathbb{N}^n$.

Let $\mathfrak{S}_n$ denote the symmetric group of degree n, and let $e = e_n$ denote its unit element. Let $\mathfrak{C}_3$ and $\mathfrak{C}_4$ be the cyclic subgroups in $\mathfrak{S}_3$ and $\mathfrak{S}_4$ given by $\mathfrak{C}_3 = \langle(123)\rangle = \{e, (123), (132)\}$ and $\mathfrak{C}_4 = \langle(1234)\rangle = \{e, (1234), (13)(24), (1432)\}$, respectively. We set $\mathfrak{C}_2 = \langle(12)\rangle$ (or $\mathfrak{C}_2 = \mathfrak{S}_2$) for convenience. The group ring $\mathbb{Z}[\mathfrak{S}_n]$ of $\mathfrak{S}_n$ over $\mathbb{Z}$ acts on a function f of n variables in a natural way by

$$(f \mid \Gamma)(x_1, \ldots, x_n) := \sum a_i f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)}),$$

where $\Gamma = \sum a_i\sigma_i \in \mathbb{Z}[\mathfrak{S}_n]$. This is a right action, that is, $f \mid (\Gamma_1\Gamma_2) = (f|\Gamma_1) \mid \Gamma_2$. For a subset H in $\mathfrak{S}_n$, we define the sum of all elements in H by

$$\Sigma_H := \sum_{\sigma\in H} \sigma \in \mathbb{Z}[\mathfrak{S}_n].$$

That is, $(f \mid \Sigma_H)(x_1, \ldots, x_n)$ is $\sum_{\sigma\in H} f(x_{\sigma^{-1}(1)}, \ldots, x_{\sigma^{-1}(n)})$. In particular, if H is a group, it is $\sum_{\sigma\in H} f(x_{\sigma(1)}, \ldots, x_{\sigma(n)})$ because $H = H^{-1}$. For positive integers $n_1, \ldots, n_j, n$ with $n_1 + \cdots + n_j = n$, we define real-valued functions with the domain $\mathbb{N}^n$ by

$$\zeta^\dagger_{(n_1,\ldots,n_j)}(\mathbf{l}_n) := \zeta^\dagger_{n_1}(l_1, \ldots, l_{n_1})\zeta^\dagger_{n_2}(l_{n_1+1}, \ldots, l_{n_1+n_2}) \cdots \zeta^\dagger_{n_j}(l_{n_1+n_2+\cdots+n_{j-1}+1}, \ldots, l_n),$$

where $\dagger \in \{*, ш\}$. For example,

$$\zeta^\dagger_{(1,1)}(\mathbf{l}_2) = \zeta^\dagger_1(l_1)\zeta^\dagger_1(l_2) \quad \text{and} \quad \zeta^\dagger_{(2,1)}(\mathbf{l}_3) = \zeta^\dagger_2(l_1, l_2)\zeta^\dagger_1(l_3).$$

We define the characteristic functions χ_n^* and $\chi_n^{ш}$ of the set $\mathbb{N}^n$ by

$$\chi_n^*(\mathbf{l}_n) = 1 \quad \text{and} \quad \chi_n^{ш}(\mathbf{l}_n) = \begin{cases} 0 & \text{if } n > 1,\ l_1 = \cdots = l_n = 1, \\ 1 & \text{otherwise,} \end{cases} \tag{1-1}$$

respectively. We have defined $\chi_1^{ш}(1)$ to be not 0 but 1, though this definition will not be used in Theorem 1.1. We will need it to hold consistency between Definitions (1-1) and (1-7); to prove Theorem 1.2, (1-7) is required.

Theorem 1.1 is stated as follows.

Theorem 1.1. *Let $\mathbf{l}_n = (l_1, \ldots, l_n)$ be an index set in $\mathbb{N}^n$, and let $L_n = w_n(\mathbf{l}_n)$ be its weight. Then we have the following identities for RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$ of $n = 2, 3,$ and 4:*

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{C}_2})(\mathbf{l}_2) = \zeta_{(1,1)}^\dagger(\mathbf{l}_2) - \chi_2^\dagger(\mathbf{l}_2)\zeta_1(L_2), \tag{1-2}$$

$$(\zeta_3^\dagger \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) = -\zeta_{(1,1,1)}^\dagger(\mathbf{l}_3) + (\zeta_{(2,1)}^\dagger \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) + \chi_3^\dagger(\mathbf{l}_3)\zeta_1(L_3), \tag{1-3}$$

$$\begin{aligned}(\zeta_4^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) = \zeta_{(1,1,1,1)}^\dagger(\mathbf{l}_4) - (\zeta_{(2,1,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) + (\zeta_{(2,2)}^\dagger \mid \Sigma_{\mathfrak{C}_4^0})(\mathbf{l}_4),\\ + (\zeta_{(3,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4) - \chi_4^\dagger(\mathbf{l}_4)\zeta_1(L_4),\end{aligned} \tag{1-4}$$

where $\dagger \in \{*, ш\}$, and $\mathfrak{C}_4^0$ in (1-4) is the subset $\{e, (1234)\}$ of $\mathfrak{C}_4$.

We note that (1-2) can be easily obtained from the harmonic relations

$$\zeta_1^*(l_1)\zeta_1^*(l_2) = \zeta_2^*(l_1, l_2) + \zeta_2^*(l_2, l_1) + \zeta_1^*(l_1 + l_2)$$

for RMZVs of harmonic type of depth 2; thus our main results are (1-3) and (1-4) (see Section 5 for their straightforward expressions).

We now recall Hoffman's identity. Let $|P|$ be the number of elements of a set P. For any partition $\Pi = \{P_1, \ldots, P_m\}$ of the set $\{1, \ldots, n\}$, we define an integer $\tilde{c}_n(\Pi)$ by

$$\tilde{c}_n(\Pi) := (-1)^{n-m} \prod_{i=1}^{m} (|P_i| - 1)!. \tag{1-5}$$

For $\dagger \in \{*, ш\}$, we define a real number $\zeta^\dagger(\mathbf{l}_n; \Pi)$ by

$$\zeta^\dagger(\mathbf{l}_n; \Pi) := \prod_{i=1}^{m} \chi^\dagger(\mathbf{l}_n; P_i)\biggl(\sum_{p \in P_i} l_p\biggr), \tag{1-6}$$

where

$$\chi^\dagger(\mathbf{l}_n; P_i) := \begin{cases} 0 & \text{if } \dagger = ш,\ |P_i| > 1, \text{ and } l_p = 1 \text{ for all } p \in P_i, \\ 1 & \text{otherwise.} \end{cases} \tag{1-7}$$

For example,

$$\chi^{ш}((2,1,1);\{2,3\})=0 \quad \text{and} \quad \chi^{ш}((2,1,1);\{1,3\})=\chi^{ш}((2,1,1);\{3\})=1.$$

We note that $\chi^{\dagger}(\mathbf{l}_n; P_i) = \chi^{\dagger}_j(l_{p_1}, \dots, l_{p_j})$ if $P_i = \{p_1, \dots, p_j\}$. For any index $\mathbf{l}_n = (l_1, \dots, l_n)$ in $\mathbb{N}^n$, Hoffman [2015, Theorem 2.3] proved the following identity involving symmetric sums of RMZVs of harmonic type:

$$(\zeta^{\dagger}_n \mid \Sigma_{\mathfrak{S}_n})(\mathbf{l}_n) = \sum_{\text{partitions } \Pi \text{ of } \{1,\dots,n\}} \tilde{c}_n(\Pi)\zeta^{\dagger}(\mathbf{l}_n; \Pi), \tag{1-8}$$

where $\dagger = *$. In the case that $l_i > 1$ for all i, he proved (1-8) in [Hoffman 1992, Theorem 2.2]. (In this case, $\zeta_n(\mathbf{l}_n) = \zeta^*_n(\mathbf{l}_n) = \zeta^{ш}_n(\mathbf{l}_n)$.)

Theorem 1.2 is stated as follows.

Theorem 1.2. *Identity (1-8) for $\dagger = ш$ holds.*

Corollary 1.3 gives a connection between identities involving cyclic sums and symmetric sums of RMZVs, for depth less than 5.

Corollary 1.3. *Let $\dagger \in \{*, ш\}$. Identity (1-2) yields (1-8) for $n = 2$, identities (1-2) and (1-3) yield (1-8) for $n = 3$, and identities (1-2), (1-3), and (1-4) yield (1-8) for $n = 4$.*

Remark 1.4. Hoffman proved (1-8) for $\dagger = *$ under a general algebraic setup, i.e., the harmonic algebra $\mathfrak{H}^1_*$ that will be introduced in Section 2. (To be more precise, he used the algebra of quasisymmetric functions that is isomorphic to $\mathfrak{H}^1_*$.) The constant terms of the polynomials $Z^*_{\mathbf{l}_n}(T)$ defined in [Ihara et al. 2006] are RMZVs $\zeta^*_n(\mathbf{l}_n)$, and the polynomials $Z^*_{\mathbf{l}_n}(T)$ have the same harmonic relation structure as RMZVs $\zeta^*_n(\mathbf{l}_n)$ (see Section 2 for details). Thus, (1-8) for $\dagger = *$ also holds in the case of $Z^*_{\mathbf{l}_n}(T)$. This fact will be necessary to prove Theorem 1.2.

We now briefly explain how Theorem 1.1, Theorem 1.2, and Corollary 1.3 can be proved. We first prove the identities in Theorem 1.1 for $\dagger = *$ from harmonic relations of RMZVs $\zeta^*_n(\mathbf{l}_n)$. Ihara et al. [2006, Theorem 1] gave a class of relations over $\mathbb{Q}$ between RMZVs $\zeta^*_n(\mathbf{l}_n)$ and $\zeta^{ш}_n(\mathbf{l}_n)$, which we call renormalization relations. Using renormalization relations, we derive the identities in Theorem 1.1 for $\dagger = ш$ from those for $\dagger = *$, and we complete the proof of Theorem 1.1. Similarly, we prove Theorem 1.2 by combining the renormalization relations and (1-8) for $\dagger = *$ in which $\zeta^*_n(\mathbf{l}_n)$ are replaced by $Z^*_{\mathbf{l}_n}(T)$. We show Corollary 1.3 by focusing on the fact that $\mathfrak{C}_n$ is a subgroup of $\mathfrak{S}_n$, i.e., $(\zeta^{\dagger}_n \mid \Sigma_{\mathfrak{C}_n})(\mathbf{l}_n)$ is a partial sum of $(\zeta^{\dagger}_n \mid \Sigma_{\mathfrak{S}_n})(\mathbf{l}_n)$.

It is worth noting that Theorem 1.1 gives the following property, which is an analog of the parity property [Borwein and Girgensohn 1996; Euler 1776; Ihara et al. 2006; Tsumura 2004]; any cyclic sum of RMZVs of depth less than 5, or

$$(\zeta_n^{\dagger} \mid \Sigma_{\mathfrak{C}_n})(\mathbf{l}_n) = \sum_{j=1}^{n} \zeta_n^{\dagger}(l_j, \ldots, l_n, l_1, \ldots, l_{j-1}) \tag{1-9}$$

for $n = 2, 3, 4$ and $\dagger \in \{*, ш\}$, is a rational linear combination of the Riemann zeta value $\zeta_1(l_1 + \cdots + l_n)$ and products of RMZVs of smaller depth and weight. It appears that the existence of such a property for depth greater than 4 is an open problem. (The case of symmetric sums of general depth easily follows from (1-8); there is a stronger property from (1-8), such that any symmetric sum can be written in terms of only Riemann zeta values.) It is also worth noting that Hoffman and Ohno [2003] studied a class of relations involving

$$\sum_{j=1}^{n} \zeta_n(l_j + 1, l_{j+1}, \ldots, l_n, l_1, \ldots, l_{j-1}),$$

whose form is quite similar to (1-9), but the first indices differ.

The paper is organized as follows. In Section 2, we review some facts of RMZVs by referring to [Hoffman 1997; Ihara et al. 2006]. Sections 3 and 4 have two and three subsections, respectively. Sections 3.1 and 3.2 are devoted to calculating harmonic relations for RMZVs $\zeta_n^*(\mathbf{l}_n)$ and renormalization relations between RMZVs $\zeta_n^*(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, respectively, for depth less than 5. We then prove Theorem 1.1 in Section 4.1, Theorem 1.2 in Section 4.2, and Corollary 1.3 in Section 4.3. We give some examples of Theorems 1.1 and 1.2 in Section 5.

Remark 1.5. (i) Although the ideas of the proofs are the same, the computational complexity of proving (1-4) is much greater than that required to prove (1-2) and (1-3). We recommend that, on first reading, those readers who are interested only in the ideas skip over the statements relating to the proof of (1-4) (or statements in the case of depth 4).

(ii) This paper is an expansion of Section 2.1 in [Machide 2012]. The remainder of the results of that article has been amplified in [Machide 2015].

2. Preparation

Let $\mathfrak{H} = \mathbb{Q}\langle x, y\rangle$ be the noncommutative polynomial algebra over $\mathbb{Q}$ in two indeterminates x and y, and let $\mathfrak{H}^0$ and $\mathfrak{H}^1$ be its subalgebras $\mathbb{Q} + x\mathfrak{H}y$ and $\mathbb{Q} + \mathfrak{H}y$, respectively. These algebras satisfy the inclusion relations $\mathfrak{H}^0 \subset \mathfrak{H}^1 \subset \mathfrak{H}$. Let z_l denote $x^{l-1}y$ for any integer $l \geq 1$. Every word $w = w_0 y$ in the set $\{x, y\}$ with terminal letter y is expressed as $w = z_{l_1} \cdots z_{l_n}$ uniquely, and so $\mathfrak{H}^1$ is the free algebra generated by z_l $(l = 1, 2, 3, \ldots)$. We define the harmonic product $*$ on $\mathfrak{H}^1$

inductively by

$$1 * w = w * 1 = w, \tag{2-1}$$

$$z_k w_1 * z_l w_2 = z_k(w_1 * z_l w_2) + z_l(z_k w_1 * w_2) + z_{k+l}(w_1 * w_2), \tag{2-2}$$

for any integers $k, l \geq 1$ and words $w, w_1, w_2 \in \mathfrak{H}^1$, and then extend it by $\mathbb{Q}$-bilinearity. This product gives the subalgebras $\mathfrak{H}^0$ and $\mathfrak{H}^1$ structures of commutative $\mathbb{Q}$-algebras [Hoffman 1997], which we denote by $\mathfrak{H}^0_*$ and $\mathfrak{H}^1_*$, respectively; note that $\mathfrak{H}^0_*$ is a subalgebra of $\mathfrak{H}^1_*$. In a similar way, we can define the shuffle product ш on $\mathfrak{H}^1$ and the commutative $\mathbb{Q}$-algebras $\mathfrak{H}^0_{ш}$ and $\mathfrak{H}^1_{ш}$ (see [Ihara et al. 2006; Reutenauer 1993] for details).

Let $Z : \mathfrak{H}^0 \to \mathbb{R}$ be the $\mathbb{Q}$-linear map (evaluation map) given by

$$Z(z_{l_1} \cdots z_{l_n}) = \zeta_n(\mathbf{l}_n) \quad (z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^0). \tag{2-3}$$

We know from [Hoffman 1997] that Z is homomorphic on both products $*$ and ш, that is,

$$Z(w_1 * w_2) = Z(w_1 \,ш\, w_2) = Z(w_1)Z(w_2)$$

for $w_1, w_2 \in \mathfrak{H}^0$. Let $\mathbb{R}[T]$ be the polynomial ring in a single indeterminate with real coefficients. Through the isomorphisms $\mathfrak{H}^1_* \simeq \mathfrak{H}^0_*[y]$ and $\mathfrak{H}^1_{ш} \simeq \mathfrak{H}^0_{ш}[y]$, which were proved in [Hoffman 1997] and [Reutenauer 1993], respectively, Ihara et al. [2006, Proposition 1] considered the algebra homomorphisms

$$Z^* : \mathfrak{H}^1_* \to \mathbb{R}[T] \quad \text{and} \quad Z^{ш} : \mathfrak{H}^1_{ш} \to \mathbb{R}[T],$$

respectively, which are uniquely characterized by the property that they extend the evaluation map Z and send y to T. For any word $w = z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^1$, we denote by $Z^*_{\mathbf{l}_n}(T)$ and $Z^{ш}_{\mathbf{l}_n}(T)$ the images under the maps Z^* and $Z^{ш}$, respectively, of the word w, that is,

$$Z^*_{l_1,\ldots,l_n}(T) = Z^*(z_{l_1} \cdots z_{l_n}) \quad \text{and} \quad Z^{ш}_{l_1,\ldots,l_n}(T) = Z^{ш}(z_{l_1} \cdots z_{l_n}). \tag{2-4}$$

(The notation $Z^*_{\mathbf{l}_n}(T)$ and $Z^{ш}_{\mathbf{l}_n}(T)$ will be used when we focus on the variable T and the corresponding index set $\mathbf{l}_n$ of the word $z_{l_1} \cdots z_{l_n}$.) Then the RMZVs $\zeta^*(\mathbf{l}_n)$ and $\zeta^{ш}(\mathbf{l}_n)$ of the harmonic and shuffle types are defined as

$$\zeta^*(l_1, \ldots, l_n) := Z^*_{l_1,\ldots,l_n}(0) \quad \text{and} \quad \zeta^{ш}(l_1, \ldots, l_n) := Z^{ш}_{l_1,\ldots,l_n}(0), \tag{2-5}$$

respectively. Obviously, $\zeta^*_n(\mathbf{l}_n) = \zeta^{ш}_n(\mathbf{l}_n) = \zeta_n(\mathbf{l}_n)$ if $l_1 > 1$. We have

$$Z^*(z_{k_1} \cdots z_{k_m} * z_{l_1} \cdots z_{l_n}) = Z^*(z_{k_1} \cdots z_{k_m})Z^*(z_{l_1} \cdots z_{l_n})$$

for index sets $(k_1, \ldots, k_m)$ and $(l_1, \ldots, l_n)$, since Z^* is homomorphic, and so we see from the first equations of (2-4) and (2-5) that the RMZVs $\zeta^*_n(\mathbf{l}_n)$ satisfy the harmonic relations. In Section 3.1, we will calculate these relations in detail for depth less than 5. (We can also see that the RMZVs $\zeta_n^{ш}(\mathbf{l}_n)$ satisfy the shuffle relations since $Z^{ш}$ is homomorphic, but we will not discuss this in the present paper.)

Let $A(u)=\sum_{k=0}^{\infty}\gamma_k u^k$ be the Taylor expansion of $e^{\gamma u}\Gamma(1+u)$ near $u=0$, where γ is Euler's constant and $\Gamma(x)$ is the gamma function. The renormalization map $\rho:\mathbb{R}[T]\to\mathbb{R}[T]$ is an $\mathbb{R}$-linear map defined by

$$\rho(e^{Tu})=A(u)e^{Tu}. \tag{2-6}$$

That is, images $\rho(T^m)$ are determined by comparing the coefficients of u^m on both sides of (2-6), and expressed as

$$\rho(T^m)=m!\sum_{i=0}^{m}\gamma_i\frac{T^{m-i}}{(m-i)!}\quad (m=0,1,2,\dots). \tag{2-7}$$

Then the renormalization formula proved by Ihara et al. [2006, Theorem 1] is

$$\rho(Z^{*}_{\mathbf{l}_n}(T))=Z^{ш}_{\mathbf{l}_n}(T). \tag{2-8}$$

Combining (2-5) and (2-8) with $T=0$, we can obtain relations between RMZVs $\zeta_n^{*}(\mathbf{l}_n)$ and $\zeta_n^{ш}(\mathbf{l}_n)$, or renormalization relations. In Section 3.2, we will calculate these relations in detail for depth less than 5.

3. Relations

3.1. ***Harmonic relations.*** We begin by defining the notation that we will use to state the harmonic relations of RMZVs $\zeta_n^{*}(\mathbf{l}_n)$ of depth less than 5 in terms of real-valued functions.

We first define analogs of the weight map $w_n:\mathbb{N}^n\to\mathbb{N}$ of depth n. For positive integers $n_1,\dots,n_j,n$ with $n_1+\cdots+n_j=n$, we define the map $w_{(n_1,\dots,n_j)}$ from $\mathbb{N}^n$ to $\mathbb{N}^j$ by

$$w_{(n_1,\dots,n_j)}(\mathbf{l}_n):=(w_{n_1}(l_1,\dots,l_{n_1}),\dots,w_{n_j}(l_{n_1+n_2+\cdots+n_{j-1}+1},\dots,l_n)). \tag{3-1}$$

For example, $w_{(2,1)}(l_1,l_2,l_3)=(l_1+l_2,l_3)$ and $w_{(1,2,1)}(l_1,l_2,l_3,l_4)=(l_1,l_2+l_3,l_4)$. We define a subset U_3 in $\mathfrak{S}_3$ as

$$U_3=\{e_3,(23),(123)\}, \tag{3-2}$$

and subsets U_4, V_4^0, V_4, W_4^0, W_4^1, W_4, and X_4 in $\mathfrak{S}_4$ as

$$U_4=\{e_4,(34),(234),(1234)\}, \tag{3-3}$$

$$\begin{aligned} V_4^0&=\{(23),(1243)\},\\ V_4&=\{e_4,(13)(24),(123),(243)\}\cup V_4^0, \end{aligned} \tag{3-4}$$

$$\begin{aligned} W_4^0 &= \{(23), (24)\}, \\ W_4^1 &= \{(34), (1234), (1243), (1324)\} \cup W_4^0, \\ W_4 &= \{e_4, (13)(24), (123), (124), (234), (243)\} \cup W_4^1, \end{aligned} \tag{3-5}$$

$$X_4 = \{(14), (23)\} \cup \mathfrak{C}_4. \tag{3-6}$$

We have the inclusion relations $V_4^0 \subset V_4$ and $W_4^0 \subset W_4^1 \subset W_4$. We denote by $\mathfrak{A}_3$ and $\mathfrak{A}_4$ the alternating groups of degree 3 and 4, respectively. Note that $\mathfrak{A}_3 = \mathfrak{C}_3$. Functional composition $\circ$ satisfies the distributive law, i.e.,

$$\Big(\sum_i f_i\Big) \circ \Big(\sum_j g_j\Big) = \sum_{i,j} f_i \circ g_j,$$

where f_i are real-valued functions with the domain $\mathbb{N}^n$, and g_j are vector-valued functions with a same domain whose images are included in $\mathbb{N}^n$. The notation $f \circ g \mid \sigma$ is unambiguous since $(f \circ g) \mid \sigma = f \circ (g \mid \sigma)$.

Remark 3.1. For integers j, n with $1 \le j \le n-1$, let $\mathrm{sh}_j^{(n)}$ be the shuffle elements given in [Ihara et al. 2006], which are elements in $\mathbb{Z}[\mathfrak{S}_n]$ and defined as

$$\mathrm{sh}_j^{(n)} := \sum_{\substack{\sigma \in \mathfrak{S}_n \\ \left(\substack{\sigma(1) < \cdots < \sigma(j) \\ \sigma(j+1) < \cdots < \sigma(n)}\right)}} \sigma.$$

The elements Σ_{U_3}, Σ_{U_4}, and Σ_{V_4} are equal to $\mathrm{sh}_2^{(3)}$, $\mathrm{sh}_3^{(4)}$, and $\mathrm{sh}_2^{(4)}$, respectively. The element Σ_{W_4} cannot be written in terms of only a shuffle element, but it is equal to $\mathrm{sh}_2^{(4)} \Sigma_{\langle(34)\rangle} = \Sigma_{V_4} \Sigma_{\langle(34)\rangle}$ as we will see in (3-36), below.

The harmonic relations we desire are listed below.

Proposition 3.2 (case of depth 2). *We have*

$$\zeta^*_{(1,1)} = \zeta^*_2 \mid \Sigma_{\mathfrak{C}_2} + \zeta_1 \circ w_2. \tag{3-7}$$

Proposition 3.3 (case of depth 3). *We have*

$$\zeta^*_{(2,1)} = \zeta^*_3 \mid \Sigma_{U_3} + \zeta^*_2 \circ (w_{(2,1)} \mid (123) + w_{(1,2)}), \tag{3-8}$$

$$\zeta^*_{(1,1,1)} = \zeta^*_3 \mid \Sigma_{\mathfrak{S}_3} + \zeta^*_2 \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3. \tag{3-9}$$

Proposition 3.4 (case of depth 4). *We have*

$$\zeta^*_{(3,1)} = \zeta^*_4 \mid \Sigma_{U_4} + \zeta^*_3 \circ ((w_{(2,1,1)} + w_{(1,2,1)}) \mid (234) + w_{(1,1,2)}), \tag{3-10}$$

$$\begin{aligned} \zeta^*_{(2,2)} = \zeta^*_4 \mid \Sigma_{V_4} &+ \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0} \\ &+ \zeta^*_2 \circ w_{(2,2)} \mid (23), \end{aligned} \tag{3-11}$$

$$\begin{aligned} \zeta^*_{(2,1,1)} = \zeta^*_4 \mid \Sigma_{W_4} \\ + \zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}} \end{aligned} \tag{3-12}$$

$$+\zeta_2^*\circ\big(w_{(2,2)}\mid\Sigma_{W_4^0}+w_{(3,1)}\mid(24)+w_{(1,3)}\big),$$

$$\begin{aligned}\zeta^*_{(1,1,1,1)}&=\zeta_4^*\mid\Sigma_{\mathfrak{S}_4}+\zeta_3^*\circ(w_{(2,1,1)}+w_{(1,2,1)}+w_{(1,1,2)})\mid\Sigma_{\mathfrak{A}_4}\\&\quad+\zeta_2^*\circ(w_{(2,2)}\mid\Sigma_{X_4}+(w_{(3,1)}+w_{(1,3)})\mid\Sigma_{\mathfrak{C}_4})+\zeta_1\circ w_4,\end{aligned}\tag{3-13}$$

where $W^1_{4,\sigma}$ $(\sigma\in\{(34),(1234),(1324)\})$ *in* (3-12) *mean the subsets* $W_4^1\setminus\{\sigma\}$.

We will show Lemmas 3.5, 3.6, and 3.7 to prove Propositions 3.2, 3.3, and 3.4, respectively. These lemmas calculate the harmonic products of the generators z_l of $\mathfrak{H}^1_*$ for the corresponding depths.

Lemma 3.5 (case of depth 2). *For positive integers* l_1, l_2, *we have*

$$z_{l_1}*z_{l_2}=z_{l_1}z_{l_2}+z_{l_2}z_{l_1}+z_{l_1+l_2}.\tag{3-14}$$

Lemma 3.6 (case of depth 3). *For positive integers* l_1, l_2, l_3, *we have*

$$z_{l_1}z_{l_2}*z_{l_3}=z_{l_1}z_{l_2}z_{l_3}+z_{l_1}z_{l_3}z_{l_2}+z_{l_3}z_{l_1}z_{l_2}+z_{l_1+l_3}z_{l_2}+z_{l_1}z_{l_2+l_3},\tag{3-15}$$

$$\begin{aligned}z_{l_1}*z_{l_2}*z_{l_3}&=z_{l_1}z_{l_2}z_{l_3}+z_{l_1}z_{l_3}z_{l_2}+z_{l_2}z_{l_1}z_{l_3}+z_{l_2}z_{l_3}z_{l_1}+z_{l_3}z_{l_1}z_{l_2}\\&\quad+z_{l_3}z_{l_2}z_{l_1}+z_{l_1+l_2}z_{l_3}+z_{l_1+l_3}z_{l_2}+z_{l_2+l_3}z_{l_1}\\&\quad+z_{l_1}z_{l_2+l_3}+z_{l_2}z_{l_1+l_3}+z_{l_3}z_{l_1+l_2}+z_{l_1+l_2+l_3}.\end{aligned}\tag{3-16}$$

Lemma 3.7 (case of depth 4). *For positive integers* l_1, l_2, l_3, l_4, *we have*

$$\begin{aligned}z_{l_1}z_{l_2}z_{l_3}*z_{l_4}&=z_{l_1}z_{l_2}z_{l_3}z_{l_4}+z_{l_1}z_{l_2}z_{l_4}z_{l_3}+z_{l_1}z_{l_4}z_{l_2}z_{l_3}+z_{l_4}z_{l_1}z_{l_2}z_{l_3}\\&\quad+z_{l_1+l_4}z_{l_2}z_{l_3}+z_{l_1}z_{l_2+l_4}z_{l_3}+z_{l_1}z_{l_2}z_{l_3+l_4},\end{aligned}\tag{3-17}$$

$$\begin{aligned}z_{l_1}z_{l_2}*z_{l_3}z_{l_4}&=z_{l_1}z_{l_2}z_{l_3}z_{l_4}+z_{l_1}z_{l_3}z_{l_2}z_{l_4}+z_{l_1}z_{l_3}z_{l_4}z_{l_2}\\&\quad+z_{l_3}z_{l_1}z_{l_2}z_{l_4}+z_{l_3}z_{l_1}z_{l_4}z_{l_2}+z_{l_3}z_{l_4}z_{l_1}z_{l_2}\\&\quad+z_{l_1+l_3}z_{l_2}z_{l_4}+z_{l_1+l_3}z_{l_4}z_{l_2}+z_{l_1}z_{l_2+l_3}z_{l_4}\\&\quad+z_{l_3}z_{l_1+l_4}z_{l_2}+z_{l_1}z_{l_3}z_{l_2+l_4}+z_{l_3}z_{l_1}z_{l_2+l_4}+z_{l_1+l_3}z_{l_2+l_4},\end{aligned}\tag{3-18}$$

$$\begin{aligned}z_{l_1}z_{l_2}*z_{l_3}*z_{l_4}&=z_{l_1}z_{l_2}*z_{l_3}z_{l_4}+z_{l_1}z_{l_2}*z_{l_4}z_{l_3}\\&\quad+z_{l_3+l_4}z_{l_1}z_{l_2}+z_{l_1}z_{l_3+l_4}z_{l_2}+z_{l_1}z_{l_2}z_{l_3+l_4}\\&\quad+z_{l_1+l_3+l_4}z_{l_2}+z_{l_1}z_{l_2+l_3+l_4},\end{aligned}\tag{3-19}$$

$$\begin{aligned}z_{l_1}*z_{l_2}*z_{l_3}*z_{l_4}&=z_{l_1}z_{l_2}*z_{l_3}*z_{l_4}+z_{l_2}z_{l_1}*z_{l_3}*z_{l_4}\\&\quad+z_{l_1+l_2}z_{l_3}z_{l_4}+z_{l_1+l_2}z_{l_4}z_{l_3}+z_{l_3}z_{l_1+l_2}z_{l_4}+z_{l_4}z_{l_1+l_2}z_{l_3}\\&\quad+z_{l_3}z_{l_4}z_{l_1+l_2}+z_{l_4}z_{l_3}z_{l_1+l_2}+z_{l_1+l_2}z_{l_3+l_4}+z_{l_3+l_4}z_{l_1+l_2}\\&\quad+z_{l_1+l_2+l_3}z_{l_4}+z_{l_1+l_2+l_4}z_{l_3}+z_{l_3}z_{l_1+l_2+l_4}+z_{l_4}z_{l_1+l_2+l_3}\\&\quad+z_{l_1+l_2+l_3+l_4}.\end{aligned}\tag{3-20}$$

Proof of Lemma 3.5. Identity (3-14) follows from Equations (2-1) and (2-2) with $w_1=w_2=1$. □

Proof of Lemma 3.6. We see from (2-1), (2-2), and (3-14) that

$$\begin{aligned}z_{l_1}z_{l_2}*z_{l_3}&=z_{l_1}(z_{l_2}*z_{l_3})+z_{l_3}(z_{l_1}z_{l_2}*1)+z_{l_1+l_3}(z_{l_2}*1)\\&=z_{l_1}(z_{l_2}z_{l_3}+z_{l_3}z_{l_2}+z_{l_2+l_3})+z_{l_3}z_{l_1}z_{l_2}+z_{l_1+l_3}z_{l_2}\\&=z_{l_1}z_{l_2}z_{l_3}+z_{l_1}z_{l_3}z_{l_2}+z_{l_3}z_{l_1}z_{l_2}+z_{l_1+l_3}z_{l_2}+z_{l_1}z_{l_2+l_3},\end{aligned}$$

which proves (3-15). We see from (3-14) and (3-15) that

$$\begin{aligned}z_{l_1}*z_{l_2}*z_{l_3}&=(z_{l_1}z_{l_2}+z_{l_2}z_{l_1}+z_{l_1+l_2})*z_{l_3}\\&=z_{l_1}z_{l_2}*z_{l_3}+z_{l_2}z_{l_1}*z_{l_3}+z_{l_1+l_2}*z_{l_3}\\&=z_{l_1}z_{l_2}z_{l_3}+z_{l_1}z_{l_3}z_{l_2}+z_{l_3}z_{l_1}z_{l_2}+z_{l_1+l_3}z_{l_2}+z_{l_1}z_{l_2+l_3}\\&\quad+z_{l_2}z_{l_1}z_{l_3}+z_{l_2}z_{l_3}z_{l_1}+z_{l_3}z_{l_2}z_{l_1}+z_{l_2+l_3}z_{l_1}+_{l_2}z_{l_1+l_3}\\&\quad+z_{l_1+l_2}z_{l_3}+z_{l_3}z_{l_1+l_2}+z_{l_1+l_2+l_3},\end{aligned}$$

which proves (3-16), and completes the proof. □

Proof of Lemma 3.7. We see from (2-1), (2-2), and (3-15) that

$$\begin{aligned}z_{l_1}z_{l_2}z_{l_3}*z_{l_4}&=z_{l_1}(z_{l_2}z_{l_3}*z_{l_4})+z_{l_4}(z_{l_1}z_{l_2}z_{l_3}*1)+z_{l_1+l_4}(z_{l_2}z_{l_3}*1)\\&=z_1(z_{l_2}z_{l_3}z_{l_4}+z_{l_2}z_{l_4}z_{l_3}+z_{l_4}z_{l_2}z_{l_3}+z_{l_2+l_4}z_{l_3}+z_{l_2}z_{l_3+l_4})\\&\quad+z_{l_4}z_{l_1}z_{l_2}z_{l_3}+z_{l_1+l_4}z_{l_2}z_{l_3}\\&=z_{l_1}z_{l_2}z_{l_3}z_{l_4}+z_{l_1}z_{l_2}z_{l_4}z_{l_3}+z_{l_1}z_{l_4}z_{l_2}z_{l_3}+z_{l_4}z_{l_1}z_{l_2}z_{l_3}\\&\quad+z_{l_1+l_4}z_{l_2}z_{l_3}+z_{l_1}z_{l_2+l_4}z_{l_3}+z_{l_1}z_{l_2}z_{l_3+l_4},\end{aligned}$$

which proves (3-17). We see from (2-2), (3-14), and (3-15) that

$$\begin{aligned}z_{l_1}z_{l_2}*z_{l_3}z_{l_4}&=z_{l_1}(z_{l_2}*z_{l_3}z_{l_4})+z_{l_3}(z_{l_1}z_{l_2}*z_{l_4})+z_{l_1+l_3}(z_{l_2}*z_{l_4})\\&=z_{l_1}(z_{l_3}z_{l_4}z_{l_2}+z_{l_3}z_{l_2}z_{l_4}+z_{l_2}z_{l_3}z_{l_4}+z_{l_3+l_2}z_{l_4}+z_{l_3}z_{l_4+l_2})\\&\quad+z_{l_3}(z_{l_1}z_{l_2}z_{l_4}+z_{l_1}z_{l_4}z_{l_2}+z_{l_4}z_{l_1}z_{l_2}+z_{l_1+l_4}z_{l_2}+z_{l_1}z_{l_2+l_4})\\&\quad+z_{l_1+l_3}(z_{l_2}z_{l_4}+z_{l_4}z_{l_2}+z_{l_2+l_4})\\&=z_{l_1}z_{l_2}z_{l_3}z_{l_4}+z_{l_1}z_{l_3}z_{l_2}z_{l_4}+z_{l_1}z_{l_3}z_{l_4}z_{l_2}+z_{l_3}z_{l_1}z_{l_2}z_{l_4}\\&\quad+z_{l_3}z_{l_1}z_{l_4}z_{l_2}+z_{l_3}z_{l_4}z_{l_1}z_{l_2}+z_{l_1+l_3}z_{l_2}z_{l_4}+z_{l_1+l_3}z_{l_4}z_{l_2}\\&\quad+z_{l_1}z_{l_2+l_3}z_{l_4}+z_{l_3}z_{l_1+l_4}z_{l_2}+z_{l_1}z_{l_3}z_{l_2+l_4}+z_{l_3}z_{l_1}z_{l_2+l_4}+z_{l_1+l_3}z_{l_2+l_4},\end{aligned}$$

which proves (3-18). We see from (3-14) and (3-15) that

$$\begin{aligned}z_{l_1}z_{l_2}*z_{l_3}*z_{l_4}&=z_{l_1}z_{l_2}*(z_{l_3}z_{l_4}+z_{l_4}z_{l_3}+z_{l_3+l_4})\\&=z_{l_1}z_{l_2}*z_{l_3}z_{l_4}+z_{l_1}z_{l_2}*z_{l_4}z_{l_3}+z_{l_1}z_{l_2}*z_{l_3+l_4}\\&=z_{l_1}z_{l_2}*z_{l_3}z_{l_4}+z_{l_1}z_{l_2}*z_{l_4}z_{l_3}\\&\quad+z_{l_1}z_{l_2}z_{l_3+l_4}+z_{l_1}z_{l_3+l_4}z_{l_2}+z_{l_3+l_4}z_{l_1}z_{l_2}+z_{l_1+l_3+l_4}z_{l_2}+z_{l_1}z_{l_2+l_3+l_4},\end{aligned}$$

which proves (3-19). We see from (3-14) and (3-16) that

$$\begin{aligned} z_{l_1} * z_{l_2} * z_{l_3} * z_{l_4} &= (z_{l_1} z_{l_2} + z_{l_2} z_{l_1} + z_{l_1+l_2}) * z_{l_3} * z_{l_4} \\ &= z_{l_1} z_{l_2} * z_{l_3} * z_{l_4} + z_{l_2} z_{l_1} * z_{l_3} * z_{l_4} + z_{l_1+l_2} * z_{l_3} * z_{l_4} \\ &= z_{l_1} z_{l_2} * z_{l_3} * z_{l_4} + z_{l_2} z_{l_1} * z_{l_3} * z_{l_4} \\ &\quad + z_{l_1+l_2} z_{l_3} z_{l_4} + z_{l_1+l_2} z_{l_4} z_{l_3} + z_{l_3} z_{l_1+l_2} z_{l_4} + z_{l_4} z_{l_1+l_2} z_{l_3} \\ &\quad + z_{l_3} z_{l_4} z_{l_1+l_2} + z_{l_4} z_{l_3} z_{l_1+l_2} + z_{l_1+l_2+l_3} z_{l_4} + z_{l_1+l_2+l_4} z_{l_3} + z_{l_3+l_4} z_{l_1+l_2} \\ &\quad + z_{l_1+l_2} z_{l_3+l_4} + z_{l_3} z_{l_1+l_2+l_4} + z_{l_4} z_{l_1+l_2+l_3} + z_{l_1+l_2+l_3+l_4}, \end{aligned}$$

which proves (3-20), and completes the proof. □

We are now in a position to prove Propositions 3.2 and 3.3.

Proof of Proposition 3.2. Let $\mathbf{l}_2 = (l_1, l_2)$ be an index set in $\mathbb{N}^2$. Applying the map Z^* to both sides of (3-14) and substituting $T = 0$, we obtain

$$\begin{aligned} \zeta_1^*(l_1)\zeta_1^*(l_2) &= \zeta_2^*(l_1, l_2) + \zeta_2^*(l_2, l_1) + \zeta_1^*(l_1 + l_2) \\ &= \sum_{\sigma \in \mathfrak{C}_2} \zeta_2^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}) + \zeta_1^*(l_1 + l_2). \end{aligned}$$

We thus have

$$\zeta_{(1,1)}^*(\mathbf{l}_2) = (\zeta_2^* \mid \Sigma_{\mathfrak{C}_2})(\mathbf{l}_2) + \zeta_1^* \circ w_2(\mathbf{l}_2),$$

which proves (3-7) because $\mathbf{l}_2$ is arbitrary and $\zeta_1^* \circ w_2(\mathbf{l}_2) = \zeta_1 \circ w_2(\mathbf{l}_2)$ by virtue of $w_2(\mathbf{l}_2) = l_1 + l_2 \geq 2$. □

Proof of Proposition 3.3. Let $\mathbf{l}_3 = (l_1, l_2, l_3)$ be an index set in $\mathbb{N}^3$. Applying the map Z^* to both sides of (3-15) and substituting $T = 0$, we obtain

$$\begin{aligned} &\zeta_2^*(l_1, l_2)\zeta_1^*(l_3) \\ &\quad = \zeta_3^*(l_1, l_2, l_3) + \zeta_3^*(l_1, l_3, l_2) + \zeta_3^*(l_3, l_1, l_2) + \zeta_2^*(l_1 + l_3, l_2) + \zeta_2^*(l_1, l_2 + l_3) \\ &\quad = \sum_{\sigma \in U_3} \zeta_3^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}, l_{\sigma^{-1}(3)}) + \zeta_2^*(l_{\tau^{-1}(1)} + l_{\tau^{-1}(2)}, l_{\tau^{-1}(3)}) + \zeta_2^*(l_1, l_2 + l_3), \end{aligned}$$

where $\tau = (123)$. We thus have

$$\zeta_{(2,1)}^*(\mathbf{l}_3) = (\zeta_3^* \mid \Sigma_{U_3})(\mathbf{l}_3) + (\zeta_2^* \circ w_{(2,1)} \mid (123))(\mathbf{l}_3) + \zeta_2^* \circ w_{(1,2)}(\mathbf{l}_3),$$

which proves (3-8). In a similar way, we obtain from (3-16) that

$$\zeta_{(1,1,1)}^*(\mathbf{l}_3) = (\zeta_3^* \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) + (\zeta_2^* \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3) + \zeta_1^* \circ w_3(\mathbf{l}_3),$$

which proves (3-9). □

We require another lemma for the proof of Proposition 3.4, since the proof is more complicated than those of Propositions 3.2 and 3.3.

Lemma 3.8. *Let* $\mathrm{id} = \mathrm{id}_4$ *mean the identity map on* $\mathbb{N}^4$. *We have the following equations in maps with the domain* $\mathbb{N}^4$:

(i)

$$w_{(2,2)} \mid (23)\Sigma_{\langle(34)\rangle} = w_{(2,2)} \mid \Sigma_{W_4^0}, \tag{3-21}$$

$$w_{(i,j,k)} \mid \Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} = \begin{cases} w_{(i,j,k)} \mid (\Sigma_{W_4^1} - (34) - (1324)) & ((i,j,k) \in I), \\ w_{(i,j,k)} \mid (\Sigma_{W_4^1} - (24) - (1234)) & ((i,j,k) = (1,2,1)), \end{cases} \tag{3-22}$$

$$\mathrm{id} \mid \Sigma_{V_4}\Sigma_{\langle(34)\rangle} = \mathrm{id} \mid \Sigma_{W_4}, \tag{3-23}$$

where I *in* (3-22) *means the set* $\{(2,1,1),(1,1,2)\}$.

(ii)

$$w_{(3,1)} \mid (24)\Sigma_{\langle(12)\rangle} = w_{(3,1)} \mid (\Sigma_{\mathfrak{C}_4} - e - (1234)), \tag{3-24}$$

$$w_{(1,3)} \mid \Sigma_{\langle(12)\rangle} = w_{(1,3)} \mid (\Sigma_{\mathfrak{C}_4} - (13)(24) - (1234)), \tag{3-25}$$

$$w_{(2,2)} \mid \Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} = w_{(2,2)} \mid (\Sigma_{X_4} - e - (13)(24)), \tag{3-26}$$

$$w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} = w_{(2,1,1)} \mid (\Sigma_{\mathfrak{A}_4} - e - (12)(34)), \tag{3-27}$$

$$w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}}\Sigma_{\langle(12)\rangle} = w_{(1,2,1)} \mid (\Sigma_{\mathfrak{A}_4} - (123) - (134)), \tag{3-28}$$

$$w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} = w_{(1,1,2)} \mid (\Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23)), \tag{3-29}$$

$$\mathrm{id} \mid \Sigma_{W_4}\Sigma_{\langle(12)\rangle} = \mathrm{id} \mid \Sigma_{\mathfrak{S}_4}. \tag{3-30}$$

We now prove Proposition 3.4. We will then discuss a proof of Lemma 3.8.

Proof of Proposition 3.4. Let $\mathbf{l}_4 = (l_1, l_2, l_3, l_4)$ be an index set in $\mathbb{N}^4$. Applying the map Z^* to both sides of (3-17) and substituting $T = 0$, we obtain

$$\begin{aligned} &\zeta_3^*(l_1,l_2,l_3)\zeta_1^*(l_4) \\ &\quad= \zeta_4^*(l_1,l_2,l_3,l_4) + \zeta_4^*(l_1,l_2,l_4,l_3) + \zeta_4^*(l_1,l_4,l_2,l_3) + \zeta_4^*(l_4,l_1,l_2,l_3) \\ &\qquad\quad + \zeta_3^*(l_1+l_4,l_2,l_3) + \zeta_3^*(l_1,l_2+l_4,l_3) + \zeta_3^*(l_1,l_2,l_3+l_4) \\ &\quad= \sum_{\sigma\in U_4} \zeta_4^*(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}, l_{\sigma^{-1}(3)}, l_{\sigma^{-1}(4)}) \\ &\qquad\quad + \zeta_3^*(l_{\tau^{-1}(1)} + l_{\tau^{-1}(2)}, l_{\tau^{-1}(3)}, l_{\tau^{-1}(4)}) \\ &\qquad\quad + \zeta_3^*(l_{\tau^{-1}(1)}, l_{\tau^{-1}(2)} + l_{\tau^{-1}(3)}, l_{\tau^{-1}(4)}) + \zeta_3^*(l_1,l_2,l_3+l_4), \end{aligned}$$

where $\tau = (234)$. We thus have

$$\zeta^*_{(3,1)}(\mathbf{l}_4) = (\zeta_4^* \mid \Sigma_{U_4})(\mathbf{l}_4) + \zeta_3^* \circ ((w_{(2,1,1)} + w_{(1,2,1)}) \mid (234) + w_{(1,1,2)})(\mathbf{l}_4),$$

which proves (3-10). Similarly, we have by (3-18),

$$\zeta^*_{(2,2)}(\mathbf{l}_4) = (\zeta^*_4 \mid \Sigma_{V_4})(\mathbf{l}_4) \\ +(\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0})(\mathbf{l}_4) + (\zeta^*_2 \circ w_{(2,2)} \mid (23))(\mathbf{l}_4),$$

which proves (3-11).

As we calculated above by using (3-17) and (3-18), we can deduce from (3-19) and (3-20) that

$$\zeta^*_{(2,1,1)} = \zeta^*_{(2,2)} \mid \Sigma_{\langle(34)\rangle} + \zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) \\ + w_{(1,1,2)} \mid (34)) + \zeta^*_2 \circ (w_{(3,1)} \mid (24) + w_{(1,3)}) \tag{3-31}$$

and

$$\zeta^*_{(1,1,1,1)} = \zeta^*_{(2,1,1)} \mid \Sigma_{\langle(12)\rangle} \\ +\zeta^*_3 \circ \big(w_{(2,1,1)} \mid (e + (12)(34)) \\ +w_{(1,2,1)} \mid ((123)+(134)) + w_{(1,1,2)} \mid ((13)(24)+(14)(23))\big) \\ +\zeta^*_2 \circ \big(w_{(2,2)} \mid (e + (13)(24)) + w_{(3,1)} \mid (e + (1234)) \\ +w_{(1,3)} \mid ((13)(24) + (1234))\big) \\ +\zeta_1 \circ w_4, \tag{3-32}$$

respectively. Combining (3-11) and the equations of Lemma 3.8(i), we obtain

$$\zeta^*_{(2,2)} \mid \Sigma_{\langle(34)\rangle} \\ = \zeta^*_4 \mid \Sigma_{V_4} \Sigma_{\langle(34)\rangle} \\ +\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{V_4^0} \Sigma_{\langle(34)\rangle} \\ +\zeta^*_2 \circ w_{(2,2)} \mid (23) \Sigma_{\langle(34)\rangle} \\ = \zeta^*_4 \mid \Sigma_{W_4} \\ +\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\ +\zeta^*_2 \circ w_{(2,2)} \mid \Sigma_{W_4^0}, \tag{3-33}$$

where $W^1_{4,\{\sigma,\tau\}}$ denotes $W^1_4 \setminus \{\sigma, \tau\}$. A straightforward calculation shows that

$$\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\ + \zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) + w_{(1,1,2)} \mid (34)) \\ = \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_4} \\ - \zeta^*_3 \circ (w_{(2,1,1)} \mid (34) + w_{(1,2,1)} \mid (1234) + w_{(1,1,2)} \mid (1324)) \\ = \zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\big)$$

and so, substituting (3-33) into the right-hand side of (3-31) gives

$$\begin{aligned}
\zeta^*_{(2,1,1)} &= \zeta^*_4 \mid \Sigma_{W_4} \\
&\quad +\zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,1,2)}) \mid \Sigma_{W^1_{4,\{(34),(1324)\}}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,\{(24),(1234)\}}}\big) \\
&\quad +\zeta^*_2 \circ w_{(2,2)} \mid \Sigma_{W^0_4} \\
&\quad +\zeta^*_3 \circ (w_{(2,1,1)} \mid (1324) + w_{(1,2,1)} \mid (24) + w_{(1,1,2)} \mid (34)) \\
&\quad +\zeta^*_2 \circ (w_{(3,1)} \mid (24) + w_{(1,3)}) \\
&= \zeta^*_4 \mid \Sigma_{W_4} \\
&\quad +\zeta^*_3 \circ \big(w_{(2,1,1)} \mid \Sigma_{W^1_{4,(34)}} + w_{(1,2,1)} \mid \Sigma_{W^1_{4,(1234)}} + w_{(1,1,2)} \mid \Sigma_{W^1_{4,(1324)}}\big) \\
&\quad +\zeta^*_2 \circ (w_{(2,2)} \mid \Sigma_{W^0_4} + w_{(3,1)} \mid (24) + w_{(1,3)}),
\end{aligned}$$

which proves (3-12). Similarly, combining (3-12) and the equations of Lemma 3.8(ii), we obtain

$$\begin{aligned}
&\zeta^*_{(2,1,1)} \mid \Sigma_{\langle(12)\rangle} \\
&\quad = \zeta^*_4 \mid \Sigma_{\mathfrak{S}_4} \\
&\qquad +\zeta^*_3 \circ \big(w_{(2,1,1)} \mid (\Sigma_{\mathfrak{A}_4} - e - (12)(34)) + w_{(1,2,1)} \mid (\Sigma_{\mathfrak{A}_4} - (123) - (134)) \\
&\qquad\qquad + w_{(1,1,2)} \mid (\Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23))\big) \\
&\qquad +\zeta^*_2 \circ \big(w_{(2,2)} \mid (\Sigma_{X_4} - e - (13)(24)) + w_{(3,1)} \mid (\Sigma_{\mathfrak{C}_4} - e - (1234)) \\
&\qquad\qquad + w_{(1,3)} \mid (\Sigma_{\mathfrak{C}_4} - (13)(24) - (1234))\big).
\end{aligned}$$

Substituting this into the right-hand side of (3-32) proves (3-13). □

We will show Lemma 3.8 for the completeness of the proof of Proposition 3.4.

For a subgroup H in $\mathfrak{S}_4$, we define an equivalence relation $\equiv$ on $\mathfrak{S}_4$ such that $\sigma \equiv \tau \mod H$ if and only if $\sigma\tau^{-1} \in H$, and we denote by $[\sigma]_H$ the equivalence class of σ. Note that $[\sigma]_H$ is the right coset $H\sigma$ of $\mathfrak{S}_4$. Table 1 below gives all the equivalence classes in $\mathfrak{S}_4$ modulo certain subgroups, where we denote by $\langle\sigma_1, \dots, \sigma_i\rangle$ the subgroup generated by permutations $\sigma_1, \dots, \sigma_i$. (We have already used $\langle\sigma\rangle$ to denote a cyclic subgroup.) We extend the congruence relation $\equiv$ on $\mathfrak{S}_4$ to that on its group ring $\mathbb{Z}[\mathfrak{S}_4]$, as follows. Let $\sum_{i=1}^m a_i\sigma_i$ and $\sum_{j=1}^n b_j\tau_j$ be elements in $\mathbb{Z}[\mathfrak{S}_4]$. Without loss of generality, we may assume that $\sigma_a \neq \sigma_b$ and $\tau_a \neq \tau_b$ if $a \neq b$. We then say that

$$\sum_{i=1}^m a_i\sigma_i \equiv \sum_{j=1}^n b_j\tau_j \mod H$$

if and only if $m = n$ and there is a permutation $\rho \in \mathfrak{S}_m$ such that $a_i = b_{\rho(i)}$ and $\sigma_i \equiv \tau_{\rho(i)} \mod H$ $(i = 1, \dots, m)$. The equivalence classes in Table 1 will be necessary when we prove some congruence equations in $\mathbb{Z}[\mathfrak{S}_4]$.

The following congruence equations in $\mathbb{Z}[\mathfrak{S}_4]$ are useful for proving Lemma 3.8.

Lemma 3.9. *The following congruence equations hold*:

(i)

$$(23)\Sigma_{\langle(34)\rangle} \equiv \Sigma_{W_4^0} \mod \langle(12), (34)\rangle, \tag{3-34}$$

$$\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} \equiv \begin{cases} \Sigma_{W_4^1} - (34) - (1324) \mod \langle(12)\rangle \text{ or } \mod \langle(34)\rangle, \\ \Sigma_{W_4^1} - (24) - (1234) \mod \langle(23)\rangle, \end{cases} \tag{3-35}$$

$$\Sigma_{V_4}\Sigma_{\langle(34)\rangle} \equiv \Sigma_{W_4} \mod \langle e\rangle. \tag{3-36}$$

(ii)

$$(24)\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{C}_4} - e - (1234) \mod \langle(12), (123)\rangle, \tag{3-37}$$

$$\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{C}_4} - (13)(24) - (1234) \mod \langle(23), (234)\rangle, \tag{3-38}$$

$$\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{X_4} - e - (13)(24) \mod \langle(12), (34)\rangle, \tag{3-39}$$

$$\Sigma_{W^1_{4,(34)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - e - (12)(34) \mod \langle(12)\rangle, \tag{3-40}$$

$$\Sigma_{W^1_{4,(1234)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - (123) - (134) \mod \langle(23)\rangle, \tag{3-41}$$

$$\Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{A}_4} - (13)(24) - (14)(23) \mod \langle(34)\rangle, \tag{3-42}$$

$$\Sigma_{W_4}\Sigma_{\langle(12)\rangle} \equiv \Sigma_{\mathfrak{S}_4} \mod \langle e\rangle. \tag{3-43}$$

Proof. Before proving the congruence equations, we introduce an identity in $\mathbb{Z}[\mathfrak{S}_n]$, which immediately follows from the definition:

$$\Sigma_H\Sigma_K = \Sigma_{H_1}\Sigma_K + \cdots + \Sigma_{H_n}\Sigma_K, \tag{3-44}$$

where H and K are subsets in $\mathfrak{S}_n$ such that $H_1, \ldots, H_n$ are a partition of H (i.e., a set of subsets of H satisfying $\bigcup_{i=1}^n H_i = H$ and $H_i \cap H_j = \phi$ for $i \neq j$).

We first prove the congruence equations stated in (i). We obtain from $\Sigma_{\langle(34)\rangle} = e + (34)$ that

$$(23)\Sigma_{\langle(34)\rangle} = (23) + (234). \tag{3-45}$$

Since $\{(24), (124), (234), (1234)\}$ is an equivalence class modulo $\langle(12), (34)\rangle$ as we see in Table 1,

$$(234) \equiv (24) \mod \langle(12), (34)\rangle.$$

Thus, noting the definition of W_4^0 in (3-5), we have

$$(23)\Sigma_{\langle(34)\rangle} \equiv (23) + (24) = \Sigma_{W_4^0} \mod \langle(12), (34)\rangle,$$

mod	All equivalence classes
$\langle(12), (123)\rangle$	$\{e, (12), (13), (23), (123), (132)\}$, $\{(14), (14)(23), (142), (143), (1423), (1432)\}$, $\{(24), (13)(24), (124), (243), (1243), (1324)\}$, $\{(34), (12)(34), (134), (234), (1234), (1342)\}$.
$\langle(23), (234)\rangle$	$\{e, (23), (24), (34), (234), (243)\}$, $\{(12), (12)(34), (132), (142), (1342), (1432)\}$, $\{(13), (13)(24), (123), (143), (1243), (1423)\}$, $\{(14), (14)(23), (124), (134), (1234), (1324)\}$.
$\langle(12), (34)\rangle$	$\{e, (12), (34), (12)(34)\}$, $\{(13), (132), (143), (1432)\}$, $\{(14), (134), (142), (1342)\}$, $\{(23), (123), (243), (1243)\}$, $\{(24), (124), (234), (1234)\}$, $\{(13)(24), (14)(23), (1324), (1423)\}$.
$\langle(12)\rangle$	$\{\mathrm{e}, (12)\}$, $\{(13), (132)\}$, $\{(14), (142)\}$, $\{(23), (123)\}$, $\{(24), (124)\}$, $\{(34), (12)(34)\}$, $\{(13)(24), (1324)\}$, $\{(14)(23), (1423)\}$, $\{(134), (1342)\}$, $\{(143), (1432)\}$, $\{(234), (1234)\}$, $\{(243), (1243)\}$.
$\langle(23)\rangle$	$\{\mathrm{e}, (23)\}$, $\{(12), (132)\}$, $\{(13), (123)\}$, $\{(14), (14)(23)\}$, $\{(24), (243)\}$, $\{(34), (234)\}$, $\{(12)(34), (1342)\}$, $\{(13)(24), (1243)\}$, $\{(124), (1324)\}$, $\{(134), (1234)\}$, $\{(142), (1432)\}$, $\{(143), (1423)\}$.
$\langle(34)\rangle$	$\{\mathrm{e}, (34)\}$, $\{(12), (12)(34)\}$, $\{(13), (143)\}$, $\{(14), (134)\}$, $\{(23), (243)\}$, $\{(24), (234)\}$, $\{(13)(24), (1423)\}$, $\{(14)(23), (1324)\}$, $\{(123), (1243)\}$, $\{(124), (1234)\}$, $\{(132), (1432)\}$, $\{(142), (1342)\}$.
$\langle(13)(24)\rangle$	$\{\mathrm{e}, (13)(24)\}$, $\{(12), (1423)\}$, $\{(13), (24)\}$, $\{(14), (1243)\}$, $\{(23), (1342)\}$, $\{(34), (1324)\}$, $\{(12)(34), (14)(23)\}$, $\{(123), (142)\}$, $\{(124), (143)\}$, $\{(132), (234)\}$, $\{(134), (243)\}$, $\{(1234), (1432)\}$.

Table 1. All equivalence classes (or all right cosets $H\sigma$) in $\mathfrak{S}_4$ modulo subgroups H.

which proves (3-34). A calculation shows that

$$(1243)\,\Sigma_{\langle(34)\rangle} = (1243) + (124), \tag{3-46}$$

and so we see from (3-44), (3-45), and (3-46) that

$$\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} = (23)\,\Sigma_{\langle(34)\rangle} + (1243)\,\Sigma_{\langle(34)\rangle} = (23) + (124) + (234) + (1243). \tag{3-47}$$

Using (3-47) and the equivalence classes modulo $\langle(12)\rangle$, $\langle(23)\rangle$, and $\langle(34)\rangle$ in Table 1, we obtain

$$\begin{aligned}\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle} &\equiv \begin{cases}(23)+(24)+(1234)+(1243) & \bmod \langle(12)\rangle \quad \text{or} \quad \bmod \langle(34)\rangle,\\ (23)+(34)+(1243)+(1324) & \bmod \langle(23)\rangle,\end{cases}\\ &= \begin{cases}\Sigma_{W_4^1} - (34) - (1324) & \bmod \langle(12)\rangle \quad \text{or} \quad \bmod \langle(34)\rangle,\\ \Sigma_{W_4^1} - (24) - (1234) & \bmod \langle(23)\rangle,\end{cases}\end{aligned}$$

which proves (3-35). Direct calculations show that

$$\begin{aligned}(13)(24)\,\Sigma_{\langle(34)\rangle} &= (13)(24)+(1324),\\ (123)\,\Sigma_{\langle(34)\rangle} &= (123)+(1234),\\ (243)\,\Sigma_{\langle(34)\rangle} &= (243)+(24),\end{aligned}$$

which together with (3-47) yields

$$\begin{aligned}\Sigma_{V_4}\Sigma_{\langle(34)\rangle} &= \Sigma_{\{e,(13)(24),(123),(243)\}}\Sigma_{\langle(34)\rangle}+\Sigma_{V_4^0}\Sigma_{\langle(34)\rangle}\\ &= e+(34)+(13)(24)+(1324)+(123)+(1234)+(243)+(24)\\ &\qquad +(23)+(124)+(234)+(1243).\end{aligned}$$

We obtain (3-36) because the right-hand side of this equation is Σ_{W_4}, by definition.

We next prove the congruence equations stated in (ii). We easily see that

$$(24)\,\Sigma_{\langle(12)\rangle}=(24)+(142) \quad\text{and}\quad \Sigma_{\langle(12)\rangle}=e+(12). \tag{3-48}$$

Using (3-48) and the equivalence classes modulo $\langle(12),(123)\rangle$ and $\langle(23),(234)\rangle$ in Table 1, we obtain

$$\begin{aligned}(24)\,\Sigma_{\langle(12)\rangle} &\equiv (13)(24)+(1432) = \Sigma_{\mathfrak{C}_4}-e-(1234) \mod \langle(12),(123)\rangle,\\ \Sigma_{\langle(12)\rangle} &\equiv e+(1432) = \Sigma_{\mathfrak{C}_4}-(13)(24)-(1234) \mod \langle(23),(234)\rangle,\end{aligned}$$

which prove (3-37) and (3-38), respectively. A direct calculation shows that

$$(23)\,\Sigma_{\langle(12)\rangle}=(23)+(132), \tag{3-49}$$

and so we see from (3-48) and (3-49) that

$$\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle}=(23)\,\Sigma_{\langle(12)\rangle}+(24)\,\Sigma_{\langle(12)\rangle}=(23)+(24)+(132)+(142). \tag{3-50}$$

Using (3-50) and the equivalence classes modulo $\langle(12),(34)\rangle$ in Table 1, we obtain

$$\begin{aligned}\Sigma_{W_4^0}\Sigma_{\langle(12)\rangle} &\equiv (23)+(1234)+(1432)+(14)\\ &= \Sigma_{X_4}-e-(13)(24) \mod \langle(12),(34)\rangle,\end{aligned}$$

which proves (3-39). Direct calculations show that

$$\begin{aligned}(34)\,\Sigma_{\langle(12)\rangle} &= (34)+(12)(34),\\ (1234)\,\Sigma_{\langle(12)\rangle} &= (1234)+(134),\\ (1243)\,\Sigma_{\langle(12)\rangle} &= (1243)+(143),\\ (1324)\,\Sigma_{\langle(12)\rangle} &= (1324)+(14)(23),\end{aligned} \tag{3-51}$$

and so we see from (3-50) and (3-51) that

$$\begin{aligned}\Sigma_{W_4^1}\Sigma_{\langle(12)\rangle} &= \Sigma_{\{(34),(1234),(1243),(1324)\}}\Sigma_{\langle(12)\rangle} + \Sigma_{W_4^0}\Sigma_{\langle(12)\rangle}\\ &= (34)+(12)(34)+(1234)+(134)+(1243)+(143)+(1324)+(14)(23)\\ &\qquad +(23)+(24)+(132)+(142),\end{aligned}$$

which can be restated as

$$\Sigma_{W_4^1}\Sigma_{\langle(12)\rangle} = (23)+(24)+(34)+(12)(34)+(14)(23)+(132)+(134) \\ +(142)+(143)+(1234)+(1243)+(1324). \tag{3-52}$$

Equation (3-52) together with the first equation of (3-51) gives

$$\begin{aligned}\Sigma_{W_{4,(34)}^1}\Sigma_{\langle(12)\rangle} &= \Sigma_{W_4^1\setminus\{(34)\}}\Sigma_{\langle(12)\rangle}\\ &= \Sigma_{W_4^1}\Sigma_{\langle(12)\rangle} - (34)\Sigma_{\langle(12)\rangle}\\ &= (23)+(24)+(14)(23)\\ &\quad +(132)+(134)+(142)+(143)+(1234)+(1243)+(1324).\end{aligned}$$

Using this equation and the equivalence classes modulo $\langle(12)\rangle$ in Table 1, we obtain

$$\Sigma_{W_{4,(34)}^1}\Sigma_{\langle(12)\rangle} \equiv (123)+(124)+(14)(23)+(132)+(134)+(142)+(143) \\ +(234)+(243)+(13)(24) \mod \langle(12)\rangle. \tag{3-53}$$

Since $(ijk)=(ik)(ij)$ is an even permutation for a tuple (i,j,k) of distinct integers i,j,k, and since $\mathfrak{A}_4$ consists of even permutations in $\mathfrak{S}_4$ and $|\mathfrak{A}_4|=12$, we can express $\Sigma_{\mathfrak{A}_4}$ as

$$\Sigma_{\mathfrak{A}_4} = e+(12)(34)+(13)(24)+(14)(23) \\ +(123)+(124)+(132)+(134)+(142)+(143)+(234)+(243). \tag{3-54}$$

Combining (3-53) and (3-54) proves (3-40). Similarly, (3-52) together with the second equation of (3-51) and the equivalence classes modulo $\langle(23)\rangle$ in Table 1 yields

$$\begin{aligned}\Sigma_{W_{4,(1234)}^1}\Sigma_{\langle(12)\rangle} &= (23)+(24)+(34)+(12)(34)+(14)(23)\\ &\quad +(132)+(142)+(143)+(1243)+(1324)\\ &\equiv e+(243)+(234)+(12)(34)+(14)(23)\\ &\quad +(132)+(142)+(143)+(13)(24)+(124)\\ &= \Sigma_{\mathfrak{A}_4}-(123)-(134) \mod \langle(23)\rangle,\end{aligned} \tag{3-55}$$

and (3-52) together with the fourth equation of (3-51) and the equivalence classes modulo $\langle(34)\rangle$ in Table 1 yields

$$\begin{aligned}\text{(3-56)}\quad \Sigma_{W^1_{4,(1324)}}\Sigma_{\langle(12)\rangle} &= (23)+(24)+(34)+(12)(34)\\ &\quad +(132)+(134)+(142)+(143)+(1234)+(1243)\\ &\equiv (243)+(234)+e+(12)(34)\\ &\quad +(132)+(134)+(142)+(143)+(124)+(123)\\ &= \Sigma_{\mathfrak{A}_4}-(13)(24)-(14)(23) \mod \langle(34)\rangle.\end{aligned}$$

Equations (3-55) and (3-56) prove (3-41) and (3-42), respectively. Direct calculations show that

$$\text{(3-57)}\quad \begin{aligned}(13)(24)\,\Sigma_{\langle(12)\rangle} &= (13)(24)+(1423),\\ (123)\,\Sigma_{\langle(12)\rangle} &= (123)+(13),\\ (124)\,\Sigma_{\langle(12)\rangle} &= (124)+(14),\\ (234)\,\Sigma_{\langle(12)\rangle} &= (234)+(1342),\\ (243)\,\Sigma_{\langle(12)\rangle} &= (243)+(1432),\end{aligned}$$

and so we can see from (3-52) and (3-57) that

$$\begin{aligned}\text{(3-58)}\quad \Sigma_{W_4}\Sigma_{\langle(12)\rangle} &= \Sigma_{\{e,(13)(24),(123),(124),(234),(243)\}}\Sigma_{\langle(12)\rangle}+\Sigma_{W^1_4}\Sigma_{\langle(12)\rangle}\\ &= \Sigma_{\mathfrak{S}_4},\end{aligned}$$

which proves (3-43), and completes the proof. □

The following statement holds: the maps $w_{(3,1)}$, $w_{(1,3)}$, $w_{(2,2)}$, $w_{(2,1,1)}$, $w_{(1,2,1)}$, and $w_{(1,1,2)}$ are invariant under the subgroups

$$\langle(12),(123)\rangle,\ \langle(23),(234)\rangle,\ \langle(12),(34)\rangle,\ \langle(12)\rangle,\ \langle(23)\rangle,\ \langle(34)\rangle,$$

respectively. In fact, this statement immediately follows from (3-1) and the fact that w_n is invariant under $\mathfrak{S}_n$, i.e., $(w_n\,|\,\sigma)(\mathbf{l}_n)=w_n(\mathbf{l}_n)$ for any $\sigma\in\mathfrak{S}_n$. Note that $\langle(12),(123)\rangle$ and $\langle(23),(234)\rangle$ are equivalent to the symmetric groups on $\{1,2,3\}$ and $\{2,3,4\}$, respectively.

We are now able to prove Lemma 3.8.

Proof of Lemma 3.8. We can obtain (3-21) by using (3-34) because of the invariance of $w_{(2,2)}$ under $\langle(12),(34)\rangle$. Similarly, we can obtain the equations from (3-22) through (3-30) by using the congruence equations from (3-35) through (3-43), respectively. □

3.2. *Renormalization relations.* For any real-valued functions $f_1, \dots, f_j$ of n variables, we define the product $f_1 \cdots f_j$ of the functions by using the multiplication in the real number field such that

$$(f_1 \cdots f_j)(x_1, \dots, x_n) := f_1(x_1, \dots, x_n) \times \cdots \times f_j(x_1, \dots, x_n).$$

For real-valued functions $g_{n_1}, \dots, g_{n_j}$ such that each g_{n_i} has n_i variables, we define the function $g_{n_1} \otimes \cdots \otimes g_{n_j}$ of $n = n_1 + \cdots + n_j$ variables by

$$\begin{aligned}&g_{n_1} \otimes g_{n_2} \otimes \cdots \otimes g_{n_j}(x_1, \dots, x_n) :=\\ &\quad g_{n_1}(x_1, \dots, x_{n_1}) \times g_{n_2}(x_{n_1+1}, \dots, x_{n_1+n_2}) \times \cdots \times g_{n_j}(x_{n_1+n_2+\cdots+n_{j-1}+1}, \dots, x_n).\end{aligned}$$

Note that $\zeta^{\dagger}_{(n_1,n_2,\dots,n_j)} = \zeta^{\dagger}_{n_1} \otimes \zeta^{\dagger}_{n_2} \otimes \cdots \otimes \zeta^{\dagger}_{n_j}$. We define a characteristic function $\check{\chi}^{ш}_n$ of the set $\mathbb{N}^n$ by

$$\check{\chi}^{ш}_n(\mathbf{l}_n) := \begin{cases} 1 & \text{if } l_1 = \cdots = l_n = 1, \\ 0 & \text{otherwise.} \end{cases} \tag{3-59}$$

For example, $\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2$ is the two-variable function such that

$$\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2) = \check{\chi}^{ш}_2(l_1, l_2) \times \zeta_1 \circ w_2(l_1, l_2) \;=\; \check{\chi}^{ш}_2(l_1, l_2)\zeta_1(l_1 + l_2),$$

and $(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1$ is the three-variable function such that

$$(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1(\mathbf{l}_3) = \check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2) \times \zeta^{ш}_1(l_3) = \check{\chi}^{ш}_2(l_1, l_2)\zeta_1(l_1 + l_2)\zeta^{ш}_1(l_3).$$

The renormalization relations for depth less than 5 are written in terms of real-valued functions, as follows.

Proposition 3.10. *We have*

$$\zeta^*_1 = \zeta^{ш}_1, \tag{3-60}$$

$$\zeta^*_2 = \zeta^{ш}_2 - \tfrac{1}{2}\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2, \tag{3-61}$$

$$\zeta^*_3 = \zeta^{ш}_3 - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1 + \tfrac{1}{3}\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3, \tag{3-62}$$

$$\zeta^*_4 = \zeta^{ш}_4 - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 + \tfrac{1}{16}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4. \tag{3-63}$$

We require two lemmas to prove Proposition 3.10.

Lemma 3.11. *Let $P(T) = \sum_{j=0}^{n} a_j T^j$ be a polynomial whose degree n is less than 5. Then the constant term of $\rho(P(T)) - P(T)$ is*

$$\rho(P(T))\,|_{T=0} - P(0) = \begin{cases} 0 & (n < 2), \\ a_2\zeta_1(2) & (n = 2), \\ a_2\zeta_1(2) - 2a_3\zeta_1(3) & (n = 3), \\ a_2\zeta_1(2) - 2a_3\zeta_1(3) + \frac{27}{2}a_4\zeta_1(4) & (n = 4). \end{cases} \tag{3-64}$$

Lemma 3.12. *Let n be an integer with $1 \le n \le 4$, and let $\mathbf{l}_n = (l_1, \dots, l_n) \in \mathbb{N}^n$. Then*

$$Z^*_{\mathbf{l}_1}(T) \approx 0, \tag{3-65}$$

$$Z^*_{\mathbf{l}_2}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2(\mathbf{l}_2)T^2, \tag{3-66}$$

$$Z^*_{\mathbf{l}_3}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3)T^2 + \tfrac{1}{6}\check{\chi}^{ш}_3(\mathbf{l}_3)T^3, \tag{3-67}$$

$$Z^*_{\mathbf{l}_4}(T) \approx \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_2(\mathbf{l}_4)T^2 + \tfrac{1}{6}\check{\chi}^{ш}_3 \otimes \zeta^*_1(\mathbf{l}_4)T^3 + \tfrac{1}{24}\check{\chi}^{ш}_4(\mathbf{l}_4)T^4, \tag{3-68}$$

where $\approx$ means the congruence relation on $\mathbb{R}[T]$ modulo $\mathbb{R}T + \mathbb{R}$, i.e., $P(T) \approx Q(T)$ if and only if $\deg(P(T) - Q(T)) < 2$.

We will now prove Proposition 3.10. We will then discuss proofs of Lemmas 3.11 and 3.12.

Proof of Proposition 3.10. We first introduce an identity for proving (3-60), (3-61), (3-62), and (3-63): for any index set $\mathbf{l}_n = (l_1, \dots, l_n)$ in $\mathbb{N}^n$ with $n \ge 2$,

$$\check{\chi}^{ш}_n(\mathbf{l}_n)\zeta_1(n) = (\check{\chi}^{ш}_n \cdot \zeta_1 \circ w_n)(\mathbf{l}_n), \tag{3-69}$$

which can be rewritten in terms of real-valued functions with the domain $\mathbb{N}^n$ as

$$\zeta_1(n)\check{\chi}^{ш}_n = \check{\chi}^{ш}_n \cdot \zeta_1 \circ w_n.$$

Identity (3-69) is obtained by the fact that $\check{\chi}^{ш}_n(\mathbf{l}_n) = 0$ unless $l_1 = \cdots = l_n = 1$, and the fact that $\zeta_1(n) = \zeta_1(l_1 + \cdots + l_n) = \zeta_1(w_n(\mathbf{l}_n)) = \zeta_1 \circ w_n(\mathbf{l}_n)$ if $l_1 = \cdots = l_n = 1$ and $n \ge 2$.

It follows from (3-64) and (3-65) that

$$\rho(Z^*_{\mathbf{l}_1}(T))\,|_{T=0} - Z^*_{\mathbf{l}_1}(0) = 0.$$

Using (2-5) and (2-8) with $T = 0$, we can restate this identity as

$$\zeta^{ш}_1(\mathbf{l}_1) - \zeta^*_1(\mathbf{l}_1) = 0,$$

which proves (3-60). Similarly, we obtain from (3-64) and (3-66) that

$$\zeta^{ш}_2(\mathbf{l}_2) - \zeta^*_2(\mathbf{l}_2) = \tfrac{1}{2}\check{\chi}^{ш}_2(\mathbf{l}_2)\zeta_1(2),$$

which proves (3-61) since $\check{\chi}^{ш}_2(\mathbf{l}_2)\zeta_1(2) = \check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(\mathbf{l}_2)$ by (3-69). We can obtain from (3-64) and (3-67) that

$$\zeta^{ш}_3(\mathbf{l}_3) - \zeta^*_3(\mathbf{l}_3) = \tfrac{1}{2}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3) \cdot \zeta_1(2) - \tfrac{1}{3}\check{\chi}^{ш}_3(\mathbf{l}_3)\zeta_1(3),$$

which proves (3-62) since $\check{\chi}^{ш}_3(\mathbf{l}_3)\zeta_1(3) = \check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3(\mathbf{l}_3)$ and

$$\begin{aligned}\check{\chi}^{ш}_2 \otimes \zeta^*_1(\mathbf{l}_3) \cdot \zeta_1(2) &= (\check{\chi}^{ш}_2(l_1, l_2)\zeta^*_1(l_3))\zeta_1(2) = (\check{\chi}^{ш}_2(l_1, l_2)\zeta_1(2))\zeta^*_1(l_3) \\ &\overset{\substack{(3\text{-}60)\\(3\text{-}69)}}{=} (\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2(l_1, l_2)) \cdot \zeta^{ш}_1(l_3) = (\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_1(\mathbf{l}_3).\end{aligned}$$

We can obtain from (3-64) and (3-68) that

(3-70) $$\zeta_4^{\text{ш}}(\mathbf{l}_4)-\zeta_4^{*}(\mathbf{l}_4) = \tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{3}\check{\chi}_3^{\text{ш}}\otimes\zeta_1^{*}(\mathbf{l}_4)\cdot\zeta_1(3)+\tfrac{9}{16}\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)\zeta_1(4).$$

The first term on the right-hand side of (3-70) can be calculated as

(3-71) $$\tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)=\tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{\text{ш}}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{5}{8}\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)\zeta_1(4).$$

In fact, we see from (3-61) and (3-69) that $\zeta_2^{*}(l_3,l_4)=\zeta_2^{\text{ш}}(l_3,l_4)-\frac{1}{2}\check{\chi}_2^{\text{ш}}(l_3,l_4)\zeta_1(2)$, and so

$$\begin{aligned}\tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{*}(\mathbf{l}_4)\cdot\zeta_1(2)&=\tfrac{1}{2}\check{\chi}_2^{\text{ш}}(l_1,l_2)\zeta_2^{*}(l_3,l_4)\zeta_1(2)\\&=\tfrac{1}{2}\check{\chi}_2^{\text{ш}}(l_1,l_2)\zeta_2^{\text{ш}}(l_3,l_4)\zeta_1(2)-\tfrac{1}{4}\check{\chi}_2^{\text{ш}}(l_1,l_2)\check{\chi}_2^{\text{ш}}(l_3,l_4)\zeta_1(2)^2\\&=\tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{\text{ш}}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{4}\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)\zeta_1(2)^2,\end{aligned}$$

where we note that, by definition, $\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)=\check{\chi}_2^{\text{ш}}(l_1,l_2)\check{\chi}_2^{\text{ш}}(l_3,l_4)$. This equality proves (3-71) because

(3-72) $$\zeta_1(2)^2=\tfrac{5}{2}\zeta_1(4),$$

which follows from Euler's results $\zeta_1(2)=\pi^2/6$ and $\zeta_1(4)=\pi^2/90$. Since

$$\check{\chi}_3^{\text{ш}}\otimes\zeta_1^{*}(\mathbf{l}_4)=\check{\chi}_3^{\text{ш}}\otimes\zeta_1^{\text{ш}}(\mathbf{l}_4)$$

by (3-60), combining (3-70) and (3-71) gives

(3-73) $$\zeta_4^{\text{ш}}(\mathbf{l}_4)-\zeta_4^{*}(\mathbf{l}_4) = \tfrac{1}{2}\check{\chi}_2^{\text{ш}}\otimes\zeta_2^{\text{ш}}(\mathbf{l}_4)\cdot\zeta_1(2)-\tfrac{1}{3}\check{\chi}_3^{\text{ш}}\otimes\zeta_1^{\text{ш}}(\mathbf{l}_4)\cdot\zeta_1(3)-\tfrac{1}{16}\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)\zeta_1(4).$$

By (3-69), the right-hand side of (3-73) can be rewritten as

(3-74) (RHS of (3-73))
$$\begin{aligned}&=\tfrac{1}{2}(\zeta_1(2)\check{\chi}_2^{\text{ш}})\otimes\zeta_2^{\text{ш}}(\mathbf{l}_4)-\tfrac{1}{3}(\zeta_1(3)\check{\chi}_3^{\text{ш}})\otimes\zeta_1^{\text{ш}}(\mathbf{l}_4)-\tfrac{1}{16}\check{\chi}_4^{\text{ш}}(\mathbf{l}_4)\zeta_1(4)\\&=\tfrac{1}{2}(\check{\chi}_2^{\text{ш}}\cdot\zeta_1\circ w_2)\otimes\zeta_2^{\text{ш}}(\mathbf{l}_4)-\tfrac{1}{3}(\check{\chi}_3^{\text{ш}}\cdot\zeta_1\circ w_3)\otimes\zeta_1^{\text{ш}}(\mathbf{l}_4)-\tfrac{1}{16}\check{\chi}_4^{\text{ш}}\cdot\zeta_1\circ w_4(\mathbf{l}_4).\end{aligned}$$

Equating (3-73) and (3-74), we obtain (3-63). □

We will now show Lemmas 3.11 and 3.12 for the completeness of the proof of Proposition 3.10.

Proof of Lemma 3.11. Let O denote the Landau symbol. By definition,

(3-75) $$A(u)=\sum_{k=0}^{\infty}\gamma_k u^k = \exp\biggl(\sum_{m=2}^{\infty}\frac{(-1)^m\zeta_1(m)}{m}u^m\biggr)$$

near $u=0$. Thus,

$$\begin{aligned}A(u)&=1+\left(\frac{\zeta_1(2)}{2}u^2-\frac{\zeta_1(3)}{3}u^3+\frac{\zeta_1(4)}{4}u^4+O(u^5)\right)+\frac{1}{2}\left(\frac{\zeta_1(2)}{2}u^2+O(u^3)\right)^2+\cdots\\&=1+\frac{\zeta_1(2)}{2}u^2-\frac{\zeta_1(3)}{3}u^3+\left(\frac{\zeta_1(4)}{4}+\frac{\zeta_1(2)^2}{8}\right)u^4+O(u^5),\end{aligned}$$

and so

$$\gamma_0=1,\quad \gamma_1=0,\quad \gamma_2=\frac{\zeta_1(2)}{2},\quad \gamma_3=-\frac{\zeta_1(3)}{3},$$
$$\gamma_4=\frac{2\zeta_1(4)+\zeta_1(2)^2}{8}=\frac{9\zeta_1(4)}{16}$$

where we have used (3-72) for the last equality. Therefore, we see from (2-7) that

$$\rho(T^j)=\begin{cases}1 & (j=0),\\ T & (j=1),\\ T^2+\zeta_1(2) & (j=2),\\ T^3+3\zeta_1(2)T-2\zeta_1(3) & (j=3),\\ T^4+6\zeta_1(2)T^2-8\zeta_1(3)T+\frac{27}{2}\zeta_1(4) & (j=4),\end{cases}$$

and so $\rho(1)\,|_{T=0}=1$, $\rho(T)\,|_{T=0}=0$, $\rho(T^2)\,|_{T=0}=\zeta_1(2)$, $\rho(T^3)\,|_{T=0}=-2\zeta_1(3)$, and $\rho(T^4)\,|_{T=0}=27\zeta_1(4)/2$. Since

$$\rho(P(T))\,|_{T=0}-P(0)=\sum_{j=0}^{n}a_j\rho(T^j)\,|_{T=0}-a_0,$$

we obtain (3-64). □

Proof of Lemma 3.12. We first recall a result in [Ihara et al. 2006] that will be required to prove Lemma 3.12. Let $\mathrm{reg}_*^T:\mathfrak{H}_*^1(\simeq\mathfrak{H}_*^0[y])\to\mathfrak{H}_*^0[T]$ be the algebra homomorphism defined in [Ihara et al. 2006, Section 3], which is characterized by the property that it is the identity on $\mathfrak{H}^0$ and sends y to T. Let $\mathrm{reg}_*:\mathfrak{H}_*^1\to\mathfrak{H}_*^0$ be the algebra homomorphism obtained by specializing reg_*^T to $T=0$. It immediately follows that

$$Z(\mathrm{reg}_*(z_{k_1}\cdots z_{k_n}))=Z^*(z_{k_1}\cdots z_{k_n})\,|_{T=0}\;=\;\zeta_n^*(k_1,\ldots,k_n)$$

for positive integers $k_1,\ldots,k_n$, since $Z^*:\mathfrak{H}_*^1\to\mathbb{R}[T]$ is the homomorphism characterized by the property that it extends the evaluation map $Z:\mathfrak{H}^0\to\mathbb{R}$ and sends y to T. Ihara et al. [2006, Corollary 5] showed that

$$w = \sum_{j=0}^{m} \frac{1}{j!} \mathrm{reg}_{*}(y^{m-j} w_0) * y^{*j}, \tag{3-76}$$

for $w \in \mathfrak{H}^1$, $w_0 \in \mathfrak{H}^0$, and $m \geq 0$ with $w = y^m w_0$.

We set $w = z_{l_1} \cdots z_{l_n} \in \mathfrak{H}^1$ for the given index set $\mathbf{l}_n$. The element w_0 can be written as $w_0 = z_{l_{m+1}} \cdots z_{l_n}$, where $l_{m+1} \geq 2$. (We set $w_0 = 1$ if $l_1 = \cdots = l_n = 1$.) Let $\{1\}^k$ denote k repetitions of 1. Applying Z^* to both sides of (3-76) gives

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{m} \frac{T^j}{j!} \zeta^*_{n-j}(\{1\}^{m-j}, l_{m+1}, \dots, l_n), \tag{3-77}$$

where we define $\zeta^*_0(\phi) = 1$ for the case that $j = m = n$. For an integer j with $0 \leq j \leq m$, we see from (3-59) and $l_1 = \cdots = l_m = 1$ that $\check{\chi}^{ш}_j(l_1, \dots, l_j) = 1$, and so

$$\begin{aligned} \zeta^*_{n-j}(\{1\}^{m-j}, l_{m+1}, \dots, l_n) &= \zeta^*_{n-j}(l_{j+1}, \dots, l_m, l_{m+1}, \dots, l_n) \\ &= \check{\chi}^{ш}_j(l_1, \dots, l_j)\zeta^*_{n-j}(l_{j+1}, \dots, l_n) \\ &= \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n), \end{aligned} \tag{3-78}$$

where we define $\check{\chi}^{ш}_0(\phi) = 1$ and $\check{\chi}^{ш}_0 \otimes \zeta^*_n(\mathbf{l}_n) = \zeta^*_n(\mathbf{l}_n)$. Combining (3-77) and (3-78), we obtain

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{m} \frac{T^j}{j!} \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n). \tag{3-79}$$

Since $l_{m+1} \geq 2$, it follows from (3-59) that $\check{\chi}^{ш}_j(l_1, \dots, l_j) = 0$ if $m < j \leq n$. Thus, (3-79) can be rewritten as

$$Z^*_{\mathbf{l}_n}(T) = \sum_{j=0}^{n} \frac{T^j}{j!} \check{\chi}^{ш}_j \otimes \zeta^*_{n-j}(\mathbf{l}_n). \tag{3-80}$$

Identities (3-65), (3-66), (3-67), and (3-68) are obtained from (3-80) for $n = 1, 2, 3$, and 4, respectively. □

4. Proofs

4.1. ***Proof of Theorem 1.1.*** Before proving Theorem 1.1 we introduce the following identity, which can be easily obtained by definitions (1-1) and (3-59): For $n \geq 2$,

$$\chi^{ш}_n + \check{\chi}^{ш}_n = I_n, \tag{4-1}$$

where 1_n is the constant function whose value is 1. Identity (4-1) dose not hold when $n=1$, but we will not need this case.

We now prove (1-2) and (1-3) in Theorem 1.1.

Proof of (1-2). By (3-7), we easily obtain

$$\zeta_2^* \mid \Sigma_{\mathfrak{C}_2} = \zeta_{(1,1)}^* - \zeta_1 \circ w_2, \tag{4-2}$$

which proves (1-2) for $\dagger = *$.

We can deduce the following identities from (3-60) and (3-61):

$$\zeta_{(1,1)}^* = \zeta_{(1,1)}^{ш} \quad \text{and} \quad \zeta_2^* \mid \Sigma_{\mathfrak{C}_2} = \zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} - \check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2, \tag{4-3}$$

where we have used in the second identity the property that $\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2$ is invariant under $\mathfrak{S}_2$, or $\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2 \mid \Sigma_{\mathfrak{C}_2} = 2\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2$. Substituting (4-3) into (4-2) gives

$$\zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} - \check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2 = \zeta_{(1,1)}^{ш} - \zeta_1 \circ w_2.$$

By (4-1) with $n=2$, we can write this identity as

$$\zeta_2^{ш} \mid \Sigma_{\mathfrak{C}_2} = \zeta_{(1,1)}^{ш} - \chi_2^{ш} \cdot \zeta_1^{ш} \circ w_2, \tag{4-4}$$

which proves (1-2) for $\dagger = ш$. ∎

Proof of (1-3). Since $\mathfrak{C}_3 = \{e, (123), (132)\}$ and $U_3 = \{e, (23), (123)\}$, direct calculations give the following equations in $\mathbb{Z}[\mathfrak{S}_3]$:

$$(123)\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{C}_3}, \qquad \Sigma_{U_3}\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{S}_3} + \Sigma_{\mathfrak{C}_3}.$$

We thus see from (3-8) that

$$\zeta_{(2,1)}^* \mid \Sigma_{\mathfrak{C}_3} = \zeta_3^* \mid (\Sigma_{\mathfrak{S}_3} + \Sigma_{\mathfrak{C}_3}) + \zeta_2^* \circ (w_{(2,1)} + w_{(1,2)}) \mid \Sigma_{\mathfrak{C}_3}. \tag{4-5}$$

Subtracting (4-5) from (3-9), we obtain $\zeta_{(1,1,1)}^* - \zeta_{(2,1)}^* \mid \Sigma_{\mathfrak{C}_3} = -\zeta_3^* \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3$. This identity is equivalent to

$$\zeta_3^* \mid \Sigma_{\mathfrak{C}_3} = -\zeta_{(1,1,1)}^* + \zeta_{(2,1)}^* \mid \Sigma_{\mathfrak{C}_3} + \zeta_1 \circ w_3, \tag{4-6}$$

which proves (1-3) for $\dagger = *$.

We can deduce the following identities from (3-60), (3-61), and (3-62):

$$\zeta_{(1,1,1)}^* = \zeta_{(1,1,1)}^{ш}, \tag{4-7}$$

$$\zeta_{(2,1)}^* \mid \Sigma_{\mathfrak{C}_3} = \zeta_{(2,1)}^{ш} \mid \Sigma_{\mathfrak{C}_3} - \tfrac{1}{2}(\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{ш} \mid \Sigma_{\mathfrak{C}_3}, \tag{4-8}$$

$$\zeta_3^* \mid \Sigma_{\mathfrak{C}_3} = \zeta_3^{ш} \mid \Sigma_{\mathfrak{C}_3} - \tfrac{1}{2}(\check{\chi}_2^{ш} \cdot \zeta_1 \circ w_2) \otimes \zeta_1^{ш} \mid \Sigma_{\mathfrak{C}_3} + \check{\chi}_3^{ш} \cdot \zeta_1 \circ w_3, \tag{4-9}$$

where we have used in the third identity the property that $\check{\chi}_3^{ш} \cdot (\zeta_1 \circ w_3)$ is invariant under $\mathfrak{S}_3$. By (4-1) with $n=3$, substituting (4-7), (4-8), and (4-9) into (4-6) yields

$$\zeta_3^{ш} \mid \Sigma_{\mathfrak{C}_3} = -\zeta_{(1,1,1)}^{ш} + \zeta_{(2,1)}^{ш} \mid \Sigma_{\mathfrak{C}_3} + \chi_3^{ш} \cdot \zeta_1 \circ w_3. \tag{4-10}$$

Identity (4-10) proves (1-3) for $\dagger = ш$, and we complete the proof. □

We now prepare two lemmas before proving (1-4), because the proof of (1-4) is more complicated than those of (1-2) and (1-3). The identities of Lemma 4.1 (resp. Lemma 4.2) correspond to (4-5) (resp. (4-7), (4-8), and (4-9)) in the proof of (1-3).

Lemma 4.1. *We have*

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} &= \zeta^*_4 \mid (2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4})\\ &\quad + \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4} - (23)\Sigma_{\mathfrak{C}_4}),\end{aligned} \tag{4-11}$$

$$\begin{aligned}\zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} &= \zeta^*_4 \mid (\Sigma_{\mathfrak{C}_4} + (14)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4})\\ &\quad + \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (23)\Sigma_{\mathfrak{C}_4}\\ &\quad + \zeta^*_2 \circ w_{(2,2)} \mid (23)\Sigma_{\mathfrak{C}^0_4},\end{aligned} \tag{4-12}$$

$$\begin{aligned}\zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} &= \zeta^*_4 \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4})\\ &\quad + \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4})\\ &\quad + \zeta^*_2 \circ (w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\Sigma_{\mathfrak{C}^0_4}) + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).\end{aligned} \tag{4-13}$$

Lemma 4.2. *We have*

$$\zeta^*_{(1,1,1,1)} = \zeta^{ш}_{(1,1,1,1)}, \tag{4-14}$$

$$\zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_{(1,1)} \mid \Sigma_{\mathfrak{C}_4}, \tag{4-15}$$

$$\zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} = \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4} + \tfrac{5}{4}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4, \tag{4-16}$$

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} &= \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_{(1,1)} \mid \Sigma_{\mathfrak{C}_4}\\ &\quad + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-17}$$

$$\begin{aligned}\zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} &= \zeta^{ш}_4 \mid \Sigma_{\mathfrak{C}_4} - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4}\\ &\quad + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4} + \tfrac{1}{4}\check{\chi}^{ш}_4 \cdot \zeta_1 \circ w_4.\end{aligned} \tag{4-18}$$

We now prove (1-4). We will then discuss proofs of Lemmas 4.1 and 4.2.

Proof of identity (1-4). Direct calculations show that

$$\begin{aligned}\Sigma_{\mathfrak{C}_4} &= e + (13)(24) + (1234) + (1432),\\ (12)\Sigma_{\mathfrak{C}_4} &= (12) + (143) + (234) + (1324),\\ (13)\Sigma_{\mathfrak{C}_4} &= (13) + (24) + (12)(34) + (14)(23),\\ (14)\Sigma_{\mathfrak{C}_4} &= (14) + (123) + (243) + (1342),\\ (23)\Sigma_{\mathfrak{C}_4} &= (23) + (134) + (142) + (1243),\\ (34)\Sigma_{\mathfrak{C}_4} &= (34) + (124) + (132) + (1423),\end{aligned} \tag{4-19}$$

from which we see that

$$\Sigma_{\mathfrak{S}_4} = (e + (12) + (13) + (14) + (23) + (34))\,\Sigma_{\mathfrak{C}_4}, \tag{4-20}$$

i.e., $\{\mathfrak{C}_4, (12)\mathfrak{C}_4, (13)\mathfrak{C}_4, (14)\mathfrak{C}_4, (23)\mathfrak{C}_4, (34)\mathfrak{C}_4\}$ gives a left $\mathfrak{C}_4$-coset decomposition of $\mathfrak{S}_4$. By (4-20), the sum of (4-11) and (4-12) yields

$$\begin{aligned}\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} = \zeta^*_4 \mid (\Sigma_{\mathfrak{S}_4} + 2\Sigma_{\mathfrak{C}_4} - (13)\,\Sigma_{\mathfrak{C}_4})&\\ +\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4} - (13)\,\Sigma_{\mathfrak{C}_4})&\\ +\zeta^*_2 \circ w_{(2,2)} \mid (23)\,\Sigma_{\mathfrak{C}^0_4}.&\end{aligned}$$

Subtracting (4-13) from this identity, we obtain

$$\begin{aligned}&\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4}\\ &\quad = -\zeta^*_4 \mid (\Sigma_{\mathfrak{S}_4} - \Sigma_{\mathfrak{C}_4})\\ &\qquad - \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid \Sigma_{\mathfrak{A}_4}\\ &\qquad - \zeta^*_2 \circ (w_{(2,2)} \mid \Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}^0_4}) + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).\end{aligned} \tag{4-21}$$

We see from (3-6) and the equivalence classes modulo $\langle(12), (34)\rangle$ in Table 1 that $\Sigma_{X_4} = (14) + (23) + \Sigma_{\mathfrak{C}_4} \equiv (23) + (134) + \Sigma_{\mathfrak{C}_4} = (23)\,\Sigma_{\mathfrak{C}^0_4} + \Sigma_{\mathfrak{C}_4} \mod \langle(12), (34)\rangle$, and so

$$w_{(2,2)} \mid \Sigma_{X_4} = w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}^0_4}).$$

Thus the sum of (3-13) and (4-21) yields

$$\zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(1,1,1,1)} = \zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} + \zeta_1 \circ w_4,$$

which is equivalent to

$$\zeta^*_4 \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_{(1,1,1,1)} - \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^*_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} + \zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} - \zeta_1 \circ w_4. \tag{4-22}$$

Identity (4-22) proves (1-4) for $\dagger = *$.

Combining (4-14)–(4-17) (or considering (4-14) − (4-15) + (4-16) + (4-17), roughly speaking), we can restate the right-hand side of (4-22) as

$$\begin{aligned}(\text{RHS of (4-22)}) = \zeta^{ш}_{(1,1,1,1)} - \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} + \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}&\\ - \tfrac{1}{2}(\check{\chi}^{ш}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{ш}_2 \mid \Sigma_{\mathfrak{C}_4}&\\ + \tfrac{1}{3}(\check{\chi}^{ш}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{ш}_1 \mid \Sigma_{\mathfrak{C}_4}&\\ + (\tfrac{5}{4}\check{\chi}^{ш}_4 - I_4) \cdot (\zeta_1 \circ w_4).&\end{aligned} \tag{4-23}$$

Equating (4-18), (4-22), and (4-23), we obtain

$$\zeta^{ш}_4 \mid \Sigma_{\mathfrak{C}_4} = \zeta^{ш}_{(1,1,1,1)} - \zeta^{ш}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} + \zeta^{ш}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4} + \zeta^{ш}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} + (\check{\chi}^{ш}_4 - I_4) \cdot (\zeta_1 \circ w_4),$$

which, together with (4-1) of $n=4$, proves (1-4) for $\dagger=\text{ш}$, and we complete the proof. □

We will show Lemmas 4.1 and 4.2 for the completeness of the proof of (1-4). We first prove Lemma 4.2.

Proof of Lemma 4.2. We easily see from (3-60) that $\zeta^{*}_{(\{1\}^n)}=\zeta^{\text{ш}}_{(\{1\}^n)}$, which with $n=4$ proves (4-14). Multiplying both sides of (3-61) by $\zeta^{*}_{(1,1)}=\zeta^{\text{ш}}_{(1,1)}$ from the right, in the sense of the operator $\otimes$, gives

$$\zeta^{*}_{(2,1,1)}=\zeta^{\text{ш}}_{(2,1,1)}-\tfrac{1}{2}(\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)\otimes\zeta^{\text{ш}}_{(1,1)}.$$

Applying $\Sigma_{\mathfrak{C}_4}$ to both sides of this equation, we obtain (4-15). We can similarly obtain (4-17), by using (3-62) and $\zeta^{*}_1=\zeta^{\text{ш}}_1$ instead of (3-61) and $\zeta^{*}_{(1,1)}=\zeta^{\text{ш}}_{(1,1)}$, respectively. We also obtain (4-18) by applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-63), since $\check{\chi}^{\text{ш}}_4\cdot(\zeta_1\circ w_4)$ is invariant under $\mathfrak{S}_4$.

We prove (4-16). We easily see that $f\otimes g\mid(13)(24)=g\otimes f$ for any functions f and g of two variables, and so we obtain from (3-61) and $\zeta^{*}_{(2,2)}=\zeta^{*}_2\otimes\zeta^{*}_2(=\zeta^{*\otimes 2}_2)$ that

$$\begin{aligned}\text{(4-24)}\quad \zeta^{*}_{(2,2)}&=(\zeta^{\text{ш}}_2-\tfrac{1}{2}\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)\otimes(\zeta^{\text{ш}}_2-\tfrac{1}{2}\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)\\&=\zeta^{\text{ш}}_{(2,2)}-\tfrac{1}{2}(\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)\otimes\zeta^{\text{ш}}_2\mid(e+(13)(24))+\tfrac{1}{4}(\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)^{\otimes 2}.\end{aligned}$$

We see from (3-69), (3-72), and $\check{\chi}^{\text{ш}\otimes 2}_2=\check{\chi}^{\text{ш}}_4$ that

$$(\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)^{\otimes 2}=\zeta_1(2)^2\check{\chi}^{\text{ш}\otimes 2}_2=\tfrac{5}{2}\zeta_1(4)\check{\chi}^{\text{ш}}_4=\tfrac{5}{2}\check{\chi}^{\text{ш}}_4\cdot\zeta_1\circ w_4,$$

by which we can restate (4-24) as

$$\text{(4-25)}\quad \zeta^{*}_{(2,2)}=\zeta^{\text{ш}}_{(2,2)}-\tfrac{1}{2}(\check{\chi}^{\text{ш}}_2\cdot\zeta_1\circ w_2)\otimes\zeta^{\text{ш}}_2\mid\Sigma_{\langle(13)(24)\rangle}+\tfrac{5}{8}\check{\chi}^{\text{ш}}_4\cdot\zeta_1\circ w_4.$$

Since $\mathfrak{C}^0_4=\{e,(1234)\}\subset\mathfrak{C}_4=\{e,(1234),(13)(24),(1432)\}$,

$$\text{(4-26)}\qquad \Sigma_{\langle(13)(24)\rangle}\Sigma_{\mathfrak{C}^0_4}=\Sigma_{\mathfrak{C}_4}.$$

Applying $\Sigma_{\mathfrak{C}^0_4}$ to both sides of (4-25), we obtain (4-16), and this completes the proof. □

We now prove Lemma 4.1.

Proof of Lemma 4.1. Let $\sigma\in\{(12),(23),(34)\}$. By the equivalence classes modulo $\langle\sigma\rangle$ in Table 1 and straightforward calculations, (3-54) yields

$$\text{(4-27)}\qquad 2\Sigma_{\mathfrak{A}_4}\equiv\Sigma_{\mathfrak{S}_4}\mod\langle\sigma\rangle,$$

and (4-19) yields

$$\begin{aligned} &\Sigma_{\mathfrak{C}_4} \equiv (12)\Sigma_{\mathfrak{C}_4}, \quad (13)\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (14)\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4} \mod \langle (12)\rangle, \\ &\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4}, \quad (12)\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (13)\Sigma_{\mathfrak{C}_4} \equiv (14)\Sigma_{\mathfrak{C}_4} \mod \langle (23)\rangle, \\ &\Sigma_{\mathfrak{C}_4} \equiv (34)\Sigma_{\mathfrak{C}_4}, \quad (12)\Sigma_{\mathfrak{C}_4} \equiv (13)\Sigma_{\mathfrak{C}_4}, \quad (14)\Sigma_{\mathfrak{C}_4} \equiv (23)\Sigma_{\mathfrak{C}_4} \mod \langle (34)\rangle. \end{aligned} \tag{4-28}$$

Thus, we deduce from (4-20) that

$$\Sigma_{\mathfrak{A}_4} \equiv \alpha\Sigma_{\mathfrak{C}_4} + \beta\Sigma_{\mathfrak{C}_4} + \gamma\Sigma_{\mathfrak{C}_4} \mod \langle \sigma\rangle, \tag{4-29}$$

where (α, β, γ) is a 3-tuple of $\{e, (12), (13), (14), (23), (34)\}$ such that

$$\begin{cases} \alpha \in \{e, (12)\},\ \beta \in \{(13), (34)\},\ \gamma \in \{(14), (23)\} & (\sigma = (12)), \\ \alpha \in \{e, (23)\},\ \beta \in \{(12), (34)\},\ \gamma \in \{(13), (14)\} & (\sigma = (23)), \\ \alpha \in \{e, (34)\},\ \beta \in \{(12), (13)\},\ \gamma \in \{(14), (23)\} & (\sigma = (34)). \end{cases} \tag{4-30}$$

We now prove (4-11). Since either $g\mathfrak{C}_4 = h\mathfrak{C}_4$ or $g\mathfrak{C}_4 \cap h\mathfrak{C}_4 = \phi$ for any $g, h \in \mathfrak{S}_4$, we can see from the first and second equations of (4-19) that

$$(1234)\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4} \quad \text{and} \quad (234)\Sigma_{\mathfrak{C}_4} = (12)\Sigma_{\mathfrak{C}_4},$$

respectively. By (3-3) and (3-44), we obtain

$$\Sigma_{U_4}\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4} + (234)\Sigma_{\mathfrak{C}_4} + (1234)\Sigma_{\mathfrak{C}_4} = 2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4}$$

Thus, applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-10) yields

$$\begin{aligned} \zeta^*_{(3,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_4 \mid (2\Sigma_{\mathfrak{C}_4} + (12)\Sigma_{\mathfrak{C}_4} + (34)\Sigma_{\mathfrak{C}_4}) \\ + \zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,2,1)}) \mid (12)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid \Sigma_{\mathfrak{C}_4}\big). \end{aligned} \tag{4-31}$$

We know from (4-29) and (4-30) that

$$\Sigma_{\mathfrak{A}_4} \equiv (13)\Sigma_{\mathfrak{C}_4} + (23)\Sigma_{\mathfrak{C}_4} + \begin{cases} (12)\Sigma_{\mathfrak{C}_4} & \mod \langle (12)\rangle \quad \text{or} \quad \mod \langle (23)\rangle, \\ \Sigma_{\mathfrak{C}_4} & \mod \langle (34)\rangle. \end{cases}$$

Since $w_{(2,1,1)}$, $w_{(1,2,1)}$, and $w_{(1,1,2)}$ are invariant under $\langle (12)\rangle$, $\langle (23)\rangle$, and $\langle (34)\rangle$, respectively, we have

$$\begin{aligned} w_{(i,j,k)} \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4} - (23)\Sigma_{\mathfrak{C}_4}) \\ \equiv \begin{cases} w_{(i,j,k)} \mid (12)\Sigma_{\mathfrak{C}_4} & ((i, j, k) = (2, 1, 1), (1, 2, 1)), \\ w_{(i,j,k)} \mid \Sigma_{\mathfrak{C}_4} & ((i, j, k) = (1, 1, 2)), \end{cases} \end{aligned}$$

and so

$$\begin{aligned} \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (\Sigma_{\mathfrak{A}_4} - (13)\Sigma_{\mathfrak{C}_4} - (23)\Sigma_{\mathfrak{C}_4}) \\ = \zeta^*_3 \circ \big((w_{(2,1,1)} + w_{(1,2,1)}) \mid (12)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid \Sigma_{\mathfrak{C}_4}\big). \end{aligned} \tag{4-32}$$

Combining (4-31) and (4-32), we obtain (4-11).

We can easily see that

$$\Sigma_{V_4^0} = (23)\,\Sigma_{\langle(13)(24)\rangle} \quad \text{and} \quad \Sigma_{V_4} = (e+(123)+(23))\,\Sigma_{\langle(13)(24)\rangle},$$

which together with (4-26) give

$$\Sigma_{V_4^0}\Sigma_{\mathfrak{C}_4^0} = (23)\,\Sigma_{\mathfrak{C}_4} \quad \text{and} \quad \Sigma_{V_4}\Sigma_{\mathfrak{C}_4^0} = \Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4},$$

respectively, where we note that $(123)\,\Sigma_{\mathfrak{C}_4} = (14)\,\Sigma_{\mathfrak{C}_4}$ by the fourth equation of (4-19). Thus, applying $\Sigma_{\mathfrak{C}_4^0}$ to both sides of (3-11), we obtain (4-12).

Lastly, we prove (4-13). We can obtain the following identity by applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-12):

$$\begin{aligned}
\text{(4-33)}\quad \zeta^*_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4} = \zeta^*_4 \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\,\Sigma_{\mathfrak{C}_4})& \\
+\zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4})& \\
-\zeta^*_3 \circ (w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4})& \\
+\zeta^*_2 \circ (w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4^0}) + (w_{(3,1)} + w_{(1,3)}) \mid \Sigma_{\mathfrak{C}_4}).&
\end{aligned}$$

(We will prove (4-33) in Lemma 4.3 below because the proof is not short.) We can also obtain by (4-28)

$$(23)\,\Sigma_{\mathfrak{C}_4} + (13)\,\Sigma_{\mathfrak{C}_4} \equiv (14)\,\Sigma_{\mathfrak{C}_4} + \begin{cases} (34)\,\Sigma_{\mathfrak{C}_4} & \operatorname{mod} \langle(12)\rangle, \\ \Sigma_{\mathfrak{C}_4} & \operatorname{mod} \langle(23)\rangle, \\ (12)\,\Sigma_{\mathfrak{C}_4} & \operatorname{mod} \langle(34)\rangle. \end{cases}$$

Thus, $(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (13)\,\Sigma_{\mathfrak{C}_4}$ can be expressed as

$$\begin{aligned}
&(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (13)\,\Sigma_{\mathfrak{C}_4} \\
&\qquad = (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (-(23)\,\Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4}) \\
&\qquad\qquad + (w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4}).
\end{aligned}$$

Adding $-(w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4})$ to both sides of this equation, and then multiplying both sides by -1, we obtain

$$\begin{aligned}
\text{(4-34)}\quad \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} - (13)\,\Sigma_{\mathfrak{C}_4})& \\
= \zeta^*_3 \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4})& \\
-\zeta^*_3 \circ (w_{(2,1,1)} \mid (34)\,\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (12)\,\Sigma_{\mathfrak{C}_4}).&
\end{aligned}$$

Combining (4-33) and (4-34) proves (4-13). □

Lemma 4.3. (i) *Let* $\sigma \in \{(12), (23), (34)\}$. *The following congruence equations hold*:

$$\text{(4-35)}\qquad \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} \equiv \Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4^0} \quad \operatorname{mod} \langle(12), (34)\rangle,$$

$$\Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4} \mod \langle\sigma\rangle, \tag{4-36}$$

$$\Sigma_{W_4}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\,\Sigma_{\mathfrak{C}_4} \mod \langle e\rangle. \tag{4-37}$$

(ii) *Identity* (4-33) *holds.*

Proof. We first prove the assertion (i). We see from (3-5) and (3-44) that

$$\Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = (23)\,\Sigma_{\mathfrak{C}_4} + (24)\,\Sigma_{\mathfrak{C}_4}$$

and from the third equation of (4-19), we see that

$$(24)\,\Sigma_{\mathfrak{C}_4} = (13)\,\Sigma_{\mathfrak{C}_4}.$$

We thus obtain

$$\Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = (13)\,\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4}. \tag{4-38}$$

Equation (4-38) proves (4-35), since

$$\begin{aligned}(13)\,\Sigma_{\mathfrak{C}_4} &\equiv (1432) + (1234) + e + (13)(24) \;=\; \Sigma_{\mathfrak{C}_4} \mod \langle(12),(34)\rangle,\\ (23)\,\Sigma_{\mathfrak{C}_4} &\equiv 2((23) + (134)) \;=\; 2(23)\,\Sigma_{\mathfrak{C}_4^0} \mod \langle(12),(34)\rangle,\end{aligned}$$

which can be seen from the equivalence classes modulo $\langle(12),(34)\rangle$ in Table 1. By virtue of (4-20), calculations similar to (4-38) show that

$$\begin{aligned}\Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} &= \Sigma_{\{(34),(1234),(1243),(1324)\}}\Sigma_{\mathfrak{C}_4} + \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4}\\ &= ((34)\,\Sigma_{\mathfrak{C}_4} + \Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4} + (12)\,\Sigma_{\mathfrak{C}_4}) + ((13)\,\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4})\\ &= \Sigma_{\mathfrak{C}_4} + (12)\,\Sigma_{\mathfrak{C}_4} + (13)\,\Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4} + (34)\,\Sigma_{\mathfrak{C}_4}\\ &= \Sigma_{\mathfrak{S}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-39}$$

and

$$\begin{aligned}\Sigma_{W_4}\Sigma_{\mathfrak{C}_4} &= \Sigma_{\{e_4,(13)(24),(123),(124),(234),(243)\}}\Sigma_{\mathfrak{C}_4} + \Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4}\\ &= (\Sigma_{\mathfrak{C}_4} + \Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4} + (34)\,\Sigma_{\mathfrak{C}_4} + (12)\,\Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4})\\ &\qquad + (\Sigma_{\mathfrak{S}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4})\\ &= \Sigma_{\mathfrak{S}_4} + 2\Sigma_{\mathfrak{C}_4} + (12)\,\Sigma_{\mathfrak{C}_4} + (14)\,\Sigma_{\mathfrak{C}_4} + (23)\,\Sigma_{\mathfrak{C}_4} + (34)\,\Sigma_{\mathfrak{C}_4}\\ &= 2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\,\Sigma_{\mathfrak{C}_4}.\end{aligned} \tag{4-40}$$

Then we obtain (4-36) by (4-27) and (4-39), and obtain (4-37) by (4-40).

We now prove the assertion (ii), or (4-33). We can deduce from (4-35), (4-36), and (4-37) that

$$\zeta_2^* \circ w_{(2,2)} \mid \Sigma_{W_4^0}\Sigma_{\mathfrak{C}_4} = \zeta_2^* \circ w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\,\Sigma_{\mathfrak{C}_4^0}), \tag{4-41}$$

$$\zeta_3^* \circ w_{(i,j,k)} \mid \Sigma_{W_4^1}\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(i,j,k)} \mid (2\Sigma_{\mathfrak{A}_4} + (23)\,\Sigma_{\mathfrak{C}_4} - (14)\,\Sigma_{\mathfrak{C}_4}), \tag{4-42}$$

$$\zeta_4^* \mid \Sigma_{W_4}\Sigma_{\mathfrak{C}_4} = \zeta_4^* \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4}), \tag{4-43}$$

respectively, where $(i, j, k) \in \{(2, 1, 1), (1, 2, 1), (1, 1, 2)\}$. Applying $\Sigma_{\mathfrak{C}_4}$ to both sides of (3-12) and substituting (4-41), (4-42), and (4-43) into it, we obtain

$$\begin{aligned} \zeta_{(2,1,1)}^* \mid \Sigma_{\mathfrak{C}_4} = \zeta_4^* \mid (2\Sigma_{\mathfrak{S}_4} + \Sigma_{\mathfrak{C}_4} - (13)\Sigma_{\mathfrak{C}_4}) \\ +\zeta_3^* \circ (w_{(2,1,1)} + w_{(1,2,1)} + w_{(1,1,2)}) \mid (2\Sigma_{\mathfrak{A}_4} + (23)\Sigma_{\mathfrak{C}_4} - (14)\Sigma_{\mathfrak{C}_4}) \\ -\zeta_3^* \circ (w_{(2,1,1)} \mid (34)\Sigma_{\mathfrak{C}_4} + w_{(1,2,1)} \mid (1234)\Sigma_{\mathfrak{C}_4} + w_{(1,1,2)} \mid (1324)\Sigma_{\mathfrak{C}_4}) \\ +\zeta_2^* \circ (w_{(2,2)} \mid (\Sigma_{\mathfrak{C}_4} + 2(23)\Sigma_{\mathfrak{C}_4^0}) + w_{(3,1)} \mid (24)\Sigma_{\mathfrak{C}_4} + w_{(1,3)} \mid \Sigma_{\mathfrak{C}_4}). \end{aligned} \tag{4-44}$$

We see from the third equation of (4-19) and the equivalence classes modulo $\langle(12), (123)\rangle$ in Table 1 that

$$\begin{aligned} (24)\Sigma_{\mathfrak{C}_4} &= (13) + (24) + (12)(34) + (14)(23) \\ &\equiv e + (13)(24) + (1234) + (1432) = \Sigma_{\mathfrak{C}_4} \mod \langle(12), (123)\rangle, \end{aligned}$$

and so

$$\zeta_2^* \circ w_{(3,1)} \mid (24)\Sigma_{\mathfrak{C}_4} = \zeta_2^* \circ w_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}. \tag{4-45}$$

We also have

$$\zeta_3^* \circ w_{(1,2,1)} \mid (1234)\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(1,2,1)} \mid \Sigma_{\mathfrak{C}_4}, \tag{4-46}$$

$$\zeta_3^* \circ w_{(1,1,2)} \mid (1324)\Sigma_{\mathfrak{C}_4} = \zeta_3^* \circ w_{(1,1,2)} \mid (12)\Sigma_{\mathfrak{C}_4}, \tag{4-47}$$

since $(1234)\Sigma_{\mathfrak{C}_4} = \Sigma_{\mathfrak{C}_4}$ and $(1324)\Sigma_{\mathfrak{C}_4} = (12)\Sigma_{\mathfrak{C}_4}$ by the first and second equations of (4-19), respectively. Combining (4-44), (4-45), (4-46), and (4-47), we obtain (4-33). □

4.2. *Proof of Theorem 1.2.* We denote by $\mathcal{P}(A)$ the set of partitions of a set A; if A is the empty set ϕ, we set $\mathcal{P}(\phi) = \{\phi\}$. We denote by $\mathbb{N}_n$ the subset $\{1, 2, \ldots, n\}$ in the set $\mathbb{N}$.

For $\dagger \in \{*, ш\}$, $\mathbf{l}_n = (l_1, \ldots, l_n) \in \mathbb{N}^n$, and $\Pi = \{P_1, \ldots, P_m\} \in \mathcal{P}(\mathbb{N}_n)$, we define a polynomial $Z^\dagger_{\mathbf{l}_n;\Pi}(T)$ with real coefficients by

$$Z^\dagger_{\mathbf{l}_n;\Pi}(T) := \prod_{i=1}^{m} \chi^\dagger(\mathbf{l}_n;\, P_i) Z^\dagger_{l_{P_i}}(T), \tag{4-48}$$

where $l_{P_i} = \sum_{p\in P_i} l_p$. For example,

$$Z^\dagger_{(2,1,1);\{\{1,2,3\}\}}(T) = Z^\dagger_{2+1+1}(T) = \zeta_1(4)$$

and

$$Z^\dagger_{(2,1,1);\{\{1,2\},\{3\}\}}(T) = Z^\dagger_{2+1}(T) Z^\dagger_1(T) = \zeta_1(3)T,$$

where note that $\chi^{\dagger}((2,1,1);\{1,2,3\}) = \chi^{\dagger}((2,1,1);\{1,2\}) = \chi^{\dagger}((2,1,1);\{3\}) = 1$ by the definition (1-7). Since $Z_k^{\dagger}(T) = \zeta_1^{\dagger}(k)$ for $k \geq 2$, we can see from (1-6) and (4-48) that the difference between $Z^{\dagger}_{\mathbf{l}_n;\Pi}(T)$ and $\zeta^{\dagger}(\mathbf{l}_n;\Pi)$ depends on only the difference between $Z_1^{\dagger}(T) = T$ and $\zeta_1^{\dagger}(1) = 0$, and so

$$Z^{\dagger}_{\mathbf{l}_n;\Pi}(T)\Big|_{T=0} = \zeta^{\dagger}(\mathbf{l}_n;\Pi).$$

By the correspondence between $\mathfrak{H}^1$ and the algebra of quasisymmetric functions, which is given by

$$z_{l_1}\cdots z_{l_n} \longleftrightarrow M_{(l_1,\dots,l_n)} := \sum_{i_1<\cdots<i_n} t_{i_1}^{l_1}\cdots t_{i_n}^{l_n} \in \operatorname{proj}\lim_{p} \mathbb{Z}[t_1,\dots,t_p],$$

we can restate [Hoffman 2015, Theorem 2.3] as

$$\sum_{\sigma\in\mathfrak{S}_n} z_{l_{\sigma(1)}}\cdots z_{l_{\sigma(n)}} = \sum_{\Pi=\{P_1,\dots,P_m\}\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) z_{l_{P_1}} * \cdots * z_{l_{P_m}}. \tag{4-49}$$

We see from (1-7) that $\chi^*(\mathbf{l}_n;P) = 1$ for any $\mathbf{l}_n \in \mathbb{N}^n$ and $P \subset \mathbb{N}_n$. Thus, applying Z^* to both sides of (4-49) yields the following identity (4-50). Since $\mathfrak{S}_n^{-1} = \mathfrak{S}_n$, (4-50) with $T = 0$ proves (1-8) for $\dagger = *$.

Theorem 4.4 (see [Hoffman 2015, Theorem 2.3]). *For any index set $\mathbf{l}_n = (l_1,\dots,l_n)$ in $\mathbb{N}^n$,*

$$\sum_{\sigma\in\mathfrak{S}_n} Z^*_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^*_{\mathbf{l}_n;\Pi}(T). \tag{4-50}$$

We may show (4-51) in order to prove (1-8) for $\dagger = ш$, or Theorem 1.2.

Proposition 4.5. *For any index set $\mathbf{l}_n = (l_1,\dots,l_n)$ in $\mathbb{N}^n$,*

$$\rho\Bigg(\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^*_{\mathbf{l}_n;\Pi}(T)\Bigg) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\mathbf{l}_n;\Pi}(T). \tag{4-51}$$

In fact, we can easily prove Theorem 1.2, as follows.

Proof of Theorem 1.2. We see from (2-8) that

$$\rho\Bigg(\sum_{\sigma\in\mathfrak{S}_n} Z^*_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T)\Bigg) = \sum_{\sigma\in\mathfrak{S}_n} Z^{ш}_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T). \tag{4-52}$$

By (4-51) and (4-52), applying ρ to both sides of (4-50) yields

$$\sum_{\sigma\in\mathfrak{S}_n} Z^{ш}_{l_{\sigma(1)},\dots,l_{\sigma(n)}}(T) = \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\mathbf{l}_n;\Pi}(T) \tag{4-53}$$

which with $T = 0$ proves (1-8) for $\dagger = ш$. □

For subsets A and B in $\mathbb{N}_n$, we define a subset $\mathcal{P}_B(A)$ in $\mathcal{P}(A)$ by

$$\mathcal{P}_B(A) := \{\Pi = \{P_1, \ldots, P_m\} \in \mathcal{P}(A) \mid P_i \not\subset B \text{ for all } i\}.$$

For example, if $A = \{1, 2, 3\}$ and $B = \{3, 4\}$, then

$$\mathcal{P}_B(A) = \big\{\{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{1, 2, 3\}\}\big\},$$

where

$$\mathcal{P}(A) = \big\{\{\{1\}, \{2\}, \{3\}\}, \{\{1\}, \{2, 3\}\}, \{\{2\}, \{1, 3\}\}, \{\{3\}, \{1, 2\}\}, \{\{1, 2, 3\}\}\big\}.$$

We note that $\mathcal{P}_B(A) = \phi$ if $A = \phi$ (or $\mathcal{P}(A) = \{\phi\}$), because the empty set ϕ is a subset of any set, i.e., $\phi \subset B$. We denote by $A^{c_n} = A^c$ the complement of A in $\mathbb{N}_n$, and by $\sqcup$ the disjoint union.

We will show (4-51) for the completeness of the proof of Theorem 1.2. For this, we will require Lemmas 4.7, 4.8, and 4.9.

Remark 4.6. The condition $B \neq \mathbb{N}_n$ in Lemma 4.7 is necessary for taking an element in $\mathcal{P}_B(A^c)$. In fact, if $B = \mathbb{N}_n$, then $P \subset B$ for any subset P in A^c, and so $\mathcal{P}_B(A^c) = \phi$. That is, (4-54) in Lemma 4.7 does not hold in the case that $B = \mathbb{N}_n$.

Lemma 4.7. *For any subset $B \subset \mathbb{N}_n$ with $B \neq \mathbb{N}_n$, we have*

$$\bigsqcup_{A \subset B} \{\Xi \sqcup \Delta \mid (\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}_B(A^c)\} = \mathcal{P}(\mathbb{N}_n), \tag{4-54}$$

where the disjoint union $\bigsqcup_{A \subset B}$ ranges over all subsets in B, which include the empty set ϕ.

We require some notation to state Lemma 4.8. Let A be a subset in $\mathbb{N}_n$, and let $\Xi = \{P_1, \ldots, P_g\}$ be a partition in $\mathcal{P}(A)$. We can define a partition in $\mathcal{P}(\mathbb{N}_s)$ that is induced from A and Ξ, as follows. Let $a_1 < \cdots < a_s$ be the increasing sequence of integers such that $A = \{a_1, \ldots, a_s\}$. Let σ_A be the permutation of $\mathfrak{S}_n$ that is uniquely determined by

$$\sigma_A^{-1}(i) = a_i \ (i = 1, \ldots, s) \quad \text{and} \quad \sigma_A^{-1}(s+1) < \cdots < \sigma_A^{-1}(n);$$

by the definition, $\sigma(A) = \{\sigma_A(a_1), \ldots, \sigma_A(a_s)\} = \{1, \ldots, s\} = \mathbb{N}_s$. We then define the partition induced from A and Ξ as

$$\sigma_A(\Xi) := \{\sigma_A(P_1), \ldots, \sigma_A(P_g)\} \in \mathcal{P}(\mathbb{N}_s).$$

We define $\sigma_A(\Xi) = \phi$ if $A = \Xi = \phi$.

Lemma 4.8. *Let A, B, and (Ξ, Δ) be as in Lemma 4.7, i.e., let A and B be subsets with $A \subset B \neq \mathbb{N}_n$, and let (Ξ, Δ) be an element in $\mathcal{P}(A) \times \mathcal{P}_B(A^c)$. We define $|\phi| = 0$ and $\tilde{c}_0(\phi) = Z^{\dagger}_{\phi;\phi}(T) = 1$.*

(i) *We have*

$$\tilde{c}_n(\Xi \cup \Delta) = \tilde{c}_{|A|}(\Xi)\tilde{c}_{|A^c|}(\Delta). \tag{4-55}$$

(ii) *Let* $\dagger \in \{*, ш\}$, *and let* $\mathbf{l}_n = (l_1, \ldots, l_n) \in \mathbb{N}^n$ *with* $\mathbf{l}_n \neq (\{1\}^n)$. *If* $B = \{j \in \mathbb{N}_n \mid l_j = 1\}$, *then*

$$Z^{\dagger}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T) = \left(\prod_{i=1}^{h} \zeta_1(l_{Q_i})\right) Z^{\dagger}_{\{1\}^{|A|};\sigma_A(\Xi)}(T), \tag{4-56}$$

where $Q_1, \ldots, Q_h$ *mean the parts of* Δ (*i.e.*, $\Delta = \{Q_1, \ldots, Q_h\}$).

Lemma 4.9. *For a positive integer* n, *we have*

$$\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\{1\}^n;\Pi}(T) = \rho^{-1}(T^n), \tag{4-57}$$

$$\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T) = T^n. \tag{4-58}$$

We now prove Proposition 4.5. We will then discuss proofs of Lemmas 4.7–4.9.

Proof of Proposition 4.5. Let $B = \{j \in \mathbb{N}_n \mid l_j = 1\} \subset \mathbb{N}_n$. We suppose that $B = \mathbb{N}_n$. Then, $\mathbf{l}_n = (\{1\}^n)$, and so, we can see from Lemma 4.9 that

$$\begin{aligned} \rho\left(\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\{1\}^n;\Pi}(T)\right) &\overset{(4\text{-}57)}{=} \rho(\rho^{-1}(T^n)) \;=\; T^n \\ &\overset{(4\text{-}58)}{=} \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{ш}_{\{1\}^n;\Pi}(T), \end{aligned} \tag{4-59}$$

which proves (4-51) for $B = \mathbb{N}_n$.

We suppose that $B \neq \mathbb{N}_n$. Let A be a subset in B. Then we have

$$\{\sigma_A(\Xi) \mid \Xi \in \mathcal{P}(A)\} = \{\Xi' \mid \Xi' \in \mathcal{P}(\mathbb{N}_{|A|})\}, \tag{4-60}$$

because the restriction of the permutation σ_A to the subset A is a bijection from A to $\mathbb{N}_{|A|}$. From the definition (1-5) we easily see that $\tilde{c}_{|A|}(\Xi) = \tilde{c}_{|A|}(\sigma_A(\Xi))$. Thus,

$$\begin{aligned} &\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{*}_{\mathbf{l}_n;\Pi}(T) \\ &\quad\overset{(\text{Lemma 4.7})}{=} \sum_{A\subset B} \sum_{\substack{\Xi\in\mathcal{P}(A)\\ \Delta\in\mathcal{P}_B(A^c)}} \tilde{c}_n(\Xi\cup\Delta) Z^{*}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T) \\ &\quad\overset{(\text{Lemma 4.8})}{=} \sum_{A\subset B} \sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\left(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\right) \sum_{\Xi\in\mathcal{P}(A)} \tilde{c}_{|A|}(\Xi) Z^{*}_{\{1\}^{|A|};\sigma_A(\Xi)}(T) \end{aligned}$$

$$\stackrel{(4\text{-}60)}{=} \sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)\sum_{\Xi'\in\mathcal{P}(\mathbb{N}_{|A|})}\tilde{c}_{|A|}(\Xi')Z^{*}_{\{1\}^{|A|};\Xi'}(T)$$

$$\stackrel{(4\text{-}57)}{=} \sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)\rho^{-1}(T^{|A|}),$$

where $Q_1, \ldots, Q_h$ mean the parts of Δ. Therefore,

$$\begin{aligned}(4\text{-}61)\quad \rho\biggl(\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^{*}_{\mathbf{l}_n;\Pi}(T)\biggr) &= \sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)(\rho^{-1}(T^{|A|}))\\ &= \sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)T^{|A|}.\end{aligned}$$

By using Lemma 4.7, Lemma 4.8, and (4-60), and by using (4-58) instead of (4-57), we can similarly prove

$$(4\text{-}62)\quad \sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^{\text{III}}_{\mathbf{l}_n;\Pi}(T) = \sum_{A\subset B}\sum_{\Delta\in\mathcal{P}_B(A^c)} \tilde{c}_{|A^c|}(\Delta)\biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)T^{|A|}.$$

Equating (4-61) and (4-62), we obtain (4-51) for $B \neq \mathbb{N}_n$. □

We prove Lemmas 4.7 and 4.8.

Proof of Lemma 4.7. Let A be a subset in B, and let (Ξ, Δ) be an element in $\mathcal{P}(A)\times\mathcal{P}_B(A^c)$. It follows from $\mathcal{P}_B(A^c)\subset\mathcal{P}(A^c)$ that $(\Xi,\Delta)\in\mathcal{P}(A)\times\mathcal{P}(A^c)$, which together with $A\sqcup A^c=\mathbb{N}_n$ yields $\Xi\sqcup\Delta\in\mathcal{P}(\mathbb{N}_n)$. Thus, the left-hand side of (4-54) is included in the right-hand side.

Let $\Pi=\{P_1,\ldots,P_m\}$ be a partition in $\mathcal{P}(\mathbb{N}_n)$. We can reorder $P_1,\ldots,P_m$ such that

$$(4\text{-}63)\qquad P_j\subset B\,(j=1,\ldots,g)\quad\text{and}\quad P_j\not\subset B\,(j=g+1,\ldots,m).$$

We define

$$\Xi := \{P_1,\ldots,P_g\},\quad \Delta := \{P_{g+1},\ldots,P_m\}\quad\text{and}\quad A := P_1\cup\cdots\cup P_g,$$

where A and Ξ mean the empty set ϕ if $g=0$. By definition, it is obvious that $A\subset B$, $\Pi=\Xi\sqcup\Delta$, $\Xi\in\mathcal{P}(A)$, and $\Delta\in\mathcal{P}(A^c)$. We assume that $\Delta\notin\mathcal{P}_B(A^c)$. Then, either $\mathcal{P}_B(A^c)=\phi$, or there is an integer i such that $g<i\le m$ and $P_i\subset B$. We can see from (4-63) that the latter case does not occur, and so $\mathcal{P}_B(A^c)=\phi$. Thus, the simplest partition $\{A^c\}$ of A^c does not belong to $\mathcal{P}_B(A^c)$, which yields

that $A^c \subset B$. Since $A \subset B$, we have $\mathbb{N}_n = A \cup A^c \subset B$, i.e., $B = \mathbb{N}_n$, which is a contradiction to the condition $B \neq \mathbb{N}_n$. Therefore, $\Delta \in \mathcal{P}_B(A^c)$, and we can conclude that

$$A \subset B, \quad \Pi = \Xi \cup \Delta \quad \text{and} \quad (\Xi, \Delta) \in \mathcal{P}(A) \times \mathcal{P}_B(A^c).$$

This fact proves that the right-hand side of (4-54) is included in the left-hand side, since Π is arbitrary.

We should show the disjointness of the left-hand side of (4-54) in order to finish the proof. Assume that there are subsets $A_1, A_2 \subset B$ with $A_1 \neq A_2$ such that

$$\begin{aligned}\phi \neq \{\Xi_1 \sqcup \Delta_1 \mid (\Xi_1, \Delta_1) \in \mathcal{P}(A_1) \times \mathcal{P}_B(A_1^c)\}\\ \cap \{\Xi_2 \sqcup \Delta_2 \mid (\Xi_2, \Delta_2) \in \mathcal{P}(A_2) \times \mathcal{P}_B(A_2^c)\}.\end{aligned} \tag{4-64}$$

We can take elements $(\Xi_j, \Delta_j) \in \mathcal{P}(A_j) \times \mathcal{P}_B(A_j^c)$ $(j = 1, 2)$ such that

$$\Xi_1 \sqcup \Delta_1 = \Xi_2 \sqcup \Delta_2. \tag{4-65}$$

Let $P_1 \in \Xi_1$. We easily see that $P_1 \subset B$, since $\Xi_1 \in \mathcal{P}(A_1)$ and $A_1 \subset B$. By (4-65), there is a subset $P_2 \in \Xi_2 \sqcup \Delta_2$ such that $P_1 = P_2$. If $P_2 \in \Delta_2$, then $P_2 \not\subset B$, which contradicts $P_1 \subset B$. We thus have $P_1 = P_2 \in \Xi_2$, and so $\Xi_1 \subset \Xi_2$ since P_1 is arbitrary. Similarly, we can prove $\Xi_2 \subset \Xi_1$, and we conclude that $\Xi_1 = \Xi_2$. Since Ξ_j is a partition of A_j for each $j = 1, 2$, we can obtain $A_1 = A_2$, which contradicts the assumption $A_1 \neq A_2$. Therefore, there are no subsets $A_1, A_2 \subset B$ with $A_1 \neq A_2$ such as (4-64), which proves that the left-hand side of (4-54) satisfies the disjointness. $\square$

Proof of Lemma 4.8. Let $P_1, \dots, P_g$ be the parts of Ξ, and let $Q_1, \dots, Q_h$ be those of Δ. Since $n = |A| + |A^c|$ and $\Xi \cup \Delta = \{P_1, \dots, P_g, Q_1, \dots, Q_h\}$, we see from (1-5) that

$$\begin{aligned}\tilde{c}_n(\Xi \cup \Delta) &= (-1)^{n-(g+h)} \left(\prod_{i=1}^{g} (|P_i| - 1)! \right) \left(\prod_{i=1}^{h} (|Q_i| - 1)! \right)\\ &= (-1)^{|A|-g} \left(\prod_{i=1}^{g} (|P_i| - 1)! \right) (-1)^{|A^c|-h} \left(\prod_{i=1}^{h} (|Q_i| - 1)! \right)\\ &= \tilde{c}_{|A|}(\Xi) \tilde{c}_{|A^c|}(\Delta),\end{aligned}$$

which proves (4-55). We next prove (4-56). By $\Delta \in \mathcal{P}_B(A^c)$, any part Q in Δ satisfies $Q \not\subset B = \{j \in \mathbb{N}_n \mid l_j = 1\}$, which yields that $\chi^\dagger(\mathbf{l}_n; Q) = 1$ and

$Z^{\dagger}_{l_Q}(T)=\zeta_1(l_Q)$. Thus, we can see from (4-48) that

$$Z^{\dagger}_{\mathbf{l}_n;\,\Xi\cup\Delta}(T)=\biggl(\prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)\biggr)\biggl(\prod_{i=1}^{h}\chi^{\dagger}(\mathbf{l}_n;\,Q_i)Z^{\dagger}_{l_{Q_i}}(T)\biggr) = \biggl(\prod_{i=1}^{h}\zeta_1(l_{Q_i})\biggr)\biggl(\prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)\biggr). \tag{4-66}$$

For any part $P=\{p_1,\dots,p_a\}$ in Ξ, every index l_{p_q} is 1, since $A\subset B$ and $\Xi\in\mathcal{P}(A)$. By this fact, we obtain

$$\chi^{\dagger}(\mathbf{l}_n;\,P)=\chi^{\dagger}_a(l_{p_1},\dots,l_{p_a})=\chi^{\dagger}_a(\{1\}^a)=\chi^{\dagger}(\{1\}^{|A|};\,\sigma_A(P))$$

and

$$Z^{\dagger}_{l_P}(T)=Z^{\dagger}_{\sum_{p\in P}l_p}(T)=Z^{\dagger}_{|P|}(T)=Z^{\dagger}_{|\sigma_A(P)|}(T).$$

Therefore,

$$\prod_{i=1}^{g}\chi^{\dagger}(\mathbf{l}_n;\,P_i)Z^{\dagger}_{l_{P_i}}(T)=\prod_{i=1}^{g}\chi^{\dagger}(\{1\}^{|A|};\,\sigma_A(P_i))Z^{\dagger}_{|\sigma_A(P_i)|}(T) = Z^{\dagger}_{\{1\}^{|A|};\{\sigma_A(P_1),\dots,\sigma_A(P_g)\}}(T). \tag{4-67}$$

Combining (4-66) and (4-67) proves (4-56). $\square$

From Theorems 7.12 and 7.13 in [Stanley 2013], we can obtain the following identity in formal power series:

$$\exp\biggl(u_1u+u_2\frac{u^2}{2}+u_3\frac{u^3}{3}+\cdots\biggr) = 1+\sum_{n=1}^{\infty}u^n\sum_{\substack{i_1,i_2,\dots,i_n\ge 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{u_1^{i_1}u_2^{i_2}\cdots u_n^{i_n}}{1^{i_1}i_1!\,2^{i_2}i_2!\,\cdots n^{i_n}i_n!}, \tag{4-68}$$

where $u,u_1,u_2,\dots$ are variables. (We can also prove (4-68) by a direct calculation of the Taylor expansion of the exponential function e^x.)

We require the following identity (4-69) to prove Lemma 4.9.

Lemma 4.10. *We have*

$$\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^{*}_{\{1\}^n;\Pi}(T)=n!\widetilde{\gamma}_n(T), \tag{4-69}$$

where we define

$$\widetilde{\gamma}_n(T):=(-1)^n\sum_{\substack{i_1,i_2,\dots,i_n\ge 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{(-1)^{i_1+i_2+\cdots+i_n}}{i_1!\,i_2!\,\cdots i_n!}T^{i_1}\prod_{a=2}^{n}\biggl(\frac{\zeta(a)}{a}\biggr)^{i_a}. \tag{4-70}$$

Proof of Lemma 4.10. For a partition $\Pi = \{P_1, \dots, P_g\}$ in $\mathcal{P}(\mathbb{N}_n)$ and a positive integer a, we denote by $N_a(\Pi)$ the number of the parts P_j whose cardinalities equal a, i.e.,

$$N_a(\Pi) := |\{j \in \{1, \dots, g\} \mid |P_j| = a\}|.$$

For example, $N_1(\Pi) = 2$, $N_2(\Pi) = 1$ and $N_3(\Pi) = N_4(\Pi) = 0$ if

$$\Pi = \{\{1\}, \{2\}, \{3, 4\}\} \in \mathcal{P}(\mathbb{N}_4).$$

We note that

$$g = N_1(\Pi) + \cdots + N_n(\Pi) \quad \text{and} \quad n = 1 \cdot N_1(\Pi) + \cdots + n \cdot N_n(\Pi)$$

and that

$$\prod_{i=1}^{g} (|P_i| - 1)! Z^*_{|P_i|}(T) = \prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{N_a(\Pi)}.$$

Noting $\chi^*(\{1\}^n; P_i) = 1$, we obtain from (1-5) and (4-48) that

$$\begin{aligned}\tilde{c}_n(\Pi) Z^*_{\{1\}^n;\Pi}(T) &= (-1)^{n-g} \prod_{i=1}^{g} (|P_i| - 1)! Z^*_{|P_i|}(T) \\ &= (-1)^{n-(N_1(\Pi)+\cdots+N_n(\Pi))} \prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{N_a(\Pi)}.\end{aligned}$$

Thus,

$$\begin{aligned}(4\text{-}71) \qquad & \sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1+\cdots+n\cdot i_n=n)}} \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi)=i_a (\forall a))}} \tilde{c}_n(\Pi) Z^*_{\{1\}^n;\Pi}(T) \\ &= \sum_{\substack{i_1,\dots,i_n \ge 0 \\ (1\cdot i_1+\cdots+n\cdot i_n=n)}} (-1)^{n-(i_1+\cdots+i_n)} \left(\prod_{a=1}^{n} ((a-1)! Z^*_a(T))^{i_a} \right) \sum_{\substack{\Pi \in \mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi)=i_a (\forall a))}} 1.\end{aligned}$$

Let m be an integer with $a i_a < m$. We can choose i_a disjoint subsets

$$Q_1, \dots, Q_{i_a} \subset \mathbb{N}_m,$$

with $|Q_1| = \cdots = |Q_{i_a}| = a$ in

$$\binom{m}{a i_a} \cdot \binom{a i_a}{a} \binom{a i_a - a}{a} \cdots \binom{a}{a} \cdot \frac{1}{i_a!}$$

ways, as follows. First, we choose $a i_a$ integers $N = \{k_1, \dots, k_{a i_a}\}$ from $\mathbb{N}_m$ in $\binom{m}{a i_a}$ ways. Then we select a integers Q_1 from N, select a integers Q_2 from $N \setminus Q_1$, and so on: these combinations are

$$\binom{ai_a}{a}\binom{ai_a-a}{a}\cdots\binom{a}{a}.$$

Finally, we divide it by $i_a!$ to ignore the order of $Q_1,\ldots,Q_{i_a}$, and we reach the desired result. Any partition in

$$\{\Pi\in\mathcal{P}(\mathbb{N}_n)\mid N_a(\Pi)=i_a\ (\text{for all } a)\}$$

can be uniquely obtained by choosing i_1 disjoint subsets

$$Q_1^{(1)},\ldots,Q_{i_1}^{(1)}\quad\text{with }\ |Q_j^{(1)}|=1\ (\text{for all } j)$$

from the set $\mathbb{N}_n$, choosing i_2 disjoint subsets

$$Q_1^{(2)},\ldots,Q_{i_2}^{(2)}\quad\text{with }\ |Q_j^{(2)}|=2\ (\text{for all } j)$$

from the set $\mathbb{N}_n\setminus(Q_1^{(1)}\cup\cdots\cup Q_{i_1}^{(1)})$, and repeating it. Thus,

$$\begin{aligned}
&|\{\Pi\in\mathcal{P}(\mathbb{N}_n)\mid N_a(\Pi)=i_a\ (\text{for all } a)\}|\\
&\quad=\prod_{a=1}^{n}\binom{n-1\cdot i_1-\cdots-(a-1)\cdot i_{a-1}}{ai_a}\cdot\binom{ai_a}{a}\binom{ai_a-a}{a}\cdots\binom{a}{a}\cdot\frac{1}{i_a!}\\
&\quad=\prod_{a=1}^{n}\binom{n-1\cdot i_1-\cdots-(a-1)\cdot i_{a-1}}{ai_a}(ai_a)!\frac{1}{(a!)^{i_a}i_a!}\\
&\quad=\prod_{a=1}^{n}\frac{(n-1\cdot i_1-\cdots-(a-1)\cdot i_{a-1})!}{(n-1\cdot i_1-\cdots-a\cdot i_a)!}\frac{1}{(a!)^{i_a}i_a!}\\
&\quad=n!\prod_{a=1}^{n}\frac{1}{(a!)^{i_a}i_a!},
\end{aligned}$$

which is equivalent to

$$\sum_{\substack{\Pi\in\mathcal{P}(\mathbb{N}_n)\\(N_a(\Pi)=i_a(\forall a))}}1=n!\prod_{a=1}^{n}\frac{1}{(a!)^{i_a}i_a!}.$$

Therefore, (4-71) can be rewritten as

$$\sum_{\substack{i_1,\ldots,i_n\ge 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}\ \sum_{\substack{\Pi\in\mathcal{P}(\mathbb{N}_n)\\(N_a(\Pi)=i_a(\forall a))}}\tilde{c}_n(\Pi)Z^*_{\{1\}^n;\Pi}(T)
=n!(-1)^n\sum_{\substack{i_1,\ldots,i_n\ge 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}(-1)^{i_1+\cdots+i_n}\prod_{a=1}^{n}\frac{1}{i_a!}\left(\frac{Z^*_a(T)}{a}\right)^{i_a}. \tag{4-72}$$

By (3-80), $Z^*_a(T)$ equals T if $a=1$ and $\zeta_1(a)$ if $a>1$, and so combining (4-70) and (4-72) yields

$$\sum_{\substack{i_1,\dots,i_n\ge 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}\ \sum_{\substack{\Pi\in\mathcal{P}(\mathbb{N}_n)\\(N_a(\Pi)=i_a\,(\forall a))}} \tilde{c}_n(\Pi)Z^*_{\{1\}^n;\Pi}(T)=n!\widetilde{\gamma}_n(T). \tag{4-73}$$

It is obvious that $\mathcal{P}(\mathbb{N}_n)$ can be divided into disjoint subsets as follows:

$$\mathcal{P}(\mathbb{N}_n)=\bigsqcup_{\substack{i_1,\dots,i_n\ge 0\\(1\cdot i_1+\cdots+n\cdot i_n=n)}}\{\Pi\in\mathcal{P}(\mathbb{N}_n)\mid N_a(\Pi)=i_a\ (a=1,\dots,n)\}. \tag{4-74}$$

Thus, the left-hand sides of (4-69) and (4-73) are equal, and we obtain (4-69). □

We now prove Lemma 4.9.

Proof of Lemma 4.9. By (3-75) we have

$$\begin{aligned}A(u)^{-1}e^{Tu}&=\exp\biggl(-\sum_{m=2}^{\infty}\frac{(-1)^m\zeta_1(m)}{m}u^m\biggr)e^{Tu}\\&=\exp\biggl((-1)^2Tu+\sum_{m=2}^{\infty}\frac{(-1)^{m+1}\zeta_1(m)}{m}u^m\biggr),\end{aligned}$$

which, together with (4-68) for $u_1=(-1)^2T$ and $u_m=(-1)^{m+1}\zeta_1(m)$ $(m\ge 2)$, yields

$$A(u)^{-1}e^{Tu}=1+\sum_{n=1}^{\infty}u^n(-1)^n\sum_{\substack{i_1,i_2,\dots,i_n\ge 0\\(1\cdot i_1+2\cdot i_2+\cdots+n\cdot i_n=n)}}\frac{(-1)^{i_1+i_2+\cdots+i_n}}{i_1!\,i_2!\cdots i_n!}\,T^{i_1}\prod_{a=2}^{n}\biggl(\frac{\zeta(a)}{a}\biggr)^{i_a}.$$

Thus, by Lemma 4.10,

$$A(u)^{-1}e^{Tu}=1+\sum_{n=1}^{\infty}\frac{u^n}{n!}\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)}\tilde{c}_n(\Pi)Z^*_{\{1\}^n;\Pi}(T) \tag{4-75}$$

Since the renormalization map ρ is an $\mathbb{R}$-linear map from $\mathbb{R}[T]$ to $\mathbb{R}[T]$, we can see from (2-6) that the inverse ρ^{-1} is determined by

$$\sum_{n=0}^{\infty}\frac{u^n}{n!}\rho^{-1}(T^n)=\rho^{-1}\left(e^{Tu}\right)=\rho^{-1}\left(A(u)^{-1}\rho(e^{Tu})\right)=A(u)^{-1}e^{Tu}. \tag{4-76}$$

Equating (4-75) and (4-76), and comparing the coefficients of u^n $(n\ge 1)$, we obtain (4-57).

Let $i_1,\dots,i_n$ be nonnegative integers with $1\cdot i_1+\cdots+n\cdot i_n=n$, and let $\Pi=\{P_1,\dots,P_g\}$ be a partition in $\mathcal{P}(\mathbb{N}_n)$ with $g=i_1+\cdots+i_n$ and $N_a(\Pi)=i_a$

($a \in \mathbb{N}_n$). Noting (1-7) for $\dagger = \text{III}$, we can obtain

$$\begin{aligned}\tilde{c}_n(\Pi) Z^{\text{III}}_{\{1\}^n;\Pi}(T) &= (-1)^{n-g} \prod_{i=1}^{g} (|P_i|-1)!\,\chi^{\text{III}}(\{1\}^n;\, P_i) Z^{\text{III}}_{|P_i|}(T) \\ &= \begin{cases} (-1)^{n-g} \prod_{i=1}^{g} T & (|P_i| = 1 \text{ for all } i\ ), \\ 0 & (\exists\, i \ \text{ such that } |P_i| > 1\ ), \end{cases}\end{aligned}$$

and so

$$\tilde{c}_n(\Pi) Z^{\text{III}}_{\{1\}^n;\Pi}(T) = \begin{cases} T^n & \text{if } \Pi = \big\{\underbrace{\{1\},\ldots,\{1\}}_{n}\big\}, \\ 0 & \text{otherwise.} \end{cases}$$

Thus, by (4-74),

$$\begin{aligned}\sum_{\Pi\in\mathcal{P}(\mathbb{N}_n)} \tilde{c}_n(\Pi) Z^{\text{III}}_{\{1\}^n;\Pi}(T) &= \sum_{\substack{i_1,\ldots,i_n\ge 0 \\ (1\cdot i_1+\cdots+n\cdot i_n=n)}} \sum_{\substack{\Pi\in\mathcal{P}(\mathbb{N}_n) \\ (N_a(\Pi)=i_a\,(\forall a))}} \tilde{c}_n(\Pi) Z^{\text{III}}_{\{1\}^n;\Pi}(T) \\ &= \tilde{c}_n(\Pi) Z^{\text{III}}_{\{1\}^n;\Pi}(T)\,\Big|_{\,\Pi=\big\{\underbrace{\{1\},\ldots,\{1\}}_{n}\big\}} \\ &= T^n,\end{aligned}$$

which proves (4-58). □

4.3. ***Proof of Corollary 1.3.*** Let $\mathcal{P}_n$ be the set of partitions of $\{1,\ldots,n\}$, i.e., $\mathcal{P}_n = \mathcal{P}(\mathbb{N}_n)$. Let $\mathcal{P}_{n;m}$ be the subset of $\mathcal{P}_n$ which consists of partitions $\Pi = \{P_1,\ldots,P_m\}$ such that the number of the parts is m. Note that $\mathcal{P}_n = \bigsqcup_{j=1}^{n} \mathcal{P}_{n;j}$, In what follows, we identify a partition

$$\Pi = \{\{n_1^{(1)},\ldots,n_{a_1}^{(1)}\},\ldots,\{n_1^{(m)},\ldots,n_{a_m}^{(m)}\}\}$$

with

$$n_1^{(1)}\ldots n_{a_1}^{(1)}|\ldots|n_1^{(m)}\ldots n_{a_m}^{(m)}.$$

For example,

$$\{\{1,2,3\}\} = 123, \quad \{\{1,2\},\{3\}\} = 12|3, \quad \text{and } \{\{1\},\{2\},\{3\}\} = 1|2|3.$$

Let n and n' be positive integers with $n < n'$. For convenience, we embed $\mathfrak{S}_n$ into $\mathfrak{S}_{n'}$ in the following way: a permutation

$$\begin{pmatrix} 1 & \ldots & n \\ j_1 & \ldots & j_n \end{pmatrix} \quad \text{of } \mathfrak{S}_n$$

is identified with the permutation

$$\begin{pmatrix} 1 & \ldots & n & n+1 & \ldots & n' \\ j_1 & \ldots & j_n & n+1 & \ldots & n' \end{pmatrix} \quad \text{of } \mathfrak{S}_{n'},$$

which fixes integers between $n+1$ and n'.

To prove Corollary 1.3, we require the following three lemmas, which state that certain sums of values $\zeta^{\dagger}_{(n_1,\dots,n_j)}(\mathbf{l}_n)$ can be written in terms of values $\zeta^{\dagger}(\mathbf{l}_n;\Pi)$, for depths 2, 3, and 4. We assume that $\mathbf{l}_n=(l_1,\dots,l_n)\in\mathbb{N}^n$ and $\dagger\in\{*,ш\}$ in the lemmas.

Lemma 4.11 (case of depth 2).

$$\zeta^{\dagger}_{(1,1)}(\mathbf{l}_2)=\sum_{\Pi\in\mathcal{P}_{2;2}}\zeta^{\dagger}(\mathbf{l}_2;\Pi),\tag{4-77}$$

$$(\chi^{\dagger}_2\cdot\zeta_1\circ w_2)(\mathbf{l}_2)=\sum_{\Pi\in\mathcal{P}_{2;1}}\zeta^{\dagger}(\mathbf{l}_2;\Pi).\tag{4-78}$$

Lemma 4.12 (case of depth 3).

$$(\zeta^{\dagger}_{(1,1,1)}\mid\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=2\sum_{\Pi\in\mathcal{P}_{3;3}}\zeta^{\dagger}(\mathbf{l}_3;\Pi),\tag{4-79}$$

$$(\zeta^{\dagger}_{(2,1)}\mid\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=3\sum_{\Pi\in\mathcal{P}_{3;3}}\zeta^{\dagger}(\mathbf{l}_3;\Pi)-\sum_{\Pi\in\mathcal{P}_{3;2}}\zeta^{\dagger}(\mathbf{l}_3;\Pi),\tag{4-80}$$

$$(\chi^{\dagger}_3\cdot\zeta_1\circ w_3\mid\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3)=2\sum_{\Pi\in\mathcal{P}_{3;1}}\zeta^{\dagger}(\mathbf{l}_3;\Pi).\tag{4-81}$$

Lemma 4.13 (case of depth 4).

$$(\zeta^{\dagger}_{(1,1,1,1)}|\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi),\tag{4-82}$$

$$(\zeta^{\dagger}_{(2,1,1)}|\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=12\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-2\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi),\tag{4-83}$$

$$(\zeta^{\dagger}_{(2,2)}|\Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=3\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)+\sum_{\Pi\in\mathcal{P}^{(2,2)}_{4;2}}\zeta^{\dagger}(\mathbf{l}_4;\Pi),\tag{4-84}$$

$$(\zeta^{\dagger}_{(3,1)}|\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=4\sum_{\Pi\in\mathcal{P}_{4;4}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)-2\sum_{\Pi\in\mathcal{P}_{4;3}}\zeta^{\dagger}(\mathbf{l}_4;\Pi)+2\sum_{\Pi\in\mathcal{P}^{(3,1)}_{4;2}}\zeta^{\dagger}(\mathbf{l}_4;\Pi),\tag{4-85}$$

$$(\chi^{\dagger}_4\cdot\zeta_1\circ w_4|\Sigma_{\mathfrak{S}_3})(\mathbf{l}_4)=6\sum_{\Pi\in\mathcal{P}_{4;1}}\zeta^{\dagger}(\mathbf{l}_4;\Pi),\tag{4-86}$$

where $\mathcal{P}^{(2,2)}_{4;2}$ and $\mathcal{P}^{(3,1)}_{4;2}$ in (4-84) and (4-85) are subsets in $\mathcal{P}_{4;2}$ defined by

$$\mathcal{P}^{(2,2)}_{4;2}:=\{12|34,\ 13|24,\ 14|23\}\quad\textit{and}\quad\mathcal{P}^{(3,1)}_{4;2}:=\{123|4,\ 124|3,\ 134|2,\ 234|1\},$$

respectively. Note that $\mathcal{P}_{4;2}=\mathcal{P}^{(2,2)}_{4;2}\cup\mathcal{P}^{(3,1)}_{4;2}$.

We will prove Corollary 1.3 before discussing proofs of Lemmas 4.11, 4.12, and 4.13. We will divide the proof of Corollary 1.3 into three for the cases of $n = 2$, 3, and 4.

Proof of Corollary 1.3 for $n = 2$. Substituting (4-77) and (4-78) into the right-hand side of (1-2) yields

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_2) = \sum_{\Pi \in \mathcal{P}_{2;2}} \zeta^\dagger(\mathbf{l}_2; \Pi) - \sum_{\Pi \in \mathcal{P}_{2;1}} \zeta^\dagger(\mathbf{l}_2; \Pi),$$

since $\mathfrak{C}_2 = \mathfrak{S}_2$ and

$$\chi_2^\dagger(\mathbf{l}_2)\zeta_1(L_2) = \chi_2^\dagger(\mathbf{l}_2)\zeta_1(l_1 + l_2) = \chi_2^\dagger(\mathbf{l}_2) \cdot \zeta_1 \circ w_2(\mathbf{l}_2) = (\chi_2^\dagger \cdot \zeta_1 \circ w_2)(\mathbf{l}_2).$$

We have by definition (see (1-5))

$$\tilde{c}_2(\Pi) = \begin{cases} 1 & (\Pi \in \mathcal{P}_{2;2}), \\ -1 & (\Pi \in \mathcal{P}_{2;1}), \end{cases}$$

and thus we obtain by $\mathcal{P}_2 = \bigcup_{m=1}^{2} \mathcal{P}_{2;m}$

$$(\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_2) = \sum_{\Pi \in \mathcal{P}_2} \tilde{c}_2(\Pi)\zeta^\dagger(\mathbf{l}_2; \Pi),$$

which proves (1-8) for $n = 2$. □

Proof of Corollary 1.3 for $n = 3$. Applying $\Sigma_{\mathfrak{S}_2}$ to both sides of (1-3), we obtain

$$(4\text{-}87)\quad \begin{aligned} &(\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) \\ &\quad = -(\zeta_{(1,1,1)}^\dagger \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) + (\zeta_{(2,1)}^\dagger \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) + (\chi_3^\dagger \cdot \zeta_1 \circ w_3 \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3), \end{aligned}$$

where we have used $\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2}$ on the left-hand side of (4-87). Substituting (4-79), (4-80), and (4-81) into the right-hand side of (4-87) yields

$$(\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_3) = \sum_{\Pi \in \mathcal{P}_{3;3}} \zeta^\dagger(\mathbf{l}_3; \Pi) - \sum_{\Pi \in \mathcal{P}_{3;2}} \zeta^\dagger(\mathbf{l}_3; \Pi) + 2\sum_{\Pi \in \mathcal{P}_{3;1}} \zeta^\dagger(\mathbf{l}_3; \Pi),$$

which proves (1-8) for $n = 3$, since

$$\tilde{c}_3(\Pi) = \begin{cases} 1 & (\Pi \in \mathcal{P}_{3;3}), \\ -1 & (\Pi \in \mathcal{P}_{3;2}), \\ 2 & (\Pi \in \mathcal{P}_{3;1}), \end{cases}$$

and $\mathcal{P}_3 = \bigcup_{m=1}^{3} \mathcal{P}_{3;m}$. □

Proof of Corollary 1.3 for $n = 4$. We can see from the fourth and sixth equations in (4-19) that $(14)\Sigma_{\mathfrak{C}_4} = (123)\Sigma_{\mathfrak{C}_4}$ and $(34)\Sigma_{\mathfrak{C}_4} = (132)\Sigma_{\mathfrak{C}_4}$, respectively, and so it follows from (4-20) that $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4}$. Taking the inverses of both sides of this equation gives $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}$. Thus, applying $\Sigma_{\mathfrak{S}_3}$ to both sides of (1-4) and combining the identities in Lemma 4.13 (or considering (4-82) − (4-83) + (4-84) + (4-85) − (4-86)), we can obtain

$$\begin{aligned}(\zeta_4^\dagger \mid \Sigma_{\mathfrak{S}_4})(\mathbf{l}_4) &= \sum_{\Pi\in\mathcal{P}_{4;4}} \zeta^\dagger(\mathbf{l}_4;\Pi) - \sum_{\Pi\in\mathcal{P}_{4;3}} \zeta^\dagger(\mathbf{l}_4;\Pi) + \sum_{\Pi\in\mathcal{P}_{4;2}^{(2,2)}} \zeta^\dagger(\mathbf{l}_4;\Pi) + 2\sum_{\Pi\in\mathcal{P}_{4;2}^{(3,1)}} \zeta^\dagger(\mathbf{l}_4;\Pi) \\ &\qquad -6\sum_{\Pi\in\mathcal{P}_{4;1}} \zeta^\dagger(\mathbf{l}_4;\Pi),\end{aligned}$$

which proves (1-8) for $n=4$, since

$$\tilde{c}_4(\Pi) = \begin{cases} 1 & (\Pi\in\mathcal{P}_{4;4}), \\ -1 & (\Pi\in\mathcal{P}_{4;3}), \\ 1 & (\Pi\in\mathcal{P}_{4;2}^{(2,2)}), \\ 2 & (\Pi\in\mathcal{P}_{4;2}^{(3,1)}), \\ -6 & (\Pi\in\mathcal{P}_{4;1}), \end{cases}$$

$\mathcal{P}_{4;2} = \mathcal{P}_{4;2}^{(2,2)} \cup \mathcal{P}_{4;2}^{(3,1)}$ and $\mathcal{P}_4 = \bigcup_{m=1}^4 \mathcal{P}_{4;m}$. □

We see from (1-1) that $\chi_1^\dagger(k)=1$ for any positive integer k, and so

$$\chi_1^\dagger(k)\zeta_1^\dagger(k) = \zeta_1^\dagger(k).$$

Note that $\zeta_1(k)=\zeta_1^\dagger(k)$ for $k\geq 2$. These facts will be used repeatedly below.

We now give proofs of Lemmas 4.11, 4.12, and 4.13.

Proof of Lemma 4.11. We have $\mathcal{P}_{2;2}=\{1|2\}$ and $\mathcal{P}_{2;1}=\{12\}$ by definition. Thus,

$$\sum_{\Pi\in\mathcal{P}_{2;2}} \zeta^\dagger(\mathbf{l}_2;\Pi) = \chi_1^\dagger(l_1)\zeta_1^\dagger(l_1)\chi_1^\dagger(l_2)\zeta_1^\dagger(l_2) = \zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2) = \zeta_{(1,1)}^\dagger(\mathbf{l}_2),$$

$$\sum_{\Pi\in\mathcal{P}_{2;1}} \zeta^\dagger(\mathbf{l}_2;\Pi) = \chi_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_1+l_2) = \chi_2^\dagger(l_1,l_2)\zeta_1(l_1+l_2) = (\chi_2^\dagger\cdot\zeta_1\circ w_2)(\mathbf{l}_2),$$

which prove (4-77) and (4-78), respectively. □

Proof of Lemma 4.12. We have

$$\mathcal{P}_{3;3}=\{1|2|3\},\quad \mathcal{P}_{3;2}=\{12|3,\,13|2,\,23|1\},\quad \mathcal{P}_{3;1}=\{123\},$$

by definition. In particular, $\mathcal{P}_{3;2}$ is expressed as $\bigcup_{\sigma\in\mathfrak{C}_3}\{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)\}$, and so

$$\sum_{\Pi\in\mathcal{P}_{3;2}} \zeta^\dagger(\mathbf{l}_3;\Pi) = \sum_{\sigma\in\mathfrak{C}_3} \chi_2^\dagger(l_{\sigma^{-1}(1)},l_{\sigma^{-1}(2)})\zeta_1^\dagger(l_{\sigma^{-1}(1)}+l_{\sigma^{-1}(2)})\chi_1^\dagger(l_{\sigma^{-1}(3)})\zeta_1^\dagger(l_{\sigma^{-1}(3)})$$

$$= \sum_{\sigma \in \mathfrak{C}_3} \chi_2^\dagger(l_{\sigma^{-1}(1)}, l_{\sigma^{-1}(2)}) \zeta_1(l_{\sigma^{-1}(1)} + l_{\sigma^{-1}(2)}) \zeta_1^\dagger(l_{\sigma^{-1}(3)}).$$

Thus, we can obtain

$$\sum_{\Pi \in \mathcal{P}_{3;3}} \zeta^\dagger(\mathbf{l}_3; \Pi) = \zeta^\dagger_{(1,1,1)}(\mathbf{l}_3), \tag{4-88}$$

$$\sum_{\Pi \in \mathcal{P}_{3;2}} \zeta^\dagger(\mathbf{l}_3; \Pi) = ((\chi_2^\dagger \cdot \zeta_1 \circ w_2) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_3), \tag{4-89}$$

$$\sum_{\Pi \in \mathcal{P}_{3;1}} \zeta^\dagger(\mathbf{l}_3; \Pi) = (\chi_3^\dagger \cdot \zeta_1 \circ w_3)(\mathbf{l}_3). \tag{4-90}$$

Since $\zeta^\dagger_{(1,1,1)}$ is invariant under $\mathfrak{S}_3$, we have $(\zeta^\dagger_{(1,1,1)} \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) = 2\zeta^\dagger_{(1,1,1)}(\mathbf{l}_3)$, which together with (4-88) proves (4-79). Similarly, we have $(\chi_3^\dagger \cdot \zeta_1 \circ w_3 \mid \Sigma_{\mathfrak{S}_2})(\mathbf{l}_3) = 2\chi_3^\dagger \cdot \zeta_1 \circ w_3(\mathbf{l}_3)$, which together with (4-90) proves (4-81). We know from (1-2) that

$$\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2} = \zeta^\dagger_{(1,1)} - \chi_2^\dagger \cdot \zeta_1 \circ w_2, \tag{4-91}$$

and so

$$\begin{aligned} \zeta^\dagger_{(2,1)} \mid \Sigma_{\mathfrak{S}_2} &= \zeta_2^\dagger \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{S}_2} \\ &= (\zeta_2^\dagger \mid \Sigma_{\mathfrak{S}_2}) \otimes \zeta_1^\dagger \\ &= (\zeta^\dagger_{(1,1)} - \chi_2^\dagger \cdot \zeta_1 \circ w_2) \otimes \zeta_1^\dagger \\ &= \zeta^\dagger_{(1,1,1)} - (\chi_2^\dagger \cdot \zeta_1 \circ w_2) \otimes \zeta_1^\dagger. \end{aligned}$$

Since $\mathfrak{C}_3\mathfrak{S}_2 = \mathfrak{S}_2\mathfrak{C}_3$, we have

$$\begin{aligned} \zeta^\dagger_{(2,1)} \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{S}_2} &= (\zeta^\dagger_{(2,1)} \mid \Sigma_{\mathfrak{S}_2}) | \Sigma_{\mathfrak{C}_3} \\ &= (\zeta^\dagger_{(1,1,1)} - (\chi_2^\dagger \cdot \zeta_1 \circ w_2) \otimes \zeta_1^\dagger) \mid \Sigma_{\mathfrak{C}_3} \\ &= 3\zeta^\dagger_{(1,1,1)} - (\chi_2^\dagger \cdot \zeta_1 \circ w_2) \otimes \zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_3}, \end{aligned}$$

which together with (4-88) and (4-89) proves (4-80). This completes the proof. $\square$

Proof of Lemma 4.13. Let $\mathfrak{A}_4^0$ be the subset of $\mathfrak{A}_4$ given by

$$\mathfrak{A}_4^0 = \{e, (13)(24), (123), (132), (142), (234)\} = \langle (13)(24) \rangle \mathfrak{C}_3.$$

Note that $\Sigma_{\mathfrak{A}_4} = \Sigma_{\langle (12)(34) \rangle} \Sigma_{\mathfrak{A}_4^0}$. From the definitions of $\mathcal{P}_{4;m}$ and $\mathcal{P}_{4;2}^{(i,j)}$ and some straightforward calculations, we can see that

$$\mathcal{P}_{4;4} = \{1|2|3|4\},$$

$$\mathcal{P}_{4;3} = \bigcup_{\sigma\in\mathfrak{A}_4^0} \{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)|\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;2}^{(2,2)} = \bigcup_{\sigma\in\mathfrak{C}_3} \{\sigma^{-1}(1)\sigma^{-1}(2)|\sigma^{-1}(3)\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;2}^{(3,1)} = \bigcup_{\sigma\in\mathfrak{C}_4} \{\sigma^{-1}(1)\sigma^{-1}(2)\sigma^{-1}(3)|\sigma^{-1}(4)\},$$

$$\mathcal{P}_{4;1} = \{1234\}.$$

Thus, we can obtain

$$\sum_{\Pi\in\mathcal{P}_{4;4}} \zeta^{\dagger}(\mathbf{l}_4;\Pi) = \zeta^{\dagger}_{(1,1,1,1)}(\mathbf{l}_4), \tag{4-92}$$

$$\sum_{\Pi\in\mathcal{P}_{4;3}} \zeta^{\dagger}(\mathbf{l}_4;\Pi) = ((\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0})(\mathbf{l}_4), \tag{4-93}$$

$$\sum_{\Pi\in\mathcal{P}_{4;2}^{(2,2)}} \zeta^{\dagger}(\mathbf{l}_4;\Pi) = ((\chi_2^{\dagger}\cdot\zeta_1\circ w_2)^{\otimes 2} \mid \Sigma_{\mathfrak{C}_3})(\mathbf{l}_4), \tag{4-94}$$

$$\sum_{\Pi\in\mathcal{P}_{4;2}^{(3,1)}} \zeta^{\dagger}(\mathbf{l}_4;\Pi) = ((\chi_3^{\dagger}\cdot\zeta_1\circ w_3)\otimes\zeta_1^{\dagger} \mid \Sigma_{\mathfrak{C}_4})(\mathbf{l}_4), \tag{4-95}$$

$$\sum_{\Pi\in\mathcal{P}_{4;1}} \zeta^{\dagger}(\mathbf{l}_4;\Pi) = (\chi_4^{\dagger}\cdot\zeta_1\circ w_4)(\mathbf{l}_4). \tag{4-96}$$

Since $\zeta^{\dagger}_{(1,1,1,1)}$ and $\chi_4^{\dagger}\cdot\zeta_1\circ w_4$ are invariant under $\mathfrak{S}_4$, we have

$$(\zeta^{\dagger}_{(1,1,1,1)} \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_4) = 6\zeta^{\dagger}_{(1,1,1,1)}(\mathbf{l}_4) \quad\text{and}\quad (\chi_4^{\dagger}\cdot\zeta_1\circ w_4 \mid \Sigma_{\mathfrak{S}_3})(\mathbf{l}_4) = 6\chi_4^{\dagger}\cdot\zeta_1\circ w_4(\mathbf{l}_4),$$

which together with (4-92) and (4-96) prove (4-82) and (4-86), respectively.

We now prove (4-83). A direct calculation shows that $\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{S}_2}\Sigma_{\mathfrak{A}_4}$, and so, by (4-91),

$$\begin{aligned} \zeta^{\dagger}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} &= (\zeta^{\dagger}_{(2,1,1)} \mid \Sigma_{\mathfrak{S}_2}) | \Sigma_{\mathfrak{A}_4} \\ &= (\zeta_2^{\dagger} \mid \Sigma_{\mathfrak{S}_2})\otimes\zeta^{\dagger}_{(1,1)} | \Sigma_{\mathfrak{A}_4} \\ &= (\zeta^{\dagger}_{(1,1,1,1)} - (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)}) \mid \Sigma_{\mathfrak{A}_4} \\ &= 12\zeta^{\dagger}_{(1,1,1,1)} - (\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4}. \end{aligned} \tag{4-97}$$

Since $\Sigma_{\mathfrak{A}_4} = \Sigma_{\langle(12)(34)\rangle}\Sigma_{\mathfrak{A}_4^0}$ and $(\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)}$ is invariant under $\langle(12)(34)\rangle$,

$$(\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4} = 2(\chi_2^{\dagger}\cdot\zeta_1\circ w_2)\otimes\zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}_4^0}. \tag{4-98}$$

Combining (4-97) and (4-98), we obtain

$$\zeta^{\dagger}_{(2,1,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = 12\zeta^{\dagger}_{(1,1,1,1)} - 2(\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}^0_4},$$

which, together with (4-92) and (4-93), proves (4-83).

We now prove (4-84). For this, we require the identity

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3} = (\zeta^{\dagger}_2 \mid \Sigma_{\langle(12)\rangle} \otimes \zeta^{\dagger}_2 \mid \Sigma_{\langle(34)\rangle})|\Sigma_{\mathfrak{C}_3}, \tag{4-99}$$

which can be verified as follows. A direct calculation shows that

$$\begin{aligned}\Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} &= (e+(12)+(34)+(12)(34))(e+(123)+(132)) \\ &= e+(12)+(13)+(23)+(34)+(12)(34) \\ &\qquad +(123)+(132)+(143)+(243)+(1243)+(1432).\end{aligned}$$

From this equation and the equivalence classes modulo $\langle(13)(24)\rangle$ in Table 1, we see that

$$\Sigma_{\mathfrak{S}_4} \equiv 2\Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} \quad \bmod \langle(13)(24)\rangle.$$

We also see from $\Sigma_{\mathfrak{S}_4} = \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}$ and (4-26) that $\Sigma_{\mathfrak{S}_4} \equiv 2\Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3} \bmod \langle(13)(24)\rangle$, and so

$$\Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3} \equiv \Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} \quad \bmod \langle(13)(24)\rangle.$$

Since $\zeta^{\dagger}_{(2,2)}$ is invariant under $\langle(13)(24)\rangle$ and since $\Sigma_{\langle(12),(34)\rangle} = \Sigma_{\langle(12)\rangle}\Sigma_{\langle(34)\rangle}$,

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3} = \zeta^{\dagger}_{(2,2)} \mid \Sigma_{\langle(12),(34)\rangle}\Sigma_{\mathfrak{C}_3} \;=\; (\zeta^{\dagger}_2 \mid \Sigma_{\langle(12)\rangle} \otimes \zeta^{\dagger}_2 \mid \Sigma_{\langle(34)\rangle})|\Sigma_{\mathfrak{C}_3},$$

which verifies (4-99). Then, by (4-91), the right-hand side of (4-99) can be calculated as

(RHS of (4-99))

$$\begin{aligned}&= (\zeta^{\dagger}_{(1,1)} - \chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) \otimes (\zeta^{\dagger}_{(1,1)} - \chi^{\dagger}_2 \cdot \zeta_1 \circ w_2)|\Sigma_{\mathfrak{C}_3} \\ &= \{\zeta^{\dagger}_{(1,1,1,1)} - (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{\dagger}_{(1,1)} - \zeta^{\dagger}_{(1,1)} \otimes (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) + (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2)^{\otimes 2}\} \mid \Sigma_{\mathfrak{C}_3} \\ &= 3\zeta^{\dagger}_{(1,1,1,1)} - (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{\dagger}_{(1,1)} \mid \Sigma_{\langle(13)(24)\rangle}\Sigma_{\mathfrak{C}_3} + (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2)^{\otimes 2} \mid \Sigma_{\mathfrak{C}_3}.\end{aligned}$$

It holds that $\Sigma_{\langle(13)(24)\rangle}\Sigma_{\mathfrak{C}_3} = \Sigma_{\mathfrak{A}^0_4}$, and so (4-99) can be restated as

$$\zeta^{\dagger}_{(2,2)} \mid \Sigma_{\mathfrak{C}^0_4}\Sigma_{\mathfrak{S}_3} = 3\zeta^{\dagger}_{(1,1,1,1)} - (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2) \otimes \zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}^0_4} + (\chi^{\dagger}_2 \cdot \zeta_1 \circ w_2)^{\otimes 2} \mid \Sigma_{\mathfrak{C}_3},$$

which, together with (4-92), (4-93), and (4-94), proves (4-84).

We lastly prove (4-85) in a similar way to (4-84). We require the identity

$$\begin{aligned}&\zeta^{\dagger}_{(3,1)} \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} \\ &\quad = 4\zeta^{\dagger}_{(1,1,1,1)}\zeta_1^{\dagger\otimes 4} - 2(\chi^{\dagger}_2 \cdot \zeta^{\dagger}_2 \circ w_2) \otimes \zeta^{\dagger}_{(1,1)} \mid \Sigma_{\mathfrak{A}^0_4} + 2(\chi^{\dagger}_3 \cdot \zeta_1 \circ w_3) \otimes \zeta^{\dagger}_1 \mid \Sigma_{\mathfrak{C}_4},\end{aligned} \tag{4-100}$$

which can be verified as follows. Identity (1-8) for $n=3$ can be restated as

$$\zeta_3^\dagger \mid \Sigma_{\mathfrak{S}_3} = \zeta_{(1,1,1)}^\dagger - (\chi_2^\dagger\cdot\zeta_1^\dagger\circ w_2)\otimes\zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_3} + 2\chi_3^\dagger\cdot\zeta_1\circ w_3, \tag{4-101}$$

because of (4-88), (4-89), and (4-90). A direct calculation shows that

$$\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} = e+(14)+(34)+(13)(24)+(123)+(124)+(132)+(243) \\ +(1234)+(1342)+(1423)+(1432),$$

and so we see from the equivalence classes modulo $\langle(12),(34)\rangle$ in Table 1 that

$$\Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} \equiv 2\Sigma_{\mathfrak{A}_4^0} \mod \langle(12),(34)\rangle.$$

Since $\Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3} = \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4}$, we thus have

$$\begin{aligned}
&\zeta_{(3,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4}\Sigma_{\mathfrak{S}_3}\\
&\quad= \zeta_3^\dagger\otimes\zeta_1^\dagger \mid \Sigma_{\mathfrak{S}_3}\Sigma_{\mathfrak{C}_4} = (\zeta_3^\dagger|\Sigma_{\mathfrak{S}_3})\otimes\zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4}\\
&\quad\overset{(4\text{-}101)}{=} \zeta_{(1,1,1,1)}^\dagger \mid \Sigma_{\mathfrak{C}_4} - (\chi_2^\dagger\cdot\zeta_1^\dagger\circ w_2)\otimes\zeta_{(1,1)}^\dagger \mid \Sigma_{\mathfrak{C}_3}\Sigma_{\mathfrak{C}_4} + 2(\chi_3^\dagger\cdot\zeta_1\circ w_3)\otimes\zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4}\\
&\quad= 4\zeta_{(1,1,1,1)}^\dagger - 2(\chi_2^\dagger\cdot\zeta_2^\dagger\circ w_2)\otimes\zeta_{(1,1)}^\dagger \mid \Sigma_{\mathfrak{A}_4^0} + 2(\chi_3^\dagger\cdot\zeta_1\circ w_3)\otimes\zeta_1^\dagger \mid \Sigma_{\mathfrak{C}_4},
\end{aligned}$$

which verifies (4-100). Then, combining (4-92), (4-93), (4-95), and (4-100), we obtain (4-85), which completes the proof. □

Remark 4.14. We can find that (1-2) and (1-3) are used to show Lemma 4.12; this lemma is required for the proof of (1-8) for $n=3$. Thus, not only (1-3) but also (1-2) are necessary to prove (1-8) for $n=3$. Similarly, we can find that (1-2), (1-3), and (1-4) are used to show Lemma 4.13, and thus not only (1-4) but also (1-2) and (1-3) are necessary to prove (1-8) for $n=4$.

5. Examples

We list examples of (1-3) and (1-4) in Table 2 and Table 4, respectively. We also list examples of (1-8) for $n=3$ and $n=4$ in Table 3 and Table 5, respectively, for comparison. The examples treat the case of weight less than 7. We omit examples of (1-2) and (1-8) for $n=2$ because they are essentially the harmonic relations. The following straightforward expressions of (1-3) and (1-4) are convenient for calculating the examples in Table 2 and Table 4:

$$\begin{aligned}
&\zeta_3^\dagger(l_1,l_2,l_3)+\zeta_3^\dagger(l_2,l_3,l_1)+\zeta_3^\dagger(l_3,l_1,l_2)\\
&\qquad= -\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\\
&\qquad\quad+\zeta_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_3)+\zeta_2^\dagger(l_2,l_3)\zeta_1^\dagger(l_1)+\zeta_2^\dagger(l_3,l_1)\zeta_1^\dagger(l_2)\\
&\qquad\quad+\chi_3^\dagger(l_1,l_2,l_3)\zeta_1(l_1+l_2+l_3),
\end{aligned} \tag{5-1}$$

Index set	Linear relation	
(1,1,1)	$3\zeta_3^\dagger(1,1,1)=\chi_3^\dagger(1,1,1)\zeta_1(3)$	(d3-1)
(1,1,2)	$\zeta_3^\dagger(1,1,2)+\zeta_3^\dagger(1,2,1)+\zeta_3(2,1,1)=\zeta_2^\dagger(1,1)\zeta_1(2)+\zeta_1(4)$	(d3-2)
(1,1,3)	$\zeta_3^\dagger(1,1,3)+\zeta_3^\dagger(1,3,1)+\zeta_3(3,1,1)=\zeta_2^\dagger(1,1)\zeta_1(3)+\zeta_1(5)$	(d3-3)
(1,2,2)	$\zeta_3^\dagger(1,2,2)+\zeta_3(2,2,1)+\zeta_3(2,1,2)=-\zeta_1(2)\zeta_1(3)+\zeta_1(5)$	(d3-4)
(1,1,4)	$\zeta_3^\dagger(1,1,4)+\zeta_3^\dagger(1,4,1)+\zeta_3(4,1,1)=\zeta_2^\dagger(1,1)\zeta_1(4)+\zeta_1(6)$	(d3-5)
(1,2,3)	$\zeta_3^\dagger(1,2,3)+\zeta_3(2,3,1)+\zeta_3(3,1,2)$ $=\zeta_2^\dagger(1,2)\zeta_1(3)+\zeta_2(3,1)\zeta_1(2)+\zeta_1(6)$	(d3-6)
(1,3,2)	$\zeta_3^\dagger(1,3,2)+\zeta_3(3,2,1)+\zeta_3(2,1,3)$ $=\zeta_2^\dagger(1,3)\zeta_1(2)+\zeta_2(2,1)\zeta_1(3)+\zeta_1(6)$	(d3-7)
(2,2,2)	$3\zeta_3(2,2,2)=-\zeta_1(2)^3+3\zeta_2(2,2)\zeta_1(2)+\zeta_1(6)$	(d3-8)

Table 2. Examples of (1-3) (or (5-1)).

(5-2) $$\begin{aligned}\zeta_4^\dagger(l_1,l_2,l_3,l_4)+\zeta_4^\dagger(l_2,l_3,l_4,l_1)+\zeta_4^\dagger(l_3,l_4,l_1,l_2)+\zeta_4^\dagger(l_4,l_1,l_2,l_3)\\ =\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\zeta_1^\dagger(l_4)\\ -\zeta_2^\dagger(l_1,l_2)\zeta_1^\dagger(l_3)\zeta_1^\dagger(l_4)-\zeta_2^\dagger(l_2,l_3)\zeta_1^\dagger(l_4)\zeta_1^\dagger(l_1)\\ -\zeta_2^\dagger(l_3,l_4)\zeta_1^\dagger(l_1)\zeta_1^\dagger(l_2)-\zeta_2^\dagger(l_4,l_1)\zeta_1^\dagger(l_2)\zeta_1^\dagger(l_3)\\ +\zeta_2^\dagger(l_1,l_2)\zeta_2^\dagger(l_3,l_4)+\zeta_2^\dagger(l_2,l_3)\zeta_2^\dagger(l_4,l_1)\\ +\zeta_3^\dagger(l_1,l_2,l_3)\zeta_1^\dagger(l_4)+\zeta_3^\dagger(l_2,l_3,l_4)\zeta_1^\dagger(l_1)\\ +\zeta_3^\dagger(l_3,l_4,l_1)\zeta_1^\dagger(l_2)+\zeta_3^\dagger(l_4,l_1,l_2)\zeta_1^\dagger(l_3)\\ -\chi_4^\dagger(l_1,l_2,l_3,l_4)\zeta_1(l_1+l_2+l_3+l_4).\end{aligned}$$

We have used $\zeta_1^\dagger(1)=0$ for all equations in the tables, and $\zeta_2^\dagger(1,k)+\zeta_2^\dagger(k,1)=-\zeta_1(k+1)$ $(k>1)$ for (d3-4), (d4-2), and (d4-3).

As was mentioned in Section 1, it holds that

(5-3) $$\zeta_2^{ш}(1,1)=0 \qquad\text{and}\qquad \zeta_2^*(1,1)=-\tfrac{1}{2}\zeta_1(2),$$

which follows from (3-14) with $l_1=l_2=1$, or $2z_1z_1=z_1*z_1-z_2$. In fact, applying Z^* to both sides of this equation, we obtain $2Z^*_{1,1}(T)=T^2-\zeta_1(2)$, which together with (2-8) and (3-64) gives (5-3). Lastly, we derive the following equations from (d3-1) and (d4-1) as applications of examples:

(5-4) $$\zeta_3^{ш}(1,1,1)=\zeta_4^{ш}(1,1,1,1)=0,$$

(5-5) $$\zeta_3^*(1,1,1)=\tfrac{1}{3}\zeta_1(3),$$

(5-6) $$\zeta_4^*(1,1,1,1)=\tfrac{1}{16}\zeta_1(4).$$

Index set	Linear relation	
(1,1,1)	$6\zeta_3^\dagger(1,1,1) = 2\chi_3^\dagger(1,1,1)\zeta_1(3)$	(d3′-1)
(1,1,2)	$2(\zeta_3^\dagger(1,1,2)+\zeta_3^\dagger(1,2,1)+\zeta_3(2,1,1))$ $= -\chi_2^\dagger(1,1)\zeta_1(2)^2 + 2\zeta_1(4)$	(d3′-2)
(1,1,3)	$2(\zeta_3^\dagger(1,1,3)+\zeta_3^\dagger(1,3,1)+\zeta_3(3,1,1))$ $= -\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(3) + 2\zeta_1(5)$	(d3′-3)
(1,2,2)	$2(\zeta_3^\dagger(1,2,2)+\zeta_3(2,2,1)+\zeta_3(2,1,2))$ $= -2\zeta_1(2)\zeta_1(3) + 2\zeta_1(5)$	(d3′-4)
(1,1,4)	$2(\zeta_3^\dagger(1,1,4)+\zeta_3^\dagger(1,4,1)+\zeta_3(4,1,1))$ $= -\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(4) + 2\zeta_1(6)$	(d3′-5)
(1,2,3)	$\zeta_3^\dagger(1,2,3)+\zeta_3^\dagger(1,3,2)+\zeta_3(2,1,3)+\zeta_3(2,3,1)$ $+\zeta_3(3,1,2)+\zeta_3(3,2,1)$ $= -(\zeta_1(2)\zeta_1(4)+\zeta_1(3)^2) + 2\zeta_1(6)$	(d3′-6)
(2,2,2)	$6\zeta_3(2,2,2) = \zeta_1(2)^3 - 3\zeta_1(2)\zeta_1(4) + 2\zeta_1(6)$	(d3′-7)

Table 3. Examples of (1-8) for $n = 3$.

Index set	Linear relation	
(1,1,1,1)	$4\zeta_4^\dagger(1,1,1,1) = 2\zeta_2^\dagger(1,1)^2 - \chi_4^\dagger(1,1,1,1)\zeta_1(4)$	(d4-1)
(1,1,1,2)	$\zeta_4^\dagger(1,1,1,2)+\zeta_4^\dagger(1,1,2,1)+\zeta_4^\dagger(1,2,1,1)+\zeta_4(2,1,1,1)$ $= -\zeta_2^\dagger(1,1)\zeta_1(3) + \zeta_3^\dagger(1,1,1)\zeta_1(2) - \zeta_1(5)$	(d4-2)
(1,1,1,3)	$\zeta_4^\dagger(1,1,1,3)+\zeta_4^\dagger(1,1,3,1)+\zeta_4^\dagger(1,3,1,1)+\zeta_4(3,1,1,1)$ $= -\zeta_2^\dagger(1,1)\zeta_1(4) + \zeta_3^\dagger(1,1,1)\zeta_1(3) - \zeta_1(6)$	(d4-3)
(1,1,2,2)	$\zeta_4^\dagger(1,1,2,2)+\zeta_4(1,2,2,1)+\zeta_4^\dagger(2,2,1,1)+\zeta_4^\dagger(2,1,1,2)$ $= -\zeta_2^\dagger(1,1)\zeta_1(2)^2 + \zeta_2^\dagger(1,1)\zeta_2(2,2) + \zeta_2^\dagger(1,2)\zeta_2(2,1)$ $+(\zeta_3^\dagger(1,1,2)+\zeta_3(2,1,1))\zeta_1(2) - \zeta_1(6)$	(d4-4)
(1,2,1,2)	$2(\zeta_4^\dagger(1,2,1,2)+\zeta_4(2,1,2,1))$ $= \zeta_2^\dagger(1,2)^2 + \zeta_2(2,1)^2 + 2\zeta_3^\dagger(1,2,1)\zeta_1(2) - \zeta_1(6)$	(d4-5)

Table 4. Examples of (1-4) (or (5-2)).

We can easily obtain (5-4) from (d3-1) and (d4-1) for $\dagger = ш$, since

$$\chi_3^{ш}(1,1,1) = \chi_4^{ш}(1,1,1,1) = 0 \quad \text{and} \quad \zeta_2^{ш}(1,1) = 0.$$

We can also obtain (5-5) from (d3-1) for $\dagger = *$, since $\chi_3^*(1,1,1) = 1$. Since $\chi_4^*(1,1,1,1) = 1$ and $\zeta_2^*(1,1) = -\zeta_1(2)/2$, we obtain from (d4-1) for $\dagger = *$ that

$$4\zeta_4^*(1,1,1,1) = 2\zeta_2^*(1,1)^2 - \zeta_1(4) = \tfrac{1}{2}\zeta_1(2)^2 - \zeta_1(4),$$

which together with (3-72) proves (5-6).

Index set	Linear relation
(1,1,1,1)	$24\zeta_4^\dagger(1,1,1,1) = 3\chi_2^\dagger(1,1)\zeta_1(2)^2 - 6\chi_4^\dagger(1,1,1,1)\zeta_1(4)$ (d4′-1)
(1,1,1,2)	$6(\zeta_4^\dagger(1,1,1,2)+\zeta_4^\dagger(1,1,2,1)+\zeta_4^\dagger(1,2,1,1)+\zeta_4(2,1,1,1)) = 3\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(3)+2\chi_3^\dagger(1,1,1)\zeta_1(2)\zeta_1(3)-6\zeta_1(5)$ (d4′-2)
(1,1,1,3)	$6(\zeta_4^\dagger(1,1,1,3)+\zeta_4^\dagger(1,1,3,1)+\zeta_4^\dagger(1,3,1,1)+\zeta_4(3,1,1,1)) = 3\chi_2^\dagger(1,1)\zeta_1(2)\zeta_1(4)+2\chi_3^\dagger(1,1,1)\zeta_1(3)^2-6\zeta_1(6)$ (d4′-3)
(1,1,2,2)	$4(\zeta_4^\dagger(1,1,2,2)+\zeta_4(1,2,1,2)+\zeta_4(1,2,2,1)+\zeta_4^\dagger(2,1,1,2) +\zeta_4^\dagger(2,1,2,1)+\zeta_4^\dagger(2,2,1,1)) = -\chi_2^\dagger(1,1)\zeta_1(2)^3+(\chi_2^\dagger(1,1)+4)\zeta_1(2)\zeta_1(4) +2\zeta_1(3)^2-6\zeta_1(6)$ (d4′-4)

Table 5. Examples of (1-8) for $n = 4$.

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